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Symmetry structure of integrable non-evolutionary equations *

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Abstract

We study a class of evolutionary partial differential systems with two components related to second order (in time) non-evolutionary equations of odd order in spatial variable. We develop the formal diagonalisation method in symbolic representation, which enables us to derive an explicit set of necessary conditions of existence of higher symmetries. Using these conditions we globally classify all such homogeneous integrable systems, i.e. systems which possess a hierarchy of infinitely many higher symmetries.

Key words: Integrable PDEs, Symmetry approach, symbolic representation, formal diagonalisation.

1 Introduction.

Classification of integrable equations of a given family of nonlinear equations is an important topic in the field of soliton theory. There are many approaches to this problem, among which the symmetry approach has proved to be very efficient and powerful method. The symmetry approach has been used to classify large classes of integrable nonlinear partial differential equations and difference-differential equations such as scalar evolutionary equations, Volterra and Toda type equations, hyperbolic equations etc. We refer the reader to the recent review paper [1] and the references therein for details.

This paper is devoted to a problem of classification of systems of integrable evolutionary equations with two components. The first work in this direction had been carried out by Mikhailov, Shabat and Yamilov [2, 3, 4]. They classified integrable second order systems of the form

$$\mathbf{u}_t = \mathbf{A}(\mathbf{u})\mathbf{u}_{xx} + F(\mathbf{u}, \mathbf{u}_x), \quad \det(A(\mathbf{u})) \neq 0, \quad \mathbf{u} = (u, v)^T,$$

where the 2×2 matrix $A(\mathbf{u})$ can be reduced to

$$A(\mathbf{u}) = \text{diag}(a, b), \quad a = -1, \quad b = 1,$$

if the system possesses higher order conservation laws. Later, Svinolupov [5] studied the Burgers type equations, i.e. the case $a = b = 1$ (see also the review paper [6]). After these two special cases, attention was turned to the more general case: $a \neq \pm b$ and $ab \neq 0$ [7] and to the systems of higher order. In principle, the symmetry approach can be applied to classification of higher order systems. Although the method is algorithmic, the computation involved grows very rapidly with the order of the system, so further achievement in this direction had to wait for the development of the appropriate computer algebra.

Recently, several research groups have developed the computer programmes to carry out the task of classification of polynomial integrable evolutionary systems with two components. Foursov and Olver considered the symmetrically-coupled evolutionary equations of lower order [8]. After linear transformation, the leading linear part of the systems can always be diagonalised. Tsuchida and Wolf studied polynomial homogeneous integrable evolutionary systems of mixed scalar and vector dependent variables [9] of order 2 and 3, where the authors compared their comprehensive results with the existing ones. The linear parts of systems they considered are diagonal including zero eigenvalues. Their classification methods have led to interesting new integrable equations, but they could not claim that their lists are complete in the sense that there are no other integrable systems of the fixed order in certain classes since the methods they used were based on searching symmetries of specific orders and forms. The symbolic representation combined with applications of number theory [10] is the solution to this problem at least for subclass of homogeneous

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polynomial evolutionary systems. It enables us to translate the question of integrability into the problem of divisibility of certain symmetric polynomials, which can be solved by using number theoretical methods. A series papers have been published on it including the complete classification of second order two components evolutionary systems [11] and Bakirov-type systems [12, 13].

The ultimate goal of classification of integrable equations is to obtain the classification result for integrable equations of any order, i.e. the global classification result. So far the only global result has been obtained for scalar homogeneous evolutionary equations [14, 15]. In this paper, based on the symbolic representation in [16], we develop the formal diagonalisation method so that we can globally classify homogeneous two-component evolutionary systems whose leading linear terms are not explicitly diagonal. We demonstrate our method classifying integrable systems of the form

$$\begin{cases} u_t = \partial_x^r v, \\ v_t = \partial_x^{2n+1-r} u + F(u, u_x, u_{xx}, \dots, \partial_x^{2n-r} u, v, v_x, v_{xx}, \dots, \partial_x^{2n-r} v), \end{cases} \quad n = 1, 2, 3, \dots, \quad r \in \{0, 1, \dots, n\}, \quad (1)$$

of arbitrary order (i.e. for any integer $n > 0$). Here F is a homogeneous differential polynomial in u, v and their x -derivatives. If function F does not depend on $v, v_x, \dots, \partial_x^{r-1} v$ then we can exclude v from system (1) and obtain non-evolutionary equations of odd order

$$u_{tt} = \partial_x^{2n+1} u + K(u, u_x, u_{xx}, \dots, \partial_x^{2n} u, u_t, u_{tx}, u_{txx}, \dots, \partial_x^{2n} u_t), \quad K = D_x^r(F), \quad n = 1, 2, 3, \dots \quad (2)$$

The even-order non-evolutionary equations of form (2) have been studied using the perturbative symmetry approach in symbolic representation in [16]. The famous Boussinesq equation [17]

$$u_{tt} = u_{xxxx} + (u^2)_{xx}$$

belongs to this class. As a result, integrable equations of orders 4 and 6 were classified as well as three new integrable equations of order 10 were found. The approach developed in [16] is also suitable for odd order equations. However, there is a difficulty to formulate the explicit necessary conditions for this case. Besides, it can not be used for global classification. The special subclass of non-evolutionary equations was also considered in [20], where a family of partially integrable equations with only one higher symmetry was constructed.

In this paper in the framework of the perturbative symmetry approach in the symbolic representation we derive an explicit set of necessary conditions of existence of a hierarchy of higher symmetries for systems (1). These conditions are obtained after diagonalisation of the linear part of system (1) via a non-local linear transformation of dependent variables u, v . We prove the ultimate classification result: If a homogeneous system (1) with positive weight of u possesses a hierarchy of infinitely many higher symmetries, then up to re-scaling it one of the following two equations:

$$\begin{cases} u_t = v_x, \\ v_t = u_{xx} + 3uv_x + vu_x - 3u^2u_x, \end{cases} \quad \begin{cases} u_t = v_x, \\ v_t = (\partial_x + u)^{2n}(u) - v^2, \end{cases} \quad n = 1, 2, 3, \dots$$

These systems can be rewritten in the form of non-evolutionary equations if we introduce a new variable $u = w_x$:

$$\begin{aligned} w_{tt} &= w_{xxx} + 3w_x w_{t,x} + w_{xx} w_t - 3w_x^2 w_{xx}, \\ w_{tt} &= (\partial_x + w_x)^{2n}(w_x) - w_t^2 \end{aligned}$$

The first equation is known to be integrable [18]. The second equation can be brought into linear $f_{tt} = \partial_x^{2n+1} f$ by the Cole-Hopf transformation $w = \log(f)$.

The paper is organized as follows: in section 2 we remind basic definitions and notations of the symbolic representation [16]; in section 3 we develop the formal diagonalisation formalism for systems (1); using the formal diagonalisation method we obtain explicit formulae for the (approximate) symmetries of system (1) and derive necessary conditions for the existence of an infinite hierarchy of higher symmetries; in section 4 we apply these conditions to classify integrable homogeneous systems (1) for all orders $n = 1, 2, 3, \dots$ and for all possible representations $r = 0, 1, \dots, n$.

2 Symbolic representation over the ring of differential polynomials.

In this section we remind basic definitions and notations of the ring of differential polynomials and symbolic representation (for more details see e.g. [11, 16, 19]).

Throughout the whole paper we assume that all functions, such as F and K in equation (1) and (2), are differential polynomials in variables $u, v, u_x, v_x, u_{xx}, \dots$. We introduce an infinite sequence of *dynamical variables* $\{u_0, v_0, u_1, v_1, u_2, v_2, \dots\}$ by the identification

$$u_0 = u, v_0 = v, u_n = \partial_x^n u, v_n = \partial_x^n v. \quad (3)$$

We often omit the zero index and write u and v instead of u_0 and v_0 .

We denote \mathcal{R} the ring of polynomials over \mathbb{C} of infinite number of dynamical variables. We also assume that $1 \notin \mathcal{R}$. Elements of the ring \mathcal{R} are finite sums of monomials with complex coefficients and therefore each element depends on a finite number of the dynamical variables. The degree of a monomial is defined as a total power, i.e. the sum of all powers of dynamical variables that contribute to the monomial. Let \mathcal{R}^n denote the set of polynomials in \mathcal{R} of degree n . The ring \mathcal{R} has a natural gradation

$$\mathcal{R} = \bigoplus_{n \in \mathbb{Z}_+} \mathcal{R}^n, \quad \mathcal{R}^n \cdot \mathcal{R}^m \subset \mathcal{R}^{n+m}.$$

Elements of \mathcal{R}^1 are linear functions of the dynamical variables, \mathcal{R}^2 quadratic, etc. We suppose that $1 \notin \mathcal{R}$. It is convenient to define a ‘‘little-oh’’ order symbol $o(\mathcal{R}^n)$. We say that $f = o(\mathcal{R}^n)$ if $f \in \bigoplus_{k > n} \mathcal{R}^k$, i.e. the degree of every monomial of f is bigger than n .

Let μ, ν be two positive rational numbers¹, which we call the weights of u and v respectively and denote $W(u) = \mu, W(v) = \nu$. We define the weights of dynamical variables (3) as $W(u_i) = \mu + i$ and $W(v_j) = \nu + j$. The weight of a monomial is the sum of the weights of dynamical variables that contribute to the monomial. We say that a polynomial $f \in \mathcal{R}$ is a homogeneous polynomial of weight λ (and write $W(f) = \lambda$) if every its monomial is of weight λ . The *weighted* gradation of \mathcal{R} is defined as

$$\mathcal{R} = \bigoplus_{m, n, s \in \mathbb{Z}_{\geq 0}} \mathcal{R}_{m\mu + n\nu + s}, \quad \mathcal{R}_p \cdot \mathcal{R}_q \subset \mathcal{R}_{p+q},$$

where \mathcal{R}_p is the set of polynomials in \mathcal{R} of weight p . In fact, we can define a degree-weighted gradation of the ring \mathcal{R}

$$\mathcal{R} = \bigoplus_{n, m, s \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_+} \mathcal{R}_{m\mu + n\nu + s}^r, \quad \mathcal{R}_p^r \cdot \mathcal{R}_q^m \subset \mathcal{R}_{p+q}^{r+m},$$

where \mathcal{R}_p^r is the set of polynomials in \mathcal{R} of degree r and weight p . Note that the condition $\mu, \nu > 0$ makes each such subspace to be finitely generated. For example, if $\mu = 1, \nu = 2$ then

$$\mathcal{R}_4^2 = \text{span}\{uu_2, u_1^2, u_1v, uv_1, v^2\}$$

The ring \mathcal{R} is a differential ring with a derivation defined as

$$D_x = \sum_{k \geq 0} \left(u_{k+1} \frac{\partial}{\partial u_k} + v_{k+1} \frac{\partial}{\partial v_k} \right). \quad (4)$$

Since $1 \notin \mathcal{R}$, the kernel of the linear map $D_x : \mathcal{R} \mapsto \text{Im } D_x \subset \mathcal{R}$ is empty and therefore D_x^{-1} is defined uniquely on $\text{Im } D_x$. It is easy to verify that

$$D_x : \mathcal{R}_p^n \mapsto \mathcal{R}_{p+1}^n.$$

To any element $g \in \mathcal{R}$ we associate differential operators $g_{*,u}$ and $g_{*,v}$ called Fréchet derivatives with respect to u and v and defined as

$$g_{*,u} = \sum_{k \geq 0} \frac{\partial g}{\partial u_k} D_x^k, \quad g_{*,v} = \sum_{k \geq 0} \frac{\partial g}{\partial v_k} D_x^k.$$

Now we define the symbolic representation $\hat{\mathcal{R}}$ of the ring \mathcal{R} . A symbolic representation of a monomial

$$u_0^{n_0} u_1^{n_1} \dots u_p^{n_p} v_0^{m_0} v_1^{m_1} \dots v_q^{m_q}, \quad n_0 + n_1 + \dots + n_p = n, \quad m_0 + m_1 + \dots + m_q = m$$

is defined as:

$$\begin{aligned} & u_0^{n_0} u_1^{n_1} \dots u_p^{n_p} v_0^{m_0} v_1^{m_1} \dots v_q^{m_q} \mapsto \\ & \rightarrow u^n v^m \langle \xi_1^0 \xi_2^0 \dots \xi_{n_0}^0 \xi_{n_0+1}^1 \dots \xi_{n_0+n_1}^1 \dots \xi_n^p \rangle_{\xi} \langle \zeta_1^0 \zeta_2^0 \dots \zeta_{m_0}^0 \zeta_{m_0+1}^1 \dots \zeta_{m_0+m_1}^1 \dots \zeta_m^q \rangle_{\zeta}, \end{aligned} \quad (5)$$

¹In principle, the weights could be any rational numbers including zero, but in this paper we consider only positive weights.

where triangular brackets $\langle \rangle_\xi$ and $\langle \rangle_\zeta$ denote the averaging over group Σ_n of permutations of n elements ξ_1, \dots, ξ_n , i.e.

$$\langle c(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m) \rangle_\xi = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} c(\sigma(\xi_1), \dots, \sigma(\xi_n), \zeta_1, \dots, \zeta_m)$$

and group Σ_m of m elements ζ_1, \dots, ζ_m respectively. Later we refer to this as symmetrisation operation. For example, linear monomials u_n, v_m are represented by

$$u_n \rightarrow u\xi_1^n, \quad v_m \rightarrow v\zeta_1^m \quad (6)$$

and quadratic monomials $u_n u_m, u_n v_m, v_n v_m$ have the following symbols

$$u_n u_m \rightarrow \frac{u^2}{2} (\xi_1^n \xi_2^m + \xi_1^m \xi_2^n), \quad u_n v_m \rightarrow uv(\xi_1^n \zeta_1^m), \quad v_n v_m \rightarrow \frac{v^2}{2} (\zeta_1^n \zeta_2^m + \zeta_1^m \zeta_2^n). \quad (7)$$

To the sum of two elements of the ring corresponds the sum of their symbols. To the product of two elements $f, g \in \mathcal{R}$ with symbols $f \rightarrow u^n v^m a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m)$ and $g \rightarrow u^p v^q b(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_q)$ corresponds:

$$fg \rightarrow u^{n+p} v^{m+q} \langle \langle a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m) b(\xi_{n+1}, \dots, \xi_{n+p}, \zeta_{m+1}, \dots, \zeta_{m+q}) \rangle_\xi \rangle_\zeta, \quad (8)$$

where the symmetrisation operation is taken with respect to permutations of all arguments ξ and ζ . It is easy to see that the symbolic representations of quadratic (7) and general (5) monomials immediately follows from (6) and (8).

If $f \in \mathcal{R}$ has a symbol $f \rightarrow u^n v^m a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m)$, then the symbolic representation for its N -th derivative $D_x^N(f)$ is:

$$D_x^N(f) \rightarrow u^n v^m (\xi_1 + \xi_2 + \dots + \xi_n + \zeta_1 + \zeta_2 + \dots + \zeta_m)^N a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m).$$

We will assign a symbol η to the operator D_x in the symbolic representation with obvious action rule

$$\eta^N (u^n v^m a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m)) = u^n v^m (\xi_1 + \xi_2 + \dots + \xi_n + \zeta_1 + \zeta_2 + \dots + \zeta_m)^N a(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m)$$

Note that N in the above formula can be not only positive integer but in principal any rational number (or even a complex number) as long as such calculation has any sense apart from formal. In the next section we will need N to be half-integer numbers which correspond to fractional derivation in x -space.

If $g \in \mathcal{R}$ and $g \rightarrow u^n v^m a_{n,m}(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_m)$ then for the symbol of its Fréchet derivatives $g_{*,u}$ and $g_{*,v}$ we have

$$g_{*,u} \rightarrow nu^{n-1} v^m a_{n,m}(\xi_1, \dots, \xi_{n-1}, \eta, \zeta_1, \dots, \zeta_m), \quad g_{*,v} \rightarrow mu^n v^{m-1} a_{n,m}(\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_{m-1}, \eta).$$

Thus we obtained the symbolic representation $\hat{\mathcal{R}}$ of the differential ring \mathcal{R} .

3 Structure of symmetries and approximate symmetries.

Consider a family of homogeneous polynomial systems (1), i.e.

$$\begin{cases} u_t = \partial_x^r v, \\ v_t = \partial_x^{2n+1-r} u + F(u, u_1, \dots, u_{2n-r}, v, v_1, \dots, v_{2n-r}), \end{cases} \quad n = 1, 2, 3, \dots, \quad r \in \{0, 1, \dots, n\}. \quad (9)$$

Definition 1. A pair of differential polynomials G and M is called a symmetry of an evolution system $u_t = f, \quad v_t = g$, where f, g, G and M are all in ring \mathcal{R} , if system $u_\tau = G, \quad v_\tau = M$ is compatible with the given system.

This definition is equivalent to the Lie bracket between $\mathbf{a} = (f, g)^{\text{tr}}$ and $\mathbf{b} = (G, M)^{\text{tr}}$ vanishing, where the bracket is defined as

$$[\mathbf{a}, \mathbf{b}] = \begin{pmatrix} f_{*,u}(G) + f_{*,v}(M) - G_{*,u}(f) - G_{*,v}(g) \\ g_{*,u}(G) + g_{*,v}(M) - M_{*,u}(f) - M_{*,v}(g) \end{pmatrix}. \quad (10)$$

The compatibility condition for system (9) can be written as

$$D_t(G) = D_x^r v_\tau = D_x^r M, \quad D_t(M) = D_\tau(u_{2n+1-r} + F), \quad (11)$$

where evolutionary derivations D_t and D_τ are

$$D_t = \sum_{j \geq 0} \left(v_{j+r} \frac{\partial}{\partial u_j} + D_x^j (u_{2n+1-r} + F) \frac{\partial}{\partial v_j} \right), \quad D_\tau = \sum_{j \geq 0} \left(D_x^j(G) \frac{\partial}{\partial u_j} + D_x^{j-r} D_t(G) \frac{\partial}{\partial v_j} \right).$$

Eliminating M from these equations, we obtain

$$D_x^{-r} D_t^2(G) = D_\tau(u_{2n+1-r} + F),$$

Thus the symmetry of system (9) is completely determined by its first component $G \in \mathcal{R}$.

Definition 2. We say that a differential polynomial $G \in \mathcal{R}$ generates an approximate symmetry of degree p of system (9) if G satisfies the equation:

$$D_x^{-r} D_t^2(G) - D_\tau(u_{2n+1-r} + F) = o(\mathcal{R}^p).$$

Notice that any system (9) has an infinite hierarchy of approximate symmetries of degree 1. These are simply the symmetries of its linear part $u_t = v_r$, $v_t = u_{2n+1-r}$. The requirement of the existence of approximate symmetries of degree 2 is very restrictive. Integrable equations by definition possess infinite hierarchies of approximate symmetries of any degree. We will prove later in this paper that the existence of infinite hierarchy of approximate symmetries of degree 2 and the existence of at least one exact symmetry for systems (9) implies integrability just as in the case of scalar evolutionary equations [11].

We now derive the necessary and sufficient conditions of the existence of approximate symmetries of degree 2. It is convenient to do this in the symbolic representation.

System (9) in the symbolic representation takes the form

$$\begin{cases} u_t = v\zeta_1^r, \\ v_t = u\xi_1^{2n+1-r} + \sum_{k \geq 2} \sum_{i=0}^k u^i v^{k-i} a_{i,k-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{k-i}). \end{cases} \quad (12)$$

Let us assume that the weight of variable u , denoted by w , is positive, i.e. $W(u) = w > 0$. Since the system is homogeneous, we have

$$\begin{cases} w + W(D_t) = W(v) + r \\ W(v) + W(D_t) = 2n + 1 - r + w, \end{cases}$$

which leads to $W(D_t) = n + \frac{1}{2}$ and $W(v) = w + n + \frac{1}{2} - r$.

Homogeneous polynomials $a_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)$ are symmetric with respect to variables ξ_1, \dots, ξ_i and ζ_1, \dots, ζ_j . The degrees of $a_{i,j}$ are given by

$$\deg(a_{i,j}) = 2n + 1 - r + w - iW(u) - jW(v) = (2-j)n - (i+j-1)w + (j-1)r + \frac{2-j}{2}. \quad (13)$$

The degree of $a_{i,j}$ determined above must be a non-negative integer, otherwise, we put $a_{i,j} = 0$. The sum in (12) terminates due to the assumption $W(u) = w > 0$.

The symmetry of system (9) always starts with linear terms. A homogeneous symmetry of (9) in the symbolic representation starts either with $u\xi_1^m$ or with $v\zeta_1^{m+r}$. Without loss of generality we have

$$u_\tau = G = u\xi_1^m + \sum_{s \geq 2} \sum_{j=0}^s u^j v^{s-j} A_{j,s-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{s-j}), \quad m > 1 \quad (14)$$

or

$$u_\tau = G = v\zeta_1^{m+r} + \sum_{s \geq 2} \sum_{j=0}^s u^j v^{s-j} A_{j,s-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{s-j}), \quad m > 0 \quad (15)$$

We call the symmetries of the form (14) as **type I** symmetries and those of the form (15) – **type II** symmetries. Without causing confusion, we call integer m in (14) or (15) the order of the corresponding symmetry. Functions $A_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)$ in (14) and (15) are homogeneous polynomials in their variables, symmetric with respect to arguments ξ_1, \dots, ξ_i and ζ_1, \dots, ζ_j . These functions can be explicitly determined in the terms of system (12) from the compatibility conditions.

3.1 Formal diagonalisation.

Let us first concentrate on how to compute the Lie bracket defined as (10) between the linear part of system (12) denoted by K^1 , i.e.

$$K^1 = \begin{pmatrix} v\zeta_1^r \\ u\xi_1^{2n+1-r} \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \eta^r \\ \eta^{2n+1-r} & 0 \end{pmatrix} \quad (16)$$

and any pair of differential polynomials. We know its symbolic representation takes simple and elegant form if matrix L is diagonal [11]. Inspired by this, we shall formally diagonalise matrix L , produce the required formula in new variables and then transform back to the original variables.

Notice that matrix L has two eigenvalues $\pm\eta^{n+\frac{1}{2}}$. Therefore, there exists a linear transformation

$$T = \begin{pmatrix} 1 & 1 \\ \eta^{n+\frac{1}{2}-r} & -\eta^{n+\frac{1}{2}-r} \end{pmatrix}$$

such that

$$T^{-1}LT = \text{diag}(\eta^{n+\frac{1}{2}}, -\eta^{n+\frac{1}{2}}).$$

Let us introduce new variables \hat{u} and \hat{v}

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \hat{u} + \hat{v} \\ \hat{u}\xi_1^{n+\frac{1}{2}-r} - \hat{v}\zeta_1^{n+\frac{1}{2}-r} \end{pmatrix}.$$

Equally, we have

$$\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = T^{-1} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u + \eta^{-n-\frac{1}{2}+r}(v) \\ u - \eta^{-n-\frac{1}{2}+r}(v) \end{pmatrix}.$$

The new variables \hat{u} and \hat{v} have the same weights, i.e. $W(\hat{u}) = W(\hat{v}) = W(u)$. Without causing confusion we assign the same symbols ξ and ζ for the symbolic representation of the ring generated by \hat{u} , \hat{v} and their derivatives. The exponents of symbols can be half-integer, which corresponds to half-differentiation in x -space.

Before we work out the exact form of system (12) in variables \hat{u} and \hat{v} , we prove two useful propositions.

Proposition 1. *Under the transformation T , an arbitrary polynomial $u^i v^j b_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)$ takes the form*

$$u^i v^j b_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) = \sum_{p=0}^i \sum_{q=0}^j \hat{u}^{p+q} \hat{v}^{i+j-p-q} C_i^p C_j^q (-1)^{j-q} \quad (17)$$

$$\langle \langle b_{i,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+1}, \dots, \zeta_{i+j-p-q}) (\xi_{p+1} \cdots \xi_{p+q} \zeta_{i-p+1} \cdots \zeta_{i+j-p-q})^{n+\frac{1}{2}-r} \rangle_{\xi} \rangle_{\zeta},$$

where C_i^j are binomial coefficients defined by $C_i^j = \frac{i!}{j!(i-j)!}$.

Proof. We prove the statement by induction on both i and j . It is easy to see that linear terms $u\xi_1^i$, $v\zeta_1^j$ transform as

$$u\xi_1^i = \eta^i (\hat{u} + \hat{v}) = \hat{u}\xi_1^i + \hat{v}\zeta_1^i, \quad v\zeta_1^j = \eta^j (\hat{u}\xi_1^{n+\frac{1}{2}-r} - \hat{v}\zeta_1^{n+\frac{1}{2}-r}) = \hat{u}\xi_1^{n+\frac{1}{2}-r+j} - \hat{v}\zeta_1^{n+\frac{1}{2}-r+j}.$$

The procedure is: we first substitute ξ_1 and ζ_1 by η , the symbol of the total x -derivative D_x , u and v by $\hat{u} + \hat{v}$ and $\hat{u}\xi_1^{n+\frac{1}{2}-r} - \hat{v}\zeta_1^{n+\frac{1}{2}-r}$; and then we compute the action of η on $\hat{u} + \hat{v}$ and $\hat{u}\xi_1^{n+\frac{1}{2}-r} - \hat{v}\zeta_1^{n+\frac{1}{2}-r}$. For the higher degree terms, we apply this procedure for every element in the arguments. Assume formula (17) without symmetrisation is true for arbitrary i and j . We have

$$\begin{aligned} & u^{i+1} v^j b_{i+1,j}(\xi_1, \dots, \xi_i, \xi_{i+1}, \zeta_1, \dots, \zeta_j) \\ &= \sum_{p=0}^i \sum_{q=0}^j \hat{u}^{p+q} \hat{v}^{i+j-p-q} C_i^p C_j^q (-1)^{j-q} (\xi_{p+1} \cdots \xi_{p+q} \zeta_{i-p+1} \cdots \zeta_{i+j-p-q})^{n+\frac{1}{2}-r} \\ & \quad b_{i+1,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \eta, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+1}, \dots, \zeta_{i+j-p-q}) (\hat{u} + \hat{v}) \\ &= \sum_{p=0}^i \sum_{q=0}^j \hat{u}^{p+q+1} \hat{v}^{i+j-p-q} C_i^p C_j^q (-1)^{j-q} (\xi_{p+2} \cdots \xi_{p+q+1} \zeta_{i-p+1} \cdots \zeta_{i+j-p-q})^{n+\frac{1}{2}-r} \\ & \quad b_{i+1,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_{p+q+1}, \zeta_{i-p+1}, \dots, \zeta_{i+j-p-q}) \\ &+ \sum_{p=0}^i \sum_{q=0}^j \hat{u}^{p+q} \hat{v}^{i+j-p-q+1} C_i^p C_j^q (-1)^{j-q} (\xi_{p+1} \cdots \xi_{p+q} \zeta_{i-p+2} \cdots \zeta_{i+j-p-q+1})^{n+\frac{1}{2}-r} \\ & \quad b_{i+1,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \zeta_{i+1-p}, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+2}, \dots, \zeta_{i+j-p-q+1}) \\ &= \sum_{p=0}^{i+1} \sum_{q=0}^j \hat{u}^{p+q} \hat{v}^{i+1+j-p-q} C_{i+1}^p C_j^q (-1)^{j-q} (\xi_{p+1} \cdots \xi_{p+q} \zeta_{i-p+2} \cdots \zeta_{i+1+j-p-q})^{n+\frac{1}{2}-r} \\ & \quad b_{i+1,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i+1-p}, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+2}, \dots, \zeta_{i+1+j-p-q}) \end{aligned}$$

and similarly we can prove the formula is valid for $j+1$. Finally, we need to symmetrise the expression with respect to permutations of arguments ξ and ζ . \diamond

In the same manner, we can work out how a polynomial in \hat{u} , \hat{v} and their derivatives changes under the transformation T^{-1} . Here we only give the formula.

Proposition 2. Under the transformation T^{-1} , an arbitrary polynomial $\hat{u}^i \hat{v}^j \hat{b}_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)$ takes the form

$$\hat{u}^i \hat{v}^j \hat{b}_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) = \frac{1}{2^{i+j}} \sum_{p=0}^i \sum_{q=0}^j u^{p+q} v^{i+j-p-q} C_i^p C_j^q (-1)^{j-q} \quad (18)$$

$$\langle \langle \hat{b}_{i,j}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+1}, \dots, \zeta_{i+j-p-q}) (\zeta_1 \cdots \zeta_{i+j-p-q})^{-n-\frac{1}{2}+r} \rangle \rangle_{\xi},$$

where C_i^j are binomial coefficients.

These two propositions immediately lead to the following result.

Proposition 3. For any integer $s \geq 1$, if

$$\sum_{i=0}^s u^i v^{s-i} b_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) = \sum_{l=0}^s \hat{u}^l \hat{v}^{s-l} \hat{b}_{l,s-l}(\xi_1, \dots, \xi_l, \zeta_1, \dots, \zeta_{s-l}),$$

then

$$\hat{b}_{l,s-l}(\xi_1, \dots, \xi_l, \zeta_1, \dots, \zeta_{s-l}) = \sum_{i=0}^s \sum_{p=\max\{0, l-s+i\}}^{\min\{i, l\}} C_i^p C_{s-i}^{l-p} (-1)^{s-i-l+p} \quad (19)$$

$$\langle \langle b_{i,s-i}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_l, \zeta_{i-p+1}, \dots, \zeta_{s-l}) (\xi_{p+1} \cdots \xi_j \zeta_{i-p+1} \cdots \zeta_{s-l})^{n+\frac{1}{2}-r} \rangle \rangle_{\xi}$$

and

$$b_{j,s-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{s-j}) = \frac{1}{2^s} \sum_{i=0}^s \sum_{p=\max\{0, j-s+i\}}^{\min\{i, j\}} C_i^p C_{s-i}^{j-p} (-1)^{s-i-j+p} \quad (20)$$

$$\langle \langle \hat{b}_{i,s-i}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_j, \zeta_{i-p+1}, \dots, \zeta_{s-j}) (\zeta_1 \cdots \zeta_{s-j})^{-n-\frac{1}{2}+r} \rangle \rangle_{\xi},$$

where C_i^j are binomial coefficients defined in Proposition 1.

Proof. Applying formulae (17) in Proposition 1 to every term in the sum and collecting the coefficients at every monomial $\hat{u}^i \hat{v}^j$, we obtain

$$\begin{aligned} \sum_{i=0}^s u^i v^{s-i} b_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) &= \sum_{i=0}^s \sum_{p=0}^i \sum_{q=0}^{s-i} \hat{u}^{p+q} \hat{v}^{s-p-q} C_i^p C_{s-i}^q (-1)^{s-i-q} \\ &\langle \langle b_{i,s-i}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_{p+q}, \zeta_{i-p+1}, \dots, \zeta_{s-p-q}) (\xi_{p+1} \cdots \xi_{p+q} \zeta_{i-p+1} \cdots \zeta_{s-p-q})^{n+\frac{1}{2}-r} \rangle \rangle_{\xi} \\ &= \sum_{i=0}^s \sum_{l=0}^s \sum_{p=\max\{0, l-s+i\}}^{\min\{i, l\}} \hat{u}^l \hat{v}^{s-l} C_i^p C_{s-i}^{l-p} (-1)^{s-i-l+p} \\ &\langle \langle b_{i,s-i}(\xi_1, \dots, \xi_p, \zeta_1, \dots, \zeta_{i-p}, \xi_{p+1}, \dots, \xi_l, \zeta_{i-p+1}, \dots, \zeta_{s-l}) (\xi_{p+1} \cdots \xi_l \zeta_{i-p+1} \cdots \zeta_{s-l})^{n+\frac{1}{2}-r} \rangle \rangle_{\xi}, \end{aligned}$$

which leads to formula (19). Similarly, using formulae (18) in Proposition 2, we can prove formula (20). \diamond

Formula (19) in Proposition 3 tells us how a polynomial of degree s changes under transformation T . The following result is an immediate corollary.

Corollary 1. In variables \hat{u} and \hat{v} , system (12) takes the form

$$\begin{cases} \hat{u}_t = \hat{u} \xi_1^{n+\frac{1}{2}} + \frac{1}{2} \sum_{s \geq 2} \sum_{i=0}^s \hat{u}^i \hat{v}^{s-i} \frac{\hat{a}_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i})}{(\xi_1 + \cdots + \xi_i + \zeta_1 + \cdots + \zeta_{s-i})^{n+\frac{1}{2}-r}} \\ \hat{v}_t = -\hat{v} \xi_1^{n+\frac{1}{2}} - \frac{1}{2} \sum_{s \geq 2} \sum_{i=0}^s \hat{u}^i \hat{v}^{s-i} \frac{\hat{a}_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i})}{(\xi_1 + \cdots + \xi_i + \zeta_1 + \cdots + \zeta_{s-i})^{n+\frac{1}{2}-r}} \end{cases}, \quad (21)$$

where $\hat{a}_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i})$ are defined in terms of $a_{j,s-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{s-j})$, $j = 0, \dots, s$ as in formula (19) in Proposition 3.

Proof. We perform transformation T to system (12): the linear part becomes diagonal and we simply write the system as

$$\begin{cases} \hat{u}_t = \hat{u} \xi_1^{n+\frac{1}{2}} + \frac{1}{2\eta^{n+\frac{1}{2}-r}} \left(\sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} a_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right) \\ \hat{v}_t = -\hat{v} \xi_1^{n+\frac{1}{2}} - \frac{1}{2\eta^{n+\frac{1}{2}-r}} \left(\sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} a_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right) \end{cases}. \quad (22)$$

Applying Proposition 3, we obtain system (21) as stated. \diamond

For the future references we explicitly write down the relation between the quadratic terms of (12) and of (21):

$$\hat{a}_{2,0}(\xi_1, \xi_2) = a_{2,0}(\xi_1, \xi_2) + a_{0,2}(\xi_1, \xi_2)(\xi_1 \xi_2)^{n+\frac{1}{2}-r} + \frac{1}{2}a_{1,1}(\xi_1, \xi_2)\xi_2^{n+\frac{1}{2}-r} + \frac{1}{2}a_{1,1}(\xi_2, \xi_1)\xi_1^{n+\frac{1}{2}-r}, \quad (23)$$

$$\hat{a}_{1,1}(\xi_1, \xi_2) = 2a_{2,0}(\xi_1, \xi_2) - 2a_{0,2}(\xi_1, \xi_2)(\xi_1 \xi_2)^{n+\frac{1}{2}-r} + a_{1,1}(\xi_2, \xi_1)\xi_1^{n+\frac{1}{2}-r} - a_{1,1}(\xi_1, \xi_2)\xi_2^{n+\frac{1}{2}-r}, \quad (24)$$

$$\hat{a}_{0,2}(\xi_1, \xi_2) = a_{2,0}(\xi_1, \xi_2) + a_{0,2}(\xi_1, \xi_2)(\xi_1 \xi_2)^{n+\frac{1}{2}-r} - \frac{1}{2}a_{1,1}(\xi_1, \xi_2)\xi_2^{n+\frac{1}{2}-r} - \frac{1}{2}a_{1,1}(\xi_2, \xi_1)\xi_1^{n+\frac{1}{2}-r} \quad (25)$$

We now express the symmetries (14) and (15) of system (12) in variables \hat{u}, \hat{v} .

Let us start with **type I** symmetries (14):

$$u_\tau = u\xi_1^m + g, \quad \text{with } g = \sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} A_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}).$$

Due to the compatibility condition (11), its second component is of the form

$$v_\tau = v\zeta_1^m + \frac{1}{\eta^r} (D_t(g)),$$

where the action of the operator D_t is defined in the symbolic representation as

$$D_t(g) = g_{*,u}(v\zeta_1^r) + g_{*,v} \left(u\xi_1^{2n+1-r} + \sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} a_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right).$$

In variables \hat{u} and \hat{v} , a **type I** symmetry takes the form

$$\begin{cases} \hat{u}_\tau = \hat{u}\xi_1^m + \frac{1}{2\eta^{n+\frac{1}{2}}} \left[\eta^{n+\frac{1}{2}} + \hat{D}_t \right] (\hat{g}) \\ \hat{v}_\tau = \hat{v}\zeta_1^m + \frac{1}{2\eta^{n+\frac{1}{2}}} \left[\eta^{n+\frac{1}{2}} - \hat{D}_t \right] (\hat{g}) \end{cases}. \quad (26)$$

Here \hat{g} stands for the transformed g under transformation T , whose exact expression can be obtained by applying Proposition 3. Notation \hat{D}_t denotes the operator of total derivation with respect to t due to diagonal system (21). Hence

$$\hat{D}_t(\hat{g}) = \hat{g}_{*,\hat{u}}(\hat{u}_t) + \hat{g}_{*,\hat{v}}(\hat{v}_t).$$

For **type II** symmetries (15):

$$u_\tau = v\zeta_1^{m+r} + g, \quad \text{with } g = \sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} A_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}),$$

its second component is of the form

$$v_\tau = u\xi_1^{2n+m+1-r} + \eta^m \left(\sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} a_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right) + \eta^{-r} D_t(g).$$

Using the same notations as described for **type I** symmetries, in variables \hat{u} and \hat{v} a **type II** symmetry takes the form

$$\begin{cases} \hat{u}_\tau = \hat{u}\xi_1^{n+m+\frac{1}{2}} + \frac{1}{2\eta^{n+\frac{1}{2}}} \left[\eta^{n+\frac{1}{2}} + \hat{D}_t \right] \hat{g} + \frac{1}{2}\eta^{m+r-n-\frac{1}{2}} \left(\sum_{s \geq 2} \sum_{i=0}^s \hat{u}^i \hat{v}^{s-i} \hat{a}_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right) \\ \hat{v}_\tau = -\hat{v}\zeta_1^{n+m+\frac{1}{2}} + \frac{1}{2\eta^{n+\frac{1}{2}}} \left[\eta^{n+\frac{1}{2}} - \hat{D}_t \right] \hat{g} - \frac{1}{2}\eta^{m+r-n-\frac{1}{2}} \left(\sum_{s \geq 2} \sum_{i=0}^s \hat{u}^i \hat{v}^{s-i} \hat{a}_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \right) \end{cases}. \quad (27)$$

3.2 Approximate symmetries and integrability conditions.

We derive the compatibility conditions of (14) or (15) and (12) via the compatibility conditions of their transformed forms (26) or (27) and (21). For simplicity, we rewrite them in the following compact forms correspondingly:

$$\begin{cases} u_\tau = u\Omega(\xi_1) + e^1 \\ v_\tau = v\Omega(\zeta_1) + e^2 \end{cases} \quad e^j = \sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} e_{i,s-i}^j(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}), \quad j = 1, 2 \quad (28)$$

$$\begin{cases} u_\tau = u\Omega(\xi_1) + e^1 \\ v_\tau = -v\Omega(\zeta_1) + e^2 \end{cases} \quad (29)$$

$$\begin{cases} u_t = u\omega(\xi_1) + p \\ v_t = -v\omega(\zeta_1) - p \end{cases} \quad p = \sum_{s \geq 2} \sum_{i=0}^s u^i v^{s-i} p_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}). \quad (30)$$

Now we introduce a useful notation

$$\mathcal{G}_{i,s-i}^\omega(c_1, c_2; \xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) = c_1 \omega \left(\sum_{l=1}^i \xi_l + \sum_{k=1}^{s-i} \zeta_k \right) - c_1 \sum_{l=1}^i \omega(\xi_l) - c_2 \sum_{k=1}^{s-i} \omega(\zeta_k). \quad (31)$$

Proposition 4. *System (28) or system (29) is compatible with (30) of degree 2 if and only if the quadratic terms in (28) or system (29) are related to the quadratic terms in (30) as follows:*

$$\begin{aligned} e_{2,0}^1(\xi_1, \xi_2) &= \frac{\mathcal{G}_{2,0}^\Omega(1, c; \xi_1, \xi_2)}{\mathcal{G}_{2,0}^\omega(1, -1; \xi_1, \xi_2)} p_{2,0}(\xi_1, \xi_2), & e_{2,0}^2(\xi_1, \xi_2) &= -\frac{\mathcal{G}_{0,2}^\Omega(c, 1; \xi_1, \xi_2)}{\mathcal{G}_{0,2}^\omega(-1, 1; \xi_1, \xi_2)} p_{2,0}(\xi_1, \xi_2), \\ e_{1,1}^1(\xi_1, \zeta_1) &= \frac{\mathcal{G}_{1,1}^\Omega(1, c; \xi_1, \zeta_1)}{\mathcal{G}_{1,1}^\omega(1, -1; \xi_1, \zeta_1)} p_{1,1}(\xi_1, \zeta_1), & e_{1,1}^2(\xi_1, \zeta_1) &= -\frac{\mathcal{G}_{1,1}^\Omega(c, 1; \zeta_1, \xi_1)}{\mathcal{G}_{1,1}^\omega(-1, 1; \zeta_1, \xi_1)} p_{1,1}(\xi_1, \zeta_1), \\ e_{0,2}^1(\zeta_1, \zeta_2) &= \frac{\mathcal{G}_{0,2}^\Omega(1, c; \zeta_1, \zeta_2)}{\mathcal{G}_{0,2}^\omega(1, -1; \zeta_1, \zeta_2)} p_{0,2}(\zeta_1, \zeta_2), & e_{0,2}^2(\zeta_1, \zeta_2) &= -\frac{\mathcal{G}_{2,0}^\Omega(c, 1; \zeta_1, \zeta_2)}{\mathcal{G}_{2,0}^\omega(-1, 1; \zeta_1, \zeta_2)} p_{0,2}(\zeta_1, \zeta_2), \end{aligned}$$

where $c = 1$ for the terms in (28) and $c = -1$ for the terms in (29).

Proof. Recall that for any polynomial pair

$$P = u^i v^{s-i} a_{i,s-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) \quad \text{and} \quad Q = u^j v^{r-j} b_{j,r-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{r-j}),$$

where $s \geq 2$, $r \geq 2$, $0 \leq i \leq s$ and $0 \leq j \leq r$ are integers, we have [11]

$$\left[\begin{pmatrix} c_1 u \omega(\xi_1) \\ c_2 v \omega(\zeta_1) \end{pmatrix}, \begin{pmatrix} P \\ Q \end{pmatrix} \right] = \begin{pmatrix} \mathcal{G}_{i,s-i}^\omega(c_1, c_2; \xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{s-i}) P \\ \mathcal{G}_{r-j,j}^\omega(c_2, c_1; \zeta_1, \dots, \zeta_{r-j}, \xi_1, \dots, \xi_j) Q \end{pmatrix}. \quad (32)$$

The compatibility conditions of (28) and (30) up to degree 2 read as

$$\begin{aligned} & \left[\begin{pmatrix} u \omega(\xi_1) \\ -v \omega(\zeta_1) \end{pmatrix}, \begin{pmatrix} \sum_{i=0}^2 u^i v^{2-i} e_{i,2-i}^1(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{2-i}) \\ \sum_{i=0}^2 u^i v^{2-i} e_{i,2-i}^2(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{2-i}) \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} u \Omega(\xi_1) \\ v \Omega(\zeta_1) \end{pmatrix}, \begin{pmatrix} \sum_{i=0}^2 u^i v^{2-i} p_{i,2-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{2-i}) \\ -\sum_{i=0}^2 u^i v^{2-i} p_{i,2-i}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_{2-i}) \end{pmatrix} \right] \end{aligned}$$

Using formula (32) and collecting the terms at every monomial $u^i v^{2-i}$ for $i \in \{0, 1, 2\}$, we obtain the formula for the quadratic terms in (28) with $c = 1$. Similarly, we can derive the quadratic terms in (29) as in the statement. \diamond

In fact, the results of Proposition 4 can be generated to the terms of higher degree $i + j > 2$, namely, one can derive that

$$e_{i,j}^1(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) = \frac{\mathcal{G}_{i,j}^\Omega(1, c; \xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)}{\mathcal{G}_{i,j}^\omega(1, -1; \xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j)} p_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) + L_{i,j}^1 \quad (33)$$

$$e_{i,j}^2(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) = -\frac{\mathcal{G}_{j,i}^\Omega(c, 1; \zeta_1, \dots, \zeta_j, \xi_1, \dots, \xi_i)}{\mathcal{G}_{j,i}^\omega(-1, 1; \zeta_1, \dots, \zeta_j, \xi_1, \dots, \xi_i)} p_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) + L_{i,j}^2, \quad (34)$$

where $c = 1$ for the terms in (28) and $c = -1$ for the terms in (29). Expressions $L_{i,j}^1$ and $L_{i,j}^2$ depend only on the terms of degree lower than $i + j$ in system (30). We will not write out them explicitly.

Now we are ready to derive the compatibility conditions of (14) or (15) and (12). Let us introduce the following notations:

$$N_s^{(m)}(\xi_1, \dots, \xi_s) = \left(\sum_{l=1}^s \xi_l \right)^m - \sum_{l=1}^s \xi_l^m \quad (35)$$

$$S_{s,i}^{(n)}(\xi_1, \dots, \xi_s) = (\xi_1 + \dots + \xi_s)^{2n+1} - \left(\sum_{l=1}^i \xi_l^{n+\frac{1}{2}} - \sum_{l=i+1}^s \xi_l^{n+\frac{1}{2}} \right)^2, \quad (36)$$

$$M_{s,i}^{(m)}(\xi_1, \dots, \xi_s) = (\xi_1 + \dots + \xi_s)^m \left(\sum_{l=1}^i \xi_l^{n+\frac{1}{2}} - \sum_{l=i+1}^s \xi_l^{n+\frac{1}{2}} \right) - \sum_{l=1}^i \xi_l^{n+m+\frac{1}{2}} + \sum_{l=i+1}^s \xi_l^{n+m+\frac{1}{2}}. \quad (37)$$

Theorem 1. Expression (14) and expression (15) are approximate symmetries of degree 2 of system (12) if and only if functions $A_{2,0}$, $A_{1,1}$, $A_{0,2}$ given by formulae

$$A_{2,0}(\xi_1, \xi_2) = \frac{1}{8} \left[2\hat{A}_{2,0}(\xi_1, \xi_2) + 2\hat{A}_{0,2}(\xi_1, \xi_2) + \hat{A}_{1,1}(\xi_1, \xi_2) + \hat{A}_{1,1}(\xi_2, \xi_1) \right], \quad (38)$$

$$A_{1,1}(\xi_1, \xi_2) = \frac{1}{4\xi_2^{n+\frac{1}{2}-r}} \left[2\hat{A}_{2,0}(\xi_1, \xi_2) - 2\hat{A}_{0,2}(\xi_1, \xi_2) - \hat{A}_{1,1}(\xi_1, \xi_2) + \hat{A}_{1,1}(\xi_2, \xi_1) \right], \quad (39)$$

$$A_{0,2}(\xi_1, \xi_2) = \frac{1}{8(\xi_1\xi_2)^{n+\frac{1}{2}-r}} \left[2\hat{A}_{2,0}(\xi_1, \xi_2) + 2\hat{A}_{0,2}(\xi_1, \xi_2) - \hat{A}_{1,1}(\xi_1, \xi_2) - \hat{A}_{1,1}(\xi_2, \xi_1) \right]. \quad (40)$$

are polynomials in ξ_1, ξ_2 , where $\hat{A}_{2,0}, \hat{A}_{1,1}, \hat{A}_{0,2}$ are determined as follows:

1. For the quadratic terms in (14),

$$\begin{aligned} \hat{A}_{2,0}(\xi_1, \xi_2) &= \frac{(\xi_1 + \xi_2)^r N_2^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{2,0}(\xi_1, \xi_2), & \hat{A}_{1,1}(\xi_1, \xi_2) &= \frac{(\xi_1 + \xi_2)^r N_2^{(m)}(\xi_1, \xi_2)}{S_{2,1}^{(n)}(\xi_1, \xi_2)} \hat{a}_{1,1}(\xi_1, \xi_2), \\ \hat{A}_{0,2}(\xi_1, \xi_2) &= \frac{(\xi_1 + \xi_2)^r N_2^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{0,2}(\xi_1, \xi_2). \end{aligned} \quad (41)$$

2. For the quadratic terms in (15),

$$\begin{aligned} \hat{A}_{2,0}(\xi_1, \xi_2) &= \frac{(\xi_1 + \xi_2)^r M_{2,2}^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{2,0}(\xi_1, \xi_2), & \hat{A}_{1,1}(\xi_1, \xi_2) &= \frac{(\xi_1 + \xi_2)^r M_{2,1}^{(m)}(\xi_1, \xi_2)}{S_{2,1}^{(n)}(\xi_1, \xi_2)} \hat{a}_{1,1}(\xi_1, \xi_2), \\ \hat{A}_{0,2}(\xi_1, \xi_2) &= -\frac{(\xi_1 + \xi_2)^r M_{2,2}^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{0,2}(\xi_1, \xi_2). \end{aligned} \quad (42)$$

Here $\hat{a}_{2,0}, \hat{a}_{1,1}, \hat{a}_{0,2}$ are given in terms of $a_{2,0}, a_{1,1}, a_{0,2}$ by (23), (24) and (25).

Proof. Expression (14) and (15) are approximate symmetries of degree 2 of (12) if and only if they are compatible with (12) up to degree 2. Therefore the diagonal forms of these symmetries (26) and (27) must be compatible with the diagonal form of our system (21). The latter is of the form of (30) with $\omega(x) = x^{n+\frac{1}{2}}$, while (26) and (27) are of the form (28) with $\Omega(x) = x^m$ and (29) with $\Omega(x) = x^{n+m+\frac{1}{2}}$ respectively. From Proposition 3, we know the relation of $A_{2,0}, A_{1,1}, A_{0,2}$ and $\hat{A}_{2,0}, \hat{A}_{1,1}, \hat{A}_{0,2}$ as well as the relation of $\hat{a}_{2,0}, \hat{a}_{1,1}, \hat{a}_{0,2}$ and $a_{2,0}, a_{1,1}, a_{0,2}$. Now we need to show that $\hat{A}_{2,0}, \hat{A}_{1,1}, \hat{A}_{0,2}$ are determined as stated.

Consider first the case of **type I** symmetries (14). Applying Proposition 4 in the case of system (30) and (28) with $\omega(x) = x^{n+\frac{1}{2}}$ and $\Omega(x) = x^m$, we obtain (for the coefficient at u^2v^0):

$$e_{2,0}^1(\xi_1, \xi_2) = \frac{N_2^{(m)}(\xi_1, \xi_2)}{(\xi_1 + \xi_2)^{n+\frac{1}{2}} - \xi_1^{n+\frac{1}{2}} - \xi_2^{n+\frac{1}{2}}} p_{2,0}(\xi_1, \xi_2).$$

Comparing system (28) to system (26) and system (30) to system (21), we have

$$e_{2,0}^1(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^{n+\frac{1}{2}} + \xi_1^{n+\frac{1}{2}} + \xi_2^{n+\frac{1}{2}}}{2(\xi_1 + \xi_2)^{n+\frac{1}{2}}} \hat{A}_{2,0}(\xi_1, \xi_2) \quad \text{and} \quad p_{2,0}(\xi_1, \xi_2) = \frac{\hat{a}_{2,0}(\xi_1, \xi_2)}{2(\xi_1 + \xi_2)^{n+\frac{1}{2}-r}}.$$

Substituting these into the previous formula, we get

$$\hat{A}_{2,0}(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^r N_2^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{2,0}(\xi_1, \xi_2),$$

the first formula of (41). The other two relations can be obtained in the same way.

Consider now the case of **type II** symmetries (15). Using Proposition 4 in the case of systems (30) and (29) with $\omega(x) = x^{n+\frac{1}{2}}$ and $\Omega(x) = x^{n+m+\frac{1}{2}}$ we find (for the coefficient at u^2v^0):

$$e_{2,0}^1(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^{m+n+\frac{1}{2}} - \xi_1^{m+n+\frac{1}{2}} - \xi_2^{m+n+\frac{1}{2}}}{(\xi_1 + \xi_2)^{n+\frac{1}{2}} - \xi_1^{n+\frac{1}{2}} - \xi_2^{n+\frac{1}{2}}} p_{2,0}(\xi_1, \xi_2).$$

Comparing system (29) to system (27), we know

$$e_{2,0}^1(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^{n+\frac{1}{2}} + \xi_1^{n+\frac{1}{2}} + \xi_2^{n+\frac{1}{2}}}{2(\xi_1 + \xi_2)^{n+\frac{1}{2}}} \hat{A}_{2,0}(\xi_1, \xi_2) + \frac{(\xi_1 + \xi_2)^{m+r}}{2(\xi_1 + \xi_2)^{n+\frac{1}{2}}} \hat{a}_{2,0}(\xi_1, \xi_2).$$

This leads to

$$\hat{A}_{2,0}(\xi_1, \xi_2) = \frac{(\xi_1 + \xi_2)^r M_{2,2}^{(m)}(\xi_1, \xi_2)}{S_{2,2}^{(n)}(\xi_1, \xi_2)} \hat{a}_{2,0}(\xi_1, \xi_2),$$

the first formula of (42). The other two formula can be obtained similarly. \diamond

The study of symmetries of (12) is the study of the divisibility of certain special functions. If we introduce variables x_1, x_2 instead of ξ_1, ξ_2 by relations

$$x_1 = \xi_1^{\frac{1}{2}}, \quad x_2 = \xi_2^{\frac{1}{2}}$$

then functions $\hat{a}_{2,0}(x_1^2, x_2^2)$, $\hat{a}_{1,1}(x_1^2, x_2^2)$, $\hat{a}_{0,2}(x_1^2, x_2^2)$, $S_{2,i}^{(n)}(x_1^2, x_2^2)$ and $M_{2,i}^{(m)}(x_1^2, x_2^2)$ become *polynomials* in x_1, x_2 (see formulae (23), (24), (25), (36) and (37)).

Corollary 2. *If (14) and (15) are approximate symmetries of (12) of degree 2 then functions $\hat{A}_{2,0}(x_1^2, x_2^2)$, $\hat{A}_{1,1}(x_1^2, x_2^2)$, and $\hat{A}_{0,2}(x_1^2, x_2^2)$ defined by (41) and (42) are polynomials in x_1, x_2 .*

Proof. From Theorem 1, if (14) and (15) are approximate symmetries of (12) of degree 2 then $A_{2,0}(x_1^2, x_2^2)$, $A_{1,1}(x_1^2, x_2^2)$, and $A_{0,2}(x_1^2, x_2^2)$ are polynomials in x_1, x_2 (in fact these functions are polynomials in x_1^2, x_2^2). We know $\hat{A}_{2,0}(x_1^2, x_2^2)$, $\hat{A}_{1,1}(x_1^2, x_2^2)$, $\hat{A}_{0,2}(x_1^2, x_2^2)$ can be expressed as polynomials in $A_{2,0}, A_{1,1}, A_{0,2}$ with coefficients being polynomials in x_1, x_2 according to Proposition 3. Therefore $\hat{A}_{2,0}$, $\hat{A}_{1,1}$, and $\hat{A}_{0,2}$ are polynomials in x_1, x_2 . \diamond

We are able to obtain explicit recursive relations for determining the higher degree terms in (14) and (15). Similar as formula (33) and (34), we present the strictures of these expressions without the full details. From the proof of Theorem 1, we only need to show how functions $\hat{A}_{i,s-i}$ are related to functions $\hat{a}_{i,s-i}$, where $r \leq s$ and these can be obtained by using formula (33) and (34) and comparing the notations in (28), (29) and (30) with the ones in (26), (27) and (21).

Suppose that (14) or (15) is an approximate symmetry of (12) of degree $p \geq 3$. Then for any $3 \leq s \leq p$ and any $i = 0, \dots, s$ we have

$$\text{Type I: } \hat{A}_{i,s-i}(\xi_1, \dots, \xi_s) = \frac{(\sum_{k=1}^s \xi_k)^r N_s^{(m)}(\xi_1, \dots, \xi_s)}{S_{s,i}^{(n)}(\xi_1, \dots, \xi_s)} \hat{a}_{i,s-i}(\xi_1, \dots, \xi_s) + L_i^{(s)}, \quad (43)$$

$$\text{Type II: } \hat{A}_{i,s-i}(\xi_1, \dots, \xi_s) = \frac{(\sum_{k=1}^s \xi_k)^r M_{s,i}^{(m)}(\xi_1, \dots, \xi_s)}{S_{s,i}^{(n)}(\xi_1, \dots, \xi_s)} \hat{a}_{i,s-i}(\xi_1, \dots, \xi_s) + \tilde{L}_i^{(s)}, \quad (44)$$

where functions $\hat{A}_{i,s-i}$ and $\hat{a}_{i,s-i}$ are given in terms of $A_{i,s-i}$ and $a_{i,s-i}$ according to Proposition 3; $L_i^{(s)}$ and $\tilde{L}_i^{(s)}$ depend only on lower degree terms $a_{i,j}$, $i+j < s$, and $N_s^{(m)}$, $S_{s,i}^{(n)}$ and $M_{s,i}^{(m)}$ are defined as (35), (36) and (37).

Again if we introduce variables

$$x_1 = \xi_1^{\frac{1}{2}}, \dots, x_s = \xi_s^{\frac{1}{2}},$$

functions $\hat{a}_{i,s-i}(x_1^2, \dots, x_s^2)$, $\hat{A}_{i,s-i}(x_1^2, \dots, x_s^2)$, $S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)$, and $M_{s,i}^{(m)}(x_1^2, \dots, x_s^2)$ all become *polynomials* in variables x_1, \dots, x_s .

4 Global classification result.

In this section, we state and prove the global classification result of system (9) in the sense that we list out all the integrable equations of arbitrary order. We begin with an crucial theorem leading to global classification. In order to prove the theorem, we prove the irreducibility of a family of special polynomials.

Proposition 5. *For any $i = 0, \dots, s$, polynomials*

$$S_{s,i}^{(n)}(x_1^2, \dots, x_s^2) = (x_1^2 + \dots + x_s^2)^{2n+1} - (x_1^{2n+1} + \dots + x_i^{2n+1} - x_{i+1}^{2n+1} - \dots - x_s^{2n+1})^2$$

are irreducible polynomials in x_1, \dots, x_s over \mathbb{C} when $s \geq 3$ and $n \geq 1$.

Proof. It suffices to prove that the polynomial $S_{s,s}^{(n)}(x_1^2, \dots, x_s^2)$, cf. (36), is irreducible over \mathbb{C} since all the rest can be obtained from it by taking negation of $s - i$ arguments x_{i+1}, \dots, x_s .

Let us first suppose that $s \geq 4$. If $S_{s,s}^{(n)}$ is reducible then the projective hyperspace given by $S_{s,s}^{(n)} = 0$ consists of two components. These components intersect in infinite number of points, which should be singularities of the hyperspace. These singular points can be computed by setting the derivatives of $S_{s,s}^{(n)}$ with respect to x_j , $j = 1, \dots, s$ equal to zero. We obtain:

$$\frac{\partial S_{s,s}^{(n)}}{\partial x_j} = 2(2n+1)x_j [x_j^{2n-1}(x_1^{2n+1} + \dots + x_s^{2n+1}) - (x_1^2 + \dots + x_s^2)^{2n}] = 0, \quad j = 1, \dots, s.$$

So the singular points are either the solutions

$$x_1^2 + \dots + x_s^2 = 0, \quad x_1^{2n+1} + \dots + x_s^{2n+1} = 0 \quad (45)$$

or the solutions of

$$x_1^{2n-1} = x_2^{2n-1} = \dots = x_s^{2n-1} = x, \quad (x_1^2 + \dots + x_s^2)^{2n-1} = x^2. \quad (46)$$

The singular points from (45) are kink points, which do not contribute to factorization. From (46), it follows that

$$(x_1^2)^{2n-1} = (x_2^2)^{2n-1} = \dots = (x_s^2)^{2n-1} = (x_1^2 + \dots + x_s^2)^{2n-1}. \quad (47)$$

Hence the coordinates differ by $(4n-2)$ -th root of unity. Thus we get finitely many singular points. Thus the assumption that $S_{s,s}^{(n)}$ is reducible is false.

When $s = 3$, system (47) writes as

$$(x_1^2)^{2n-1} = (x_2^2)^{2n-1} = (x_3^2)^{2n-1} = (x_1^2 + x_2^2 + x_3^2)^{2n-1}.$$

By taking $x_3 = 1$, we have that x_1^2, x_2^2 and w are $2n-1$ -st roots of unity such that $w = x_1^2 + x_2^2 + 1$. Note that four complex numbers of the same absolute value can only add up to zero if they form the sides of a parallelogram with equal sides. Therefore either $w = 1$ or $x_i^2 = -1$, $i = 1, 2$. If $w = 1$, then $x_1^2 = -x_2^2$ leading to $(x_1^2)^{2n-1} = -(x_2^2)^{2n-1}$, contradicting to $(x_1^2)^{2n-1} = (x_2^2)^{2n-1}$. If $x_i^2 = -1$, we have $(x_i^2)^{2n-1} = -1$, contradicting to $(x_2^2)^{2n-1} = (x_3^2)^{2n-1} = 1$. Hence, $S_{3,3}^{(n)}$ is irreducible. \diamond

The irreducibility of this family of polynomials leads to the following useful result in testing integrability.

Corollary 3. *If there are no quadratic terms in homogeneous system (12) with $W(u) > 0$, i.e.*

$$a_{2,0}(\xi_1, \xi_2) = a_{1,1}(\xi_1, \zeta_1) = a_{0,2}(\zeta_1, \zeta_2) = 0,$$

and $a_{i,j} \neq 0$ for some i, j , $i + j > 2$, then system (12) does not possess higher symmetries of any order.

Proof. Without loss of generality, we assume that some of the terms in (12) of degree $s \geq 3$ are not equal zero and all terms of degree less than s are equal zero. Suppose that such system possesses a type I symmetry (14) of order $m > 1$. Then it is easy to see from formulae (41) and (43) that $A_{i,j}(\xi_1, \dots, \xi_i, \zeta_1, \dots, \zeta_j) = 0$, $i + j < s$ and

$$\hat{A}_{i,s-i}(x_1^2, \dots, x_s^2) = \frac{(x_1^2 + \dots + x_s^2)^r N_s^{(m)}(x_1^2, \dots, x_s^2)}{S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)} \hat{a}_{i,s-i}(x_1^2, \dots, x_s^2), \quad i = 0, \dots, s.$$

The left hand side of the above formula must be polynomial in x_1, \dots, x_s . From Proposition 5 we know that polynomials $S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)$, $i = 0, \dots, s$ are irreducible polynomials over \mathbb{C} and these polynomials do not divide

$$(x_1^2 + \dots + x_s^2)^r N_s^{(m)}(x_1^2, \dots, x_s^2)$$

for any $m > 1$ and $r \geq 0$. Therefore $S_{s,i}^{(n)}$ must divide $\hat{a}_{i,s-i}(x_1^2, \dots, x_s^2)$, what is impossible since

$$\deg(\hat{a}_{i,s-i}(x_1^2, \dots, x_s^2)) = 4n + 2 - 2(s-1)W(u) < \deg(S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)) = 4n + 2, \quad i = 0, \dots, s.$$

Thus system (12) does not possess any type I symmetry. The consideration in the case of type II symmetries is similar. \diamond

We are now ready to prove the following important theorem in classification of integrable homogeneous systems of the form (12).

Theorem 2. Consider homogeneous system (12) with $W(u) > 0$. Assume that it possesses an m -th order type I or type II symmetry of the form (14) or (15). Suppose there is another system of the same weight and of the same form

$$\begin{cases} u_t = v\zeta_1^r, \\ v_t = u\zeta_1^{2n+1-r} + \sum_{k \geq 2} \sum_{i=0}^k u^i v^{k-i} b_{i,k-i}(\xi_1, \dots, \xi_k, \zeta_1, \dots, \zeta_{k-i}), \end{cases} \quad (48)$$

whose quadratic terms equal to those of (12), that is, $b_{i,2-i}(x, y) = a_{i,2-i}(x, y)$, $i = 0, 1, 2$. Then if system (48) possesses an m -th order type I or type II symmetry, then equation (48) and (12) are equal and sharing the same type I or type II symmetry of order m .

Proof. We will prove the statement by induction. Consider the case of symmetries of type I. Let us suppose that equation (48) possesses a type I symmetry of the form

$$u_\tau = u\xi_1^m + \sum_{s \geq 2} \sum_{j=0}^s u^j v^{s-j} B_{j,s-j}(\xi_1, \dots, \xi_j, \zeta_1, \dots, \zeta_{s-j}) \quad (49)$$

The quadratic terms of this symmetry coincide with those of (14) $B_{i,2-i}(x, y) = A_{i,2-i}(x, y)$, $i = 0, 1, 2$ because the quadratic terms of (48) and (12) coincide. Let us prove that the terms of degree 3 in (12) and (48) coincide. According to formula (43) we have

$$\hat{A}_{i,3-i}(x_1^2, x_2^2, x_3^2) = \frac{(x_1^2 + x_2^2 + x_3^2)^r N_3^{(m)}(x_1^2, x_2^2, x_3^2)}{S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)} \hat{a}_{i,3-i}(x_1^2, x_2^2, x_3^2) + L_i^{(3)}, \quad i = 0, 1, 2, 3$$

for the coefficients of the symmetry (14) of (12) and

$$\hat{B}_{i,3-i}(x_1^2, x_2^2, x_3^2) = \frac{(x_1^2 + x_2^2 + x_3^2)^r N_3^{(m)}(x_1^2, x_2^2, x_3^2)}{S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)} \hat{b}_{i,3-i}(x_1^2, x_2^2, x_3^2) + \tilde{L}_i^{(3)}, \quad i = 0, 1, 2, 3$$

for the coefficients of the symmetry (49) of (48).

Notice that $L_i^{(3)} = \tilde{L}_i^{(3)}$, $i = 0, 1, 2, 3$ since they only depend on the linear and quadratic terms of equations (12), (48) and their symmetries (14) and (49), which are equal by the assumption. Therefore

$$\begin{aligned} \hat{B}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{A}_{i,3-i}(x_1^2, x_2^2, x_3^2) &= \\ &= \frac{(x_1^2 + x_2^2 + x_3^2)^r N_3^{(m)}(x_1^2, x_2^2, x_3^2)}{S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)} \left(\hat{b}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{a}_{i,3-i}(x_1^2, x_2^2, x_3^2) \right). \end{aligned}$$

Its left hand side must be a polynomial in x_1, x_2, x_3 . In proposition 5 we proved that polynomials $S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)$ are irreducible polynomials over \mathbb{C} . It is easy to see that it does not divide $(x_1^2 + x_2^2 + x_3^2)^r N_3^{(m)}(x_1^2, x_2^2, x_3^2)$ for any $n \geq 1$, $m > 1$ and $i = 0, 1, 2, 3$. Therefore $S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)$ must divide $\hat{b}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{a}_{i,3-i}(x_1^2, x_2^2, x_3^2)$. But this is impossible since, for $i \in \{0, 1, 2, 3\}$,

$$\deg(\hat{b}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{a}_{i,3-i}(x_1^2, x_2^2, x_3^2)) = 2(2n + 1 - 2W(u) - r) < \deg(S_{3,i}^{(n)}(x_1^2, x_2^2, x_3^2)) = 4n + 2$$

due to the assumption $W(u) > 0$ and $r \geq 0$. Hence

$$\hat{b}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{a}_{i,3-i}(x_1^2, x_2^2, x_3^2) = 0, \quad \hat{B}_{i,3-i}(x_1^2, x_2^2, x_3^2) - \hat{A}_{i,3-i}(x_1^2, x_2^2, x_3^2) = 0,$$

that is, $b_{i,3-i}(\xi_1, \xi_2, \xi_3) = a_{i,3-i}(\xi_1, \xi_2, \xi_3)$, $B_{i,3-i}(\xi_1, \xi_2, \xi_3) = A_{i,3-i}(\xi_1, \xi_2, \xi_3)$.

Suppose we have proved that the terms of degree $s - 1$ coincide. For terms of degree s , we have, according to formula (43),

$$\hat{A}_{i,s-i}(x_1^2, \dots, x_s^2) = \frac{(x_1^2 + \dots + x_s^2)^r N_s^{(m)}(\xi_1, \dots, \xi_s)}{S_{s,i}^{(n)}(\xi_1, \dots, \xi_s)} \hat{a}_{i,s-i}(x_1^2, \dots, x_s^2) + L_i^{(s)}, \quad i = 0, \dots, s$$

for the coefficients of the symmetry (14) of (12) and

$$\hat{B}_{i,s-i}(x_1^2, \dots, x_s^2) = \frac{(x_1^2 + \dots + x_s^2)^r N_s^{(m)}(\xi_1, \dots, \xi_s)}{S_{s,i}^{(n)}(\xi_1, \dots, \xi_s)} \hat{b}_{i,s-i}(x_1^2, \dots, x_s^2) + \tilde{L}_i^{(s)}, \quad i = 0, \dots, s$$

for the coefficients of the symmetry (49) of (48). Terms $L_i^{(s)}$ and $\tilde{L}_i^{(s)}$ depend only on coefficients of degree less than s and therefore they are equal $L_i^{(s)} = \tilde{L}_i^{(s)}$, $i = 0, \dots, s$. Hence

$$\hat{B}_{i,s-i}(x_1^2, \dots, x_s^2) - \hat{A}_{i,s-i}(x_1^2, \dots, x_s^2) = \frac{(x_1^2 + \dots + x_s^2)^r N_s^{(m)}(\xi_1, \dots, \xi_s)}{S_{s,i}^{(n)}(\xi_1, \dots, \xi_s)} \left(\hat{b}_{i,s-i}(x_1^2, \dots, x_s^2) - \hat{a}_{i,s-i}(x_1^2, \dots, x_s^2) \right).$$

Again the left hand side of the last formula must be a polynomial in x_1, \dots, x_s and therefore $S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)$ must divide $\hat{b}_{i,s-i}(x_1^2, \dots, x_s^2) - \hat{a}_{i,s-i}(x_1^2, \dots, x_s^2)$. This is impossible since

$$\deg(\hat{b}_{i,s-i}(x_1^2, \dots, x_s^2) - \hat{a}_{i,s-i}(x_1^2, \dots, x_s^2)) = 4n + 2 - 2(s-1)W(u) < \deg(S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)) = 4n + 2, \quad i = 0, \dots, s$$

using the assumption $W(u) > 0$. Hence $b_{i,s-i} = a_{i,s-i}$ and $B_{i,s-i} = A_{i,s-i}$ for $i = 0, \dots, s$. Thus we proved the statement for type I symmetries. The proof in the case of symmetries of type II is analogous. \diamond

Finally, we state and prove the global classification result of homogeneous system (9).

Theorem 3. *If a homogeneous system (9) with $W(u) = w > 0$ possesses a hierarchy of infinitely many higher symmetries, then it is one of the systems in the following list up to re-scaling $u \rightarrow \alpha u, v \rightarrow \beta v, t \rightarrow \gamma t, x \rightarrow \delta x$, where $\alpha, \beta, \gamma, \delta$ are constant:*

$$\begin{cases} u_t = v_1, \\ v_t = u_2 + 3uv_1 + vu_1 - 3u^2u_1, \end{cases} \quad (50)$$

$$\begin{cases} u_t = v_1, \\ v_t = (D_x + u)^{2n}(u) - v^2, \quad n = 1, 2, 3, \dots \end{cases} \quad (51)$$

Proof. We prove the theorem in symbolic representation. We know that system (9) in symbolic representation takes the form of (12). In Corollary 3 we have proved that if (9) possesses higher symmetries then the differential polynomial F in (9) must contain quadratic terms and therefore $w = W(u) > 0$ is either integer or half-integer. Indeed,

$$\begin{aligned} \mathcal{R}_{2n+1-r+w}^2 &= \text{span}\{u_i u_j | i + j = 2n + 1 - r - w, i, j \in \mathbb{N}_0\} \bigoplus \\ &\quad \text{span}\{u_i v_j | i + j = n + \frac{1}{2} - w, i, j \in \mathbb{N}_0\} \bigoplus \text{span}\{v_i v_j | i + j = r - w, i, j \in \mathbb{N}_0\}, \end{aligned}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Therefore if differential polynomial F in (9) contains quadratic terms then the weight of variable u is either integer or half-integer, i.e. either $w \in \mathbb{N}$ or $w = s - \frac{1}{2}$, $s \in \mathbb{N}$. Furthermore, from the Theorem 2 it follows that it suffice to classify integrable systems (9) up to quadratic terms. If system (9) or rather say (12) possesses a hierarchy of infinitely many higher symmetries, it possesses infinitely many approximate symmetries of degree 2 of type I or type II. We first assume that these are of type I of the form

$$u_{\tau_j} = u \xi_1^{m_j} + u^2 A_{2,0}^{(j)}(\xi_1, \xi_2) + uv A_{1,1}^{(j)}(\xi_1, \zeta_1) + v^2 A_{0,2}^{(j)}(\zeta_1, \zeta_2) + o(\mathcal{R}^2), \quad j = 1, 2, 3, \dots$$

where $1 < m_1 < m_2 < \dots < m_j < m_{j+1} < \dots$. From Theorem 1 and Corollary 2, it follows that

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{N_2^{(m_j)}(x_1^2, x_2^2)}{S_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{2,0}(x_1^2, x_2^2), & \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{N_2^{(m_j)}(x_1^2, x_2^2)}{S_{2,1}^{(n)}(x_1^2, x_2^2)} \hat{a}_{1,1}(x_1^2, x_2^2) \\ \hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{N_2^{(m_j)}(x_1^2, x_2^2)}{S_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{0,2}(x_1^2, x_2^2) \end{aligned}$$

must be polynomials in x_1, x_2 for any m_j , $j = 1, 2, \dots$. Notice that polynomials $S_{2,i}^{(n)}(x_1^2, x_2^2)$ and $N_2^{(m_j)}(x_1^2, x_2^2)$ can be factorized as

$$S_{2,i}^{(n)}(x_1^2, x_2^2) = x_1^2 x_2^2 \hat{S}_{2,i}^{(n)}(x_1^2, x_2^2), \quad N_2^{(m_j)}(x_1^2, x_2^2) = x_1^2 x_2^2 \hat{N}_2^{(m_j)}(x_1^2, x_2^2),$$

which leads to

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{\hat{N}_2^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{2,0}(x_1^2, x_2^2), & \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{\hat{N}_2^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)} \hat{a}_{1,1}(x_1^2, x_2^2) \\ \hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{\hat{N}_2^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{0,2}(x_1^2, x_2^2). \end{aligned}$$

We can easily check $\gcd((x_1^2 + x_2^2)^r, S_{2,i}^{(n)}(x_1^2, x_2^2)) = \gcd((x_1^2 + x_2^2)^r, \hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)) = 1$. Moreover, from Lemma 1 in Appendix A it follows that if there exist infinitely many m_j such that $\hat{A}_{i,2-i}$, $i = 0, 1, 2$ are polynomial, then $m_j = 2, 3, 4, \dots$ and

$$\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) | \hat{a}_{2,0}(x_1^2, x_2^2), \quad \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2) | \hat{a}_{1,1}(x_1^2, x_2^2), \quad \hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) | \hat{a}_{0,2}(x_1^2, x_2^2). \quad (52)$$

Counting the degrees of these polynomials we obtain that the division is possible only if

$$\deg(\hat{a}_{i,2-i}) = 4n + 2 - 2r - 2w \geq \deg(\hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)) = 4n - 2, \quad i = 0, 1, 2$$

implying $r + w \leq 2$. Since $w > 0$ is either integer or half-integer, there are only four possible cases as follows:

1. $w = 2, \quad r = 0$;
2. $w = \frac{3}{2}, \quad r = 0$;
3. $w = 1, \quad r = 0, 1$;
4. $w = \frac{1}{2}, \quad r = 0, 1$.

Let us compute the degrees of quadratic terms in the system we are considering. Applying formula (13), we have $\deg(a_{1,1}(\xi_1, \xi_2)) = n - w + \frac{1}{2}$, $\deg(a_{0,2}(\xi_1, \xi_2)) = r - w$, $\deg(a_{2,0}(\xi_1, \xi_2)) = 2n + 1 - w - r$. We know that all these numbers should be positive integers, which enable us to determine the possible quadratic terms.

Case 1. The only possible quadratic term in equation (12) is $a_{2,0}$. This leads to

$$\hat{a}_{2,0}(x_1^2, x_2^2) = \hat{a}_{0,2}(x_1^2, x_2^2) = a_{2,0}(x_1^2, x_2^2), \quad \hat{a}_{1,1}(x_1^2, x_2^2) = 2a_{2,0}(x_1^2, x_2^2).$$

From formula (52) we see that both $\hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)$ must divide $a_{2,0}(x_1^2, x_2^2)$. It is easy to check that

$$\gcd(\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2), \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)) = 1.$$

So we have $\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2) | a_{2,0}(x_1^2, x_2^2)$. But this is impossible since

$$\deg(a_{2,0}(x_1^2, x_2^2)) = 4n - 2 < \deg(\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)) = 8n - 4, \quad \text{for all } n > 0.$$

Case 2. The only possible quadratic term in equation (12) is $a_{1,1}$. Notice that polynomial $\hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)$, $\hat{a}_{2,0}(x_1^2, x_2^2)$ and $\hat{a}_{0,2}(x_1^2, x_2^2)$ are symmetric and polynomial $\hat{a}_{1,1}(x_1^2, x_2^2)$ is antisymmetric with respect to x_1, x_2 . Due to (52), we have

$$\hat{a}_{2,0}(x_1^2, x_2^2) = -\hat{a}_{0,2}(x_1^2, x_2^2) = c_1(x_1 + x_2) \hat{S}_{2,2}^{(n)}(x_1^2, x_2^2), \quad \hat{a}_{1,1} = c_2(x_1 - x_2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2), \quad (53)$$

where c_1, c_2 are arbitrary constants. Inverting formulae (23), (24) and (25) or using directly (20) for $a_{1,1}$, we find that

$$a_{1,1}(x_1^2, x_2^2) = \frac{1}{4x_2^{2n+1}} (2\hat{a}_{2,0}(x_1^2, x_2^2) - 2\hat{a}_{0,2}(x_1^2, x_2^2) + \hat{a}_{1,1}(x_2^2, x_1^2) - \hat{a}_{1,1}(x_1^2, x_2^2)), \quad (54)$$

which must be a polynomial in x_1, x_2 . Substituting (53) into (54), we see that this can only happen when $c_1 = c_2 = 0$ implying $a_{1,1} = 0$. Hence our equation does not contain quadratic terms and therefore does not possess higher symmetries.

Case 3. If $r = 0$, the only possible quadratic term in equation (12) is $a_{2,0}$. This leads to

$$\hat{a}_{2,0}(x_1^2, x_2^2) = \hat{a}_{0,2}(x_1^2, x_2^2) = a_{2,0}(x_1^2, x_2^2), \quad \hat{a}_{1,1}(x_1^2, x_2^2) = 2a_{2,0}(x_1^2, x_2^2).$$

Similar to Case 1, we obtain that $\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)$ must divide $a_{2,0}(x_1^2, x_2^2)$, which can only happen if

$$\deg(a_{2,0}(x_1^2, x_2^2)) = 4n \geq \deg(\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)) = 8n - 4,$$

that is, $4n \leq 4$ implying $n = 1$. Therefore, there exists a constant c such that

$$a_{2,0}(x_1^2, x_2^2) = c \hat{S}_{2,2}^{(n)}(x_1^2, x_2^2) \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2) = c (9x_1^4 + 9x_2^4 + 14x_1^2 x_2^2).$$

This leads to

$$\hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) = \hat{A}_{0,2}(x_1^2, x_2^2) = c(3x_1^2 + 3x_2^2 + 2x_1x_2)\hat{N}_2^{(m_j)}, \quad \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) = c(3x_1^2 + 3x_2^2 - 2x_1x_2)\hat{N}_2^{(m_j)}.$$

From formulae (38), (39) and (40) we find that

$$A_{2,0}^{(j)}(x_1^2, x_2^2) = 3c(x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2), \quad A_{1,1}^{(j)}(x_1^2, x_2^2) = 0 \quad A_{0,2}^{(j)}(x_1^2, x_2^2) = 2c\frac{\hat{N}_2^{(m_j)}(x_1^2, x_2^2)}{x_1^2x_2^2}.$$

It is easy to see that $x_1^2x_2^2$ does not divide $\hat{N}_2^{(m_j)}(x_1^2, x_2^2)$ for any integer $m_j > 1$, so $A_{0,2}(x_1^2, x_2^2)$ is not a polynomial unless $c = 0$ what implies $a_{2,0} = 0$. So system (12) has no quadratic terms and therefore is not integrable.

If $r = 1$, then we have $a_{1,1} = 0$ and therefore

$$\begin{aligned} \hat{a}_{2,0}(x_1^2, x_2^2) &= \hat{a}_{0,2}(x_1^2, x_2^2) = a_{2,0}(x_1^2, x_2^2) + a_{0,2}(x_1^2, x_2^2)(x_1x_2)^{2n+1} \\ \hat{a}_{1,1}(x_1^2, x_2^2) &= 2a_{2,0}(x_1^2, x_2^2) - 2a_{0,2}(x_1^2, x_2^2)(x_1x_2)^{2n+1} \end{aligned}$$

It is easy to see that $\deg(\hat{a}_{i,2-i}(x_1^2, x_2^2)) = \deg(\hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)) = 4n - 2$, $i = 0, 1, 2$. It follows from (52) that

$$\hat{a}_{2,0}(x_1^2, x_2^2) = \hat{a}_{0,2}(x_1^2, x_2^2) = c_1\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2), \quad \hat{a}_{1,1}(x_1^2, x_2^2) = c_2\hat{S}_{2,1}^{(n)}(x_1^2, x_2^2), \quad (55)$$

where c_1, c_2 are constant. Inverting formulae (23), (24) and (25) or using directly formula (20) for the quadratic terms, we find

$$\begin{aligned} a_{2,0}(x_1^2, x_2^2) &= \frac{1}{8} (4\hat{a}_{2,0}(x_1^2, x_2^2) + \hat{a}_{1,1}(x_1^2, x_2^2) + \hat{a}_{1,1}(x_2^2, x_1^2)), \\ a_{0,2}(x_1^2, x_2^2) &= \frac{1}{8x_1^{2n-1}x_2^{2n-1}} (4\hat{a}_{2,0}(x_1^2, x_2^2) - \hat{a}_{1,1}(x_1^2, x_2^2) - \hat{a}_{1,1}(x_2^2, x_1^2)) \end{aligned}$$

Substituting (55) into the above formulae we find that $a_{0,2}(x_1^2, x_2^2)$ is a polynomial in x_1^2, x_2^2 if and only if $c_2 = 2c_1$. Hence $a_{0,2}(x_1^2, x_2^2) = -2c_1$. Without loss of generality we can put $c_1 = \frac{1}{2}$, so

$$a_{0,2}(x_1^2, x_2^2) = -1, \quad a_{2,0}(x_1^2, x_2^2) = \frac{1}{4} \sum_{j=1}^{2n} C_{2n+1}^j (x_1^2)^{j-1} (x_2^2)^{2n-j}. \quad (56)$$

Therefore

$$\hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) = \hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) = \frac{1}{2}(x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2), \quad \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) = (x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2)$$

and leads to

$$A_{2,0}^{(j)}(x_1^2, x_2^2) = \frac{1}{2}(x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2), \quad A_{1,1}^{(j)}(x_1^2, x_2^2) = A_{0,2}^{(j)}(x_1^2, x_2^2) = 0.$$

Thus system (12) with quadratic terms (56) and $r = 1$ possesses infinitely many type I approximate symmetries of degree 2 of order $m_j > 1$.

Case 4. The only possible quadratic term in equation (12) is $a_{1,1}$. If $r = 0$, then $\deg(\hat{a}_{i,2-i}(x_1^2, x_2^2)) = 4n + 1$, $i = 0, 1, 2$. Notice that polynomials $\hat{S}_{2,i}^{(n)}(x_1^2, x_2^2)$, $\hat{a}_{2,0}(x_1^2, x_2^2)$ and $\hat{a}_{0,2}(x_1^2, x_2^2)$ are symmetric while polynomial $\hat{a}_{1,1}(x_1^2, x_2^2)$ is antisymmetric with respect to x_1, x_2 . Due to (52), we have

$$\hat{a}_{2,0}(x_1^2, x_2^2) = -\hat{a}_{0,2}(x_1^2, x_2^2) = (c_1(x_1^3 + x_2^3) + c_2x_1x_2(x_1 + x_2))\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2), \quad (57)$$

$$\hat{a}_{1,1}(x_1^2, x_2^2) = (d_1(x_1^3 - x_2^3) + d_2x_1x_2(x_1 - x_2))\hat{S}_{2,1}^{(n)}(x_1^2, x_2^2), \quad (58)$$

where c_1, c_2, d_1, d_2 are arbitrary constants. Substituting these expressions into (54) as in Case 2, we find that $\hat{a}_{1,1}(x_1^2, x_2^2)$ is a non-vanishing polynomial in even powers of x_1, x_2 only when $n = 1$ and $d_1 = 2c_1$, $d_2 = -\frac{4}{3}c_1$, $c_2 = \frac{2}{3}c_1$. Without loss of generality we can choose $c_1 = \frac{3}{2}$. It follows that

$$a_{1,1}(x_1^2, x_2) = 11x_1^2 + 9x_2^2$$

Therefore for $\hat{A}_{i,2-i}$ we obtain

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= -\hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) = \left(\frac{3}{2}(x_1^3 + x_2^3) + x_1x_2(x_1 + x_2) \right) \hat{N}_2^{(m_j)}(x_1^2, x_2^2) \\ \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) &= (3(x_1^3 - x_2^3) - 2x_1x_2(x_1 - x_2)) \hat{N}_2^{(m_j)}(x_1^2, x_2^2) \end{aligned}$$

and hence

$$A_{1,1}^{(j)}(x_1^2, x_2^2) = \frac{\hat{N}_2^{(m_j)}(x_1^2, x_2^2)}{x_2^2} (3x_2^2 + 2x_1^2), \quad A_{2,0}^{(j)}(x_1^2, x_2^2) = A_{0,2}^{(j)}(x_1^2, x_2^2) = 0.$$

It is clear that $A_{1,1}^{(j)}(x_1^2, x_2^2)$ is not polynomial since x_2^2 does not divide $\hat{N}_2^{(m_j)}(x_1^2, x_2^2)$ for any $m_j > 1$. Thus equation (12) does not possess an infinite many type I approximate symmetries of degree 2.

If $r = 1$, then $\deg(\hat{a}_{i,2-i}(x_1^2, x_2^2)) = 4n - 1$, $i = 0, 1, 2$. Similar as Case 2, we obtain the formula (53). Inverting (23), (24) and (25) we find that

$$a_{1,1}(x_1^2, x_2^2) = \frac{1}{4x_2^{2n-1}} (2\hat{a}_{2,0}(x_1^2, x_2^2) - 2\hat{a}_{0,2}(x_1^2, x_2^2) + \hat{a}_{1,1}(x_2^2, x_1^2) - \hat{a}_{1,1}(x_1^2, x_2^2)), \quad (59)$$

Substituting (53) into (59), we find that $a_{1,1}(x_1^2, x_2^2)$ is a non-zero polynomial in even powers of x_1, x_2 if and only $c_2 = 2c_1$ and $n = 1$. Without loss of generality we choose $c_1 = \frac{1}{2}$, $c_2 = 1$ and hence

$$a_{1,1}(x_1^2, x_2^2) = x_1^2 + 3x_2^2. \quad (60)$$

For $\hat{A}_{i,2-i}^{(j)}(x_1^2, x_2^2)$ we therefore obtain

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= -\hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) = \frac{1}{2}(x_1 + x_2)(x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2), \\ \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) &= (x_1 - x_2)(x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2) \end{aligned}$$

and hence

$$A_{1,1}^{(j)}(x_1^2, x_2^2) = (x_1^2 + x_2^2)\hat{N}_2^{(m_j)}(x_1^2, x_2^2), \quad A_{2,0}^{(j)}(x_1^2, x_2^2) = A_{0,2}^{(j)}(x_1^2, x_2^2) = 0.$$

Thus we proved that system (12) with the quadratic terms (60) and $n = 1, r = 1$ possesses infinitely many type I approximate symmetries of degree 2 of any order $m_j > 1$.

Summarizing the results in the above four cases, we proved that if system (9) possesses infinitely many type I approximate symmetries of degree 2 of any order $m_j > 1$, then it is up re-scaling one of the systems in the following list:

$$\begin{cases} u_t = v_1 \\ v_t = u_2 + u_1v + 3uv_1 + h, \quad h = o(\mathcal{R}^2) \end{cases} \quad (61)$$

$$\begin{cases} u_t = v_1 \\ v_t = u_{2n} - v^2 + \frac{1}{2} \sum_{j=1}^{2n} C_{2n+1}^j u_{j-1} u_{2n-j} + h, \quad h = o(\mathcal{R}^2) \end{cases} \quad (62)$$

Let us now assume that system (12) possesses infinitely many approximate symmetries of degree 2 of type II of the form

$$u_{\tau_j} = v\zeta_1^{r+m_j} + u^2 A_{2,0}^{(j)}(\xi_1, \xi_2) + uv A_{1,1}^{(j)}(\xi_1, \zeta_1) + v^2 A_{0,2}^{(j)}(\zeta_1, \zeta_2) + o(\mathcal{R}^2), \quad j = 1, 2, 3, \dots$$

and $1 < m_1 < m_2 < \dots < m_j < m_{j+1} < \dots$. From Theorem 1 and Corollary 2, it follows that

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{M_{2,2}^{(m_j)}(x_1^2, x_2^2)}{S_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{2,0}(x_1^2, x_2^2), \quad \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) = (x_1^2 + x_2^2)^r \frac{M_{2,1}^{(m_j)}(x_1^2, x_2^2)}{S_{2,1}^{(n)}(x_1^2, x_2^2)} \hat{a}_{1,1}(x_1^2, x_2^2) \\ \hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) &= -(x_1^2 + x_2^2)^r \frac{M_{2,2}^{(m_j)}(x_1^2, x_2^2)}{S_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{0,2}(x_1^2, x_2^2) \end{aligned}$$

must be polynomials in x_1, x_2 for every m_j . Notice that polynomials $M_{2,i}^{(m_j)}(x_1^2, x_2^2)$ can be factorized as

$$M_{2,i}^{(m_j)}(x_1^2, x_2^2) = x_1^2 x_2^2 \hat{M}_{2,i}^{(m_j)}(x_1^2, x_2^2),$$

which leads to

$$\begin{aligned} \hat{A}_{2,0}^{(j)}(x_1^2, x_2^2) &= (x_1^2 + x_2^2)^r \frac{\hat{M}_{2,2}^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{2,0}(x_1^2, x_2^2), \quad \hat{A}_{1,1}^{(j)}(x_1^2, x_2^2) = (x_1^2 + x_2^2)^r \frac{\hat{M}_{2,1}^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)} \hat{a}_{1,1}(x_1^2, x_2^2) \\ \hat{A}_{0,2}^{(j)}(x_1^2, x_2^2) &= -(x_1^2 + x_2^2)^r \frac{\hat{M}_{2,2}^{(m_j)}(x_1^2, x_2^2)}{\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)} \hat{a}_{0,2}(x_1^2, x_2^2) \end{aligned}$$

from Lemma 2 in Appendix A, if there exist infinitely many m_j such that the above expressions are polynomials in x_1, x_2 , then $m_j = 1, 2, 3, 4, \dots$ and

$$\hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)|\hat{a}_{2,0}(x_1^2, x_2^2), \quad \hat{S}_{2,2}^{(n)}(x_1^2, x_2^2)|\hat{a}_{0,2}(x_1^2, x_2^2) \quad \hat{S}_{2,1}^{(n)}(x_1^2, x_2^2)|\hat{a}_{1,1}(x_1^2, x_2^2). \quad (63)$$

We then repeat what we did for the case of type I symmetries and we obtain the similar conclusion: If system (12) possesses infinitely many type II approximate symmetries of degree 2 of any order $m_j > 1$, then it is up re-scaling either (61) or (62).

By now we have classified all systems (12) possessing infinite many approximate symmetries of degree 2. We know that system (50) possesses infinitely many higher symmetries of any order of form (14) and (15). Notice that the quadratic terms of (50) and of the system (61) coincide. By Theorem 2, if system (61) possesses an exact symmetry of some order m for some h in its right hand side, it must coincide with the system (50), that is, $h = -3u^2u_1$. Similarly, quadratic terms of (51) and of the system (62) coincide and system (51) possesses higher symmetries of any order. Therefore, if system (62) possesses an exact symmetry of some order for some h then it coincides with (51). \diamond

System (51) is equivalent to a linear equation $w_{tt} = \partial_x^{2n+1}w$ under the Cole-Hopf transformation $u = (\log w)_x$. This system possesses infinitely many type I symmetries

$$u_{\tau_m} = D_x(D_x + u)^{m-1}u, \quad m = 2, 3, \dots$$

and type II symmetries

$$u_{\tau_m} = D_x(v + D_t)(D_x + u)^{m-1}u.$$

These symmetries correspond to the symmetries $w_{\tau_m} = w_m$ and $w_{\tau_m} = w_{tm}$ of $w_{tt} = w_{2n+1}$.

5 Conclusion and Discussion

This is the first paper dealing with the global classification of the systems with two components. The success of global classification relies on so-called implicit function theorem in [11]. The theorem implies that the existence of infinitely many approximate symmetries of low degree together with one symmetry leads to integrability. The proof of the theorem is straightforward and purely algebraic. However, to check the conditions of the theorem requires to prove the irreducibility of a family of polynomials. In this paper, Theorem 2 can be viewed as another version of it. The required irreducibility of polynomials $S_{s,i}^{(n)}(x_1^2, \dots, x_s^2)$ defined by (36) is proved by algebraic geometrical arguments in Proposition 5.

The formal diagonalisation approach proposed in this paper can be applied to large classes of systems with the diagonalisable linear part. This type of equations is generic and it has not been treated by other classification methods. Our approach enables us to explicitly write down the necessary conditions for integrability. We are able to prove that these conditions are indeed sufficient. Resolution of these conditions leads us to the ultimate classification result.

We have not applied this approach to the systems with nilpotent linear part yet. However, we do not expect any difficulties in this case. The final goal will be to combine these two classes and to obtain the global classification of homogeneous polynomial evolutionary systems of two components as it has been done for the scalar evolutionary equations [11].

Appendix A

Here we prove the lemmas required in the proof of Theorem 3. which is based on the theorem of Lech-Mahler and its straightforward corollary.

Theorem 4 (Lech, Mahler). *Let $A_1, A_2, \dots, A_n \in \mathbb{C}$ and a_1, a_2, \dots, a_n be non-zero complex numbers. Suppose that none of the ratios A_i/A_j with $i \neq j$ is a root of unity. Then the equation*

$$a_1A_1^m + a_2A_2^m + \dots + a_nA_n^m = 0$$

in the unknown integer m has finitely many solutions.

Corollary 4. *Let $A, B, C \in \mathbb{C}$ and a, b, c be non-zero complex numbers. Suppose that the equation*

$$aA^m + bB^m + cC^m = 0$$

has infinitely many integers m as solution. Then A/B and A/C are both roots of unity.

We simplify our notations by writing $S^+ = S_{2,2}^{(n)}$, $S^- = S_{2,1}^{(n)}$, $M_{m_j}^+ = M_{2,2}^{(m_j)}$ and $M_{m_j}^- = M_{2,1}^{(m_j)}$.

Lemma 1. *Consider an infinite sequence of polynomials*

$$N_2^{(m_j)}(x_1^2, x_2^2) = (x_1^2 + x_2^2)^{m_j} - x_1^{2m_j} - x_2^{2m_j}, \quad 1 < m_1 < m_2 < \dots < m_j < m_{j+1} < \dots \quad (64)$$

If there exist infinite many m_j such that each $N_2^{(m_j)}(x_1^2, x_2^2)$ has nontrivial common divisor with the polynomial $S^\pm(x_1^2, x_2^2)$ defined by (36), Then $m_j = 2, 3, 4, \dots$ and $\gcd\{S^\pm(x_1^2, x_2^2), N_2^{(2)}(x_1^2, x_2^2), N_2^{(3)}(x_1^2, x_2^2), \dots\} = x_1^2 x_2^2$.

Proof. Since the polynomials $S^\pm(x_1^2, x_2^2)$ and $N_2^{(m_j)}(x_1^2, x_2^2)$ are homogeneous polynomials in x_1, x_2 it is convenient to consider them in affine coordinate $\frac{x_1}{x_2}$. If they have nontrivial common divisor, these exist q as a common root, that is,

$$\begin{aligned} S^\pm(q) &= (1 + q^2)^{2n+1} - (1 \pm q^{2n+1})^2 = 0 \\ N_2^{(m_j)}(q) &= (1 + q^2)^{m_j} - 1 - q^{2m_j} = 0, \end{aligned}$$

which is required to be satisfied for infinitely many integers $m_j > 1$. Applying the Lech-Mahler theorem to it we find that this is possible if one of the listed cases holds:

- $q = 0$ (double root), $m_j = 2, 3, 4, \dots$,
- $q = \pm i$, $m_j = 1 \pmod{2}$,
- $q = e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}$, $m_j = 5 \pmod{6}$,
- $q = e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}$ (double roots), $m_j = 1 \pmod{6}$.

The last three sets of roots do not satisfy the equation $S^\pm(q) = 0$ for any $n > 0$, while $q = 0$ is a double roots of $S^\pm(q)$. Lemma is proved. \diamond

Lemma 2. *Consider an infinite sequence of polynomials*

$$\begin{aligned} M_{m_j}^\pm(x_1^2, x_2^2) &= (x_1^2 + x_2^2)(x_1^{2n+1} \pm x_2^{2n+1}) - x_1^{2n+2m+1} - (\pm x_2^{2n+2m+1}), \\ 0 &< m_1 < m_2 < \dots < m_j < m_{j+1} < \dots \end{aligned} \quad (65)$$

If there exist infinite many m_j such that each $M_{m_j}^+(x_1^2, x_2^2)$ (or $M_{m_j}^-(x_1^2, x_2^2)$) has nontrivial common divisor with the polynomial $S^+(x_1^2, x_2^2)$ (or $S^-(x_1^2, x_2^2)$) defined by (36), Then $m_j = 1, 2, 3, 4, \dots$ and

$$\begin{aligned} \gcd\{S^+(x_1^2, x_2^2), M_1^+(x_1^2, x_2^2), M_2^+(x_1^2, x_2^2), \dots\} &= x_1^2 x_2^2; \\ \gcd\{S^-(x_1^2, x_2^2), M_1^-(x_1^2, x_2^2), M_2^-(x_1^2, x_2^2), \dots\} &= x_1^2 x_2^2. \end{aligned}$$

Proof. Similar as in the proof of Lemma 1, we consider polynomials $S^\pm(x_1^2, x_2^2)$ and $M_{m_j}^\pm(x_1^2, x_2^2)$ in affine coordinate $\frac{x_1}{x_2}$. If $S^+(x_1^2, x_2^2)$ and $M_{m_j}^+(x_1^2, x_2^2)$ have nontrivial common divisor, these exist q as a common root, that is,

$$\begin{aligned} S^+(q) &= (1 + q^2)^{2n+1} - 1 - q^{4n+2} - 2q^{2n+1} = 0 \\ M_{m_j}^+(q) &= (1 + q^2)^{m_j} (1 + q^{2n+1}) - (q^2)^{m_j} q^{2n+1} - 1 = 0, \end{aligned}$$

which is required to hold for infinitely many positive integers m_j . We apply the Lech-Mahler theorem to it and obtain:

- $q = 0$ (double root), $m_j \in \mathbb{Z}_+$,
- $q^{2n+1} = -1$, $q^{2m_j} = 1$,
- $q = e^{\frac{\pi i}{3}}, e^{\frac{5\pi i}{3}}$ and $n = 1 \pmod{3}$, $m_j = 0 \pmod{3}$ or $n = 2 \pmod{3}$, $m_j = 0 \pmod{6}$, $1 \pmod{6}$ or $n = 0 \pmod{3}$, $m_j = 0 \pmod{6}$, $5 \pmod{6}$,
- $q = e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}$ and $n = 1 \pmod{3}$, $m_j = 0 \pmod{6}$ or $n = 2 \pmod{3}$, $m_j = 0 \pmod{6}$, $4 \pmod{6}$ or $n = 0 \pmod{3}$, $m_j = 0 \pmod{6}$, $2 \pmod{6}$.

It is easy to check that the roots from the last three cases are not the roots of $S^+(q)$ for any integer $n \geq 1$, while $q = 0$ is a double root of $S^+(q)$. Therefore, we have $m_j = 1, 2, 3, \dots$ and $\gcd\{S^+(x_1^2, x_2^2), M_1^+(x_1^2, x_2^2), M_2^+(x_1^2, x_2^2), \dots\} = x_1^2 x_2^2$. Similarly, we can prove the second half of the lemma. \diamond

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