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Integrability of the russian doll BCS model

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Abstract

We show that integrability of the BCS model extends beyond Richardson's model (where all Cooper pair scatterings have equal coupling) to that of the russian doll BCS model for which the couplings have a particular phase dependence that breaks time-reversal symmetry. This model is shown to be integrable using the quantum inverse scattering method, and the exact solution is obtained by means of the algebraic Bethe ansatz. The inverse problem of expressing local operators in terms of the global operators of the monodromy matrix is solved. This result is used to find a determinant formulation of a correlation function for fluctuations in the Cooper pair occupation numbers. These results are used to undertake exact numerical analysis for small systems at half-filling.

1 Introduction

Experimental work conducted by Ralph, Black and Tinkham [1, 2] was successful in determining the discrete energy spectra of aluminium nanograins with a fixed number of electrons. Their results surprisingly showed remnants of superconducting behaviour. Significant differences between the spectra of grains with even or odd number of electrons suggested that strong pairing correlations characteristic of superconductivity were still present, and quantum fluctuations significant. The Bardeen, Cooper and Schrieffer (BCS) model of superconductivity [3] can be applied in this situation, but the usual mean-field approach is not suitable for two reasons: firstly the number of electrons must be fixed, and secondly the strong quantum fluctuations destroy the validity of the assumption that operators may be averaged.

Fortunately predictions from Richardson's model, for which all scatterings between pairs are assumed to have equal coupling, give an excellent match with the experimental results discussed above [4]. In particular, this model is exactly solvable. The solution was originally obtained and analysed in the 1960s by Richardson and Sherman in the context of nuclear physics [5,6]. The model has also been shown to be integrable [7], and by recasting this result in the context of the quantum inverse scattering method, the solution was subsequently obtained by means of the algebraic Bethe ansatz method [8,9]. The latter approach has the advantage that it is possible to derive exact expressions for form factors and correlation functions [8, 10, 11]. These considerations have already been used to produce several generalisations of Richardson's model [12–14], and to study other pairing Hamiltonians [15, 16]. Reviews on these topics can be found in [4,11,17]. Exact studies of these BCS models are also important for studying pairing correlations in systems of ultracold dilute Fermi gases [18].

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Here we study the russian doll BCS model, a one-parameter generalisation of Richardson's model. It differs from Richardson's model by the inclusion of phases (indexed by a parameter α) in the pair scattering couplings, which break time-reversal symmetry. Richardson's model is reproduced in the limit as the parameter α is taken to zero. The terminology 'russian doll' refers to the russian craft of making nested wooden dolls, known as *matryoshka*. Inside each doll one finds a smaller version of the doll. A russian doll renormalisation group flow is one which displays a cyclic nature rather than flowing to a fixed point. The russian doll BCS model was proposed in [19] as an example of a many-body system exhibiting a russian doll renormalisation group flow. In §2, the model is shown to be integrable via the quantum inverse scattering method and its exact spectrum is obtained via the algebraic Bethe ansatz method. Our derivation shows that the Hamiltonian is embedded in the transfer matrix as the second-order term in the expansion about an infinite spectral parameter. This construction is in contrast to the proof of integrability for Richardson's model given in [11] in that the quasi-classical limit is not taken. In this sense there is no connection between the russian doll BCS model and Gaudin magnets (cf. [17]). The effect of the parameter α on the spectrum of the model is discussed in §3. In particular we will show that α influences the degeneracies of the energy bands in the strong coupling limit. In §4 we solve the inverse problem. This result is used in §5 to obtain an exact formula for a superconducting pairing correlation function. Numerical analysis of our results show that the ground state of the model is qualitatively independent of α in terms of the fluctuations in Cooper pair number for the single particle energy levels. Increasing α does however lead to a quantitative suppression of the fluctuations across all levels. Our conclusions are given in §6.

2 Integrability of the russian doll BCS model and its exact solution

The physical properties of a metallic nanograin with pairing interactions are described by the BCS Hamiltonian [4]

$$H_{\mathrm{BCS}} = \sum_{j=1}^{\mathcal{L}} \varepsilon_j n_j - \sum_{j,k=1}^{\mathcal{L}} g_{jk} c^\dagger_{k+} c^\dagger_{k-} c_{j-} c_{j+}. \tag{1}$$

Here, j=1, . . . , $\mathcal{L}$ labels a shell of doubly degenerate single particle energy levels with energies ε_j, and $n_j = c^\dagger_{j+}c_{j+} + c^\dagger_{j-}c_{j-}$ is the fermion number operator for level j. The operators $c_{j\pm}$, $c^\dagger_{j\pm}$ are the annihilation and creation operators for the fermions at level j. The labels $\pm$ refer to pairs of time-reversed states.

An important aspect of the Hamiltonian (1) is the blocking effect. For any unpaired fermion at level j the action of the pairing interaction is zero since only paired fermions are scattered. This means that the Hilbert space can be decoupled into a product of paired and unpaired fermion states in which the action of the Hamiltonian on the space for the unpaired fermions is automatically diagonal in the natural basis. In view of the blocking effect, it is convenient to introduce hard-core boson operators $b_j = c_{j-}c_{j+}$, $b^\dagger_j = c^\dagger_{j+}c^\dagger_{j-}$ which satisfy the relations

$$(b^\dagger_j)^2 = 0, \qquad [b_j,\, b^\dagger_k] = \delta_{jk}(1 - 2b^\dagger_j b_j), \qquad [b_j,\, b_k] = [b^\dagger_j,\, b^\dagger_k] = 0$$

on the space excluding single particle states. We also set $N_j = b^\dagger_j b_j$ to be the Cooper pair number operator for the jth level. In this setting the hard-core boson operators realise the

$su(2)$ algebra in the pseudo-spin representation, through the identification

$$S^- = b, \qquad S^+ = b^\dagger, \qquad S^z = \frac{1}{2}(2N - I). \tag{2}$$

The pseudo-spin operators satisfy the following commutation relations:

$$[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z. \tag{3}$$

It is now well known that the Hamiltonian (1) can be solved exactly in the case when $g_{jk} = g$ is constant for all $1 \leq j, k \leq \mathcal{L}$. This is referred to as Richardson's model. The solution was obtained in [5]. It was subsequently shown that the model is also integrable through an explicit construction for a set of conserved operators [7]. Both of these results can be obtained from the formulation of the model through the quantum inverse scattering method [8,9]. Here we will adopt this approach to establish that there exists a more general manifold of integrability. We will show that the following Hamiltonian

$$H_{\mathrm{RD}} = 2\sum_{j=1}^{\mathcal{L}} \varepsilon_j N_j - g\sum_{j<k}^{\mathcal{L}} (e^{i\alpha} b_k^\dagger b_j + e^{-i\alpha} b_j^\dagger b_k) \tag{4}$$

is integrable for arbitrary values of α. This model includes Richardson's model in the limit $\alpha \to 0$, up to the addition of a multiple of the $u(1)$ operator $N = \sum_{j=1}^{\mathcal{L}} N_j$ for the total number of Cooper pairs. It coincides with the russian doll BCS model recently studied in [19].

To establish the integrability of (4), we begin with the R-matrix $R(u)$ that satisfies the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \tag{5}$$

and has the form [20, 21]

$$\begin{aligned} R(u) &= \frac{1}{u+i\eta}(uI \otimes I + i\eta P) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{6}$$

where $b(u) = u/(u + i\eta)$, $c(u) = \eta/(u + i\eta)$ and η is taken to be an arbitrary real parameter. Above, P is the permutation operator. It satisfies

$$P(x \otimes y) = y \otimes x$$

for all vectors x and y. An important property of the R-matrix that will be exploited later is that it satisfies the unitarity condition

$$R(u)R(-u) = I \otimes I. \tag{7}$$

The R-matrix is $gl(2)$-invariant in that

$$[R(u),\, \mathfrak{g} \otimes \mathfrak{g}] = 0 \tag{8}$$

where $\mathfrak{g}$ is any 2×2 matrix. We introduce the Lax operator $L(u)$ in terms of the $su(2)$ Lie algebra with generators S^z and $S^{\pm}$ [22, 23],

$$L(u) = \frac{1}{u + i\eta} \begin{pmatrix} uI + i\eta(S^z + I/2) & i\eta S^- \\ i\eta S^+ & uI + i\eta(I/2 - S^z) \end{pmatrix}, \tag{9}$$

subject to the commutation relations (3). Then the following holds:

$$R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v). \tag{10}$$

When the $su(2)$ algebra takes the spin 1/2 representation (2) the resulting L-operator is equivalent to that given by the R-matrix (6).

We take $\mathfrak{g} = \exp(-i\alpha\sigma)$ with $\sigma = \mathrm{diag}(1, -1)$ and use (9) to construct the monodromy matrix

$$\begin{aligned} T(u) &= \mathfrak{g}_0 L_{0\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \cdots L_{02}(u - \varepsilon_2) L_{01}(u - \varepsilon_1) \\ &= \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \end{aligned}$$

Above, 0 labels the auxiliary space, while the tensor components of the physical space are labelled $1, \ldots, \mathcal{L}$. Thus the entries $A(u)$, $B(u)$, $C(u)$, $D(u)$ of $T(u)$ are elements of the $\mathcal{L}$-fold tensor algebra of $su(2)$. As a consequence of (8) and (10) we have the relation

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \tag{11}$$

We next define the transfer matrix

$$\begin{aligned} t(u) &= \mathrm{tr}_0 T(u) \\ &= A(u) + D(u). \end{aligned}$$

Here, tr_0 denotes the trace taken over the auxiliary space. It can be shown using (11) that the transfer matrix $t(u)$ generates a one-parameter family of commuting operators, viz.

$$[t(u),\, t(v)] = 0 \qquad \forall\, u, v \in \mathbb{C}.$$

Next we specialise the representation of the $su(2)$ algebra to that given by (2). Following the method of the algebraic Bethe ansatz (see for example [11]) we can deduce that the transfer matrix eigenvalues are

$$\Lambda(u) = \exp(-i\alpha) \prod_{k=1}^{\mathcal{L}} \frac{u - \varepsilon_k}{u - \varepsilon_k + i\eta} \prod_{j=1}^{N} \frac{u - v_j + 3i\eta/2}{u - v_j + i\eta/2} + \exp(i\alpha) \prod_{j=1}^{N} \frac{u - v_j - i\eta/2}{u - v_j + i\eta/2}$$

subject to the constraints of the Bethe ansatz equations

$$\exp(-2i\alpha) \prod_{k=1}^{\mathcal{L}} \frac{v_i - \varepsilon_k - i\eta/2}{v_i - \varepsilon_k + i\eta/2} = \prod_{j \neq i}^{N} \frac{v_i - v_j - i\eta}{v_i - v_j + i\eta}. \tag{12}$$

It is convenient to define an equivalent transfer matrix

$$\tilde{t}(u) = \left(\prod_{k=1}^{\mathcal{L}} \frac{u - \varepsilon_k + i\eta}{u - \varepsilon_k + i\eta/2} \right) t(u),$$

which has eigenvalues

$$\tilde{\Lambda}(u) = \tilde{a}(u) \prod_{j=1}^{N} \frac{u - v_j + 3i\eta/2}{u - v_j + i\eta/2} + \tilde{d}(u) \prod_{j=1}^{N} \frac{u - v_j - i\eta/2}{u - v_j + i\eta/2}$$

where

$$\tilde{a}(u) = \exp(-i\alpha) \prod_{k=1}^{\mathcal{L}} \frac{u - \varepsilon_k}{u - \varepsilon_k + i\eta/2} \tag{13}$$

$$\tilde{d}(u) = \exp(i\alpha) \prod_{k=1}^{\mathcal{L}} \frac{u - \varepsilon_k + i\eta}{u - \varepsilon_k + i\eta/2}. \tag{14}$$

Analogously we define

$$\tilde{A}(u) = \left(\prod_{k=1}^{\mathcal{L}} \frac{u - \varepsilon_k + i\eta}{u - \varepsilon_k + i\eta/2} \right) A(u)$$

and similarly for $\tilde{B}(u)$, $\tilde{C}(u)$ and $\tilde{D}(u)$.

A series expansion for $\tilde{t}(u)$ yields a set of mutually commuting operators. Here we choose to expand about the point at infinity:

$$\tilde{t}(u) = \sum_{k=0}^{\infty} \tilde{t}^{(k)} u^{-k}$$

such that

$$[\tilde{t}^{(k)}, \tilde{t}^{(l)}] = 0 \qquad \forall\, k,\, l.$$

It is straightforward to deduce the leading terms:

$$\begin{aligned} \tilde{t}(u) \sim 2\cos(\alpha) \Bigg(& I + \eta u^{-1} \tan(\alpha)(N - \mathcal{L}/2) - \frac{i\eta^2 u^{-2}}{2} \tan(\alpha)(N - \mathcal{L}/2) \\ & + \eta u^{-2} \tan(\alpha) \left[\sum_{i=1}^{\mathcal{L}} \varepsilon_i N_i - \frac{1}{2} \sum_{i=1}^{\mathcal{L}} \varepsilon_i \right] \\ & - \eta^2 u^{-2} \left[\frac{1}{2}(N - \mathcal{L}/2)^2 - \frac{\mathcal{L}}{8} + \frac{1}{2\cos(\alpha)} \sum_{j<k}^{\mathcal{L}} (e^{i\alpha} b_k^\dagger b_j + e^{-i\alpha} b_j^\dagger b_k) \right] \Bigg). \end{aligned} \tag{15}$$

Expanding $\tilde{\Lambda}(u)$ in powers of u^{-1} we find

$$\begin{aligned} \tilde{\Lambda}(u) \sim 2\cos(\alpha) \Bigg(& 1 + \eta u^{-1} \tan(\alpha)(N - \mathcal{L}/2) - \frac{i\eta^2 u^{-2}}{2} \tan(\alpha)(N - \mathcal{L}/2) \\ & + \eta u^{-2} \tan(\alpha) \left[\sum_{i=1}^{N} v_i - \frac{1}{2} \sum_{i=1}^{\mathcal{L}} \varepsilon_i \right] \\ & - \eta^2 u^{-2} \left[\frac{1}{2}(N - \mathcal{L}/2)^2 - \frac{N}{2} - \frac{\mathcal{L}}{8} \right] \Bigg). \end{aligned}$$

Comparing this expression with the expansion (15) we immediately see that the exact solution for the energies of the Hamiltonian (4) with $g = \eta/\sin(\alpha)$ is

$$E = 2\sum_{j=1}^{N} v_j + gN\cos(\alpha),$$

where the $\{v_j\}$ are solutions to (12). Making the change of variable

$$\alpha \rightarrow \eta\alpha$$

and taking the limit $\eta \rightarrow 0$ this reproduces the known exact solution [5], given that

$$\sum_{j=1}^{\mathcal{L}} b_j^\dagger b_j = N.$$

As a set of conserved operators for the Hamiltonian (4) we take $\{t(\varepsilon_k) : k = 1, \ldots, \mathcal{L}\}$, which naturally generalise those of [7], and arise in the solution of the inverse problem below. These operators clearly satisfy

$$[H_{RD}, t(\varepsilon_k)] = [t(\varepsilon_l), t(\varepsilon_k)] = 0, \qquad \forall\, k, l = 1, \ldots, \mathcal{L}.$$

Hence the above construction shows that the russian doll BCS model is both integrable and exactly solvable.

We note that the unitary time-reversal transformation of (4) is equivalent to the change of variable $\alpha \rightarrow -\alpha$ with g and each ε_j fixed. It is easily seen that the change $\alpha \rightarrow -\alpha$ and $\eta \rightarrow -\eta$ leaves g, E and each of the Bethe ansatz equations (12) invariant. We mention also that in the above the single particle energies ε_j are entirely arbitrary. In particular, it is not necessary to have $\varepsilon_j < \varepsilon_k$ for $j < k$.

3 Energy spectrum

To study the behaviour of the spectrum of (4) as α varies between 0 and $\pi/2$, we solved the Bethe ansatz equations (12) for the picket fence model in which the energy levels are equally spaced, that is $\varepsilon_j = j - (\mathcal{L}+1)/2$. With this convention the Fermi level is zero for a system at half-filling. The blocking effect simplifies the calculation of the energy of the excited states, since a state consisting of say N' Cooper pairs and two free electrons blocking levels m and n has energy

$$E = 2\sum_{j=1}^{N'} v_j + gN'\cos(\alpha) + \varepsilon_m + \varepsilon_n$$

where the $\{v\} \equiv \{v_1, \ldots, v_{N'}\}$ satisfy the following Bethe ansatz equations

$$\exp(-2i\alpha) \prod_{\substack{k=1 \\ k\neq m,n}}^{\mathcal{L}} \frac{v_j - \varepsilon_k - i\eta/2}{v_j - \varepsilon_k + i\eta/2} = \prod_{l\neq j}^{N'} \frac{v_j - v_l - i\eta}{v_j - v_l + i\eta}.$$

For a fixed number of particles the excitations of the model (4) are classified according to the initial distribution of the Cooper pairs and blocked energy levels at $g = 0$, thereby suggesting the initial configuration of the Bethe ansatz roots. For example, a solution of the Bethe ansatz equations for the ground state at half-filling is found using the initial conditions $v_j = \varepsilon_j\ ,\ j = 1, \ldots, \mathcal{L}/2$, whereas an excited state formed by promoting a Cooper pair above the Fermi level is obtained from the initial conditions $v_j = \varepsilon_j\ ,\ j = 1, \ldots, (\mathcal{L}-2)/2$ and $v_{\mathcal{L}/2} = \varepsilon_{(\mathcal{L}+2)/2}$.

We solved the Bethe ansatz equations iteratively starting with each possible configuration of the roots at $g = 0$ and $\alpha = 0$. Using this method we calculated the excited state spectrum for a model with three Cooper pairs at half-filling ($\mathcal{L} = 6$), a total of 141 states if the particle-hole symmetry is not taken into account. Figure 1 shows the energy levels as a function of the coupling g for three different values of α. For small α and large coupling g, the energy levels form narrow, well-separated bands in agreement with the results of Richardson's model (e.g. see [24]). As α increases the energy levels move, the original bands split apart, and new bands are formed. Throughout, the gap between the ground state energy and the first group of excited states remains, though it decreases as α increases. There is also a set of states for which the energy of each state does not depend on α. These are the states that consist of one Cooper pair and all levels apart from levels l and m are blocked. The single Bethe ansatz equation has two solutions:

$$2v_{\pm} = \varepsilon_l + \varepsilon_m \pm \left(g^2 + (\varepsilon_l - \varepsilon_m)^2\right)^{1/2} - g\cos(\alpha).$$

Consequently the energy for this state is

$$E_{\pm} = \varepsilon_l + \varepsilon_m \pm \left(g^2 + (\varepsilon_l - \varepsilon_m)^2\right)^{1/2} + \sum_{i \in \mathcal{B}} \varepsilon_i,$$

where $\mathcal{B}$ indexes the set of blocked energy levels.

Despite the fact that the Bethe ansatz equations have to be solved numerically to obtain the root positions for nonzero values of g, Gaudin [25] and Sierra et al. [26] have made some conjectures as to the behaviour of the BCS roots at very large g for Richardson's model. Numerical studies indicate that as g increases some of the roots form complex conjugate pairs in a complicated pattern which may involve roots forming pairs, splitting apart, and forming new pairs, or complex conjugate pairs separating and becoming independent real roots again. The root behaviour of the ground state of a model with an even number of Cooper pairs is very simple: all roots form complex conjugate pairs and become infinite at large coupling. For an odd number of Cooper pairs the roots behave in the same way apart from the root initially closest to the Fermi level, which remains finite. Generally most roots tend to infinity at very large g whether or not they have formed a complex conjugate pair. The number of the remaining finite roots can be thought of as the number of elementary excitations of the excited state. There is a formula due to Gaudin [25] that predicts the number of states with a particular number of finite roots at $g = \infty$, and an algorithm due to Sierra et al. [26] that matches the initial configuration of roots (or distribution of Cooper pairs over the available energy levels) to a final state.

It is clear from the splitting of the energy bands shown in Figure 1 that Gaudin's formula and the algorithm due to Sierra et al. are not valid for nonzero α. In fact the russian doll BCS model exhibits the same general root behaviour as Richardson's model (as shown in Figures 2a-c) but many more roots tend to infinity as $g \to \infty$. In each of the figures the large g configuration of Bethe ansatz roots consists of two real roots and one complex conjugate

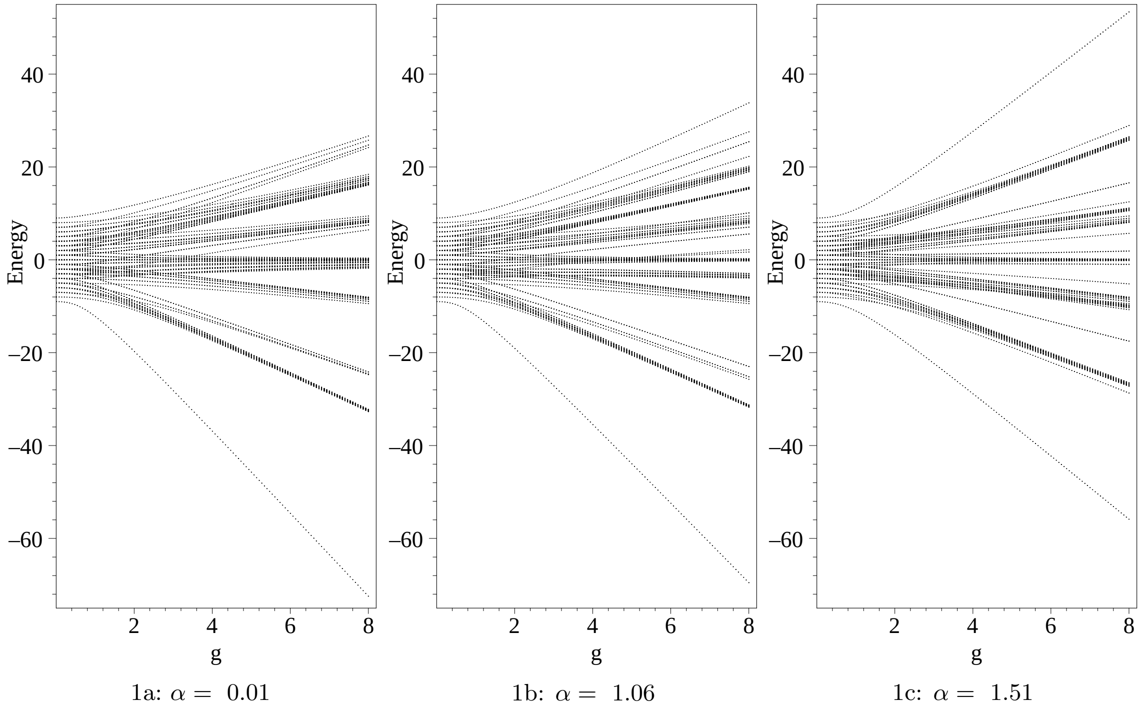

Figure 1: The behaviour of the energy levels as α is increased. The results shown are for the picket fence model of $\mathcal{L} = 6$ single particle energy levels at half-filling. As the coupling g increases, the energy levels coalesce into bands. Variations in α lead to different degeneracies of the bands. Note that in each case the ground state is non-degenerate and there is no level crossing between the ground and first excited energy level as g increases.

pair of roots. For $\alpha = 0.01$, Figure 2a shows there are two finite real roots, in agreement with the algorithm which predicts two elementary excitations [26]. At α close to $\pi/2$ (see Figure 2c), it appears that a single real root remains finite, but upon close examination at $g \approx 100$, the root is seen to be very slowly diverging. Thus there are no longer any finite roots in this particular case.

4 Solution of the inverse problem

In preparation for the next section we would like to be able to express any local operator in terms of the elements of the monodromy matrix. This is known as the inverse problem, and its solution will enable the form factors of the local operators to be determined. The results of this section are the generalisation of those in [27] to the case where the operator $\mathfrak{g}$ is included in the definition of the monodromy and transfer matrices (cf. [8]).

Hereafter we set $R_{jk} \equiv R_{jk}(\varepsilon_j - \varepsilon_k)$ for notational ease. First we note that

$$T(\varepsilon_j) = R_{j(j-1)} \cdots R_{j1} P_{0j} \mathfrak{g}_j R_{j\mathcal{L}} \cdots R_{j(j+1)}$$

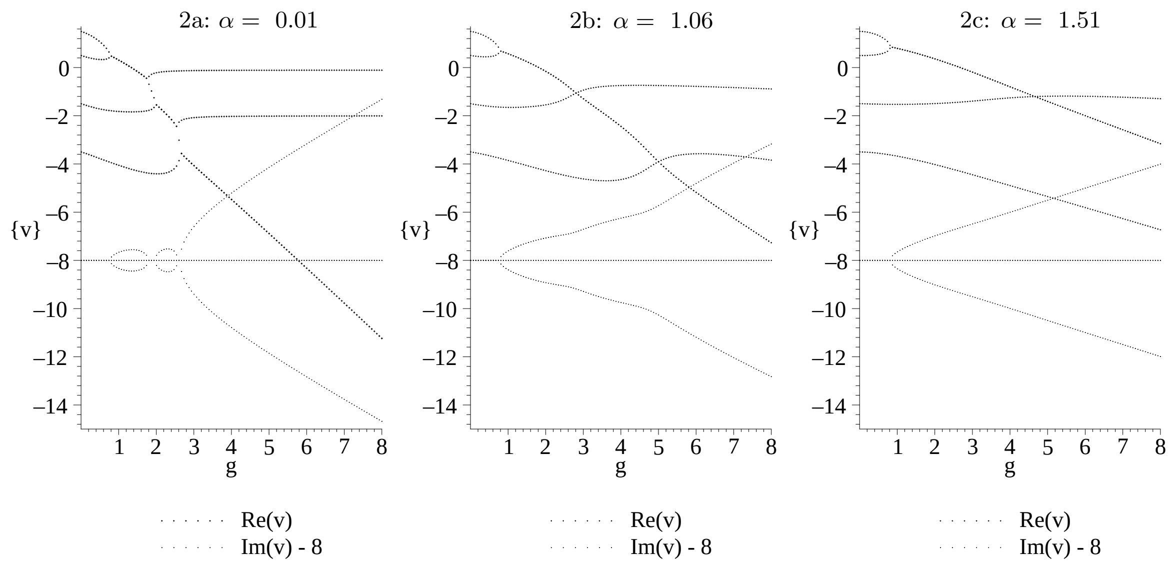

Figure 2: The evolution of the real and imaginary parts of the roots $\{v\}$ with increasing g. The three plots shown are for fixed values of α. In each case the roots correspond to the excited state of the $\mathcal{L}=8$ model with $N=4$ Cooper pairs and initial conditions $\{v\}=\{\varepsilon_1,\varepsilon_3,\varepsilon_5,\varepsilon_6\}$.

or equivalently

$$P_{0j}=R_{j1}^{-1}\cdots R_{j(j-1)}^{-1}T(\varepsilon_j)R_{j(j+1)}^{-1}\cdots R_{j\mathcal{L}}^{-1}\mathfrak{g}_j^{-1}. \tag{16}$$

It is readily deduced that

$$t(\varepsilon_j)=R_{j(j-1)}\cdots R_{j1}\mathfrak{g}_jR_{j\mathcal{L}}\cdots R_{j(j+1)}. \tag{17}$$

At this point we appeal to the unitarity condition (7) which allows us to write

$$t(\varepsilon_j)=R_{(j-1)j}^{-1}\cdots R_{1j}^{-1}\mathfrak{g}_jR_{j\mathcal{L}}\cdots R_{j(j+1)}.$$

Defining

$$\Gamma_{jk}=R_{1k}R_{2k}\cdots R_{jk},$$

we then have

$$t(\varepsilon_j)=\Gamma_{(j-1)j}^{-1}\mathfrak{g}_jR_{j\mathcal{L}}\cdots R_{j(j+1)},$$

which leads to the formula

$$\prod_{j=1}^{k}t(\varepsilon_j)=\left(\prod_{j=1}^{k}\mathfrak{g}_j\right)\Gamma_{k\mathcal{L}}\Gamma_{k(\mathcal{L}-1)}\cdots\Gamma_{k(k+1)}. \tag{18}$$

Equation (18) is proved by induction, noting first that

$$\begin{aligned} t(\varepsilon_1) &= \mathfrak{g}_1R_{1\mathcal{L}}R_{1(\mathcal{L}-1)}\cdots R_{12} \\ &= \mathfrak{g}_1\Gamma_{1\mathcal{L}}\Gamma_{1(\mathcal{L}-1)}\cdots\Gamma_{12}. \end{aligned}$$

Employing equation (18) we find

$$\left(\prod_{j=1}^{k-1} t^{-1}(\varepsilon_j)\right) P_{0k} \left(\prod_{j=1}^{k-1} t(\varepsilon_j)\right) = \Gamma^{-1}_{(k-1)k} P_{0k} \Gamma_{(k-1)k}. \tag{19}$$

Combining (16) and (17) we have

$$P_{0k} = R^{-1}_{k1} \cdots R^{-1}_{k(k-1)} T(\varepsilon_k) t^{-1}(\varepsilon_k) R_{k(k-1)} \cdots R_{k1}$$

from which we obtain the following expression

$$T(\varepsilon_k) t^{-1}(\varepsilon_k) = \Gamma^{-1}_{(k-1)k} P_{0k} \Gamma_{(k-1)k}.$$

Thus using (19) we have

$$P_{0k} = \left(\prod_{j=1}^{k-1} t(\varepsilon_j)\right) T(\varepsilon_k) \left(\prod_{j=1}^{k} t^{-1}(\varepsilon_j)\right). \tag{20}$$

Finally, since

$$P_{0k} = \begin{pmatrix} N_k & b_k \\ b_k^\dagger & I - N_k \end{pmatrix}$$

formula (20) leads to expressions for the local operators b_k, $b_k^\dagger$, N_k terms of the global operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$ of the algebraic Bethe ansatz.

5 Pair occupation fluctuations

In this section we use the notation $\langle\chi\rangle = \langle\{v\}|\chi|\{v\}\rangle \,/\, \langle\{v\}|\{v\}\rangle$ for any operator χ where $|\{v\}\rangle$ is a Bethe state associated with the roots $\{v\} \equiv \{v_1, \ldots, v_N\}$. For the BCS model it is of interest to study the behaviour of the correlation function C_k describing the fluctuations in the Cooper pair occupation numbers for each level k:

$$\begin{aligned} C_k^2 &= \langle N_k^2\rangle - \langle N_k\rangle^2 \\ &= \langle I - N_k\rangle\langle N_k\rangle. \end{aligned}$$

For Richardson's model ($\alpha = 0$), this correlation function has been studied in [4, 10].

Solving the inverse scattering problem allows the correlator to be expressed in terms of the form factor of the global operator $D(u)$ via

$$\langle\{v\}|1 - N_k|\{v\}\rangle = \langle\{v\}|D(\varepsilon_k) t^{-1}(\varepsilon_k)|\{v\}\rangle.$$

Fortunately the Slavnov formula [28] (see also [11, 22, 29]) for the scalar products of Bethe states leads to explicit determinant representation for the form factors of the operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$. We follow [11], in which the Slavnov formula is described for $\tilde{t}(u - i\eta/2)$ together with form factors for the elements of the associated monodromy matrix (up to a rescaling of η). We define it here in terms of the shifted functions

$$a(u) = \tilde{a}(u - i\eta/2), \qquad d(u) = \tilde{d}(u - i\eta/2)$$

where $\tilde{a}(u)$ and $\tilde{d}(u)$ are given by (13) and (14) respectively.

Below both sets of parameters $\{w_i\}$ and $\{v_i\}$ are assumed to be solutions of the Bethe ansatz equations (12). The scalar products of the Bethe states are obtained from

$$\langle\{w\}|\{v\}\rangle = \frac{\det F}{\prod_{k>l}^{N}(v_k - v_l)\prod_{i<j}^{N}(w_i - w_j)},$$

where F is an $N \times N$ matrix with entries

$$F_{ij} = \frac{i\eta d(w_i)}{(v_j - w_i)}\left(a(v_j)\prod_{k\neq i}^{N}(v_j - w_k + i\eta) - d(v_j)\prod_{k\neq i}^{N}(v_j - w_k - i\eta)\right),$$

unless $\{w\} = \{v\}$, in which case the diagonal entries become

$$\begin{aligned} F_{ii} &= d(v_i)\left(\sum_{l\neq i}^{N}\frac{a(v_i)}{v_i - v_l + i\eta}\prod_{k=1}^{N}(v_i - v_k + i\eta) + \sum_{l\neq i}^{N}\frac{d(v_i)}{v_i - v_l - i\eta}\prod_{k=1}^{N}(v_i - v_k - i\eta)\right) \\ &\quad + d(v_i)\left(a'(v_i)\prod_{k=1}^{N}(v_i - v_k + i\eta) + d'(v_i)\prod_{k=1}^{N}(v_i - v_k - i\eta)\right), \end{aligned} \tag{21}$$

where the prime denotes the derivative. Note that (21) corrects typographical errors appearing in equation (46) of [11]. The matrix elements of the operator $\tilde{D}(u)$ were derived in [11] and read

$$\langle\{w\}|\tilde{D}(u)|\{v\}\rangle = \frac{\Theta(u)\tilde{d}(u)}{\prod_{k>l}^{N}(v_k - v_l)\prod_{i<j}^{N}(w_i - w_j)}\left(\prod_{i=1}^{N}\frac{u - w_i - i\eta/2}{u - w_i + i\eta/2}\right)\det\left(F + Q(u)\right)$$

where

$$\Theta(u) = \prod_{k=1}^{N}\frac{u - w_k + i\eta/2}{u - v_k + i\eta/2}$$

and

$$Q_{ij}(u) = \frac{i\eta d(w_i)d(v_j)}{(u - w_i + i\eta/2)(u - w_i - i\eta/2)}\left[1 - \frac{\tilde{a}(u)}{\tilde{d}(u)}\prod_{k\neq i}^{N}\frac{u - w_k + 3i\eta/2}{u - w_k - i\eta/2}\right]\prod_{l=1}^{N}(v_j - v_l - i\eta).$$

In the limit $u \to \varepsilon_k$ the term within the square brackets becomes equal to unity since $\tilde{a}(\varepsilon_k) = 0$, and the expression for the form factor of the local operator N_k between identical Bethe eigenstates is also very simple:

$$\langle N_k\rangle = 1 - \frac{\det(F + Q(\varepsilon_k))}{\det F}.$$

Figure 3 shows the behaviour of C_k for the ground state of the $\mathcal{L} = 6$ system at half-filling, for three values of the coupling strength g as the parameter α is increased from just above zero to just below $\pi/2$.

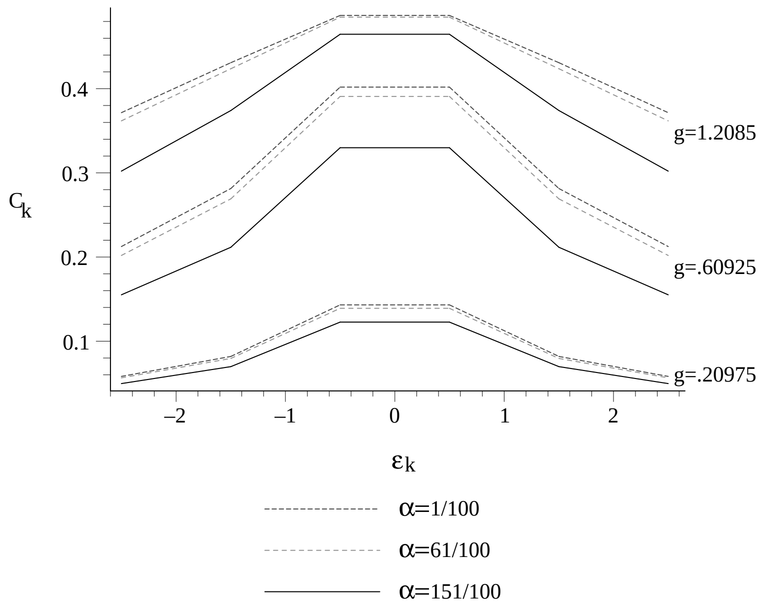

Figure 3: The correlation function C_k for various choices of α and the coupling strength g. The results shown are for the ground state of the $\mathcal{L} = 6$ model at half-filling. In each case the distribution of C_k peaks about the Fermi level. For fixed g, increasing values of α suppress the magnitude of C_k across all single particle energies ε_k.

6 Conclusion

We have demonstrated the integrability of the russian doll BCS model [19], which generalises Richardson's model by the inclusion of phases in the pair scattering couplings, parameterised by a variable α. We have applied the algebraic Bethe ansatz to find the exact solution of the model, and have also solved the inverse problem for the computation of form factors and correlation functions. In particular, an explicit expression for the one-point correlator characterising Cooper pair occupation fluctuations was obtained. Analysis of this result indicates that the ground state structure is qualitatively independent of α, and increasing α suppresses the fluctuations across all single particle energy levels. On the other hand, our analysis of the energy spectrum shows that the degeneracies of the energy bands in the strong coupling limit is dependent on α. An open problem is to determine the extent to which α affects the thermodynamic properties of the model. The quantum transfer matrix method and the solution of the associated nonlinear integral equations constitute powerful techniques for exactly calculating the free energy of many integrable systems. These techniques have successfully been applied to the cases of the XXZ model [30] and an integrable spin ladder [31]. As we have shown that the Hamiltonian is embedded in the transfer matrix as the second-order term in the series expansion about infinite spectral parameter, we hope that our results are a first step in formulating the quantum transfer matrix method for the russian doll BCS model.

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