A Bethe ansatz solvable model for superpositions of Cooper pairs and condensed molecular bosons

K. E. Hibberd[♠], C. Dunning[♢] and J. Links[♠]
[♠]Centre for Mathematical Physics,
The University of Queensland, 4072, Australia
[♢]Institute of Mathematics, Statistics and Actuarial Science
The University of Kent, U.K.

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Abstract

We introduce a general Hamiltonian describing coherent superpositions of Cooper pairs and condensed molecular bosons. For particular choices of the coupling parameters, the model is integrable. One integrable manifold, as well as the Bethe ansatz solution, was found by *Dukelsky et al., Phys. Rev. Lett.* **93** *(2004) 050403*. Here we show that there is a second integrable manifold, established using the boundary Quantum Inverse Scattering Method. In this manner we obtain the exact solution by means of the algebraic Bethe ansatz. In the case where the Cooper pair energies are degenerate we examine the relationship between the spectrum of these integrable Hamiltonians and the quasi-exactly solvable spectrum of particular Schrödinger operators. For the solution we derive here the potential of the Schrödinger operator is given in terms of hyperbolic functions. For the solution derived by *Dukelsky et al., loc. cit.* the potential is sextic and the wavefunctions obey $\mathcal{PT}$-symmetric boundary conditions. This latter case provides a novel example of an integrable Hermitian Hamiltonian acting on a Fock space whose states map into a Hilbert space of $\mathcal{PT}$-symmetric wavefunctions defined on a contour in the complex plane.

1 Introduction

One of the most interesting developments in the physics of ultra-cold gases has been the production of molecular condensates from fermionic constituents [1, 2]. Due to the property of Pauli blocking, it turns out that molecular condensates formed from fermionic atoms are generally more resistant to decay compared to the case of molecules formed from bosonic atoms. Furthermore, such systems open possibilities to experimentally probe the nature of the crossover between BCS and BEC type physics [3].

In order to theoretically model such systems, it is useful to have integrable systems with exact solutions which allow for investigations beyond the limits of perturbative and mean-field treatments.

Recently, Dukelsky et al. [4, 5] have proposed such a model and derived an exact solution using the methods of Richardson-Gaudin type spin models [6]. In their approach, a bosonic degree of freedom describing the molecular condensate is introduced by taking the infinite spin limit of one realisation of particular spin operators following a procedure of Gaudin [7]. The remaining spin operators are then realised in terms of paired creation and annihilation operators, which may be either bosonic or fermionic.

The aim of the present work is to derive a similar, although significantly distinct, model which also admits an exact solution. The approach we adopt is to use the boundary Quantum Inverse Scattering Method (QISM) as developed by Sklyanin [8]. We construct a doubled monodromy matrix from the spin realisation of the Yang–Baxter algebra in the usual way but with a bosonic operator valued solution of the reflection equations, which is obtained by dressing a boundary K-matrix with the bosonic realisation of the Yang–Baxter algebra. For simplicity, here we only consider the case where the spin operators are realised in terms of Cooper pairs of spin 1/2 particles, although generalisations to higher spin particles are possible. Through this construction we yield a family of commuting transfer matrices which give rise to an abstract integrable system. Following the method used to establish the integrability of the Russian Doll BCS model [9], we obtain the conserved charges by expanding the transfer matrix in inverse powers of the spectral parameter. Finally, by taking the quasi-classical limit (e.g. see [10]) we obtain a Hamiltonian that is expressible in terms of the conserved charges, thus establishing integrability. We also undertake the algebraic Bethe ansatz to derive the associated exact solution.

The integrable Hamiltonian that we will derive, and a particular case of the ones given in [4, 5], both belong to the following class of Hamiltonians:

$$H = UN_b^2 + \omega N_b + \sum_{j=1}^{M} \varepsilon_j n_j + \sum_{j=1}^{M} g_j (b^\dagger \mathcal{S}_j^- + b\mathcal{S}_j^+) \tag{1}$$

where the ε_j denote M two-fold degenerate energy levels for a system of fermions, ω is the single energy level for condensed molecular bosons, U is the interaction energy associated with S-wave scattering of the molecular bosons, and g_j are the interaction strengths for scattering between Cooper pairs and molecular bosons. In this article, the spin operators represent Cooper pairs through a realisation in terms of canonical Fermi operators $\mathcal{S}_j^- = c_{j-}c_{j+}$, $\mathcal{S}_j^+ = c_{j+}^\dagger c_{j-}^\dagger$, $\mathcal{S}_j^z = (n_{j-} + n_{j+} - 1)/2$ where $n_{j+} = c_{j+}^\dagger c_{j+}$, $n_{j-} = c_{j-}^\dagger c_{j-}$ such that the following $su(2)$ relations are satisfied:

$$[\mathcal{S}_j^-, \mathcal{S}_j^+] = 2\mathcal{S}_j^z, \quad [\mathcal{S}_j^z, \mathcal{S}_j^\pm] = \pm\mathcal{S}_j^\pm.$$

Further, $b, b^\dagger$ are canonical Bose operators obeying $[b, b^\dagger] = 1$. We adopt the notation $N_b = b^\dagger b$, $n_j = n_{j-} + n_{j+}$, $N_j = n_{j+}n_{j-}$. It can be verified that the total "bosonic" particle number (i.e. the number of Cooper pairs plus the number of molecular bosons) $N = N_b + \sum_{j=1}^{M} N_j$ commutes with the Hamiltonian and is thus conserved. We note that for any j the change of variable

$$g_j \to -g_j \tag{2}$$

is a unitary transformation which is induced by the automorphism

$$\mathcal{S}_j \to -\mathcal{S}_j, \quad \mathcal{S}_j^+ \to -\mathcal{S}_j^+, \quad \mathcal{S}_j^z \to \mathcal{S}_j^z.$$

Similar to BCS models [6, 9, 10] the Hamiltonian (1) displays Pauli blocking. For any unpaired fermion at energy level ε_j the action of the interactions with coupling g_j is zero. This means that the Hilbert space can be decoupled into a tensor product of paired and unpaired fermion states for which the action of the Hamiltonian on the subsystem of unpaired fermions is automatically diagonal in the natural Fock basis. Below we let $\mathcal{I}$ denote the index set for the unblocked levels and let $\mathcal{L} \leq M$ denote the cardinality of $\mathcal{I}$.

It was shown in [4, 5] that

$$U = 0, \qquad g_j = g \quad \forall\, j \tag{3}$$

is an integrable manifold for the Hamiltonian (1). For this case the energy eigenvalues are given by

$$E \;=\; g\sum_{j=1}^{N} x_j + \sum_{k\notin\mathcal{I}} \varepsilon_k$$

where the parameters x_j are solutions of the Bethe ansatz equations

$$\frac{\omega}{2g} - \frac{1}{2}x_j - \frac{1}{4}\sum_{k\in\mathcal{I}} \frac{2}{2\varepsilon_k/g - x_j} = -\sum_{k\neq j,\, k=1}^{N} \frac{1}{x_k - x_j}. \tag{4}$$

In the first part of the paper we will show that there is a second manifold of integrability which holds when

$$\varepsilon_j = \frac{g_j^2}{2U}, \qquad \frac{\omega}{U} = (\mathcal{L} - 2N) \tag{5}$$

and we will derive the corresponding Bethe ansatz solution. We remark at this point that there are two types of elementary excitations for the model, those for which Cooper pairs are broken and those for which they are not. Clearly for a non-pair breaking excitation $\mathcal{L} - 2N$ remains constant. For a pair breaking excitation N decreases by one and $\mathcal{L}$ decreases by two, such that $\mathcal{L} - 2N$ remains constant. Hence the coupling constraints defined by (5) are independent of the type of excitation.

In the second part of the paper we will show that for particular submanifolds of both (3) and (5) there exists in each case a mapping of the energy spectrum into part of the spectrum of a quasi-exactly solvable (QES) Schrödinger equation. Quasi-exactly solvable problems are characterised by having a finite number of wave functions and eigenvalues that can be found algebraically [11, 12]. The submanifolds of (3) and (5) we consider map onto the QES sectors of two Schrödinger equations. The case of (3) is particularly interesting because it naturally maps onto a $\mathcal{PT}$-symmetric [13, 14] Schrödinger problem.

2 Boundary QISM and construction of the Hamiltonian

We will construct a family of commuting transfer matrices $t(u)$ via the boundary QISM [8] from which the Hamiltonian will be obtained. For the operator $t(u)$ acting on a Hilbert space of physical states, we require

$$[t(u), t(v)] = 0, \quad \forall u, v \in \mathbb{C}.$$

A consequence of constructing the transfer matrix in this way is that by taking the series expansion

$$t(u) = \sum_{k=-\infty}^{\infty} t_k u^k$$

we have

$$[t_k, t_j] = 0, \quad \forall\, k, j,$$

which represent constants of the motion. Below we demonstrate how the Hamiltonian is expressible as a function of these constants of motion and as a consequence deduce that the resulting model is integrable.

We begin with the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$$

and use the $su(2)$ invariant R-matrix solution which has the form

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $b(u) = u/u+\eta, \quad c(u) = \eta/u+\eta$ where η is an arbitrary complex parameter. We require an L-operator, a realisation of the Yang–Baxter algebra acting on local physical spaces, which here is given by a spin 1/2 realisation of the $su(2)$ Lie algebra

$$L_j(u) \;=\; I + \frac{\eta}{u} S_j \tag{6}$$

for

$$S_j = \begin{pmatrix} N_j - \frac{1}{2} & \mathcal{S}_j^- \\ \mathcal{S}_j^+ & \frac{1}{2} - N_j \end{pmatrix}.$$

We note that for

$$\hat{L}_j(u) = I - \frac{\eta}{u-\eta} S_j,$$

we have

$$L_j(u)\hat{L}_j(u) = \left(1 - \frac{3\eta^2}{4u(u-\eta)}\right) I. \tag{7}$$

The L-operator so defined satisfies

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v). \tag{8}$$

We also take the following realisation of the Yang–Baxter algebra in terms of the molecular boson operators b, $b^\dagger$

$$\begin{aligned} J(u) &= \begin{pmatrix} 1 + \eta u + \eta^2(N_b + 1) & \eta b \\ \eta b^\dagger & 1 \end{pmatrix}, \\ \hat{J}(u) &= \begin{pmatrix} 1 & -\eta b \\ -\eta b^\dagger & 1 + \eta u + \eta^2 N_b \end{pmatrix}, \end{aligned}$$

where we also have

$$J(u)\hat{J}(u) = (1+\eta u)I. \tag{9}$$

The operator $J(u)$ obeys the following relation

$$R_{12}(u-v)J_1(u)J_2(v) = J_2(v)J_1(u)R_{12}(u-v). \tag{10}$$

For further details on these realisations of the Yang–Baxter algebra, the reader is referred to [10]. It will be useful to define

$$X = \begin{pmatrix} \eta(N_b+1) & -b(1+\eta^2 N_b) \\ b^\dagger & -\eta N_b \end{pmatrix}. \tag{11}$$

We construct the doubled monodromy matrix [8] in terms of the local operators for Cooper pairs and molecular bosons. For a given index set $\mathcal{I}$ for the unblocked levels with cardinality $\mathcal{L}$, we first perform a relabelling such that $\mathcal{I} \to \mathcal{I}' = \{1, 2, \ldots, \mathcal{L}-1, \mathcal{L}\}$. The doubled monodromy matrix is given by

$$T(u) \;=\; L_1(u-\epsilon_1)\ldots L_{\mathcal{L}}(u-\epsilon_{\mathcal{L}})J(u)K\hat{J}(-u)\hat{L}_{\mathcal{L}}(-\epsilon_{\mathcal{L}}-u)\ldots \hat{L}_1(-\epsilon_1-u) \tag{12}$$

where

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the boundary K-matrix. Note that here we will choose the boundary K-matrices, usually denoted $K_+,\ K_-$, to be equal. These are not the most general K-matrices that can be chosen, but this choice is sufficient for the considerations of the present work. The K-matrix satisfies the equation

$$R_{12}(u-v)(K\otimes I)R_{12}(u+v)(I\otimes K) = (K\otimes I)R_{12}(u+v)(K\otimes I)R_{12}(u-v). \tag{13}$$

The contribution $J(u)K\hat{J}(-u)$ to the doubled monodromy matrix is a dressing of the boundary K-matrix, which itself can be considered as an operator valued boundary K-matrix. We remark that operator valued boundary K-matrices have been used previously for the purpose of embedding Kondo impurities in integrable one-dimensional $t-J$ models [15, 16].

From (7,8,9,10,13) it follows that the doubled monodromy matrix satisfies

$$R_{12}(u-v)(T(u)\otimes I)R_{12}(u+v)(I\otimes T(v)) = (I\otimes T(v))R_{12}(u+v)(T(u)\otimes I)R_{12}(u-v). \tag{14}$$

Taking the expansion of the doubled monodromy matrix in inverse powers of u, we can express it as

$$T(u) \;\approx\; uT_0 + u^0 T_1 + u^{-1}T_2 + u^{-2}T_3$$

where

$$\begin{aligned} T_0 &= \eta I \\ T_1 &= K + 2\eta X + 2\eta^2 \sum_j^{\mathcal{L}} S_j \\ T_2 &= \eta\sum_i^{\mathcal{L}}(S_iK + KS_i) + 2\eta^2\sum_i^{\mathcal{L}}(S_iX + XS_i) - \eta^3\sum_i^{\mathcal{L}} S_i + \eta^3\left(\sum_{j<i}^{\mathcal{L}} S_jS_i + \sum_{j>i}^{\mathcal{L}} S_jS_i + \sum_{i,j}^{\mathcal{L}} S_jS_i\right) \end{aligned}$$

$$
\begin{aligned}
T_3 &= \eta\sum_i^{\mathcal{L}}\epsilon_i(S_iK-KS_i)+2\eta^2\sum_i^{\mathcal{L}}\epsilon_i(S_iX-XS_i)+\eta^2\left(\sum_{j<i}^{\mathcal{L}}S_jS_iK+\sum_{j>i}^{\mathcal{L}}KS_jS_i+\sum_{i,j}^{\mathcal{L}}S_jKS_i\right)\\
&\quad +2\eta^2\sum_i^{\mathcal{L}}\epsilon_i^2S_i-\eta^2\sum_i^{\mathcal{L}}KS_i+O(\eta^3).
\end{aligned}
$$

The transfer matrix is given by

$$
t(u) \;=\; \mathrm{Tr}[KT(u)] \tag{15}
$$

which provides a family of commuting matrices:

$$
[t(u),\, t(v)] = 0 \qquad \forall\, u, v.
$$

A series expansion of the transfer matrix provides a set of mutually commuting operators,

$$
\begin{aligned}
t(u) &\approx ut_0+u^0t_1+u^{-1}t_2+u^{-2}t_3,\\
&[t_k,t_j]=0.
\end{aligned}
$$

We find

$$
\begin{aligned}
t_0 &= 0\\
t_1 &= 2+2\eta^2(2N_b+1)+2\eta^2\sum_j^{\mathcal{L}}(2N_j-1)\\
t_2 &= 0\\
t_3 &= 4\eta^2\sum_j^{\mathcal{L}}\epsilon_j(\mathcal{S}_j^-b^\dagger+\mathcal{S}_j^+b)+2\eta^2\sum_i^{\mathcal{L}}\epsilon_i^2(2N_i-1)+\eta^2\sum_{j<i}^{\mathcal{L}}\left[2(N_j-\frac{1}{2})(N_i-\frac{1}{2})+\mathcal{S}_j^-\mathcal{S}_i^++\mathcal{S}_j^+\mathcal{S}_i^-\right]\\
&\quad +\eta^2\sum_{j>i}^{\mathcal{L}}\left[2(N_j-\frac{1}{2})(N_i-\frac{1}{2})+\mathcal{S}_j^-\mathcal{S}_i^++\mathcal{S}_j^+\mathcal{S}_i^-\right]+\eta^2\sum_{j,i}^{\mathcal{L}}\left[2(N_j-\frac{1}{2})(N_i-\frac{1}{2})-\mathcal{S}_j^-\mathcal{S}_i^+-\mathcal{S}_j^+\mathcal{S}_i^-\right]\\
&\quad +O(\eta^3).
\end{aligned}
$$

The Hamiltonian (1) is related to t_3 by

$$
H=\lim_{\eta\to 0}\frac{U}{4\eta^2}\left(t_3+2\eta^2\sum_{i=1}^{\mathcal{L}}\epsilon_i^2-4\eta^2N(N-\mathcal{L})-\frac{\eta^2\mathcal{L}(2\mathcal{L}-3)}{2}\right) \tag{16}
$$

after making the substitutions

$$
\varepsilon_i=\frac{U\epsilon_i^2}{2},\quad g_i=U\epsilon_i,\quad \omega=U(\mathcal{L}-2N) \tag{17}
$$

where $i\in\mathcal{I}'=\{1,\ldots\mathcal{L}\}$ label the unblocked levels. The above conditions may be easily relabelled in terms of the original index set $\mathcal{I}$.

3 Algebraic Bethe ansatz solution

Sklyanin's boundary QISM for the doubled monodromy matrix [8] was adapted for the Gaudin magnet in [17, 18] to obtain a solution using the algebraic Bethe ansatz (ABA) method. Using this general scheme, we outline the boundary ABA to develop a solution for the Hamiltonian (1) subject to the constraints (5).

With the above definition for the transfer matrix (15), we solve the eigenvalue problem

$$t(u)\Psi = \Lambda(u)\Psi$$

by the boundary ABA method. Representing the doubled monodromy matrix as

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{18}$$

the transfer matrix is given by

$$t(u)\Psi = (A(u) - D(u))\Psi.$$

The relations resulting from (14) are quite complicated in terms of which contributions are relevant to the diagonalisation of the transfer matrix. For this reason, we define a new $\hat{A}(u)$ through

$$\hat{A}(u) = (2u + \eta)A(u) - \eta D(u).$$

We can now rewrite the relevant relations arising from (14) in a more convenient form:

$$\begin{aligned} \hat{A}(u)C(v) &= \frac{(u-v+\eta)(u+v+2\eta)}{(u+v+\eta)(u-v)}C(v)\hat{A}(u) - \frac{2\eta(u+\eta)}{(2v+\eta)(u-v)}C(u)\hat{A}(v) \\ &+ \frac{4v\eta(u+\eta)}{(2v+\eta)(u+v+\eta)}C(u)D(v) \end{aligned} \tag{19}$$

$$\begin{aligned} D(u)C(v) &= \frac{(u+v)(u-v-\eta)}{(u+v+\eta)(u-v)}C(v)D(u) + \frac{2v\eta}{(2v+\eta)(u-v)}C(u)D(v) \\ &- \frac{\eta}{(u+v+\eta)(2v+\eta)}C(u)\hat{A}(v). \end{aligned} \tag{20}$$

Correspondingly, the transfer matrix may now be expressed as

$$t(u) = \frac{1}{2u+\eta}\hat{A}(u) - \frac{2u}{2u+\eta}D(u).$$

For the construction of the eigenstates, the reference state is chosen to be

$$|\Omega\rangle = |0\rangle \bigotimes_{i=1}^{\mathcal{L}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \tag{21}$$

such that $B(u)|\Omega\rangle = 0$. We find that the action of $\hat{A}(u),\ D(u)$ on this reference state is

$$\hat{A}(u)|\Omega\rangle = 2(u+\eta)\alpha(u)\hat{\alpha}(-u)|\Omega\rangle \tag{22}$$

$$D(u)|\Omega\rangle = -\delta(u)\hat{\delta}(-u)|\Omega\rangle \tag{23}$$

where

$$
\begin{aligned}
\alpha(u)|\Omega\rangle &= (1+\eta u+\eta^2)\prod_{i=1}^{\mathcal{L}}\left(1-\frac{\eta}{2(u-\epsilon_i)}\right)|\Omega\rangle \\
\hat{\alpha}(-u)|\Omega\rangle &= \prod_{i=1}^{\mathcal{L}}\left(1-\frac{\eta}{2(u+\epsilon_i)}\right)|\Omega\rangle \\
\delta(u)|\Omega\rangle &= \prod_{i=1}^{\mathcal{L}}\left(1+\frac{\eta}{2(u-\epsilon_i)}\right)|\Omega\rangle \\
\hat{\delta}(-u)|\Omega\rangle &= (1-\eta u)\prod_{i=1}^{\mathcal{L}}\left(1+\frac{\eta}{2(u+\epsilon_i)}\right)|\Omega\rangle.
\end{aligned}
$$

Taking the Bethe ansatz states to be a product of the creation operators on the reference state,

$$
\Psi = \prod_{\alpha=1}^{N} C(v_\alpha)|\Omega\rangle,
$$

we find that for

$$
t(u)\Psi = \Lambda(u)\Psi + \text{“unwanted terms”}
$$

the co-efficient $\Lambda(u)$ is given by

$$
\begin{aligned}
\Lambda(u) &= \frac{2(u+\eta)(1+\eta u+\eta^2)}{2u+\eta}\prod_{\alpha=1}^{N}\frac{(u-v_\alpha+\eta)(u+v_\alpha+2\eta)}{(u+v_\alpha+\eta)(u-v_\alpha)}\prod_{i=1}^{\mathcal{L}}\frac{(u-\epsilon_i-\eta/2)(u+\epsilon_i+\eta/2)}{(u-\epsilon_i)(u+\epsilon_i+\eta)} \\
&+ \frac{2u(1-\eta u)}{2u+\eta}\prod_{\alpha=1}^{N}\frac{(u+v_\alpha)(u-v_\alpha-\eta)}{(u+v_\alpha+\eta)(u-v_\alpha)}\prod_{i=1}^{\mathcal{L}}\frac{(u-\epsilon_i+\eta/2)(u+\epsilon_i+3\eta/2)}{(u-\epsilon_i)(u+\epsilon_i+\eta)}. \qquad (24)
\end{aligned}
$$

Cancellation of the unwanted terms above, which is equivalent to imposing that $\Lambda(u)$ has no poles, leads to the Bethe ansatz equations

$$
\frac{v_\beta(1-\eta v_\beta)}{(v_\beta+\eta)(1+\eta v_\beta+\eta^2)}\prod_{i=1}^{\mathcal{L}}\frac{(v_\beta+\epsilon_i+3\eta/2)(v_\beta-\epsilon_i+\eta/2)}{(v_\beta+\epsilon_i+\eta/2)(v_\beta-\epsilon_i-\eta/2)} = \prod_{\alpha\neq\beta,\alpha=1}^{N}\frac{(v_\beta-v_\alpha+\eta)(v_\beta+v_\alpha+2\eta)}{(v_\beta+v_\alpha)(v_\beta-v_\alpha-\eta)}, \quad \beta = 1\ldots N. \qquad (25)
$$

3.1 Results in the quasi-classical limit

Expanding the eigenvalue of the transfer matrix (24) in inverse powers of u we obtain

$$
\begin{aligned}
\Lambda(u) &\approx -2\eta^2\mathcal{L}+4\eta^2N+2+2\eta^2+\frac{\eta^2}{u^2}\left[\mathcal{L}^2-4\mathcal{L}N-2\sum_{i=1}^{\mathcal{L}}\epsilon_i^2+4N^2-\frac{3\mathcal{L}}{2}+4\sum_{\alpha=1}^{N}v_\alpha^2\right] \\
&= \lambda_0 u+\lambda_1 u^0+\lambda_2 u^{-1}+\lambda_3 u^{-2}
\end{aligned}
$$

and in particular

$$\lambda_3 \quad = \quad \eta^2 \left[4 \sum_{\alpha=1}^{N} v_\alpha^2 - 2 \sum_{i=1}^{\mathcal{L}} \epsilon_i^2 + 4N^2 - 4\mathcal{L}N + \mathcal{L}^2 - \frac{3\mathcal{L}}{2} \right].$$

In view of (16) this leads us to the energy of the Hamiltonian being

$$E = U \sum_{\alpha=1}^{N} v_\alpha^2 + \sum_{k \notin \mathcal{I}} \varepsilon_k$$

where $\mathcal{I}$ again denotes the index set for the unblocked levels. The corresponding Bethe ansatz equations arising from (25) in the quasi-classical limit are

$$-1 - \frac{1}{2v_\beta^2} + \sum_{k \in \mathcal{I}} \frac{1}{(v_\beta^2 - 2\varepsilon_k/U)} \quad = \quad \sum_{\alpha \neq \beta, \alpha=1}^{N} \frac{2}{(v_\beta^2 - v_\alpha^2)}. \tag{26}$$

4 Mapping to a Schrödinger equation

For the remainder of the paper we will restrict to a degenerate case for which ε_j is independent of j and consider only non-pair breaking excitations such that $\mathcal{L} = M$. For both exact solutions given by the Bethe ansatz equations (BAE) (4,26), we will demonstrate that the spectrum of the model, with a suitable scaling, can be mapped into that of a one-dimensional Schrödinger equation.

4.1 A first mapping

We begin with the BAE (26) derived in the previous section. First observe through (17) that the variable U determines the overall scaling of the model. We set $U = -1$ (so the S-wave scattering is attractive) and also

$$-2\varepsilon_j = \gamma, \forall j, \qquad v_\beta^2 = x_\beta.$$

The BAE are now

$$-1 - \frac{1}{2x_\beta} + \frac{M}{x_\beta - \gamma} = \sum_{\alpha \neq \beta, \alpha=1}^{N} \frac{2}{x_\beta - x_\alpha}.$$

It is convenient to shift the roots, $x_\beta \to x_\beta + \gamma/2$. Then the BAE become

$$-1 - \frac{1}{2(x_\beta + \gamma/2)} + \frac{M}{x_\beta - \gamma/2} = \sum_{\alpha \neq \beta, \alpha=1}^{N} \frac{2}{x_\beta - x_\alpha}. \tag{27}$$

Following the procedures taken in [19,20], we choose the following ansatz for an ordinary differential equation (ODE):

$$F(y) = [y^2 - (\gamma/2)^2]Q''(y) - \left[-\frac{1}{2}(y - \gamma/2) + M(y + \gamma/2) - y^2 + (\gamma/2)^2 \right] Q'(y). \tag{28}$$

For $Q(y) = \prod_{\beta=1}^{N}(y - x_\beta)$, we can check that $F(x_\beta) = 0$ using the BAE (27) and noting

$$
\begin{aligned}
Q'(y) &= Q(y)\sum_{\alpha=1}^{N}\frac{1}{y - x_\alpha} \\
Q''(y) &= Q(y)\sum_{\beta=1}^{N}\sum_{\alpha\neq\beta,\alpha=1}^{N}\frac{1}{(y - x_\beta)(y - x_\alpha)}.
\end{aligned}
$$

The degree of the polynomial $Q(y)$ is N, so in the RHS of (28) the highest order term is $N+1$. Thus we can fix $F(y)$ to be

$$F(y) = (Ay + B)Q(y).$$

It can be shown that the leading order term gives

$$A = N.$$

Next consider

$$
\begin{aligned}
\frac{Q'(\gamma/2)}{Q(\gamma/2)} &= -\sum_{\alpha=1}^{N}\frac{1}{x_\alpha - \gamma/2} \\
\frac{Q'(-\gamma/2)}{Q(-\gamma/2)} &= -\sum_{\alpha=1}^{N}\frac{1}{x_\alpha + \gamma/2}.
\end{aligned}
$$

From the ODE (28), we also see that

$$
\begin{aligned}
\frac{Q'(\gamma/2)}{Q(\gamma/2)} &= -\frac{N}{2M} - \frac{B}{M\gamma} \\
\frac{Q'(-\gamma/2)}{Q(-\gamma/2)} &= N - \frac{2B}{\gamma}.
\end{aligned}
$$

Rearranging for B we find

$$
\begin{aligned}
B &= -\frac{M\gamma}{2}\frac{Q'(\gamma/2)}{Q(\gamma/2)} - \frac{\gamma}{4}\frac{Q'(-\gamma/2)}{Q(-\gamma/2)} \\
&= \frac{M\gamma}{2}\sum_{\beta=1}^{N}\frac{1}{x_\beta - \gamma/2} + \frac{\gamma}{4}\sum_{\beta=1}^{N}\frac{1}{x_\beta + \gamma/2}.
\end{aligned}
$$

We can simplify B using an identity from the BAE. Taking the BAE (27), multiplying by x_β and taking the sum over β gives us

$$-\sum_{\beta=1}^{N}x_\beta + (-1/2 + M)N + \frac{\gamma}{4}\sum_{\beta=1}^{N}\frac{1}{x_\beta + \gamma/2} + \frac{M\gamma}{2}\sum_{\beta=1}^{N}\frac{1}{x_\beta - \gamma/2} = N(N-1). \tag{29}$$

Using this identity, we obtain B as follows

$$B = -E - N\left(M - N + \frac{1}{2}\right) - \frac{N\gamma}{2},$$

where for the energy we have

$$E = -\sum_{\beta=1}^{N} v_\beta^2 = -\frac{N\gamma}{2} - \sum_{\beta=1}^{N} x_\beta.$$

Hence $Q(y)$ satisfies the differential equation

$$\begin{aligned}\left[Ny - E \right. &- \left. N\left(M - N + \frac{1}{2} + \frac{\gamma}{2}\right)\right] Q(y) = (y^2 - (\gamma/2)^2)Q''(y) \\ &- \left[-\frac{1}{2}(y - \gamma/2) + M(y + \gamma/2) - y^2 + (\gamma/2)^2\right] Q'(y). \end{aligned} \tag{30}$$

Now we look to put the ODE (30) into the form of a Schrödinger equation. We start with the substitution $y = -(\gamma \cosh x)/2$, so that

$$\begin{aligned} \frac{dQ}{dx} &= -\frac{\gamma}{2}\sinh(x)\frac{dQ}{dy}, \\ \frac{d^2Q}{dy^2} &= \frac{4}{\gamma^2 \sinh^2(x)}\left(\frac{d^2Q}{dx^2} - \coth(x)\frac{dQ}{dx}\right). \end{aligned}$$

Now (30) becomes

$$\begin{aligned} &\frac{d^2Q}{dx^2} - \frac{\sinh(x)}{2(\cosh(x)+1)}\left[1 + 2M + \gamma(\cosh(x)+1)\right]\frac{dQ}{dx} \\ = &\left[-\frac{N\gamma}{2}(\cosh(x)+1) - E - N\left(M - N + \frac{1}{2}\right)\right] Q(x). \end{aligned}$$

Next we put

$$\Psi(x) = \exp(-g(x))Q(x)$$

and we need to find $g(x)$ such that upon substituting $\Psi(x)$ into the ODE (28), the $\Psi'(x)$ term cancels. (Note that the $\Psi(x)$ here is not related to the state vector Ψ introduced earlier in the ABA section.) For

$$\begin{aligned} Q'(x) &= \exp(g(x))[g'(x)\Psi(x) + \Psi'(x)] \\ Q''(x) &= \exp(g(x))[(g''(x) + g'(x)^2)\Psi(x) + 2g'(x)\Psi'(x) + \Psi''(x)], \end{aligned}$$

we find that the contribution to $\Psi'(x)$ (which we need to eliminate) is as follows:

$$g'(x) = \frac{\sinh(x)}{4(\cosh(x)+1)}\left[1 + 2M + \gamma(\cosh(x)+1)\right].$$

This is easily solved for $g(x)$:

$$g(x) = \frac{1}{4}[\gamma\cosh(x) + (2M+1)\ln(\cosh(x)+1)].$$

We now obtain

$$\Psi(x) = (\cosh(x)+1)^{-(M/2+1/4)} \exp\left(-\frac{\gamma}{4}\cosh(x)\right) Q(y)$$

and also note

$$g''(x) = \frac{\gamma\cosh(x)(1+\cosh(x)) + 2M+1}{4(\cosh(x)+1)}.$$

Next we can rewrite the ODE (28) as a Schrödinger equation

$$-\Psi''(x) + V(x)\Psi(x) = E\Psi(x)$$

where

$$\begin{aligned} V(x) &= -N\left(\frac{\gamma}{2}(\cosh(x)+1) + M - N + \frac{1}{2}\right) + \frac{(2M+1)^2}{16} - \frac{(2M+1)\gamma}{8} \\ &\quad - \frac{(2M+1)(2M+3)}{8(\cosh(x)+1)} + \frac{(2M-1)\gamma\cosh(x)}{8} + \frac{\gamma^2\sinh^2(x)}{16}, \end{aligned} \tag{31}$$

and

$$\Psi(x) = (\cosh(x)+1)^{-(M/2+1/4)} \exp\left(-\frac{\gamma}{4}\cosh(x)\right) \prod_{\alpha=1}^{N}\left(\frac{\gamma}{2}\cosh x + x_\alpha\right). \tag{32}$$

This potential belongs to a class of known quasi-exactly solvable potentials [11, 21].

4.2 A second mapping

We can perform a similar mapping for the case of the solution (4). Setting $\varepsilon_j = \epsilon$, $g = -1$ and performing a shift in the roots $x_j \to x_j + \epsilon$, the BAE are

$$2\epsilon - \omega - x_j + \frac{M}{x_j} = -\sum_{k\neq j, k=1}^{N} \frac{2}{(x_k - x_j)}$$

where the energy is given by

$$E = -\sum_{j=1}^{N} x_j - 2\epsilon N. \tag{33}$$

We choose the following ansatz for the ODE:

$$F(y) = yQ''(y) + \left[(\omega - 2\epsilon)y + y^2 - M\right] Q'(y). \tag{34}$$

For $Q(y) = \prod_{j=1}^{N}(y - x_j)$, we can easily check that $F(x_j) = 0$ using the BAE. Now set

$$F(y) = (Ay + B)Q(y). \tag{35}$$

It can be shown that the leading order term gives

$$A = N.$$

Next consider

$$\begin{aligned} \frac{Q'(0)}{Q(0)} &= -\sum_{j=1}^{N} \frac{1}{x_j} \\ \frac{Q''(0)}{Q(0)} &= \sum_{j=1}^{N} \sum_{k\leq j,k=1}^{N} \frac{1}{x_j x_k}. \end{aligned}$$

Also

$$F(0) = B = -MQ'(0) = M \sum_{j=1}^{N} \frac{1}{x_j}.$$

We simplify B using an identity from the BAE. Taking the summation over j in the BAE, we obtain B as follows

$$B = N\omega - E.$$

Hence $Q(y)$ satisfies the differential equation

$$yQ''(y) + \left[(\omega - 2\epsilon)y + y^2 - M\right] Q'(y) + \left[-Ny - N\omega + E\right] Q(y) = 0. \tag{36}$$

We make the substitution $y = x^2/4$ so that

$$\begin{aligned} \frac{dQ}{dx} &= \frac{x}{2}\frac{dQ}{dy}, \\ \frac{d^2Q}{dy^2} &= -\frac{4}{x^3}\frac{dQ}{dx} + \frac{4}{x^2}\frac{d^2Q}{dx^2}. \end{aligned}$$

Now the ODE (36) becomes

$$\left[-\frac{Nx^2}{4} - N\omega + E\right] Q(x) + \frac{d^2Q}{dx^2} + \left[\frac{x^3}{8} - \frac{2M+1}{x} - \frac{(\omega - 2\epsilon)x}{2}\right] \frac{dQ}{dx} = 0.$$

Next we put

$$\Psi(x) = \exp(-g(x))Q(x)$$

and substituting $\Psi(x)$ into the ODE, the contribution to $\Psi'(x)$ is given by

$$2g'(x) = \left[-\frac{x^3}{8} + \frac{(2M+1)}{x} + \frac{(\omega - 2\epsilon)x}{2}\right],$$

which we solve for $g(x)$ to obtain

$$g(x) = \frac{(2M+1)}{2} \ln x - \frac{x^4}{64} - \frac{x^2(\omega - 2\epsilon)}{8}.$$

Setting $z = x/2$, $\beta = \omega - 2\epsilon$, we can now rewrite the ODE (36) as a Schrödinger equation

$$-\Psi''(z) + V(z)\Psi(z) = \bar{E}\Psi(z), \tag{37}$$

where

$$\begin{aligned} V(z) &= z^6 + 2\beta z^4 + \left\{\beta^2 + 2(1+2N-M)\right\} z^2 + \frac{(M+3/2)(M+1/2)}{z^2} && (38) \\ \bar{E} &= -2\beta(2N-M) - 4\sum_{j=1}^{N} x_j, \\ \Psi(z) &= z^{-(2M+1)/2} \exp\left(\frac{z^4}{4} + \frac{\beta z^2}{2}\right) \prod_{\alpha=1}^{N} \left(z^2 - x_\alpha\right) && (39) \end{aligned}$$

and the parameters x_j are roots of the BAE (33).

The quasi-exact solvability of the potential (38) has previously been discussed in [12] (see Chapter 2.2). However some care is needed in order to properly embed the eigenfunctions into a Hilbert space of states. Specifically, the difference between the standard Schrödinger problem and the one above lies in the boundary conditions imposed on the eigenfunctions in order to obtain the Hilbert space. In the standard case the wave functions are required to be square-integrable, and thus decay at $\pm\infty$ along the real axis. The wave function (39) is not square-integrable on the real axis, since at large real $|z|$ it clearly blows up. However it is square-integrable if we instead define it on a contour which lies in wedges of the complex plane of open angle $\pi/4$ centered about $\arg z = -\pi/4$ and $\arg z = -3\pi/4$, and distorted to pass below the origin since $(M+3/2)(M+1/2) \neq 0$ [26]. These are $\mathcal{PT}$-symmetric [13,14] boundary conditions, and our QES potential (38) leads to the angular-momentum generalisation of the $\mathcal{PT}$-symmetric Schrödinger equation studied in [27]. It is interesting to note that for (38), the QES spectrum is necessarily real, since it has been derived from a mapping of a particular integrable manifold of the Hermitian Hamiltonian (1). By contrast, the potential studied in [27] admits a region of broken $\mathcal{PT}$-symmetry where the QES spectrum becomes complex. We always map to the region of unbroken $\mathcal{PT}$-symmetry because of the constraint $M > 0$.

Finally, we remark that the QES potential (38) has an interesting duality property under the transformation $\beta \to -\beta$, $z \to iz$. The quasi-exactly solvable sector of the spectrum maps onto itself via $\bar{E} \to -\bar{E}$ [23, 24] while the non-QES sectors remain unrelated. In our picture this duality is a trivial consequence of the freedom we have to redefine the Bose operators $b \to -b$ and $b^\dagger \to -b^\dagger$. If we send $\omega \to -\omega$ and $\epsilon \to \epsilon$ the Hamiltonian becomes $-H$ and the energy eigenvalues change sign. Hence the QES energy levels become $-\bar{E}$.

5 Conclusion

The general Hamiltonian (1) has two distinct integrable manifolds in parameter space which are given by (3,5) and exact Bethe ansatz solutions given by (4,26) respectively. For a certain submanifold in each instance, we have shown that the eigenspectrum and eigenstates can be mapped to the algebraic sector of a QES Schrödinger potential. It is surprising that for one solution the mapping is to $\mathcal{PT}$-symmetric eigenstates defined on a contour in the complex plane, while for the other case the eigenstates are defined on the real axis. The implications of this curious result in relation to the different physical properties of the Hamiltonians warrant further investigation.

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