Primitive ideals and automorphisms of quantum matrices

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Abstract

Let q be a nonzero complex number that is not a root of unity. We give a criterion for $\langle 0 \rangle$ to be a primitive ideal of the algebra $\mathcal{O}_q(M_{m,n})$ of quantum matrices. Next, we describe all height one primes of $\mathcal{O}_q(M_{m,n})$; these two problems are actually interlinked since it turns out that $\langle 0 \rangle$ is a primitive ideal of $\mathcal{O}_q(M_{m,n})$ whenever $\mathcal{O}_q(M_{m,n})$ has only finitely many height one primes. Finally, we compute the automorphism group of $\mathcal{O}_q(M_{m,n})$ in the case where $m \neq n$. In order to do this, we first study the action of this group on the prime spectrum of $\mathcal{O}_q(M_{m,n})$. Then, by using the preferred basis of $\mathcal{O}_q(M_{m,n})$ and PBW bases, we prove that the automorphism group of $\mathcal{O}_q(M_{m,n})$ is isomorphic to the torus $(\mathbb{C}^*)^{m+n-1}$ when $m \neq n$ and $(m,n) \neq (1,3),(3,1)$.

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Introduction

The automorphism group of a polynomial algebra $\mathbb{C}[X_1, \ldots, X_n]$, for $n \geq 2$, is in general far from being understood. In the case $n = 2$ the group was described explicitly by Jung, [12]. However, it is only recently that Shestakov and Umirbaev, [19], have proved that the well-known Nagata automorphism of $\mathbb{C}[X_1, X_2, X_3]$ is wild; that is, the Nagata automorphism cannot be written as a product of elementary automorphisms.

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In this paper, we are interested in the quantum case. More precisely, when q is a nonzero complex number that is not a root of unity, we study the automorphism group of the algebra $\mathcal{O}_q(M_{m,n})$ of $m \times n$ quantum matrices which in turn is a non-commutative deformation of a polynomial ring in $m \times n$ indeterminates. We denote by $Y_{i,\alpha}$, $(i,\alpha) \in [\![1,m]\!] \times [\![1,n]\!]$, the canonical generators of $\mathcal{O}_q(M_{m,n})$. It is well-known that the group $\mathcal{H} := (\mathbb{C}^*)^{m+n}$ acts on $\mathcal{O}_q(M_{m,n})$ by $\mathbb{C}$-automorphisms via:

$$(a_1, \ldots, a_m; b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad ((i,\alpha) \in [\![1,m]\!] \times [\![1,n]\!]).$$

(Note that this is not a faithful action; for example, $(a, \ldots, a; 1, \ldots, 1)$ and $(1, \ldots, 1; a, \ldots, a)$ have the same action on $\mathcal{O}_q(M_{m,n})$, multiplying each $Y_{i,\alpha}$ by a. This explains why the automorphism group is an $(m+n-1)$-torus rather than an $(m+n)$-torus.)

It was observed by Alev and Chamarie, [1], that quantization implies rigidity and so puts limits on the automorphism group of quantized algebras. This explains why it has been possible to compute the automorphism group of several quantum algebras at least in the generic case. For instance, Alev and Chamarie, [1], have described the automorphism group of the algebra of 2×2 quantum matrices, Alev and Dumas, [2], the automorphism group of the positive part of the quantized enveloping algebra of a complex simple Lie algebra of type A_2, the first author, [13], the automorphism group of the positive part of the quantized enveloping algebra of a complex simple Lie algebra of type B_2, ...

One method to study the automorphism group of an algebra is to use the invariance of the set of height one primes under the action of the automorphism group. This method was used successfully by Rigal, [18], to compute the automorphism group of quantized Weyl algebras and next by Gomez-Torrecillas and El Kaoutit, [6], to calculate the automorphism group of the coordinate ring of quantum symplectic spaces. We also use this method in the present paper. However, in the cases of quantized Weyl algebras and coordinate ring of quantum symplectic spaces, the number of height one primes is finite (because of the choice of parameters) and so the restrictions on the automorphisms are very strong. In the case of the algebra of quantum matrices, the set of height one primes is in general not finite; so the situation is substantially more complicated.

We start by considering when the algebra $\mathcal{O}_q(M_{m,n})$ has only finitely many height one primes. It turns out that, because of the stratification theorem of Goodearl and Letzter, see [3] for example, this situation arises exactly when $\langle 0 \rangle$ is a primitive ideal of $\mathcal{O}_q(M_{m,n})$. Thus, the first section of this paper is devoted to this question: we establish a criterion for the ideal $\langle 0 \rangle$ to be primitive in $\mathcal{O}_q(M_{m,n})$. More precisely, we prove that $\langle 0 \rangle$ is a primitive ideal of $\mathcal{O}_q(M_{m,n})$ if and only if $v_2(m) \neq v_2(n)$, where $v_2(k)$ denotes the 2-adic valuation of a positive integer k. This shows, for example, that $\langle 0 \rangle$ is a primitive ideal of $\mathcal{O}_q(M_{2,3})$.

This criterion together with the stratification theorem of Goodearl and Letzter shows that, if $v_2(m) \neq v_2(n)$, then $\mathcal{O}_q(M_{m,n})$ has only finitely many height one primes and it turns out that they are all $\mathcal{H}$-invariant. On the other hand, if $v_2(m) = v_2(n)$, then $\mathcal{O}_q(M_{m,n})$ has infinitely many height one primes. A finite number (those that are $\mathcal{H}$-invariant) are already known from previous work of the authors and Rigal, [14]. In the second part of this paper, we provide an explicit generator for every height one prime of $\mathcal{O}_q(M_{m,n})$. (Note that, since $\mathcal{O}_q(M_{m,n})$ is a Noetherian (non-commutative) UFD, every height one prime of $\mathcal{O}_q(M_{m,n})$ is generated by a normal element, [14].) Hence, in this second part, we describe all normal elements of $\mathcal{O}_q(M_{m,n})$ that generate a prime ideal.

Finally, in the third section, we investigate the automorphism group of $\mathcal{O}_q(M_{m,n})$ when $m \neq n$. Using the description of the height one primes of $\mathcal{O}_q(M_{m,n})$ together with graded arguments, we show that all height one primes that are $\mathcal{H}$-invariant, except possibly one, are invariant under every automorphism of $\mathcal{O}_q(M_{m,n})$. Next, by using the preferred basis of $\mathcal{O}_q(M_{m,n})$ and certain PBW bases, we are able to prove that the automorphism group of $\mathcal{O}_q(M_{m,n})$ is isomorphic to the torus $(\mathbb{C}^*)^{m+n-1}$ when $m \neq n$ and $(m,n) \neq (1,3),(3,1)$. This latter restriction occurs because of the existence of a rogue automorphism for quantum 3-space (which may be viewed as 1×3 quantum matrices), see [1]. This is the only exception, and our analysis recovers the Alev-Chamarie result for this exceptional case.

In the case where $m = n$, the algebra $O_q(M_n)$ has a homogeneous central element of degree n, the quantum determinant. The existence of this element considerably complicates the computation of the automorphism group, since the graded arguments that we use in the non-squared case do not put strong limits on the automorphism group and its action on the height one primes in the square case. In addition, transposition provides an automorphism that also clouds the analysis. The conjecture is that the automorphim group of $\mathcal{O}_q(M_n)$ is generated by a torus and the transposition automorphism. This has been verified by Alev and Chamarie, [1], in the 2×2 case. Our present methods can recover the Alev-Chamarie result, but, as yet, we are unable to deal with the general case, although we do have partial results. We intend to return to this question in a subsequent paper.

1 A criterion for quantum matrices to be primitive.

In this section, we use the $\mathcal{H}$-stratification theory of Goodearl and Letzter together with the deleting derivations theory of Cauchon in order to characterize the integers m and n such that $\langle 0 \rangle$ is a primitive ideal in $\mathcal{O}_q(M_{m,n})$.

1.1 The $\mathcal{H}$-stratification of the prime spectrum of $\mathcal{O}_q(M_{m,n})$.

Throughout this paper, we use the following conventions.

• If I is a finite set, $|I|$ denotes its cardinality.
• $[\![a, b]\!] := \{i \in \mathbb{N} \mid a \leq i \leq b\}$.
• $\mathbb{C}$ denotes the field of complex numbers and we set $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.
• $q \in \mathbb{C}^*$ **is not a root of unity**.
• m, n denote positive integers.
• $R = \mathcal{O}_q(M_{m,n})$ is the quantization of the ring of regular functions on $m \times n$ matrices with entries in $\mathbb{C}$; it is the $\mathbb{C}$-algebra generated by the $m \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i \leq m$ and $1 \leq \alpha \leq n$, subject to the following relations:

$$\begin{aligned}
&Y_{i,\beta}Y_{i,\alpha} = q^{-1}Y_{i,\alpha}Y_{i,\beta}, && (\alpha < \beta);\\
&Y_{j,\alpha}Y_{i,\alpha} = q^{-1}Y_{i,\alpha}Y_{j,\alpha}, && (i < j);\\
&Y_{j,\beta}Y_{i,\alpha} = Y_{i,\alpha}Y_{j,\beta}, && (i < j, \alpha > \beta);\\
&Y_{j,\beta}Y_{i,\alpha} = Y_{i,\alpha}Y_{j,\beta} - (q - q^{-1})Y_{i,\beta}Y_{j,\alpha}, && (i < j, \alpha < \beta).
\end{aligned}$$

It is well-known that R can be presented as an iterated Ore extension over $\mathbb{C}$, with the generators $Y_{i,\alpha}$ adjoined in lexicographic order. Thus the ring R is a Noetherian domain; we denote by F its skew-field of fractions. Moreover, since q is not a root of unity, it follows from [10, Theorem 3.2] that all prime ideals of R are completely prime. We denote by $\mathrm{Spec}(R)$ the set of (completely) prime ideals of R.
• It is well-known that the algebras $\mathcal{O}_q(M_{m,n})$ and $\mathcal{O}_q(M_{n,m})$ are isomorphic. Hence, we assume that $m \leq n$.
• It is easy to check that the group $\mathcal{H} := (\mathbb{C}^*)^{m+n}$ acts on R by $\mathbb{C}$-algebra automorphisms via:

$$(a_1, \ldots, a_m, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \text{for all} \quad (i, \alpha) \in [\![1, m]\!] \times [\![1, n]\!].$$

An $\mathcal{H}$-eigenvector x of R is a nonzero element $x \in R$ such that $h.x \in \mathbb{C}^*x$ for each $h \in \mathcal{H}$. An ideal I of R is said to be $\mathcal{H}$-invariant if $h.I = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}\text{-Spec}(R)$ the set of $\mathcal{H}$-invariant prime ideals of R. Since q is not a root of unity, it follows from [9, 5.7] that $\mathcal{H}\text{-Spec}(R)$ is a finite set. Note that $\langle 0 \rangle$ is an $\mathcal{H}$-invariant prime ideal of R, since R is a domain.

The action of $\mathcal{H}$ on R allows us to use the $\mathcal{H}$-stratification theory of Goodearl and Letzter, see [3, II.2], to constuct a partition of $\mathrm{Spec}(R)$ as follows. If J is an $\mathcal{H}$-invariant

prime ideal of R, we denote by $\mathrm{Spec}_J(R)$ the $\mathcal{H}$-stratum of $\mathrm{Spec}(R)$ associated to J. Recall that $\mathrm{Spec}_J(R) := \{P \in \mathrm{Spec}(R) \mid \bigcap_{h\in\mathcal{H}} h.P = J\}$. Then the $\mathcal{H}$-strata $\mathrm{Spec}_J(R)$, with $J \in \mathcal{H}\text{-Spec}(R)$, form a partition of $\mathrm{Spec}(R)$, see [3]:

$$\mathrm{Spec}(R) \;=\; \bigsqcup_{J\in\mathcal{H}\text{-Spec}(R)} \mathrm{Spec}_J(R). \tag{1}$$

Naturally, this partition induces a partition of the set $\mathrm{Prim}(R)$ of all (left) primitive ideals of R as follows. For all $J \in \mathcal{H}\text{-Spec}(R)$, we set $\mathrm{Prim}_J(R) := \mathrm{Spec}_J(R) \cap \mathrm{Prim}(R)$. Then it is obvious that the $\mathcal{H}$-strata $\mathrm{Prim}_J(R)$ ($J \in \mathcal{H}\text{-Spec}(R)$) form a partition of $\mathrm{Prim}(R)$:

$$\mathrm{Prim}(R) = \bigsqcup_{J\in\mathcal{H}\text{-Spec}(R)} \mathrm{Prim}_J(R).$$

One of the reasons that makes the $\mathcal{H}$-stratification interesting is that it provides a powerful tool for recognizing primitive ideals. Indeed, since $\mathbb{C}$ is uncountable and since the Noetherian domain R is generated as an algebra by a finite number of elements, it follows from [3, Proposition II.7.16] that the algebra R satisfies the Nullstellensatz over $\mathbb{C}$, see [3, II.7.14]. Further the set of $\mathcal{H}$-invariant prime ideals of R is finite. Thus [3, Theorem II.8.4] implies that $\mathrm{Prim}_J(R)$ ($J \in \mathcal{H}\text{-Spec}(R)$) coincides with the set of those primes in $\mathrm{Spec}_J(R)$ that are maximal in $\mathrm{Spec}_J(R)$. Hence, we deduce the following criterion for R to be a primitive ring.

Proposition 1.1 *R is a primitive ring if and only if the $\langle 0\rangle$-stratum of $\mathrm{Spec}(R)$ is reduced to $\langle 0\rangle$; that is, $\langle 0\rangle$ is the only prime ideal in $\mathrm{Spec}_{\langle 0\rangle}(R)$.*

1.2 Dimension of the $\langle 0\rangle$-stratum of $\mathrm{Spec}(R)$.

In this section, we show that the $\langle 0\rangle$-stratum $\mathrm{Spec}_{\langle 0\rangle}(\mathcal{O}_q(M_{m,n}))$ is Zariski-homeomorphic to the prime spectrum of a commutative Laurent polynomial ring in $\dim(\ker(B))$ indeterminates, where B is a $mn \times mn$ matrix with entries in $\mathbb{C}$. First, we describe the matrix B explicitly and then we will compute the dimension of the kernel of B.

First recall, see [5], that the theory of deleting derivations can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}]\dots[Y_{m,n}; \sigma_{m,n}, \delta_{m,n}]$ (where the indices are increasing for the lexicographic order $\leq$). The corresponding deleting derivations algorithm is called the standard deleting derivations algorithm. Before recalling its construction, we need to introduce some notation.

- We denote by $\leq_s$ the lexicographic ordering on $\mathbb{N}^2$. We often call it the standard ordering on $\mathbb{N}^2$. Recall that $(i,\alpha) \leq_s (j,\beta)$ if and only if $[(i<j)$ or $(i=j$ and $\alpha \leq \beta)]$.
- We set $E_s = (\llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket \cup \{(m,n+1)\}) \setminus \{(1,1)\}$.
- Let $(j,\beta) \in E_s$. If $(j,\beta) \neq (m,n+1)$, then $(j,\beta)^+$ denotes the least element (relative to $\leq_s$) of the set $\{(i,\alpha) \in E_s \mid (j,\beta) <_s (i,\alpha)\}$.

As described in [5], the standard deleting derivations algorithm constructs, for each $r \in E_s$, a family $(Y_{i,\alpha}^{(r)})_{(i,\alpha)\in\llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket}$ of elements of $F := \mathrm{Frac}(R)$, defined as follows.

1. If $r = (m,n+1)$, then $Y_{i,\alpha}^{(m,n+1)} = Y_{i,\alpha}$ for all $(i,\alpha) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket$.

2. Assume that $r = (j,\beta) <_s (m,n+1)$ and that the $Y_{i,\alpha}^{(r^+)}$ $((i,\alpha) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket)$ are already constructed. Then, it follows from [4, Théorème 3.2.1] that $Y_{j,\beta}^{(r^+)} \neq 0$ and, for all $(i,\alpha) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket$, we have:

$$Y_{i,\alpha}^{(r)} = \begin{cases} Y_{i,\alpha}^{(r^+)} - Y_{i,\beta}^{(r^+)} \left(Y_{j,\beta}^{(r^+)}\right)^{-1} Y_{j,\alpha}^{(r^+)} & \text{if } i<j \text{ and } \alpha<\beta \\ Y_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

As in [4], we denote by $\overline{R}$ the subalgebra of $\mathrm{Frac}(R)$ generated by the indeterminates obtained at the end of this algorithm, that is, we denote by $\overline{R}$ the subalgebra of $\mathrm{Frac}(R)$ generated by the $T_{i,\alpha} := Y_{i,\alpha}^{(1,2)}$ for each $(i,\alpha) \in \llbracket 1,m \rrbracket \times \llbracket 1,n \rrbracket$.

Let $N \in \mathbb{N}^*$ and let $\Lambda = (\Lambda_{i,j})$ be a multiplicatively antisymmetric $N \times N$ matrix over $\mathbb{C}^*$; that is, $\Lambda_{i,i} = 1$ and $\Lambda_{j,i} = \Lambda_{i,j}^{-1}$ for all $i,j \in \llbracket 1,N \rrbracket$. We denote by $\mathbb{C}_\Lambda[T_1,\ldots,T_N]$ the corresponding quantum affine space; that is, the $\mathbb{C}$-algebra generated by the N indeterminates $T_1,\ldots,T_N$ subject to the relations $T_iT_j = \Lambda_{i,j}T_jT_i$ for all $i,j \in \llbracket 1,N \rrbracket$. In [5, Section 2.2], Cauchon has shown that $\overline{R}$ can be viewed as the quantum affine space generated by the indeterminates $T_{i,\alpha}$ for $(i,\alpha) \in \llbracket 1,n \rrbracket^2$, subject to the following relations.

$$\begin{aligned} &T_{i,\beta}T_{i,\alpha} = q^{-1}T_{i,\alpha}T_{i,\beta}, && (\alpha<\beta); \\ &T_{j,\alpha}T_{i,\alpha} = q^{-1}T_{i,\alpha}T_{j,\alpha}, && (i<j); \\ &T_{j,\beta}T_{i,\alpha} = T_{i,\alpha}T_{j,\beta}, && (i<j, \alpha>\beta); \\ &T_{j,\beta}T_{i,\alpha} = T_{i,\alpha}T_{j,\beta}, && (i<j, \alpha<\beta). \end{aligned}$$

Hence $\overline{R} = \mathbb{C}_\Lambda[T_{1,1}, T_{1,2}, \ldots, T_{m,n}]$, where Λ denotes the $mn \times mn$ matrix defined as follows. We set

$$A := \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ -1 & 0 & 1 & \ldots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \ldots & -1 & 0 & 1 \\ -1 & \ldots & \ldots & -1 & 0 \end{pmatrix} \in \mathcal{M}_m(\mathbb{C}),$$

and

$$B := \begin{pmatrix} A & I_m & I_m & \ldots & I_m \\ -I_m & A & I_m & \ldots & I_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -I_m & \ldots & -I_m & A & I_m \\ -I_m & \ldots & \ldots & -I_m & A \end{pmatrix} \in \mathcal{M}_{mn}(\mathbb{C}),$$

where I_m denotes the identity matrix of $\mathcal{M}_m$. Then Λ is the $mn \times mn$ matrix whose entries are defined by $\Lambda_{k,l} = q^{b_{k,l}}$ for all $k, l \in [\![1, mn]\!]$.

It follows from [4, Théorèmes 5.1.1, 5.5.1, 5.5.2] that $\mathrm{Spec}_{\langle 0 \rangle}(R)$ is Zariski-homeomorphic to the prime spectrum of the quantum torus $P(\Lambda) := \overline{R}\Sigma^{-1}$, where Σ denotes the multiplicative system of $\overline{R}$ generated by the normal elements $T_{i,\alpha}$ with $(i, \alpha) \in [\![1, m]\!] \times [\![1, n]\!]$. Next, $\mathrm{Spec}(P(\Lambda))$ is Zariski-homeomorphic via extension and contraction to the prime spectrum of the centre $Z(P(\Lambda))$ of $P(\Lambda)$, by [8, Corollary 1.5]. Further, $Z(P(\Lambda))$ turns out to be a Laurent polynomial ring. To make this result precise, we need to introduce the following notation.

If $\underline{s} = (s_{1,1}, s_{1,2}, \ldots, s_{m,n}) \in \mathbb{Z}^{mn}$, then we set $T^{\underline{s}} := T_{1,1}^{s_{1,1}} T_{1,2}^{s_{1,2}} \ldots T_{m,n}^{s_{m,n}} \in P(\Lambda)$.

As in [8], we denote by $\sigma : \mathbb{Z}^{mn} \times \mathbb{Z}^{mn} \to \mathbb{C}^*$ the antisymmetric bicharacter defined by

$$\sigma(\underline{s}, \underline{t}) := \prod_{k,l=1}^{mn} \Lambda_{k,l}^{s_k t_l} \quad \text{for all} \quad \underline{s}, \underline{t} \in \mathbb{Z}^{mn}.$$

Then it follows from [8, 1.3] that the centre $Z(P(\Lambda))$ of $P(\Lambda)$ is a Laurent polynomial ring in the variables $(T^{\underline{b_1}})^{\pm 1}, \ldots, (T^{\underline{b_r}})^{\pm 1}$, where $(\underline{b_1}, \ldots, \underline{b_r})$ is any basis of $S := \{\underline{s} \in \mathbb{Z}^{mn} \mid \sigma(\underline{s}, -) \equiv 1\}$. Since q is not a root of unity, easy computations show that $\underline{s} \in S$ if and only if $B^t \underline{s}^t = 0$. Hence the centre $Z(P(\Lambda))$ of $P(\Lambda)$ is a Laurent polynomial ring in $\ker(B^t)$ indeterminates; so we have just proved the following statement.

Proposition 1.2 $\mathrm{Spec}_{\langle 0 \rangle}(\mathcal{O}_q(M_{m,n}))$ *is Zariski-homeomorphic to the prime spectrum of a commutative Laurent polynomial ring in* $\dim(\ker(B^t)) = \dim(\ker(B))$ *indeterminates.*

We deduce from Propositions 1.1 and 1.2 the following criterion for R to be a primitive ring.

Corollary 1.3 *R is a primitive ring if and only if B is invertible.*

We now compute the dimension of the kernel of B. First, straightforward computations show that

$$\dim(\ker(B)) = m - \operatorname{rk}\left[(A+I)^n + (A-I)^n\right].$$

If i is a positive integer greater than or equal to 2, we denote by $v_2(i)$ the 2-adic valuation of i; that is, m is the largest integer such that $2^m | i$.

One can easily check the following result.

Proposition 1.4

$$\dim(\ker(B)) = \begin{cases} 0 & \text{if } v_2(m) \neq v_2(n) \\ m \wedge n & \text{otherwise} \end{cases}$$

where $m \wedge n$ denotes the greatest common divisor of m and n.

Proof. Set $e_k := \exp\left(i\left(\frac{(2k+1)\pi}{m}\right)\right)$ for all $k \in [\![0, m-1]\!]$; that is, the e_k are the mth root of -1. It is easy to show that there exists $U \in \mathrm{GL}_m(\mathbb{C})$ such that $U^{-1}AU = \operatorname{diag}(\frac{e_0+1}{e_0-1}, \ldots, \frac{e_{m-1}+1}{e_{m-1}-1})$. Hence we have

$$U^{-1}\left[(A+I)^n + (A-I)^n\right]U =$$

$$\operatorname{diag}\left(\left(\frac{2e_0}{e_0-1}\right)^n + \left(\frac{2}{e_0-1}\right)^n, \ldots, \left(\frac{2e_{m-1}}{e_{m-1}-1}\right)^n + \left(\frac{2}{e_{m-1}-1}\right)^n\right).$$

Since $U \in \mathrm{GL}_m(\mathbb{C})$,

$$\begin{aligned} \operatorname{rk}\left((A+I)^n + (A-I)^n\right) &= \operatorname{rk}\left(U^{-1}\left[(A+I)^n + (A-I)^n\right]U\right) \\ &= \left| \left\{k \in [\![0, m-1]\!] \mid \left(\tfrac{2e_k}{e_k-1}\right)^n + \left(\tfrac{2}{e_k-1}\right)^n \neq 0\right\} \right|, \end{aligned}$$

so that

$$\dim(\ker(B)) = \left| \left\{k \in [\![0, m-1]\!] \mid \left(\tfrac{2e_k}{e_k-1}\right)^n + \left(\tfrac{2}{e_k-1}\right)^n = 0\right\} \right|.$$

Now, easy calculations show that $\left(\frac{2e_k}{e_k-1}\right)^n + \left(\frac{2}{e_k-1}\right)^n = 0$ if and only if $e_k^n = -1$; that is, if and only if $\exp\left(i\left(\frac{(2k+1)n\pi}{m}\right)\right) = -1$, that is, if and only if $m | (2k+1)n$ and $\frac{(2k+1)n}{m}$ is an odd integer. Hence,

$$\dim(\ker(B)) = \left| \left\{k \in [\![0, m-1]\!] \mid m | (2k+1)n \text{ and } \tfrac{(2k+1)n}{m} \text{is odd}\right\} \right|.$$

We now distinguish between two cases.

• First, assume that $v_2(m) \neq v_2(n)$. Suppose that there exists $k \in [\![0, m-1]\!]$ such that $m|(2k+1)n$. Then $v_2(m) \leq v_2(n)$. This implies that $v_2(m) < v_2(n)$, since $v_2(m) \neq v_2(n)$; and so $\frac{(2k+1)n}{m}$ is even. Hence there does not exist $k \in [\![0, m-1]\!]$ such that $m|(2k+1)n$ and $\frac{(2k+1)n}{m}$ is odd . This proves that, in this case,

$$\dim(\ker(B)) = \Big| \{k \in [\![0, m-1]\!] \mid m|(2k+1)n \text{ and } \tfrac{(2k+1)n}{m} \text{ is odd } \} \Big| = 0,$$

as desired.

• Next, assume that $v_2(m) = v_2(n) = \alpha$. Write $n = 2^\alpha v$ and $m = 2^\alpha u$ with u, v odd. It follows that

$$\begin{aligned} \dim(\ker(B)) &= \Big| \{k \in [\![0, m-1]\!] \mid u|(2k+1)v \text{ and } \tfrac{(2k+1)v}{u} \text{ is odd } \} \Big| \\ &= \Big| \{k \in [\![0, m-1]\!] \mid u|(2k+1)v\} \Big|. \end{aligned}$$

Set $d' = u \wedge v$ and $u = d'u'$. Note that d' and u' are odd, since u and v are odd. This implies that $\dim(\ker(B)) = \Big| \{k \in [\![0, m-1]\!] \mid u'|(2k+1)\} \Big|$, the number of odd multiples of u' less that $2m - 1 = 2^{\alpha+1}d'u' - 1$. Hence $\dim(\ker(B)) = 2^\alpha d' = m \wedge n$, as required. □

Proposition 1.4 together with Proposition 1.2 and Corollary 1.3 yields the following results.

Theorem 1.5 $\mathrm{Spec}_{\langle 0 \rangle}(\mathcal{O}_q(M_{m,n}))$ *is Zariski-homeomorphic to the prime spectrum of a commutative Laurent polynomial ring in* $\alpha_{m,n}$ *indeterminates, where*

$$\alpha_{m,n} = \begin{cases} 0 & \text{if } v_2(m) \neq v_2(n) \\ m \wedge n & \text{otherwise.} \end{cases}$$

Theorem 1.6 $\mathcal{O}_q(M_{m,n})$ *is a primitive ring if and only if* $v_2(m) \neq v_2(n)$.

For example, $\mathcal{O}_q(M_2)$ is not primitive, while $\mathcal{O}_q(M_{2,3})$ is primitive.

2 Height one primes in quantum matrices.

In this section, we investigate height one primes of $R = \mathcal{O}_q(M_{m,n})$. Every height one prime ideal of R is generated by a normal element, since R is a Noetherian UFD, [14]. In this section, we describe explicitly the normal elements that generate the height one prime ideals. Some of them are already known. Indeed, the height one primes that are $\mathcal{H}$-invariant have been described in [14]. Hence, we mainly focus our attention on the other height one primes: these belong to the $\langle 0 \rangle$-stratum of $\mathrm{Spec}(\mathcal{O}_q(M_{m,n}))$.

2.1 Height one primes of $\mathcal{O}_q(M_{m,n})$ that are $\mathcal{H}$-invariant.

The algebra $\mathcal{O}_q(M_n)$ has a special element, $\det_q$, the *quantum determinant*, defined by

$$\det_q := \sum_\sigma (-q)^{l(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)},$$

where the sum is taken over the permutations of $\{1, \ldots, n\}$ and $l(\sigma)$ is the usual length function on such permutations. The quantum determinant is a central element of $\mathcal{O}_q(M_n)$, see, for example, [17, Theorem 4.6.1]. If I is a t-element subset of $\{1, \ldots, m\}$ and Γ is a t-element subset of $\{1, \ldots, n\}$, then the quantum determinant of the subalgebra of $\mathcal{O}_q(M_{m,n})$ generated by $Y_{i,\alpha}$, with $i \in I$ and $\alpha \in \Gamma$, is denoted by $[I \mid \Gamma]$. The elements $[I \mid \Gamma]$ are the *quantum minors* of $\mathcal{O}_q(M_{m,n})$. Note that the quantum minors are $\mathcal{H}$-eigenvectors; and so every ideal generated by a family of quantum minors is $\mathcal{H}$-invariant.

It follows from [14, Proposition 4.2] that there are exactly $m + n - 1$ $\mathcal{H}$-invariant prime ideals in R that have height one. To make this result precise, let us introduce some notation.

For $1 \leq i \leq n + m - 1$, let b_i be the quantum minor defined as follows.

$$b_i := \begin{cases} [1, \ldots, i \mid n - i + 1, \ldots, n] & \text{if } 1 \leq i \leq m \\ [1, \ldots, m \mid n - i + 1, \ldots, n + m - i] & \text{if } m < i \leq n \\ [i - n + 1, \ldots, m \mid 1, \ldots, m + n - i] & \text{if } n < i \leq m + n - 1 \end{cases}$$

Note that the b_i with $m \leq i \leq n$ are precisely the $m \times m$ minors of $\mathcal{O}_q(M_n)$ that have consecutive column indices. They are homogeneous of degree m. This fact will be used several times later.

Proposition 2.1 ([14], Proposition 4.2) *There are precisely $m+n-1$ height one prime ideals that are $\mathcal{H}$-invariant in $\mathcal{O}_q(M_{m,n})$. They are the ideals generated by $b_1, \ldots, b_{m+n-1}$.*

The quantum minors b_i are normal elements of R. Moreover they belong to the algebra $\overline{R}$ obtained from R by the standard deleting derivations algorithm, see Section 1.2. Indeed, every quantum minor b_i can be expressed as a product of the canonical generators $T_{i,\alpha}$ of $\overline{R}$ as follows.

Lemma 2.2 *For $1 \leq i \leq m + n - 1$, we have*

$$b_i = \begin{cases} T_{1,n-i+1} T_{2,n-i+2} \ldots T_{i,n} & \text{if } 1 \leq i \leq m \\ T_{1,n-i+1} T_{2,n-i+2} \ldots T_{m,n+m-i} & \text{if } m < i \leq n \\ T_{i-n+1,1} T_{i-n+2,2} \ldots T_{m,m+n-i} & \text{if } n < i \leq n + m - 1 \end{cases}$$

Proof. This lemma is a consequence of [5, Proposition 5.2.2]. □

Recall that two elements a, b of $\mathrm{Frac}(R) = \mathrm{Frac}(\overline{R})$ are said to q-commute if there exists an integer $\bullet$ such that $ab = q^{\bullet}ba$. Since the $T_{i,\alpha}$ q-commute pairwise, it follows from the previous Lemma 2.2 that the b_j q-commute with the $T_{i,\alpha}$ and that the b_j also q-commute pairwise. Sometimes, it will not be necessary to know exactly the integers that appear in the power of q. However, at some points, we will need the following commutation relations that can be easily deduced from Lemma 2.2 and from the commutation relations between the $T_{i,\alpha}$.

Corollary 2.3 *Assume that $m < n$. Then,*

1. *for all $m \leq i < j \leq n$,*
$$b_i b_j = q^{\alpha_{ij}} b_j b_i,$$
where $\alpha_{ij} := \Big| \{n-i+1, \ldots, m+n-i\} \cap \{n-j+1, \ldots, m+n-j\} \Big| - m$,

2. *for all $i \in \{1, \ldots, m-1\}$,*
$$b_n b_i = q^{\alpha_i} b_i b_n \quad \text{and} \quad b_n b_{m+n-i} = b_{m+n-i} b_n,$$
where $\alpha_i := \Big| \{n-i+1, \ldots, n\} \cap \{m+1, \ldots, n\} \Big|$.

2.2 The $\langle 0 \rangle$-stratum of $\mathrm{Spec}(\mathcal{O}_q(M_{m,n}))$.

First, it follows from Theorem 1.5 that the $\langle 0 \rangle$-stratum of $\mathrm{Spec}(\mathcal{O}_q(M_{m,n}))$ is reduced to $\langle 0 \rangle$ when $v_2(m) \neq v_2(n)$. Thus, throughout this section, we assume that $v_2(m) = v_2(n)$; so that $\mathrm{Spec}_{\langle 0 \rangle}(\mathcal{O}_q(M_{m,n}))$ is Zariski-homeomorphic to the prime spectrum of a commutative Laurent polynomial ring in $m \wedge n$ indeterminates. We set $d := m \wedge n$ and we denote by m' and n' the positive integers such that $m = dm'$ and $n = dn'$. Observe that m' and n' are odd, since $v_2(m) = v_2(n)$. This observation will be crucial in what follows.

In Section 1.2, we have shown that $\mathrm{Spec}_{\langle 0 \rangle}(R)$ is Zariski-homeomorphic to the prime spectrum of the quantum torus $P(\Lambda) = \overline{R}\Sigma^{-1}$, where Σ denotes the multiplicative system of $\overline{R}$ generated by the normal elements $T_{i,\alpha}$ with $(i, \alpha) \in [\![1, m]\!] \times [\![1, n]\!]$. Moreover, $\mathrm{Spec}(P(\Lambda))$ is Zariski-homeomorphic via extension and contraction to the prime spectrum of the centre $Z(P(\Lambda))$ of $P(\Lambda)$, by [8, Corollary 1.5]. Before describing the $\langle 0 \rangle$-stratum of R, we calculate the centre $Z(P(\Lambda))$ of the quantum torus $P(\Lambda)$

2.2.1 The centre of the quantum torus $P(\Lambda) = \overline{R}\Sigma^{-1}$

Recall that the quantum minors b_i belong to $\overline{R}$. Moreover, the b_i are invertible in the quantum torus $P(\Lambda) = \overline{R}\Sigma^{-1}$,

For $j \in \{1, \dots, d\}$, set

$$\Delta_j := \prod_{i=0}^{m'+n'-1} b_{id+j}^{(-1)^i}.$$

(Here we set $b_{m+n} := 1$.)

Theorem 2.4 $Z(P(\Lambda)) = \mathbb{C}[\Delta_1^{\pm 1}, \dots, \Delta_d^{\pm 1}]$

Proof. First, straightforward computations (see also [16, Theorem 2.13]) show that $\Delta_1, \dots, \Delta_d$ are central in $P(\Lambda)$, so that $Z(P(\Lambda)) \supseteq \mathbb{C}[\Delta_1^{\pm 1}, \dots, \Delta_d^{\pm 1}]$.

If $\underline{s} = (s_{1,1}, s_{1,2}, \dots, s_{m,n}) \in \mathbb{Z}^{mn}$, we set $T^{\underline{s}} := T_{1,1}^{s_{1,1}} T_{1,2}^{s_{1,2}} \dots T_{m,n}^{s_{m,n}} \in P(\Lambda)$. Since the $T_{i,\alpha}$ q-commute, we deduce from Lemma 2.2 that the central elements Δ_i can be expressed as follows:

$$\Delta_i = q^{\gamma_i} T^{\underline{u^{(i)}}},$$

where $\gamma_i \in \mathbb{Z}$ and $\underline{u^{(i)}} = (u_{1,1}^{(i)}, u_{1,2}^{(i)}, \dots, u_{m,n}^{(i)}) \in \mathbb{Z}^{mn}$ with $(u_{1,m-d+1}^{(i)}, u_{1,m-d+2}^{(i)}, \dots, u_{1,m}^{(i)}) = (0, \dots, 0, 1, 0, \dots, 0)$ (the one being in the i-th position).

As in [8], we denote by $\sigma : \mathbb{Z}^{mn} \times \mathbb{Z}^{mn} \to \mathbb{C}^*$ the antisymmetric bicharacter defined by

$$\sigma(\underline{s}, \underline{t}) = \prod_{k,l=1}^{mn} \Lambda_{k,l}^{s_k t_l}$$

for all $\underline{s}, \underline{t} \in \mathbb{Z}^{mn}$.

Then it follows from [8, 1.3] that the centre $Z(P(\Lambda))$ of $P(\Lambda)$ is a Laurent polynomial ring in the variables $(T^{\underline{b_1}})^{\pm 1}, \dots, (T^{\underline{b_r}})^{\pm 1}$, where $(\underline{b_1}, \dots, \underline{b_r})$ is any basis of $S := \{\underline{s} \in \mathbb{Z}^{mn} \mid \sigma(\underline{s}, -) \equiv 1\}$.

Now, because of Theorem 1.5, we know $r = \mathrm{rk}(S) = d$. Moreover, the Δ_i are central, so that the $\underline{u^{(i)}}$ belong to S. To conclude, observe that, since we have

$$(u_{1,m-d+1}^{(i)}, u_{1,m-d+2}^{(i)}, \dots, u_{1,m}^{(i)}) = (0, \dots, 0, 1, 0, \dots, 0),$$

the group $\mathbb{Z}^{mn} / \sum_{i=1}^{d} \mathbb{Z}\underline{u^{(i)}}$ is torsionfree. Hence, the $\underline{u^{(i)}}$ form a basis of S, and so the centre $Z(P(\Lambda))$ of $P(\Lambda)$ is a Laurent polynomial ring in the variables $\Delta_1^{\pm 1}, \dots, \Delta_d^{\pm 1}$, as desired. □

2.2.2 Height one primes in $\mathcal{O}_q(M_{m,n})$.

Let P a height one prime of $R = \mathcal{O}_q(M_{m,n})$. Because of the $\mathcal{H}$-stratification, see (1) in Section 1.1, there exists an $\mathcal{H}$-invariant prime ideal J of R such that P belongs to the $\mathcal{H}$-stratum associated to J. In particular, we have $J \subseteq P$. Since P has height one, this implies that the height of J is at most 1. Thus, two cases arise.

1. First, suppose that J has height one. In this case, $P = J$, since P has height one; so that P is an $\mathcal{H}$-invariant height one prime ideal of R. Hence it follows from Proposition 2.1 that there exists $i \in \{1, \ldots, m+n-1\}$ such that $P = \langle b_i \rangle$.

2. Next, suppose that $J = \langle 0 \rangle$. In this case, P is a height one prime ideal that belongs to the $\langle 0 \rangle$-stratum of $\mathrm{Spec}(R)$. Note that this case can only arise when $v_2(m) = v_2(n)$, since it follows from Theorem 1.5 that the $\langle 0 \rangle$-stratum of $\mathrm{Spec}(\mathcal{O}_q(M_{m,n}))$ is reduced to $\langle 0 \rangle$ when $v_2(m) \neq v_2(n)$.

Let $\mathrm{Spec}^1(R)$ denote the set of all height one primes of R. The previous discussion proves the following statement.

Proposition 2.5

1. If $v_2(m) \neq v_2(n)$, then

$$\mathrm{Spec}^1(R) = \{\langle b_i \rangle \mid i \in \{1, \ldots, m+n-1\}\}.$$

2. If $v_2(m) = v_2(n)$, then

$$\mathrm{Spec}^1(R) = \{\langle b_i \rangle \mid i \in \{1, \ldots, m+n-1\}\} \cup \mathrm{Spec}^1_{\langle 0 \rangle}(R),$$

where $\mathrm{Spec}^1_{\langle 0 \rangle}(R)$ denotes the set of all height one primes of R that belong to the $\langle 0 \rangle$-stratum of $\mathrm{Spec}(R)$.

In order to complete the previous result, we now describe, in the case where $v_2(m) = v_2(n)$, the height one primes of $\mathcal{O}_q(M_{m,n})$ that belongs to $\mathrm{Spec}_{\langle 0 \rangle}(\mathcal{O}_q(M_{m,n}))$.

Proposition 2.6 *Assume that $v_2(m) = v_2(n)$. Then, for any height one prime ideal P of $\mathcal{O}_q(M_{m,n})$ that belongs to $\mathrm{Spec}_{\langle 0 \rangle}(\mathcal{O}_q(M_{m,n}))$, there exists a unique (up to scalar) irreducible*

polynomial $V = \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} X_1^{i_1} \ldots X_d^{i_d} \in \mathbb{C}[X_1, \ldots, X_d]$ *(where* $r_i = \deg_{X_i} V$*) with* $V \neq X_i$ *for all* $i \in \{1, \ldots, d\}$ *such that* $P = \langle u \rangle$ *where*

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} \prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j}.$$

Moreover, u *is normal in* R*.*

Proof. We proceed in three steps.

• *Step 1. A generator for the extension of* P *in a localisation of* R*.*

First, observe that the prime ideals in $\mathrm{Spec}_{\langle 0 \rangle}(R)$ do not contain any b_i. Indeed, assume that this is not the case; that is, assume that there exists $P \in \mathrm{Spec}_{\langle 0 \rangle}(R)$ with $b_i \in P$ for a certain i. Then, since b_i is an $\mathcal{H}$-eigenvector, we have $b_i \in \bigcap_{h \in \mathcal{H}} h.P = \langle 0 \rangle$. This is a contradiction; and so $\mathrm{Spec}_{\langle 0 \rangle}(R) \subseteq \{P \in \mathrm{Spec}(R) \mid b_i \notin P \text{ for all } i\}$. On the other hand, if P is a prime ideal of R such that $b_i \notin P$ for all i, then $\bigcap_{h \in \mathcal{H}} h.P$ is an $\mathcal{H}$-invariant prime ideal of R that does not contain any b_i. However, because of [14, Proposition 2.9], every nonzero $\mathcal{H}$-invariant prime ideal of R contains a height one prime that is $\mathcal{H}$-invariant. In other words, every nonzero $\mathcal{H}$-invariant prime ideal of R contains a b_i. Thus $\bigcap_{h \in \mathcal{H}} h.P = \langle 0 \rangle$ and so $P \in \mathrm{Spec}_{\langle 0 \rangle}(R)$. To sum up, we have shown that $\mathrm{Spec}_{\langle 0 \rangle}(R) = \{P \in \mathrm{Spec}(R) \mid b_i \notin P \text{ for all } i\}$.

Denote by T the localisation of R with respect of the multiplicatively closed set $\mathcal{B}$ generated by the normal elements b_i. Note that the torus $\mathcal{H}$ still acts rationally by automorphisms on T, since the b_i are $\mathcal{H}$-eigenvectors, see [3, Exercise II.3.A]. Moreover, it follows from the previous study, and from classical results of non-commutative localisation theory, that the map $\varphi : P \to P\mathcal{B}^{-1}$ is an increasing bijection from $\mathrm{Spec}_{\langle 0 \rangle}(R)$ onto $\mathrm{Spec}\,(T)$.

Next, T is $\mathcal{H}$-simple; that is, the only $\mathcal{H}$-invariant ideals in T are $\langle 0 \rangle$ and T, by [14, Proposition 3.5]. The action of $\mathcal{H}$ on T is rational, see [3, Exercise II.3.A], and T is $\mathcal{H}$-simple; so [3, Corollary II.3.9] applies to T. Thus extension and contraction provide mutually inverse bijections between $\mathrm{Spec}(T)$ and $\mathrm{Spec}(Z(T))$.

Now, recall that the b_i can be expressed as products of $T_{i,\alpha}$, see Lemma 2.2; so the b_i belong to the quantum torus $P(\Lambda) = \overline{R}\Sigma^{-1}$. Moreover, it follows from [4, Théorème 3.3.1] that there exists a multiplicative system S of R such that $R \subseteq RS^{-1} = P(\Lambda) = \overline{R}\Sigma^{-1}$. Hence T is a subalgebra of the quantum torus $P(\Lambda) = RS^{-1}$, and $R \subseteq T \subseteq P(\Lambda) =$

$\overline{R}\Sigma^{-1} = RS^{-1}$. Thus $Z(T) \subseteq Z(P(\Lambda)) = \mathbb{C}[\Delta_1^{\pm 1}, \ldots, \Delta_d^{\pm 1}]$. But the Δ_i are just products of $b_i^{\pm 1}$, so that they belong to T. Hence $Z(T) = \mathbb{C}[\Delta_1^{\pm 1}, \ldots, \Delta_d^{\pm 1}]$.

Observe that Δ_j can be written as follows.

$$\Delta_j = q^{\bullet} \prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j}^{-1},$$

where as usual $\bullet$ denotes an integer, since the b_i q-commute. For $j \in \{1, \ldots, d\}$, we set

$$\Delta'_j = \left(\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right) \left(\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right)^{-1}. \tag{2}$$

Since $Z(T) = \mathbb{C}[\Delta_1^{\pm 1}, \ldots, \Delta_d^{\pm 1}]$, the centre of T is also the (commutative) Laurent polynomial ring in the indeterminates $\Delta'_1, \ldots, \Delta'_d$, that is,

$$Z(T) = \mathbb{C}[\Delta_1'^{\pm 1}, \ldots, \Delta_d'^{\pm 1}].$$

Now, let P be a height one prime of R that belongs to $\mathrm{Spec}_{\langle 0 \rangle}(R)$. It follows from the previous study that $\varphi(P) \cap Z(T)$ is a height one prime of $Z(T) = \mathbb{C}[\Delta_1'^{\pm 1}, \ldots, \Delta_d'^{\pm 1}]$. Hence there exists an irreducible polynomial $V = \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} X_1^{i_1} \ldots X_d^{i_d} \in \mathbb{C}[X_1, \ldots, X_d]$, with $r_i = \deg_{X_i} V$, and with $V \neq X_i$ for all $i \in \{1, \ldots, d\}$, such that

$$\varphi(P) \cap Z(T) = \left\langle \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} \Delta_1'^{i_1} \ldots \Delta_d'^{i_d} \right\rangle_{Z(T)}.$$

Thus,

$$\varphi(P) = \left\langle \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} \Delta_1'^{i_1} \ldots \Delta_d'^{i_d} \right\rangle_{T},$$

since extension and contraction provide mutually inverse bijections between $\mathrm{Spec}(T)$ and $\mathrm{Spec}(Z(T))$. Since the b_i are invertible in T, (2) leads to $\varphi(P) = \langle u \rangle$, where

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} \prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j}.$$

Note that u is a normal element in both R and T.

• *Step 2. We prove that u is not contained in any $\langle b_k \rangle$.*

Set

$$u_{i_1,\dots,i_d} := \prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j}$$

for all $(i_1, \dots, i_d) \in [\![0, r_1]\!] \times \cdots \times [\![0, r_d]\!]$; so that

$$u = \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\dots,i_d} u_{i_1,\dots,i_d}.$$

Since the b_k are $\mathcal{H}$-eigenvectors, it is clear that the $u_{i_1,\dots,i_d}$ are also $\mathcal{H}$-eigenvectors. Hence, for all $(i_1, \dots, i_d) \in [\![0, r_1]\!] \times \cdots \times [\![0, r_d]\!]$, there exists a (unique) character $f_{i_1,\dots,i_d} : \mathcal{H} \to \mathbb{C}^*$ such that $h.u_{i_1,\dots,i_d} = f_{i_1,\dots,i_d}(h) u_{i_1,\dots,i_d}$ for all $h \in \mathcal{H}$. The character $f_{i_1,\dots,i_d}$ is called the $\mathcal{H}$-eigenvalue of the $\mathcal{H}$-eigenvector $u_{i_1,\dots,i_d}$.

Claim 2.7 *The $u_{i_1,\dots,i_d}$ are $\mathcal{H}$-eigenvectors with pairwise distinct $\mathcal{H}$-eigenvalues.*

Proof of Claim 2.7. Let $(i_1, \dots, i_d) \in [\![0, r_1]\!] \times \cdots \times [\![0, r_d]\!]$ and $h := (g_1, \dots, g_m; h_1, \dots, h_n) \in \mathcal{H}$. Then,

$$h.u_{i_1,\dots,i_d} = g_1^{\bullet} \dots g_m^{\bullet} h_1^{\bullet} \dots h_{n-m}^{\bullet} h_{n-m+1}^{\alpha_{m-1}} \dots h_n^{\alpha_0} u_{i_1,\dots,i_d},$$

where $\bullet \in \mathbb{Z}$ and α_k is the number of quantum minors involving the $(n-k)$-th column in the product that defines $u_{i_1,\dots,i_d}$.

Now one can observe that, for all $k \in \{0, \dots, d-1\}$, the $(n-k)$-th column appears in b_{id+j} if and only if either (i) $k+1 \leq j \leq d$ and $0 \leq i \leq m'-1$, or (ii) $1 \leq j \leq k$ and $1 \leq i \leq m'$.
Hence, since m' is odd,

$$\begin{aligned} \alpha_k &= \sum_{j=1}^{k} \left(\frac{m'-1}{2} i_j + \frac{m'+1}{2} (r_j - i_j) \right) + \sum_{j=k+1}^{d} \left(\frac{m'+1}{2} i_j + \frac{m'-1}{2} (r_j - i_j) \right) \\ &= \sum_{j=1}^{k} \left(\frac{m'+1}{2} r_j - i_j \right) + \sum_{j=k+1}^{d} \left(\frac{m'-1}{2} r_j + i_j \right) \end{aligned}$$

Now, let $(i_1, \dots, i_d), (i'_1, \dots, i'_d) \in [\![0, r_1]\!] \times \cdots \times [\![0, r_d]\!]$, and assume that $u_{i_1,\dots,i_d}$ and $u_{i'_1,\dots,i'_d}$ are associated to the same $\mathcal{H}$-eigenvalue. Then, since $\mathbb{C}$ is infinite, it follows from the previous study that,

$$\sum_{j=1}^{k} \left(\frac{m'+1}{2} r_j - i_j \right) + \sum_{j=k+1}^{d} \left(\frac{m'-1}{2} r_j + i_j \right) = \sum_{j=1}^{k} \left(\frac{m'+1}{2} r_j - i'_j \right) + \sum_{j=k+1}^{d} \left(\frac{m'-1}{2} r_j + i'_j \right),$$

for all $k \in \{0, \ldots, d-1\}$. Hence,

$$\sum_{j=1}^{k} \left(i'_j - i_j\right) + \sum_{j=k+1}^{d} \left(i_j - i'_j\right) = 0,$$

for all $k \in \{0, \ldots, d-1\}$. This forces $i_j = i'_j$ for all $j \in \{1, \ldots, d\}$; so the claim is proved. $\square$

Next, we prove that u does not belong to any $\langle b_k \rangle$. Indeed, assume that $u \in \langle b_k \rangle$ for a certain $k \in \{1, \ldots, m+n-1\}$. Then, since $\langle b_k \rangle$ is an $\mathcal{H}$-invariant prime ideal of R, it follows from the previous claim and from [3, II.2.10] that:

$$\text{If } a_{i_1,\ldots,i_d} \neq 0, \text{ then } u_{i_1,\ldots,i_d} = \prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j} \in \langle b_k \rangle . \tag{3}$$

Set $k := rd + s$ with $r \in \{0, \ldots, m'+n'-1\}$ and $s \in \{1, \ldots, d\}$. We distinguish between two cases. First, assume that r is even. Since the $\langle b_i \rangle$ are pairwise distinct height one completely prime ideals of R, we deduce from (3) that $a_{i_1,\ldots,i_d} = 0$ if $i_s = 0$. This implies that $V = \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} X_1^{i_1} \ldots X_d^{i_d} = X_s V'$, with $V' \in \mathbb{C}[X_1, \ldots, X_d]$. This contradicts the facts that V is irreducible and $V \neq X_s$.

Next, assume that r is odd. Since the $\langle b_i \rangle$ are pairwise distinct height one completely prime ideals of R, we deduce from (3) that $a_{i_1,\ldots,i_d} = 0$ if $i_s = r_s$. Now this contradicts $\deg_{X_s} V = r_s$.

To sum up: u does not belong to any $\langle b_k \rangle$.

• *Step 3. We prove that P is generated by u.*

Recall from step one that $\varphi(P)$ is generated by u. Hence, it is clear that $P = \varphi(P) \cap R \supseteq \langle u \rangle$. Let now $x \in P$. It remains to prove that $x \in \langle u \rangle$. There exists $(\alpha_1, \ldots, \alpha_{m+n-1}) \in \mathbb{N}^{m+n-1}$ such that $x b_1^{\alpha_1} \ldots b_{m+n-1}^{\alpha_{m+n-1}} = ur$ with $r \in R$. Choose such a $(\alpha_1, \ldots, \alpha_{m+n-1}) \in \mathbb{N}^{m+n-1}$ minimal (for the lexicographic order). If $(\alpha_1, \ldots, \alpha_{m+n-1}) \neq 0$, there exists k such that $\alpha_k \neq 0$. Then $ur = x b_1^{\alpha_1} \ldots b_{m+n-1}^{\alpha_{m+n-1}}$ belongs to the completely prime ideal of R generated by b_k. Hence, $u \in \langle b_k \rangle$ or $r \in \langle b_k \rangle$. Because of step two, the first possibility can not happen. Hence, $r \in \langle b_k \rangle$. Since b_k is normal, we can write $r = r' b_k$ with $r' \in R$. Thus $x b_1^{\alpha_1} \ldots b_{m+n-1}^{\alpha_{m+n-1}} = ur = ur' b_k$; and so $x b_1^{\alpha_1} \ldots b_k^{\alpha_k - 1} \ldots b_{m+n-1}^{\alpha_{m+n-1}} = ur'$. This contradicts the minimality of $(\alpha_1, \ldots, \alpha_{m+n-1})$. Hence $(\alpha_1, \ldots, \alpha_{m+n-1}) = 0$; and so $x = ur \in \langle u \rangle$, as desired. $\square$

3 Automorphisms of quantum matrices

In this section, we investigate the group of automorphisms of $\mathcal{O}_q(M_{m,n})$. Using graded arguments together with the results of the previous sections, we show that, in the non-square case, every $\mathcal{H}$-invariant height one prime of $\mathcal{O}_q(M_{m,n})$, except possibly one, is invariant under every automorphism. Next, by using the preferred basis of $\mathcal{O}_q(M_{m,n})$ introduced in [7], we show that the group of automorphisms of $\mathcal{O}_q(M_{m,n})$, with $2 \leq m < n$, is isomorphic to the torus $(\mathbb{C}^*)^{m+n-1}$.

In the sequel, we will use several times the following well-known result concerning normal elements of R.

Lemma 3.1 *Let u and v two nonzero normal elements of R such that $\langle u \rangle = \langle v \rangle$. Then there exist $\lambda, \mu \in \mathbb{C}^*$ such that $u = \lambda v$ and $v = \mu u$.*

3.1 q-commutation, gradings and automorphisms.

Let $A = \oplus_{i \in \mathbb{N}} A_i$ be a $\mathbb{N}$-graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$. Assume that A is a domain generated as an algebra by x_1,..., x_n, and that $A_1 = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. We set $A_{\geq d} := \oplus_{i \geq d} A_i$. The following result was inspired by a result in [1].

Proposition 3.2 *Assume that, for all $i \in \{1, \ldots, n\}$, there exist $j \neq i$ and $q_{ij} \neq 1$ such that $x_i x_j = q_{ij} x_j x_i$. Let σ be an automorphism of A and x be a nonzero homogeneous element of degree d of A.*
Then $\sigma(x) = y_d + y_{>d}$, where $y_d \in A_d \setminus \{0\}$ and $y_{>d} \in A_{>d}$.

Proof. First, observe that it is sufficient to prove that $\sigma(A_d) \subseteq A_{\geq d}$, for every automorphism σ of A. Indeed, assume that this is the case, and let x be a nonzero homogeneous element of degree d of A. Then we can write $\sigma(x) = y_d + y_{>d}$, where $y_d \in A_d$ and $y_{>d} \in A_{>d}$. If $y_d = 0$, then $\sigma(x) = y_{>d} \in A_{>d}$, and thus $\sigma^{-1}(A_{>d})$ is not contained in $A_{>d}$. This is a contradiction.

Hence it just remains to prove that, for every automorphism σ of A, we have $\sigma(A_d) \subseteq A_{\geq d}$. Naturally, it is sufficient to prove this result when $x = x_i$ is one of the canonical generators of A. So let σ be an automorphism σ of A and $i \in \{1, \ldots, n\}$. We can write

$$\sigma(x_i) = \alpha_i + f_i,$$

where $\alpha_i \in \mathbb{C}$ and $f_i \in A_{\geq 1}$. We have to prove that $\alpha_i = 0$.

Now, by hypothesis, there exist $j \in \{1, \ldots, n\}$ and $q_{ij} \neq 1$ such that $x_i x_j = q_{ij} x_j x_i$. Set

$$\sigma(x_j) = f_j + g_j,$$

where $f_j \in A_t$, $f_j \neq 0$ and $g_j \in A_{>t}$. Applying σ to the equality $x_i x_j = q_{ij} x_j x_i$, and next identifying the homogeneous part of degree t yields $\alpha_i f_j = q_{ij} f_j \alpha_i$. Thus, $\alpha_i f_j = 0$, since $q_{ij} \neq 1$. Now, since $f_j \neq 0$, this forces $\alpha_i = 0$, as desired. □

Note that the commutativity hypothesis of the Proposition 3.2 is satisfied by the algebra $R = \mathcal{O}_q(M_{m,n})$, provided that $n \geq 2$. Indeed, the relations that define R are all quadratic, so that $R = \oplus_{i \in \mathbb{N}} R_i$ is a $\mathbb{N}$-graded algebra, the canonical generators $Y_{i,\alpha}$ of R having degree one.

Next, for all $(i, \alpha) \in [\![1, m]\!] \times [\![1, n]\!]$ and $\beta \neq \alpha$, we have $Y_{i,\alpha} Y_{i,\beta} = q^{\bullet} Y_{i,\beta} Y_{i,\alpha}$ with $\bullet = \pm 1$. Thus, the commutativity hypothesis of Proposition 3.2 is satisfied; and so one can state:

Corollary 3.3 *Let σ be an automorphism of $R = \mathcal{O}_q(M_{m,n})$ and x an homogeneous element of degree d of R.*
Then $\sigma(x) = y_d + y_{>d}$, where $y_d \in R_d \setminus \{0\}$ and $y_{>d} \in R_{>d}$.

Note, for later use, that a $t \times t$ quantum minor of R is a homogeneous element of degree t with respect to this grading of R. In the sequel, R will always be endowed with this grading.

3.2 The action of $\mathrm{Aut}(\mathcal{O}_q(M_{m,n}))$ on the set of height one primes: the non-square case.

Throughout this section, we assume that $m < n$. Our aim in this section is to show that every $\mathcal{H}$-invariant height one prime of $\mathcal{O}_q(M_{m,n})$, except possibly one, is invariant under every automorphism. In order to do this, we will distinguish between two cases.

3.2.1 The case where $n \neq 3m$.

Throughout this section, we assume that $m < n$ and $n \neq 3m$. In this case, we first show that the set of all $\mathcal{H}$-invariant height one primes of R is invariant under every automorphism of R, that is, we have:

Lemma 3.4 *Assume that $n \neq 3m$. Let σ be an automorphism of R and $i \in \{1, \ldots, m + n - 1\}$. Then there exists $j \in \{1, \ldots, m + n - 1\}$ such that $\sigma(\langle b_i \rangle) = \langle b_j \rangle$.*

Proof. First, assume that $v_2(m) \neq v_2(n)$. It follows from Proposition 2.5 that $\{\langle b_i \rangle \mid i \in [\![1, m+n-1]\!]\}$ is exactly the set of all height one primes of R and so the result is obvious in this case.

Next, assume that $v_2(m) = v_2(n)$. We use the notation of Section 2.2. In particular, d denotes the greatest common divisor of m and n, and we set $m = dm'$ and $n = dn'$. Observe that m' and n' are odd. Note further that, since $n \neq 3m$, we have $(m', n') \neq (1, 3)$. Moreover, since $m < n$, we have $m' < n'$. Since m' and n' are both odd, this forces $n' \geq 5$.

Since $\langle b_i \rangle$ is a height one prime ideal of R, its image under σ is also a height one prime ideal of R. We distinguish between two cases.

First, if $\langle \sigma(b_i) \rangle$ is $\mathcal{H}$-invariant, then it follows from Proposition 2.1 that $\langle \sigma(b_i) \rangle = \langle b_j \rangle$ for some j and so the proof is complete in this case.

Next, assume that $\langle \sigma(b_i) \rangle$ is not $\mathcal{H}$-invariant. In this case, $\langle \sigma(b_i) \rangle$ is a height one (completely) prime ideal of R which is not $\mathcal{H}$-invariant. Thus, $\langle \sigma(b_i) \rangle$ belongs to the $\langle 0 \rangle$-stratum of the prime spectrum of R, by Proposition 2.5; and so we deduce from Proposition 2.6 that there exist $(r_1, \ldots, r_d) \in \mathbb{N}^d \setminus \{0\}$ and scalars $a_{i_1,\ldots,i_d} \in \mathbb{C}$ such that $\langle \sigma(b_i) \rangle = \langle u \rangle$, where

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_d=0}^{r_d} a_{i_1,\ldots,i_d} \prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j}.$$

It follows from Proposition 2.6 that u is a normal element of R. On the other hand, $\sigma(b_i)$ is normal, since b_i is normal. Thus, we deduce from Lemma 3.1 that $u \in \mathbb{C}^* \sigma(b_i)$.

Now b_i is a quantum minor of $R = \mathcal{O}_q(M_{m,n})$, and so b_i is a homogeneous element of degree less than or equal to m. So Corollary 3.3 implies that

$$u \notin R_{>m}. \tag{4}$$

On the other hand, since $(r_1, \ldots, r_d) \neq 0$, there exists k such that $r_k \geq 1$. Now, recalling that m' and n' are odd and that $n' \geq 5$, we have:

$$\deg \left(\prod_{j=1}^{d} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{m'+n'-1} b_{id+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{m'+n'-1} b_{id+j} \right]^{r_j - i_j} \right) \geq \begin{cases} \deg\left(b_k b_{(m'+1)d+k}\right) & \text{if } i_k \geq 1 \\ \deg\left(b_{m'd+k} b_{(m'+n'-3)d+k}\right) & \text{if } i_k = 0 \end{cases}$$

Since $m < n$, we have $m' < n'$. Further, m' and n' are odd, and so $m' + 1 < n'$. Hence $m = m'd \leq m'd + k \leq (m'+1)d + k \leq n = n'd$. This implies that $b_{(m'+1)d+k}$ and $b_{m'd+k}$

are $m \times m$ quantum minors. Thus, in both cases, we get

$$\deg\left(\prod_{j=1}^{d}\left[\prod_{\substack{i=0\\ i \text{ even}}}^{m'+n'-1} b_{id+j}\right]^{i_j}\left[\prod_{\substack{i=0\\ i \text{ odd}}}^{m'+n'-1} b_{id+j}\right]^{r_j-i_j}\right) \geq m+1.$$

Thus u is a linear combination of terms of degree greater than m. This implies that $u \in R_{>m}$, contradicting (4); and so the proof is complete. □

In fact, more is true: each $\mathcal{H}$-invariant height one prime is left invariant by any automorphism, as the following result shows.

Proposition 3.5 *Assume that $n \neq 3m$ and let σ be an automorphism of R. Then, for each $i \in \{1, \ldots, m+n-1\}$, there exists $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_i$.*

Proof. Let $i \in \{1, \ldots, m+n-1\}$. It follows from Lemma 3.4 that there exists $j \in \{1, \ldots, m+n-1\}$ such that $\sigma(\langle b_i \rangle) = (\langle b_j \rangle)$. Since $\sigma(b_i)$ and b_j are normal, it follows from Lemma 3.1 that there exists $\lambda_j \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$.

Thus, there exist scalars $\lambda_1, \ldots, \lambda_{m+n-1} \in \mathbb{C}^*$ and a permutation $s \in S_{m+n-1}$ such that $\sigma(b_i) = \lambda_i b_{s(i)}$ for all $i \in \{1, \ldots, m+n-1\}$.

We will now prove that s is the identity, that is, $s(i) = i$ for all $i \in \{1, \ldots, m+n-1\}$.

First, let $i \in \{m, \ldots, n\}$, so that b_i is a $m \times m$ quantum minor. Then $b_i \in R_m$ and it follows from Corollary 3.3 that $\sigma(b_i) \in R_{\geq m}$. Hence $b_{s(i)} \in R_{\geq m}$. This implies that $b_{s(i)}$ is also a $m \times m$ quantum minor, so that $s(i) \in \{m, \ldots, n\}$. Thus s induces a permutation of $\{m, \ldots, n\}$.

We now prove with the help of a decreasing induction that $s(i) = i$ for all $i \in \{m, \ldots, n\}$.

It follows from Corollary 2.3 that $b_n b_j = q^{\bullet} b_j b_n$ with $\bullet \geq 0$ for all $j \in \{m, \ldots, n\}$. Then, applying σ leads to $b_{s(n)} b_{s(j)} = q^{\bullet} b_{s(j)} b_{s(n)}$ with $\bullet \geq 0$ for all $j \in \{m, \ldots, n\}$. Since s is a permutation of $\{m, \ldots, n\}$, this implies that $b_{s(n)} b_j = q^{\bullet} b_j b_{s(n)}$ with $\bullet \geq 0$ for all $j \in \{m, \ldots, n\}$. Now, if $s(n) \neq n$, then we have $s(n)+1 \in \{m, \ldots, n\}$ and $b_{s(n)} b_{s(n)+1} = q^{-1} b_{s(n)+1} b_{s(n)}$. This is a contradiction and so $s(n) = n$.

We now assume that $m \leq i < n$. It follows from the induction hypothesis that s induces a permutation of $\{m, \ldots, i\}$; so that $s(i) \leq i$. By using a similar argument to that in the previous paragraph, we obtain $s(i) = i$.

Hence, $s(i) = i$ for all $i \in \{m, \ldots, n\}$.

Now let $i \in \{1, \ldots, m-1\}$. Then $b_i \in R_i$; and so it follows from Proposition 3.2 that $\sigma(b_i) \in R_{\geq i} \setminus R_{>i}$. Hence $b_{s(i)} \in R_{\geq i} \setminus R_{>i}$ so that $b_{s(i)}$ is also a $i \times i$ quantum minor. This implies that either $s(i) = i$ or $s(i) = m+n-i$. Note that similar arguments show

that either $s(m+n-i) = i$ or $s(m+n-i) = m+n-i$; so s induces a permutation of $\{i, m+n-i\}$. Observe that it follows from Corollary 2.3 that $b_n b_{m+n-i} = b_{m+n-i} b_n$ and $b_n b_i = q^{\bullet} b_i b_n$ with $\bullet > 0$. Since we have already proved that $s(n) = n$, applying σ leads to $b_n b_{s(m+n-i)} = b_{s(m+n-i)} b_n$ and $b_n b_{s(i)} = q^{\bullet} b_{s(i)} b_n$ with $\bullet > 0$. This forces $s(i) = i$ and $s(m+n-i) = m+n-i$. □

3.2.2 The case where $n = 3m$.

Throughout this section, we assume that $n = 3m$. In this case, we are not able to prove directly that the set of all $\mathcal{H}$-invariant height one primes is invariant under every automorphism of $R = \mathcal{O}_q(M_{m,n})$. However, by using arguments similar to those developed in the proof of Lemma 3.4, one can establish the following weaker result.

Lemma 3.6 *Assume that $n = 3m$. Let σ be an automorphism of R and $i \in \{1, \ldots, m+n-1\}$.*
Then, either

1. *there exists $j \in \{1, \ldots, m+n-1\}$ such that $\sigma(\langle b_i \rangle) = \langle b_j \rangle$,*
   or,
2. *there exist $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ such that $\sigma(b_i) = \lambda b_{2m} + \mu b_m b_{3m}$.*

Proof. Note that $v_2(m) = v_2(n)$, since $n = 3m$. Also, observe that the greatest common divisor d of m and n is equal to m and that, if we set $m = dm'$ and $n = dn'$, then $m' = 1$ and $n' = 3$.

Since $\langle b_i \rangle$ is a height one prime ideal of R, its image under σ is also a height one prime ideal of R. We distinguish between two cases.

If $\langle \sigma(b_i) \rangle$ is $\mathcal{H}$-invariant, then it follows from Proposition 2.1 that $\sigma(\langle b_i \rangle) = \langle b_j \rangle$ for some j and so the proof is complete in this case.

Assume now that $\langle \sigma(b_i) \rangle$ is not $\mathcal{H}$-invariant. In this case, $\langle \sigma(b_i) \rangle$ is a height one (completely) prime ideal of R which is not $\mathcal{H}$-invariant. Thus it follows from Proposition 2.5 that $\langle \sigma(b_i) \rangle$ belongs to the $\langle 0 \rangle$-stratum of the prime spectrum of R; and so, recalling that $d = m$, we deduce from Proposition 2.6 that there exist $(r_1, \ldots, r_m) \in \mathbb{N}^m \setminus \{0\}$ and scalars $a_{i_1,\ldots,i_m} \in \mathbb{C}$ such that $\langle \sigma(b_i) \rangle = \langle u \rangle$ where

$$u := \sum_{i_1=0}^{r_1} \cdots \sum_{i_m=0}^{r_m} a_{i_1,\ldots,i_m} \prod_{j=1}^{m} \left[\prod_{\substack{i=0 \\ i \text{ even}}}^{3} b_{im+j} \right]^{i_j} \left[\prod_{\substack{i=0 \\ i \text{ odd}}}^{3} b_{im+j} \right]^{r_j - i_j}.$$

It follows that u is a normal element of R, by Proposition 2.6. On the other hand, $\sigma(b_i)$ is normal in R, since b_i is normal in R. . Hence, $u \in \mathbb{C}^*\sigma(b_i)$, by Lemma 3.1.

Now b_i is a quantum minor of $R = \mathcal{O}_q(M_{m,n})$; and so b_i is a homogeneous element of degree less than or equal to m. Thus, Corollary 3.3 implies that

$$u \notin R_{>m}. \tag{5}$$

On the other hand, since $(r_1, \ldots, r_m) \neq 0$, there exists k such that $r_k \geq 1$. We consider three separate cases.

• First, suppose that $k < m$, then

$$\deg\left(\prod_{j=1}^{m}\left[\prod_{\substack{i=0\\ i \text{ even}}}^{3} b_{im+j}\right]^{i_j}\left[\prod_{\substack{i=0\\ i \text{ odd}}}^{3} b_{im+j}\right]^{r_j-i_j}\right) \geq \begin{cases} \deg\left(b_k b_{2m+k}\right) & \text{if } i_k \geq 1 \\ \deg\left(b_{m+k} b_{3m+k}\right) & \text{if } i_k = 0 \end{cases}$$

Now, $m+k$ and $2m+k$ are both between m and n, since $k < m$ and $n = 3m$. Thus, b_{m+k} and b_{2m+k} are $m \times m$ quantum minors. Also, b_k and b_{3m+k} are homogeneous of degree greater than or equal to 1. Thus, in both cases, we get

$$\deg\left(\prod_{j=1}^{m}\left[\prod_{\substack{i=0\\ i \text{ even}}}^{3} b_{im+j}\right]^{i_j}\left[\prod_{\substack{i=0\\ i \text{ odd}}}^{3} b_{im+j}\right]^{r_j-i_j}\right) \geq m+1.$$

Hence u is a linear combination of terms of degree greater than m. This implies that $u \in R_{>m}$. This contradicts (5).

• Next, assume that $k = m$ and $r_m \geq 2$. In this case, one can prove, by using similar arguments, that u is a linear combination of terms of degree greater than m; so this case cannot happen.

• Finally, assume that $k = m$ and $(r_1, \ldots, r_m) = (0, \ldots, 0, 1)$. In this case, there exist $\lambda', \mu' \in \mathbb{C}$ such that

$$u = \lambda' b_{2m} + \mu' b_m b_{3m}.$$

If $\lambda' = 0$, then once again u is a linear combination of terms of degree greater than m, contradicting (5). Hence $\lambda' \neq 0$. Since we have already proved that $u \in \mathbb{C}^*\sigma(b_i)$, we see that there exist $\lambda \in \mathbb{C}^*$ and $\mu \in \mathbb{C}$ such that $\sigma(b_i) = \lambda b_{2m} + \mu b_m b_{3m}$, as desired. □

The following commutation relations can be easily deduced from Corollary 2.3.

Lemma 3.7 *For all $m \leq i \leq 3m$:*

$$b_i(\lambda b_{2m} + \mu b_m b_{3m}) = \begin{cases} q^{\alpha_{i,2m}}(\lambda b_{2m} + \mu b_m b_{3m})b_i & \text{if } i < 2m \\ q^{-\alpha_{i,2m}}(\lambda b_{2m} + \mu b_m b_{3m})b_i & \text{if } i > 2m, \end{cases}$$

where $\alpha_{i,2m} := \Big| \{3m - i + 1, \ldots, 4m - i\} \cap \{m + 1, \ldots, 2m\} \Big| - m.$

We can now obtain the analogous (but slightly weaker) result to Proposition 3.5 in the $n = 3m$ case.

Proposition 3.8 *Assume that $n = 3m$ and let σ be an automorphism of R. Then, for all $i \in \{1, \ldots, 4m-1\}$, $i \neq 2m$, there exists $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_i$.*

Proof. Let $i \in \{1, \ldots, 4m-1\}$ and assume that there exists $j \in \{1, \ldots, 4m-1\}$ such that $\sigma(\langle b_i \rangle) = \langle b_j \rangle$. Then, it follows from Lemma 3.1 that there exists $\lambda_j \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$, since $\sigma(b_i)$ and b_j are normal.

Thus, we deduce from Lemma 3.6 that, for all $i \in \{1, \ldots, 4m-1\}$, either there exist $j \in \{1, \ldots, 4m-1\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$, or there exist $\lambda_i, \mu_i \in \mathbb{C}$ with $\lambda_i \neq 0$ such that $\sigma(b_i) = \lambda_i b_{2m} + \mu_i b_m b_{3m}$.

We distinguish between two cases.

• If, for each $i \in \{1, \ldots, 4m-1\}$, there exist $j \in \{1, \ldots, 4m-1\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$, then, by using similar arguments to those in the proof of Proposition 3.5, we show that $\sigma(b_i) = \lambda_i b_i$ for all $i \in \{1, \ldots, 4m-1\}$.

• Now, assume that there exist $k \in \{1, \ldots, 4m-1\}$, and $\lambda_k, \mu_k \in \mathbb{C}$ with $\lambda_k \neq 0$, such that $\sigma(b_k) = \lambda_k b_{2m} + \mu_k b_m b_{3m}$.

•• First, we show that $k = 2m$.

Observe that, for all $i \in \{1, \ldots, m-1\} \cup \{3m+1, \ldots, 4m-1\}$, the quantum minor b_i is a homogeneous element of degree less than m. Hence, $\sigma(b_i) \notin R_{\geq m}$, by Corollary 3.3. Since $\sigma(b_k) = \lambda_k b_{2m} + \mu_k b_m b_{3m} \in R_{\geq m}$, this shows that $i \neq k$. Thus, $k \in \{m, \ldots, 3m\}$.

Next, let $i \in \{m, \ldots, 3m\}$, so that b_i is a $m \times m$ quantum minor. Then $b_i \in R_m$ and it follows from Corollary 3.3 that $\sigma(b_i) \in R_{\geq m}$. Hence, either

$$\text{there exist } j \in \{m, \ldots, 3m\} \text{ and } \lambda_i \in \mathbb{C}^* \text{ such that } \sigma(b_i) = \lambda_i b_j, \tag{6}$$

or

$$\text{there exist } \lambda_i, \mu_i \in \mathbb{C} \text{ with } \lambda_i \neq 0 \text{ such that } \sigma(b_i) = \lambda_i b_{2m} + \mu_i b_m b_{3m}. \tag{7}$$

We now prove that (7) can not happen, for all $i \in \{2m+1, \dots, 3m\}$. Indeed, assume that there exists $i \in \{2m+1, \dots, 3m\}$ such that $\sigma(b_i) = \lambda_i b_{2m} + \mu_i b_m b_{3m}$. Note that $b_i b_j = q^\bullet b_j b_i$ with $\bullet > 0$, for all $m \leq j < i$, by Corollary 2.3.

Hence, by applying σ, we obtain:

$$(\lambda_i b_{2m} + \mu_i b_m b_{3m})\sigma(b_j) \quad = \quad q^\bullet \sigma(b_j)(\lambda_i b_{2m} + \mu_i b_m b_{3m}) \tag{8}$$

with $\bullet > 0$ for all $m \leq j < i$. This implies that $\sigma(b_j)$ can not be equal to $\lambda b_{2m} + \mu b_m b_{3m}$ with $\lambda, \mu \in \mathbb{C}$, since $\lambda b_{2m} + \mu b_m b_{3m}$ commutes with $\lambda_i b_{2m} + \mu_i b_m b_{3m}$ by Lemma 3.7. Hence we deduce from (6) and (7) that, for each j such that $m \leq j < i$, there exist $l \in \{m, \dots, 3m-1\}$ and $\lambda_j \in \mathbb{C}^*$ such that $\sigma(b_j) = \lambda_j b_l$. Moreover, we deduce from (8) and Lemma 3.7 that we must have $l \in \{m, \dots, 2m-1\}$. Thus, since $i > 2m$, there exist $j \neq j'$ with $m \leq j, j' < i$ with $\sigma(b_j) \in \mathbb{C}^* \sigma(b_{j'})$. This is impossible since σ is an automorphism and $\langle b_j \rangle \neq \langle b_{j'} \rangle$, by Proposition 2.1.

Hence, for all $i \in \{2m+1, \dots, 3m\}$, there exist $j \in \{m, \dots, 3m\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$. In other words, $k \neq i$ for all $i \in \{2m+1, \dots, 3m\}$.

A similar argument shows that, for all $i \in \{m, \dots, 2m-1\}$, there exist $j \in \{m, \dots, 3m\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$, and so $k \neq i$ for all $i \in \{m, \dots, 2m-1\}$.

In conclusion, the only possibility is $k = 2m$. Hence, we have already proved that:

1. There exist $\lambda, \mu \in \mathbb{C}$, with $\lambda \neq 0$, such that $\sigma(b_{2m}) = \lambda b_{2m} + \mu b_m b_{3m}$.

2. For all $i \in \{m, \dots, 3m\}$ with $i \neq 2m$, there exist $j \in \{m, \dots, 3m\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$.

3. For all $i \in \{1, \dots, m-1\} \cup \{3m+1, \dots, 4m-1\}$, there exist $j \in \{1, \dots, 4m-1\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$.

•• We prove by induction that $\sigma(b_i) = \lambda_i b_i$, for all $i \in \{2m+1, \dots, 3m\}$.

First, we know that there exist $\lambda_{2m+1} \in \mathbb{C}^*$ and $j \in \{m, \dots, 3m\}$ such that $\sigma(b_{2m+1}) = \lambda_{2m+1} b_j$. It follows from Corollary 2.3 that $b_{2m} b_{2m+1} = q b_{2m+1} b_{2m}$. Hence, applying σ yields:

$$(\lambda b_{2m} + \mu b_m b_{3m}) b_j = q b_j (\lambda b_{2m} + \mu b_m b_{3m}).$$

In view of Lemma 3.7, this forces $j = 2m+1$, as desired.

Next, let $i \in \{2m+2, \dots, 3m\}$. It follows from the previous study that there exist $j \in \{m, \dots, 3m\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$. Moreover we deduce from the

induction hypothesis that $\sigma(b_{i-1}) = \lambda_{i-1}b_{i-1}$. Now, Corollary 2.3 shows that $b_ib_{i-1} = qb_{i-1}b_i$. Applying σ yields

$$b_jb_{i-1} = qb_{i-1}b_j.$$

In view of Corollary 2.3, this implies that $j = i$, as desired.

Hence, $\sigma(b_i) = \lambda_i b_i$, for all $i \in \{2m+1, \dots, 3m\}$. A similar argument shows that $\sigma(b_i) = \lambda_i b_i$, for all $i \in \{m, \dots, 2m-1\}$.

•• Now, let $i \in \{1, \dots, m-1\}$. Then $b_i \in R_i$; and so it follows from Proposition 3.2 that $\sigma(b_i) \in R_{\geq i} \setminus R_{>i}$. Now, there exist $j \in \{1, \dots, 4m-1\}$ and $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_j$ because of the previous study. Hence we have $b_j \in R_{\geq i} \setminus R_{>i}$, so that b_j is also a $i \times i$ quantum minor. This implies that either $j = i$ or $j = 4m - i$. Now, it follows from Corollary 2.3 that $b_{3m}b_i = q^\bullet b_i b_{3m}$ with $\bullet > 0$. Since we have already proved that $\sigma(b_{3m}) = \lambda_{3m}b_{3m}$, composing by σ leads to $b_{3m}b_j = q^\bullet b_j b_{3m}$ with $\bullet > 0$. On the other hand, it follows Corollary 2.3 that $b_{3m}b_{4m-i} = b_{4m-i}b_{3m}$. So j can not be equal to $4m - i$. Hence $j = i$ and so $\sigma(b_i) = \lambda_i b_i$, as desired.

A similar argument shows that there exist $\lambda_i \in \mathbb{C}^*$ such that $\sigma(b_i) = \lambda_i b_i$, for all $i \in \{3m+1, \dots, 4m-1\}$. □

3.3 The automorphism group of non-square quantum matrices.

Theorem 3.9 *Assume that $m < n$ and $(m,n) \neq (1,3)$. Let σ be an automorphism of $\mathcal{O}_q(M_{m,n})$. Then there exist $\mu_{i,\alpha} \in \mathbb{C}^*$ such that $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha}Y_{i,\alpha}$, for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,n]\!]$.*

Proof. We proceed by induction on n. The case $n = 2$ easily follows from Proposition 3.5. So we assume that $n \geq 3$. If $m = 1$, then $n > 3$, and so once again the result easily follows from Proposition 3.5. So we assume that $m \geq 2$. We need to see that σ acts on each generator $Y_{i,\alpha}$ by multiplication by a scalar. We do this by using preferred basis arguments. We use the language and notation of [7].

Note that, because of [7, Proposition 5.3], the quantum minor $b_n = [1 \dots m | 1 \dots m]$ commutes with $Y_{i,\alpha}$, for $1 \leq i, \alpha \leq m$, and that $b_n[R \mid C] = q^\bullet[R \mid C]b_n$ with $\bullet \geq 0$ for all other quantum minors $[R \mid C]$ of $\mathcal{O}_q(M_{m,n})$. Thus, b_n $q^\bullet$-commutes with each monomial in the preferred basis where $\bullet$ is ≥ 0, and [7, Proposition 5.3] shows that $\bullet$ is equal to zero if and only if the only quantum minors that occur in the monomial are those of $(Y_{i,\alpha})_{i,\alpha \in \{1,\dots,m\}}$. Observe further that, if $b_n = [1 \dots m | 1 \dots m]$ commutes with an element

$y \in \mathcal{O}_q(M_{m,n})$, then it must commute with each of the monomials in the expression for y in the preferred basis.

Now, let σ be an automorphism of $\mathcal{O}_q(M_{m,n})$. We will show first that $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha}$ for all $1 \leq i, \alpha \leq m$.

Set $y_{i,\alpha} := \sigma(Y_{i,\alpha})$. Since $Y_{i,\alpha}$ commutes with b_n and since σ acts on b_n by multiplication by a scalar (see Proposition 3.5 if $n \neq 3m$ or Proposition 3.8 if $n = 3m$), then $y_{i,\alpha}$ must commute with b_n. Thus, any monomial z in the expression for $y_{i,\alpha}$ in the preferred basis must also commute with b_n. This means that the only quantum minors that can occur in z are those of $(Y_{i,\alpha})_{i,\alpha \in \{1,\dots,m\}}$. In particular, $\sigma(Y_{i,\alpha})$ belongs to the subalgebra $R_{m,m}$ of $R = \mathcal{O}_q(M_{m,n})$ generated by $Y_{i,\alpha}$, $1 \leq i, \alpha \leq m$ (which is a copy of $O_q(M_{m,m})$). Hence σ induces an automorphism of $R_{m,m}$.

Moreover, σ acts on the quantum minors $b_{n+i} = [i+1, \dots, m | 1, \dots, m-i]$ by multiplication by scalars, by Proposition 3.5 (if $n \neq 3m$) or Proposition 3.8 (if $n = 3m$). In particular, there exists $\mu_{m,1} \in \mathbb{C}^*$ such that $\sigma(Y_{m,1}) = \mu_{m,1} Y_{m,1}$. Let $j \in \{1, \dots, m-1\}$. Then $Y_{j,1} Y_{m,1} = q Y_{m,1} Y_{j,1}$; so that $\sigma(Y_{j,1}) Y_{m,1} = q Y_{m,1} \sigma(Y_{j,1})$. Write $\sigma(Y_{j,1})$ in the PBW basis of $R_{m,m}$:

$$\sigma(Y_{j,1}) = \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{m,m}^{\gamma_{m,m}},$$

where Γ is a finite subset of $\mathbb{N}^{m^2}$. Hence $\sigma(Y_{j,1}) Y_{m,1} = q Y_{m,1} \sigma(Y_{j,1})$ implies that, for all $\underline{\gamma} \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$, we have $\sum_{i=1}^{m-1} \gamma_{i,1} - \sum_{i=2}^{m} \gamma_{m,i} = 1$. Thus, for all $\underline{\gamma} \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$, there exists $i \in \{1, \dots, m-1\}$ with $\gamma_{i,1} \neq 0$. Denote by J, the prime ideal of R generated by the $Y_{i,1}$, with $i \in \{1, \dots, m\}$, and, similarly, denote by J' the prime ideal of $R_{m,m}$ generated by the $Y_{i,1}$, with $i \in \{1, \dots, m\}$. Observe that $J = Y_{1,1}R + \dots + Y_{m,1}R$ and $J' = Y_{1,1}R_{m,m} + \dots + Y_{m,1}R_{m,m}$. We have just proved that $\sigma(Y_{j,1}) \in J'$ for all $j \in \{1, \dots, m\}$. Hence $\sigma(J') = J'$ and $\sigma(J) = J$.

Thus, σ induces an automorphism of $R_{m,m}/J' \simeq O_q(M_{m,m-1})$. Now, because of the induction hypothesis (and the isomorphism $O_q(M_{m,m-1}) \simeq O_q(M_{m-1,m})$), we obtain $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{m,m}^{\gamma_{m,m}}$ with $\mu_{i,\alpha} \neq 0$ and at least one of $\gamma_{k,1} > 0$ for each nonzero term of the sum. (Observe that we can apply the inductive hypothesis since $(m-1, m) \neq (1, 3)$.)

On the other hand, it follows from Proposition 3.5 or Proposition 3.8 that

$\sigma(b_{n-1}) = \lambda b_{n-1}$ with $\lambda \in \mathbb{C}^*$. Moreover, recalling that $b_{n-1} = [1 \dots m | 2 \dots m+1]$, we deduce from a transposed version of [7, Lemma 5.1] that, for all $1 \leq i, \alpha \leq m$, we have

$$b_{n-1} Y_{i,\alpha} = q^{\bullet} Y_{i,\alpha} b_{n-1}$$

with $\bullet = -1$ if $i = 1$ and 0 otherwise. Let $1 \leq i, \alpha \leq m$ with $\alpha \neq 1$. Thus we must have $b_{n-1}\sigma(Y_{i,\alpha}) = \sigma(Y_{i,\alpha}) b_{n-1}$. Since $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} Y_{1,2}^{\gamma_{1,2}} \dots Y_{m,m}^{\gamma_{m,m}}$ with at least one of $\gamma_{k,1} \neq 0$ for each nonzero term of the sum, this implies that the sum is empty and so $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha}$ for all $1 \leq i, \alpha \leq m$ with $\alpha \neq 1$.

Now σ induces an automorphism of $R_{m,m}$ that acts on the indeterminates $Y_{i,\alpha}$ ($1 \leq i \leq m$ and $1 < \alpha \leq m$) by multiplication by scalars. Moreover, it follows from Proposition 3.5 or Proposition 3.8 that σ also acts on the $b_i = [i-n+1, \dots, m \mid 1, \dots, m+n-i]$ ($n \leq i \leq m+n-1$) by multiplication by scalars. This forces $\sigma(Y_{i,1}) = \mu_{i,1} Y_{i,1}$ for all $1 \leq i \leq m$.

This establishes the following claim:

Claim 3.10 *For all $(i, \alpha) \in [\![1, m]\!]^2$, there exists $\mu_{i,\alpha} \in \mathbb{C}^*$ such that $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha}$.*

It remains to consider the case that $\alpha > m$. Let us now distinguish between two cases.

If $n = m+1$, then σ is an automorphism of $R = O_q(M_{m,m+1})$ that acts on the indeterminates $Y_{i,\alpha}$ ($1 \leq i, \alpha \leq m$) by multiplication by scalars. Further, because of Proposition 3.5 or Proposition 3.8, σ also acts on the quantum minors $b_i = [1, \dots, i \mid n-i+1, \dots, n]$ ($1 \leq i \leq m$) by multiplication by scalars. This forces $\sigma(Y_{i,m+1}) = \mu_{i,m+1} Y_{i,m+1}$ for all $1 \leq i \leq m$. This finishes the proof of Theorem 3.9 in the case where $n = m+1$.

Now, assume that $m+1 < n$. Recall that J denotes the two-sided ideal generated by the $Y_{k,1}$, $1 \leq k \leq m$. Note that $J = Y_{1,1}R + \dots + Y_{m,1}R$. It is well-known that the monomials

$$Y_{1,1}^{\gamma_{1,1}} Y_{2,1}^{\gamma_{2,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}}$$

form a PBW basis of R. Observe that $x = \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}}$ belongs to J if and only if at least one of the $\gamma_{k,1} \geq 1$ for all $\gamma \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$.

It follows from Claim 3.10 that $\sigma(J) = J$. Hence σ induces an automorphism of $R/J \simeq \mathcal{O}_q(M_{m,n-1})$. Since $m < n-1$ and $m \geq 2$, it follows from the inductive hypothesis that, for all (i, α) with $m+1 \leq \alpha \leq n$, we can write

$$\sigma(Y_{i,\alpha}) \;=\; \mu_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}} \tag{9}$$

where Γ is a finite subset of $\mathbb{N}^{mn}$ and at least one of the $\gamma_{k,1} \geq 1$ for all $\gamma \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$.

Let K the two-sided ideal generated by the $Y_{k,2}$, $1 \leq k \leq m$. Note that an element

$$x = \sum_{\underline{\gamma} \in \Gamma'} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}} \in R$$

belongs to K if and only if at least one of the $\gamma_{k,2} \geq 1$ for all $\gamma \in \Gamma'$ such that $c_{\underline{\gamma}} \neq 0$.

It follows from Claim 3.10 that $\sigma(K) = K$. Hence σ induces an automorphism of $R/K \simeq \mathcal{O}_q(M_{m,n-1})$. Since $m < n-1$ and $m \geq 2$, the inductive hypothesis applies and, for all (i,α) with $m+1 \leq \alpha \leq n$, we can write

$$\sigma(Y_{i,\alpha}) \;=\; \mu'_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma} \in \Gamma'} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}} \tag{10}$$

where Γ' is a finite subset of $\mathbb{N}^{mn}$ and at least one of the $\gamma_{k,2} \geq 1$ for all $\gamma \in \Gamma'$ such that $c_{\underline{\gamma}} \neq 0$.

Let (i,α) with $m+1 \leq \alpha \leq n$. It remains to prove that σ acts on $Y_{i,\alpha}$ by multiplication by a scalar. First, identifying the two expressions (9) and (10) of $\sigma(Y_{i,\alpha})$ in the PBW basis of R leads to:

$$\sigma(Y_{i,\alpha}) \;=\; \mu_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}} \tag{11}$$

where Γ is a finite subset of $\mathbb{N}^{mn}$ such that at least one of the $\gamma_{k,1} \geq 1$ and at least one of the $\gamma_{l,2} \geq 1$ for all $\gamma \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$.

By Proposition 3.5 (if $n \neq 3m$) or Proposition 3.8 (if $n = 3m$), the automorphism σ acts on $b_m = [1 \dots m \mid n-m+1 \dots n]$ by multiplication by a scalar. Further, it follows from a transposed version of [7, Lemma 5.1] that

$$b_m Y_{i,\alpha} = \begin{cases} q^{-1} Y_{i,\alpha} b_m & \text{if } \alpha \leq n-m \\ Y_{i,\alpha} b_m & \text{if } \alpha \geq n-m+1. \end{cases}$$

Hence

$$b_m \sigma(Y_{i,\alpha}) = \begin{cases} q^{-1} \sigma(Y_{i,\alpha}) b_m & \text{if } \alpha \leq n-m \\ \sigma(Y_{i,\alpha}) b_m & \text{if } \alpha \geq n-m+1, \end{cases}$$

Thus

$$\sum_{\underline{\gamma} \in \Gamma} c_{\underline{\gamma}} q^{-\gamma_{1,1} - \dots - \gamma_{m,1} - \dots - \gamma_{1,n-m} - \dots - \gamma_{m,n-m}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}}$$

$$= q^\bullet \sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}},$$

where $q^\bullet = -1$ if $\alpha \leq n-m$, and $q^\bullet = 0$ otherwise.

Consequently,

$$\gamma_{1,1} + \dots + \gamma_{m,1} + \dots + \gamma_{1,n-m} + \dots + \gamma_{m,n-m} \quad = \quad \begin{cases} 1 & \text{if } \alpha \leq n-m \\ 0 & \text{if } \alpha \geq n-m+1, \end{cases} \tag{12}$$

for all $\underline{\gamma} \in \Gamma$ such that $c_{\underline{\gamma}} \neq 0$, since q is not a root of unity.

On the other hand, recall that

$$\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha} + \sum_{\underline{\gamma}\in\Gamma} c_{\underline{\gamma}} Y_{1,1}^{\gamma_{1,1}} \dots Y_{m,1}^{\gamma_{m,1}} \dots Y_{1,n}^{\gamma_{1,n}} \dots Y_{m,n}^{\gamma_{m,n}}$$

where, for each nonzero term of the sum, at least one of the $\gamma_{k,1}$ is a positive integer and at least one of the $\gamma_{l,2}$ is a positive integer. Hence, $\gamma_{1,1}+\dots+\gamma_{m,1}+\dots+\gamma_{1,n-m}+\dots+\gamma_{m,n-m} \geq 2$ for all $\underline{\gamma} \in \Gamma$ with $c_{\underline{\gamma}} \neq 0$, since $n-m \geq 2$. This contradicts (12). Thus Γ must be empty for all (i,α) with $\alpha \geq m$ and so $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha}$ for all $\alpha > m$. This finishes the proof. □

Corollary 3.11 *Assume that $m < n$ and $(m,n) \neq (1,3)$. Let σ be an automorphism of $\mathcal{O}_q(M_{m,n})$. Then there exist unique nonzero complex numbers $h_1, \dots, h_m, h'_1, \dots, h'_{n-1}$ such that $\sigma(Y_{i,\alpha}) = h_i h'_\alpha Y_{i,\alpha}$ (with the convention $h'_n = 1$).*
Thus, $\mathrm{Aut}(\mathcal{O}_q(M_{m,n}))$ *is isomorphic to the torus* $(\mathbb{C}^*)^{m+n-1}$.

Proof. Let σ be an automorphism of $\mathcal{O}_q(M_{m,n})$. By Theorem 3.9, there exists a family $(\mu_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,n]\!]}$ of elements of $\mathbb{C}^*$ such that $\sigma(Y_{i,\alpha}) = \mu_{i,\alpha} Y_{i,\alpha}$ for all $(i,\alpha) \in [\![1,m]\!] \times [\![1,n]\!]$. Recall that, if $(i,\alpha),(j,\beta) \in [\![1,m]\!] \times [\![1,n]\!]$ with $i<j$ and $\alpha<\beta$, then $Y_{j,\beta}Y_{i,\alpha} = Y_{i,\alpha}Y_{j,\beta} - (q-q^{-1})Y_{i,\beta}Y_{j,\alpha}$. Hence, since σ is an automorphism, we must have $\mu_{i,\alpha}\mu_{j,\beta} = \mu_{i,\beta}\mu_{j,\alpha}$ for all $(i,\alpha),(j,\beta) \in [\![1,m]\!] \times [\![1,n]\!]$ with $i<j$ and $\alpha<\beta$. In other words, the matrix $(\mu_{i,\alpha})_{(i,\alpha)\in[\![1,m]\!]\times[\![1,n]\!]}$ has rank 1. Hence, there exist unique nonzero complex numbers $h_1, \dots, h_m, h'_1, \dots, h'_{n-1}, h'_n = 1$ such that $\mu_{i,\alpha} = h_i h'_\alpha$ for all i, α, as desired. □

Note that, in the exceptional case where $(m,n) = (1,3)$, the automorphism group of $\mathcal{O}_q(M_{1,3})$ has been computed by Alev and Chamarie, [1, Théorème 1.4.6]; in this case, the group is not isomorphic to the torus $(\mathbb{C}^*)^3$, since the second case of Lemma 3.6 does arise. Indeed, Alev and Chamarie show that any automorphism of $\mathcal{O}_q(M_{1,3})$ is of the form

$$\sigma(Y_{11}) = \mu_1 Y_{11}, \quad \sigma(Y_{12}) = \mu_2 Y_{12} + \lambda Y_{11} Y_{13}, \quad \sigma(Y_{13}) = \mu_3 Y_{13},$$

where $\mu_i \neq 0$ and λ are complex numbers; so that the automorphism group is isomorphic to the semidirect product $\mathbb{C} \times (\mathbb{C}^*)^3$. This result can easily be obtained from our analysis.

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