Prime ideals in the quantum grassmannian

S Launois, T H Lenagan and L Rigal*

Abstract

We consider quantum Schubert cells in the quantum grassmannian and give a cell decomposition of the prime spectrum via the Schubert cells. As a consequence, we show that all primes are completely prime in the generic case where the deformation parameter q is not a root of unity. There is a natural torus action of $\mathcal{H} = (\Bbbk^*)^n$ on $\mathcal{G}_q(m, n)$ and the cell decomposition of the set of $\mathcal{H}$-primes leads to a parameterisation of the $\mathcal{H}$-spectrum via certain diagrams on partitions associated to the Schubert cells. Interestingly, the same parameterisation occurs for the non-negative cells in recent studies concerning the totally non-negative grassmannian. Finally, we use the cell decomposition to establish that the quantum grassmannian satisfies normal separation and catenarity.

Introduction

Let $m \leq n$ be positive integers and let $\mathcal{O}_q(M_{m,n}(\Bbbk))$ denote the quantum deformation of the affine coordinate ring on $m \times n$ matrices, with nonzero deformation parameter q in the base field. The quantum deformation of the homogeneous coordinate ring of the grassmannian, denoted $\mathcal{O}_q(G_{m,n}(\Bbbk))$, is defined as the subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the maximal quantum minors of the generic matrix of $\mathcal{O}_q(M_{m,n}(\Bbbk))$. To simplify, these algebras will be referred to in the sequel as the algebra of quantum matrices and the quantum grassmannian, respectively.

The main goal of this work is the study of the prime spectrum of the quantum grassmannian. This algebra is naturally endowed with the action of a torus $\mathcal{H}$. Thus, according

to the philosophy of the *stratification theory* as developed by Goodearl and Letzter (see [1]), our main concern is the set of $\mathcal{H}$-prime ideals (namely, the prime ideals invariant under the action of $\mathcal{H}$). Recall that if A is an algebra and $\mathcal{H}$ a torus which acts on A by algebra automorphisms then the stratification theory suggests a study of the prime spectrum of A by means of a partition into strata, each stratum being indexed by an $\mathcal{H}$-prime ideal. For many algebras arising from the theory of quantum groups, general results have been proved about such a stratification. For example, when such an algebra is a certain kind of iterated skew polynomial extension, general results show that it has only finitely many $\mathcal{H}$-primes and that each stratum is homeomorphic to the spectrum of a suitable commutative Laurent polynomial ring. However, the algebra which interests us here is far from being such an extension and it is not even clear at the outset that it has finitely many $\mathcal{H}$-primes. For this reason, these general results do not apply and we are led to use a very different approach which has a geometric flavour. Recall that a classical approach to the study of the grassmanian variety $G_{m,n}(\Bbbk)$ is to use its partition into Schubert cells and their closures which are the so-called Schubert subvarieties of the grassmannian. Notice that, in this decomposition, Schubert cells are indexed by Young diagrams fitting in a rectangular $m \times (n-m)$ Young diagram. Our method is inspired by this classical geometric setting.

Quantum analogues of Schubert varieties (or rather of their coordinate rings) were studied in [14] in order to show that the quantum grassmannian has a certain combinatorial structure, namely the stucture of a *quantum graded algebra with a straightening law*. Subsequently, some of their properties have been established in [15]. In this paper, we define quantum Schubert cells as noncommutative dehomogenisations of quantum Schubert varieties. Using the structure of a quantum graded algebra with a straightening law enjoyed by the quantum grassmannian, we are then in position to define a partition of its prime spectrum. This partition is called a *cell decomposition* since it turns out that the set of $\mathcal{H}$-primes of a given component is in natural one-to-one correspondence with the set of $\mathcal{H}$-primes of an associated quantum Schubert cell. Hence, the description of the $\mathcal{H}$-primes of the quantum grassmannian reduces to that of the $\mathcal{H}$-primes of each of its associated quantum Schubert cells. (Here, the actions of $\mathcal{H}$ on the quantum Schubert varieties and cells are naturally induced by its action on the quantum grassmannian.)

On the other hand, we can show that a quantum Schubert cell can be identified as a subalgebra of a quantum matrix algebra, with the variables that are included sitting naturally in the Young diagram associated to that cell. As a consequence, we can establish properties for quantum Schubert cells akin to known properties of quantum matrix algebras. For example, we are able to parameterise the $\mathcal{H}$-prime ideals of a quantum Schubert cell by *Cauchon diagrams* on the corresponding Young diagram, in the same way that Cauchon was able to parameterise the $\mathcal{H}$-prime ideals in quantum matrices, see [3]. This is achieved by using the theory of *deleting derivations* as developed by Cauchon in [2]. This

theory utilizes certain changes of variables in the field of fractions of the algebra under consideration. In the case of quantum matrices, these changes of variable can be reinterpreted using quasi-determinants, see [5]. Recently, Cauchon diagrams in Young diagrams have appeared in the literature under the name Le-diagrams see, for example, [16] and [17].

By using this approach, we are able to show that there are only finitely many $\mathcal{H}$-prime ideals in $\mathcal{O}_q(G_{m,n}(\Bbbk))$. More precisely, we show that such $\mathcal{H}$-primes are in natural one-to-one correspondence with Cauchon diagrams defined on Young diagrams fitting into a rectangular $m \times (n - m)$ Young diagram. Following on from this description, we are able to calculate the number of $\mathcal{H}$-prime ideals in the quantum grassmannian.

In addition, we are able to show that prime ideals in the quantum grassmannian are completely prime, and that this algebra satisfies normal separation and, hence, is catenary. Again, the method is to establish these properties for each quantum Schubert cell and then transfer them to the quantum grassmannian.

To conclude this introduction, it should be stressed that there are very interesting connections between our results in the present paper and recent results in the theory of total positivity. More details on this are given in Section 5.

1 Basic definitions

Throughout the paper, $\Bbbk$ is a field and q is a nonzero element of $\Bbbk$ that is not a root of unity. Occasionally, we will remind the reader of this restriction in the statement of results.

In this section, we collect some basic definitions and properties about the objects we intend to study. Most proofs will be omitted since these results already appear in [10, 14, 15]. Appropriate references will be given in the text.

Let m, n be positive integers.

The quantisation of the coordinate ring of the affine variety $M_{m,n}(\Bbbk)$ of $m \times n$ matrices with entries in $\Bbbk$ is denoted $\mathcal{O}_q(M_{m,n}(\Bbbk))$. It is the $\Bbbk$-algebra generated by mn indeterminates x_{ij}, with $1 \leq i \leq m$ and $1 \leq j \leq n$, subject to the relations:

$$
\begin{aligned}
x_{ij}x_{il} &= qx_{il}x_{ij}, && \text{for } 1 \leq i \leq m, \text{ and } 1 \leq j < l \leq n \,; \\
x_{ij}x_{kj} &= qx_{kj}x_{ij}, && \text{for } 1 \leq i < k \leq m, \text{ and } 1 \leq j \leq n \,; \\
x_{ij}x_{kl} &= x_{kl}x_{ij}, && \text{for } 1 \leq k < i \leq m, \text{ and } 1 \leq j < l \leq n \,; \\
x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj}, && \text{for } 1 \leq i < k \leq m, \text{ and } 1 \leq j < l \leq n.
\end{aligned}
$$

To simplify, we write $M_n(\Bbbk)$ for $M_{n,n}(\Bbbk)$ and $\mathcal{O}_q(M_n(\Bbbk))$ for $\mathcal{O}_q(M_{n,n}(\Bbbk))$. The $m \times n$ matrix $\mathbf{X} = (x_{ij})$ is called the generic matrix associated with $\mathcal{O}_q(M_{m,n}(\Bbbk))$.

3

As is well known, there exists a $\Bbbk$-algebra *transpose isomorphism* between $\mathcal{O}_q(M_{m,n}(\Bbbk))$ and $\mathcal{O}_q(M_{n,m}(\Bbbk))$, see [14, Remark 3.1.3]. Hence, from now on, we assume that $m \leq n$, without loss of generality.

An index pair is a pair (I, J) such that $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$ are subsets with the same cardinality. Hence, an index pair is given by an integer t such that $1 \leq t \leq m$ and ordered sets $I = \{i_1 < \cdots < i_t\} \subseteq \{1, \ldots, m\}$ and $J = \{j_1 < \cdots < j_t\} \subseteq \{1, \ldots, n\}$. To any such index pair we associate the quantum minor

$$[I|J] = \sum_{\sigma \in \mathfrak{S}_t} (-q)^{\ell(\sigma)} x_{i_{\sigma(1)} j_1} \ldots x_{i_{\sigma(t)} j_t}.$$

Definition 1.1 – *The quantisation of the coordinate ring of the grassmannian of m-dimensional subspaces of $\Bbbk^n$, denoted by $\mathcal{O}_q(G_{m,n}(\Bbbk))$ and informally referred to as the $(m \times n)$ quantum grassmannian is the subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the $m \times m$ quantum minors.*

An index set is a subset $I = \{i_1 < \cdots < i_m\} \subseteq \{1, \ldots, n\}$. To any index set we associate the maximal quantum minor $[I] := [\{1, \ldots, m\}|I]$ of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ which is, thus, an element of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. The set of all index sets is denoted by $\Pi_{m,n}$. Since $\Pi_{m,n}$ is in one-to-one correspondence with the set of all maximal quantum minors of $\mathcal{O}_q(M_{m,n}(\Bbbk))$, we will often identify these two sets. We equip $\Pi_{m,n}$ with a partial order $\leq_{\mathrm{st}}$ defined in the following way. Let $I = \{i_1 < \cdots < i_m\}$ and $J = \{j_1 < \cdots < j_m\}$ be two index sets, then

$$I \leq_{\mathrm{st}} J \Longleftrightarrow i_s \leq j_s \quad \text{for} \quad 1 \leq s \leq m.$$

For example, Figure 1 shows the partial ordering on generators of $\mathcal{O}_q(G_{3,6}(\Bbbk))$.

Let A be a noetherian $\Bbbk$-algebra, and assume that the torus $\mathcal{H} := (\Bbbk^*)^r$ acts rationally on A by $\Bbbk$-algebra automorphisms. (For details concerning rational actions of tori, see [1, Chapter II.2].) A two-sided ideal I of A is said $\mathcal{H}$-invariant if $h \cdot I = I$ for all $h \in \mathcal{H}$. An $\mathcal{H}$-prime ideal of A is a proper $\mathcal{H}$-invariant ideal J of A such that whenever J contains the product of two $\mathcal{H}$-invariant ideals of A then J contains at least one of them. We denote by $\mathcal{H}\text{-Spec}(A)$ the $\mathcal{H}$-spectrum of A; that is, the set of all $\mathcal{H}$-prime ideals of A. It follows from [1, Proposition II.2.9] that every $\mathcal{H}$-prime ideal is prime when q is not a root of unity; so that in this case $\mathcal{H}\text{-Spec}(A)$ coincides with the set of all $\mathcal{H}$-invariant prime ideals of A.

There are natural torus actions on the classes of algebras that we study here, including quantum matrices, partition subalgebras of quantum matrices and quantum grassmannians. These actions are rational; and so the remarks above apply.

First, there is an action of a torus $\mathcal{H} := (\Bbbk^*)^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\Bbbk))$ given by

$$(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \circ x_{ij} := \alpha_i \beta_j x_{ij}.$$

In other words, one is able to multiply through rows and columns by nonzero scalars.

Next, there is an action of the torus $\mathcal{H} := (\Bbbk^*)^n$ on $\mathcal{O}_q(G_{m,n}(\Bbbk))$ which comes from the column action on quantum matrices. Thus, $(\alpha_1, \ldots, \alpha_n) \circ [i_1, \ldots, i_m] := \alpha_{i_1} \ldots \alpha_{i_m} [i_1, \ldots, i_m]$. We shall be interested in prime ideals left invariant under the action of this torus. The set of such prime ideals is the $\mathcal{H}$-spectrum of $\mathcal{O}_q(G_{m,n}(\Bbbk))$.
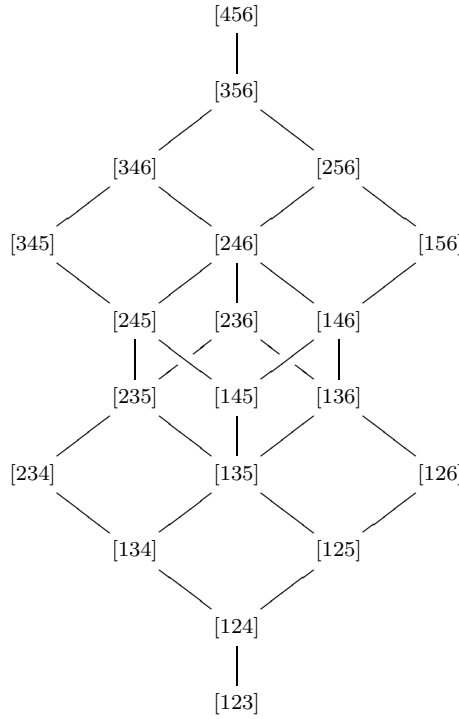
Figure 1: The partial ordering $\leq_{\mathrm{st}}$ on $\mathcal{O}_q(G_{3,6}(\Bbbk))$.

We recall the definition of quantum Schubert varieties given in [15].

Definition 1.2 – *Let $\gamma \in \Pi_{m,n}$ and put $\Pi_{m,n}^\gamma = \{\alpha \in \Pi_{m,n} \,|\, \alpha \not\geq_{\mathrm{st}} \gamma\}$. The quantum Schubert variety $S(\gamma)$ associated to γ is*

$$S(\gamma) := \mathcal{O}_q(G_{m,n}(\Bbbk))/\langle \Pi_{m,n}^\gamma \rangle.$$

(Note that $S(\gamma)$ was denoted by $\mathcal{O}_q(G_{m,n}(\Bbbk))_\gamma$ in [15].)

This definition is inspired by the classical description of the coordinate rings of Schubert varieties in the grassmannian. For more details about this matter, see [6, Section

6.3.4].

Note that each of the maximal quantum minors that generate $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is an $\mathcal{H}$-eigenvector. Thus, the $\mathcal{H}$-action on $\mathcal{O}_q(G_{m,n}(\Bbbk))$ transfers to the quantum Schubert varieties $S(\gamma)$.

In order to study properties of the quantum grassmannian, the notion of a quantum graded algebra with a straightening law (on a partially ordered set Π) was introduced in [14]. We now recall the definition of these algebras and mention various properties that we will use later.

Let A be an algebra and Π a finite subset of elements of A with a partial order $<_{\text{st}}$. A *standard monomial* on Π is an element of A which is either 1 or of the form $\alpha_1 \ldots \alpha_s$, for some $s \geq 1$, where $\alpha_1, \ldots, \alpha_s \in \Pi$ and $\alpha_1 \leq_{\text{st}} \cdots \leq_{\text{st}} \alpha_s$.

Definition 1.3 – *Let A be an $\mathbb{N}$-graded $\Bbbk$-algebra and Π a finite subset of A equipped with a partial order $<_{\text{st}}$. We say that A is a* quantum graded algebra with a straightening law *(quantum graded A.S.L. for short) on the poset $(\Pi, <_{\text{st}})$ if the following conditions are satisfied.*
(1) The elements of Π are homogeneous with positive degree.
(2) The elements of Π generate A as a $\Bbbk$-algebra.
(3) The set of standard monomials on Π is a linearly independent set.
(4) If $\alpha, \beta \in \Pi$ are not comparable for $<_{\text{st}}$, then $\alpha\beta$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\text{st}} \mu$ and $\lambda <_{\text{st}} \alpha, \beta$.
(5) For all $\alpha, \beta \in \Pi$, there exists $c_{\alpha\beta} \in \Bbbk^*$ such that $\alpha\beta - c_{\alpha\beta}\beta\alpha$ is a linear combination of terms λ or $\lambda\mu$, where $\lambda, \mu \in \Pi$, $\lambda \leq_{\text{st}} \mu$ and $\lambda <_{\text{st}} \alpha, \beta$.

By [14, Proposition 1.1.4], if A is a quantum graded A.S.L. on the partially ordered set $(\Pi, <_{\text{st}})$, then the set of standard monomials on Π forms a $\Bbbk$-basis of A. Hence, in the presence of a standard monomial basis, the structure of a quantum graded A.S.L. may be seen as providing more detailed information on the way standard monomials multiply and commute.

Example 1.4 – It is shown, in [14, Theorem 3.4.4], that $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is a quantum graded algebra with a straightening law on $(\Pi_{m,n}, \leq_{\text{st}})$.

From our point of view, one important feature of quantum graded A.S.L. is the following. Let A be a $\Bbbk$-algebra which is a quantum graded A.S.L. on the set $(\Pi, \leq_{\text{st}})$. A subset Ω of Π will be called a Π-ideal if it is an ideal of the partially ordered set $(\Pi, \leq_{\text{st}})$ in the

sense of lattice theory; that is, if it satisfies the following property: if $\alpha \in \Omega$ and if $\beta \in \Pi$, with $\beta \leq_{\mathrm{st}} \alpha$, then $\beta \in \Omega$. We can consider the quotient $A/\langle\Omega\rangle$ of A by the ideal generated by Ω. Clearly, it is still a graded algebra and it is generated by the images in $A/\langle\Omega\rangle$ of the elements of $\Pi \setminus \Omega$. The important point here is that $A/\langle\Omega\rangle$ inherits from A a natural quantum graded A.S.L. structure on $\Pi \setminus \Omega$ (or, more precisely, on the canonical image of $\Pi \setminus \Omega$ in $A/\langle\Omega\rangle$). In particular, the set of homomorphic images in $A/\langle\Omega\rangle$ of the standard monomials of A which either equal 1 or are of the form $\alpha_1 \ldots \alpha_t$ ($t \in \mathbb{N}^*$) and $\alpha_1 \notin \Omega$ form a $\Bbbk$-basis for $A/\langle\Omega\rangle$. The reader will find all the necessary details in §1.2 of [14].

Example 1.5 – Let $\gamma \in \Pi_{m,n}$. It is clear that the set $\Pi^\gamma_{m,n}$ introduced in Definition 1.2 is a $\Pi_{m,n}$-ideal. Hence, the discussion above shows that the quantum Schubert variety $S(\gamma)$ is a quantum graded A.S.L. on the canonical image in $S(\gamma)$ of $\Pi_{m,n} \setminus \Pi^\gamma_{m,n}$. In particular, the canonical images in $S(\gamma)$ of the standard monomials of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ which either equal to 1 or are of the form $[I_1] \ldots [I_t]$, for some $t \geq 1$ and with $\gamma \leq_{\mathrm{st}} [I_1]$, form a $\Bbbk$-basis of $S(\gamma)$.

Remark 1.6 – Let $\gamma \in \Pi_{m,n}$. As mentioned in Example 1.5, the quantum Schubert variety $S(\gamma)$ is a quantum graded A.S.L. on the canonical image in $S(\gamma)$ of $\Pi_{m,n} \setminus \Pi^\gamma_{m,n}$. At this point, it is worth noting that the set $\Pi_{m,n} \setminus \Pi^\gamma_{m,n}$ has a single minimal element, namely γ, and that the image of γ is a normal nonzerodivisor in $S(\gamma)$, by [14, Lemma 1.2.1].

2 Partition subalgebras of quantum matrices

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. The *partition subalgebra* $\mathcal{A}_\lambda$ of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ is defined to be the subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the variables x_{ij} with $j \leq \lambda_i$. By looking at the defining relations for quantum matrices, it is easy to see that $\mathcal{A}_\lambda$ can be presented as an iterated Ore extension with the variables x_{ij} added in lexicographic order. As a consequence, partition subalgebras are noetherian domains. Recall that there is an action of a torus $\mathcal{H} := (\Bbbk^*)^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\Bbbk))$ given by $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \circ x_{ij} := \alpha_i \beta_j x_{ij}$. This action induces an action on $\mathcal{A}_\lambda$, by restriction. Our main aim in this section is to observe that the Goodearl-Letzter stratification theory and the Cauchon theory of deleting derivations apply to partition subalgebras of quantum matrices. As a consequence, we can then exploit these theories to obtain information about the prime and $\mathcal{H}$-prime spectra of partition subalgebras.

The conditions needed to use the theories have been brought together in the notion of a (torsion-free) CGL-extension introduced in [12, Definition 3.1]; the definition is given below, for convenience.

Definition 2.1 An iterated skew polynomial extension

$$A = \Bbbk[x_1][x_2; \sigma_2, \delta_2] \dots [x_n; \sigma_n, \delta_n]$$

is said to be a *CGL extension* (after Cauchon, Goodearl and Letzter) provided that the following list of conditions is satisfied:

- With $A_j := \Bbbk[x_1][x_2; \sigma_2, \delta_2] \dots [x_j; \sigma_j, \delta_j]$ for each $1 \leq j \leq n$, each σ_j is a $\Bbbk$-algebra automorphism of A_{j-1}, each δ_j is a locally nilpotent $\Bbbk$-linear σ_j-derivation of A_{j-1}, and there exist nonroots of unity $q_j \in \Bbbk^*$ with $\sigma_j \delta_j = q_j \delta_j \sigma_j$;

- For each $i < j$ there exists a $\lambda_{ji} \in \Bbbk^*$ such that $\sigma_j(x_i) = \lambda_{ji} x_i$;

- There is a torus $\mathcal{H} = (\Bbbk^*)^r$ acting rationally on A by $\Bbbk$-algebra automorphisms;

- The x_i for $1 \leq i \leq n$ are $\mathcal{H}$-eigenvectors;

- There exist elements $h_1, \dots, h_n \in \mathcal{H}$ such that $h_j(x_i) = \sigma_j(x_i)$ for $j > i$ and such that the h_j-eigenvalue of x_j is not a root of unity.

If, in addition, the subgroup of $\Bbbk^*$ generated by the λ_{ji} is torsionfree then we will say that A is a *torsionfree CGL extension*.

For a discussion of rational actions of tori, see [1, Chapter II.2].

It is easy to check that all of these conditions are satisfied for partition subalgebras (for exactly the same reasons that quantum matrices are CGL-extensions).

Proposition 2.2 *Partition subalgebras of quantum matrix algebras are CGL-extensions and are torsion-free CGL extensions when the parameter q is not a root of unity.*

Proof: It is only necessary to show that we can introduce the variables x_{ij} that define the partition subalgebra in such a way that the resulting iterated skew polynomial extension satisfies the list of conditions above. Lexicographic ordering is suitable. ∎

Corollary 2.3 *Let $\mathcal{A}_\lambda$ be a partition subalgebra of quantum matrices and suppose that $\mathcal{A}_\lambda$ is equipped with the induced action of $\mathcal{H}$. Then $\mathcal{A}_\lambda$ has only finitely many $\mathcal{H}$-prime ideals and all prime ideals of $\mathcal{A}_\lambda$ are completely prime when the parameter q is not a root of unity.*

8

Proof: This follows immediately from the previous result and [1, Theorem II.5.12 and Theorem II.6.9]. ∎

In fact, we can be much more precise about the number of $\mathcal{H}$-primes. We will prove below that there exists a natural bijection between the $\mathcal{H}$-prime spectrum of $\mathcal{A}_\lambda$ and Cauchon diagrams defined on the Young diagram corresponding to the partition λ.

Suppose that Y_λ is the Young diagram corresponding to the partition λ. Then a *Cauchon diagram* on Y_λ is an assignment of a colour, either white or black, to each square of the diagram Y_λ in such a way that if a square is coloured black then either each square above is coloured black, or each square to the left is coloured black. These diagrams were first introduced by Cauchon, [3], in his study of the $\mathcal{H}$-prime spectrum of quantum matrices. Recently, they have occurred with the name Le-diagrams in work of Postnikov, [16], and Williams, [17].

Lemma 2.4 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$. The number of $\mathcal{H}$-prime ideals in $\mathcal{A}_\lambda$ is equal to the number of Cauchon diagrams defined on the Young diagram corresponding to the partition λ.*

Proof: Let n_λ denote the number of $\mathcal{H}$-prime ideals in $\mathcal{A}_\lambda$. First, we obtain a recurrence relation for n_λ.

The $\mathcal{H}$-prime spectrum of $\mathcal{A}_\lambda$ can be written as a disjoint union:

$$\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) = \{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) | x_{m,\lambda_m} \in J\} \sqcup \{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) | x_{m,\lambda_m} \notin J\}.$$

It follows from the complete primeness of every $\mathcal{H}$-prime ideal of $\mathcal{A}_\lambda$ that an $\mathcal{H}$-prime ideal J of $\mathcal{A}_\lambda$ that contains x_{m,λ_m} must also contain either x_{i,λ_m} for each $i \in \{1, \ldots, m\}$ or $x_{m,\alpha}$ for each $\alpha \in \{1, \ldots, \lambda_m\}$. Let I_1 be the ideal generated by x_{i,λ_m} for $i \in \{1, \ldots, m\}$, and let I_2 be the ideal generated by $x_{m,\alpha}$ for $\alpha \in \{1, \ldots, \lambda_m\}$. Set $I_3 := I_1 + I_2$. As

$$\frac{\mathcal{A}_\lambda}{I_1} \simeq \mathcal{A}_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_m-1)}, \quad \frac{\mathcal{A}_\lambda}{I_2} \simeq \mathcal{A}_{(\lambda_1,\lambda_2,\ldots,\lambda_{m-1})} \quad \text{and} \quad \frac{\mathcal{A}_\lambda}{I_3} \simeq \mathcal{A}_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_{m-1}-1)},$$

we obtain

$$\begin{aligned} n_\lambda &= n_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_m-1)} + n_{(\lambda_1,\lambda_2,\ldots,\lambda_{m-1})} - n_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_{m-1}-1)} \\ &\quad + |\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda) | x_{m,\lambda_m} \notin J\}|. \end{aligned}$$

(Even though the above isomorphisms are not always $\mathcal{H}$-equivariant, they preserve the property of being an $\mathcal{H}$-prime.)

As $\mathcal{A}_\lambda$ is a CGL extension, one can apply the theory of deleting derivations to this algebra. In particular, it follows from [2, Théorème 3.2.1] that the multiplicative system of $\mathcal{A}_\lambda$ generated by x_{m,λ_m} is an Ore set in $\mathcal{A}_\lambda$, and

$$\mathcal{A}_\lambda[x_{m,\lambda_m}^{-1}] \simeq \mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}[y^{\pm 1};\sigma],$$

where σ is the automorphism of $\mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}$ defined by $\sigma(x_{i\alpha}) = q^{-1}x_{i\alpha}$ if $i = m$ or $\alpha = \lambda_m$, and $\sigma(x_{i\alpha}) = x_{i\alpha}$ otherwise. Denote this isomorphism by ψ, and note that $\psi(x_{m,\lambda_m}) = y$. As x_{m,λ_m} is an $\mathcal{H}$-eigenvector, the action of $\mathcal{H}$ on $\mathcal{A}_\lambda$ extends to an action of $\mathcal{H}$ on $\mathcal{A}_\lambda[x_{m,\lambda_m}^{-1}]$, and so on $\mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}[y^{\pm 1};\sigma]$. It is easy to show that this action restricts to an action on $\mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}$ which coincides with the "natural" action of $\mathcal{H}$ on this algebra. Hence the isomorphism ψ induces a bijection from $\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)|x_{m,\lambda_m} \notin J\}$ to $\mathcal{H}\text{-Spec}(\mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}[y^{\pm 1};\sigma])$; and so it follows from [12, Theorem 2.3] that there exists a bijection between $\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)|x_{m,\lambda_m} \notin J\}$ and $\mathcal{H}\text{-Spec}(\mathcal{A}_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)})$. Hence

$$|\{J \in \mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)|x_{m,\lambda_m} \notin J\}| = n_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)};$$

so that

$$n_\lambda = n_{(\lambda_1-1,\lambda_2-1,\dots,\lambda_m-1)} + n_{(\lambda_1,\lambda_2,\dots,\lambda_m-1)} - n_{(\lambda_1-1,\lambda_2-1,\dots,\lambda_{m-1}-1)} + n_{(\lambda_1,\lambda_2,\dots,\lambda_{m-1},\lambda_m-1)}.$$

On the other hand, it follows from [17, Remark 4.2] that the number of Cauchon diagrams (equivalently, Le-diagrams) defined on the Young diagram corresponding to the partition λ satisfies the same recurrence. As the number of $\mathcal{H}$-prime ideals in $\mathcal{A}_{(1)}$ is equal to 2 which is also the number of Cauchon diagrams defined on the Young diagram corresponding to the partition $\lambda = (1)$, the proof is complete. ∎

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Let $\mathcal{C}_\lambda$ denote the set of Cauchon diagrams on the Young diagram Y_λ corresponding to the partition λ. We have just seen that the sets $\mathcal{H}\text{-Spec}(\mathcal{A}_\lambda)$ and $\mathcal{C}_\lambda$ have the same cardinality. In fact, there is a natural bijection between these two sets which carries over important algebraic and geometric information. This natural bijection arises by using Cauchon's theory of deleting derivations developed in [2] and [3].

As $\mathcal{A}_\lambda$ is a CGL extension, the theory of deleting derivations can be applied to the iterated Ore extension $\mathcal{A}_\lambda = k[x_{1,1}]\dots[x_{m,\lambda_m};\sigma_{m,\lambda_m},\delta_{m,\lambda_m}]$ (where the indices are increasing for the lexicographic order). Before describing the deleting derivations algorithm, we introduce some notation. Denote by $\leq_{\text{lex}}$ the lexicographic ordering on $\mathbb{N}^2$ and set $E := (\bigsqcup_{i=1}^{m}\{i\} \times \{1,\dots,\lambda_i\} \cup \{(m,\lambda_m+1)\}) \setminus \{(1,1)\}$. If $(j,\beta) \in E$ with $(j,\beta) \neq (m,\lambda_m+1)$, then $(j,\beta)^+$ denotes the least element (relative to $\leq_{\text{lex}}$) of the set $\{(i,\alpha) \in E \mid (j,\beta) < (i,\alpha)\}$.

The deleting derivations algorithm constructs, for each $r \in E$, a family of elements $x_{i,\alpha}^{(r)}$ for $\alpha \leq \lambda_i$ of $F := \mathrm{Frac}(\mathcal{A}_\lambda)$, defined as follows.

1. If $r = (m, \lambda_m + 1)$, then $x_{i,\alpha}^{(m,\lambda_m+1)} = x_{i,\alpha}$ for all (i,α) with $\alpha \leq \lambda_i$.

2. Assume that $r = (j,\beta) < (m, \lambda_m + 1)$ and that the $x_{i,\alpha}^{(r^+)}$ are already constructed. Then, it follows from [2, Théorème 3.2.1] that $x_{j,\beta}^{(r^+)} \neq 0$ and, for all (i,α), we have:

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r^+)} - x_{i,\beta}^{(r^+)} \left(x_{j,\beta}^{(r^+)} \right)^{-1} x_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

As in [2], we denote by $\overline{\mathcal{A}_\lambda}$ the subalgebra of $\mathrm{Frac}(\mathcal{A}_\lambda)$ generated by the indeterminates obtained at the end of this algorithm; that is, we denote by $\overline{\mathcal{A}_\lambda}$ the subalgebra of $\mathrm{Frac}(\mathcal{A}_\lambda)$ generated by the $t_{i,\alpha} := x_{i,\alpha}^{(1,2)}$ for each (i,α) such that $\alpha \leq \lambda_i$. Cauchon has shown that $\overline{\mathcal{A}_\lambda}$ can be viewed as the quantum affine space $\overline{\mathcal{A}_\lambda}$ generated by indeterminates t_{ij} for $j \leq \lambda_i$ with relations $t_{ij}t_{il} = qt_{il}t_{ij}$ for $j < l$, while $t_{ij}t_{kj} = qt_{kj}t_{ij}$ for $i < k$, and all other pairs commute. Observe that the torus $\mathcal{H}$ still acts by automorphisms on $\overline{\mathcal{A}_\lambda}$ via $(a_1, \ldots, a_m, b_1, \ldots, b_n).t_{ij} = a_i b_j t_{ij}$. The theory of deleting derivations allows the explicit (but technical) construction of an embedding φ, called the canonical embedding, from $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{A}_\lambda)$ into the $\mathcal{H}$-prime spectrum of $\overline{\mathcal{A}_\lambda}$. The $\mathcal{H}$-prime ideals of $\overline{\mathcal{A}_\lambda}$ are well-known: they are generated by the subsets of $\{t_{ij}\}$. If C is a Cauchon diagram defined on the Young tableau corresponding to λ, then we denote by K_C the (completely) prime ideal of $\overline{\mathcal{A}_\lambda}$ generated by the subset of indeterminates t_{ij} such that the square in position (i,j) is a black square of C.

Theorem 2.5 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Let $\mathcal{C}_\lambda$ denote the set of Cauchon diagrams defined on the Young tableau corresponding to λ.*
For every Cauchon diagram $C \in \mathcal{C}_\lambda$, there exists a unique $\mathcal{H}$-invariant (completely) prime ideal J_C of $\mathcal{A}_\lambda$ such that $\varphi(J_C) = K_C$. Moreover there is no other $\mathcal{H}$-prime in $\mathcal{A}_\lambda$; so that

$$\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{A}_\lambda) = \{J_C | C \in \mathcal{C}_\lambda\}.$$

Proof: As the sets $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{A}_\lambda)$ and $\{J_C | C \in \mathcal{C}_\lambda\}$ have the same cardinality by the previous lemma, it is sufficient to show that $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{A}_\lambda) \subseteq \{J_C | C \in \mathcal{C}_\lambda\}$. This inclusion can be obtained by following the arguments of [3, Lemmes 3.1.6 and 3.1.7]. The details are left to the interested reader. ∎

Remark 2.6 Theorem 2.5 provides more than just an explicit bijection between the $\mathcal{H}$-spectrum of $\mathcal{A}_\lambda$ and $\mathcal{C}_\lambda$. This natural bijection carries algebraic and geometric data. For example, it can be shown that the height of J_C is given by the number of black boxes of the Cauchon diagram C. Also, the dimension of the $\mathcal{H}$-stratum (in the sense of [1, Definition 2.2.1]) associated to J_C can be read off from the Cauchon diagram C.

An algebra A is said to be *catenary* if for each pair of prime ideals $Q \subseteq P$ of A all saturated chains of prime ideals between Q and P have the same length. Our next aim is to show that partition subalgebras of quantum matrix algebras are catenary. The key property that we need to establish in order to prove catenarity is the property of normal separation. Two prime ideals $Q \subsetneq P$ are said to be *normally separated* if there is an element $c \in P\backslash Q$ such that c is normal modulo Q. The algebra is *normally separated* if each such pair of prime ideals is normally separated. In our case, a result of Goodearl, see [7, Section 5], shows that it is enough to concentrate on the $\mathcal{H}$-prime ideals. Suppose that A is a $\Bbbk$-algebra with a torus $\mathcal{H}$ acting rationally. If Q is any $\mathcal{H}$-invariant ideal of A then an element c is said to be $\mathcal{H}$-*normal modulo Q* provided that there exists $h \in \mathcal{H}$ such that $ca - h(a)c \in Q$ for all $a \in A$. Goodearl observes that in this case one may choose the element c to be an $\mathcal{H}$-eigenvector. The algebra A has $\mathcal{H}$-*normal separation* provided that for each pair of $\mathcal{H}$-prime ideals $Q \subsetneq P$ there exists an element $c \in P\backslash Q$ such that c is $\mathcal{H}$-normal modulo Q.

A slightly weaker notion, also introduced by Goodearl, is that of *normal $\mathcal{H}$-separation*. The algebra A has normal $\mathcal{H}$-separation provided that for each pair of $\mathcal{H}$-primes $Q \subsetneq P$ there is an $\mathcal{H}$-eigenvector $c \in P\backslash Q$ which is normal modulo Q. Goodearl shows that in the situation that we are considering, normal $\mathcal{H}$-separation implies normal separation, see [7, Theorem 5.3].

Notice that, as explained in paragraph 5.1 of [7], the action of $\mathcal{H}$ induces a grading on A by a suitable free abelian group. Using this grading, it is easy to see that A has normal $\mathcal{H}$-separation if and only if for each pair of $\mathcal{H}$-primes $Q \subsetneq P$ there is an element $c \in P\backslash Q$ whose image in A/Q is normal and an $\mathcal{H}$-eigenvector. This fact will be freely used in the sequel.

Recall, from [12, Definition 2.5], the definition of a Cauchon extension. Let A be a domain that is a noetherian $\Bbbk$-algebra and let $R = A[X; \sigma, \delta]$ be a skew polynomial extension of A. We say that $R = A[X; \sigma, \delta]$ is a *Cauchon Extension* provided that

- σ is a $\Bbbk$-algebra automorphism of A and δ is a $\Bbbk$-linear locally nilpotent σ-derivation of A. Moreover we assume that there exists $q \in \Bbbk^*$ which is not a root of unity such that $\sigma \circ \delta = q\delta \circ \sigma$.

12

- There exists an abelian group $\mathcal{H}$ which acts on R by $\Bbbk$-algebra automorphisms such that X is an $\mathcal{H}$-eigenvector and A is $\mathcal{H}$-stable.

- σ coincides with the action on A of an element $h_0 \in \mathcal{H}$.

- Since X is an $\mathcal{H}$-eigenvector and since $h_0 \in \mathcal{H}$, there exists $\lambda_0 \in \Bbbk^*$ such that $h_0.X = \lambda_0 X$. We assume that λ_0 is not a root of unity.

- Every $\mathcal{H}$-prime ideal of A is completely prime.

Lemma 2.7 *Suppose that $R = A[X; \sigma, \delta]$ is a Cauchon extension. Moreover, assume that $\mathcal{H}$ is a torus and that the action of $\mathcal{H}$ on R is rational. If R has $\mathcal{H}$-normal separation then A has $\mathcal{H}$-normal separation.*

Proof: First, note that $\{X^n\}$ is an Ore set in R, by [2, Lemme 2.1]; and so we can form the Ore localization $\widehat{R} := RS^{-1} = S^{-1}R$. As X is an $\mathcal{H}$-eigenvector, the rational action of $\mathcal{H}$ on R extends to a rational action on $\widehat{R}$. We claim that $\widehat{R}$ has $\mathcal{H}$-normal separation. Suppose that $Q \subsetneqq P$ are $\mathcal{H}$-prime ideals of $\widehat{R}$. Then $Q \cap R \subsetneqq P \cap R$ are distinct $\mathcal{H}$-prime ideals of R. Thus, there exist an element $c \in (P \cap R)\backslash(Q \cap R)$ and an element $h \in \mathcal{H}$ such that $cr - h(r)c \in Q \cap R$ for all $r \in R$. In particular, $cX - \lambda Xc = cX - h(X)c \in Q \cap R$ for some $\lambda \in \Bbbk^*$, as X is an $\mathcal{H}$-eigenvector. From this it is easy to calculate that $(\lambda X)^{-k}c - cX^{-k} \in Q$. Now, let $y = rX^{-k}$ be an element of $\widehat{R}$. Then, working modulo Q, we calculate

$$cy = crX^{-k} = h(r)(\lambda X)^{-k}c = h(r)h(X^{-k})c = h(rX^{-k})c = h(y)c;$$

so that $\widehat{R}$ has $\mathcal{H}$-normal separation, as claimed.

For each $a \in A$, set

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a)X^{-n} \in \widehat{R}$$

(Note that $\theta(a)$ is a well-defined element of $\widehat{R}$, since δ is locally nilpotent, q is not a root of unity, and $0 \neq 1 - q \in \Bbbk$.)

The following facts are established in [2, Section 2]. The map $\theta : A \longrightarrow \widehat{R}$ is a $\Bbbk$-algebra monomorphism. Let $A[Y; \sigma]$ be a skew polynomial extension. Then θ extends to a monomorphism $\theta : A[Y; \sigma] \longrightarrow \widehat{R}$ with $\theta(Y) = X$. Set $B = \theta(A)$ and $T = \theta(A[Y; \sigma]) \subseteq \widehat{R}$. Then $T = B[X; \alpha]$, where α is the automorphism of B defined by $\alpha(\theta(a)) = \theta(\sigma(a))$.

The element X is a normal element in T, and so the set S is an Ore set in T and Cauchon shows that $TS^{-1} = S^{-1}T = \widehat{R}$. Thus, $\widehat{R} = B[X, X^{-1}; \alpha]$. Also, the $\mathcal{H}$-action transfers to B via θ, by [12, Lemma 2.6]. Note, in particular, that α coincides with the action of an element of $\mathcal{H}$ on B.

13

Thus, it is enough to show that B has $\mathcal{H}$-normal separation, given that $B[X, X^{-1}; \alpha]$ has $\mathcal{H}$-normal separation.

Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals of B. Set $\widehat{Q} = \oplus_{i \in \mathbb{Z}} Q X^i$ and $\widehat{P} = \oplus_{i \in \mathbb{Z}} P X^i$. Then $\widehat{Q} \cap B = Q$ and $\widehat{P} \cap B = P$, and it follows that $\widehat{Q} \subsetneq \widehat{P}$ are $\mathcal{H}$-prime ideals in $B[X, X^{-1}; \alpha]$, see [12, Theorem 2.3]. As $B[X, X^{-1}; \alpha]$ has $\mathcal{H}$-normal separation, there is an element $c \in \widehat{P} \backslash \widehat{Q}$ and an element $h \in \mathcal{H}$ such that $cs - h(s)c \in \widehat{Q}$, for each $s \in B[X, X^{-1}; \alpha]$. Now, write $c = \sum_{i \in \mathbb{Z}} c_i X^i$. Note that each $c_i \in P$ and at least one $c_i \notin Q$, say $c_{i_0} \notin Q$. Let $b \in B$. Then, $cb - h(b)c \in \widehat{Q}$. Therefore, $\sum_i c_i X^i b - h(b)c_i X^i \in \widehat{Q}$; and so

$$\sum_i (c_i \alpha^i(b) - h(b)c_i)X^i \in \widehat{Q}$$

As $\widehat{Q} = \oplus_{i \in \mathbb{Z}} Q X^i$, this forces $c_i \alpha^i(b) - h(b)c_i \in Q$ for each i, and, in particular, $c_{i_0} \alpha^{i_0}(b) - h(b)c_{i_0} \in Q$. As b was an arbitrary element of B, we may replace b by $\alpha^{-i}(b)$ to obtain

$$c_{i_0} b - h\alpha^{-i}(b)c_{i_0} \in Q$$

As α coincides with the action of an element of $\mathcal{H}$ on B, this produces an element $h_{i_0} \in \mathcal{H}$ such that

$$c_{i_0} b - h_{i_0}(b)c_{i_0} \in Q,$$

as required to show that B has $\mathcal{H}$-normal separation. ∎

Theorem 2.8 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ has $\mathcal{H}$-normal separation.*

Proof: Let $\mu = (n, \ldots, n)$ (m times); so that Y_μ is an $m \times n$ rectangle. Then $\mathcal{A}_\mu = \mathcal{O}_q(M_{m,n}(\Bbbk))$; and so $\mathcal{A}_\mu$ has $\mathcal{H}$-normal separation, by [3, Théorème 6.3.1]. We can construct $\mathcal{A}_\mu$ from $\mathcal{A}_\lambda$ by adding the missing variables x_{ij} in lexicographic order. At each stage, the extension is a Cauchon extension. Thus, $\mathcal{A}_\lambda$ has $\mathcal{H}$-normal separation, by repeated application of the previous lemma. ∎

Corollary 2.9 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ has normal $\mathcal{H}$-separation and normal separation.*

Proof: We have seen earlier that $\mathcal{H}$-normal separation implies normal $\mathcal{H}$-separation. Normal separation now follows from [7, Theorem 5.3]. ∎

Corollary 2.10 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and let $\mathcal{A}_\lambda$ be the corresponding partition subalgebra of generic quantum matrices. Then $\mathcal{A}_\lambda$ is catenary.*

Proof: This follows from the previous results and [18, Theorem 0.1] which states that if A is a normally separated filtered $\Bbbk$-algebra such that $\mathrm{gr}(A)$ is a noetherian connected graded $\Bbbk$-algebra with enough normal elements then $\mathrm{Spec}(A)$ is catenary. (For the notion of an algebra with enough normal elements see [19].) ∎

Note that it is also possible to deduce this result from [8, Theorem 1.6]

3 Quantum Schubert cells

Quantum Schubert cells in the quantum grassmannian are obtained from quantum Schubert varieties via the process of noncommutative dehomogenisation introduced in [10]. Recall that if $R = \oplus R_i$ is an $\mathbb{N}$-graded $\Bbbk$-algebra and x is a regular homogeneous normal element of R of degree one, then the *dehomogenisation* of R at x, written $\mathrm{Dhom}(R, x)$, is defined to be the zero degree subalgebra S_0 of the $\mathbb{Z}$-graded algebra $S := R[x^{-1}]$. If R is generated as a $\Bbbk$-algebra by $a_1, a_2, \ldots, a_s$ and each a_i has degree one, then it is easy to check that $\mathrm{Dhom}(R, x) = k[a_1 x^{-1}, \ldots, a_s x^{-1}]$.

If σ denotes the automorphism of S given by $\sigma(s) = x s x^{-1}$ for $s \in S$ then σ induces an automorphism of S_0, also denoted by σ, and there is an isomorphism

$$\theta : \mathrm{Dhom}(R, x)[y, y^{-1}; \sigma] \longrightarrow R[x^{-1}]$$

which is the identity on $\mathrm{Dhom}(R, x)$ and sends y to x.

Let $\gamma \in \Pi_{m,n}$. Recall from Remark 1.6 that $S(\gamma) = \mathcal{O}_q(G_{m,n}(\Bbbk))/\langle \Pi_{m,n}^\gamma \rangle$ and that $\overline{\gamma}$ is a homogeneous regular normal element of degree one in $S(\gamma)$. It follows that we can form the localisation $S(\gamma)[\overline{\gamma}^{-1}]$ and that $S(\gamma) \subseteq S(\gamma)[\overline{\gamma}^{-1}]$.

Definition 3.1 *The* quantum Schubert cell *associated to the quantum minor γ is denoted by $S^o(\gamma)$ and is defined to be* $\mathrm{Dhom}(S(\gamma), \overline{\gamma})$.

Remark 3.2 In the classical case when $q = 1$, it can be seen that this definition coincides with the usual definition of Schubert cells, as discussed, for example, in [4, Section 9.4]

It follows from the definition that $S^o(\gamma)$ is generated by the elements $\overline{x}\,\overline{\gamma}^{-1}$, for $x \in \Pi_{m,n} \setminus (\Pi_{m,n}^\gamma \cup \{\gamma\})$. However, these elements are not independent; so we will pick out a better generating set for the quantum Schubert cell.

This is achieved by using the quantum ladder matrix algebras introduced in [15, Section 3.1]. Let us recall the definition. To each $\gamma = (\gamma_1, \ldots, \gamma_m) \in \Pi_{m,n}$, with $1 \le \gamma_1 < \cdots <$

$\gamma_m \leq n$, we associate the substet $\mathcal{L}_\gamma$ of $\{1, \ldots, m\} \times \{1, \ldots, n\}$ defined by

$$\mathcal{L}_\gamma = \{(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid j > \gamma_{m+1-i} \quad \text{and} \quad j \neq \gamma_\ell \quad \text{for} \quad 1 \leq \ell \leq m\},$$

which we call the *ladder* associated with γ.

Consider the quantum minors m_{ij} defined by $m_{ij} := [\{\gamma_1, \ldots, \gamma_m\} \setminus \{\gamma_{m+1-i}\} \cup \{j\}]$, for each $(i,j) \in \mathcal{L}_\gamma$. These are the quantum minors that are above γ in the standard order and differ from γ in exactly one position. Denote the set of these quantum minors by $\mathcal{M}_\gamma$.

Proposition 3.3 $S^o(\gamma) = \Bbbk[\overline{m_{ij}} \, \overline{\gamma}^{-1} \mid m_{ij} \in \mathcal{M}_\gamma]$

Proof: In the proof of [15, Theorem 3.1.6] it is shown that $S(\gamma)[\overline{\gamma}^{-1}]$ is generated by the elements $\overline{\gamma}, \overline{\gamma}^{-1}$ and the $\overline{m_{ij}}$. The Schubert cell $S^o(\gamma)$ is the degree zero part of this algebra. As $\overline{\gamma}$ and $\overline{m_{ij}}$ commute up to scalars, see [15, Lemma 3.1.4(v)], it is easy to check that $S^o(\gamma)$ is generated by $\overline{m_{ij}} \, \overline{\gamma}^{-1}$, as required. $\blacksquare$

Set $\widetilde{m_{ij}} := \overline{m_{ij}} \, \overline{\gamma}^{-1}$.

Lemma 3.4 *There is an induced action of $\mathcal{H} = (\Bbbk^*)^n$ on $S^o(\gamma)$ given by*

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \circ \widetilde{m_{ij}} := \alpha_{\gamma_{m+1-i}}^{-1} \alpha_j \widetilde{m_{ij}}.$$

Proof: This follows immediately from the fact that

$$\widetilde{m_{ij}} = \overline{[\{\gamma_1, \ldots, \gamma_m\} \setminus \{\gamma_{m+1-i}\} \cup \{j\}]} \, \overline{[\gamma_1, \ldots, \gamma_m]}^{-1}.$$

$\blacksquare$

We now need to establish the commutation relations between the $\widetilde{m_{ij}}$.

Definition 3.5 *– Let $\gamma = (\gamma_1, \ldots, \gamma_m) \in \Pi_{m,n}$, with $1 \leq \gamma_1 < \cdots < \gamma_m \leq n$. The quantum ladder matrix algebra associated with γ, denoted $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$, is the $\Bbbk$-subalgebra of $\mathcal{O}_q(M_{m,n}(\Bbbk))$ generated by the elements $x_{ij} \in \mathcal{O}_q(M_{m,n}(\Bbbk))$ such that $(i,j) \in \mathcal{L}_\gamma$.*

The following example, taken from [15] will help clarify this definition.

Example 3.6 – We put $(m,n) = (3,7)$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3) = (1,3,6) \in \Pi_{3,7}$. In the 3×7 generic matrix $\mathbf{X} = (x_{ij})$ associated with $\mathcal{O}_q(M_{3,7}(\Bbbk))$, put a bullet on each row as follows: on the first row, the bullet is in column 6 because γ_3 is 6, on the second row, the bullet is in column 3 because γ_2 is 3 and on the third row, the bullet is in column 1 because $\gamma_1 = 1$.

Now, in each position which is to the left of a bullet, or which is below a bullet, put a star. To finish, place x_{ij} in any position (i, j) that has not yet been filled. We obtain

$$\begin{pmatrix} * & * & * & * & * & \bullet & x_{17} \\ * & * & \bullet & x_{24} & x_{25} & * & x_{27} \\ \bullet & x_{32} & * & x_{34} & x_{35} & * & x_{37} \end{pmatrix}.$$

By definition, the quantum ladder matrix algebra associated with $\gamma = (1, 3, 6)$ is the subalgebra of $\mathcal{O}_q(M_{3,7}(\Bbbk))$ generated by the elements $x_{17}, x_{24}, x_{25}, x_{27}, x_{32}, x_{34}, x_{35}, x_{37}$.

Notice that if we rotate the matrix above through $180°$ then the x_{ij} involved in the definition of $\mathcal{O}_q(M_{3,7,\gamma}(\Bbbk))$ sit naturally in the Young Diagram of the partition $\lambda = (4, 3, 1)$. We will return to this point later.

Lemma 3.7 *The quantum Schubert cell $S^o(\gamma)$ is isomorphic to the quantum ladder matrix algebra $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$.*

Proof: Lemma 3.1.4 of [15] shows that the commutation relations for the m_{ij} are the same as the commutation relations for corresponding variables x_{ij} in the quantum ladder matrix algebra $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$. As $\gamma m_{ij} = q m_{ij} \gamma$, for each i, j, by [15, Lemma 3.1.4(v)], it follows that the $\widetilde{m_{ij}}$ satisfy the same relations. Thus there is an epimorphism from $\mathcal{O}_q(M_{m,n,\gamma}(\Bbbk))$ onto $S^o(\gamma)$. A comparison of Gelfand-Kirillov dimensions similar to that used in [15, Theorem 3.1.6] now shows that this epimorphism is in fact an isomorphism. ∎

Theorem 3.8 *The quantum Schubert cell $S^o(\gamma)$ is (isomorphic to) a partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\Bbbk))$.*

Proof: For any n, there is an isomorphism $\delta : \mathcal{O}_q(M_n(\Bbbk)) \longrightarrow \mathcal{O}_{q^{-1}}(M_n(\Bbbk))$ defined by $\delta(x_{ij}) = x_{n+1-i,n+1-j}$, see the proof of [9, Corollary 5.9]. The isomorphism δ can be used to convert quantum ladder matrix algebras into partition subalgebras. As Schubert cells are isomorphic to quantum ladder matrix algebras, the result follows. ∎

The isomorphism described in the previous result carries over the $\mathcal{H}$-action on $S^o(\gamma)$ to the partition subalgebra, and this induced action acts via row and column multiplications. After suitable re-numbering of the components of $\mathcal{H}$, this action coincides with the action discussed at the beginning of Section 2. As a consequence of Theorem 3.8, the results obtained in Section 2 apply to quantum Schubert cells. In particular, the following results hold.

17

Theorem 3.9 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be the partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ defined by $\lambda_i + \gamma_i = n - m + i$ and let Y_λ be the corresponding Young diagram. Then the $\mathcal{H}$-prime spectrum of $S^o(\gamma)$ is in bijection with the set of Cauchon diagrams on the Young diagram, Y_λ, as described in Theorem 2.5.*

Theorem 3.10 *The quantum Schubert cell $S^o(\gamma)$ has $\mathcal{H}$-normal separation, normal $\mathcal{H}$-separation and normal separation.*

Corollary 3.11 *The quantum Schubert cell $S^o(\gamma)$ is catenary.*

4 The prime spectrum of the quantum grassmannian

In this section, we use the quantum Schubert cells to obtain information concerning the prime spectrum of the quantum grassmannian. We show that, in the generic case, where q is not a root of unity, all primes are completely prime and that there are only finitely many primes that are invariant under the natural torus action on the quantum grassmannian. By using a result of Lauren Williams, we are able to count the number of $\mathcal{H}$-primes. Also, we are able to show that the quantum grassmannian is catenary.

Note that the following result is valid for any $q \neq 0$.

Theorem 4.1 *Let P be a prime ideal of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ with $P \neq \langle \Pi \rangle$; that is, P is not the irrelevant ideal. Then there is a unique γ in Π with the property that $\gamma \notin P$ but $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$.*

Proof: If $\Pi \subseteq P$ then P is the irrelevant ideal. Otherwise, there exists $\gamma \in \Pi$ with $\gamma \notin P$. Choose such a γ that is minimal in Π with this property. Then $\lambda \in P$ for all $\lambda <_{\mathrm{st}} \gamma$.

Note that $\langle \{\lambda \mid \lambda <_{\mathrm{st}} \gamma\} \rangle \subseteq P$ and that γ is normal modulo $\langle \{\lambda \mid \lambda <_{\mathrm{st}} \gamma\} \rangle$, by [14, Lemma 1.2.1]; so that γ is normal modulo P.

Suppose that $\pi \not\geq_{\mathrm{st}} \gamma$. If $\pi <_{\mathrm{st}} \gamma$ then $\pi \in P$ by the choice of γ. If not, then π and γ are not comparable. Thus, we can write

$$\pi\gamma = \sum k_{\lambda\mu}\lambda\mu$$

with $k_{\lambda\mu} \in \Bbbk$ while $\lambda, \mu \in \Pi$ with $\lambda <_{\mathrm{st}} \gamma$, by [14, Theorem 3.3.8].

It follows that $\pi\gamma \in P$. Thus, $\pi \in P$, since $\gamma \notin P$ and γ is normal modulo P.

This shows that there is a γ with the required properties. It is easy to check that there can only be one such γ. ∎

This enables us to give a decomposition of the prime spectrum, $\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. Set $\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ to be the set of prime ideals P such that $\gamma \notin P$ while $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$. The previous result shows that

$$\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk))) = \bigsqcup_{\gamma \in \Pi} \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk))) \bigsqcup \langle \Pi \rangle \,.$$

We now re-instate our convention that q is not a root of unity.

Theorem 4.2 *Let q be a non root of unity. Then all prime ideals of the quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ are completely prime.*

Proof: Let P be a prime ideal of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. If $P = \langle \Pi \rangle$ then $\mathcal{O}_q(G_{m,n}(\Bbbk))/P \cong \Bbbk$; so P is completely prime.

Otherwise, suppose that $P \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. In this case, $\overline{P} = P/\langle \Pi^\gamma_{m,n} \rangle$ is a prime ideal in $S(\gamma) = \mathcal{O}_q(G_{m,n}(\Bbbk))/\langle \Pi^\gamma_{m,n} \rangle$ and it is enough to show that $\overline{P}$ is completely prime. Set $T := S(\gamma)[\overline{\gamma}^{-1}]$. Then $\overline{P}T$ is a prime ideal of T and $\overline{P}T \cap S(\gamma) = \overline{P}$. Thus $S(\gamma)/\overline{P} \subseteq T/\overline{P}T$ and so it is enough to show that $\overline{P}T$ is completely prime.

Now, the dehomogenisation isomorphism shows that $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$, where σ is the automorphism determined by the conjugation action of $\overline{\gamma}$, see the beginning of Section 3.

We know that $S^o(\gamma)$ is a torsionfree CGL-extension by Proposition 2.2 and Theorem 3.8. It is then easy to check that $S^o(\gamma)[y; \sigma]$ is also a torsionfree CGL-extension. Thus, all prime ideals of $S^o(\gamma)[y; \sigma]$ are completely prime, by [1, Theorem II.6.9], and it follows that all prime ideals of $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ are completely prime, as required. ∎

Of course, the decomposition of $\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ above induces a similar decomposition of $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$:

$$\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk))) = \bigsqcup_{\gamma \in \Pi} \mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk))) \bigsqcup \langle \Pi \rangle \,,$$

where $\mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is the set of $\mathcal{H}$-prime ideals P such that $\gamma \notin P$ while $\pi \in P$ for all $\pi \not\geq_{\mathrm{st}} \gamma$.

Our next task is to show that $\mathcal{H}\text{-}\mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is in natural bijection with $\mathcal{H}\text{-}\mathrm{Spec}(S^o(\gamma))$ and hence in bijection with Cauchon diagrams on the associated Young diagram Y_λ. As a consequence, we are able to calculate the size of $\mathcal{H}\text{-}\mathrm{Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))$.

Remark 4.3 Recall from the beginning of Section 3 that, for any $\gamma \in \Pi_{m,n}$, there is the dehomogenisation isomorphism

$$\theta : S^o(\gamma)[y, y^{-1}; \sigma] \longrightarrow S(\gamma)[\overline{\gamma}^{-1}],$$

19

where σ is conjugation by $\overline{\gamma}$. Hence, the action of $\mathcal{H}$ on $S(\gamma)[\overline{\gamma}^{-1}]$ transfers, via θ, to an action on $S^o(\gamma)[y, y^{-1}; \sigma]$. By Lemma 3.4, $S^o(\gamma)$ is stable under this action and it is clear that y is an $\mathcal{H}$-eigenvector. Further, let $h_0 = (\alpha_1, \ldots, \alpha_n) \in \mathcal{H}$ be such that $\alpha_i = q^2$ if $i \notin \{\gamma_1, \ldots, \gamma_m\}$ and $\alpha_i = q$ otherwise. Then, by using [15, Lemma 3.1.4(v)] and Lemma 3.4, it is easily verified that the action of h_0 on $S^o(\gamma)$ coincides with σ. In addition, $h_0(y) = q^m y$, since $h_0(\overline{\gamma}) = q^m \overline{\gamma}$. It follows that $S^o(\gamma)[y, y^{-1}; \sigma]$ satisfies Hypothesis 2.1 in [12].

Theorem 4.4 *Let $P \in \mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$; so that P is an $\mathcal{H}$-prime ideal of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ such that $\gamma \notin P$, while $\pi \in P$ for all $\pi \not\geq_{\text{st}} \gamma$. Set $T = S(\gamma)[\overline{\gamma}^{-1}] \cong S^o(\gamma)[y, y^{-1}; \sigma]$. Then the assignment $P \mapsto \overline{P}T \cap S^o(\gamma)$ defines an inclusion-preserving bijection from $\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ to $\mathcal{H}\text{-Spec}(S^o(\gamma))$, with inverse obtained by sending Q to the inverse image in $\mathcal{O}_q(G_{m,n}(\Bbbk))$ of $QT \cap S(\gamma)$. (Note, we are treating the isomorphism above as an id entification in these assignments.)*

Proof: This follows from the conjunction of two bijections. First, standard localisation theory shows that $\overline{P} = \overline{P}T \cap S(\gamma)$; and this gives a bijection between $\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ and $\mathcal{H}\text{-Spec}(T)$. For the second bijection, note that $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ and that the automorphism σ is given by the action of an element of $\mathcal{H}$, see Remark 4.3. Thus, it follows from [12, Theorem 2.3] that there is a bijection between $\mathcal{H}\text{-Spec}(T)$ and $\mathcal{H}\text{-Spec}(S^o(\gamma))$ given by intersecting an $\mathcal{H}$-prime of T with $S^o(\gamma)$. The composition of these two bijections produces the required bijection. ∎

Corollary 4.5 *$\mathcal{H}\text{-Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$ is in bijection with the Cauchon diagrams on Y_λ, where λ is the partition associated with γ.*

Proof: This follows from the previous theorem and Theorem 3.9. ∎

It follows from this corollary and the partition of the $\mathcal{H}$-spectrum of the quantum grassmannian that the $\mathcal{H}$-spectrum of the quantum grassmannian is finite. This finiteness is a crucial condition needed to investigate normal separation, Dixmier-Moeglin equivalence, etc. in the quantum case because of the stratification theory, see, for example, [7, Theorem 5.3], [1, Theorem II.8.4]. However, in this situation, we can say much more: we can say exactly how many $\mathcal{H}$-primes there are in the quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$. This is one more (the irrelevant ideal $\langle \Pi \rangle$) than the total number of Cauchon diagrams on the Young diagrams Y_λ corresponding to the partitions λ that fit into the partition $(n-m)^m$. This combinatorial problem has been solved by Lauren Williams, in [17]. The following result is obtained by setting $q = 1$ in the formula for $A_{k,n}(q)$ in [17, Theorem 4.1].

Theorem 4.6

$$|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{m,n}(\Bbbk)))| = 1 + \sum_{i=0}^{m-1} \binom{n}{i} \left((i-m)^i (m-i+1)^{n-i} - (i-m+1)^i (m-i)^{n-i} \right)$$

Proof: By using the results above, we see that, except for the irrelevant ideal, each $\mathcal{H}$-prime corresponds to a unique Cauchon diagram drawn on the Young diagram Y_λ that corresponds to the partition λ associated to the quantum minor γ which determines the cell that P is in.

In [17, Theorem 4.1], Lauren Williams has counted the number of Cauchon diagrams on the Young diagrams Y_λ that fit into the partition $(n-m)^m$; and this count, plus one, gives the number of $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. ∎

For example, $|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{2,4}))| = 34$ and $|\mathcal{H}\text{-Spec}(\mathcal{O}_q(G_{3,6}))| = 884$. (These numbers can be seen from the table in [16, Figure 23.1].)

We turn now to the questions of normal separation and catenarity. In order to establish these properties for the quantum grassmannian, we need to use the dehomogenisation isomorphism. Recall that the methods of [12] are available because of Remark 4.3.

Lemma 4.7 *Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals in $S(\gamma)$ that do not contain $\overline{\gamma}$. Then, there is an $\mathcal{H}$-eigenvector in $P \backslash Q$ that is normal modulo Q.*

Proof: Let $Q \subsetneq P$ be $\mathcal{H}$-prime ideals in $S(\gamma)$ that do not contain $\overline{\gamma}$. Set $T := S(\gamma)[\overline{\gamma}^{-1}]$ and observe that there is an induced action of the torus $\mathcal{H}$ on T, because γ is an $\mathcal{H}$-eigenvector. Note that $P = PT \cap S(\gamma)$ and $Q = QT \cap S(\gamma)$; so $QT \subsetneq PT$ are $\mathcal{H}$-prime ideals in T. Now, set $P_0 := PT \cap S^o(\gamma)$ and $Q_0 := QT \cap S^o(\gamma)$ (here, we are treating the isomorphism $T \cong S^o(\gamma)[y, y^{-1}; \sigma]$ as an identification) and note that $PT = \oplus_{n \in \mathbb{Z}} P_0 \, y^n$ and $QT = \oplus_{n \in \mathbb{Z}} Q_0 \, y^n$; so $Q_0 \subsetneq P_0$ are $\mathcal{H}$-prime ideals of $S^o(\gamma)$, see Remark 4.3 and [12, Theorem 2.3]. These observations make it clear that

$$\frac{S^o(\gamma)}{Q_0}[y, y^{-1}; \sigma] \quad \cong \quad \frac{T}{QT} \quad \cong \quad \frac{S(\gamma)}{Q}[\overline{\gamma}^{-1}].$$

As usual, $\overline{S^o(\gamma)}$ will denote $S^o(\gamma)/Q_0$, etc.

The quantum Schubert cell $S^o(\gamma)$ has $\mathcal{H}$-normal separation, by Theorem 3.10. Thus, there exists an $\mathcal{H}$-eigenvector $c \in P_0 \backslash Q_0$ and an element $h \in \mathcal{H}$ such that $ca - h(a)c \in Q_0$ for all $a \in S^o(\gamma)$. Recall that the action of σ coincides with the action of an element h_y of $\mathcal{H}$; so that $yc = h_y(c)y = \lambda cy$ for some $\lambda \in \Bbbk^*$. It follows that $\overline{c}$ is normal in T/QT. Define $\sigma_c : T/QT \longrightarrow T/QT$ by $\overline{c}\overline{t} = \sigma_c(\overline{t})\overline{c}$ for all $t \in T$. Note that $\sigma_c|_{\overline{S^o(\gamma)}} = h|_{\overline{S^o(\gamma)}}$ and that $\sigma_c(y) = \lambda^{-1}y$.

We claim that $\sigma_c(S(\gamma)/Q) = S(\gamma)/Q$; so that σ_c induces an isomorphism on this algebra. In order to see this, note that $S(\gamma)/Q$ is generated as an algebra by the images of the quantum minors $[\alpha_1, \ldots, \alpha_m]$ for $[\alpha_1, \ldots, \alpha_m] \geq \gamma$. Now, $\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1} \in \overline{S^o(\gamma)}$, because $[\alpha_1, \ldots, \alpha_m]\gamma^{-1}$ has degree zero in T so that $[\alpha_1, \ldots, \alpha_m]\gamma^{-1} \in S^o(\gamma)$. Thus, recalling that $\overline{\gamma}$ is identified with y under the isomorphisms above,

$$\begin{aligned}
\sigma_c(\overline{[\alpha_1, \ldots, \alpha_m]}) &= \sigma_c(\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1})\sigma_c(\overline{\gamma}) = h(\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1})(\lambda^{-1}y) \\
&= \mu\,\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1}(\lambda^{-1}y) = (\mu\lambda^{-1})\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1}y \\
&= (\mu\lambda^{-1})\overline{[\alpha_1, \ldots, \alpha_m]},
\end{aligned}$$

where the existence of $\mu \in \Bbbk^*$ is guaranteed because h is acting as a scalar on the element $\overline{[\alpha_1, \ldots, \alpha_m]}\overline{\gamma}^{-1} \in S^o(\gamma)/Q_0$. The claim follows.

There exists $d \geq 0$ such that $\overline{c}\,\overline{\gamma}^d \in S(\gamma)/Q$. It is obvious that $c\overline{\gamma}^d$ is an $\mathcal{H}$-eigenvector, because each of c and γ is an $\mathcal{H}$-eigenvector. Also, $c\overline{\gamma}^d \in P\backslash Q$. Finally, $\overline{c}\,\overline{\gamma}^d$ is normal in $S(\gamma)/Q$, because $S(\gamma)/Q$ is invariant under conjugation by each of $\overline{c}$ and $\overline{\gamma}$. ∎

Theorem 4.8 *The quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ has normal $\mathcal{H}$-separation and hence normal separation.*

Proof: Suppose that $Q \subsetneq P$ are $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{m,n}(\Bbbk))$. Suppose that $Q \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. If $\gamma \in P$, then P contains the $\mathcal{H}$-eigenvector γ.

Otherwise, $\gamma \notin P$ and $P \in \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\Bbbk)))$. In this case, it is enough to show that there is a $\mathcal{H}$-eigenvector in $\overline{P}\backslash\overline{Q}$ which is normal modulo $\overline{Q}$, where $\overline{P} = P/\langle\Pi_{m,n}^\gamma\rangle$ and $\overline{Q} = Q/\langle\Pi_{m,n}^\gamma\rangle$ are $\mathcal{H}$-prime ideals in $S(\gamma)$. However, this has been done in the previous lemma. ∎

Theorem 4.9 *The quantum grassmannian $\mathcal{O}_q(G_{m,n}(\Bbbk))$ is catenary.*

Proof: As in Corollary 2.10, this follows from the previous results and [18, Theorem 0.1]. ∎

Remark 4.10 It is obvious from the style of proof of the preceding results that there is now a good strategy for producing results concerning the quantum grassmannian: first, establish the corresponding results for partition subalgebras of quantum matrices, and then use the theory of quantum Schubert cells and noncommutative dehomogenisation to obtain the result in the quantum grassmannian. We leave any further developments for interested readers.

5 Concluding remark.

We end this work by stressing some important connections between the results established in Section 4 above, and recent work of Postnikov in total positivity, see [16].

Let $M_{m,n}^+(\mathbb{R})$ denote the space of $m \times n$ real matrices of rank m and whose $m \times m$ minors are nonnegative. The group $GL_m^+(\mathbb{R})$ of $m \times m$ real matrices of positive determinant act naturally on $M_{m,n}^+(\mathbb{R})$ by left multiplication. The corresponding quotient space $G_{m,n}^+(\mathbb{R}) = M_{m,n}^+(\mathbb{R})/GL_m^+(\mathbb{R})$ is the *totally nonnegative* grassmannian of m dimensional subspaces in $\mathbb{R}^n$. One can define a cellular decomposition of $G_{m,n}^+(\mathbb{R})$ by specifying, for each element of $G_{m,n}^+(\mathbb{R})$, which $m \times m$ minors are zero and which are strictly positive. The corresponding cells are called the *totally nonnegative cells* of $G_{m,n}^+(\mathbb{R})$. In [16], Postnikov shows that totally nonnegative cells in $G_{m,n}^+(\mathbb{R})$ are in bijection with the Cauchon diagrams on partitions fitting into the partition $(n-m)^m$. For further details, see Sections 3 and 6 in [16].

Hence, by the results in Section 4 above, the set of totally nonnegative cells of $G_{m,n}^+(\mathbb{R})$ is in one-to-one correspondance with the set of $\mathcal{H}$-prime ideals of $\mathcal{O}_q(G_{m,n}(\Bbbk))$ distinct from the augmentation ideal. We believe it would be interesting to understand this coincidence and we intend to pursue this theme in a subsequent paper.

References

[1] K A Brown and K R Goodearl, Lectures on algebraic quantum groups. Advanced Courses in Mathematics-CRM Barcelona. Birkhäuser Verlag, Basel, 2002

[2] G Cauchon, *Effacement des dérivations et spectres premiers des algèbres quantiques*, J Algebra 260 (2003), 476-518.

[3] G Cauchon, *Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale* J Algebra 260 (2003), no. 2, 519–569.

[4] W Fulton, Young tableaux. With applications to representation theory and geometry. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.

[5] I M Gelfand and V S Retakh, *Determinants of matrices over noncommutative rings.* (Russian) Funktsional. Anal. i Prilozhen. 25 (1991), no. 2, 13–25, 96; translation in Funct. Anal. Appl. 25 (1991), no. 2, 91–102

[6] N Gonciulea and V Lakshmibai, *Flag Varieties*, Actualités Mathématiques, Hermann, Paris, 2001

[7] K R Goodearl, *Prime spectra of quantized coordinate rings*, Interactions between ring theory and representations of algebras (Murcia), 205–237, Lecture Notes in Pure and Appl Math, 210, Dekker, New York, 2000

[8] K R Goodearl and T H Lenagan, *Catenarity in quantum algebras*, J Pure Appl Algebra 111 (1996), 123–142

[9] K R Goodearl and T H Lenagan, *Winding invariant prime ideals in quantum 3x3 matrices*, Journal of Algebra 260 (2003), 657-687

[10] A Kelly, T H Lenagan and L Rigal, *Ring theoretic properties of quantum grassmannians*, J Algebra Appl 3 (2004), no. 1, 9–30

[11] G R Krause and T H Lenagan, Growth of algebras and Gelfand-Kirillov dimension, Revised edition, Graduate Studies in Mathematics, 22, American Mathematical Society, Providence, RI, 2000

[12] S Launois, T H Lenagan and L Rigal, *Quantum unique factorisation domains* J London Math Soc 74 (2006), 321-340

[13] T H Lenagan and L Rigal, *The maximal order property for quantum determinantal rings*, Proc Edinb Math Soc (2) 46 (2003), no. 3, 513–529

[14] T H Lenagan and L Rigal, *Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum grassmannians*, J Algebra 301 (2006), no. 2, 670–702

[15] T H Lenagan and L Rigal, *Quantum analogues of Schubert varieties in the grassmannian*, to appear in Glasgow Mathematical Journal

[16] A Postnikov, *Total positivity, grassmannians, and networks*, arxiv:math.CO/0609764

[17] L K Williams, *Enumeration of totally positive Grassmann cells*, Adv. Math. 190 (2005), no. 2, 319–342

[18] A Yekutieli and J J Zhang *Rings with Auslander Dualizing Complexes*, J Algebra 213 (1999), 1-51

[19] J J Zhang, *Connected graded Gorenstein algebras with enough normal elements*, J. Algebra 189 (1997), no. 2, 390–405

S Launois:

Institute of Mathematics, Statistics and Actuarial Science,

University of Kent at Canterbury,

CT2 7NF,

UK

E-mail: S.Launois@kent.ac.uk

T H Lenagan:

Maxwell Institute for Mathematical Sciences

School of Mathematics, University of Edinburgh,

James Clerk Maxwell Building, King's Buildings, Mayfield Road,

Edinburgh EH9 3JZ, Scotland, UK

E-mail: tom@maths.ed.ac.uk

L Rigal:

Université Jean-Monnet (Saint-Étienne),

Faculté des Sciences et Techniques,

Département de Mathématiques,

23 rue du Docteur Paul Michelon,

42023 Saint-Étienne Cédex 2,

France

E-mail: Laurent.Rigal@univ-st-etienne.fr