Finkelstein-Rubinstein constraints for the Skyrme model with pion masses

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Abstract

The Skyrme model is a classical field theory modelling the strong interaction between atomic nuclei. It has to be quantized in order to compare it to nuclear physics. When the Skyrme model is semi-classically quantized it is important to take the Finkelstein-Rubinstein constraints into account. Recently, a simple formula has been derived to calculate the these constraints for Skyrmions which are well-approximated by rational maps. However, if a pion mass term is included in the model, Skyrmions of sufficiently large baryon number are no longer well-approximated by the rational map ansatz. This paper addresses the question how to calculate Finkelstein-Rubinstein constraints for Skyrme configurations which are only known numerically.

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1 Introduction

The Skyrme model is a classical model of the strong interaction between atomic nuclei [1]. In order to compare the Skyrme model with nuclear physics we have to understand the classical solutions and then quantize the model. The classical solutions have a surprisingly rich structure. Configurations in the Skyrme model are labelled by a topological winding number which can be interpreted as the baryon number B. Static minimal energy configurations for a given B are known as Skyrmions. The $B = 1$ Skyrmion has spherical symmetry, the $B = 2$ Skyrmion has axial symmetry and for $B > 2$ Skyrmions have various discrete symmetries, see [2] and references therein. The Skyrme model depends on a parameter which corresponds to the pion mass m_π. For $m_\pi = 0$ all the Skyrmions for $B \leq 22$ were found to be shell-like configurations with discrete symmetries [2]. Such configurations are very well described by the rational map ansatz [3]. However, if the value of the pion mass is increased to its physical value or higher, then for high enough baryon number shell-like solutions are no longer the minimal energy solutions [4].

In [5, 6] Adkins et al. quantized the translational and rotational zero modes of the $B = 1$ Skyrmion for zero and nonzero pion mass respectively and obtained good agreement with experiment. A subtle point is that Skyrmions can be quantized as fermions as has been shown in [7]. Solitons in scalar field theories can consistently be quantized as fermions provided that the fundamental group of configuration space has a $\mathbb{Z}_2$ subgroup generated by a loop in which two identical solitons are exchanged. All loops in configuration space give rise to so-called Finkelstein-Rubinstein constraints which depend on whether a loop in configuration space is contractible or not. In particular, symmetries of classical configurations induce loops in configurations space. After quantization, these loops give rise to constraints on the wave function.

The $B = 2$ Skyrmion with axial symmetry was quantized in [8, 9] using the zero mode quantization. Later, the approximation was improved by taking massive modes into account [10]. The $B = 3$ Skyrmion was first quantized in [11] and the $B = 4$ Skyrmion [12]. Irwin performed a zero mode quantization for $B = 4 - 9$ [13] using the monopole moduli space as an approximation for the Skyrmion moduli space. The physical predictions of the Skyrme model for various baryon numbers were also discussed in [14]. Recently, the Finkelstein-Rubinstein constraints have been calculated for Skyrmions which are well-approximated by the rational map ansatz [15], and which we shall call rational map Skyrmions. In this case, the Finkelstein-Rubinstein constraints are given by a simple formula. This formula is also valid if the Skyrme configuration can be deformed into a rational map Skyrmion while preserving the relevant symmetries. However, this is not always possible. The aim of this paper is to show how to calculate Finkelstein-Rubinstein constraints for more general configurations.

This paper is organized as follows. In section 2 we first describe the rational map ansatz. Then we discuss the Finkelstein-Rubinstein constraints. Finally, we derive some constraints on the symmetries which are compatible with a rational map Skyrmion. In section 3 we first introduce a truncated rational map ansatz which describes well-separated rational map Skyrmions. Then we calculate the Finkelstein-Rubinstein constraints for this class of Skyrme configurations. In the following section, we describe how to calculate the Finkelstein-Rubinstein constraints for a minimal energy configuration which is only known numerically. We also give an example. In section 5 we derive constraints on possible symmetries in order to make predictions about ground states for Skyrmions with even baryon number. We end with a conclusion.

2 Skyrmions and Rational Maps

In this section, we first recall the some basic facts about the Skyrme model. Then we describe the rational map ansatz. We then discuss how to quantize a Skyrmion as a fermion. Finally, we derive which symmetries are compatible with a rational map Skyrmion of a given baryon number.

2.1 The rational map ansatz

The Skyrme model is a classical field theory of pions. The basic field is the $SU(2)$ valued field $U(\mathbf{x},t)$ where $\mathbf{x} \in \mathbb{R}^3$. The static solutions can be obtained by varying the following energy

$$E = \int \left(-\frac{1}{2}\mathrm{Tr}(R_iR_i) - \frac{1}{16}\mathrm{Tr}([R_i,R_j][R_i,R_j]) - m^2\mathrm{Tr}(U-1)\right) d^3x, \qquad (1)$$

where $R_i = (\partial_i U)U^\dagger$ is a right invariant $su(2)$ valued current, and m is a parameter proportional to the pion mass m_π, [4]. In order to have finite energy, Skyrme fields have to take a constant value, $U(|\mathbf{x}| = \infty) = 1$, at infinity, and such maps are characterized by an integer-valued winding number. This topological charge is interpreted as the baryon number and is given by the following integral

$$B = -\frac{1}{24\pi^2}\int \epsilon_{ijk}\mathrm{Tr}(R_iR_jR_k)d^3x. \qquad (2)$$

We will denote the configuration space of Skyrmions by Q. Q splits into connected components Q_B labelled by the topological charge. Furthermore, the energy of configurations in Q_B is bounded below by $E \geq 12\pi^2 B$ [16].

The minimal energy solutions have been calculated in the massless case, $m = 0$, for all $B \leq 22$ [2]. The solutions are shell-like structures which are very well approximated by the rational map ansatz [3] which we will now describe.

The main idea is to write Skyrme fields which can be thought of as maps from $S^3 \to S^3$ in terms of rational maps which are holomorphic maps from $S^2 \to S^2$.

In algebraic topology, such a construction is known as a suspension. First, we introduce polar coordinates (r, θ, ϕ) and note that the angular coordinates can be related to the complex plane z by the stereographic projection $z = \mathrm{e}^{i\phi} \tan \frac{\theta}{2}$. Then the Skyrme field can be written as

$$U(r, z) = \exp\left(\frac{if(r)}{1 + |R|^2}\begin{pmatrix} 1 - |R|^2 & 2\bar{R} \\ 2R & |R|^2 - 1 \end{pmatrix}\right), \tag{3}$$

where the profile function $f(r)$ is a real function satisfying the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$. The map $R = R(z)$ is the eponymous rational map. It can be written as the quotient of two polynomials $p(z)$ and $q(z)$ which satisfy $\max(\deg(p(z)), \deg(q(z))) = B$ and $p(z)$ and $q(z)$ have no common factors. Here deg denotes the polynomial degree. The ansatz (3) can be inserted into the energy (1) and we obtain

$$E = 4\pi \int \left(r^2 f'^2 + 2B\left(f'^2 + 1\right)\sin^2 f + \mathcal{I}\frac{\sin^4 f}{r^2} + 2m^2 r^2 (1 - \cos f) \right) dr, \tag{4}$$

where

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2i\, dz\, d\bar{z}}{(1 + |z|^2)^2}. \tag{5}$$

To minimize the energy (4) one first determines the rational map which minimizes $\mathcal{I}$ and then calculates the shape function $f(r)$ numerically by solving the corresponding Euler-Lagrange equation. The rational maps which minimize $\mathcal{I}$ have been determined numerically in [2, 17] for all $B \leq 40$. Note that the restriction that $R(z)$ is a holomorphic map can be lifted and a generalized rational map ansatz can be introduced [18]. This generalized ansatz has been shown to improve the energy significantly for $B \leq 4$, and it also captures the singularity structure of Skyrmions better. However, it is difficult to use for higher baryon number, and from the point of view of discussing symmetries the original rational map ansatz is sufficient.

The rational map ansatz gives a good approximation to the energy of a Skyrmion and also gives a very accurate prediction of its symmetry [2]. By symmetry we mean that a rotation in space followed by a rotation in target space leaves the Skyrmion invariant. Namely,

$$U(\mathbf{x}) = AU\left(D(A')\mathbf{x}\right)A^\dagger, \tag{6}$$

where A and A' are $SU(2)$ matrices and $D(A')$ is the associated $SO(3)$ rotation. It is therefore important to understand how a rational map transforms under rotations in space and target space. It can be shown that

$$R(z) \mapsto \tilde{R}(z) = M_A(R(M_{A'}(z))), \tag{7}$$

where M_A and $M_{A'}$ are Möbius transformations. See [15] for further details.

2.2 Finkelstein-Rubinstein constraints

In the following, we recall the ideas of Finkelstein and Rubinstein [7] on how to quantized a scalar field theory and obtain fermions. For further details see [15, 19]. The main idea is to define a wave function on the covering space of configuration space. Recall that the configuration space Q of the Skyrme model splits into connected components labelled by the degree B, and will be denoted by Q_B. The fundamental group of each component of the configuration space Q is $\pi_1(Q_B) = \mathbb{Z}_2$. Therefore, the covering space $\tilde{Q}_B$ of each component is a double cover. In order to have fermionic quantisation we have to impose the condition that if two different points $p_1, p_2 \in \tilde{Q}_B$ correspond to the same point $p \in Q_B$ then the wave function $\psi : \tilde{Q}_B \to \mathbb{C}$ has to satisfy

$$\psi(p_1) = -\psi(p_2). \tag{8}$$

The points $p_1, p_2 \in \tilde{Q}_B$ can be interpreted as two paths in configuration space. The condition that $p_1 \neq p_2$ implies that p_1 and p_2 differ by a noncontractible loop. Every symmetry of a classical configuration gives rise to a loop in configuration space. In particular, we are interested in symmetries given by a rotation by α in space followed by a rotation by β in target space. This leads to the following constraint on the wave function ψ:

$$\exp\left(i\alpha\mathbf{n}\cdot\hat{\mathbf{J}}\right)\exp\left(i\beta\mathbf{N}\cdot\hat{\mathbf{I}}\right)\psi = \chi_{FR}\psi, \tag{9}$$

where $\mathbf{n}$ is the direction of the rotation axis in space, $\mathbf{N}$ is the rotation axis in target space, a $\hat{\mathbf{J}}$ and $\hat{\mathbf{I}}$ are the angular momentum operators in space and target space, respectively.[1] Note that rotations in target space will also be called isorotations. The Finkelstein-Rubinstein phase χ_{FR} enforces the condition (8) and satisfies

$$\chi_{FR} = \begin{cases} 1 & \text{if the induced loop is contractible,} \\ -1 & \text{otherwise.} \end{cases} \tag{10}$$

Here is a good place to summarize some important and well-known results. Giulini showed that a 2π rotation of a Skyrmion gives rise to $\chi_{FR} = (-1)$ if and only if the baryon number B is odd, [20]. Finkelstein and Rubinstein showed in [7] that a 2π rotation of a Skyrmion of degree B is homotopic to an exchange of two Skyrmions of degree B. This also implies that an exchange of two identical Skyrmions gives rise to $\chi_{FR} = (-1)$ if and only if their baryon number B is odd. In [15] it was shown that a 2π isorotation of a Skyrmion also gives rise to $\chi_{FR} = (-1)$ if and only if the baryon number B is odd. These results agree with the physical intuition since atomic nuclei can be modelled by interacting point-like fermionic particles.

[1] For a more detailed discussion on body fixed and space fixed angular momentum operators in this context see [19].

2.3 Symmetries of rational maps

Shell-like Skyrmions are described very well using the rational map ansatz. If a rational map Skyrmion of degree B is symmetric under a rotation by α followed by an isorotation by β then the Finkelstein-Rubinstein phase of this symmetry is given by

$$\chi_{FR} = (-1)^N \quad \text{where} \quad N = B\,(B\alpha - \beta)\,/(2\pi), \tag{11}$$

which has been proven in [15]. For $B > 2$ all the known Skyrmions are invariant under discrete subgroups, so they contain cyclic groups as subgroups. Let C_n^k be a cyclic group of order n which is generated by a rotation by $\alpha = 2\pi/n$ followed by an isorotation by $\beta = 2\pi k/n$ where $-n < k \leq n$. Equation (11) imposes a constraint on the values of B which are compatible with a given C_n^k symmetry. Namely, N has to be an integer. A stronger constraint can be derived if we work directly with rational maps.

Lemma 2.1 *A rational map of degree B can have a C_n^k symmetry if and only if $B \equiv 0 \mod n$ or $B \equiv k \mod n$.*

Proof:
Without loss of generality consider rational maps with boundary condition $R(\infty) = \infty$ and assume that the C_n^k symmetry corresponds to a rotation around the third axis in space followed by a rotation around the negative third axis in target space. With this choice of axes, the boundary conditions are preserved by the relevant rotation and also by the relevant isorotation, and the sign choice corresponds to the sign choice for (11) in [15]. The rational map $R(z) = p(z)/q(z)$ can be written as

$$R(z) = \frac{z^B + a_{B-1}z^{B-1} + \cdots + a_0}{b_{B-1}z^{B-1} + \cdots + b_0} \tag{12}$$

where the polynomials $p(z)$ and $q(z)$ have no common factors. The C_n^k symmetry condition is given by

$$R(z) = \mathrm{e}^{-2\pi k i/n} R(\mathrm{e}^{2\pi i/n} z). \tag{13}$$

Note that a C_n^{n-k} rotation can be interpreted as a C_n^{-k} rotation followed by a 2π isorotation. Since for a 2π isorotation the Finkelstein-Rubinstein phase is simply given by $\chi_{FR} = (-1)^B$, we can restrict our attention to $k = 0, \ldots, n-1$.

First we show existence. Let $B \equiv k \mod n$, so $B = nl + k$. Then the rational map

$$R(z) = \frac{z^k r(z^n)}{s(z^n)} \tag{14}$$

where $r(z)$ is a polynomial of degree l and $s(z)$ is a polynomial of at most degree l. For $k = 0$ the degree of $s(z)$ has to be less than l in order to respect the boundary conditions $R(\infty) = \infty$. This rational map is invariant under C_n^k. To make sure that it is a rational map of degree $B = nl + k$ the polynomials $r(z^n)$ and $s(z^n)$ are required not to have any common factors. Furthermore, for $k \neq 0$, we need to impose $s(0) \neq 0$, since the polynomial $p(z)$ has a zero at $z = 0$. For $k = 0$, we also impose the condition $s(0) \neq 0$, and we will discuss the case $s(0) = 0$ in the next paragraph. The simplest example of such a rational map is

$$R(z) = z^B. \tag{15}$$

Hence a rational map of degree B with C_n^k symmetry exists for $B \equiv k \mod n$.

Similarly, let $B \equiv 0 \mod n$, so $B = nl$. Again, we only consider $k = 0, \ldots, n-1$. Then the rational map

$$R(z) = \frac{r(z^n)}{z^{n-k} s(z^n)}, \tag{16}$$

is invariant under C_n^k. Here $r(z)$ is again a polynomial of degree l and $s(z)$ is a polynomial of at most degree $l-1$ which has no common factors with $r(z)$. Furthermore, we also require $r(0) \neq 0$. One example of such a rational map is

$$R(z) = \frac{z^B + 1}{z^{n-k}}. \tag{17}$$

Hence a rational map of degree B with C_n^k symmetry exists for $B \equiv 0 \mod n$. This completes the proof of existence. The classification of rational maps into type (14) and type (16) will become useful in section 5.

Now, we assume that the rational map (12) is invariant under C_n^k. In homogeneous coordinates, the rational map $R(z)$ is given by $[p(z), q(z)] \in \mathbb{C}P^1$ subject to the relation that $[p(z), q(z)] = [\lambda p(z), \lambda q(z)]$ for any complex number $\lambda \neq 0$. Under the symmetry C_n^k, the polynomials $p(z)$ and $q(z)$ in equation (12) transform as

$$\begin{aligned} p(z) &\mapsto \lambda \mathrm{e}^{-2\pi i k/n} p(\mathrm{e}^{2\pi i/n} z), \\ q(z) &\mapsto \lambda q(\mathrm{e}^{2\pi i/n} z), \end{aligned} \tag{18}$$

for $\lambda \neq 0$. Assuming the rational map is invariant under C_n^k leads to the following constraints on the coefficients.

$$\begin{aligned} a_m &= \lambda \mathrm{e}^{2\pi i(m-k)/n} a_m, \\ b_m &= \lambda \mathrm{e}^{2\pi i m/n} b_m. \end{aligned} \tag{19}$$

In order to have a rational map of degree B, $p(z)$ and $q(z)$ cannot have any common factors. This implies that a_0 and b_0 cannot both be zero. First, assume $a_0 \neq 0$. Then equation (19) implies that

$$\lambda = \mathrm{e}^{2\pi i k/n}. \tag{20}$$

The coefficient of the highest power in the numerator is also not allowed to vanish which implies

$$\mathrm{e}^{2\pi i B/n} = 1, \tag{21}$$

so that $B \equiv 0 \mod n$. Now, consider that $b_0 \neq 0$ which implies

$$\lambda = 1. \tag{22}$$

Again, the coefficient of the highest power in the numerator is not allowed to vanish. Therefore,

$$\mathrm{e}^{2\pi i(B-k)/n} = 1, \tag{23}$$

so that $B \equiv k \mod n$, which completes the proof. □

The lemma is more restrictive than the condition that N is an integer. For example $B(B-k) \equiv 0 \mod n$ suggests that a C_4^0 symmetry is possible for $B = 2$. However, since $B \equiv 2 \mod 4$ our lemma excludes such a symmetry.

3 A truncated rational map ansatz

In [15] the Finkelstein-Rubinstein constraints were calculated for Skyrmions which are well approximated by the rational map ansatz. In this section, we calculate the Finkelstein-Rubinstein constraints for Skyrme configurations $U(\mathbf{x})$ which are given by a truncated rational map ansatz defined as follows. Let $U_{B_i}(\mathbf{x})$ be a Skyrme configuration of degree B_i which is given by (3) and the shape function $f(r)$ is a smooth, decreasing function which satisfies $f(0) = \pi$ and $f(r) = 0$ for $r \geq L$. Then the Skyrme configuration is given by

$$U(\mathbf{x}) = \begin{cases} U_{B_i}(\mathbf{x} - \mathbf{X}_i) & \text{for } |\mathbf{x} - \mathbf{X}_i| < L, \\ 1 & \text{otherwise.} \end{cases} \tag{24}$$

From formula (2) it is obvious that the configuration $U(\mathbf{x})$ has the degree $B = \sum_i B_i$. The parameters $\mathbf{X}_i$ are the positions of the Skyrmions, and we assume that $|\mathbf{X}_i - \mathbf{X}_j| > 2L$ for $i \neq j$. Such an ansatz provides reasonable initial conditions for numerical simulations [21]. A related ansatz is the product ansatz. This ansatz produces Skyrme configurations which are closer in energy to the true solutions. Topologically, these two ansätze are equivalent. However, the product ansatz has the disadvantage that it is non-commutative, since in general $U_1 U_2 \neq U_2 U_1$ for $SU(2)$ matrices U_1 and U_2, so that it is slightly more difficult to discuss symmetries. Therefore, we restrict our attention to the truncated rational map ansatz.

Consider a configuration $U(\mathbf{x})$ which is invariant under C_n^k. The symmetry relates different Skyrmions U_{B_i} with each other. For each individual Skyrmion U_{B_i} there are two possibilities. Either the centre of this Skyrmion lies on the symmetry axis or it is one constituent of a regular n-gon of Skyrmions which transform into each other under the symmetry.

3.1 Skyrmions centred on the symmetry axis

Assume two Skyrmions with baryon number B_1 and B_2 have a common C_n^k axis of symmetry, say the x_3 axis, and are centred around the origin and the point $P = (0, 0, c)$ for $c > L$. Then the symmetry loop is homotopic to a product of two loops, each acting only on one Skyrmion. This can be seen as follows. The configuration can be written as

$$U(\mathbf{x}) = \begin{cases} U_1(\mathbf{x}) & \text{for } |\mathbf{x}| < L, \\ U_2(\mathbf{x}) & \text{for } |\mathbf{x} - (0,0,c)| < L, \\ 1 & \text{otherwise.} \end{cases} \tag{25}$$

Under a rotation, the configuration transforms as

$$A(\beta)U(D(\alpha)\mathbf{x})A^\dagger(\beta) = \begin{cases} A(\beta)U_1(D(\alpha)\mathbf{x})A^\dagger(\beta) & \text{for } |\mathbf{x}| < L, \\ A(\beta)U_2(D(\alpha)\mathbf{x})A^\dagger(\beta) & \text{for } |\mathbf{x} - (0,0,c)| < L, \\ 1 & \text{otherwise.} \end{cases} \tag{26}$$

Here $A(\beta)$ is a rotation in target space acting by conjugation and $D(\alpha)$ is a rotation around the x_3 axis. Note in particular, that the vacuum is invariant under isorotation. The map $H : [0,1] \times [0,1] \to Q_B : (s,t) \mapsto H(s,t)$ provides a homotopy such that $H(0,t)$ is a C_n^k rotation of the whole configuration while $H(1,t)$ corresponds to a loop which first rotates one Skyrmion and then the other.

$$H(s,t) = \begin{cases} A(h_1(s,t)\beta)U_1(D(h_1(s,t)\alpha)\mathbf{x})A^\dagger(h_1(s,t)\beta) & \text{for } |\mathbf{x}| < L, \\ A(h_2(s,t)\beta)U_2(D(h_2(s,t)\alpha)\mathbf{x})A^\dagger(h_2(s,t)\beta) & \text{for } |\mathbf{x} - (0,0,c)| < L, \\ 1 & \text{otherwise,} \end{cases} \tag{27}$$

where

$$h_1(s,t) = \begin{cases} t/(1 - s/2) & \text{for } 0 \leq t < 1 - s/2, \\ 1 & \text{for } s \leq t \leq 1, \end{cases} \tag{28}$$

and similarly

$$h_2(s,t) = \begin{cases} 0 & \text{for } 0 \leq t < s/2, \\ (t - s/2)/(1 - s/2) & \text{for } s/2 \leq t \leq 1. \end{cases} \tag{29}$$

Therefore, the Finkelstein-Rubinstein phase can be calculated as a product of the two loops of the individual Skyrmions. Since we are assuming that the B_1 and the B_2 Skyrmion are both well-described by the rational map ansatz we can apply formula (11) and obtain $\chi_{FR} = (-1)^N$ where

$$\begin{aligned} N &= B_1(B_1\alpha - \beta)/(2\pi) + B_2(B_2\alpha - \beta)/(2\pi), \\ &= (B_1^2 + B_2^2)\alpha/(2\pi) - (B_1 + B_2)\beta/(2\pi). \end{aligned} \tag{30}$$

Naive application of formula (11) gives an incorrect result, namely $\chi_{FR} = (-1)^{\tilde{N}}$ where

$$\tilde{N} = (B_1 + B_2)^2 \alpha/(2\pi) - (B_1 + B_2)\beta/(2\pi). \tag{31}$$

This $\tilde{N}$ is no longer well defined. Let $B_1 = 1$ and $B_2 = 1$ be invariant under a C_3^1 rotation. Then $N = 0$, but $\tilde{N} = 2/3$. Also, consider $B_1 = 1$ and $B_2 = 1$ with symmetry C_2^1 then $N = 0$, but $\tilde{N} = 1$.

In this context, it is worth mentioning another interesting ansatz for Skyrmions, namely, the multi-shell ansatz by Manton and Piette [22]. The main idea is to construct multiple concentric shells of Skyrmions, where each shell is given by the usual rational map ansatz. The multi-shell ansatz can then be written as

$$U(\mathbf{x}) = \begin{cases} \exp\left(if(r)\mathbf{n}_{R_1} \cdot \boldsymbol{\tau}\right) & \text{for } 2\pi \geq f(r) > \pi, \\ \\ \exp\left(if(r)\mathbf{n}_{R_2} \cdot \boldsymbol{\tau}\right) & \text{for } \pi \geq f(r) > 0, \end{cases} \tag{32}$$

where $f(r)$ is a monotonically decreasing function with boundary conditions $f(0) = 2\pi$ and $f(\infty) = 0$. Here R_1 and R_2 are two rational maps of degree B_1 and B_2, respectively. The degree of such a Skyrme configuration is $B = B_1 + B_2$. Note that for $f(r_0) = \pi$ the Skyrme field takes the value $U(r_0) = -1$ which is invariant under rotations and isorotations. As before, we can split a symmetry loop into two loops, each of which only acting on the inner or the outer Skyrmion, similar to (27). Therefore, formula (30) is also valid in this case.

3.2 Configurations of n Skyrmions related by symmetry

Lemma 3.1 *For n Skyrmions of degree B which are related by a C_n^k symmetry the Finkelstein-Rubinstein phase is $\chi_{FR} = (-1)^N$ where*

$$N = B\,(nB - k)\,. \tag{33}$$

Proof:

The relevant Skyrme configuration corresponds to a regular n-gon of Skyrmions. The loop L which is induced by the C_n^k symmetry has two effects on this configuration. Each Skyrmion is rotated and isorotated by C_n^k. Furthermore, the Skyrmions are exchanged via the permutation $(12\ldots n)$, keeping the orientation in space and target space fixed. Therefore, the loop L can be divided into n rotation and isorotation loops for each Skyrmion and a permutation loop, which in turn can be split up into $n-1$ exchanges of two Skyrmions. The individual Skyrmions are given by rational maps so we can apply formula (11) for each rotation and isorotation. Furthermore, Finkelstein and Rubinstein have shown that an exchange of two Skyrmions of degree B is homotopic to a 2π rotation of a Skyrmion of degree B and we can again apply formula (11). Note that N in formula (11) is only well-defined modulo 2. The Finkelstein-Rubinstein phase for the loop L is then given by

$$\chi_{FR} = \left(\chi_{FR}(\text{rotation/isorotation})\right)^n \left(\chi_{FR}(\text{single exchange})\right)^{n-1}. \tag{34}$$

Using (11) this gives rise to

$$\begin{aligned} N &= nB\left(B/n - k/n\right) + (n-1)B^2, & (35)\\ &= nB^2 - Bk, & (36)\end{aligned}$$

which completes the proof. □

A more heuristic way of understanding the result of lemma 3.1 is the following. Consider a regular n-gon of Skyrmions of degree B which transform into each other under a C_n^k symmetry. Intuitively, this configuration can be deformed into a torus of degree nB under a homotopy which preserves the C_n^k symmetry. Then the Finkelstein-Rubinstein phase can be calculated with formula (11) and we obtain

$$\chi_{FR} = (-1)^{B(nB-k)}, \tag{37}$$

as above.

3.3 General configurations

Given a general configuration in the truncated rational map ansatz, which is symmetric under C_n^k, we split up the configuration into regular n-gons of Skyrmions which transform into each other and Skyrmions which are on the symmetry axes. Assume that there are l regular n-gons of Skyrmions with degree B_i for $i = 1, \ldots, l$ and m Skyrmions of degree $\tilde{B}_i$ for $i = 1, \ldots, m$ which are located on the symmetry axis. Then the Finkelstein-Rubinstein phase for this symmetry is given by

$$\chi_{FR} = (-1)^N \quad \text{where} \quad N = \sum_{i=1}^{l} B_i\left(nB_i - k\right) + \sum_{i=1}^{m} \tilde{B}_i\left(\tilde{B}_i - k\right)/n. \tag{38}$$

This formula follows by constructing a homotopy between the C_n^k symmetry loop and a product of loops for the individual groups of Skyrmions as in section 3.1.

4 How to calculate Finkelstein-Rubinstein constraints from numerical configurations

In this section, we describe how to calculate the Finkelstein-Rubinstein constraints for a Skyrme configuration which is only known numerically.

1. Calculate the minimal energy configuration and analyze its symmetry properties.

2. Confirm the symmetry by starting with a symmetric configuration as initial condition and relaxing to the same final configuration.[2]

3. For each generator of the symmetry group identify k for C_n^k.

4. Approximate the Skyrmion by a truncated rational map ansatz with the right symmetries.

5. Calculate the Finkelstein-Rubinstein constraints using formula (38).

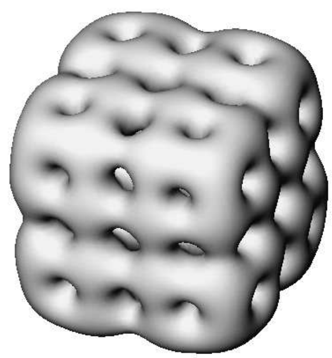

Figure 1: The $B = 32$ cube

In the following, we calculate the Finkelstein-Rubinstein constraints for the $B = 32$ cube, which is displayed in figure 1. This configuration is one of the first examples of a Skyrmion which cannot be described with the rational map ansatz [4]. The configuration has been calculated numerically in [4] (step 1). It can be approximated by a chunk of the Skyrmion crystal [23] and has cubic symmetry. Starting with a chunk of the crystal as initial conditions imposes the symmetries and corresponds to step 2. Step 3 is comparatively easy in this example since we have an analytic ansatz for the initial condition. The cubic symmetry is generated by a C_3 and a C_4 symmetry. The Skyrme field can be parametrized as $\sigma + i\tau_j\pi_j$ where τ_i are the Pauli matrices. With the choice of fields as in [4] the symmetries are

$$\begin{aligned} C_3 : &\quad (\sigma, \pi_1, \pi_2, \pi_3) \mapsto (\sigma, \pi_3, \pi_1, \pi_2), \\ C_4 : &\quad (\sigma, \pi_1, \pi_2, \pi_3) \mapsto (\sigma, \tfrac{-2\pi_1+\pi_2-2\pi_3}{3}, \tfrac{\pi_1-2\pi_2-2\pi_3}{3}, \tfrac{-2\pi_1-2\pi_2+\pi_3}{3}). \end{aligned}$$

So, the cubic symmetry is generated by C_3^1 and C_4^2. The $B = 32$ Skyrmion can be thought of as eight $B = 4$ cubes. Under the C_4^2 symmetry the top four cubes transform into each other and the bottom four cubes transform into each other

[2] This provides a homotopy from the initial condition to the final configuration which is invariant under the symmetry.

(step 4). So, we can use formula (38) with $B_i = 4$ and $l = 2$, $n = 4$ and $k = 2$ (step 5). There are no Skyrmions on the symmetry axis, $\tilde{B}_i = 0$.

$$\begin{aligned} N &= 2 * 4 * (4 * 4 - 2), \\ &= 112. \end{aligned}$$

Therefore, the Finkelstein-Rubinstein phase $\chi_{FR} = 1$ for this symmetry. Under the C_3^1 symmetry, two $B = 4$ Skyrmions are on the rotation axes and there are two groups of three $B = 4$ Skyrmions which transform into each other. Therefore, $B_i = 4$, $l = 2$, $n = 3$, $k = 1$, $\tilde{B} = 4$ and $m = 2$. So, formula (38) gives

$$\begin{aligned} N &= 2 * 4 * (4 * 3 - 1) + 2 * 4 * (4 - 1)/3, \\ &= 96. \end{aligned}$$

Therefore, the Finkelstein-Rubinstein phase is again trivial. In the following section, we show that we can derive some results from general principles, so that we only have to carry through steps $1 - 5$ for a very small subset of all the possible symmetries.

5 Symmetries and Finkelstein-Rubinstein constraints for even B

In this section we collect a set of general results about symmetries and Finkelstein-Rubinstein constraints. We start with the following lemma.

Lemma 5.1 *Negative Finkelstein-Rubinstein constraints cannot occur for C^k_{2l+1} for $l \geq 1$ if B is even.*

Proof: Applying a C^k_n symmetry n times corresponds to a 2π rotation in space followed by a $2\pi k$ rotation in target space. If B is even, then a 2π rotation (and also a 2π isorotation) is homotopic to the trivial loop in the Skyrme configuration space. In this case, the Finkelstein-Rubinstein constraints correspond to one-dimensional and hence irreducible representations of C^k_n which are obtained by mapping the generator of C^k_n to $(-1)^N$. This representation can be thought of as a homomorphism from $C^k_n \to \mathbb{Z}_2$ and therefore can only be nontrivial if n is even. □

In the following, we address the question of which symmetries can lead to negative Finkelstein-Rubinstein phases. Lemma 5.1 greatly simplifies the discussion, so we only consider even baryon number B. Furthermore, we restrict our attention to the symmetries which have been found empirically [2]. These are cyclic symmetry C_2, dihedral symmetry D_n for $n \leq 6$, the tetrahedral group T, the octahedral group O and the icosahedral group Y. Therefore, we first discuss the cyclic subgroups C_n^k for $n \leq 6$. For even baryon number B the following picture emerges for Skyrmions which are well-approximated by rational maps. Using formula (11) and lemma 5.1 the cyclic groups C_n^k can be grouped into three groups:

1. The following symmetries never lead to negative χ_{FR}, if $B \equiv 0 \mod 2$: $C_2^0,\ C_3^k,\ C_4^2,\ C_5^k,\ C_6^{2k}$.

2. The following symmetries give rise to negative χ_{FR} if $B \equiv 2 \mod 4$: $C_2^1,\ C_4^0,\ C_6^{2k+1}$.

3. The following symmetries give rise to negative χ_{FR} if $B \equiv 4 \mod 8$: $C_4^1,\ C_4^3$.

In order to understand D_n symmetry we need to examine under which conditions can there be an additional C_2 symmetry for a given realization of a C_n^k symmetry? Since 2π isorotations are always a symmetry we can restrict our attention to $j = 0$ and $j = 1$. The two different realizations of a C_n^k symmetry can be characterized by their zeros and poles at zero and infinity. A rational map of type (14) has a zero of multiplicity $k \mod n$ at $z = 0$ and a pole of multiplicity $k \mod n$ at $z = \infty$. A rational map of type (16) has a pole of multiplicity $n - k \mod n$ at $z = 0$, and a pole of order $k \mod n$ at $z = \infty$.

For our purpose, it is sufficient to discuss the case that the C_2 rotation axis is orthogonal to the C_n^k rotation axis. This generates the group D_n. For a C_2^1 symmetry it is important whether the C_2^1 isorotation axis is parallel to the C_n^k isorotation or orthogonal to it, and we will introduce the notation $(C_2^1)_\parallel$ and $(C_2^1)_\perp$. For C_2^0 such a distinction is not necessary. A C_2 symmetry around an axis orthogonal to the C_n^k symmetry axis maps $z = 0$ to $z = \infty$. In target space, $R = 0$ is mapped to $R = \infty$ if the C_2^1 axis is orthogonal to the C_n^k axis in target space, namely for $(C_2^1)_\perp$, but $R = 0$ and $R = \infty$ are invariant for $(C_2^1)_\parallel$ and for C_2^0. The numbers of zeros and poles imply that a rational map of type (14) cannot have an additional symmetry of type C_2^0 or $(C_2^1)_\parallel$. However, a $(C_2^1)_\perp$ symmetry is possible for all values of k. A rational map of type (16) can only have an additional C_2^0 or $(C_2^1)_\parallel$ symmetry if $n - k \equiv k \mod n$. An additional $(C_2^1)_\perp$ symmetry is never allowed.

Now, we can apply the above result to the case $B \equiv 0 \mod 4$. The above list shows that a negative Finkelstein-Rubinstein phase is only possible for C_4^1 and C_4^3 symmetry. A C_4^k symmetry empirically only occurs as a subgroup of a D_4 symmetry which itself might be a subgroup of the cubic group O. For $B \equiv 0 \mod 4$ the corresponding rational map can be of type (14). Then an additional C_2 symmetry is only possible for $4 - k \equiv k \mod 4$ which excludes $k = 1$ and $k = 3$. The rational map can also be of type (16) provided that $k \equiv 0 \mod 4$. Therefore, D_4 symmetry is not compatible with C_4^1 and C_4^3 so that no negative Finkelstein-Rubinstein constraints can occur for $B \equiv 0 \mod 4$.

B	symmetry	type	χ_{FR}
6	D_4	(14)	(-1)
10	D_4	(14)	(-1)
10^*	D_3	(14)	(-1)
14	D_2	(14) or (16)	(-1)
18	D_2	(14) or (16)	(-1)
22	D_5	(14)	(-1)
22^*	D_3	(14)	(-1)

Table 1: Symmetry, type and Finkelstein-Rubinstein phase for baryon number $B \equiv 2 \mod 4$.

Now, we discuss the case $B \equiv 2 \mod 4$. Negative χ_{FR} can occur for C_2^1, C_4^0 and C_6^{2k+1}. C_2^1 can occur as subgroup of D_n, T, O and Y. From the point of view of Finkelstein-Rubinstein constraints only the D_2 subgroup of T and Y contributes and similarly, only the D_4 subgroup of O contributes. Rational maps of type (14) always allow a $(C_2^1)_\perp$ symmetry which leads to $\chi_{FR} = -1$. For rational maps of type (16), the C_2 symmetry is either C_2^0 or $(C_2^1)_\parallel$, and only the latter leads to negative χ_{FR}. Note however, that a D_2 symmetry of type (16) is only possible if $k \equiv 1 \mod 2$, so that a D_2 symmetry always leads to negative χ_{FR}. In [15], the symmetries and Finkelstein-Rubinstein phases have been discussed for $B \leq 22$. The results for $B \equiv 2 \mod 4$ are displayed in table 1. Note that type (16) can only occur if $B \equiv 0 \mod n$ for the maximal C_n subgroup. All symmetries are either D_2 or are of type (14), so that for $B \equiv 2 \mod 4$ only symmetries with negative Finkelstein-Rubinstein phases have been observed. However, from symmetry arguments alone, it is not possible to exclude that the symmetries act in such a way that all the phases are positive.

5.1 Physical interpretation of the symmetry calculations

The physical interpretation of this result is as follows. In nuclear physics we are interested in the quantum ground states for a given number of nucleons. In our semi-classical approximations the ground state is given by the lowest values of the angular momentum quantum numbers J and I which are compatible with the symmetries of the classical configurations. In particular, the wave function ψ has to satisfy the symmetry condition (9) for all classical symmetries of the given Skyrmion. We decompose the wave function ψ in angular momentum eigenfunction and write $\psi = |J\rangle|I\rangle$ for a wave function with angular momentum quantum numbers J and I.

Particularly interesting is the even-even situation when there is an even number of protons and an even number of neutrons. For small nuclei the number of protons and neutrons are equal, so that even-even nuclei have $B \equiv 0 \mod 4$. In this case, experiment shows that the ground state is generally given by $|0\rangle|0\rangle$. This is only possible if the Finkelstein-Rubinstein constraints are trivial. The above discussion showed that the $|0\rangle|0\rangle$ ground state is allowed for $B \equiv 0 \mod 4$, in agreement with experiment. In this calculation we assumed that the relevant Skyrmions are well-approximated by rational maps, and that C_4 symmetry only occurs as a subgroup of a D_4 symmetry (which might itself be a subgroup of an octahedral symmetry).

For higher pion mass the configurations deviate significantly from the rational map ansatz [21], which raises the question whether it is possible to classify symmetries allowing for the more general truncated rational map ansatz. The Skyrmions would be allowed to split into groups and the Finkelstein-Rubinstein constraints have to be calculated with formula (38). Note that if Skyrmions can be thought of as being composed only of $B = 4$ Skyrmions (α particles) then formula (38) and the above symmetry discussion also imply that there are no negative Finkelstein-Rubinstein phases, so that the ground state for such Skyrmions is $|0\rangle|0\rangle$.

For odd-odd nuclei, which implies $B \equiv 2 \mod 4$ for equal number of protons and neutrons, experiment shows that the ground state is usually not given by $|0\rangle|0\rangle$. This is consistent with our observation that negative Finkelstein-Rubinstein phases occur. However, symmetry arguments alone are not sufficient to prove the occurrence of negative Finkelstein-Rubinstein phases. For odd baryon number, lemma 5.1 cannot be applied and there is little hope of finding simple rules.

6 Conclusion

In this paper we discussed how to calculate Finkelstein-Rubinstein constraints for configurations which are only known numerically. This is an important problem since recent calculations show that for large pion mass Skyrmions are not very well described by rational maps [21]. Moreover, there is mounting evidence that the pion mass in the Skyrme model should be interpreted as an effective mass with a value at least twice the physical value [24]. The calculation of the Finkelstein-Rubinstein constraints can be incorporated into the algorithm for finding the minimal energy configurations with only minor modifications.

Note that for a wide range of values of the pion mass, the rational map ansatz works well for small enough baryon number B. Therefore, the truncated rational map ansatz in its current form has a good chance of capturing all the physically relevant Skyrmions. An obvious but slightly tedious generalisation would be to allow for the constituent Skyrmions to be themselves approximated by a truncated rational map ansatz.

The paper also discussed which symmetries occur for rational map Skyrmions and when to expect negative Finkelstein-Rubinstein constraints for even baryon numbers. In particular, the Skyrme model calculations suggest the correct phenomenological trend, namely that the ground state of even-even nuclei have the ground state $|0\rangle|0\rangle$.

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