INSURANCE LOSS COVERAGE UNDER RESTRICTED RISK CLASSIFICATION

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Abstract

Insurers hope to make profit through pooling policies from a large number of individuals. Unless the risk in question is similar for all potential customers, an insurer is exposed to the possibility of adverse selection by attracting only high-risk individuals. To counter this, insurers have traditionally employed underwriting principles, identifying suitable risk factors to subdivide their potential customers into homogeneous risk groups, based on which risk-related premiums can be charged.

In reality, however, insurers may not have all the information reflecting individuals' risks due to information asymmetry or restrictions on using certain risk factors by regulators. In either case, conventional wisdom suggests that the absence of risk classification in an insurance market is likely to lead to a vicious circle: increasing premium with the prime aim to recover losses from over-subscription by high risks would lead to more low risks dropping out of the market; and eventually leading to a collapse of the whole insurance system, i.e. an adverse selection spiral. However, this concept is difficult to reconcile with the successful operation of many insurance markets, even in the presence of some restrictions on risk classification by regulators.

Theoretical analysis of polices under asymmetric information began in the 1960s and 1970s (Arrow (1963), Pauly (1974), Rothschild & Stiglitz (1976)), by which time the adverse consequences of information asymmetry had al-

ready been widely accepted. However, empirical test results of its presence are mixed and varied by markets.

Arguably from society's viewpoint, the high risks are those who most need insurance. That is, if the social purpose of insurance is to compensate the population's losses, then insuring high risks contributes more to this purpose than insuring low risks. In this case, restriction on risk classification may be reasonable, otherwise premium for high risks would be too high to be affordable. Thus, the traditional insurers' risk classification practices might be considered as contrary to this social purpose.

To highlight this issue, *"loss coverage"* was introduced in Thomas (2008) as *the expected population losses compensated by insurance.* A higher loss coverage indicates that a higher proportion of the population's expected losses can be compensated by insurance. This might be a good result for the population as a whole. A corollary of this concept is that, from a public policy perspective, adverse selection might not always be a bad thing. The author argued that a moderate degree of adverse selection could be negated by the positive influence of loss coverage. Therefore, when analysing the impact of restricting insurance risk classification, loss coverage might be a better measure than adverse selection.

In this thesis, we model the outcome in an insurance market where a pooled premium is charged as a result of an absence of risk-classification. The outcome is characterised by four quantities: equilibrium premium, adverse selection, loss coverage and social welfare. Social welfare is defined as the total expected utility of individuals (including those who buy insurance and those who do not buy insurance) at a given premium. Using a general family of demand functions (of which iso-elastic demand and negative-exponential demand are examples) with a non-decreasing demand elasticity function with respect to premium, we first analyse the case when low and high risk-groups have the same constant demand elasticity. Then we generalise the results to the case where demand elasticities could be different.

In general, equilibrium premium and adverse selection increase monotonically with demand elasticity, but loss coverage first increases and then decreases. The results are consistent with the ideas proposed by Thomas (2008, 2009) that loss coverage will be increased if a moderate degree of adverse selection is tolerated. Furthermore, we are able to show that, for iso-elastic demand with equal demand elasticities for high and low risks, social welfare moves in the same direction as loss coverage, i.e. social welfare at pooled premium is higher than at risk-differentiated premiums, when demand elasticity is less than 1. Therefore, we argue that loss coverage may be a better measure than adverse selection to quantify the impact of risk classification scheme being restricted. Moreover, (observable) loss coverage could also be a useful proxy for social welfare, which depends on unobservable utility functions. Therefore, adverse election is not always a bad thing, if demand elasticity is sufficiently low.

The research findings should add to the wider public policy debate on these issues and provide necessary research insights for informed decision making by actuaries, regulators, policyholders, insurers, policy-makers, capital providers and other stakeholders.

Contents

Motivating Examples

Many studies have been done over the past decades analysing the existence and the level of severity of adverse selection. However, this debate is still inconclusive. To make a contribution to the discussion, we introduce an alternative approach–*"loss coverage"*, i.e. the proportion of the expected population losses which can be covered by insurance, to measure the impact of risk classification scheme being restricted. A more rigorous introduction to loss coverage is given in Chapter 5.

In this chapter, we will briefly explain our motivation using a series of heuristic examples. We will show that loss coverage can increase under restricted risk classification by contrasting two alternative risk classification schemes: the *full risk classification* (under which actuarially fair premiums are charged) and the *no risk classification* (under which pooled premiums are charged).

Assume that in a population of 1000 risks, 16 losses are expected every year. There are two risk-groups. Each individual in the high risk-group of 200 individuals has a probability of loss four times higher than that of an individual in the low risk-group. All losses are assumed to be of unit size, and insurance coverage, if purchased, are also of unit size. We further assume that probability of loss is not altered by the purchase of insurance, i.e. there is no moral hazard. An individual's risk-group is fully observable to insurers and all insurers are required to use the same risk classification scheme. The equilibrium premium of insurance is determined as the premium at which insurers make zero expected profit.

Adverse selection in this thesis is defined as the ratio between *expected claim per policy* and *expected loss per risk*, which is similar to its definition in economic literature as the correlation between risk experience and insurance coverage. So when this ratio is greater than 1, we say there is adverse selection. Since at equilibrium the insurer makes neither profits nor losses, and competition between insurers in risk classification is not permitted, adverse selection does not imply insurer losses. A more rigorous definition of adverse selection is given in Chapter 4.

In the first scenario where insurers use the *full risk classification*, they charge actuarially fair premiums to members of each risk-group. We assume that the proportion of each risk-group which buys insurance under these conditions, i.e. the 'fair-premium demand', is 50%, in line with industry statistics[1]. Table 1 shows the outcome, which can be summarised as follows:

There is no adverse selection because premium charged for each group equals their corresponding level of risk. This is interpreted as the ratio of expected claim per policy (8/500) and expected loss per risk (0.016) being 1. The differentiated premiums in this example are also the break-even premiums which ensure insurers have a zero expected profit. As a result, half (8/16) of the population's expected losses can be compensated by insurance in this case. In this case, we say that the loss coverage is 0.5.

[1]Please see Footnote 2 on page 30.

Table 1: Full risk classification with no adverse selection.

	Low risk-group	High risk-group	Aggregate
Risk	0.01	0.04	0.016
Total population	800	200	1000
Expected population losses	8	8	16
Break-even premiums (**differentiated**)	0.01	0.04	0.016
Expected numbers insured	400	100	500
Expected insured losses	4	4	8
Loss coverage			0.5
Adverse selection			1

In the second scenario, suppose there is a restriction imposed on using risk classification scheme. Consequently, all insurers have to charge the same premium for both risk-groups. Table 2 shows one of the possible outcomes, which can be summarised as follows:

Table 2: No risk classification leading to adverse selection but higher loss coverage.

	Low risk-group	High risk-group	Aggregate
Risk	0.01	0.04	0.016
Total population	800	200	1000
Expected population losses	8	8	16
Break-even premiums (**pooled**)	0.02	0.02	0.02
Expected numbers insured	300	150	450
Expected insured losses	3	6	9
Loss coverage			0.5625
Adverse selection			1.25

The pooled premium of 0.02 at which insurers achieve zero expected profits in this example is calculated as the demand-weighted average of the risk premium: (300 x 0.01 + 150 x 0.04)/450=0.02. This pooled premium will be higher than the actuarially fair premium for the low risk-group as shown in Table 1 (0.02, compared with 0.01 before). So fewer of the low risks are expected to buy insurance (300, compared with 400 before). The pooled premium is cheaper for the high risks (0.02, compared with 0.04 before), so more of them are expected to buy insurance (150, compared with 100 before). In this example, because there are 4 times as many low risks as high risks in the population (800 compared with 200), and the reduction in the expected number of insured in the low risk-group (100) is greater than the increase in the expected number of insured in the high risk-group (50), the total expected number of insured reduces slightly (450, compared with 500 before). There is some adverse selection because the ratio of expected claim per policy (9/450) and expected loss per risk (0.016) is 1.25, which is greater than 1. This can also be explained by the fact that pooling premium (0.02) is higher than the population-weighted average risk (0.016), and the expected number of insured falls.

The loss coverage (0.5625), is calculated as the ratio of expected insured losses (9) to the expected population losses (16). This value is higher than the loss coverage in the case when there is no restriction on risk classification (0.5, as shown in Table 1), which indicates a higher proportion of the population's expected losses is being compensated by insurance. This is because the shift in coverage towards high risks more than outweighs the fall in the number of policies sold.

We now consider a third scenario in which the pooled premium is higher than that in the second scenario. In this time, as a result, there is heavier adverse selection. This scenario is illustrated in Table 3.

In this scenario, the pooled premium of 0.02154, is calculated as the demand-weighted average of the risk premiums: (200 x 0.01 + 125 x 0.04)/325

Table 3: No risk classification leading to adverse selection and lower loss coverage.

	Low risk-group	High risk-group	Aggregate
Risk	0.01	0.04	0.016
Total population	800	200	1000
Expected population losses	8	8	16
Break-even premiums (**pooled**)	0.02154	0.02154	0.02154
Expected numbers insured	200	125	325
Expected insured losses	2	5	7
Loss coverage			0.4375
Adverse selection			1.35

= 0.02154, which is higher than the pooled premium in the previous scenario (0.02, in Table 2). There is heavier adverse selection because the ratio of expected claim per policy (7/325) and expected loss per risk (0.016) is 1.35, which is greater greater than 1.25 as in the second scenario. This can also be explained by the fact that pooling premium (0.02154) is much higher than the population-weighted average risk (0.016), and there is a larger reduction in the total expected number insured (325, compared with 450 in the second scenario). The resulting loss coverage of 0.4375 (expected insured loss/expected population losses=7/16) is lower than 0.5 as observed in the first scenario (in Table 1). This indicates that a lower proportion of population's expected losses are compensated by insurance because the shift in coverage towards high risks is insufficient to outweigh the expected fall in the number of policies sold.

The above three examples show that charging risk differentiated premiums does not lead to adverse selection (Table 1). Charging a pooled premium leads to adverse selection, but can either increase or decrease loss coverage. A moderate degree of adverse selection can benefit the society by increasing the proportion of the population's expected losses that are compensated by insurance, i.e. higher loss coverage (Table 2). However, heavier adverse selection could reduce loss coverage and disadvantage the society (Table 3).

Note that these examples are neither derived from a particular demand function, nor rely on unrealistic assumptions. This indicates that we may be able to generalise some of these features. So in this thesis, we will develop a suitable framework to analyse these features.

Introduction

In this thesis, we mainly focus on *informational adverse selection*, i.e. adverse selection to insurers as a result of customers possessing more information about their own risks than the insurers. This information can be related to their probability of incurring losses and/or amounts of losses, which insurance buyers can withhold for their benefit. Many studies on adverse selection have been based on this asymmetry in information between insurers and their customers. There are also other interpretation of adverse selection, such as *competitive adverse selection*, which is adverse selection as a result of competition between insurers. Different insurers using different risk classification schemes to classify their potential customers is an example. In the context of our research, we assume all insurers offer the same insurance product and apply the same risk classification scheme, i.e. there is no competition between insurers in terms of risk classification practices. Therefore, we only look at potential adverse selection because of information asymmetry between insurers and their customers, i.e. informational adverse selection. In the rest of this thesis, I will refer to this simply as "adverse selection".

The initial "adverse selection" concept can be traced back to 1706 when Charles Povey set up the Trader's Exchange in London. This mutual life insurance society collected no information at all on the lives insured. For a variety of reasons, including a tendency to attract only high risk lives, this mutual life insurance society was dissolved a few years later. This illustrates some potential problems that can arise from asymmetric information. As a result, Mr Povey introduced the idea of *"giving a reasonable account of the health of the person whose life they intend to subscribe upon"* (Dickson (1960)).

This idea of "adverse selection" is also potentially present in other lines of insurance. The following statement appears in an insurance textbook written at the Wharton School (Akerlof (1970)):

"There is potential adverse selection in the fact that healthy term insurance policyholders may decide to terminate their coverage when they become older and premiums mount. This action could leave an insurer with an undue proportion of above average risks and claims might be higher than anticipated. Adverse selection appears (or at least is possible) whenever the individual or group insured has freedom to buy or not to buy, to choose the amount or plan of insurance, and to persist or to discontinue as a policyholder."

Due to the failure of Trader's Exchange, by 1725, other life insurance offices started to classify risks, i.e. identify high-risk applicants and charge differential premium rates. In other words, underwriting appeared as a necessary step in writing insurance policies.

In a competitive insurance market, if insurers can accurately classify individuals' risks and charge premiums accordingly, the insurers' expected profit on each policy is (approximately) zero, i.e. there is no adverse selection from individuals as the premiums fully reflect their risks. This can also be expressed as an "actuarially fair system", as suggested by actuaries in American Academy of Actuaries Committee (1980):

"Differences in prices among classes reflect differences in expected costs

with no intended redistribution or subsidy among classes."

Over the past 300 years, insurers have been collecting and pooling data in order to identify and assess risks. However, risk classification has often attracted controversy for a number of reasons, some of which are discussed below:

1. The general public might consider risk classification to be unfair, particularly when risk classification is based on factors which are out of the control of the individuals (Chuffart (1995)).

   Skipper & Black (2000) cited figures indicating 67% of people believe it is fair to charge higher premiums to those who smoke but only 14% believe it is fair to charge higher premiums to those with a predisposition to cancer. There have also been several fierce debates about risk classification systems which discriminate against people who are already disadvantaged (Hellman (1997), Wright *et al.* (2002), Leigh (1996)).

   Differentiation on gender is one example. In the Manhart case in the USA (Doeer (1984)), the court decided that gender-based pension fund rules were a contravention of the Equal Employment Opportunity Act. This argument was used by the advocates of unisex pension benefits in the UK (Curry & O'Connell (2004)) and it has also influenced the application of European Gender Directive. Test-Achats, the Belgian Consumer Association, brought a case arguing that Article 5(2) of the Gender Directive 2004/113/EC[2] was contrary to the principle of gender equality in Primary Community Law. As a result, from 21st December 2012, The European Court of Justice (ECJ)'s gender ruling came into

[2]This Article allows member states to permit differentiated premium rating and benefit payment based on gender.

force, which decided that insurers would no longer be allowed to charge gender specific insurance premiums.

2. Some questions asked in the underwriting process can be considered to be violating privacy, for example, questions on sexual preferences and practices for AIDS investigation; and the use of genetic test data in insurance (Harper (1992)).

3. There might also be undesirable consequences of risk classification. For example, in the UK in the 1980s, some doctors began to warn patients against taking HIV test because it might damage their chances of getting insurance (even if the result turned out to be negative). This approach was having negative public health consequences, as it discouraged people from taking HIV tests, and hence missing out opportunities on vital medical treatment (Leigh (1996), Hopegood (2000), Worsfold (1991), Hall (1991), Papworth (1991), Jones (1991), Sullivan (1991), Harris (1994)). A similar debate has affected genetic testing as well (Lapham *et al.* (1996)).

There are also situations where insurers voluntarily do not charge risk-differentiated premiums. This would happen when insurers do not make the full use of information,

1. even if it is publicly available to them, e.g. the place of residence, as such information might not be quantitatively important in improving the prediction of loss outcomes. Another reason is that the credibility of such characteristics may be limited by the extent to which such characteristics are subject to change in response to characteristics-based pricing differentials (Finkelstein & Poterba (2006)).

2. Information may be inaccessible to both individuals and insurers without incurring some cost e.g. hesitation in obtaining genetic information due to privacy and cost concerns (De Jong & Ferris (2006)).

When insurers cannot charge risk-differentiated premiums, they will charge a pooled premium calculated as a population-weighted average risk premiums which will be higher for the low-risk individuals, but lower for the high-risk individuals. This will create a cross-subsidy of funds from low risks to high risks.

As a result, more insurance is bought by higher risks and less insurance is bought by lower risks. Thus insurers end up with more high risks (in percentage terms) than implied by the pooled premium, i.e. the marginal premium cannot cover marginal costs which would give rise to adverse selection.

Initially, losses from adverse selection can be met by charging higher premiums. However, problems can arise if different risk groups react differently to premium rate changes. Normally we would expect that as price rises, demand will fall. If the demand for insurance falls uniformly across all risk groups, then the average premium rate for the group will remain stable. However, if low-risk individuals are relatively more sensitive to premium increases, then any change in premium rates might lead to an increase in adverse selection. Then the previously-calculated extra premium would be inadequate to cover expected claims. A further increase in premium rate is possible, followed by an adverse selection spiral and a breakdown of the whole insurance system (De Jong & Ferris (2006)).

Also, in most markets the number of higher risks is smaller than the number of lower risks, so the total number of risks insured falls. The usual efficiency argument focuses on this reduction in coverage, e.g. *"This reduced pool of insured individuals reflects a decrease in the efficiency of the insurance* *market"* (Dionne & Rothschild (2014)).

However, it is possible that the risk of an adverse selection spiral has been overstated. This spiral is only likely to occur under specific circumstances, e.g. when the elasticity of demand for insurance products is high. In practice, the elimination of rating factors does not always lead to disruption of the insurance market. In some cases, the elimination of a rating factor might only lead to a relatively small increase in premium rates, which can be absorbed by the market (De Jong & Ferris (2006)). Moreover, empirical test results also show that adverse selection may appear in some markets but not in others. We will discuss some empirical test results in Chapter 1.

In this thesis, we suggest that in some circumstances, there is a counter-argument to this perception of reduced efficiency. The rise in equilibrium price under pooling reflects a shift in coverage towards higher risks. If the shift in coverage is large enough, it can more than outweigh the fall in numbers insured. In these circumstances, despite fewer risks being insured under pooling, expected losses compensated by insurance – a quantity we term "loss coverage", as introduced in Thomas (2008) – can be higher. More risks being voluntarily transferred and more losses being compensated, should not be regarded as inefficient.

In Thomas (2008), using examples of two risk-groups, i.e. a low risk-group and a high risk-group, with an insurance demand function inspired by De Jong & Ferris (2006), along with realistic values for price elasticity of demand, the author demonstrated that, from a public policy perspective, adverse selection as a result of restrictions on risk classification might not always be adverse. Some degree of adverse selection may be desirable in some insurance markets, when switch in insurance coverage towards high risks, i.e. those who most need insurance, outweighs the reductions in the number of insurance sold. The author suggested the idea of loss coverage as a possible criterion for a desirable level of adverse selection. In particular, there is an increase in loss coverage if the price elasticity of demand in the low risk group is sufficiently low, compared to that of the high risk-group.

Thomas (2009) extended the model in Thomas (2008) by exploring the conditions that lead to multiple solutions to the pooled equilibrium premium and the collapse in insurance coverage, when there are restrictions on risk classification. The author found that the pooled equilibrium is related to a parameter for the elasticity of demand for insurance at an actuarially fair premium, i.e. the "fair-premium demand elasticity". The demand elasticity parameters which are required to generate a collapse in coverage in the model are higher than those values of demand elasticity which have been estimated in many empirical studies. This finding offered some explanation to why a collapse in insurance coverage has not been detected in some insurance markets under restricted risk classification.

The above loss coverage literature contrasts with economic literature on insurance risk classification, as summarised in surveys such as Hoy (2006), Einav & Finkelstein (2011) and Dionne & Rothschild (2014). Economic literature typically takes a utility-based approach: representative agents from each risk-group make purchasing decisions which maximise their expected utilities, and the outcomes of different risk classification schemes are then evaluated by a social welfare function which is a (possibly weighted) sum of expected utilities over the whole population. For example Hoy (2006) uses a utilitarian social welfare function which assigns equal weights to the utilities of all individuals. Einav & Finkelstein (2011) use a deadweight-loss concept which appears equivalent to a social welfare function with utilities cardinalized so as to weight willingness-to-pay equally across all individuals.

Our model in this thesis connects the previous loss coverage literature (Thomas (2008, 2009)) with the utility-based approach found in the economic literature (Hoy (2006), Einav & Finkelstein (2011) and Dionne & Rothschild (2014)) on insurance risk classification in three ways:

Firstly, we provide a utility-based micro-foundation for the proportional insurance demand function used in the loss coverage literature, driven by variations between individuals in their utility functions, which can explain why only a proportion of the individuals in each risk-group buy insurance at each price.

Secondly, we will define the concept of loss coverage rigorously within the utility-based framework.

Thirdly, we reconcile loss coverage to the utilitarian concept of social welfare used in the economic literature. We will show that, under certain conditions (e.g. using iso-elastic demand), if insurance premiums are small relative to wealth, maximising loss coverage maximises social welfare.

The rest of this thesis is organised as follows:

In Chapter 1, we summarise some of the existing approaches in modelling adverse selection and empirical evidence in detecting its existence in different insurance markets. In the cases when adverse selection have not been detected, we provide a brief discussion on some of the possible reasons.

In Chapter 2, we set up the model for demand for insurance along with the underlying utility-based micro-foundation.

In Chapter 3, using certain demand functions, equilibrium in the market is analysed for different risk classification schemes.

In Chapter 4, we look at adverse selection that may arise due to restrictions on the use of risk classification.

In Chapter 5, we introduce loss coverage, and analyse its features when the insurance market is in equilibrium using various demand functions under different risk classification schemes.

In Chapter 6, loss coverage is then linked to social welfare, and we present results that connect these two measures.

In Chapter 7, we discuss conditions under which loss coverage and social welfare are maximised under different risk classification schemes.

In Chapter 8, we summarise our findings in this thesis.

Chapter 1

Literature Review on Adverse Selection

In this chapter, we review some of the approaches in modelling (informational) adverse selection in literatures along with some empirical evidence on the presence of adverse selection.

1.1 Approaches to Model Adverse Selection

This section reviews some of the approaches used to model adverse selection. Since 1970s, many studies have been carried out on testing the existence of adverse selection and its implications in various insurance markets. We summarise these into two main categories: economic models and actuarial models, and we analyse them in the context of our research.

1.1.1 Economic Models

Models of adverse selection in economics tend to take a utility-based approach (e.g. Akerlof (1970), Wilson (1977, 1979), Rothschild & Stiglitz (1976)). Individuals are assumed to make decisions which maximise their expected utilities according to some assumed utility functions.

Akerlof (1970) built a model in the auto-mobile market, where there were four types of cars (good/bad/new/used) being traded. Individuals' demand is assumed to depend on two variables: price and quality of used cars. A linear utility function was proposed:

$$\begin{aligned} \text{Utility} &= \text{consumption of goods other than automobiles} \\ &+ \text{qualities of all automobiles,} \end{aligned}$$

assuming addition of an extra car adds the same amount of utility as the first. The author argued that because there is asymmetric information on the condition of cars between buyers and sellers, i.e. buyers could not decipher the difference but sellers could, the uninformed buyers' price would create an adverse selection problem that drives the high-quality cars out of the market.

Wilson (1979) extended Akerlof (1970) model by taking into account the marginal rate of substitution of used car quality for consumption when analysing utility.

Although these models were simple and were focused on markets in general, their appearance drew attention from researchers on the possible consequences of adverse selection because of asymmetric information.

The consequences in insurance market due to adverse selection have also been studied in the esteemed Rothschild & Stiglitz (1976), in which the authors proved that in a competitive insurance market, there cannot be a pooling equilibrium when only insurance buyers know their probabilities of loss.

The Rothschild-Stigltiz model assumes individuals belong to two risk-groups (a high risk-group and a low risk-group) where individuals know their risk status, and each individual faces a possible loss of the same amount. All individuals are assumed to be risk-averse and having the same utility function. Insurers are assumed to only know the proportions of the population being low risks and high risks, but they are unable to allocate individuals into different risk-groups. Moreover, each individual can only buy one contract from one insurer, i.e. insurance contracts are assumed to be exclusive.

The authors looked at price and quantity equilibrium in this study, i.e. a particular amount of insurance that an individual can buy at a given premium. If insurers charge a "pooling equilibrium premium", i.e. a demand-weighted average premium of the low and high risk-groups to both risk-groups, the market might not be stable if one insurer tries to attract only the low risks by offering a new contract with full coverage at a premium closed to the fair premium of the low risks. This is because the high risks would also prefer this new contract as it offers the full coverage at a lower price than the pooling equilibrium. Thus the presence of high risks 'distort' the pooling equilibrium in a way that both risk-groups would purchase this new contract which is offered at a premium that is insufficient to cover the amount of risks the insurer attracts.

In response to this instability, the authors suggested that insurers can offer products with different deductibles, e.g. high deductibles for low risks and low deductibles for high risks, so that the products can be designed in a way to separate the market by attracting their targeted risk-groups. Specifically, if high risks value insurance more than low risks, and so are more willing to 'pay more' for full coverage (or zero deductibles), individuals from both risk groups should buy their 'tailored' products accordingly. However, individuals from the low risk groups cannot buy a product with zero deductibles at their low-risk price, therefore, they can only be partially insured, while the high risks can be fully covered. This 'rationing' of cover is considered to be an inefficiency as a result of adverse selection. Moreover, this "separating equilibrium" might not exist as well if the proportion of high risks in the market falls below some critical level.

In conclusion, the authors referred to the existence of high risks when there is asymmetric information as *"a negative externality on the low risks"*. Thus, only when the high risks reveal their high probability of loss, both risk-groups can insure at separate contracts at fair premiums and full coverage.

Wilson (1977) generalised the Rothschild & Stiglitz (1976) model by suggesting that the *"no stationary equilibrium"* conclusion was due to insurers' *"static expectations"* about the response of other firms to changes in its own policy offer, i.e. every insurer expects other insurers will not change their policy offers in response to any changes which it may make in its own policy offer. However, responses from other insurers in the market might alter the market demand so that the profits to an insurer can change from non-negative to strictly negative. In response to this argument, the author then introduced an alternative equilibrium concept which is built upon a *"non-static expectation rule"*. This new rule says that insurers will adjust their anticipation about responses from other insurers by assuming that any policy that becomes unprofitable as a consequence of any changes in its own offer will be immediately withdrawn from the market. So insurers only offer policies which will earn non-negative profits after other firms have responded by withdrawing their unprofitable policies. Under certain assumptions, this new equilibrium which allows cross-subsidization between low and high risks is proved to exist, and there might be multiple equilibria.

Our model of insurance markets in this thesis differs from the canonical model of Rothschild & Stiglitz (1976) in two main ways.

First, in our model insurers compete only on price; they do not induce separation of risk-groups by menus of contracts offering different levels of cover priced at different rates. In this respect, our model is more in the spirit of Akerlof (1970). We justify this approach by noting that some important markets, such as life insurance, have non-exclusive contracting, and so separation via contract menus is not feasible. Furthermore, as far as we are aware, the concept of separation via contract menus is also not salient to practitioners in other markets where restrictions on risk classification apply, for example auto insurance in the European Union.[1]

Second, in our model individuals with identical probabilities of loss can have different utility functions, and so unlike the representative individuals from each risk-group in Rothschild-Stiglitz type models, they do not all make the same purchasing decision. This leads to an equilibrium where not all individuals are insured; this corresponds to the empirical reality of most voluntary insurance markets.[2]

[1]As regards life insurance, Rothschild-Stiglitz type models are inconsistent in important ways with empirical data (e.g. Cawley & Philipson (1999a)). As regards practice in other insurance markets, most recent actuarial pricing textbooks make no reference to the concept of menus of contracts as screening devices (e.g. Gray & Pitts (2012), Friedland (2013), Parodi (2014)). Other textbooks specifically recommend *against* using the level of deductible as a pricing factor (e.g. Ohlsson & Johansson (2010)).

[2]For example, in life insurance, the Life Insurance Market Research Organisation (LIMRA) states that 44% of US households have some individual life insurance (LIMRA (2013)). The American Council of Life Insurers states that 144m individual policies were in force in 2013 (American Council of Life Insurers (2014, p72)); the US adult population (aged 18 years and over) at 1 July 2013 as estimated by the US Census Bureau was 244m. In health insurance, only 14.6% of the US population has individually purchased private cover (US Census Bureau, 2015), albeit substantially more have employer group cover or Medicare or Medicaid government cover.

1.1.2 Actuarial Models

There are two main approaches in the actuarial literature. One approach, "Markov Model", initiated by Macdonald (1997), uses Markov processes (discrete and continuous) to model individual's transitions between states. The main idea is to assume a high degree of adverse selection in the sense that a small proportion of the population acquires private information, e.g. genetic test result, indicating much higher risks and this can lead to much higher transition intensities into the insured state, or a tendency to buy much higher amount of insurance. The effect of this adverse selection is then measured by the increase in pooled insurance price compared with the price if the private information did not exist.

The other approach, "Demand Model", is to assume an insurance demand function and investigate what happens under different demand elasticities for lower and higher risks. This approach was introduced by De Jong & Ferris (2006).

We look at each of these approaches in turn.

Markov Model

In Macdonald (1997), the author used a multiple-state Markov model which models the history of a single life in the context of genetic test results and insurance purchases, who is assumed to start at age x, from an original state with neither genetic test taken nor life insurance purchased. The individual can move between states (get tested and/or purchase insurance) with probabilities determined by the transition intensities.

A population is then divided into M subgroups, within each of which everyone has the same mortality as a proportion of the average. Insurers are assumed to be able to estimate the number of individuals in each subgroup but cannot identify which subgroup an individual belongs to. By observing transitions into tested and/or insured state, the expected present value of future benefit payments (or cost of adverse selection as a proportion of benefit payments without adverse selection) is calculated. Figure 1.1 shows the basic structure in this case with μ_{x+t}^{ijk} as the transition intensity of a life from subgroup i, in state j at time age $x+t$, moves to state k.

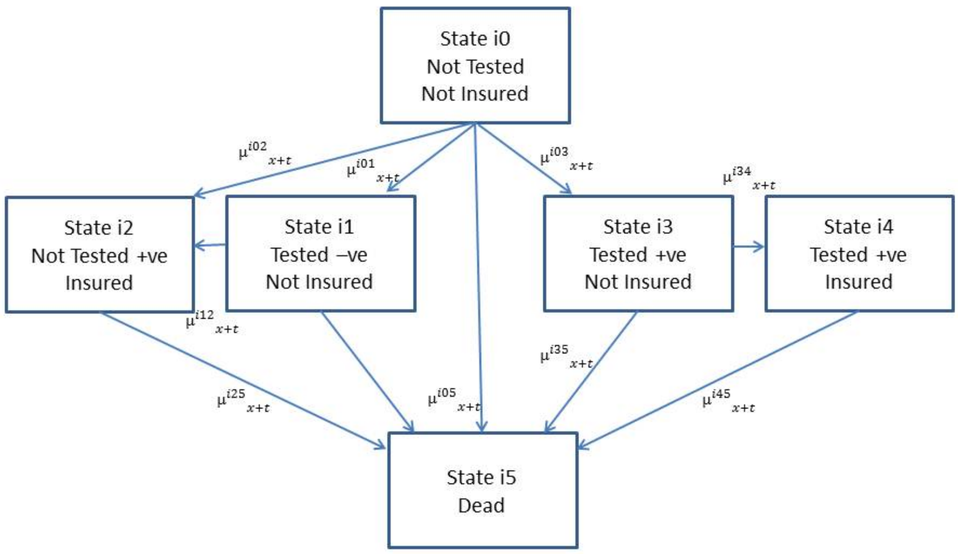

Figure 1.1: A Markov model for the ith of M subgroups from Macdonald (1997)

Using this method, potential costs of adverse selection in life insurance market due to restriction on using results from genetic tests are analysed. The author found that if life insurers could not use any result from genetic test in underwriting, additional mortality cost will be increased. However, the increase is likely to be moderate as long as adverse selection does not lead to large sums assured. To circumvent this, an imposition of upper limit on the sum assured is suggested.

Later on, the model was extended to long-term care (LTC) insurance market. In Macdonald & Pritchard (2001), using the model of Alzheimer's disease (AD) introduced in Macdonald & Pritchard (2000), the authors analysed adverse selection in LTC insurance through DNA tests for variants of the ApoE gene, the $\epsilon 4$ allele, which is an indicator of early onset of AD. By computing the expected present value of LTC benefits with respect to AD with and without adverse selection, the authors concluded that the cost of adverse selection is insignificant unless certain conditions are met.

This Markov Model approach was also extended to Critical Illness (CI) insurance market in Macdonald *et al.* (2003a) and Macdonald *et al.* (2003b). The authors focused on genes related to breast cancer (BC) and ovarian cancer (OC). Using UK population data, the probabilities that an applicant for insurance has a BRCA1 or BRCA2 mutation, i.e. the main causes of these diseases, are estimated through a series of applications of Bayes' Theorem. The costs of CI insurance in the presence of a family history of BC and OC are then estimated using a continuous-time, discrete-state model called *"This Applicant's Model"*. The cost of adverse selection is measured by any premium increase needed to absorb the potential loss. Adverse selection was found to be significant in small CI insurance markets, if the penetrance was higher than that observed in high-risk families, or if higher than average sum assured could be obtained.

Demand Model

Another type of model in the actuarial literature is to model insurance demand from lower and higher risk-groups as a function of the pooled price and investigate insurance market outcome under different demand elasticities. This approach was used in De Jong & Ferris (2006), and our research in this thesis is based on this approach.

In De Jong & Ferris (2006), to make the analysis simple, the authors ignore relevant expenses, capital costs and the insurers' profit loadings, etc. The basic model is:

- Insurers classify risks using a classification scheme g.
- An individual with risk g buys r units of insurance with $r_g = E(r|g)$.
- X is the claim cost per unit of insurance, with $\mu_g = E(X|g)$ where μ_g is the premium per unit sum insured.

Then, if insurers can fully use risk classification scheme and charge premium accordingly, insurers' expected profit should be

$$E(rX - r\mu_g|g) = 0,$$

with assumption that: given classified risk g, the units of insurance purchased r is independent to claim cost per unit of insurance X.

If insurers cannot charge premium based on risk g, instead, they use expected claim cost per unit sum assured for the population. That is $\mu = E(X)$. Then, the expected losses to insurers are:

$$E(rX - r\mu) = cov(r, X) = cov(r_g, \mu_g),$$

i.e. there will be losses if and only if there is adverse selection, and adverse selection exists if and only if r_g is positively correlated with μ_g.

As a result, if adverse selection occurs, insurers will decide to increase premium to a level where they can maintain a zero expected profit, i.e. the break even premium:

$$\epsilon(X) = \frac{E(rX)}{E(r)} = \frac{E(r_g \mu_g)}{E(r_g)}.$$

The increased premium is the expected value of claims cost, weighted by demand for insurance.

In a simple case with two risk-groups, the authors showed that losses to insurers as a result of adverse selection would be severe if the high risk-group is small but their demand for insurance is high. The problem may become worse if different risk-groups react differently to premium changes. If the low risk-group is more sensitive to premium changes, premium increase will make them leave the market quickly and lead to an increase in adverse selection (and even an adverse selection spiral).

To thoroughly understand this issue, the authors explored what would happen to the market when a single premium is charged. Equilibrium will be achieved when the premium is equal to the average claims cost per unit sum insured, averaged over all insured. Let $r_g(\pi)$ be the demand given a premium π and risk level g. Then the market will be in equilibrium when $\pi = \pi^*$, where

$$\pi^* = \frac{E[\mu_g r_g(\pi^*)]}{E[r_g(\pi^*)]} = \epsilon[\mu_g(\pi^*)].$$

To ascertain π^*, the following assumptions about demand functions for insurance r_g are made:

- Demand falls when the premium increases.
- At a given premium, people with a higher risk will buy more insurance than people with a lower risk.
- The demand is a function of the premium loading, defined as the ratio of the actual premium charged to the actuarially fair premium, i.e. $\frac{\pi}{\mu_g}$.

A specific demand function which satisfies these assumptions was introduced as:

$$r_g(\pi) = E(r|g, \pi) = d_g e^{1-(\frac{\pi}{\mu_g})^\gamma},$$

with $\gamma, \pi > 0$. When the premium rate is set equal to the expected claims cost, i.e. $\pi = \mu_g$, then the demand is d_g, and it is named as "fair premium demand". And the parameter γ controls the responsiveness of demand to changes in premium, i.e. the demand elasticity parameter.

Using this demand function, the authors showed the existence of equilibrium, and also the possibility of multiple equilibria.

In Thomas (2008) and Thomas (2009), this demand function was also used to analyse loss coverage. In this thesis, we will follow the demand approach as in De Jong & Ferris (2006) to analyse equilibrium, adverse selection, loss coverage and social welfare.

The rationale behind modelling demand function instead of utility-based framework is as follows.

In economics literatures, utility functions are often used (e.g. Akerlof (1970), Wilson (1979), Rothschild & Stiglitz (1976)), in which individuals are assumed to choose the level of insurance with the aim to maximise their expected utilities. However, utility functions are not directly observable. Researchers can only observe how much insurance individuals buy at a given premium, and how much demand changes when premium changes (i.e. demand elasticity), and so empirically validate assumptions about demand. On the other hand, utility is an individual's subjective perception of how much insurance is worth. This is much more difficult to establish. Therefore, it is more convenient to use demand functions while analysing adverse selection.

1.2 Empirical Evidence on Adverse Selection

In this section, we will review some of the empirical evidence on the presence of adverse selection. Although theoretical work on asymmetric information began in 1970s, empirical testing of the models began only in the mid-1980s.

Even though we can precisely describe the problems that may arise from the presence of adverse selection in a competitive market and even apply detailed policy recommendations for mitigating these problems, empirical results on its presence is mixed and varied by markets.

Econometric tests have been widely used to analyse adverse selection, which are typically based on quantifying the positive correlation between the level of insurance coverage and realistic risk experience. For the rest of this section, we will firstly explain a few approaches to econometric testing, and then look at some empirical evidence on adverse selection for different insurance markets in turn, i.e. "Automobile Insurance", "Annuities", "Life Insurance", "Long-Term Care" , "Health Insurance" and "Critical Illness".

1.2.1 Approaches to Econometric Testing

Finkelstein & Poterba (2004) used a regression with a single dependent variable to test the relationship between policy pricing and annuity product choice:

$$\text{Risk}_i = \alpha + \beta\text{Coverage}_i + \gamma X_i + \epsilon_i,$$

where Risk_i represents policyholder i's ex post realisation of risk, which can be the total cost to the insurer in the event of a claim. Coverage_i presents policyholder i's quantity of insurance coverage; and X_i is a vector contains all characteristics of policyholder i known to insurers. $\beta > 0$ indicates there is a positive correlation between policyholder i's ex post risk and insurance coverage, i.e. presence of adverse selection. And a large β indicates a high correlation which could lead to severe adverse selection.

Another approach is the *"Positive Correlation Test"* introduced in Chiappori & Salanie (1997) which used a bivariate regression model for insurance coverage and risk for policyholder i with

$$\begin{aligned} \text{Coverage}_i &= f(X_i) + \epsilon_i, \\ \text{Risk}_i &= g(X_i) + \mu_i, \end{aligned}$$

where X_i is defined as before. f and g are regressions. Using this model, a positive correlation between the two residuals ϵ_i and μ_i indicates a coverage-risk correlation, i.e. presence of adverse selection.

Later on, Chiappori & Salanie (2000) proposed non-parametric χ^2 tests for independence. The authors created 2^m cells for m explanatory $0-1$ variables used in risk classification. For each cell they compute a 2×2 table to count the number of individuals having combinations of two dummy variables-coverage: 1 if high and 0 if low, and risk: 1 if the policyholder had at least one accident and 0 otherwise. The conditional independence between coverage and risk is then tested for each cell. The assumption of conditional independence for a given cell is rejected if the test statistic of χ^2 test is greater than certain critical value. If independence is not rejected in any one cell, then there is no adverse selection.

Finkelstein & Poterba (2004) also used regressions to test the relationship between policy pricing and annuity product choice. A large and positive correlation indicate severe adverse selection.

As a result, using the above models, adverse selection is identified in some markets while not in others. This led to the consideration that there might be other factors which can influence the presence of adverse selection, one of which is the buyers' risk aversion.

Chiappori *et al.* (2006) extended the model from Chiappori & Salanie (2000) and highlighted the key role of risk aversion: if it is public information, then the positive correlation test could indeed prove that the relationship is between insurance coverage and risk experience; if it is private information, this need not necessarily to be true. However, as it is an intrinsic property of preferences, it cannot easily be observed by insurers.

Some other models (De Meza & Webb (2001), Pauly *et al.* (2003), Finkelstein & Poterba (2006), Finkelsein & McGarry (2006), Fang *et al.* (2008)) also took risk aversion into account: let Z_1 indicate the risk type; and Z_2 indicate risk aversion, then set the error terms from the "Positive correlation test" as:

$$\begin{aligned}
\epsilon_i &= Z_{1,i} * \pi_1 + Z_{2,i} * \pi_2 + \eta_i, \\
\mu_i &= Z_{1,i} * \rho_1 + Z_{2,i} * \rho_2 + \nu_i.
\end{aligned}$$

The positive correlation test is based on the idea that Z_1 is positively correlated with both coverage and risk, i.e. $\pi_1 > 0$ and $\rho_1 > 0$. But if Z_2 is positively correlated with coverage while negatively correlated with risk, i.e. $\pi_2 > 0$ and $\rho_2 < 0$, correlation between ϵ_i and μ_i may be zero or negative. If this is the case, then the standard positive correlation tests (e.g. Chiappori & Salanie (1997), Chiappori & Salanie (2000), Finkelstein & Poterba (2004)) might fail to reject the null hypothesis of symmetric information even in the presence of asymmetric information about risk type.

1.2.2 Automobile Insurance

Based on 1986 data from a representative insurer in Georgia, U.S., Puelz & Snow (1994) found evidence of adverse selection and market signalling in automobile collision insurance. They found that individuals with lower risks would choose contracts with higher deductibles, and contracts with higher deductibles are associated with lower average prices for coverage. However, their evidence rejects the hypothesis that high risks receive contracts subsidized by low risks. Dionne *et al.* (2001) criticised Puelz & Snow (1994) for failing to take non-linear effects into account. They concluded that once non-linearity is taken into account, insurers' risk classification scheme could be sufficient enough to eliminate adverse selection within each risk class.

Chiappori & Salanie (2000) found no evidence on the presence of risk-related adverse selection in the French automobile insurance market using various parametric and non-parametric methods. This result was reflected in the finding that accident rates for young French drivers who chose comprehensive auto-mobile insurance were not statistically different from the accident rates of those opting for the legal minimum coverage, after controlling for observable characteristics known to automobile insurers. Moreover, they also concluded that although unobserved heterogeneity of risk could probably be very important, there was no correlation between unobserved riskiness and contract choice. And many variables that one might believe to be correlated with risk appeared to be irrelevant.

Using a dataset on car insurance from an association of large French insurers, Chiappori *et al.* (2006) extended the positive correlation property to more general set-ups, in both competitive and imperfectly competitive insurance markets. In competitive markets, they showed that relevant asymmetric information did imply a positive correlation between insurance coverage and risk. In imperfectly competitive markets, the result depended on the observability of agents' risk aversion. If the agents' risk aversion was public information, then some form of positive correlation held; otherwise, the property did not necessarily hold.

1.2.3 Annuities

Finkelstein & Poterba (2004) found evidence of adverse selection between ex post mortality rates and annuity characteristics in UK annuity market, such as the timing of payments and the possibility of payments to the annuitant's estate. But they did not find evidence of substantive mortality differences by annuity size. Their results showed that not only individuals having private mortality information, but they also use this information in making annuity purchase decisions.

Later on, in Finkelstein & Poterba (2006), the authors again confirmed the presence of adverse selection in this market through the *"Unused Observables Test"* by setting the place of residence as the unused observable factor. Their result showed that social-economic characteristics of an annuitant's geographical location were correlated with both his/her survival probability and the amount of annuity he/she purchased on average.

1.2.4 Life Insurance

Various studies in the life insurance market found no material impact of adverse selection. Macdonald (1997) and Macdonald (1999) used Markov models to analyse the costs of adverse selection when insurance companies are restrained from using any genetic tests' results. The main finding was that even though restriction on using results from any genetic tests could lead to additional mortality costs, the increased demand for insurance on getting positive genetic tests' result would not have a major impact. Any costs due to adverse selection would be greatly reduced if the sum assured did not increase hugely. Thus if there were regulations to restrict the use of genetic information, a cap on sum assured would help in controlling the extent of adverse selection.

Cawley & Philipson (1999b) also found no evidence of adverse selection in the US life insurance market. However, they did identify a negative covariance between risk and quantity of insurance coverage.

Pauly *et al.* (2003) provided an empirical estimate of price elasticities of demand for term life insurance. The authors found that the elasticity, in the range of -0.3 to -0.5 was sufficiently low such that adverse selection in term life insurance was unlikely to lead to a death spiral and might not even lead to measured effects of adverse selection on total purchases.

McCarthy & Mitchell (2003) used data from insurance markets in U.S., UK and Japan, to show that life insurance buyers have lower mortality than non-insured. This might show that insurers' underwriting is very effective in helping eliminate adverse selection as a result of asymmetric information.

Viswanathan *et al.* (2007) concluded that potential adverse selection due to BRCA1/2[3] testing would not result in a significant cost to term life insurers, because their estimate of the cumulative effect of adverse selection was very small.

The lack of empirical evidence on adverse selection in life insurance market might be because of the overall decrease in mortality rates due to the fast development in pharmaceutical industry and people taking better prevention, detection and treatments on certain diseases.

However, Howard (2014) provided a different perspective. The author set up a model to analyse the actuarial implications on the individual life insurance market. Insurers are prohibited from using results of genetic tests in Canada with thirteen conditions (or genetic-lead illnesses) being included in the model. In particular, the author looked at the impact on mortality cost. The conclusion was that, the impact on insurers could be substantial. In the case when the threshold for prohibiting access to the results of genetic tests was $1 million of insurance (set by Canadian legislations), the present value of claim costs from those who tested positive in the year could be 12% of total claims. The overall mortality experience for attained ages 20-60 would increase by 36% for male and 58% for female. These potential consequences were much more than insurers could be able to absorb in the short term. However, the author also found that if the threshold was reduced to $100,000, the impact would be small enough to be negligible. Although this study was built upon various assumptions for simplification, its striking results cannot be ignored.

[3]BECA1/2 mutations are the main causes of breast cancer and ovarian cancer (Macdonald *et al.* (2003b)).

1.2.5 Long-Term Care (LTC) Insurance

In Macdonald & Pritchard (2001), the authors suggested that the cost of adverse selection in the LTC insurance due to AD (Alzheimer's disease) was insignificant unless

- the level of relative risks of the ApoE geneotypes[4] in the population was higher than observed to date;
- the LTC insurance market was small;
- $\epsilon 4$ allele carriers were more than four times likely to buy insurance; and
- a higher proportion of the population were tested for the ApoE gene.

The insignificance of adverse selection in LTC insurance market has also been confirmed by another study. Using data from a sample of Americans born before 1923, Finkelsein & McGarry (2006) found no statistically significant correlation between LTC coverage in 1995 and the use of nursing home care in the period between 1995 and 2000, even after controlling for insurers' assessment of a person's risk type.

These results may depend on the level of maturity of different markets. Compared to life insurance market, LTC market is relatively underdeveloped, so the adverse selection test results might depend on the population size, which makes the results more uncertain. Moreover, research in insurance risk in this area also depends on the development in genetic tests. The more the information available on human genetics, the more accurately research can be done on their impact on LTC insurance.

[4]i.e. $\epsilon 4$ allele which is an indicator of early onset of Alzheimer's disease.

1.2.6 Health Insurance

Similar to LTC market, there is no significant evidence suggesting the presence of adverse selection in health insurance markets. Cardon & Hendel (2001) found no evidence of asymmetric information using health insurance data from National Medical Expenditure Survey (NMES) in the U.S.. They argued that demand could be mostly explained by observables with only small and insignificant space for unobservables. Thus adverse selection was immaterial.

From another perspective, Fang *et al.* (2008), however, provided strong evidence of *"Advantageous Selection"* in the Medigap insurance market in the US by showing that people with better health are more likely to purchase supplemental coverage. But, the authors failed to find evidence to suggest variation in risk preferences, which was the primary focus of the theoretical literature on advantageous selection. The authors proposed that cognitive ability (e.g. precaution actions, planning horizons) and financial numeracy (e.g. income) which standard economic models did not accommodate could be important sources of advantageous selection.

1.2.7 Critical Illness (CI) Insurance

In CI market, information on disease-relevant genotypes is crucial in determining the impact of adverse selection. In Macdonald *et al.* (2003a), the authors used a Markov model on breast cancer (BC) and ovarian cancer (OC) with the transition intensities between different states. Using UK population data, the costs of CI insurance in the presence of BC and OC were estimated. The authors concluded that assuming BRCA1 or BRCA2 mutation was known, the extra premium in purchasing CI insurance would be as high as +1000% with the amount depending on family history, family structure, penetrance and mutation frequency. The authors argued that adverse selection was not a serious problem in UK unless:

- the CI insurance market was much smaller than the current scale in UK (could be the case in the U.S.);
- penetrance was much higher than being observed in current market; and
- higher sum assured could be obtained without disclosing family history or genetic test results.

Using similar models, Macdonald & Gutierrez (2003) analysed the impact of Adult Polycystic Kidney Disease (APKD) on CI insurance costs. The authors found that, if an individual was found out to be a APKD mutation carrier, the premium charged could be as high as 250% of the ordinary rate (i.e. the premium for those without APKD mutation). However, because APKD is rare, the aggregate cost of adverse selection would be low.

Howard (2016) also studied the impact on CI market in Canada of restricted information from the results of genetic tests. Compared to his earlier studies in life insurance market (Howard (2014)), the conclusion was: the impact on insurers could be material, but it would not be as serious as that in life insurance market. Using six conditions (or genetic-lead illnesses), prohibiting information from genetic tests could increase CI claims rates by 26% overall, with 16% for males and 41% for females[5]. And the impact on premium rates was less severe than on claim rates. The author also mentioned

[5]The reason is that only females are exposed to BRCA, the genes lead to breast cancer, which is the most significant condition. And the experiment groups contains proportionally more females.

the differences between this study on CI market and his earlier study in life insurance market (Howard (2014)): genes with the main impact on life insurance were most likely lead to death but unlikely to result in a CI claim. Thus there were fewer conditions (or trigger for claims in this case) included in this study. Another difference was although in Canada, the threshold amounts of insurance beyond which underwriters could have access to the results of genetic tests was the same for both life insurance and CI, i.e. $1 million[6], CI is much more expensive than life insurance, applicants might not apply for the maximum amount. Therefore, in this study, the amount of CI purchased was realistically assumed to be at $250,000. These two main distinctions contributed to the different degrees of severity in life insurance and CI market as a result of genetic tests results being restricted. Moreover, if the threshold for CI was further reduced (from $250,000), the impact could become even smaller.

Another interesting example to be considered is the *"over-diagnosis issue of thyroid cancer in South Korea"*. In 1999, the government of the Republic of Korea initiated a national screening program for cancer and other common diseases (including cancers for breast, cervix, liver, colon, etc). Thyroid-cancer screening was later on offered as an inexpensive fee-for-service add-on by hospitals and GPs. Thanks to the advanced development in medical technology and residences' increasing awareness of well-being, early cancer detection, especially thyroid-caner screening, has become part of the essential health checkup. As a result, thyroid cancer is now the most common type of cancer diagnosed in South Korea. However, in recent years, some physicians and researchers have raised the concern about over-diagnosis of thyroid

[6] In the UK, according to the moratorium, life insurance is capped at £500,000, and CI was capped at £300,000.

cancer and suggest that screening should be banned. Their arguments were based on the statistical evidence that the rate of thyroid-cancer diagnosis was 15 times as high in 2011 as the rate in 1993. However, this increase was mainly due to the detection of papillary thyroid cancer[7]. And, there was a great disparity between the rising incidence and the stable and low mortality rate of thyroid-cancer[8] (Ahn *et al.* (2014)). Moreover, this overdiagnosis could lead to overtreatment. The majority of patients given diagnosis of thyroid cancer would receive treatments e.g. radical thyroidectomy, which has substantial consequences and might also have lifelong complications (e.g. hypoparathyroidism and vocal-cord paralysis (Notional Evidence-based Healthcare Collaborating Agency (2013))). And the majority of tumours detected this way are 'microcarcinomas', i.e. below one centimetre in size, which is unlikely to develop or to threaten life (Ahn *et al.* (2014)). Therefore, increased screening has not saved more lives.

Simultaneously, the disease-specific medical costs for thyroid cancer has also been increasing. Hyun *et al.* (2014) reported that the cost-of-illness (including direct costs e.g. medical costs, and indirect costs e.g. loss of potential income and/or productivity) for thyroid disease has increased significantly from 2002 to 2010 (2242 hundred million won to 7622 hundred million won, a 3.4-fold increase)[9]. And the medical costs incurred by patients with thyroid disease had increased remarkably more than those incurred by patients with other endocrine diseases. Although the medical cost per patient with thyroid cancer was not very high, its high incidence and prevalence could potentially

[7] It has been estimated that at least one third of adults have small papillary thyroid cancers, the vast majority of which will never produce evident symptoms during a person's life (Harach *et al.* (1985)).

[8] Data on incidence are from the Cancer Incidence Database, Korean Central Cancer Registry; data on mortality are from the Cause of Death Database, Statistics Korea.

[9] Although in this paper, the author mentioned the exact health care costs associated with the increased diagnosis of thyroid cancer were difficult to define.

lead to significant economic complications.

Automatically, these cancer claims are payable under CI policy conditions regardless of any such threat. This has clear implications for CI claims because according to CI definitions, there is no stage zero or pre-invasive stage in the accepted staging systems of thyroid cancer (Campbell (2014)). This means that tissue is either described as “benign” or “malignant” with no pre-malignant category in the historical classification. This is considered as a big threat to CI market. To address this problem, the CI definitions should recognize the advances in medical screening and adopt exclusions to maintain the relevance and affordability of the product going forward. Alternatively, improved therapeutic guidelines and risk stratification strategies should be developed to ensure CI insurers are able to cope with the risks undertaken.

1.2.8 Explanation of the Absence of Adverse Selection

As discussed earlier, adverse selection has been identified in some insurance markets (or sections of insurance markets) but not in others. In this section, we discuss some possible explanations for its absence in some markets.

Policyholders’ Lack of Private Information

In the literature, policyholders (i.e. insurance buyers) are usually assumed to have an information advantage of insurers, and they use this information in their purchasing decisions. What if these assumptions are not true? Some possible scenarios are discussed below:

1. Absence of useful private information.
   Individual’s expected loss can be divided into two factors: the amount of loss and the probability of incurring the loss.

(a) Policyholders may not know some of the information included in both factors.

(b) Even if they can accurately estimate some of the features, they may lack the ability to transfer this information into probabilistic interpretations that can be used in analysis.

For example, a major determinant in automobile-accident risk is the total miles driven in a given year (Butler (1996)). It seems like policyholders will have more advantage as they do know their own mileage better than the insurers. However, some policyholders may not realise the importance of this factor as they do not know how to interpret accident risk in terms of the information they have.

(c) Another possibility is that there are factors which are unknown to anyone, e.g. luck.

It is the presence of these factors which make policyholders less advantaged than researchers usually assume. This leads to less adverse selection as a result.

2. Not all policyholders withhold private information.

If insurers only target certain markets where people do have private information, this might result in identification of adverse selection in some cases and vice versa.

3. Policyholders might fail to adjust their purchasing behaviours based on accurate private information that can theoretically be used to their advantage.

4. Another reason is that insurers may be able to assess policyholders'

risks more accurately than the policyholders themselves. This might be due to the complex risk classification scheme insurers use based on extensive historical data and superior knowledge compared to individuals.

The above four factors could provide some reasonable explanations to the presence of coverage-risk correlation in some markets while not in others (or subsets of a market).

Negative Correlation Between Coverage and Risk

Now we consider the case when policyholders do hold some private information and are able to use it. We focus on those factors which would favour a negative correlation between coverage and risk, which might offset the positive effect. We distinguish between the cases of risks which are negatively correlated with risk aversion, and risks which are correlated with other features.

1. Negative correlation between risk and risk aversion.

   Hemenway (1990) first proposed the concept *"Propitious Selection"* which describes the negative correlation between insurance coverage and riskiness. The theory is that the absence of adverse selection in some markets is due to the fact that high risks are also less risk averse, i.e. high risks would not demand high level of coverage as expected by insurers. On the contrary, low risks are relatively more risk averse and would still buy insurance even at a higher premium (to some extent). Thus the situation is advantageous to insurers. This negative correlation between risk and risk aversion are considered as the main factor contributing to the absence of adverse selection.

The author's idea was supported by several examples using U.S. data: for car drivers, a positive correlation between purchase of noncompulsory liability insurance and a range of health-related risk avoidance activities; also for car drivers, a positive correlation between purchase of noncompulsory liability insurance and not driving after drinking alcohol; and for motorcyclists, a positive correlation between wearing a helmet and holding medical insurance (Hemenway (1990, 1992)).

Similarly, Finkelsein & McGarry (2006) showed that those who are most cautious about their health are most likely to purchase long-term care insurance. And they are the least likely to enter nursing homes. Empirical studies on other markets where a negative correlation between insurance coverage and losses has been identified include medigap insurance (drugs coverage in the U.S.) (Fang *et al.* (2008)), health insurance in Australia (Doiron *et al.* (2007)) and commercial fire insurance (**?**).

Built on similar assumptions as Hemenway (1990), De Meza & Webb (2001) provided a more detailed analysis on the presence of propitious selection, which they named *"Advantageous Selection"*. The idea is that as premium increases, it is the least risk-averse individuals who drop out of the market; the existence of equilibrium in the market contradicts the non-existence of pooling equilibria from the standard model in Rothschild & Stiglitz (1976). Some empirical evidence also appeared to conflict with the major implications of the standard economic model of insurance. For example, De Meza & Webb (2001) provided evidence that 4.8% of U.K. credit cards has been estimated to be reported lost or stolen each year, whereas for insured cards the corresponding figure has only been 2.7%.

In a similar vein, Cawley & Philipson (1999b) found that the mortality rates of U.S. males purchasing life insurance were below that of the uninsured, even after controlling for many factors, such as income that are correlated with life expectancy. Chiappori *et al.* (2006) suggested that accident rates were lower for young French drivers choosing comprehensive insurance than for those opting for the legal minimum coverage.

Finkelsein & McGarry (2006) and Cutler *et al.* (2008) explained this advantage selection by suggesting that heterogeneity among policyholders is not only restricted to risk, but also risk aversion and other information.

2. There may also be correlations between risk and other features.

   Fang *et al.* (2008) proposed the concept of *"Cognitive Ability"* which suggests that high cognitive ability is correlated with high demand for insurance and better health (due to precaution actions e.g. regular health check with GP). There are also studies (e.g. Finkelstein & Poterba (2006)) showing insurance coverage could be associated with wealth, income, education, and socio-economic status. These theories show that people with higher income usually have lower mortality rates and are more willing to purchase all kinds of insurance (i.e. more risk averse). Pauly *et al.* (2003) gave a reasonable explanation to the different results observed in life insurance and annuity markets. For those with higher income, their demand for life insurance is a mixed result between risks being negatively correlated with risk aversion but positively correlated with insurance coverage. The net effect is that mortality rates of those who purchase life insurance would be very much the same as the general public. Thus adverse selection is not always being detected. However, higher income and lower mortality rate both increase demand for annuities, which leads to annuitants outliving the population average.

External Factors

Having discussed influences from policyholders themselves, external factors such as intermediaries, alternative products and regulations should not be ignored.

1. Intermediaries are considered as the bridge between insurers and policyholders. The fact that intermediaries have different influences in different markets contributes to the heterogeneity presented in markets. Similarly, how insurers advertise their products differ from market to market.

2. The number of alternatives and their accessibilities are also important, especially for social insurance. If there is less freedom for policyholders in choosing social insurance, some of them, if necessary, will switch their attention to private insurance where they may have more choices.

3. Influence from regulations is also crucial. Chiappori (1999) found little evidence of adverse selection in an automobile insurance market when risk classification was not regulated. He did point out that government regulations that prevent insurers from using some information that the individual has obtained could create significant adverse selection. This is called *"Regulatory Adverse Selection"*.

Questions on the presence of regulatory adverse selection have also been mentioned in other markets, e.g. health and life insurance markets. One of the areas which has been under fierce debate is the use of information from genetic tests. Effects of many genes on the likelihood of various illnesses and consequently individuals' life expectancy was discussed in Rowen *et al.* (1997). It is likely that new information from genetic tests would become readily available in the future. There are currently some regulatory restrictions on the use of genetic information for insurance pricing through prohibiting insurers from requesting and using such data, and capping the total amount of life insurance an individual can buy without providing genetic information. However, whether regulatory intervention on risk classification schemes is a good or bad thing is still inconclusive.

Hoy & Witt (2007) considered how new information on mortality risk and demand type together could influence insurance holding. By analysing the effects from different forms of regulations, the author showed that if there were sufficiently few individuals who receive negative reports about their genetic type, then a ban on using genetic information in pricing in combination with a cap which limits adverse selection was welfare improving. This result indicates that appropriate level of regulations on insurance pricing processes may have positive effect on public welfare.

The above arguments suggest that even if policyholders have accurate private information about themselves, taking advantage is not always easy due to both internal and external factors.

Moral Hazard

Even when a coverage-risk correlation has been identified, this does not necessarily indicate adverse selection, because moral hazard can also lead to this correlation.

Adverse selection deals with hidden information but moral hazard deals with hidden action. Coverage-risk correlation due to adverse selection indicates high risks choosing higher coverage (or lower deductibles). However, in terms of moral hazard, it means purchasing insurance lowers policyholders' caution and precautionary actions. Therefore, there will be higher risks as a result. As it is hard to differentiate adverse selection from moral hazard in empirical tests, some literatures on testing adverse selection remained unclear whether their models have taken moral hazard into account. Some approaches to separate adverse selection from moral hazard are:

1. Manning *et al.* (1987) used a randomized experiment on RAND Health Insurance data to test how people's spending changes given a randomly assigned level of coverage, based on which they detected the presence of moral hazard. The main initiative behind the experiment is that when coverage changes due to exogenous reasons, policyholder's action changes instead of their risks.

2. Another approach to differentiate these two is based on their distinct dynamic properties on past and future risks. Abbring *et al.* (2003) provided an explanation on how it works: Under bonus-malus systems, a single claim raises the whole distribution of future premium upwards. Therefore, under moral hazard, the greater cost of future accident should lead the insured to take more care, so that the probability of loss falls. In other words, past losses and future losses are negatively

correlated. On the other hand, under adverse selection, an insured with many past losses is likely to be a high risk, who will also have many future losses. Thus adverse selection would produce a positive correlation between past and future claims. Using similar approaches, some studies have found evidence of moral hazard (e.g. Israel (2007), Dionne *et al.* (2007)) while some others do not (e.g. Abbring *et al.* (2003)).

3. A third method was introduced in Cohen (2005) by identifying the interaction of the coverage-risk correlation with policyholder characteristics and behaviours. The author argued that the correlation found in auto-mobile insurance could be better explained in terms of adverse selection rather than moral hazard. The same logic can be applied in other markets to differentiate results from these two sources. However, compared to the previous two approaches, this is an approximate approach.

Updated Information

Looking forward into the future, it is likely that both policyholders and insurers will have access to updated information on policyholders' risks. Thus any analysis on coverage-risk correlation is not a one-off job. It should be regularly monitored based on the newest data and evidence.

One of the latest developments for insurers is the increasing availability of *"Big Data"*, especially with the unstructured data growing 15 times faster than structured data (Nair & Narayana (n.d.)). Although the insurance industry is still in the initial stages of embracing the revolution of Big Data, a research commissioned from censuswide [10] has found that 82% of UK insur-

[10] by data warehousing company Teradata

ance companies with more than £500m turnover are prioritising Big Data strategies in 2016 (**?**). The increased volume, variety (e.g. from social media) and velocity of these unstructured data along with the familiar structured data bring opportunities as well as challenges not only for insures, but also for regulators.

One example of how the impact of Big Data could affect the judgement of pooling risk characteristics and individual observation is the application of *Telematics*, i.e. the "pay as you drive" model to collect real data on how policyholders drive, e.g. how and when they drive the car, how many miles they drive and where they drive the car. This data could potentially allow for more accurate pricing on an individual level, and might also incentivise policyholders to improve driving and take precautions to reduce their insurance premiums. This might be counted as a "positive result" for those who favour personalised offers in insurance, e.g. low-risk policyholders who are reluctant to pay higher premiums to subsidise high risks.

However, Big Data could also potentially increase risk segmentation and consequently lead to customers with higher risks being unable to obtain affordable insurance coverage. Moreover, access to Big Data, especially those from social media and/or comparing websites might also enable insurers to identify customers whose insurance purchasing behaviours over time show inertia, e.g. they do not shop around. Insurers could use this information to differentiate premiums between those customers who shop around and those who do not. Customers in the latter group could face a higher premium to subsidise those in the former group.

Regulators, with the main aim to ensure well-functioning markets and customer protection, need to decide whether such behaviour should be allowed or not as this is closely related to the use of risk classifications. Allowing the use of such information might be contrary the social purpose of insurance; prohibiting those information might reinvoke discussions on the potential result of adverse selection. Moreover, the trend towards personalised offers in insurance has raised concern: *"Part of the genius of insurance is that it is based on pooled risk. That has been its social value. There is a danger of sleepwalking away from that into individual pricing."*[11]

Furthermore, the increasing availability of Big Data to insurers may also indicate that the assumed information advantages from policyholders may become less material in the future. As a result, opinions on the presence of adverse selection as a result of restriction on risk classifications may change in terms of empirical evidence.

The ever-changing nature of data and information is also relevant to *genetic discrimination in insurance*, especially in private sector. With the growing availability of high-quality genetic testing and the emergence of "personal medicine", genetic discrimination is once again moving to the forefront of the genetics policy debate.

Different options are in place around the world to resolve the genetic and insurance dilemma with most restricting insurers from introducing genetic information into their underwriting practice.[12] The options vary according

[11]by David Thomson, director of Policy and Public Affairs at the Chartered Insurance Institute, at a Teradata organisation discussion event held in the City of London. (**?**)

[12]For example, in the US, access to health insurance is mainly sponsored by employers through group plans, in which case, employers bear a large percentage of the risk of health care costs. Thus there is an incentive for employers to discriminate against those people with potential high risks and the employers are mostly likely to oppose those prohibitive regulations on genetic information. The Genetic Information Nondiscrimination Act 2008 (Public Law 110–233, 122 Stat. 881) (GINA) (Department of Health and Human Services (HHS) (2008)) and the Health Insurance Portability and Accountability Act 1996 (Public Law 104–191) (HIPAA) (The United States Congress (1996)) are two of the major pieces of American federal legislations dealing with genetic discrimination and insurance.

But Europe differs from the US in its social rule of insurance. Most European states provide some form of universal access to health care. As a result, private health insurance to local legal status, the role insurance plays in the society, and the interaction between private and public health care systems.

Last but not the least, opinions of policyholders (or even people in a certain country) are also crucial in deciding the role that genetic information plays in insurance underwriting. There are arguments for and against the genetic discrimination:

Some of the views of those who support the use of genetic discrimination in insurance underwriting are as follows:

- Those restrictive regulations might result in "adverse selection" (e.g. Macdonald (2003), Macdonald *et al.* (2006), Viswanathan *et al.* (2007)).

- They draw attention to the private and commercial nature of the arrangement between insurer and the insureds. They emphasize the fairness between a policyholder's premium and his/her potential benefits, when insurers spread out the cost of risk over time and over the pool of insured. Thus insurers require all insureds to disclose their full and truthful health information in order to protect the integrity of the overall risk pooling.

- Difficulties of distinguishing genetic health information from non-genetic health information, i.e. insurers might treat genetic information in the

is not as significant as that in the US. Considering the social purpose of insurance, there might be more supporters in the regulators' actions.

In the UK, a combination of approaches are applied. Insurers are restricted from using genetic information unless the cover exceeds a predetermined level. The "Concordat and Moratorium" restricts the ability of British insurers to use genetic information in the underwriting of life, critical illness, or income protection insurance. However, it makes exceptions for high-valued policies above a predetermined amount of insurance money as well as for certain genetic tests that meet prescribed technical, clinical, and actuarial criteria. Applicants are still allowed to disclose predictive genetic test results in their favour to override family history information (UK Department of Health and Association of British Insurers (ABI) (2014)).

same way as other kinds of health information that predict insurance risk.

Some of the views of those who opposite the use of genetic discrimination in insurance underwriting are as follows:

- Risk of irrational discrimination, i.e. the practice of distinguishing applicants on grounds other than sound actuarial principles.
- Risk of rational, but socially unjust discrimination, i.e. people diagnosed with genetic disorder such as Huntington's disease are those who most need insurance and also who are most likely to be rejected for insurance if genetic information is considered (Ashcroft (2007)).
- The lack of reliable empirical data on the impact of the use of genetic information by insurers (Rothstein & Joly (2009)).
- Introducing extra data and/or parameters into the model also introducing additional levels of complexity and risk, e.g. data being misused by insurers and other third parties.
- Confidentiality issues.
- Genetic information is qualitatively different from other forms of medical information and thus requires special treatment and protection.
- The possibility that at-risk individuals knowingly forgoing genetic testing to avoid jeopardizing their coverage.
- The possibility that insurers require getting genetic testing as a condition for applicants to get coverage.

- The possibility that genetic information and potential risks being disclosed to family members who are unwilling to know in the first place.

This debate will continue with more data and evidence becoming available over time. It is not possible to provide a concrete conclusion as for now on the pros and cons of using genetic information for insurance underwriting. However, we make the following observations. For the private insurance industry to remain robust and stable, some form of risk classification is needed and must be tolerated by all stakeholders. Insurers, on the other hand, must be aware of the limit on the use of genetic information, and recognize how their practices can create public apprehension and controversy. Insurers must respond to this challenge by increasing transparency and demonstrating the true scientific rationale behind their underwriting practices.

Chapter 2

Demand for Insurance

In this chapter, we will provide a utility-based micro-foundation for the proportional insurance demand function, driven by variations between individuals in their utility functions, which can explain why only a proportion of the individuals in each risk-group buy insurance at any given price (which for the case of risk-differentiated premiums in the motivation examples was defined as the "fair-premium demand").

The rest of this chapter is organised as follows. Section 2.1 considers a single risk-group, where all individuals have the same probabilities of loss, but who have a range of utility functions. This set-up leads to a proportional demand between 0 and 1, representing the proportion of individuals from the risk-group who buy insurance at a given premium. Examples and analysis of two demand functions, i.e. iso-elastic demand and negative-exponential demand are given in Section 2.2 and 2.3 respectively.

2.1 Insurance Demand for a Single Risk-group

2.1.1 Utility of Wealth and Certainty Equivalence

Consider an individual with an initial wealth W, who is exposed to the risk of losing an amount of L with probability μ. Suppose preference for wealth is driven by the utility function $U(w)$, which is increasing in wealth w, i.e. $U'(w) > 0$.

Individuals are typically also assumed to be risk-averse i.e. $U''(w) < 0$. This provides the motivation for insurance purchase at an actuarially fair price, and initially we shall discuss individuals for whom the assumption holds. But we shall see later that our theory of insurance demand does *not* require that *all* individuals are risk-averse. Figure 2.1 shows an example of a utility function $U(w)$ with $U'(w) > 0$ and $U''(w) < 0$:

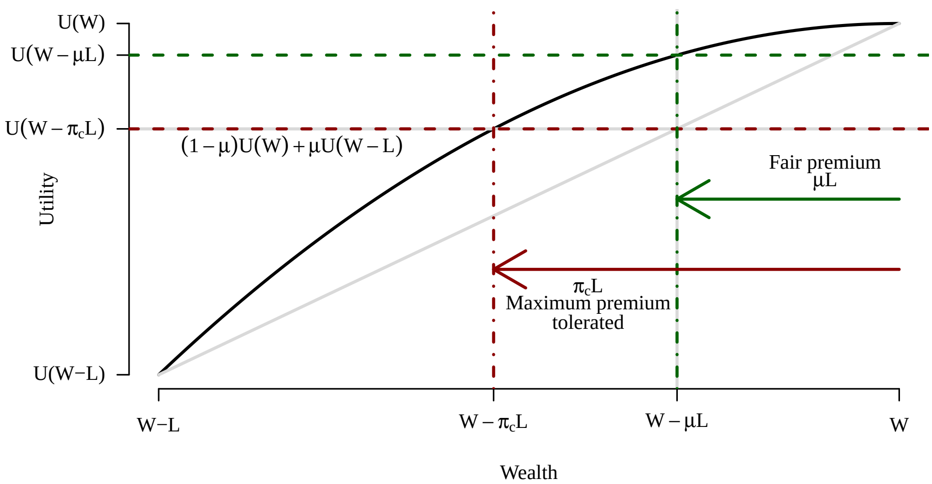

Figure 2.1: Insurance purchasing decision based on an individual's utility of wealth.

If no insurance is bought, occurrence of the risk event will reduce the individual's wealth from W to $(W-L)$ with probability μ. Hence the individual's expected utility, without insurance, is given by:

$$(1-\mu)U(W) + \mu\, U(W-L). \tag{2.1}$$

If, however, the individual has the option to insure against the risk at premium rate π per unit of loss and chooses to buy insurance for full cover, the individual's expected utility is:

$$U(W-\pi L), \tag{2.2}$$

because the individual's wealth diminishes immediately by the amount of premium, but there is no further uncertainty as the loss is insured.

An individual will choose to buy insurance if the expected utility is higher with insurance than without it, i.e.

$$U(W-\pi L) > (1-\mu)U(W) + \mu\, U(W-L). \tag{2.3}$$

In particular, individuals with concave utility functions will buy insurance at the actuarially fair premium $\pi = \mu$. Furthermore, these individuals will be prepared to purchase insurance up to the premium level π_c, where:

$$U(W-\pi_c L) = (1-\mu)U(W) + \mu\, U(W-L), \tag{2.4}$$

which is also known as the certainty-equivalence principle. This is depicted in Figure 2.1.

2.1.2 Heterogeneity in Insurance Purchasing Behaviour

In the above model, all individuals with the same utility function and probability of loss either buy insurance or they do not, based on whether or not the premium being charged, π, exceeds π_c. However, in real insurance markets, we typically observe that not all individuals with the same probability of loss make the same purchasing decision[1]. How can this variation in insurance purchasing decisions be explained?

One plausible explanation suggested by a number of authors (e.g. Finkelsein & McGarry (2006), Cutler *et al.* (2008)) is that risk preferences vary between individuals[2]. To formulate this variability, let us assume a population of individuals, all with the same risk μ but who may have different utility functions. Suppose for simplicity that utility functions belong to a family parameterized by a real number γ. So a particular individual's utility function can be denoted by $U_\gamma(w)$.

Further suppose that an individual's utility function parameter γ is sampled randomly from an underlying random variable Γ with distribution function $F_\Gamma(\gamma)$. So, a particular individual's utility function, $U_\gamma(w)$, is a random quantity[3], the randomness being induced by $F_\Gamma(\gamma)$.

Based on this formulation, an individual will choose to buy insurance if and only if the following condition is satisfied for the combination of the

[1] Please see Footnote 2 on page 30.

[2] Another possible explanation of some individuals' non-purchasing may be that an actuarially fair premium is, in practice, not offered, i.e. premium loadings for expenses and profit. However, heterogeneity in risk aversion is a more flexible explanation, in that it can explain why different individuals with the same probabilities of loss and offered the same premiums may make different decisions.

[3] We must be careful not to call the function $U_\gamma(w)$ a random variable. We shall have no need of any of the metric structure of spaces of functions that this would entail.

offered premium π and their particular utility function $U_\gamma(w)$:

$$U_\gamma(W - \pi L) > (1 - \mu)U_\gamma(W) + \mu\, U_\gamma(W - L), \tag{2.5}$$

Note that all individuals are behaving deterministically, given their knowledge.

Although utility functions of different individuals can have different origins and scales, certainty-equivalent decisions are independent of these choices. So without loss of generality, we will assume that all individuals have the same utility at the "end points" $W - L$ and W. And for clarity, we will suppress the subscript γ for the utility at the "end points" and write $U(W)$ and $U(W - L)$ as they are the same for all individuals. We can then write Equation (2.5) as:

$$U_\gamma(W - \pi L) > u_c \text{ where} \tag{2.6}$$

$$u_c = (1 - \mu)U(W) + \mu\, U(W - L) \text{ is a constant.} \tag{2.7}$$

This says that an individual insures if the utility from insurance exceeds a critical value u_c. Note that u_c is the same for all individuals who are exposed to the same probability of loss.

Figure 2.2 provides a graphical representation showing utility functions of four individuals with the same probability of loss μ. The concave utility curves, with points A, B and C, represent risk-averse individuals, where higher concavity represents higher risk-aversion. We also show a convex utility curve, with point D, which represents a risk-loving (or risk-neglecting) individual. (As mentioned previously, the model does not require that all individuals are risk-averse.) For the individual at point A, the utility with insurance, $U_{\gamma_A}(W - \pi L)$, exceeds the critical value u_c, where γ_A is the indi-

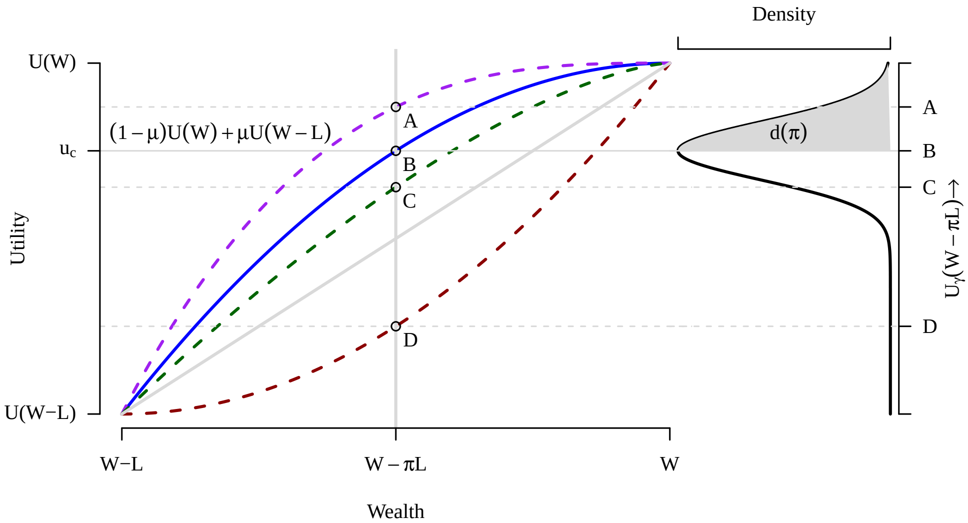

Figure 2.2: Heterogeneous utility functions within a risk-group, leading to proportional insurance demand.

vidual's utility function parameter. So the individual buys insurance. For the individuals at points C and D, the inverse applies, so they do not purchase insurance. The individual at point B is indifferent.

The utility at the fixed wealth $(W - \pi L)$ is a random variable, that we denote by $U_\Gamma(W - \pi L)$. The distribution function of $U_\Gamma(W - \pi L)$ is induced by that of Γ and we denote it by $G_\Gamma(\gamma)$. The corresponding probability density function of the utilities at that level of wealth is shown in the rotated plot on the right-hand side of Figure 2.2.

Now assume that the insurer cannot observe individuals' utility functions. Then, for given offered premium π, all the insurer can observe of insurance purchasing behaviour is the proportion of individuals within a risk-group of risk μ who buy insurance. We call this a demand function and denote it by

$d(\pi)$. We have:

$$d(\pi) = P\left[U_\Gamma(W - \pi L) > u_c\right] = 1 - G_\Gamma(u_c). \tag{2.8}$$

Insurance purchase is denoted by the shaded area, $d(\pi)$, under the density graph for $U_\Gamma(W - \pi L)$.

We note the following three properties of demand for insurance:

(a) $d(\pi)$, denotes a proportion, as $0 \leq d(\pi) \leq 1$ is a valid probability.

(b) $d(\pi)$ is non-increasing in π, i.e. demand for insurance cannot increase when premium increases. This can be shown as follows: For utility functions with $U'(w) > 0$, if $\pi_1 < \pi_2$, the random variable $U_\Gamma(W - \pi_1 L)$ is statewise dominant[4] over the random variable $U_\Gamma(W - \pi_2 L)$. So,

$$\begin{aligned}\pi_1 < \pi_2 &\Rightarrow P\left[U_\Gamma(W - \pi_1 L) > u_c\right] \geq P\left[U_\Gamma(W - \pi_2 L) > u_c\right] \\ &\Rightarrow d(\pi_1) \geq d(\pi_2).\end{aligned} \tag{2.9}$$

(c) Each individual's decision is completely deterministic, given their knowledge of their own utility function. But all the insurer sees is that any particular individual buys insurance with probability $d(\pi)$ and does not buy insurance with probability $(1 - d(\pi))$. In other words, the insurer observes stochastic behaviour. If, for a particular individual we define the function Q to be $Q = 1$ if they buy insurance or $Q = 0$ if they do not. Then Q is deterministic in the eyes of the individual, and a Bernoulli random variable with parameter $d(\pi)$ in the eyes for the insurer. A full probabilistic model accounting for these different levels of

[4]One random variable is statewise dominant over a second if the first is at least as high as the second in all states of nature, with strict inequality for at least one state. It is an absolute form of dominance.

information is given in Appendix A.1.

As noted earlier, certainty equivalent decisions do not depend on the origins and scales of utility functions, so we can standardise the utility functions such that all individuals have the same utilities $U(W)$ and $U(W-L)$ at the "end points" W and $W-L$. The following standardisation is convenient:

$$U(W) = 1, \tag{2.10}$$

$$U(W-L) = 0. \tag{2.11}$$

The constant u_c in Equation 2.8 then becomes $(1-\mu)$, and so the demand for insurance is:

$$d(\pi) = P[U_\Gamma(W - \pi\, L) > 1 - \mu]. \tag{2.12}$$

In practice, demand for insurance can be empirically observed and estimated from data, while individuals' utility functions are not observable. The micro-foundations of the insurance demand described above suggests a possible mechanism by which observable demand for insurance is generated by the unobserved utility functions of individuals, which are assumed to follow a probabilistic law. Different choices of utility functions and distributions of the parameter Γ give rise to a wide variety of demand functions $d(\pi)$. It is not our purpose to discuss either elicitation of any particular utility function, or inference on the distribution of Γ, which would present formidable challenges. Instead we propose to work directly with (proportional) demand for insurance, $d(\pi)$, which may be observed in practice.

We now define a related concept, the (point price) elasticity of insurance demand, as follows:

$$\epsilon(\pi) = -\frac{\partial \log d(\pi)}{\partial \log \pi}, \text{ or, equivalently:} \tag{2.13}$$

$$d(\pi) = \tau \exp\left[-\int_{\mu}^{\pi} \epsilon(s) d\log s\right], \tag{2.14}$$

where $\tau = d(\mu)$ is the "fair-premium demand" for insurance.

The expression in Equation 2.14 has the benefit that we do not have to impose any regularity conditions, like continuity, on the demand function $d(\pi)$. We require only that the demand elasticity $\epsilon(\pi)$ in Equation 2.13 is non-negative, so that the demand $d(\pi)$ in Equation 2.14 is non-increasing in premium π.

2.2 Iso-elastic Demand

In this section, we will firstly use iso-elastic demand as an example to demonstrate the link from specific distributions of risk preferences to specific proportional demand for insurance, where individuals are exposed to the same probability of loss.

Suppose $W = L = 1$ with a power utility function:

$$U_\gamma(w) = w^\gamma, \tag{2.15}$$

so that $U_\gamma(0) = 0$ and $U_\gamma(1) = 1$. This particular form of utility function leads to:

$$\text{relative risk aversion coefficient: } -w\frac{U''_\gamma(w)}{U'_\gamma(w)} = 1 - \gamma. \tag{2.16}$$

So the heterogeneity in preferences between individuals can be modelled through the randomness of the risk aversion parameter γ. As outlined in sub-section 2.1.2, we define a positive random variable Γ, and individual risk preferences γ are then instances drawn from the distribution of Γ.

Inserting this utility function into Equation 2.12, the demand for insurance at a given premium $\pi < 1$ is then:

$$\begin{aligned}
d(\pi) &= P\left[U_\Gamma(1-\pi) > 1-\mu\right], && (2.17)\\
&= P\left[(1-\pi)^\Gamma > 1-\mu\right], && (2.18)\\
&= P\left[\Gamma\log(1-\pi) > \log(1-\mu)\right], \text{ as log is monotonic,} && (2.19)\\
&= P\left[\Gamma < \frac{\log(1-\mu)}{\log(1-\pi)}\right], \text{ as } \log(1-\pi) < 0. && (2.20)
\end{aligned}$$

So, given an observed (proportional) insurance demand function, which is non-increasing in premium, π, the underlying random variable Γ has the following distribution function:

$$F_\Gamma(\gamma) = P\left[\Gamma < \gamma\right] = d(1-(1-\mu)^{1/\gamma}). \tag{2.21}$$

Since $1-(1-\mu)^{1/\gamma} \approx \mu/\gamma$ for small μ using first order Taylor approximation, we might in some circumstances approximate equation (2.21) by the simpler:

$$F_\Gamma(\gamma) = P\left[\Gamma < \gamma\right] = d\left(\frac{\mu}{\gamma}\right). \tag{2.22}$$

Note that $F_\Gamma(\gamma)$ is a non-decreasing function and lies between 0 and 1.

Of course, for F_Γ to be a valid distribution function, we would also require $\lim_{\gamma\to 0} F_\Gamma(\gamma) = 0$ and $\lim_{\gamma\to\infty} F_\Gamma(\gamma) = 1$, or equivalently, $\lim_{\pi\to\infty} d(\pi) = 0$ and $\lim_{\pi\to 0} d(\pi) = 1$, which appear to be reasonable assumptions. However, empirical observations are unlikely to be available for these extreme cases, so it is only possible to model insurance purchasing behaviour over the range of premiums observed in the market, with appropriate extrapolations at the limiting extremes.

Therefore, given an observed proportional insurance demand, $d(\pi)$, which is a valid probability and non-increasing in π, heterogeneity of risk preferences driven by the power utility function and characterised by the random parameter Γ with the distribution function given by Equation 2.21 produces the observed demand for insurance for small premiums.

To get iso-elastic demand function, suppose Γ has the following distribution:

$$F_\Gamma(\gamma) = P\left[\Gamma \leq \gamma\right] = \begin{cases} 0 & \text{if } \gamma < 0 \\ \tau\,\gamma^\lambda & \text{if } 0 \leq \gamma \leq (1/\tau)^{1/\lambda} \\ 1 & \text{if } \gamma > (1/\tau)^{1/\lambda}, \end{cases} \tag{2.23}$$

where τ and λ are positive parameters. Note that $\tau = \lambda = 1$ leads to a uniform distribution. λ controls the shape of the distribution function and τ controls the range over which Γ takes its values.[5]

Based on this distribution for Γ, the demand for insurance in Equation 2.12 takes the form:

$$d(\pi) = \tau\left(\frac{\mu}{\pi}\right)^\lambda, \tag{2.24}$$

which is named as *"iso-elastic demand"*. This demand function is subject to a cap of 1, because proportional demand can not be greater than 100%. The constant demand elasticity is:

$$\epsilon(\pi) = -\frac{\partial \log(d(\pi))}{\partial \log \pi} = \lambda. \tag{2.25}$$

The parameter τ can also be interpreted as the *fair-premium demand*,

[5]This is a generalised version of the Kumaraswamy distribution, which in its standard form takes values only over [0,1] (Kumaraswamy (1980)).

that is the demand when an actuarially fair premium is charged.

The motivation examples given at the beginning of this thesis can then be shown to correspond to this iso-elastic demand function, with fair-premium demand $\tau = 0.5$ and constant demand elasticity $\lambda = 0.435$ for both risk-groups. These are reasonable parameters.[6]

This equation specifies demand as a function of the "premium loading" (π/μ). When the premium loading is high (insurance is expensive), demand is low, and vice versa. The "iso-elastic" terminology reflects that the price elasticity of demand is the same everywhere along the demand curve.

Figure 2.3 shows examples of iso-elastic demand functions as functions of premium π, for risk $\mu = 0.01$ and different values of the elasticity parameter $\lambda = 0.4, 0.8$ and 1.2. We observe that, for a given value of λ, $d(\pi)$ decreases with π. For a given value of premium π, $d(\pi)$ is smaller when λ is larger. This reflects the feature that a large value of λ leads to a higher sensitivity to premium changes.

An important point to note here is that power utility function of the form given in Equation 2.15 is concave only if the risk aversion parameter γ is less than 1. Such a constraint can be imposed on random variable Γ by setting $\tau = 1$ in Equation 2.23. Then the third branch of Equation 2.23 implies that $d(\pi) = 1$ for $\pi < \mu$, which corresponds to the standard assumption in the economics literature that all individuals are risk-averse and hence will buy insurance for premiums not exceeding their probability of loss. By permitting some individuals to be 'risk-lovers', the model better represents the partial take-up of insurance which is observed in practice. Although 'risk-loving' or 'risk-seeking' are the usual descriptions, 'risk-neglecting' might be a more

[6] Approximately half the population has some life insurance (see Footnote 2). For yearly renewable term insurance in the US, demand elasticity has been estimated at 0.4 to 0.5 (Pauly *et al.* (2003)). A questionnaire survey about life insurance purchasing decisions produced an estimate of 0.66 (Viswanathan *et al.* (2006)).

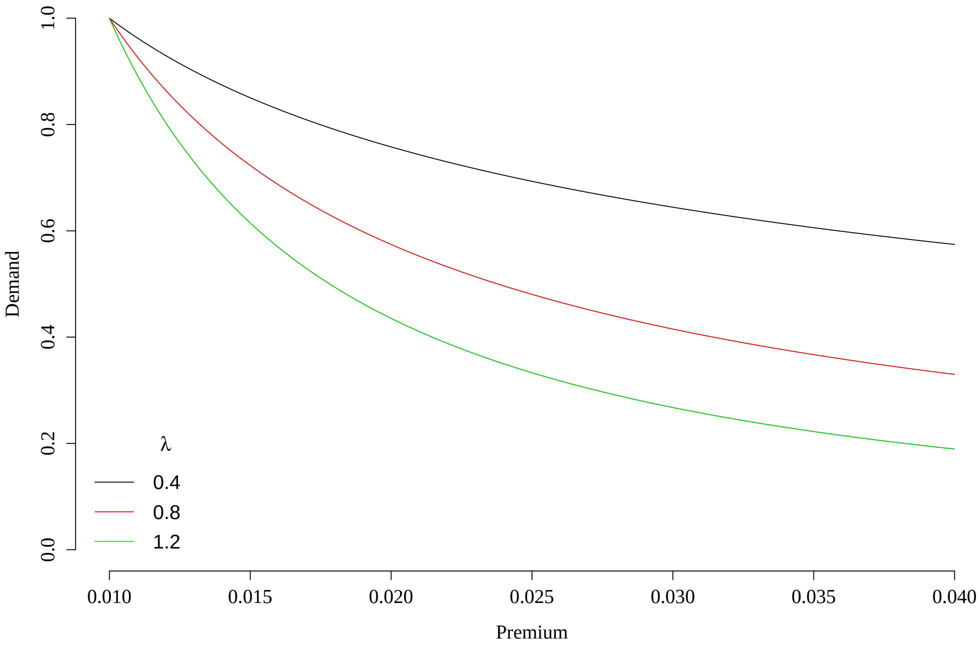

Figure 2.3: Plot of demand functions with respect to a given premium with $\tau = 1, \mu = 0.01, \lambda = 0.4, 0.8$ and 1.2.

realistic one.

In the above formulation, we assumed that the elasticity of demand from a risk-group is the same for all values of the premium π. The simplicity of the assumption is appealing. Any variations in the underlying elasticity parameters can be assumed to be smoothed using a single estimate, which can be reasonably justified if the variations are indeed small. An example is shown in the left-hand plot of Figure 2.4 with $\lambda = 0.5$.

The constant demand elasticity, say λ_i for risk-group i, could take different values for different risk-groups; for example with two risk-groups, λ_i

is typically higher for the higher risk-group. One example of this is shown in the right-hand plot of Figure 2.4, where we consider two risk-groups with $\mu_1 = 0.01$ and $\mu_2 = 0.04$ and the corresponding demand elasticities are $\lambda_1 = 0.2$ and $\lambda_2 = 0.8$ respectively.

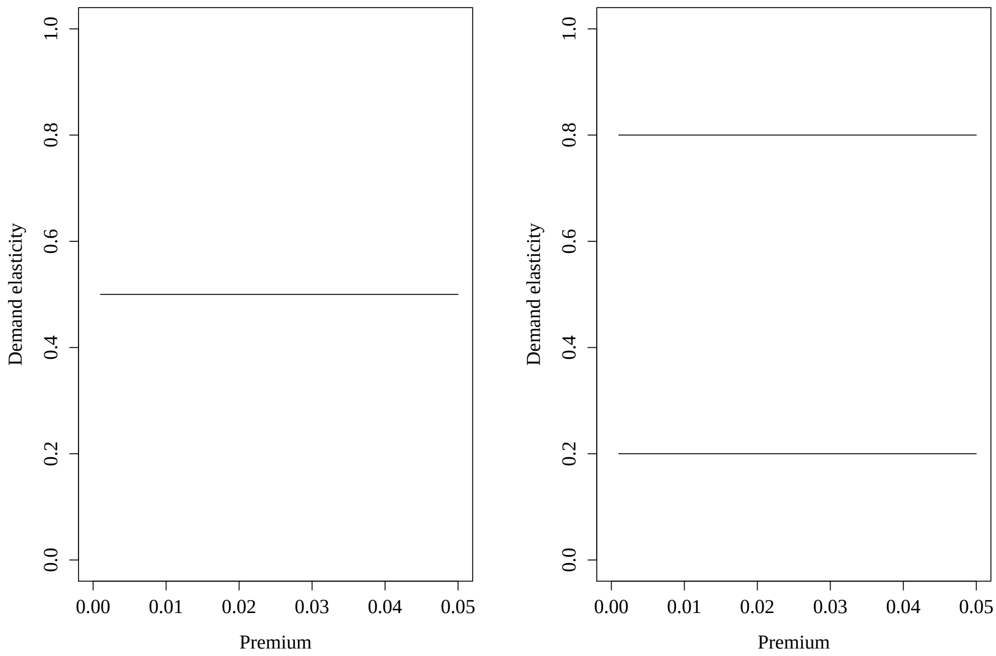

Figure 2.4: Plot of demand elasticity with respect to a given premium-the case of iso-elastic demand

2.3 Negative-exponential Demand

The assumptions of iso-elastic demand with constant demand elasticity allows mathematical tractability, but can be criticised as unrealistic. For most goods and services, we might expect demand elasticity to increase with price, because of the income effect: for high risks, a higher price of insurance represents a larger part of the consumer's total budget constraint, so the response to a small proportional change in that price should be higher. This suggests that demand elasticity should be modelled as an increasing function of the premium. One approach is shown in Figure 2.5. In this case, the linear relationship shown between demand elasticity and premium is posited to apply identically for the higher and low risk-groups.

The particular linear relationship in Figure 2.5, where the straight line representing demand elasticity passes through the origin, arises where the fair-premium demand elasticities λ_i vary in proportional to the corresponding fair premiums μ_i (for risk-groups $i = 1, 2,$), that is

$$\frac{\lambda_1}{\mu_1} = \frac{\lambda_2}{\mu_2}. \tag{2.26}$$

A suitable model for demand elasticity is then

$$\epsilon(\pi) = \frac{\lambda_1}{\mu_1}\pi = \frac{\lambda_2}{\mu_2}\pi, \tag{2.27}$$

which in turn leads to the following demand function:

$$d(\pi) = \tau \exp\left[\left(1 - \frac{\pi}{\mu}\right)\lambda\right]. \tag{2.28}$$

A further possibility is for demand elasticity as a function of premium to be a non-linear curve fitted as follows:

$$\epsilon(\pi) = k\pi^n, \text{ where } \frac{\lambda_1}{\mu_1^n} = \frac{\lambda_2}{\mu_2^n} = k, \text{ for some } n = \frac{\log(\lambda_2/\lambda_1)}{\log(\mu_2/\mu_1)}, \tag{2.29}$$

such that $\epsilon(\mu_1) = \lambda_1$ and $\epsilon(\mu_2) = \lambda_2$ if there are two risk-groups in the population. The parameter n can be thought of as the "elasticity of elasticity"

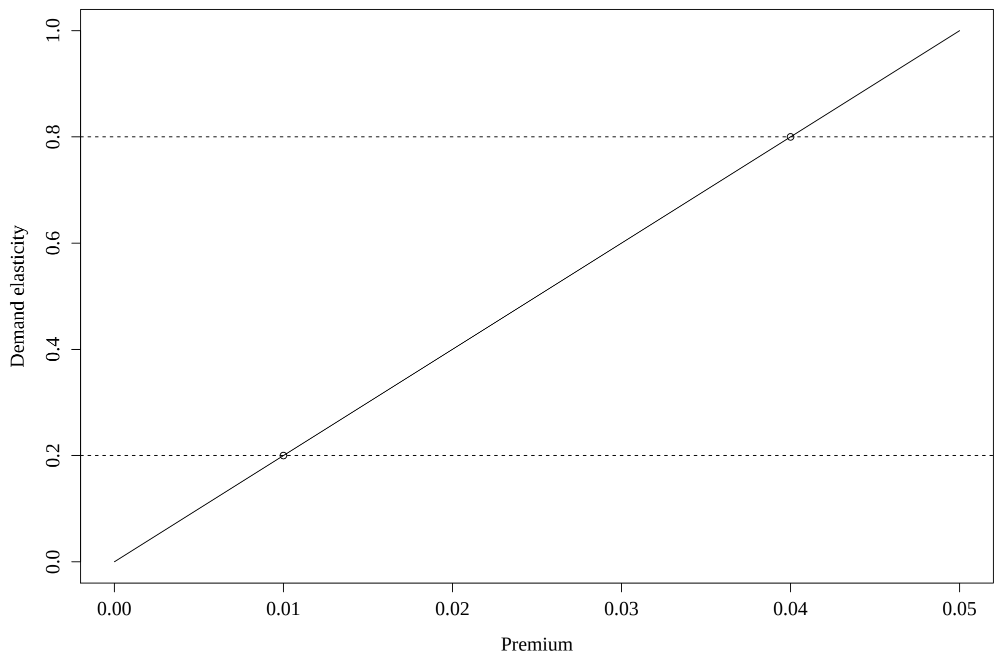

Figure 2.5: Plot of demand elasticity with respect to a given premium-the case of negative-exponential demand

of demand, which we shall call the *"second-order elasticity"*.

The form of demand elasticity in Equation 2.29 leads to the following class of demand functions:

$$d(\pi) = \tau \exp\left[\left\{1 - \left(\frac{\pi}{\mu}\right)^n\right\}\frac{\lambda}{n}\right], \tag{2.30}$$

with τ again representing fair-premium demand for insurance. We call this *"negative-exponential demand"*.

This demand function can be derived from the following distribution of risk preferences:

Suppose $Y \sim \text{Weibull}(\lambda, n)$ distribution, in the following form:

$$P\,[\,Y > y\,] = \exp\left[-\frac{\lambda}{n}\,y^n\right], \quad \lambda, n, y > 0. \tag{2.31}$$

Now define:

$$Z = Y\,|\,Y > \psi\ , \quad \text{where } \psi = \left[1 + \frac{n}{\lambda}\log\tau\right]^{1/n} > 0. \tag{2.32}$$

So,

$$P\,[\,Z > z\,] = \begin{cases} 1 & \text{if } z \le \psi; \\ \frac{P[Y>z]}{P[Y>\psi]} = \exp\left[\frac{\lambda}{n}\,(\psi^n - z^n)\right] & \text{if } z > \psi. \end{cases} \tag{2.33}$$

Next, define $\Gamma = 1/Z$, so that:

$$F_\Gamma(\gamma) = P\,[\,\Gamma < \gamma\,] = P\,[\,Z > 1/\gamma\,] = \begin{cases} 1 & \text{if } 1/\gamma \le \psi; \\ \exp\left[\frac{\lambda}{n}\,(\psi^n - \gamma^{-n})\right] & \text{if } 1/\gamma > \psi. \end{cases} \tag{2.34}$$

By Equation 2.22:

$$F_\Gamma(\gamma) = P\,[\Gamma < \gamma] = d\left(\frac{\mu}{\gamma}\right) \tag{2.35}$$

which leads to the form of negative-exponential demand in Equation 2.30.

Note: For this demand function, second-order elasticity $n = 1$ corresponds to demand elasticity as a straight-line function of premium as in Equation 2.28. $n \to 0$ corresponds to iso-elastic demand. Because when $n \to 0$, Equation 2.29 becomes:

$$\epsilon(\pi) = \lambda, \tag{2.36}$$

and the demand function becomes:

$$d(\pi) = \tau \left(\frac{\mu}{\pi}\right)^{\lambda}, \tag{2.37}$$

which is the iso-elastic demand function. The proof is given in Theorem A.2.1 in Appendix A.

And when $n \neq 1$, demand elasticity becomes a curved function of premium. Two possible graphical representations of the negative-exponential demand function in Equation 2.30 with $n \neq 1$ are shown in Figure 2.6:

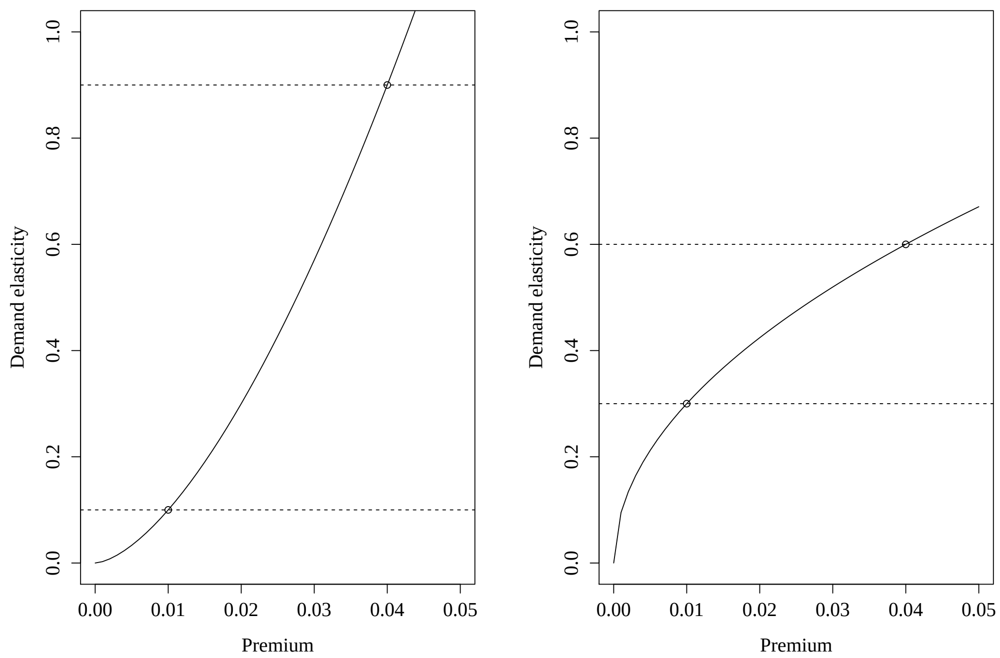

Figure 2.6: Plot of demand elasticity with respect to a given premium-the case of negative-exponential demand

- In the left-hand plot, $\mu_1 = 0.01$, $\mu_2 = 0.04$, $\epsilon(\mu_1) = 0.1$ and $\epsilon(\mu_2) = 0.9$

so that $n = \log(9)/\log(4) \approx 1.585$ and $\lambda_2 > \lambda_1$.

- In the right-hand plot, $\mu_1 = 0.01$, $\mu_2 = 0.04$, $\epsilon(\mu_1) = 0.3$ and $\epsilon(\mu_2) = 0.6$ so that $n = \log(2)/\log(4) = 0.5$ and $\lambda_2 < \lambda_1$.

2.4 Summary

In this chapter, we provide a utility-based micro-foundation for the proportional insurance demand function by introducing heterogeneity in individuals' utility functions. As observed in many insurance markets, not all individuals choose to buy insurance at any given premium because of this heterogeneity. In our model, individuals make decisions completely deterministically on the basis of certainty-equivalent utility calculations, but the insurer observes apparently stochastic decision-making, resulting in a proportional insurance demand function.

In the next chapter, we introduce generalised framework to allow individuals to belong to different risk-groups having different loss probabilities, built upon which equilibrium in the insurance market under given risk classification is analysed.

Chapter 3

Risk Classification and Equilibrium in Insurance Markets

In the previous chapter, we have developed a framework for insurance demand based on heterogeneous risk preferences of individuals who have the same wealth W and the same probabilities of loss amount L. In this chapter, we provide a generalised framework to allow individuals to belong to different risk-groups having different loss probabilities. Then, based on the assumption that insurers will charge premium(s) at which they break even under different risk classifications, we look at different types of equilibria.

3.1 Framework for Insurance Risk Classification

For simplicity, we assume all wealth and losses are of unit amount, that is $W = L = 1$. Whist we will consider a variety of risk classifications in this thesis, we always assume that as a result of regulation or otherwise, the risk classification is common to all insurers. Competition between insurers each using different risk classifications will not be considered in this thesis.

Suppose that a population can be sub-divided into m distinct risk-groups, based on information which is fully observable by insurers. Let $\mu_1, \mu_2, ..., \mu_m$ be the underlying probabilities of loss, of an individual in each of the risk-groups. Without loss of generality, we assume, the risk-groups are indexed in an increasing order of risk, i.e. $0 < \mu_1 < \mu_2 < ... < \mu_m < 1$.

Let μ be a random variable denoting the probability of loss for an individual chosen at random from the whole population, such that $P[\mu = \mu_i] = p_i$ for $i = 1, 2, ..., m$. In other words, the proportion of the population belonging to risk-group i is p_i.

Suppose insurers charge premiums $\pi_1, \pi_2, \ldots, \pi_m$ for the respective risk-groups. Initially we do not impose any constraints on the order or size of insurance premiums, so that the insurers are free to charge any premiums to any risk-group. Based on the framework developed in Chapter 2, we denote the demand for insurance for risk-group i, given offered premium π_i, by $d_i(\pi_i)$, where $0 \leq d_i(\pi_i) \leq 1$ and $d_i(\pi_i)$ is non-increasing in π_i.

Let the insurance purchasing decision of an individual chosen at random from the whole population be represented by the indicator random variable Q, taking the value of 1 if insurance is purchased; and 0 otherwise. Then conditional on the risk-group, Q is a Bernoulli random variable defined by:

$$E[Q|\mu = \mu_i] = P[Q = 1|\mu = \mu_i] = d_i(\pi_i). \tag{3.1}$$

Then the expected population demand for insurance is the unconditional expected value of Q:

$$E[Q] = \sum_{i=1}^{m} E[Q|\mu = \mu_i]P[\mu = \mu_i] = \sum_{i=1}^{m} d_i(\pi_i)p_i. \tag{3.2}$$

$E[Q]$ corresponds to a unit version of the fifth row of the tables in the motivating examples earlier in this thesis.

Now suppose that the occurrence of a loss event for an individual chosen at random from the whole population is represented by the indicator random variable, X, taking the value of 1 if a loss event has occurred; and 0 otherwise. Then X is a Bernoulli random variable defined as:

$$E[X|\mu = \mu_i] = P[X = 1|\mu = \mu_i] = \mu_i. \tag{3.3}$$

Then the expected population loss is the unconditional expected value of X:

$$E[X] = \sum_{i=1}^{m} E[X|\mu = \mu_i]\, P[\mu = \mu_i] = \sum_{i=1}^{m} \mu_i p_i. \tag{3.4}$$

$E[X]$ corresponds to a unit version of the third row of the tables in the motivating examples earlier in this thesis.

We assume that Q and X are independent, conditional on $\mu = \mu_i$. That is, the level of risk may influence the decision to buy insurance, but there is no moral hazard; insured individuals in any risk-group are not more likely to suffer the loss event than uninsured individuals. Then the expected claims outgo for insurers is:

$$\begin{aligned}
E[QX] &= \sum_{i=1}^{m} E[QX|\mu = \mu_i]\, P[\mu = \mu_i], \\
&= \sum_{i=1}^{m} E[Q|\mu = \mu_i]\, E[X|\mu = \mu_i]\, P[\mu = \mu_i], \\
&= \sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i. \qquad (3.5)
\end{aligned}$$

Finally, for an individual chosen at random from the whole population, define random variable Π, as the premium paid by that individual. As premiums are only paid by individuals who purchase insurance, $\Pi = Q\Pi$. And since everybody in risk-group i is offered the same premium π_i, we have:

$$E[\Pi|\mu = \mu_i] = E[Q\Pi|\mu = \mu_i] = E[Q|\mu = \mu_i]\pi_i = d_i(\pi_i)\pi_i. \qquad (3.6)$$

Then the unconditional expected premium income is:

$$E[\Pi] = \sum_{i=1}^{m} E[\Pi|\mu = \mu_i]\, P[\mu = \mu_i] = \sum_{i=1}^{m} d_i(\pi_i)\pi_i p_i. \qquad (3.7)$$

$E[\Pi]$ corresponds to the final column of the fourth row in the tables in the motivating examples earlier in this thesis. Since individuals who do not buy insurance pay premium zero, we can also write $E[\Pi] = E[Q\Pi]$.

The expected profit for insurers, as a function of risk classification $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$, is then :

$$\rho(\underline{\pi}) = E[\Pi] - E[QX] = \sum_{i=1}^{m} d_i(\pi_i)\pi_i p_i - \sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i. \qquad (3.8)$$

A full probabilistic model, of heterogeneity in insurance purchasing behaviour leading to a framework within which insurance risk classification is provided in Appendix A.1.

3.2 Equilibrium in the Insurance Market

Equilibrium is achieved when the expected profit for insurers is zero. In other words, $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$ denotes an equilibrium, if it satisfies the *equilibrium condition*:

$$\rho(\underline{\pi}) = E[\Pi] - E[QX] = 0, \tag{3.9}$$

$$\Rightarrow \sum_{i=1}^{m} d_i(\pi_i)\pi_i p_i = \sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i. \tag{3.10}$$

For brevity, we firstly confine our attention to two obvious, and opposing, risk classifications : *full risk classification* and *no risk classification*. We aim to analyse equilibrium premium(s) in each case.

In this thesis, we assume that an equilibrium has been reached and that insurers use the same premium strategy that break even. We do not consider how equilibrium was reached, or whether profits or losses were made along the way. We model the insurance market as a “timeless equilibrium”, “equilibrium” in the sense that it focuses on the steady state where all insurers' profits and losses are competed away. And “timeless” in the sense that when a pooled premium is charged on different risk-groups due to a restriction on risk classification, it glosses over any sequence of profits and losses which occur as insurers adjust the pooled premium towards the equilibrium level. Whilst risk classification is restricted, the level of pooled premiums is not.

3.2.1 Full Risk Classification

An obvious solution to Equation 3.10 is to set premiums equal to the respective loss probabilities, i.e. $\pi_i = \mu_i$ for $i = 1, 2, ..., m$. We call this particular set of equilibrium the *risk-differentiated premiums* under *full risk classification.* In this case, the expected population demand for insurance is

$$E[Q] = \sum_{i=1}^{m} p_i \tau_i, \tag{3.11}$$

where τ_i is the fair-premium demand $d_i(\mu_i)$.

Also, when $\pi_i = \mu_i$ for $i = 1, ..., m$, the expected premium and expected claim are equal and given by:

$$E[\Pi] = E[QX] = \sum_{i=1}^{m} p_i \tau_i \mu_i. \tag{3.12}$$

We define the concept of *expected claim per policy*:

$$\frac{E[QX]}{E[Q]} = \frac{\sum_{i=1}^{m} p_i \tau_i \mu_i}{\sum_{i=1}^{m} p_i \tau_i}, \tag{3.13}$$

$$= \sum_{i=1}^{m} \alpha_i \mu_i, \tag{3.14}$$

where α_i is defined as the *fair-premium demand-share* for risk-group i,

$$\alpha_i = \frac{p_i \tau_i}{\sum_{j=1}^{m} p_j \tau_j}, i = 1, 2, ..., m. \tag{3.15}$$

So the *expected claim per policy at risk-differentiated premiums is actually a weighted-average of true risks of all risk-groups.* Intuitively speaking, when there are proportionally more high risks in the population, we would expect the average claim amount to increase. We will come back to this concept later in this thesis.

3.2.2 No Risk Classification

At the other end of the spectrum is the *pooled* equilibrium where risk classification is banned and so all risk-groups are charged the same premium π_0, i.e. $\pi_i = \pi_0$ for $i = 1, 2, \ldots, m$. As mentioned in the Introduction and demonstrated in the Motivating examples, charging pooled equilibrium premium leads to adverse selection in terms of increase in pooled premium and reduction in total expected number of insured. However, because insurers are assumed to adjust the pooled premium to whatever level is necessary to ensure equilibrium, and competition between insurers in risk classification is not permitted, adverse selection does not imply insurer losses.

We have the following result:

Result 3.1. *When all the insurers are banned from using any risk classification, there exists at least one pooled premium at which the expected profit for insurers is zero. However, its uniqueness is not guaranteed.*

The existence of a pooled equilibrium can be demonstrated as follows. Setting the pooled premium $\pi_0 = \mu_1$, the probability of loss for the lowest risk-group, leads to $\rho(\mu_1) \leq 0$.[1] Setting the pooled premium at the highest level of risk, i.e. $\pi_0 = \mu_m$, gives $\rho(\mu_m) \geq 0$. Assuming insurance demand to be a continuous function of premium, therefore, there exists at least one root $\pi_0 \in [\mu_1, \mu_m]$ which gives a pooled equilibrium, i.e. $\rho(\pi_0) = 0$. However, depending on the type of the demand function, uniqueness of the pooled equilibrium is not guaranteed, i.e. there might be multiple solutions. We will look at multiple solutions to equilibrium premium in detail in Section 3.6.

[1] For notational convenience, we specify only one argument for multivariate functions if all arguments are equal, e.g. we write $\rho(\pi)$ for $\rho(\pi, \pi, \ldots, \pi)$.

The expected population demand for insurance, in this case of pooled equilibrium becomes

$$E[Q] = \sum_{i=1}^{m} d_i(\pi_0) p_i. \tag{3.16}$$

For equilibrium, according to Equation 3.9, $\rho(\pi_0) = 0$, i.e. the expected premium and expected claim need to be equal, so

$$\rho(\pi_0) = \sum_{i=1}^{m} d_i(\pi_0) p_i (\pi_0 - \mu_i) = 0, \tag{3.17}$$

$$\Leftrightarrow \sum_{i=1}^{m} d_i(\pi_0) p_i \mu_i = \sum_{i=1}^{m} d_i(\pi_0) p_i \pi_0. \tag{3.18}$$

In this case, the expected claim per policy will be:

$$\frac{E[QX]}{E[Q]} = \frac{E[Q\Pi]}{E[Q]} = \pi_0. \tag{3.19}$$

We name this premium, π_0, the *pooled equilibrium premium.* It corresponds to the break-even premiums in Table 2 and 3 in the motivating examples earlier in this thesis.

3.2.3 Partial Risk Classification

Between the two extreme cases of risk classifications, i.e. *full risk classification* and *no risk classification*, there is also *partial risk classification*, in which insurers can observe different risk groups and can charge any premiums, subject to the equilibrium condition $\rho(\underline{\pi}) = 0$ with no specific constraints on the premiums.

Partial risk classification might be considered to be a more realistic strategy, because insurers might be allowed to differentiate premiums to some extent, but may not be able to fully reflect the differences between different risk-groups. We will look at *partial risk classification* in detail in Chapter 7.

In the rest of this chapter, we will analyse equilibrium for three types of demand functions, i.e. iso-elastic demand, negative-exponential demand and then any general demand function. In particular, we will focus on the case when there is no risk classification, i.e. the same pooled premium is charged to all risk-groups.

3.3 Iso-elastic Demand

For iso-elastic demand, we will firstly assume there are only two risk-groups in a population, i.e. a low risk-group and a high risk-group. We look at pooled equilibrium premium when these two groups have the same demand elasticity, i.e. $\lambda_1 = \lambda_2$, and also when they have different demand elasticities, i.e. λ_1 and λ_2 are not necessarily the same. Then, we extend the analysis to the case of more risk-groups.

3.3.1 Two Risk-groups: Equal Demand Elasticity

In this sub-section, we discuss equilibrium when the low risk-group and the high risk-group have the same elasticity of demand, i.e. $\lambda_1 = \lambda_2 = \lambda$. Our main result in this case is:

Result 3.2. *When the low risk-group and the high risk-group have equal elasticity of demand, there is a unique pooled equilibrium premium.*

This is because solving Equation 3.17 gives:

$$\pi_0 = \frac{\alpha_1 \mu_1^{\lambda+1} + \alpha_2 \mu_2^{\lambda+1}}{\alpha_1 \mu_1^{\lambda} + \alpha_2 \mu_2^{\lambda}}, \tag{3.20}$$

i.e. **there is a unique equilibrium premium.** It is illustrative to express π_0 as a weighted average of the true risks μ_1 and μ_2:

$$\pi_0 = \theta\mu_1 + (1-\theta)\mu_2, \tag{3.21}$$

where

$$\theta = \frac{\alpha_1}{\alpha_1 + \alpha_2 \left(\frac{\mu_2}{\mu_1}\right)^{\lambda}}, \tag{3.22}$$

$$\alpha_i = \frac{p_i\tau_i}{p_1\tau_1 + p_2\tau_2}, \; i = 1,2 \tag{3.23}$$

where α_i is the fair-premium demand share as defined in Equation 3.15.

Figure 3.1 shows an example of a unique equilibrium premium when both risk-groups have the same demand elasticity. It demonstrates that the expected total profit for insurers monotonically increases with the pooled premium. When the premium is very low, e.g. $\pi = \mu_1 = 0.01$ in this example, the expected total profit for insurers is negative provided at least one high risk buys insurance. And when the premium is very high, e.g. $\pi = \mu_2 = 0.04$ in this example, the expected total profit for insurers is positive provided at least one low risk buy insurance. Because expected total profit is a continuous function of premium, there is at least one pooled equilibrium premium at which insurers break even. And in this scenario of equal demand elasticity, there is a unique pooled equilibrium premium.

Note that π_0 in Equation 3.20 does not depend directly on the individual values of the population fractions (p_1, p_2) and fair-premium demands (τ_1, τ_2), but only indirectly on these parameters through the demand-shares (α_1, α_2). In other words, populations with the same true risks (μ_1, μ_2) and demand-shares (α_1, α_2) have the same equilibrium premium, even if the underlying

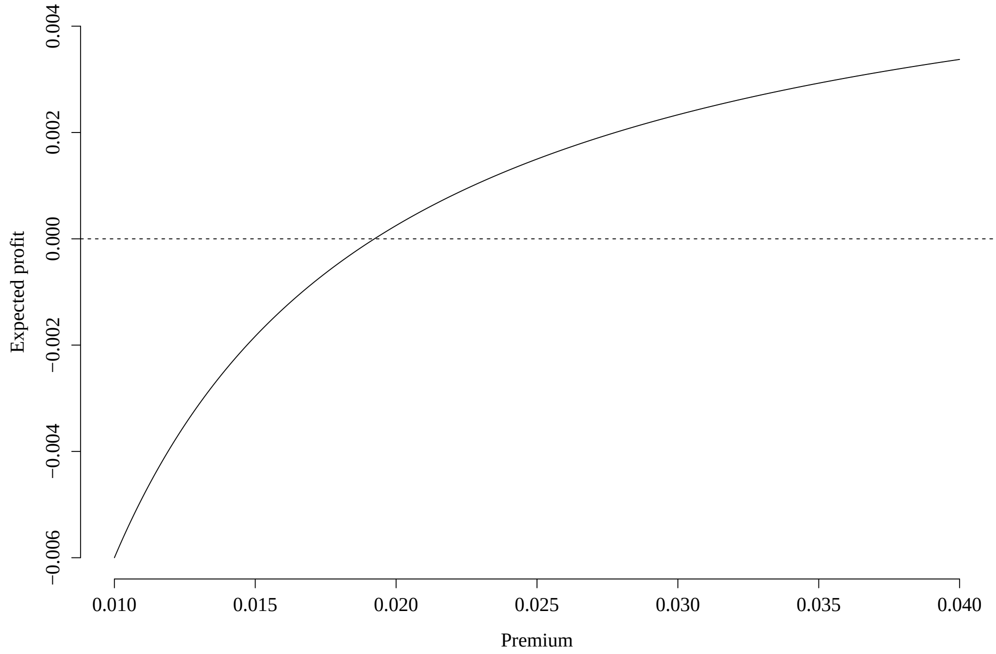

Figure 3.1: Expected profit as a function of pooled premium for a population with $(\mu_1, \mu_2) = (0.01, 0.04), \alpha_1 = 90\%, \alpha_2 = 10\%$ and $\lambda_1 = \lambda_2 = 1$.

(p_1, p_2) and (τ_1, τ_2) are different.

Figure 3.2 plots the pooled equilibrium premium against demand elasticity, λ, for two different population structures with the same true risks $(\mu_1, \mu_2) = (0.01, 0.04)$ but different fair-premium demand-shares (α_1, α_2).

The following results follow directly from Equation 3.20, and are illustrated in Figure 3.2.

Result 3.3.

$$\lim_{\lambda \to 0} \pi_0 = \alpha_1 \mu_1 + \alpha_2 \mu_2. \tag{3.24}$$

Intuitively, if demand is inelastic, changing the premium makes no difference, and so the equilibrium premium will be the same as the expected claim per

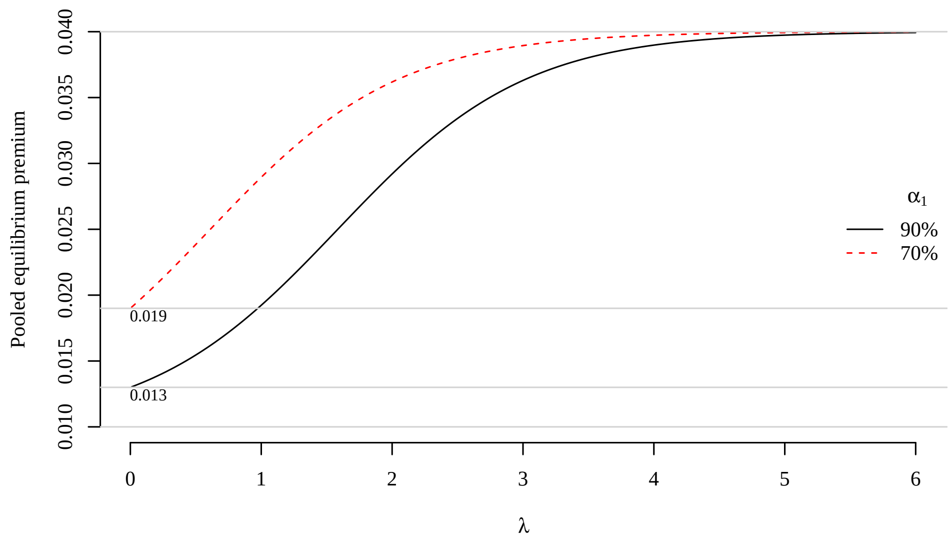

Figure 3.2: Pooled equilibrium premium as a function of λ for two populations with the same $(\mu_1, \mu_2) = (0.01, 0.04)$ but different values of (α_1, α_2).

policy if risk-differentiated premiums were charged. In Figure 3.2, this is 0.013 and 0.019 for fair-premium demand-shares of $\alpha_1 = 0.9$ and $\alpha_1 = 0.7$ respectively.

Result 3.4.

$$\pi_0 \textit{ is an increasing function of } \lambda. \tag{3.25}$$

Intuitively, an increase in demand elasticity means that at any premium between μ_1 and μ_2, there will be less demand than before from low risks and more demand than before from high risks; the premium for which profits on low risks exactly balance losses on high risks will therefore be higher. In Figure 3.2, both curves slope upwards. In Equation 3.22, increasing λ reduces the weight θ on low-risk, resulting in an increase in the equilibrium premium π_0.

Result 3.5.

$$\lim_{\lambda \to \infty} \pi_0 = \mu_2. \tag{3.26}$$

Intuitively, if demand elasticity is very high, demand from the low risk-group falls to zero for any premium above their true risk μ_1. The only remaining insureds are then all high risks, so the equilibrium premium must move to $\pi_0 = \mu_2$. In Figure 3.2, both curves converge to $\mu_2 = 0.04$ as λ increases.

Result 3.6.

$$\pi_0 \textit{ is a decreasing function of } \alpha_1. \tag{3.27}$$

Intuitively, if the fair-premium demand-share α_1 of the lower risk-group increases, we would expect the equilibrium premium to fall. In Figure 3.2, the curve for $\alpha_1 = 90\%$ lies below the curve for $\alpha_1 = 70\%$.

3.3.2 More Risk-groups: Equal Demand Elasticity

Some of the results on equilibrium premium for two risk-groups in the previous sub-section can be easily generalised to m risk-groups (where $m \geq 2$). In particular, we can derive an explicit form of the pooled equilibrium premium for the case when all m risk-groups have the same demand elasticity, i.e. $\lambda_1 = \lambda_2 = ... = \lambda_m = \lambda$. Our key result is:

Result 3.7. *When all m risk-groups have the same demand elasticity λ, there is a unique pooled equilibrium premium.*

Based on iso-elastic demand, the equilibrium condition defined in Equation 3.17 gives:

$$\sum_{i=1}^{m} p_i \tau_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda} (\pi_0 - \mu_i) = 0, \text{ or, equivalently:} \tag{3.28}$$

$$\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda} = \sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda+1}, \text{ where } \alpha_i = \frac{p_i \tau_i}{\sum_{j=1}^{m} p_j \tau_j} \text{ for } i = 1, 2, ..., m. \tag{3.29}$$

Solving Equation 3.29 gives:

$$\pi_0 = \frac{\sum_{i=1}^{m} \alpha_i \mu_i^{\lambda+1}}{\sum_{i=1}^{m} \alpha_i \mu_i^{\lambda}}. \tag{3.30}$$

Using similar logic as in sub-section 3.3.1, we have the following main results for more than two risk-groups:

Result 3.8.

$$\pi_0 \geq \sum_{i=1}^{m} \alpha_i \mu_i. \tag{3.31}$$

In other words, given an iso-elastic demand, when there are many risk-groups, pooled equilibrium premium is never smaller than the expected claim per policy under *full risk classification*.

Result 3.9.

$$\pi_0 \uparrow \lambda. \tag{3.32}$$

In other words, pooled equilibrium premium increases with demand elasticity λ.

Result 3.10.

$$\lim_{\lambda \to \infty} \pi_0 = \mu_m. \tag{3.33}$$

In other words, pooled equilibrium premium is capped at the level of the highest risk when demand elasticity is very large.

Equation 3.29 provides another perspective to the equilibrium condition. Consider a random variable, V, taking values $v_i = \frac{\mu_i}{\pi_0}$ with probabilities α_i for $i = 1, 2, ..., m$. Then, Equation 3.29 says that, under equilibrium, the random variable V satisfies:

$$E[V^{\lambda}] = E[V^{\lambda+1}]. \tag{3.34}$$

This is an important result which we will refer to in later chapters on loss coverage and social welfare.

3.3.3 Two Risk-groups: Different Demand Elasticities

In practice, however, it might not always be the case that both the low risk-group and the high risk-group will have the same demand elasticity. Therefore, in this sub-section, we consider the case when the low risk-group and the high risk-group have different elasticities of demand, i.e. $\lambda_1 \neq \lambda_2$.

If we substitute iso-elastic demand function at an equilibrium premium, π_0,

$$d_i(\pi_0) = \tau_i \left(\frac{\mu_i}{\pi_0} \right)^{\lambda_i} \tag{3.35}$$

into the equilibrium condition in Equation 3.17, we get the following equation:

$$\lambda_1 \log \left(\frac{\pi_0}{\mu_1} \right) + \lambda_2 \log \left(\frac{\mu_2}{\pi_0} \right) = \log \left(\frac{\alpha_1(\pi_0 - \mu_1)}{\alpha_2(\mu_2 - \pi_0)} \right). \tag{3.36}$$

After rearranging the above equation, we have:

$$\lambda_2 = -\frac{\log(\frac{\pi_0}{\mu_1})}{\log(\frac{\mu_2}{\pi_0})}\lambda_1 + \frac{\log \left(\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)} \right)}{\log(\frac{\mu_2}{\pi_0})}. \tag{3.37}$$

We provide two illustrative plots in Figure 3.3 showing the relationship between equilibrium premium, π_0, and demand elasticities of both low and high risk-groups, λ_1 and λ_2. A contour plot of equilibrium premium, π_0, for $\alpha_1 = 70\%$ is the left plot, and $\alpha_1 = 90\%$ is the right plot. In both plots, with λ_1 and λ_2 on x and y axes respectively, the straight lines represent different levels of equilibrium premiums for any given combinations of λ_1 and λ_2. If we focus on any one particular straight line, i.e. for a particular equilibrium premium, π_0, we notice that λ_1 and λ_2 are linearly related as shown in Equation 3.37. So we have the following results on pooled equilibrium premium:

Result 3.11. *The same equilibrium premium can be attained by populations with different demand elasticities, as long as these are linearly related as per Equation 3.37. Therefore, to reach the same equilibrium premium, an increase/decrease in λ_1 means a decrease/increase in λ_2.*

Result 3.12.

$$\lim_{(\lambda_1,\lambda_2)\to(0,0)} \pi_0 = \alpha_1\mu_1 + \alpha_2\mu_2. \tag{3.38}$$

This follows directly from Equation 3.36. Intuitively, if demand is inelastic, the equilibrium pooled premium will be close to the expected claim under fair premiums.

Result 3.13.

$$\pi_0 \geq \alpha_1\mu_1 + \alpha_2\mu_2. \tag{3.39}$$

This follows from the fact that $\mu_1 \leq \pi_0 \leq \mu_2$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and the relationship in Equation 3.36. Intuitively, the equilibrium pooled premium is never smaller than the expected claim under fair premiums.

Result 3.14.

$$\lim_{(\lambda_1,\lambda_2)\to(\infty,\lambda_2)} \pi_0 = \mu_2, \tag{3.40}$$

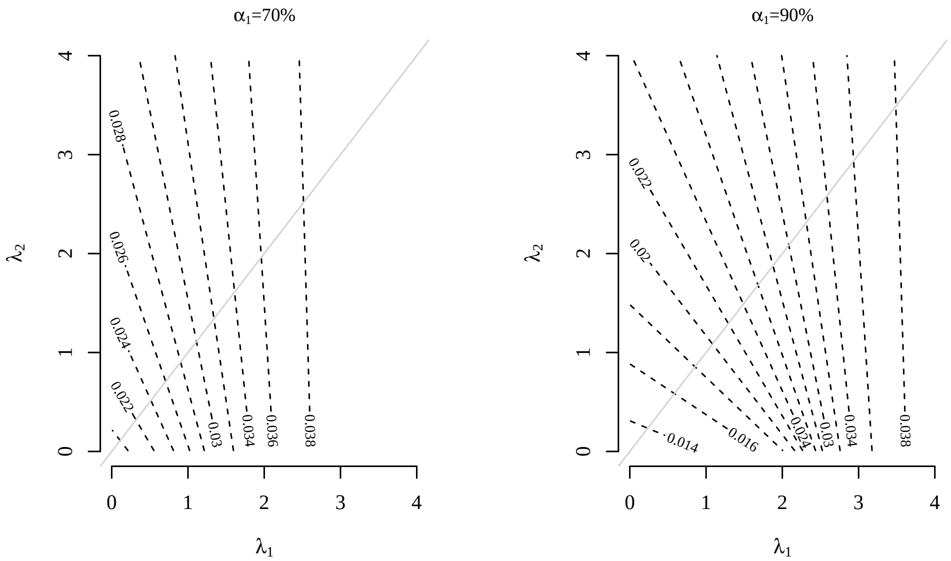

Figure 3.3: Equilibrium premium as a function of (λ_1, λ_2) for $\alpha_1 = 70\%$ and $\alpha_1 = 90\%$, when $(\mu_1, \mu_2) = (0.01, 0.04)$.

which again follows from Equation 3.36. Intuitively, high demand elasticities lead to an equilibrium where only high risks purchase insurance.

Result 3.15. *Given* π_0*:*

$$\frac{\log\left(\pi_0/\mu_1\right)}{\log\left(\mu_2/\pi_0\right)} \textit{ is an increasing function of } \pi_0, \tag{3.41}$$

i.e. the (absolute value of the) slope of the line, in Equation 3.36 increases with π_0. Intuitively, a higher equilibrium premium π_0 is consistent with higher sensitivity to λ_2 and lower sensitivity to λ_1. In the limit, as $\pi_0 \to \mu_2$, the straight line in Equation 3.36 becomes perpendicular to the λ_1-axis, as can be seen from Figure 3.3.

Result 3.16. *Given* π_0*:*

$$\lim_{\lambda_1 \to 0} \lambda_2 = \frac{\log\left(\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)}\right)}{\log\left(\frac{\mu_2}{\pi_0}\right)} \textit{ is an increasing function of } \pi_0, \tag{3.42}$$

i.e. the intercept on the λ_2-axis in the plots of Figure 3.3 increases with π_0. Intuitively, if the low-risk group is insensitive to premiums, a higher equilibrium premium π_0 is consistent with higher demand elasticity λ_2 for of high risks, because this increases the demand from that group at any premium $\pi_0 < \mu_2$.

Result 3.17. *Given π_0, changing the fair-premium demand-share α_1 results in parallel shifts of the straight lines given in Equation 3.36, as the slopes remain unchanged while the intercepts are adjusted accordingly.*

In Figure 3.3, changing α_1 from 70% to 90% has the effect of translating the contours towards the top-right corner. It also confirms that increasing the fair-premium demand-share α_1 results in a decrease in equilibrium premium, because the impact of the low risk-group increases.

3.4 Negative-exponential Demand

So far, we have only considered constant demand elasticities (as a function of premium), either for all individuals in the population, or for all individuals belonging to a particular risk-group. However, it can be argued that demand elasticities should actually be increasing functions of premiums (instead of being a constant), to reflect the income effect on demand. The argument being that for high risks, a higher price of insurance represents a larger part of the consumer's total budget constraint, so the response to a small proportional change in that price should be higher. In this section, we generalise our analysis to allow for different demand elasticity functions, $\epsilon_i(\pi)$ for different risk-groups $i = 1, 2, ..., m$.

Recall from Section 2.3, negative-exponential demand has the following form:

$$d_i(\pi) = \tau_i \exp\left[\left\{1 - \left(\frac{\pi}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right],\ i = 1, 2, ..., m, \tag{3.43}$$

with demand elasticity function being:

$$\epsilon_i(\pi) = \lambda_i\left(\frac{\pi}{\mu_i}\right)^n. \tag{3.44}$$

3.4.1 Two Risk-groups: Equal Demand Elasticity

We firstly assume that there are two risk-groups in a population, i.e. a low risk-group and a high risk-group with probabilities of loss $\mu_1 < \mu_2$. We also assume that the fair-premium demand elasticities are given by

$$\epsilon_i(\mu_i) = \lambda_i,\ i = 1, 2. \tag{3.45}$$

Then there exists an n such that

$$\epsilon(\pi) = \lambda_1\left(\frac{\pi}{\mu_1}\right)^n = \lambda_2\left(\frac{\pi}{\mu_2}\right)^n, \tag{3.46}$$

$$\Rightarrow \frac{\lambda_2}{\lambda_1} = \left(\frac{\mu_2}{\mu_1}\right)^n = \beta^n, \tag{3.47}$$

where $\beta = \frac{\mu_2}{\mu_1}$, i.e. the relative risk between the high risks and the low risks. This means that everyone's demand elasticity varies in the same way in response to variations in the premium.

To examine non-constant demand elasticities, firstly we fix the second-order elasticity n for the two risk-groups. The two demand functions are linked in the following way:

$$\frac{\lambda_1}{\mu_1^n} = \frac{\lambda_2}{\mu_2^n} = k. \tag{3.48}$$

This ensures that, $d_i(\mu_i) = \tau_i$ and $\epsilon_i(\mu_i) = \lambda_i$ as required, and also:

$$\epsilon(\pi) = \epsilon_i(\pi) = k\pi^n, \text{ for } i = 1, 2. \tag{3.49}$$

This implies that varying demand elasticity of one risk-group affects the other, due to the relationship: $\lambda_2 = \lambda_1\beta^n$ where $\beta = \mu_2/\mu_1$. In this section we will assume that, the parameters, n, μ_1 and μ_2 (and thus β) are fixed, while we study the impact of changing the demand elasticity of the low risk-group λ_1 (with corresponding changes in λ_2).

Suppose proportion of population for the two risk-groups are p_1 and p_2 respectively. Now if the same premium is charged for both risk-groups, the equilibrium premium, π_0, should satisfy the equilibrium condition, $\rho(\pi_0) = 0$.

We denote demand elasticities at pooled equilibrium premium π_0 by λ_0, therefore,

$$\epsilon(\pi_0) = \lambda_0 = \frac{\lambda_1}{\mu_1^n}\pi_0^n = \frac{\lambda_2}{\mu_2^n}\pi_0^n. \tag{3.50}$$

Within this framework, the pooled equilibrium premium is unique and is given by:

$$\pi_0 = \frac{\alpha_1 \exp\left[\left\{1-\left(\frac{\pi_0}{\mu_1}\right)^n\right\}\frac{\lambda_1}{n}\right]\mu_1 + \alpha_2 \exp\left[\left\{1-\left(\frac{\pi_0}{\mu_2}\right)^n\right\}\frac{\lambda_2}{n}\right]\mu_2}{\alpha_1 \exp\left[\left\{1-\left(\frac{\pi_0}{\mu_1}\right)^n\right\}\frac{\lambda_1}{n}\right] + \alpha_2 \exp\left[\left\{1-\left(\frac{\pi_0}{\mu_2}\right)^n\right\}\frac{\lambda_2}{n}\right]}, \tag{3.51}$$

$$= \frac{\alpha_1 \exp\left[\left\{1-\frac{\lambda_0}{\lambda_1}\right\}\frac{\lambda_1}{n}\right]\mu_1 + \alpha_2 \exp\left[\left\{1-\frac{\lambda_0}{\lambda_2}\right\}\frac{\lambda_2}{n}\right]\mu_2}{\alpha_1 \exp\left[\left\{1-\frac{\lambda_0}{\lambda_1}\right\}\frac{\lambda_1}{n}\right] + \alpha_2 \exp\left[\left\{1-\frac{\lambda_0}{\lambda_2}\right\}\frac{\lambda_2}{n}\right]}, \text{ using Equation 3.50,} \tag{3.52}$$

$$= \frac{\alpha_1 e^{\frac{\lambda_1}{n}}\mu_1 + \alpha_2 e^{\frac{\lambda_2}{n}}\mu_2}{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}}, \tag{3.53}$$

$$= u\mu_1 + (1-u)\mu_2, \tag{3.54}$$

where

$$u = \frac{\alpha_1}{\alpha_1 + \alpha_2 e^{\frac{\lambda_1(\beta^n - 1)}{n}}}. \tag{3.55}$$

We have the following results:

Result 3.18.

$$\lim_{\lambda_1 \to 0} \pi_0 = \alpha_1\mu_1 + \alpha_2\mu_2. \tag{3.56}$$

In other words, the equilibrium pooled premium has a minimum when $\lambda_1 \to 0$. This result says that for non-elastic demand elasticity, the pooled equilibrium premium is the same as the expected claim per policy if risk-differentiated premiums were charged. This is a direct consequence of $\lim_{\lambda_1 \to 0} u = \alpha_1$ in Equation 3.55.

Result 3.19.

$$\lim_{\lambda_1 \to \infty} \pi_0 = \mu_2. \tag{3.57}$$

In other words, the equilibrium pooled premium has a maximum when $\lambda_1 \to +\infty$. This result shows that if demand is very elastic, then the increased premium due to pooling becomes unattractive to low risks and only the high risks buy insurance. This is a direct consequence of $\lim_{\lambda_1 \to \infty} u = 0$ in Equation 3.55.

Result 3.20.

$$u \textit{ is a decreasing function of } \lambda_1. \tag{3.58}$$

In other words, the proportion of insured low-risk individuals falls as the demand elasticity increases, which also explains the limiting values of u. This again follows from Equation 3.55 with a detailed proof given in Theorem B.2.1 in Appendix B.2.

Result 3.21.

$$\pi_0 \textit{ is an increasing function of demand elasticity parameter } \lambda_1. \tag{3.59}$$

This result is intuitive, because when demand elasticity becomes larger, there will be less demand from the low risks and more demand from the high risks. Thus the premium that will exactly balance the profits from the low risks and the losses from the high risks will be higher, which also explains the limiting values of π_0. The proof is given as Theorem B.2.2 in Appendix B.2.

Result 3.22. *When* $n > 0$,

$$\lambda_0 \textit{ is an increasing function of } \pi_0. \tag{3.60}$$

This is also intuitive because demand elasticity, i.e. sensitivity to insurance premium changes, is expected to increase with premium. The proof is given as Theorem B.2.3 in Appendix B.2.

Result 3.23. *For* $n > 0$,

$$\lambda_0 \textit{ is an increasing function of } \lambda_1. \tag{3.61}$$

In other words, when $n > 0$, the demand elasticity at pooled equilibrium premium increases as the demand elasticity parameter of low-risk (and high-risk) increases. Again this is intuitive because we have already shown that equilibrium premium increases with λ_1, and we are dealing with demand elasticity functions which are increasing with premium. This result is proved in Theorem B.2.4 in Appendix B.2.

3.4.2 More Risk-groups: Equal Demand Elasticity

The results on equilibrium premium for two risk-groups can be easily generalised to m risk-groups (where $m \geq 2$). In particular, we focus on the case when all m risk-groups have the same demand elasticity at pooled equilibrium premium, i.e. $\lambda_1 \left(\frac{\pi_0}{\mu_1}\right)^n = \lambda_2 \left(\frac{\pi_0}{\mu_2}\right)^n = ... = \lambda_m \left(\frac{\pi_0}{\mu_m}\right)^n$ (because there is an explicit form of pooled equilibrium premium in this case). Our key result is:

Result 3.24. *When all m risk-groups have the same demand elasticity at pooled equilibrium premium, there is a unique pooled equilibrium premium.*

Based on negative-exponential demand, the equilibrium condition defined in Equation 3.17 gives:

$$\sum_{i=1}^{m} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_0}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right](\pi_0 - \mu_i) = 0, \tag{3.62}$$

$$\Rightarrow \sum_{i=1}^{m} \alpha_i \exp\left(\frac{\lambda_i}{n}\right)\pi_0 = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{\lambda_i}{n}\right)\mu_i, \tag{3.63}$$

where

$$\alpha_i = \frac{p_i \tau_i}{\sum_{j=1}^{m} p_j \tau_j} \text{ for } i = 1, 2, ..., m. \tag{3.64}$$

And the unique pooled equilibrium premium, π_0, is given by:

$$\pi_0 = \frac{\sum_{i=1}^{m} \alpha_i \exp\left(\frac{\lambda_i}{n}\right) \mu_i}{\sum_{i=1}^{m} \alpha_i \exp\left(\frac{\lambda_i}{n}\right)}. \tag{3.65}$$

Results on pooled equilibrium premium in sub-section 3.4.1 can also be extended to the case of m risk-groups. Using the same approach in proving features of pooled equilibrium premium in sub-section 3.4.1, we have the following main results on π_0 for more risk-groups:

Result 3.25.

$$\pi_0 \geq \sum_{i=1}^{m} \alpha_i \mu_i. \tag{3.66}$$

In other words, when there are many risk-groups, pooled equilibrium premium given a negative-exponential demand is never smaller than the expected claim per policy under *full risk classification*.

Result 3.26.

$$\pi_0 \uparrow \lambda_1. \tag{3.67}$$

In other words, pooled equilibrium premium increases with demand elasticity parameter λ_1.

Result 3.27.

$$\lim_{\lambda_1 \to \infty} \pi_0 = \mu_m. \tag{3.68}$$

In other words, pooled equilibrium premium is capped at the level of the highest risk when demand elasticity is very large.

3.5 General Demand

We now turn our focus on any general demand function.

Using Equation 2.14, the general demand for insurance, for risk-group i, is:

$$d_i(\pi) = \tau_i \exp\Big[- \int_{\mu_i}^{\pi} \epsilon_i(s) d\log s \Big], \text{ for } i = 1, 2, ..., m. \tag{3.69}$$

Under this formulation, the equilibrium condition under risk pooling, i.e. $\rho(\pi_0) = 0$, gives:

$$\sum_{i=1}^{m} p_i \tau_i \exp\Big[- \int_{\mu_i}^{\pi_0} \epsilon_i(s) d\log s \Big](\pi_0 - \mu_i) = 0, \text{ or, equivalently:} \tag{3.70}$$

$$\sum_{i=1}^{m} \alpha_i \exp\Big[\int_{\pi_0}^{\mu_i} \epsilon_i(s) d\log s \Big](\pi_0 - \mu_i) = 0; \tag{3.71}$$

in which, the term:

$$\int_{\pi_0}^{\mu_i} \epsilon_i(s) d\log s, \tag{3.72}$$

can be interpreted using the concept of *arc elasticity of demand*, denoted by $\eta_i(a, b)$ and defined in Vazquez (1995) as follows:

$$\eta_i(a, b) = \frac{\int_a^b \epsilon_i(s) d\log s}{\int_a^b d\log s}. \tag{3.73}$$

Arc elasticity, $\eta_i(a, b)$, can be interpreted as the average value of (point) elasticity of demand, $\epsilon_i(s)$, over the price logarithmic arc from price a to price b. So, in our case, we can define:

$$\lambda_i = \eta_i(\pi_0, \mu_i) = \frac{\int_{\pi_0}^{\mu_i} \epsilon_i(s) d\log s}{\int_{\pi_0}^{\mu_i} d\log s}, \text{ for } i = 1, 2, ..., m. \tag{3.74}$$

Equation 3.71 can then be rewritten using arc elasticities as follows:

$$\sum_{i=1}^{m} \alpha_i \exp\Big[\lambda_i \int_{\pi_0}^{\mu_i} d\log s \Big](\pi_0 - \mu_i) = 0, \text{ or, equivalently:} \tag{3.75}$$

$$\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0} \right)^{\lambda_i} (\pi_0 - \mu_i) = 0. \tag{3.76}$$

Note that Equation 3.76 is equivalent to the equilibrium condition for iso-elastic demand (in Equation 3.28) in the case of m risk-groups ($m \geq 2$) where demand elasticity for each group might potentially be different. We summarise this observation in the following result:

Result 3.28. *At equilibrium, the formulation of demand elasticities as general functions of premium is equivalent to the formulation for iso-elastic demand, but with relevant arc elasticities in place of fixed elasticities.*

This result shows that for any given general demand function with demand elasticity as a function of premium, the equilibrium condition can be interpreted using the iso-elastic formulation but with appropriate arc elasticities in place of fixed elasticities.

Therefore, some results of pooled equilibrium premium on iso-elastic demand can also be interpreted for general demand in terms of arc elasticities.

In particular, $\pi_0 \geq \sum_{i=1}^{2} \alpha_i \mu_i$, i.e. pooled equilibrium premium given a general demand is no smaller than the expected claim per policy under *full risk classification*.

3.6 Multiple Equilibria

In sub-section 3.2.2, we noted that although there exits an equilibrium premium which satisfies the equilibrium condition in Equation 3.17, uniqueness of pooled equilibrium premium is not guaranteed. In this section, we look at the possibility of multiple solutions to the equilibrium condition, i.e. *multiple equilibria*, if there are two risk-groups with possibly different demand elasticities. Iso-elastic demand and negative-exponential demand are used as examples to demonstrate the necessary conditions that lead to multiple equilibria. Our main result is:

Result 3.29. *When the low risk-group and the high risk-group might not have the same elasticities of demand, multiple equilibria can arise. To have multiple equilibria, two conditions must be satisfied, which are:*

(a) demand elasticity for the low risk-group is substantially higher than demand elasticity for the high risk-group; and

(b) the low risk-group has a fair-premium demand-share within a very narrow range of very high values.

The first condition is the opposite of what we would expect in practice, because of the income effect on demand (i.e. for high risks, a higher price of insurance represents a larger part of the consumer's total budget constraint, so the response to a small proportional change in that price should be higher). And loosely speaking, the second condition means that the high risk-group must be very small relative to the total population.

These two conditions are practically ruled out by economic considerations, which make it unlikely to appear in practical situations. Therefore, multiple equilibria are unlikely to be troublesome in any practical application.

3.6.1 Iso-elastic Demand

In this sub-section, we use iso-elastic demand to analyse the conditions required for the existence of multiple equilibria.

Figure 3.4 shows an example where there are more than one solution (multiple equilibra) to the equilibrium condition in Equation 3.17. This figure demonstrates how expected profit behaves with respect to premium. In this example, given a much higher proportion of low risks in a population in addition to having a very large constant demand elasticity, there are three equilibrium premiums (i.e. three premiums at which the expected profit for insurers is zero). The smallest equilibrium premium, is close to the risk-differentiated premium for the low risk-group, $\mu_1 = 0.01$. The largest equilibrium premium is close to the risk-differentiated premium for the high risk-group, $\mu_2 = 0.04$. And there is a third equilibrium premium that is located in between the other two.

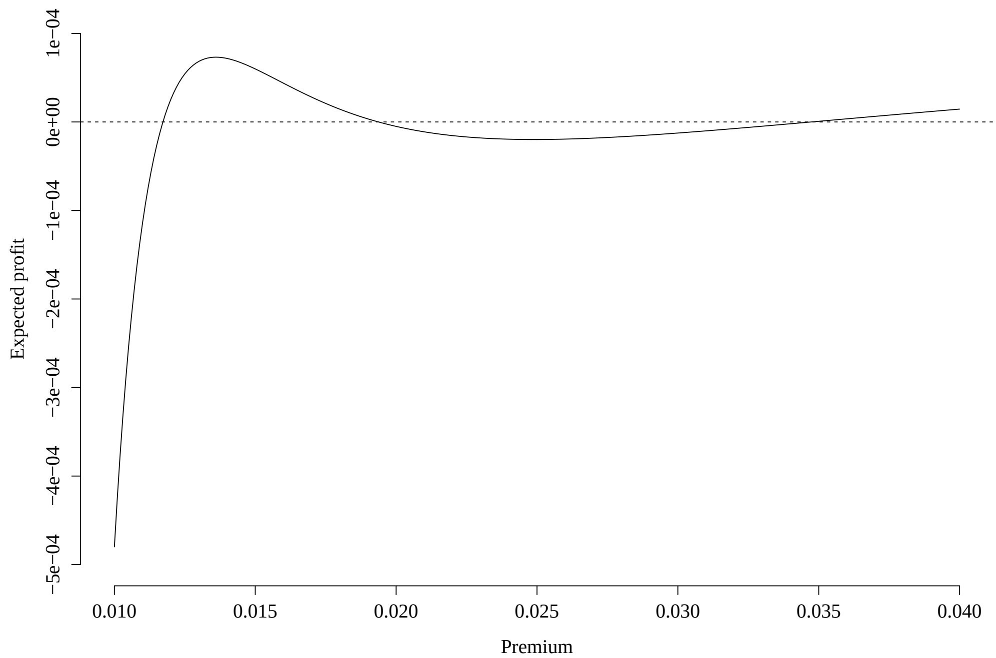

Figure 3.4: Expected profit as a function of premium for a population with $(\mu_1, \mu_2) = (0.01, 0.04), \alpha_1 = 99.2\%, \alpha_2 = 0.8\%$ and $\lambda_1 = 5, \lambda_2 = 1$.

Thus, for any choice of demand function, it is important that we determine whether or not multiple equilibria can arise, and if they can, whether or not they are associated with realistic parameters.

First we provide some examples of multiple equilibria in Figure 3.5. This uses $(\mu_1, \mu_2) = (0.01, 0.04)$ and an extreme divergence of elasticity parameters: $(\lambda_1, \lambda_2) = (5, 1)$. The figure shows plots of the expected profit curves, $\rho(\pi)$, for a narrow range of very high values [98.8%, 99.6%] of the fair-premium demand-share for low risks, α_1. We note the following patterns:

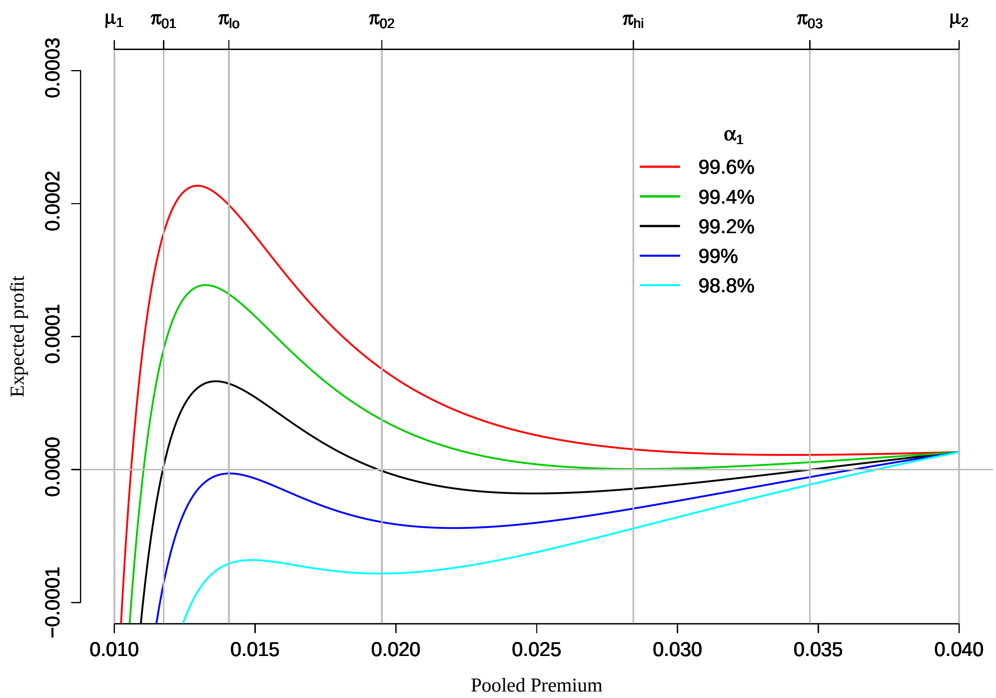

Figure 3.5: Profit function for $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\lambda_1, \lambda_2) = (5, 1)$ for various values of α_1 in the range $[98.8\%, 99.6\%]$

.

For $\alpha_1 < 99\%$, there is a unique equilibrium, close to μ_2.

For $\alpha_1 = 99\%$, in addition to an equilibrium close to μ_2, the profit curve attains a local maximum, which is also a root, at π_{lo}.

For $\alpha_1 = 99.4\%$, the profit curve has an equilibrium below π_{lo}, with another root at π_{hi}.

For $\alpha_1 > 99.4\%$, there is only one equilibrium close to μ_1.

For $99\% < \alpha_1 < 99.4\%$, the profit curve has 3 roots, $(\pi_{01}, \pi_{02}, \pi_{03})$, where

$\mu_1 < \pi_{01} < \pi_{\mathrm{lo}} < \pi_{02} < \pi_{\mathrm{hi}} < \pi_{03} < \mu_2$.

This example highlights that multiple equilibria are only possible for a narrow range $[\alpha_{\mathrm{lo}}, \alpha_{\mathrm{hi}}]$ of high values of the fair-premium demand-share for low risks, α_1, which in this particular example is $[99.0\%, 99.4\%]$. Moreover, the divergence of the elasticity parameters $(\lambda_1, \lambda_2) = (5, 1)$ in the example is also extreme, and probably implausible, because of the income effect on demand mentioned earlier.

These observations are captured in the following result:

Result 3.30. *For pooled premium, given* (μ_1, μ_2) *and* (λ_1, λ_2)*, multiple equilibria exist if*

$$\lambda_2 - \lambda_1 < -\frac{\sqrt{\mu_2} + \sqrt{\mu_1}}{\sqrt{\mu_2} - \sqrt{\mu_1}}, \quad \textit{and} \tag{3.77}$$

$$\frac{a(\pi_{lo})}{1 + a(\pi_{lo})} = \alpha_{lo} < \alpha_1 < \alpha_{hi} = \frac{a(\pi_{hi})}{1 + a(\pi_{hi})}, \quad \textit{where} \tag{3.78}$$

$$a(\pi) = \left(\frac{\mu_2 - \pi}{\pi - \mu_1}\right)\left(\frac{\mu_2^{\lambda_2}}{\mu_1^{\lambda_1}}\right)\pi^{-(\lambda_2 - \lambda_1)}, \quad \textit{and} \tag{3.79}$$

$$(\pi_{lo}, \pi_{hi}) \textit{ solves: } \pi^2 - \left(\mu_1 + \mu_2 + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\right)\pi + \mu_1\mu_2 = 0. \tag{3.80}$$

A proof and a discussion of Result 3.30 are provided in Theorem B.1.1 in Appendix B.

Note that, if there are multiple equilibria, there can be a maximum of 3 equilibria, because $a(\pi_0)$ is monotonic in π_0 over $(\pi_{\mathrm{lo}}, \pi_{\mathrm{hi}})$.

To illustrate the details, we provide a graphical representation of Result 3.30 in Figure 3.6, for the case when $(\mu_1, \mu_2) = (0.01, 0.04)$. The graph shows the two conditions that must be satisfied for multiple equilibria to arise:

(a) *Elasticity condition*: $\lambda_2 - \lambda_1 < -3$ is the condition in Equation 3.77 for this particular example with $\mu_1 = 0.01, \mu_2 = 0.04$. Multiple equilibria are possible for pairs of (λ_1, λ_2) in the region below the straight line $\lambda_2 - \lambda_1 = -3$ (but only if the demand-share condition in Condition (b) below is also satisfied).

   For example, the highlighted point $(\lambda_1, \lambda_2) = (5, 1)$ lies within the region where multiple equilibria are possible. This is consistent with our findings in the example in Figure 3.5.

(b) *Demand-share condition*: The figure also shows the contour plots of $(\alpha_{\text{lo}}, \alpha_{\text{hi}})$ as given in Equation 3.78, within the region identified in Condition (a). For any (λ_1, λ_2) point in the region below the straight line, multiple equilibria arise only if the fair-premium demand-share for low risks α_1 lies between the α_{lo} and α_{hi} specified for that point.

For the particular example of $(\lambda_1, \lambda_2) = (5, 1)$, $\alpha_{\text{lo}} = 99\%$ and $\alpha_{\text{hi}} = 99.4\%$ respectively, which matches with the boundaries identified in Figure 3.5.

Result 3.30 highlights that multiple equilibria arise only for extreme population structures. Specifically, Equation 3.78 requires the fair-premium demand-share for low risks, α_1, to be in a narrow range of high values. So provided α_1 is *less* than the lower end α_{lo} of this range, we can rule out multiple equilibria. The followings are corollaries to the main result:

Corollary 3.1. *Given* (μ_1, μ_2)*, define* $c = \frac{\sqrt{\mu_2}+\sqrt{\mu_1}}{\sqrt{\mu_2}-\sqrt{\mu_1}}$*. Then if:*

$$\alpha_1 < \frac{\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}{1 + \left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}, \tag{3.81}$$

there is a unique equilibrium.

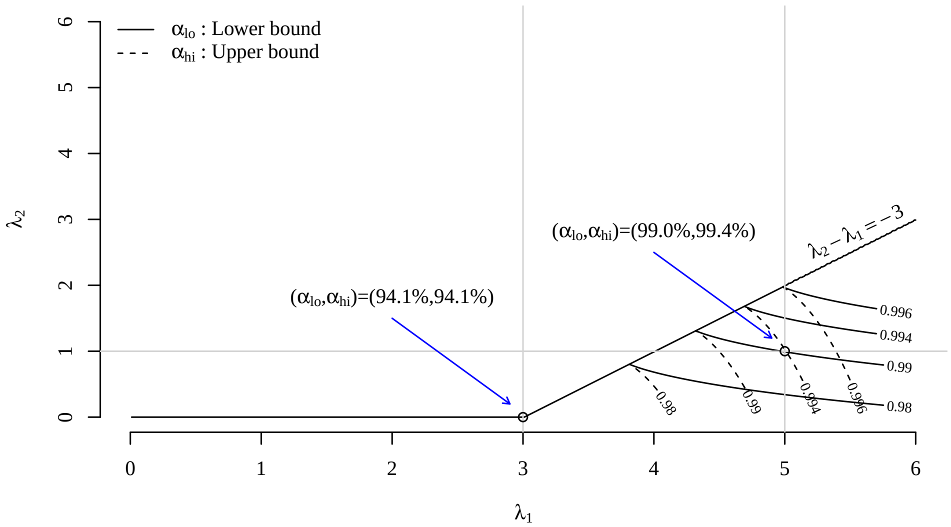

Figure 3.6: Region on the (λ_1, λ_2)-plane where multiple equilibria are possible for $(\mu_1, \mu_2) = (0.01, 0.04)$. The plot also shows the lower and upper bounds of α_1 required for multiple equilibria.

The proof is provided in Theorem B.1.2 in Appendix B.

In terms of the example in Figure 3.6, Corollary 3.1 says that for $\alpha_1 < 94.1\%$, multiple equilibria are not possible, no matter how extreme the elasticities (λ_1, λ_2). Note that values of α_{lo} and α_{hi} shown on the contour plots in the figure never fall below 94.1%, which is actually the value of $\alpha_{\text{lo}} = \alpha_{\text{hi}}$ for $(\lambda_1, \lambda_2) = (3, 0)$.

We are now in a position to consider, in detail, an example of a population for which multiple equilibria are possible. Following the same approach as in Figure 3.3 in sub-section 3.3.3, with $(\mu_1, \mu_2) = (0.01, 0.04)$, Figure 3.7 shows the contour plot of the pooled equilibrium premium when the fair-premium demand-share $\alpha_1 = 99.2\%$. Both plots in Figure 3.7 show the same example, with the right-hand plot zooming into the multiple equilibria region. The plots also show the boundary condition necessary for multiple equilibria: $\lambda_2 - \lambda_1 = -3$, as per Equation 3.77 of Result 3.30.

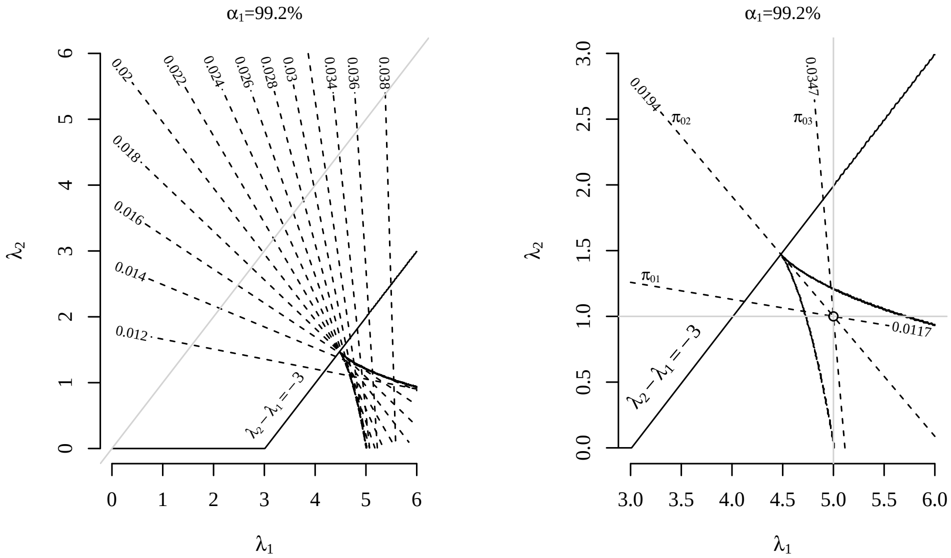

Figure 3.7: Contour plot of pooled equilibrium premium as a function of (λ_1, λ_2) for $\alpha_1 = 99.2\%$, when $(\mu_1, \mu_2) = (0.01, 0.04)$.

According to Figure 3.6, $(\lambda_1, \lambda_2) = (5, 1)$ falls within the multiple equilibria region and $\alpha_1 = 99.2\%$ falls between $(\alpha_{\text{lo}}, \alpha_{\text{hi}}) = (99\%, 99.4\%)$. So for such a combination, multiple equilibria should occur. Multiple equilibria are clearly visible in the left-hand plot of Figure 3.7 where the straight-line contours of equilibrium premiums intersect.

Focussing on the particular case of $(\lambda_1, \lambda_2) = (5, 1)$, according to Figure 3.5, there are multiple equilibria at $(\pi_{01}, \pi_{02}, \pi_{03})$. The right-hand plot of Figure 3.7 shows 3 straight lines, each representing one of the equilibrium premiums, $(\pi_{01}, \pi_{02}, \pi_{03})$, all intersecting at $(\lambda_1, \lambda_2) = (5, 1)$.

Result 3.30 says that multiple equilibria arise only if we have *both* an extreme population structure *and* an implausible divergence of demand elasticities. Corollary 3.1 gives a condition on α_1 that guarantees a unique equilibrium irrespective of the demand elasticities.

Provided equilibrium is unique, an important relationship between the equilibrium pooled premium and demand elasticities is given below:

Corollary 3.2. *If there is a unique equilibrium, then the equilibrium pooled premium is an increasing function of the individual demand elasticities.*

A proof is provided in Theorem B.1.3 in Appendix B.

3.6.2 Negative-exponential Demand

In this sub-section, we explore multiple equilibria using negative-exponential demand. Here we show some explicit results on the conditions which lead to multiple equilibria when the *second order elasticity* $n = 1$ in negative-exponential demand. Note that the following approach in determining multiple equilibria can be generalised for any $n \geq 0$. We only focus on $n = 1$ because the result can be explicitly presented.

When $n = 1$, the negative-exponential demand function in Equation 2.30 becomes:

$$d_i(\pi) = \tau_i \exp\left[\left(1 - \frac{\pi}{\mu_i}\right)\lambda_i\right], \text{ for } i = 1, 2, \tag{3.82}$$

with the demand elasticity function being:

$$\epsilon_i(\pi) = \lambda_i \frac{\pi}{\mu_i}. \tag{3.83}$$

So, in this case, the demand elasticity function is a linearly increasing function of premium.

The equilibrium condition in Equation 3.10 can be written as:

$$\left(1-\frac{\pi_0}{\mu_2}\right)\lambda_2=\left(1-\frac{\pi_0}{\mu_1}\right)\lambda_1+\log\left[\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)}\right], \tag{3.84}$$

$$\Rightarrow \lambda_2=-\frac{\left(\frac{\pi_0}{\mu_1}-1\right)}{\left(1-\frac{\pi_0}{\mu_2}\right)}\lambda_1+\frac{\log\left[\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)}\right]}{\left(1-\frac{\pi_0}{\mu_2}\right)}. \tag{3.85}$$

Figure 3.8 shows an example where there are more than one solution (i.e. multiple equilibria) to Equation 3.84. This figure demonstrates how expected profit behaves with respect to premium. In this example, there are three equilibrium premiums (i.e. three premiums at which the expected profit for insurers is zero). The smallest equilibrium premium, is close to the risk-differentiated premium for the low risk-group, $\mu_1 = 0.01$. The largest equilibrium premium, is close to the risk-differentiated premium for the high risk-group, $\mu_2 = 0.04$. And there is a third equilibrium premium that is located somewhere in between the other two.

Our key result on multiple equilibria, Result 3.29, in the case of negative-exponential demand, is formally summarised as follows:

Result 3.31. *For pooled premium, given* (μ_1, μ_2) *and* (λ_1, λ_2)*, multiple equilibria exist if*

$$\frac{\lambda_2}{\mu_2}-\frac{\lambda_1}{\mu_1}<-\frac{4}{\mu_2-\mu_1}, \quad \textit{and} \tag{3.86}$$

$$\frac{a(\pi_{lo})}{1+a(\pi_{lo})}=\alpha_{lo}<\alpha_1<\alpha_{hi}=\frac{a(\pi_{hi})}{1+a(\pi_{hi})}, \quad \textit{where} \tag{3.87}$$

$$a(\pi)=\left(\frac{\mu_2-\pi}{\pi-\mu_1}\right)\exp\left[\lambda_2-\lambda_1-(\frac{\lambda_2}{\mu_2}-\frac{\lambda_1}{\mu_1})\pi\right], \quad \textit{and} \tag{3.88}$$

$$(\pi_{lo},\pi_{hi}) \textit{ are solutions to: } \pi^2-(\mu_1+\mu_2)\pi+\mu_1\mu_2-\frac{\mu_2-\mu_1}{\frac{\lambda_2}{\mu_2}-\frac{\lambda_1}{\mu_1}}=0. \tag{3.89}$$

This result is formally proved in Theorem B.2.5 in Appendix B.2.

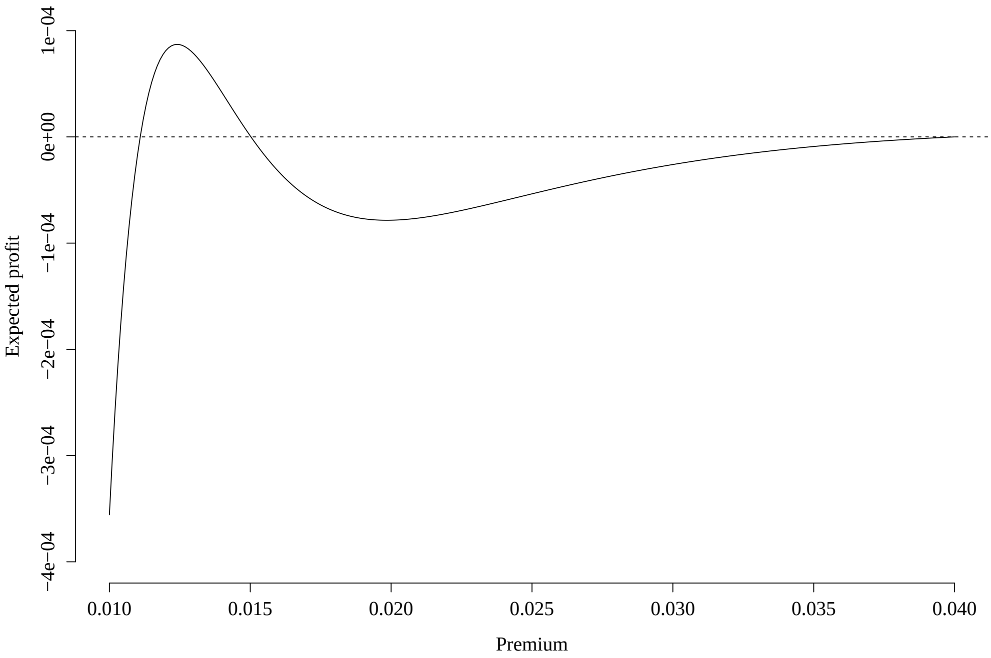

Figure 3.8: Total profit for a population with $(\mu_1, \mu_2) = (0.01, 0.04), \alpha_1 = 99.75\%, \alpha_2 = 0.25\%$ and $\lambda_1 = 5, \lambda_2 = 3$.

We begin the analysis with some examples of multiple equilibria in Figure 3.9. This uses $(\mu_1, \mu_2) = (0.01, 0.04)$ and a very divergence of elasticity parameters: $(\lambda_1, \lambda_2) = (2, 0.5)$. The figure shows plots of the expected profit curves, $\rho(\pi)$, for a narrow range of very high values [92.3%, 97.8%] of the fair-premium demand-share for low risks, α_1. We note the following patterns:

For $\alpha_1 < 94.7\%$, there is a unique equilibrium, close to μ_2.

For $\alpha_1 = 94.7\%$, there is another equilibrium premium, at π_{lo}, in addition to an equilibrium premium close to μ_2.

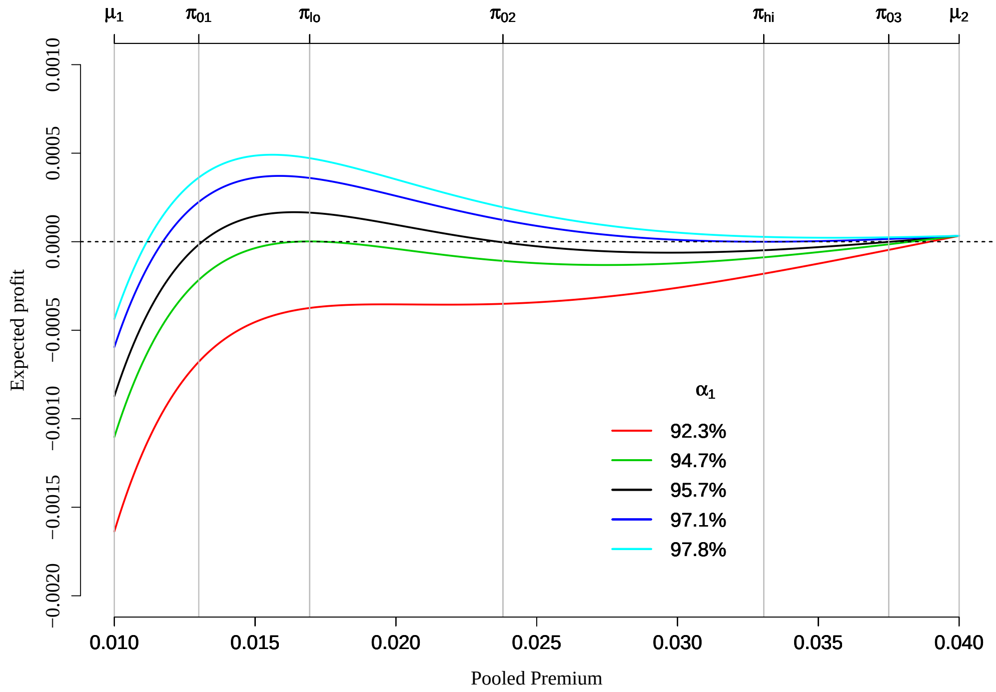

Figure 3.9: Profit function for $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\lambda_1, \lambda_2) = (2, 0.5)$, but different values of α_1 in the range $[92.3\%, 97.8\%]$

.

For $\alpha_1 = 97.1\%$, there is another equilibrium premium, at π_{hi}, in addition to an equilibrium premium close to μ_1.

For $\alpha_1 > 97.1\%$, there is only one equilibrium, close to μ_1.

For $94.7\% < \alpha_1 < 97.1\%$, there are three equilibrium premia, $(\pi_{01}, \pi_{02}, \pi_{03})$, where $\mu_1 < \pi_{01} < \pi_{lo} < \pi_{02} < \pi_{hi} < \pi_{03} < \mu_2$.

We argue that, based on our results, multiple equilibrium premia is unlikely to happen in practical world because: it is unlikely that elasticity of demand from the high risk-group would be that much smaller than that from the low risk-group. This is due to the income effect on demand mentioned before.

And it is unlikely that the fair-premium demand share will fall into such a narrow range. In this particular example, that range is $[94.7\%, 97.1\%]$.

We state the following two corollaries of the main result:

Corollary 3.3. *For pooled premium, given* (μ_1, μ_2)*, if* $\alpha_1 < \frac{e^2}{1+e^2}$ *then there is a unique equilibrium.*

This result is formally proved in Theorem B.2.6 in Appendix B.2.

Corollary 3.4. *For pooled premium, given* (μ_1, μ_2)*, if there is a unique equilibrium, then the equilibrium premium is an increasing function of the individual demand elasticities.*

This result is formally proved in Theorem B.2.7 in Appendix B.2.

3.7 Summary

In this chapter, we introduced a framework to firstly define equilibrium premium (i.e. the premium at which insurers break even) under different risk classifications, and then analysed some of its features using iso-elastic demand and negative-exponential demand before generalising to general demand.

In particular, we have focused on the case when there is no risk classification, i.e. insurers charge the same (pooled equilibrium) premium to all risk-groups. Using illustrative examples, we found that: when all risk-groups have the same demand elasticity at the pooled equilibrium premium, there is a unique pooled equilibrium premium, and it is located between the weighted-average of the population's true risk and the highest risk.

When different risk-groups do not necessarily have the same demand elasticities, there might be multiple equilibria (i.e. multiple solutions to the equilibrium premium). However, we have shown that the conditions, under which multiple equilibria can occur, are unlikely to happen in practice. In any case, all our analyses in the subsequent chapters are valid for any equilibrium premium, unique or otherwise.

Chapter 4

Adverse Selection

In the economic literature, adverse selection is usually defined by the correlation between risk and insurance coverage (e.g. Chiappori & Salanie (1997)). A positive correlation is typically considered as a sign of 'adverse' selection (e.g. for a survey see Cohen & Siegelman (2010)), because if people with high risks require more insurance coverage, this might be considered 'adverse' to insurers. Using the notations introduced in Chapter 3, this can be quantified by the covariance between the random variables Q and X, i.e. $E[QX] - E[Q]E[X]$. We prefer to use the ratio rather than the difference, so our definition is:

$$\text{Adverse selection} = \frac{E[QX]}{E[Q]E[X]}. \tag{4.1}$$

This intuitive definition indicates that adverse selection is a ratio of the *expected claim per policy* to the *expected loss per risk*, where the risk is randomly chosen from the whole population (both insured and uninsured).

To compare the impact of a given premium strategy to the case of charging risk-differentiated premiums (i.e. under *full risk classification*), we define **adverse selection ratio** as the ratio between adverse selection at a given premium strategy and adverse selection at risk-differentiated premiums, i.e.

$$R(\underline{\pi}) = \frac{\text{adverse selection at a given risk classification scheme}}{\text{adverse selection at the } \textit{full risk classification}}. \tag{4.2}$$

In particular, when a pooled premium is charged to different risk groups due to a restriction on risk classification scheme, Equation 4.2 can be rewritten as

$$R(\pi_0) = \frac{\pi_0}{\sum_{i=1}^{m} \alpha_i \mu_i}, \tag{4.3}$$

since $\pi_0 = \frac{E[QX]}{E[Q]}$.

Therefore, whether there is adverse selection as a result of risk classification scheme being restricted, depends on the size of the pooled equilibrium premium compared to the weighted-average of the population's true risk.

For both iso-elastic demand (in Section 3.3) and negative-exponential demand (in Section 3.4), we have proved that $\pi_0 \geq \alpha_1\mu_1 + \alpha_2\mu_2$ when there are two risk-groups. This result can be generalised to more risk-groups, i.e. $\pi_0 \geq \sum_{i=1}^{m} \alpha_i \mu_i$, with $m \geq 2$ (in sub-section 3.3.2 and 3.4.2), i.e. the pooled equilibrium premium under *no risk classification* is always greater than the expected claim per policy under *full risk classification*.

Moreover, in section 3.5, we have also proved that for any given general demand function with demand elasticity as a non-decreasing function of premium, its equilibrium condition at a given risk classification scheme can be represented by the equilibrium condition of iso-elastic demand, but with appropriate arc elasticities substituted for fixed elasticities. And, pooled equilibrium premium from a general demand is always greater than the expected claim per policy under full-risk-classification scheme. So using Equation 4.3, we reach the main finding on adverse selection:

Result 4.1. *When risk classification is banned and a pooled equilibrium premium π_0 is charged across different risk-groups, there is always adverse selection, i.e. $R(\pi_0) \geq 1$, and the level of adverse selection also increases with demand elasticity.*

Therefore, adverse selection might not be a good measure of the impact of pooling on society as a whole, because it cannot differentiate between cases where pooling leads to a larger or smaller fraction of society's losses being compensated by insurance.

Figure 4.1 shows adverse selection ratio for iso-elastic demand for two populations with the same underlying risks $(\mu_1, \mu_2) = (0.01, 0.04)$ with equal demand elasticity, but different values of α_1. We observe that regardless of the population split between the low and the high risk-groups, adverse selection ratio is always greater than 1. Moreover, adverse selection ratio, $R(\pi_0)$, is an increasing function of demand elasticity, because pooled premium π_0 is an increasing function of demand elasticity (by Result 3.9).

The adverse selection ratio, $R(\pi_0)$ has an upper boundary, $\frac{\mu_2}{\alpha_1\mu_1+\alpha_2\mu_2}$ (in the case of two risk-groups), when demand elasticity from both risk-groups is extremely large (and $\frac{\mu_m}{\sum_{i=1}^{m}\alpha_i\mu_i}$ in the case of m risk-groups with $m \geq 2$). This is because, at high demand elasticities, $\pi_0 \to \mu_2$. Therefore, in order to break even, insurers have to raise the pooled premium to the same level as the highest risk.

Moreover, when the fair-premium demand-share of the low risk-group tends to be very large, the limiting value increases. This is due to the fact that a large number of low risks will reduce their demand for insurance when premium goes up. This impact from population split between risk-groups

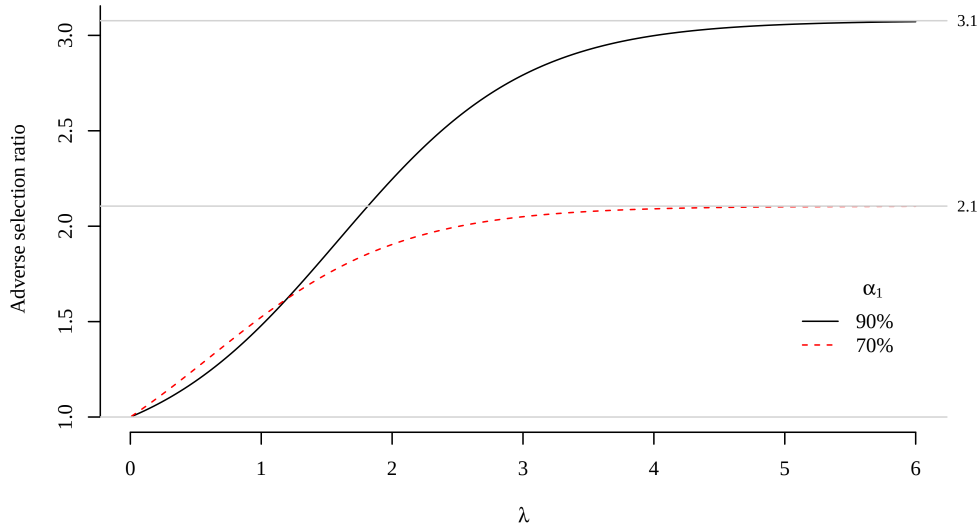

Figure 4.1: Adverse selection ratio as a function of λ for two populations with the same $(\mu_1, \mu_2) = (0.01, 0.04)$ but different values of α_1.

is illustrated in Figure 4.1, where the population with larger proportion of low risks (the dark line) has a higher adverse selection ratio than the population with fewer low risks (the dashed line) when demand elasticity is higher.

To summarise, pooling always leads to adverse selection. Therefore this metric is unable to distinguish between cases where pooling gives a better outcome for society as a whole (Table 2 in the motivating examples earlier in this thesis) and cases where pooling gives a worse outcome for society as a whole (Table 3 in the motivating examples earlier in this thesis). This leads us to the concept of loss coverage ratio discussed in the next chapter.

Chapter 5

Loss Coverage

Loss coverage was first introduced in Thomas (2008) as *"the population's expected losses compensated by insurance"* and is heuristically characterised in the motivating examples early in this thesis. In this chapter, we will formally define loss coverage and analyse its features.

Our main finding is:

Loss coverage might be a better metric to measure the impact of pooling. When a moderate level of adverse selection is tolerated, loss coverage for pooled premium exceeds that for risk-differentiated premiums, i.e. pooling can benefit the society as a whole. In particular, at pooled equilibrium, for any demand elasticity functions, as long as the elasticities of the low risk-groups (i.e. who pay more than their fair actuarial premium) do not exceed 1 and the elasticities of the high risk-groups (i.e. who pay less than their fair actuarial premium) exceed that of the low risk-groups, then loss coverage under pooling is bigger than under full risk classification.

5.1 Framework for Loss Coverage

Loss coverage can be formally defined within the model framework developed in Chapter 2 as the expected insurance claims outgo, or expected population losses compensated by insurance, at equilibrium, i.e. $E[QX]$. So:

$$\text{Loss coverage: } LC(\underline{\pi}) = E[QX] \tag{5.1}$$

$$= \sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i, \tag{5.2}$$

where $\underline{\pi}$ are premiums charged to different risk-groups which satisfy the equilibrium condition in Equation 3.10. Loss coverage can also be thought of as *risk-weighted* insurance demand.

In particular, the loss coverage at risk-differentiated premiums under *full risk classification* is:

$$LC(\underline{\mu}) = \sum_{i=1}^{m} d_i(\mu_i)\mu_i p_i. \tag{5.3}$$

Specifically, to compare the relative merits of different risk classifications, we define a reference level of loss coverage using the level under risk-differentiated premiums as follows:

$$\text{Loss coverage ratio: } C(\underline{\pi}) = \frac{LC(\underline{\pi})}{LC(\underline{\mu})}. \tag{5.4}$$

Here we do not impose any constraint on the order or size of insurance premiums, so that the insurers are free to charge any premiums to any risk-group, as long as the premiums achieve equilibrium in the market.

To analyse the impact of pooling, we consider the ratio of loss coverage for pooled premium, π_0, and loss coverage for risk-differentiated premiums,

$\underline{\mu}$. We define **loss coverage ratio at pooled premium** as:

$$C(\pi_0) = \frac{LC(\pi_0)}{LC(\underline{\mu})}. \tag{5.5}$$

Therefore, loss coverage ratio, $C(\pi_0) > 1$, indicates that pooling is better than *full risk classification*, in terms of a higher proportion of the population's losses being compensated by insurance under pooling.

In the following sections, we will analyse loss coverage at pooled equilibrium premium using iso-elastic demand. Then we extend the analysis to a more general demand with negative-exponential demand. For each demand function, we will look at the scenario when a population can be sub-divided into two risk-groups, i.e. a low risk-group and a high risk-group, and then generalise the analysis to more risk-groups. For each of the above scenarios, we further explore the case when all the risk-groups have the same demand elasticity, and the case when each risk-group could have their own different demand elasticities.

5.2 Iso-elastic Demand

We start our analysis of loss coverage ratio using iso-elastic demand. Firstly, let us consider the simple case of iso-elastic demand where there are two risk-groups, i.e. a low risk-group and a high risk-group, and they have the same demand elasticity, i.e. $\lambda_1 = \lambda_2 = \lambda$.

5.2.1 Two Risk-groups: Equal Demand Elasticity

Using iso-elastic demand in Equation 2.24, loss coverage at pooled equilibrium premium π_0 is given by:

$$LC(\pi_0) = \sum_{i=1}^{2} \tau_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda} \mu_i p_i = \sum_{i=1}^{2} \frac{\mu_i^{\lambda+1}}{\pi_0^{\lambda}} \tau_i p_i. \tag{5.6}$$

Then the loss coverage ratio at pooled equilibrium premium becomes:

$$C(\pi_0) = C(\lambda) = \frac{1}{\pi_0^{\lambda}} \frac{\alpha_1 \mu_1^{\lambda+1} + \alpha_2 \mu_2^{\lambda+1}}{\alpha_1 \mu_1 + \alpha_2 \mu_2}, \tag{5.7}$$

where

$$\pi_0 = \frac{\alpha_1 \mu_1^{\lambda+1} + \alpha_2 \mu_2^{\lambda+1}}{\alpha_1 \mu_1^{\lambda} + \alpha_2 \mu_2^{\lambda}},$$

which is the unique pooled equilibrium premium given in Equation 3.20.

Note that, loss coverage ratio, $C(\lambda)$, can be expressed as:

$$\begin{aligned} C(\lambda) &= \frac{[w\beta^{1-\lambda} + (1-w)]^{\lambda}[w + (1-w)\beta^{\lambda}]^{1-\lambda}}{\beta^{\lambda(1-\lambda)}}, \text{ where} && (5.8) \\ w &= \frac{\alpha_1 \mu_1}{\alpha_1 \mu_1 + \alpha_2 \mu_2}, \text{ and} && (5.9) \\ \beta &= \frac{\mu_2}{\mu_1} > 1. && (5.10) \end{aligned}$$

Figure 5.1 shows the loss coverage ratio for four population structures. Both plots in Figure 5.1 show the same example, with the right-hand plot zooming over the range $0 < \lambda < 1$. We have the following results:

Result 5.1.

$$\lim_{\lambda \to 0} C(\lambda) = 1. \tag{5.11}$$

When demand for insurance becomes very insensitive to premium changes, i.e. $\lambda \to 0$, both the low risks and the high risks' demand for insurance

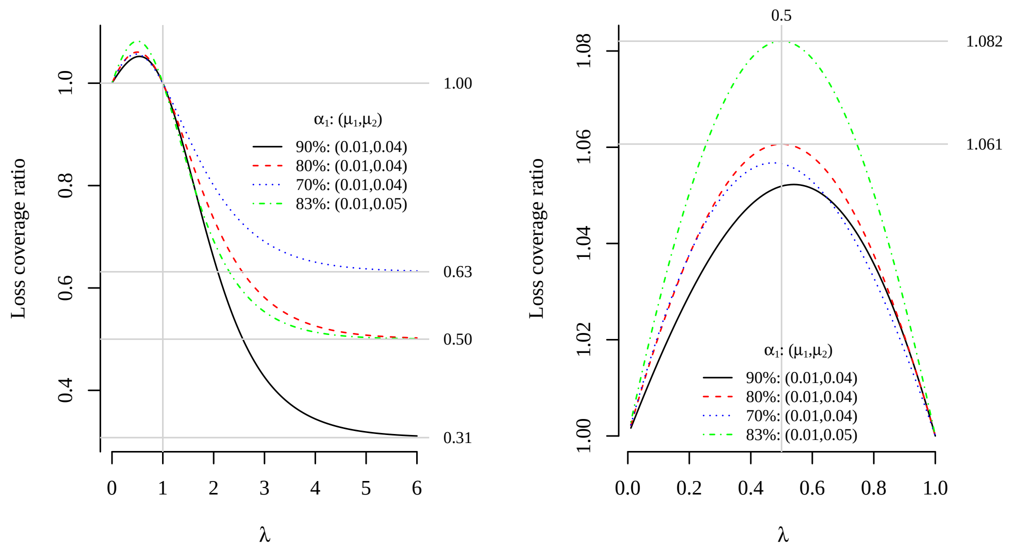

Figure 5.1: Loss coverage ratio as a function of λ for four population structures.

will be indifferent from the case when they face risk-differentiated premiums. Therefore, pooling gives the same level of loss coverage as risk-differentiated premiums.

Result 5.2.

$$\lim_{\lambda \to \infty} C(\lambda) = \frac{\alpha_2 \mu_2}{\alpha_1 \mu_1 + \alpha_2 \mu_2}. \tag{5.12}$$

If demand for insurance becomes extremely elastic to premium changes, i.e. $\lambda \to \infty$, and the pooled premium is higher than μ_1 which is the risk-differentiated premium of the low risks, only the high risks will buy insurance. To ensure a zero expected profit, insurers will increase the pooled premium π_0 to μ_2, i.e. the risk-differentiated premium for high risks. As a result, the loss coverage ratio will tend to a lower boundary.

The left-hand plot of Figure 5.1 also shows the lower limit of $C(\lambda)$ increases when the low risk-group's fair-premium demand-share decreases (or high risk-group's fair-premium demand-share increases). This means that when there are relatively more high risks in the population, more risk will be compensated by insurance when the demand becomes very elastic.

Result 5.3. *For $\lambda > 0$,*

$$\lambda \lesseqgtr 1 \Leftrightarrow C(\lambda) \gtreqless 1. \tag{5.13}$$

We see that, when demand elasticity is less than 1, loss coverage ratio is greater than or equal to 1. The proof of this result is given in Theorem C.1.1 in Appendix C.

Result 5.4. *In particular when $0 < \lambda < 1$, loss coverage ratio has a maximum value, i.e.* $\max_{w,\lambda} C = \frac{1}{2}\left(\sqrt[4]{\frac{\mu_2}{\mu_1}} + \sqrt[4]{\frac{\mu_1}{\mu_2}}\right) = \frac{1}{2}\left(\sqrt[4]{\beta} + \frac{1}{\sqrt[4]{\beta}}\right)$.

This maximum value is reached when $\lambda = 0.5$ and $w = 0.5$. Moreover, loss coverage ratio also increases with the relative risk, β, which is demonstrated in the right-hand-side plot in Figure 5.1 where the green dashed curve (with $\lambda = 0.5, w = 0.5, \beta = 5$) is higher than the red dashed curve (with $\lambda = 0.5, w = 0.5, \beta = 4$). This implies that a pooled premium might be highly beneficial in the presence of a small group with very high risk exposure. Hoy (2006) obtained a similar result based on social welfare, so there are at least two different normative justifications for pooling very different insurance risks. The proof of this result is given in Theorem C.1.2 in Appendix C.

5.2.2 More Risk-groups: Equal Demand Elasticity

Result 5.3 can be further generalised to m risk-groups (where $m \geq 2$). As before, we are looking at cases when all m risk-groups have the same demand elasticity, i.e. $\lambda_1 = \lambda_2 = ... = \lambda_m = \lambda$.

In this case, the loss coverage ratio, as defined in Equation 5.5, comparing loss coverage at pooled premium to loss coverage at risk-differentiated premiums, is:

$$C(\pi_0) = \frac{LC(\pi_0)}{LC(\underline{\mu})} = \frac{\sum_{i=1}^m \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^\lambda \mu_i}{\sum_{i=1}^m \alpha_i \mu_i} = \frac{\sum_{i=1}^m \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda+1}}{\sum_{i=1}^m \alpha_i \left(\frac{\mu_i}{\pi_0}\right)} = \frac{E[V^{\lambda+1}]}{E[V]} = \frac{E[V^\lambda]}{E[V]}, \tag{5.14}$$

where V is defined in sub-section 3.3.2 as a random variable taking values $v_i = \frac{\mu_i}{\pi_0}$. Recall that equilibrium condition leads to the relationship that $E\Big[V^\lambda\Big] = E\Big[V^{\lambda+1}\Big]$ with probabilities α_i for $i = 1, 2, ..., m$. Under this setup, we have the following general result:

Result 5.5. *Suppose there are m risk-groups with risks $\mu_1 < \mu_2 < ... < \mu_m$ and they have the same iso-elastic demand elasticity λ. Then $\lambda \lesseqqgtr 1 \Rightarrow C(\pi_0) \gtreqqless 1$.*

This result states the relationship between the common iso-elastic demand elasticity λ of m risk-groups, and loss coverage ratio when those risk-groups are pooled. In particular, this result confirms that if the common iso-elastic demand elasticity is less than 1, then loss coverage under pooling is higher than or equal to risk differentiated premiums. This result is proved in Theorem C.2.1 in Appendix C.2.

5.2.3 Two Risk-groups: Different Demand Elasticities

If the demand elasticities of the low and high risk-groups are possibly different, then the loss coverage ratio becomes:

$$C(\lambda_1, \lambda_2) = \frac{\alpha_1\mu_1(\frac{\mu_1}{\pi_0})^{\lambda_1} + \alpha_2\mu_2(\frac{\mu_2}{\pi_0})^{\lambda_2}}{\alpha_1\mu_1 + \alpha_2\mu_2}, \tag{5.15}$$

where π_0 is an equilibrium premium that satisfies the equilibrium condition in Equation 3.37. We can also express loss coverage ratio in Equation 5.15 in either of the following forms:

$$\begin{aligned}
\log C &= -\lambda_1 \log(\frac{\pi_0}{\mu_1}) + \log k_1, && (5.16)\\
\text{where } k_1 &= \frac{\alpha_1(\mu_2-\mu_1)\pi_0}{(\alpha_1\mu_1+\alpha_2\mu_2)(\mu_2-\pi_0)}; \text{ or} && (5.17)\\
\log C &= \lambda_2 \log(\frac{\mu_2}{\pi_0}) + \log k_2, && (5.18)\\
\text{where } k_2 &= \frac{\alpha_2(\mu_2-\mu_1)\pi_0}{(\alpha_1\mu_1+\alpha_2\mu_2)(\pi_0-\mu_1)}. && (5.19)
\end{aligned}$$

Figure 5.2 shows the graphical representations of Equations 5.16 and 5.18, for different values of α_1 when $(\mu_1, \mu_2) = (0.01, 0.04)$. We have the following results:

Result 5.6. *Given an equilibrium premium, π_0, the loss coverage ratio is a log-linear function of either λ_1 or λ_2. And the loss coverage ratio is an increasing function of λ_2, and a decreasing function of λ_1.*

In Result 3.11, we have proved that to maintain the same equilibrium premium, an increase in λ_2 means a decrease in λ_1. And increasing λ_2 and decreasing λ_1 will lead to increase in demand from both risk-groups. Therefore, the loss coverage ratio will increase as a result. In particular, the loss coverage ratio is maximised when $\lambda_1 = 0$ and takes the value of k_1; and minimised when $\lambda_2 = 0$ and takes the value of k_2.

Result 5.7. *For a given value of λ_2, the loss coverage ratio is a decreasing function of λ_1.*

Interpretation of this result is that, all else being fixed, including λ_2, the equilibrium pooled premium can only increase if λ_1 increases. Higher demand

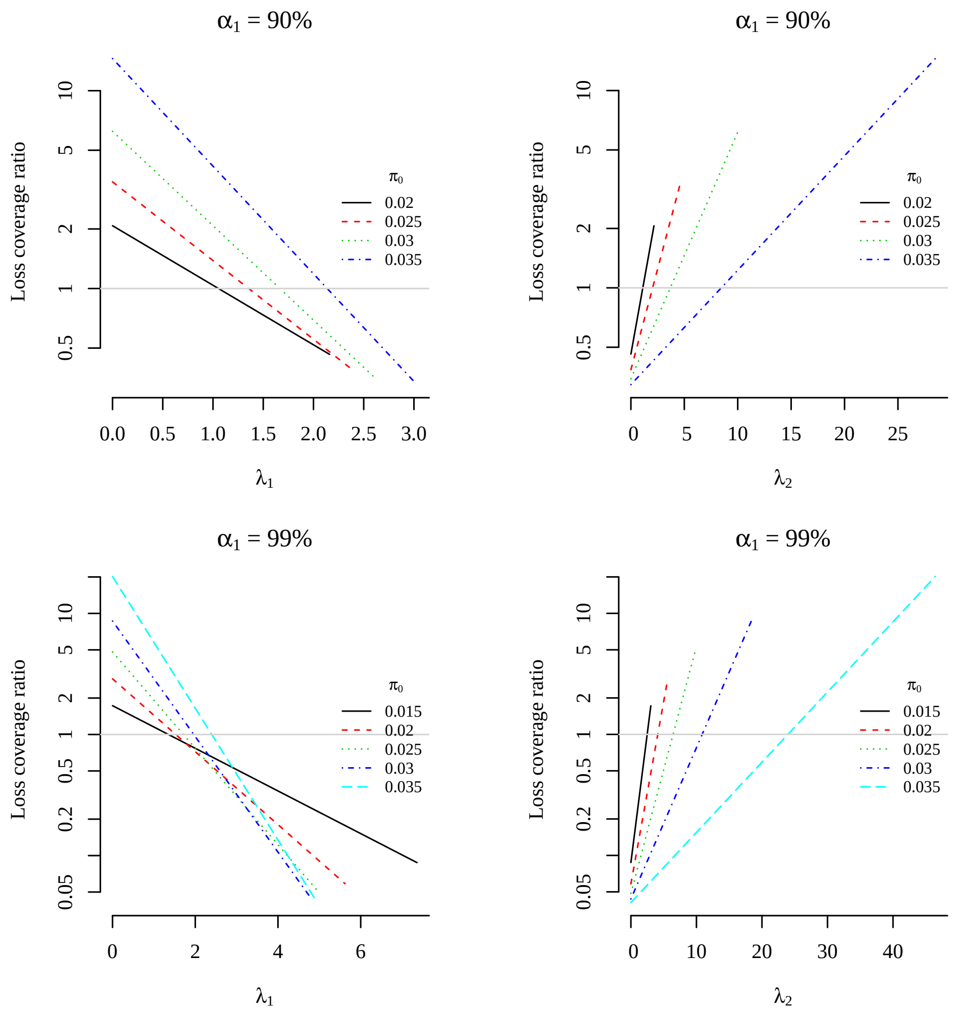

Figure 5.2: Loss coverage ratio (log scale) as functions of λ_1 and λ_2 for different values of equilibrium premiums, when $(\mu_1, \mu_2) = (0.01, 0.04)$ and $\alpha_1 = 90\%$ and 99%.

elasticity and higher equilibrium pooled premium both imply a fall in low-risk demand. A higher equilibrium pooled premium also reduces high-risk demand. Since demand from both risk-groups falls, the loss coverage ratio falls. The proof is given in Theorem C.1.3 in Appendix C.

Result 5.8. *For a given value of λ_1, loss coverage ratio is an increasing function of λ_2 if $\lambda_1 \leq \frac{\mu_2}{\alpha_1(\mu_2-\mu_1)}$.*

This is illustrated in the left panel of Figure 5.2, where loss coverage ratio is increasing with pooled equilibrium premium for small values of λ_1. However, for some large value of λ_1, the crossover of the lines for different equilibrium pooled premiums implies a non-monotonic ordering of premiums by loss coverage ratio. This effect arises because all else being fixed, including λ_1, the pooled equilibrium premium can only increase if λ_2 increases. But for high risks, an increase in premium and increase in elasticity have opposite effects on demand. The sum of these effects plus the fall in low-risk demand determine the change in the loss coverage ratio, which can either rise or fall depending on the demand elasticity of the low risks. The proof is given in Theorem C.1.4 in Appendix C.

Focusing on demand elasticities less than 1, Figure 5.3 demarcates the regions where the loss coverage ratio is greater than or less than 1. We make the following observations:

(a) For $0 < \lambda_1 < \lambda_2 < 1$, loss coverage ratio exceeds 1, i.e. when high risks have a higher demand elasticity than the low risks, we have a higher loss coverage ratio. A proof is given in Theorem C.1.5 in Appendix C.

(b) For $0 < \lambda_2 < \lambda_1 < 1$, the curve showing loss coverage ratio of 1 becomes increasingly more convex up to certain limit, as β increases.

In other words, as the relative risk increases, more combinations of (λ_1, λ_2) produce loss coverage ratio greater than 1.

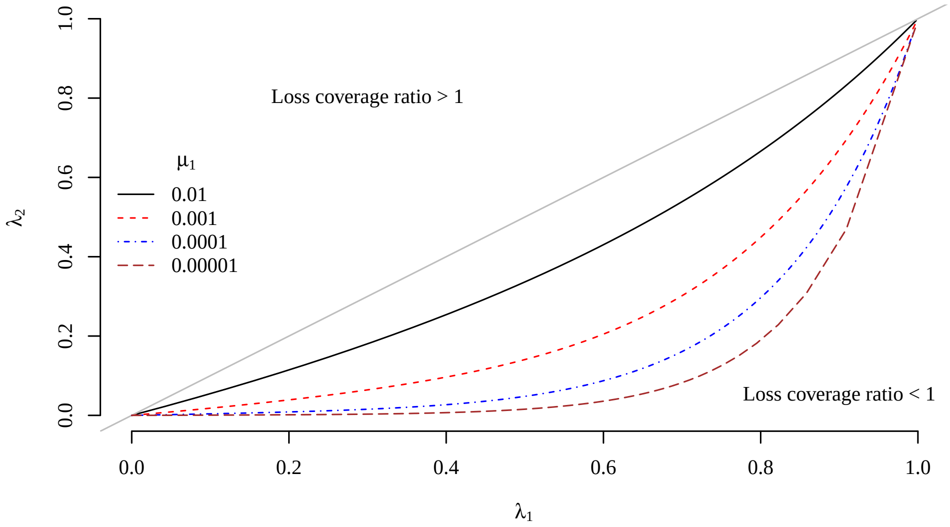

Figure 5.3: Curves demarcating the regions where loss coverage ratio is greater than and less than 1 for different values of μ_1 when $\alpha_1 = 90\%$ and $\mu_2 = 0.04$.

5.2.4 More Risk-groups: Different Demand Elasticities

We now look at loss coverage ratio when each risk-group has iso-elastic demand but where the demand elasticities can be different for different risk-groups, i.e.

$$\epsilon_i(\pi) = \lambda_i \text{ for } i = 1, 2, ..., m; \tag{5.20}$$

where higher risk-groups are likely to have higher demand elasticities because of the income effect on demand. We will formalise the relationship between λ_is later in this section. Under this formulation, the equilibrium condition under risk pooling, i.e. $\rho(\pi_0) = 0$, gives:

$$\sum_{i=1}^{m} p_i \tau_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda_i} (\pi_0 - \mu_i) = 0, \text{ or, equivalently:} \tag{5.21}$$

$$\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda_i} = \sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda_i+1}. \tag{5.22}$$

As in sub-section 3.3.2, we define a random variable V taking values $v_i = \frac{\mu_i}{\pi_0}$ with probabilities α_i for $i = 1, 2, ..., m$. Now, define a function $f(v)$, such that:

$$f(v_i) = \lambda_i, \text{ for } i = 1, 2, ..., m. \tag{5.23}$$

Then the equilibrium condition, in Equation 5.22, can be re-framed as:

$$E\Big[V^{f(V)}\Big] = E\Big[V^{f(V)+1}\Big]. \tag{5.24}$$

Loss coverage ratio that compares loss coverage under pooling to loss coverage under *full risk classification* is:

$$C(\pi_0) = \frac{\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda_i} \mu_i}{\sum_{i=1}^{m} \alpha_i \mu_i} = \frac{\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)^{\lambda_i+1}}{\sum_{i=1}^{m} \alpha_i \left(\frac{\mu_i}{\pi_0}\right)} = \frac{E[V^{f(V)+1}]}{E[V]} = \frac{E[V^{f(V)}]}{E[V]}. \tag{5.25}$$

Under this setting, we have the following result:

Result 5.9. *Suppose there are m risk-groups with risks $\mu_1 < \mu_2 < ... < \mu_m$ with iso-elastic demand elasticities $\lambda_1, \lambda_2, ..., \lambda_m$ respectively. Define $\lambda_{lo} = \max_{v \le 1} f(v)$ and $\lambda_{hi} = \min_{v>1} f(v)$. Then if $\lambda_{lo} < 1$ and $\lambda_{hi} \ge \lambda_{lo}$, loss coverage at pooled equilibrium premium is higher than or equal to loss coverage* *at risk-differentiated premiums, i.e.* $C(\pi_0) \geq 1$.

The formal proof is given in Theorem C.2.2 in Appendix C.2.

Under pooled equilibrium, λ_{lo} is the maximum of the demand elasticities of all those low risk-groups who pay more premium, π_0, than their actuarially fair risk, μ. So, $\lambda_{\text{lo}} < 1$ signifies that, for these low risk-groups, their iso-elastic demand elasticities should be less than 1, which is in line with empirical evidence in many insurance markets (which will be mentioned in Table 5.1).

On the other hand, λ_{hi}, is the minimum of the demand elasticities of all those high risk-groups who pay less premium, π_0, under pooling, than their actuarially fair risk, μ. The interpretation of the second condition, $\lambda_{\text{hi}} \geq \lambda_{\text{lo}}$ is that, for those higher risk-groups, the demand elasticities should be larger than those of the lower risk-groups, which is consistent with what we expect from the income effect on demand.

In summary, as long as the iso-elastic demand elasticities of the different risk-groups satisfy the two conditions: $\lambda_{\text{lo}} < 1$ and $\lambda_{\text{hi}} \geq \lambda_{\text{lo}}$, then loss coverage under pooling is greater than or equal to under *full risk classification*.

The following special cases of Result 5.9 are worth noting:

- If the iso-elastic demand elasticities are the same for all risk-groups, i.e. $\lambda_i = \lambda$ for $i = 1, 2, ..., m$, then by definition $\lambda_{\text{lo}} = \lambda_{\text{hi}} = \lambda$, and so $\lambda_{\text{lo}} = \lambda < 1$ gives $C(\pi_0) \geq 1$, which reinforces the result in sub-section 5.2.2 for the case when $\lambda < 1$.

- For the special case of: $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m < 1$, the two conditions, $\lambda_{\text{lo}} < 1$ and $\lambda_{\text{hi}} \geq \lambda_{\text{lo}}$, are trivially satisfied and hence in this case: $C(\pi_0) \geq 1$.

- For the case of 2 risk-groups, $\lambda_{\text{lo}} = \lambda_1$ and $\lambda_{\text{hi}} = \lambda_2$, and the conditions on the demand elasticities translate to $\lambda_1 < 1$ and $\lambda_2 \geq \lambda_1$.

  This two risk-group case is illustrated in Figure 5.4, where $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\alpha_1, \alpha_2) = (90\%, 10\%)$. The two axes represents λ_1 and λ_2. The figure demarcates the region of $C(\pi_0) > 1$ (shaded green) from the region of $C(\pi_0) < 1$ (shaded pink) by the boundary curve $C(\pi_0) = 1$ (in red).

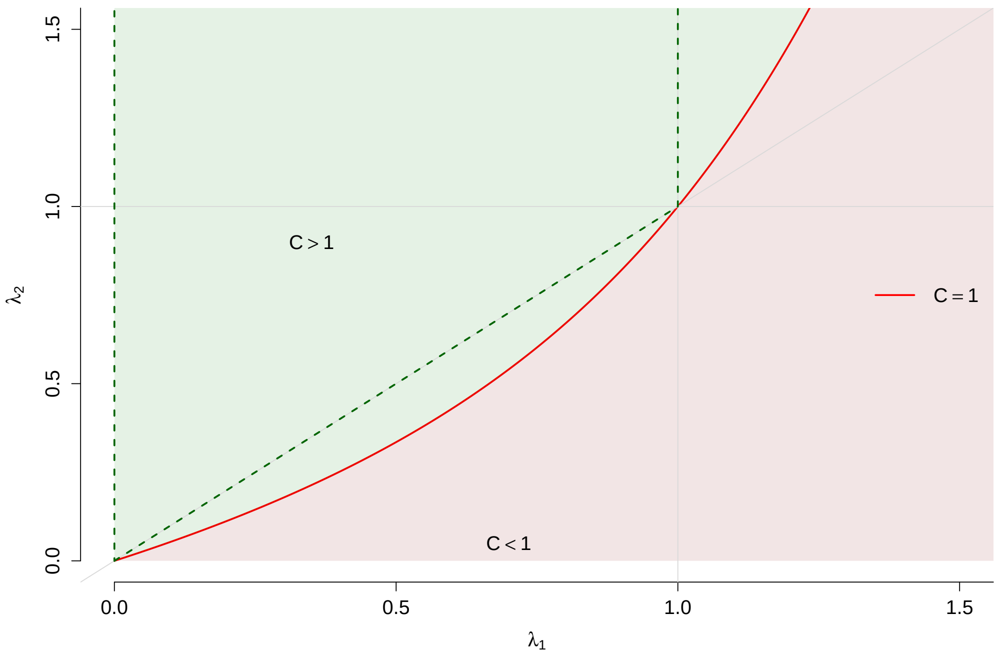

Figure 5.4: Curve demarcating the regions where loss coverage ratio is greater than and less than 1 when $(\mu_1, \mu_2) = (0.01, 0.04)$ and $(\alpha_1, \alpha_2) = (90\%, 10\%)$.

The conditions on λ_1 and λ_2 say that in the region above the $\lambda_1 = \lambda_2$ diagonal and $\lambda_1 < 1$, demarcated by the green dashed borders, loss coverage under pooling is always higher than that under *full risk classification*. This is true irrespective of the relative sizes and relative risks of the high and low risk populations.

Figure 5.4 also highlights another important point that Result 5.9 focuses only on loss coverage inside the region demarcated by green dashes. Outside this region, loss coverage under pooling can be higher or lower than under *full risk-classification* (higher in the green segments to the right of the dashed green lines; lower throughout the red area towards the right). The position of the $C(\pi_0) = 1$ curve which demarcates the red and green areas changes slightly with relative population sizes and relative risks, which have been shown in Figure 5.3 where loss coverage ratios at different relative risks are plotted. The region demarcated by green dashes is the only region for which we obtain a universal result (i.e. one which holds independent of relative sizes and risks of high and low risk populations). Fortuitously, empirical evidence and economic rationale imply that realistic values of (λ_1, λ_2) may often fall within this region.

5.2.5 Summary for Iso-elastic Demand

As a summary, for iso-elastic demand with equal demand elasticities in all risk-groups, $\lambda_1 = \lambda_2 = ... = \lambda$, the loss coverage ratio at pooled equilibrium premium (LCR) can be characterised as follows.

1. Under pooling, if $\lambda < 1$ then LCR ≥ 1.
2. As λ increases from zero, LCR increases to a maximum at around $\lambda = 0.5$; then decreases to 1 when $\lambda = 1$; and then flattens out at a

lower limit for high values of λ, where the only remaining insureds are high risks.

3. When there are two risk-group, a low risk-group and a high risk-group, the maximum value of LCR, attained for λ about 0.5, depends on the relative risk, $\beta = \mu_2/\mu_1$. A higher β gives a higher maximum value of LCR.

For iso-elastic demand with different demand elasticities λ_i in risk-group i, respectively, LCR can be characterised as follows:

1. When there are two risk-groups, i.e. a low risk-group and a high risk-group:
   (a) Given λ_2, LCR is a decreasing function of λ_1.
   (b) On the other hand, given λ_1, LCR is not necessarily a monotonic function of λ_2.
   (c) For $\lambda_1 < 1$ and $\lambda_2 > \lambda_1$, LCR is always greater than 1.
   (d) For other values of λ_1 and λ_2, LCR > 1 if λ_1 is 'sufficiently low' compared with λ_2. The value of λ_1 which is 'sufficiently low' may be greater or less than λ_2. We did not find any general conditions on (λ_1 , λ_2) that guaranteed LCR > 1.
   (e) As relative risk β increases, more combinations of (λ_1, λ_2) result in LCR > 1.
2. When there are more risk-groups: if $\lambda_{\text{lo}} < 1$ and $\lambda_{\text{hi}} \geq \lambda_{\text{lo}}$, then LCR is always greater than or equal to 1.

We suggest loss coverage — the expected losses compensated by insurance for the whole population — as a reasonable metric for the social efficacy of insurance. If this is accepted, and if our iso-elastic model of insurance demand is reasonable, then pooling will be beneficial:

(a) in the equal elasticities case, whenever $\lambda < 1$; and

(b) in the different elasticities case, if λ_{lo} is sufficiently low, compared to λ_{hi}.

5.3 General Demand

In this sub-section, we generalise our previous analysis on loss coverage ratio (using iso-elastic demand) by assuming a general demand function with demand elasticity being a non-decreasing function of premium. We have the following main finding:

Result 5.10. *Suppose there are m risk-groups with risks $\mu_1 < \mu_2 < ... < \mu_m$ and demand elasticities $\epsilon_1(\pi), \epsilon_2(\pi), ..., \epsilon_m(\pi)$, such that $\lambda_1, \lambda_2, ..., \lambda_m$ are the respective arc elasticities under pooled equilibrium. Define $\lambda_{lo} = \max_{v \le 1} f(v)$ and $\lambda_{hi} = \min_{v>1} f(v)$. Then $\lambda_{lo} < 1$ and $\lambda_{hi} \ge \lambda_{lo} \Rightarrow C(\pi_0) \ge 1$.*

Recall that arc elasticities are introduced in Equation 3.73 as

$$\eta_i(a, b) = \frac{\int_a^b \epsilon_i(s) d\log s}{\int_a^b d\log s}. \tag{5.26}$$

At equilibrium, if we define:

$$\lambda_i = \eta_i(\pi_0, \mu_i) = \frac{\int_{\pi_0}^{\mu_i} \epsilon_i(s) d\log s}{\int_{\pi_0}^{\mu_i} d\log s}, \text{ for } i = 1, 2, ...m, \tag{5.27}$$

then we have seen in Result 3.28 that, at equilibrium, the formulation of demand elasticities as general functions of premium is equivalent to the case of iso-elastic demand with relevant arc elasticities in place of fixed elasticities.

Result 5.10 says that under pooled equilibrium, as long as the arc elasticities of the low risk-groups (paying more than their actuarial premium) do not exceed 1 and the arc elasticities of the high risk-groups (paying less than their fair actuarial premium) exceed that of the low risk-groups, the loss coverage under pooling is bigger than under *full risk classification*.

For the special case of $\epsilon_i(\pi)$ being non-decreasing functions of premium π and bounded above by 1, where $\epsilon_1(\pi) \leq \epsilon_2(\pi) \leq ... \leq \epsilon_m(\pi)$ the required conditions for the above result are automatically satisfied. This is because the arc elasticities of demand, being an average of the underlying point elasticities, satisfy: $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m < 1$, which then implies that $\lambda_{\text{lo}} < 1$ and $\lambda_{\text{hi}} \geq \lambda_{\text{lo}}$ and thus $C(\pi_0) \geq 1$.

Note that our previous key results on loss coverage ratio (at pooled equilibrium premium) using iso-elastic demand are a special case of this result using general demand. Therefore, we can conclude that loss coverage under pooling is bigger than that under *full risk classification* as long as arc elasticities of different risk-groups satisfy certain conditions; there are no further requirements on the form of the demand function.

The results obtained in this section on general demand suggest that loss coverage will be higher under pooling than under *full risk classification*, if

1. elasticity (or arc elasticity, if elasticity is not constant) is less than 1 for all lower risk-groups; and

2. elasticity (or arc elasticity, if elasticity is not constant) for all higher risk-groups exceeds that for all lower risk-groups,

where arc elasticities are logarithmic averages of demand elasticities over the arc, from true risk price to the equilibrium pooled price.

Are these conditions likely to be satisfied in the real world?

For the first condition, Table 5.1 shows some relevant empirical estimates for insurance demand elasticities.[1] It can be seen that most estimates are of magnitude significantly less than 1. Whilst the various contexts in which these estimates were made may not correspond closely to the set-up in this thesis, the figures are at least suggestive of the possibility that the first condition may often be satisfied.

Table 5.1: Estimates of demand elasticity for various insurance markets

Market & country	Demand elasticities	Authors
Term life insurance, USA	-0.66	Viswanathan *et al.* (2006)
Yearly renewable term life, USA	-0.4 to -0.5	Pauly *et al.* (2003)
Whole life insurance, USA	-0.71 to -0.92	**?**
Health insurance, USA	0 to -0.2	Chernew *et al.* (1997), Blumberg *et al.* (2001), Buchmueller & Ohri (2006)
Health insurance, Australia	-0.35 to -0.50	Butler (1999)
Farm crop insurance, USA	-0.32 to -0.73	**?**

For the second condition, we know of no empirical evidence that insurance demand elasticities are higher (or lower) for higher risks. However, as noted earlier in the thesis, this condition may be plausible in that it is consistent with the income effect on demand.

5.3.1 Negative-exponential Demand Example

In this sub-section, using negative-exponential demand as an example of general demand, we demonstrate some results on loss coverage ratio, and

[1] Demand elasticity is defined as a positive constant in this thesis for convenience, but estimates in empirical papers are generally given with the negative sign, so Table 5.1 quotes them in that form.

compare the results to those derived from iso-elastic demand. Recall from Chapter 2 that negative-exponential demand is:

$$d_i(\pi) = \tau_i \exp\left[\left\{1 - \left(\frac{\pi}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right], \tag{5.28}$$

with demand elasticity as

$$\epsilon_i(\pi) = \lambda_i \left(\frac{\pi}{\mu_i}\right)^n, \text{ for } i = 1, ..., m. \tag{5.29}$$

The parameter n corresponds to the "elasticity of elasticity", and hence is called *"second-order elasticity"*. We assume n is the same for all risk-groups.

As with iso-elastic demand, we start with the scenario that a population can be subdivided into two risk-groups, $m = 2$, i.e. a low risk-group and a high risk-group.

Equal demand elasticity at a given premium π means:

$$\lambda_1 \left(\frac{\pi}{\mu_1}\right)^n = \lambda_2 \left(\frac{\pi}{\mu_2}\right)^n. \tag{5.30}$$

In particular, at pooled equilibrium premium, π_0, we have the following relationship:

$$\lambda_0 = \lambda_1 \left(\frac{\pi_0}{\mu_1}\right)^n = \lambda_2 \left(\frac{\pi_0}{\mu_2}\right)^n. \tag{5.31}$$

Then loss coverage ratio using negative-exponential demand is:

$$\begin{aligned}
C(\pi_0) &= \frac{d_1(\pi_0)p_1 + d_2(\pi_0)p_2}{d_1(\mu_1)p_1\mu_1 + d_2(\mu_2)p_2\mu_2}\pi_0, && (5.32)\\
&= \frac{\alpha_1 \exp\left[\frac{\lambda_1-\lambda_0}{n}\right] + \alpha_2 \exp\left[\frac{\lambda_2-\lambda_0}{n}\right]}{\alpha_1\mu_1 + \alpha_2\mu_2}\pi_0, && (5.33)\\
&= \frac{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}}{\alpha_1\mu_1 + \alpha_2\mu_2} e^{-\frac{\lambda_0}{n}}\pi_0, && (5.34)
\end{aligned}$$

where π_0 is the pooled equilibrium premium introduced in Equation 3.53.

Figure 5.5 shows an example of how loss coverage ratio behaves with respect to equilibrium premium for negative-exponential demand functions with different values of n. Recall that:

- When $n \to 0$, we have the iso-elastic demand with a constant demand elasticity function with respect to premium.
- When $n = 1$, the demand has a demand elasticity function which is linearly increasing with premium.

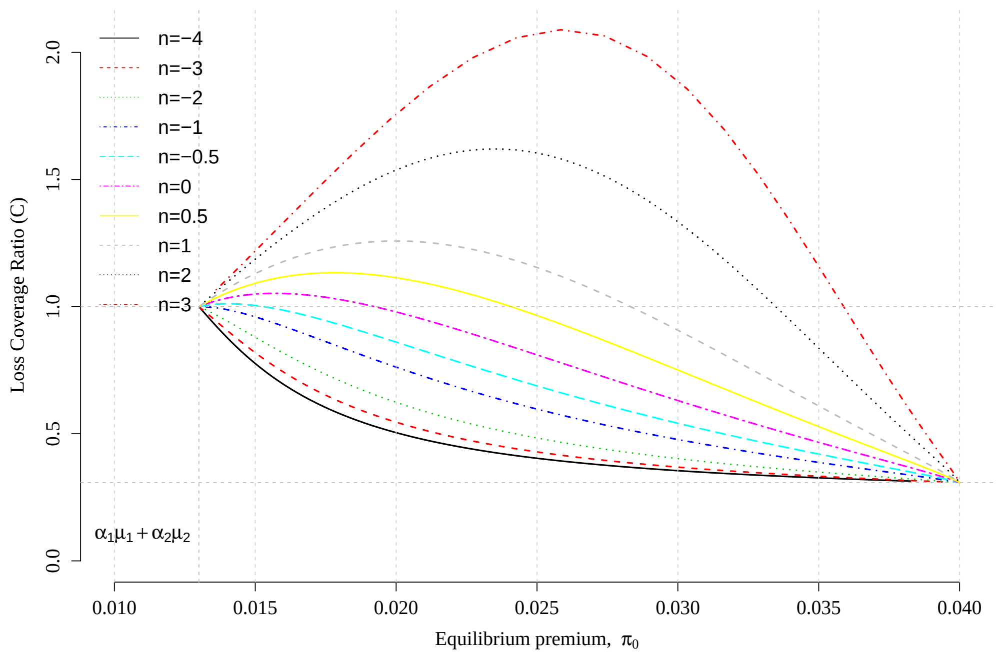

Figure 5.5: Plot of the loss coverage ratio C as a function of equilibrium premium π_0 for negative-exponential demand function of different n with $\alpha_1 = 90\%, \alpha_2 = 10\%, \mu_1 = 0.01, \mu_2 = 0.04$.

We observe the following:

(a) When $n > -1$, as the equilibrium premium increases from its minimum (i.e. $\alpha_1\mu_1 + \alpha_2\mu_2$), initially the loss coverage ratios of various negative-exponential demand with different values of n all start increasing from 1, then they continue to increase till reaching their corresponding maximums. At higher equilibrium premium, the loss coverage ratios decrease, and end up at the same level which is less than 1 when the equilibrium premium reaches its maximum of μ_2 (which is 0.04 in this example).

(b) When $n \leq -1$, loss coverage ratios are never greater than 1, i.e. when individuals have demand elasticity function that decreases more than proportionately to premium, loss coverage of pooled equilibrium premium is never greater than the loss coverage of risk-differentiated premiums.

(c) Given an equilibrium premium, loss coverage ratio increases with *"second-order elasticity"*, n, i.e. loss coverage is higher for individuals for whom demand elasticity is very sensitive to premium.

We now look at each of the above observations in details with the following results:

Result 5.11.

$$\lim_{\pi_0 \to \alpha_1\mu_1 + \alpha_2\mu_2} C(\pi_0) = 1 \textit{ for } n \in R. \tag{5.35}$$

In other words, at the minimum pooled equilibrium premium, loss coverage ratio goes to 1 for negative-exponential demand function with non-decreasing demand elasticity function of premium.

This is a direct result from Equation 5.34, and using the result that $\lim_{\lambda_1 \to 0} \pi_0 = \alpha_1\mu_1 + \alpha_2\mu_2$ from Result 3.18 in sub-section 3.4.1. Note that when $n \to 0$, the negative-exponential demand function becomes iso-elastic demand, and the result on loss coverage ratio for iso-elastic demand has been proved in sub-section 5.2.1.

Result 5.12.

$$\lim_{\pi_0 \to \mu_2} C(\pi_0) = \frac{\alpha_2\mu_2}{\alpha_1\mu_1 + \alpha_2\mu_2} < 1 \textit{ for } n \in R. \tag{5.36}$$

In other words, at the maximum pooled equilibrium premium, loss coverage ratio with different values of n (i.e. second-order elasticity) tends to the same lower limit.

This result can be proved using the same approach as in Result 3.19 in sub-section 3.4.1. When demand elasticity becomes very large, only the high risks still buy insurance, so, only high risks will be covered by insurance.

Result 5.13.

$$C(\pi_0) \leq 1, \ \textit{when } n \leq -1. \tag{5.37}$$

In other words, for negative-exponential demand with demand elasticity decreasing more than proportionately to the premium, loss coverage at pooled equilibrium premium is always smaller than loss coverage at risk-differentiated premiums. This result is proved in Theorem D.2.1 in Appendix D.

Note that demand elasticity functions which fall as a function of premium are not realistic, because of the income effect on demand (i.e. at higher prices the cost of insurance represents a larger part of consumers' total budget constraint, so their elasticity of demand for insurance is expected to be higher), but we include these cases for completeness.

Result 5.14. *For $n > -1$,*

$$\frac{\partial}{\partial \pi_0} C(\pi_0)|_{\lambda_1 \to 0} > 0. \tag{5.38}$$

This result indicates that for those demand elasticity functions which are non-decreasing function of premium or decreasing less than proportionately to premium, when demand elasticities are small, loss coverage is an increasing function of pooled equilibrium premium. Therefore, loss coverage at pooled equilibrium premium is higher than loss coverage at risk-differentiated premiums if demand elasticity is small. This result is proved in Theorem D.2.2 in Appendix D.

Result 5.15. *For a given equilibrium premium π_0, the loss coverage ratio is an increasing function of the second-order elasticity.*

This result indicates that given a pooled market equilibrium, loss coverage under pooling is higher if the individual's demand elasticity is very sensitive to premium. This result is proved in D.2.3 in Appendix D.

Result 5.16. *For any $n \geq 0$, the loss coverage ratio is maximised for:*

$$\lambda_0^* = 1 + \frac{[\{u + (1-u)\beta^n\} - \{u + (1-u)\beta\}^n]}{n \frac{\partial}{\partial \lambda_1} \log \pi_0}, \tag{5.39}$$

where

$$u = \frac{\alpha_1}{\alpha_1 + \alpha_2 e^{\frac{\lambda_1(\beta^n - 1)}{n}}} \tag{5.40}$$

defined in Equation 3.55.

Figure 5.6 shows the graph of loss coverage ratio as functions of demand elasticity at pooled equilibrium premium, λ_0 for $n \geq 0$.

Note that both Figures 5.5 and 5.6 show loss coverage ratio for negative-exponential demand for different values of n, but Figure 5.5 is plotted against equilibrium premium π_0 while Figure 5.6 is plotted against demand elasticity λ_0.

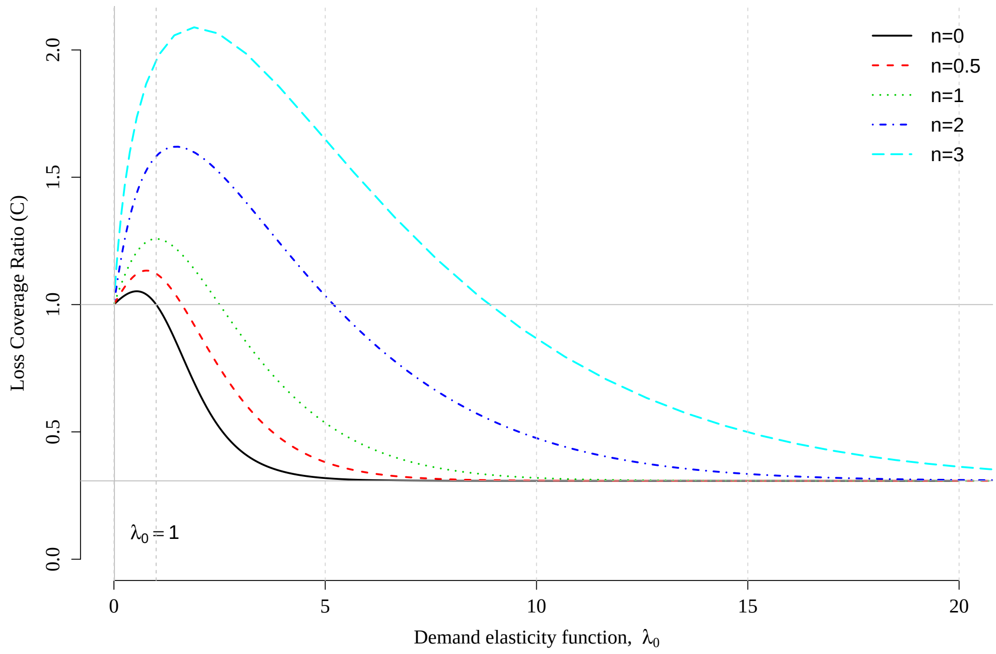

Figure 5.6: Plot of the loss coverage ratio C as a function of demand elasticity λ_0 for different demand function with $\alpha_1 = 90\%, \alpha_2 = 10\%, \mu_1 = 0.01, \mu_2 = 0.04$.

We observe that:

- For $0 < n < 1$, $\lambda_0^* < 1$. This result is shown in Figure 5.6 in which the black curve (with $n = 0$) and the red dashed curve (with $n = 0.5 < 1$) achieves their maximums when $\lambda_0 < 1$. Note that iso-elastic demand is a special case when $n \to 0$.
- For $n = 1$, $\lambda_0^* = 1$. This result is shown in Figure 5.6 in which the green dashed curve (with $n = 1$) achieves its maximum when $\lambda_0 = 1$. This

is the case when negative-exponential demand has a demand elasticity that is linearly increasing with premium.

- For $n > 1$, $\lambda_0^* > 1$. This result is shown in Figure 5.6 in which the two blue dashed curves (with $n = 2$ and 3) achieves their maximums when $\lambda_0 > 1$.

The proofs are given in Theorem D.2.4 in Appendix D.

5.4 Summary

In this chapter, we formally define the concept of loss coverage which is the population's expected losses compensated by insurance using examples of iso-elastic demand, negative-exponential demand and general demand. Loss coverage ratio is then defined as the ratio of loss coverage at a given premium strategy to the loss coverage at risk-differentiated premiums.

We focus on the case when risk classification regime is banned so that insurers charge a pooled premium to all the risk-groups. We then analyse the loss coverage ratio for different scenarios.

Our main finding is: Loss coverage might be a better metric to measure the impact of pooling. When a moderate level of adverse selection is tolerated, loss coverage is higher under pooling if the shift in coverage towards higher risks more than compensates for the fall in number of risks insured, i.e. pooling can benefit the society as a whole. This is specifically the case when the demand elasticities of the low risk-groups are less than 1, and the demand elasticities of the high risk-groups are higher than those of the low risk-groups (which is consistent with some empirical findings).

Chapter 6

Social Welfare and Loss Coverage

In this chapter, we reconcile loss coverage to the utilitarian concept of social welfare. We will show that, **under iso-elastic demand, if insurance premiums are small relative to wealth, maximising loss coverage maximises social welfare**.

Our approach to social welfare is in the same spirit as Hoy (2006): we assume cardinal and interpersonally comparable utilities, and assign equal weights to the utilities of all individuals. This equal-weights approach is based on the Harsanyi (1955) “veil of ignorance” argument: that is, behind the (hypothetical) “veil of ignorance”, where one does not know what position in society (e.g. high risk or low risk) one occupies, the appropriate probability to assign to being any individual is $1/m$, where m is the number of individuals in society. Alternative risk classifications can then be compared by comparing expected utility in each scheme for the (hypothetical) individual utility-maximiser behind the ‘veil of ignorance’.

In Hoy (2006), the author found that, following the simple *Rothschild-*

Stiglitz-Wilson (RSW) model (Rothschild & Stiglitz (1976), Wilson (1977)) for insurance, if the population of the high risks was 'sufficiently small', the expected welfare might be enhanced by regulatory adverse selection, i.e. the society can benefit from some adverse selection. Although, according to the author, it was difficult to quantify how 'small' this high risks' population should be, the suggestion that adverse selection should not always be eliminated has opened a new area of discussion.

Based on the model framework in Section 2.1, suppose an individual is selected at random from the whole population. The individual's expected utility can be written as follows:

$$\begin{aligned} &\text{Social Welfare} \qquad (6.1)\\ &= E\left[\,Q\,U_\Gamma(W-\Pi L)+(1-Q)\left[(1-X)\,U_\Gamma(W)+X\,U_\Gamma(W-L)\right]\right] \end{aligned}$$

where the first part represents random utility if insurance is purchased; and the second part is the random utility if insurance is not purchased. Recall the definition of Q and X:

Q : is a Bernoulli random variable which takes the value of 1 if insurance is purchased; 0 otherwise,

X : is a Bernoulli random variable, taking the value of 1 if a loss event occurs and 0 otherwise.

As certainty equivalent decisions do not depend on the origins and scales of utility functions, in Section 2.1, we assumed without loss of generality, that utilities for all individuals are the same at the 'end-points', W and $W-L$. But, this argument cannot be directly extended to Equation (6.1), because individuals' utilities can differ for identical levels of wealth, which has direct implications for social welfare.

However, without any standardisation, Equation (6.1) is susceptible to being dominated by a 'utility monster' who derives more utility from a given level of wealth than all other individuals combined (Bailey (1997), Nozick (1974)). So we propose to continue standardising utility functions so that all utilities are the same at 'end-points', W and $W - L$, as before. This standardisation implies that the same 'disutility of uninsured loss' $[U(W) - U(W - L)]$ is assigned to all individuals, but utility if insurance is purchased $U_\Gamma(W - \Pi L)$ differs between individuals. Under this standardisation, social welfare, denoted by S can be expressed as:

$$S = E\left[\, Q\, U_\Gamma(W - \Pi L) + (1 - Q)\left[(1 - X)\, U(W) + X\, U(W - L)\right]\right]. \quad (6.2)$$

To derive an expression for S, we consider the constituent parts of Equation (6.2) separately. Here we sketch the argument, the full probabilistic model is in Appendix A.3. First:

$$\begin{aligned} &E\left[\, Q\, U_\Gamma(W - \Pi L)\,\right] \\ &= \sum_{i=1}^{m} E[\, Q\, U_\Gamma(W - \pi_i L) \mid \mu = \mu_i\,]\, P[\mu = \mu_i], && (6.3)\\ &= \sum_{i=1}^{m} \{E[\, U_\Gamma(W - \pi_i L) \mid U_\Gamma(W - \pi_i L) > u_{c_i}\,,\, \mu = \mu_i\,] && (6.4)\\ &\qquad \times P\left[U_\Gamma(W - \pi_i L) > u_{c_i} \mid \mu = \mu_i\,\right]\, p_i\}\,, \\ &= \sum_{i=1}^{m} U_i^*(W - \pi_i L)\, d_i(\pi_i)\, p_i, \quad \text{using Equation (2.8)}, && (6.5) \end{aligned}$$

where $u_{c_i} = (1 - \mu_i)U(W) + \mu_i U(W - L)$ (as defined in Equation (2.7)) and $U_i^*(W - \pi_i L) = E[\, U_\Gamma(W - \pi_i L) \mid U_\Gamma(W - \pi_i L) > u_{c_i}\,,\, \mu = \mu_i\,]$ represents the expected utility of individuals purchasing insurance in risk-group i.

Using the assumption that all individuals have the same utilities $U(W)$ and $U(W-L)$ at wealth levels W and $W-L$, and that the random variables Q and X are independent given a risk-group, the second part of Equation (6.2) becomes:

$$
\begin{aligned}
& E\,[\,(1-Q)\,[(1-X)\,U(W) + X\,U(W-L)]\,] \\
&= \sum_{i=1}^{m} E[\,(1-Q)\,[(1-X)\,U(W) + X\,U(W-L)] \mid \mu = \mu_i\,]\,P[\mu = \mu_i], && (6.6)\\
&= \sum_{i=1}^{m} [(1-d_i(\pi_i))\,\{(1-\mu_i)U(W) + \mu_i\,U(W-L)\}]\,p_i. && (6.7)
\end{aligned}
$$

Combining Equations (6.5) and (6.7), we get the following expression for social welfare:

$$
\begin{aligned}
S = & \sum_{i=1}^{m} \Bigg[\underbrace{d_i(\pi_i)\,U_i^*(W-\pi_i\,L)}_{\text{Insured population}} && (6.8)\\
& \qquad + \underbrace{(1-d_i(\pi_i))\,\{(1-\mu_i)U(W) + \mu_i\,U(W-L)\}}_{\text{Uninsured population}} \Bigg] p_i, \\
= & \underbrace{\sum_{i=1}^{m} [(1-\mu_i)U(W) + \mu_i\,U(W-L)]\;p_i}_{\text{Constant as a function of } \pi_i} && (6.9)\\
& + \underbrace{\left(\sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i\right) \times [U(W) - U(W-L)]}_{\text{Loss coverage}\times\text{Positive multiplier}} \\
& - \underbrace{\sum_{i=1}^{m} d_i(\pi_i)\,[U(W) - U_i^*(W-\pi_i L)]\;p_i\,.}_{\text{Adjustment factor to account for premiums}} \\
= & \text{ Constant} + \textbf{Loss Coverage} \times \text{Positive multiplier} && (6.10)\\
& \qquad - \text{Premium adjustment factor.}
\end{aligned}
$$

A regulator or a policymaker aiming to maximise social welfare, will be interested in choosing a risk-classification $\underline{\pi} = (\pi_1, \pi_2, ..., \pi_m)$ which maximises S. However, social welfare depends on unobservable utility functions, which makes it difficult to implement. On the other hand, loss coverage depends solely on observable quantities and Equation 6.10 shows that social welfare and loss coverage are directly related. So, it will be useful if it can be shown that both measures, social welfare and loss coverage, agree on the choice of risk-classification under certain assumptions. A regulator or policymaker can then use loss coverage as a proxy for social welfare.

6.1 Iso-elastic Demand Example: Equal Demand Elasticity

In this section, we will demonstrate the relationship between loss coverage and social welfare using iso-elastic demand.

Using the convenient standardisation of $U(W) = 1$ and $U(W - L) = 0$ as defined in Equations 2.10 and 2.11, along with the assumption that $W = L = 1$, and noting that social welfare S is a function of the risk-classification $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$, Equation (6.2) becomes:

$$S(\underline{\pi}) = E\left[\, Q\, U_\Gamma(1 - \Pi) + (1 - Q)(1 - X)\,\right], \tag{6.11}$$

$$= E\left[\, Q\, \{U_\Gamma(1 - \Pi) - (1 - X)\,\}\,\right] + K, \tag{6.12}$$

where $K = E[1 - X]$ is a constant as it does not depend on $\underline{\pi}$.

Using the particular form of utility function $U(w) = w^\gamma$, we have:

$$S(\underline{\pi}) = E\left[\, Q\, \left\{(1-\Pi)^\Gamma - (1-X)\,\right\}\right] + K, \tag{6.13}$$

$$\approx E\,[\, Q\,(1 - \Gamma\,\Pi - 1 + X\,)\,] + K, \tag{6.14}$$

using first order Taylor approximation and assuming insurance premium is sufficiently small compared to individuals' wealth.

We introduce $\hat{S}(\underline{\pi})$ as this approximation of $S(\underline{\pi})$. Hence,

$$\hat{S}(\underline{\pi}) = E\,[\, Q(X - \Gamma\,\Pi)\,] + K, \tag{6.15}$$

$$= E\,[\, QX\,] - E\,[\, Q\,\Gamma\,\Pi\,] + K, \tag{6.16}$$

$$= LC(\underline{\pi}) - PA(\underline{\pi}) + K, \tag{6.17}$$

where $LC(\underline{\pi}) = E\,[\, QX\,]$ is loss coverage and $PA(\underline{\pi}) = E\,[\, Q\,\Gamma\,\Pi\,]$ is the premium adjustment factor under the risk-classification $\underline{\pi}$. We have already analysed $LC(\underline{\pi})$ in Chapter 5, so we focus on $PA(\underline{\pi})$ here.

Firstly, recall that Q, is a Bernoulli random variable which takes the value of 1 if insurance is purchased and 0 otherwise. And from Equation 2.22, given a risk-group i, insurance is purchased when $\Gamma_i < \frac{\mu_i}{\pi_i}$, where the random variable $\Gamma_i = [\,\Gamma \mid \mu = \mu_i\,]$. Hence:

$$[\, Q \;\mid\; \mu = \mu_i\,] = I\left[\Gamma_i < \frac{\mu_i}{\pi_i}\right]. \tag{6.18}$$

So:

$$PA(\underline{\pi}) = \sum_{i=1}^{m} E\,[\,Q\,\Gamma\,\Pi \mid \mu = \mu_i\,]\; P[\mu = \mu_i], \tag{6.19}$$

$$= \sum_{i=1}^{m} E\left[I\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right]\,\Gamma_i\,\pi_i\right]\, p_i, \tag{6.20}$$

$$= \sum_{i=1}^{m} E\left[\Gamma_i\, I\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right]\right]\, \pi_i\, p_i, \tag{6.21}$$

where Γ_i has the same cumulative distribution function as given in Equation (2.23):

$$P\left[\Gamma_i \leq \gamma\right] = \begin{cases} 0 & \text{if } \gamma < 0 \\ \tau_i\,\gamma^{\lambda_i} & \text{if } 0 \leq \gamma \leq (1/\tau_i)^{1/\lambda_i} \\ 1 & \text{if } \gamma > (1/\tau_i)^{1/\lambda_i}, \end{cases} \tag{6.22}$$

Now suppose all risk-groups have equal demand elasticities, i.e. $\lambda_1 = \lambda_2 = ... = \lambda_m = \lambda$.

Our main finding in the case of equal demand elasticities is:

Result 6.1. *For iso-elastic demand, assuming insurance premium is sufficiently small compared to individuals' wealth, ranking risk classifications by (observable) loss coverage always gives the same ordering as ranking by (unobservable) utilitarian social welfare. In particular, in the case of equal demand elasticities, loss coverage and social welfare, point to the same conclusion that pooling under no risk classification provides a greater or equal social efficacy of insurance compared to risk-differentiated premiums under full risk classification, when demand elasticity $\lambda < 1$; and vice versa.*

This result can be derived from Equation 6.21 as follows:

$$
\begin{aligned}
PA(\underline{\pi}) &= \sum_{i=1}^{m} \left[\int_0^{\frac{\mu_i}{\pi_i}} \gamma \tau_i \lambda \gamma^{\lambda-1} d\gamma \right] \pi_i p_i, && (6.23)\\
&= \frac{\lambda}{(\lambda+1)} \sum_{i=1}^{m} \left(\frac{\mu_i}{\pi_i} \right)^{\lambda+1} \tau_i \pi_i p_i, && (6.24)\\
&= \frac{\lambda}{(\lambda+1)} \sum_{i=1}^{m} \frac{\mu_i^{\lambda+1}}{\pi_i^{\lambda}} \tau_i p_i, && (6.25)\\
&= \frac{\lambda}{(\lambda+1)} LC(\underline{\pi}), \quad \text{by the definition of loss coverage.} && (6.26)
\end{aligned}
$$

Hence social welfare in Equation (6.17) becomes:

$$
\hat{S}(\underline{\pi}) = \frac{1}{\lambda+1} LC(\underline{\pi}) + K. \tag{6.27}
$$

The right-hand side of Equation (6.27) can be interpreted as follows:

The second term $K = E[1 - X]$ corresponds to expected utility in the absence of the institution of insurance (recall that we have standardised $U(W) = 1$, $U(W - L) = 0$, and X is the loss for an individual drawn at random from the population).

The first term represents an increase in expected utility, attributable to the institution of insurance; this allows for the expectations of both utility of benefits received, and disutility of premiums paid. If λ is small (corresponding to inelastic demand and high risk aversion), the premiums paid are relatively unimportant, so the increase in expected utility is a large fraction of the loss coverage [1]. If λ is large (corresponding to elastic demand and low risk aversion), the premiums paid are important, so the increase in expected

[1] The fraction $1/(\lambda + 1)$ can also be viewed as a fraction of the loss coverage $LC(\underline{\pi}) = E[QX]$ which 'counts' as an offset against the uninsured losses X which appear in $K = E[1 - X]$, where the offset is in on a welfare scale and includes allowance for both benefits and premiums.

utility is only a small fraction of the loss coverage.

It follows from Equation (6.27) that for any pair of risk classifications $\underline{\pi}_1$ and $\underline{\pi}_2$, we can write

$$\hat{S}(\underline{\pi}_1) \geq \hat{S}(\underline{\pi}_2) \Leftrightarrow LC(\underline{\pi}_1) \geq LC(\underline{\pi}_2) \tag{6.28}$$

and clearly the contrapositive (i.e with both inequalities reversed) also holds. In other words: **for iso-elastic demand, assuming insurance premium is sufficiently small compared to individuals' wealth, ranking risk classifications by loss coverage always gives the same ordering as ranking by social welfare.** So a policymaker or regulator can implement a risk classification which gives higher (observable) loss coverage, with the comfort of knowledge that this also gives higher (unobservable) social welfare.

Equation 6.28 holds for *any* pair of risk classifications which satisfy the equilibrium condition in Equation 3.10. This includes schemes where premiums are partly (but not fully) risk-differentiated (which will be discussed in Chapter 7), as well as the polar cases of pooling and actuarially fair premiums. Where the comparison is between the polar cases, combining Equation 6.28 with Equation 5.13 (i.e. $\lambda \lesseqgtr 1 \Leftrightarrow C(\pi_0) \gtreqless 1$) shows that for iso-elastic demand, pooling gives higher or equal social welfare than actuarially fair premiums whenever demand elasticity is less than one, and *vice versa*.

6.2 Iso-elastic Demand Example: Different Demand Elasticities

If all risk-groups have iso-elastic demand, but with possibly different demand elasticities, then Equation 6.21 can be expressed as:

$$PA(\underline{\pi}) = \sum_{i=1}^{m} E\left[\Gamma_i \, I\left[\Gamma_i \leq \frac{\mu_i}{\pi_i}\right]\right] p_i \pi_i, \tag{6.29}$$

$$= \sum_{i=1}^{m} \left[\int_0^{\frac{\mu_i}{\pi_i}} \gamma \tau_i \lambda_i \gamma^{\lambda_i - 1} d\gamma\right] p_i \pi_i, \tag{6.30}$$

$$= \sum_{i=1}^{m} \left(\frac{\lambda_i}{1+\lambda_i}\right) p_i \tau_i \mu_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} = \sum_{i=1}^{m} \left(\frac{\lambda_i}{1+\lambda_i}\right) d_i(\pi_i) p_i \mu_i. \tag{6.31}$$

Recall from Section 5.2:

$$LC(\underline{\pi}) = \sum_{i=1}^{n} p_i \tau_i \mu_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} = \sum_{i=1}^{n} d_i(\pi_i) p_i \mu_i. \tag{6.32}$$

It follows from Equation 6.17 that social welfare in this case becomes:

$$\hat{S}(\underline{\pi}) = \sum_{i=1}^{n} \left(\frac{1}{1+\lambda_i}\right) d_i(\pi_i) p_i \mu_i + K, \tag{6.33}$$

where $K = E[1 - X]$ is a constant as it does not depend on $\underline{\pi}$.

If we want to compare social welfare at pooled equilibrium premium (under *no risk classification*) to social welfare at risk-differentiated premiums (under *full risk classification*), we define **"social welfare ratio"** at pooled equilibrium premium as:

$$\widehat{SW}(\pi_0) = \frac{\hat{S}(\pi_0)}{\hat{S}(\underline{\mu})}. \tag{6.34}$$

To demonstrate some features of social welfare ratio at pooled equilibrium premium, we use an example of two risk-groups in the population, i.e. $m = 2$, a low risk-group and a high risk-group.

Figure 6.1 shows an example of the social welfare ratio (at pooled equilibrium premium) given different demand elasticities for the low (λ_1 on the x-axis) and the high risk-groups (λ_2 on the y-axis).

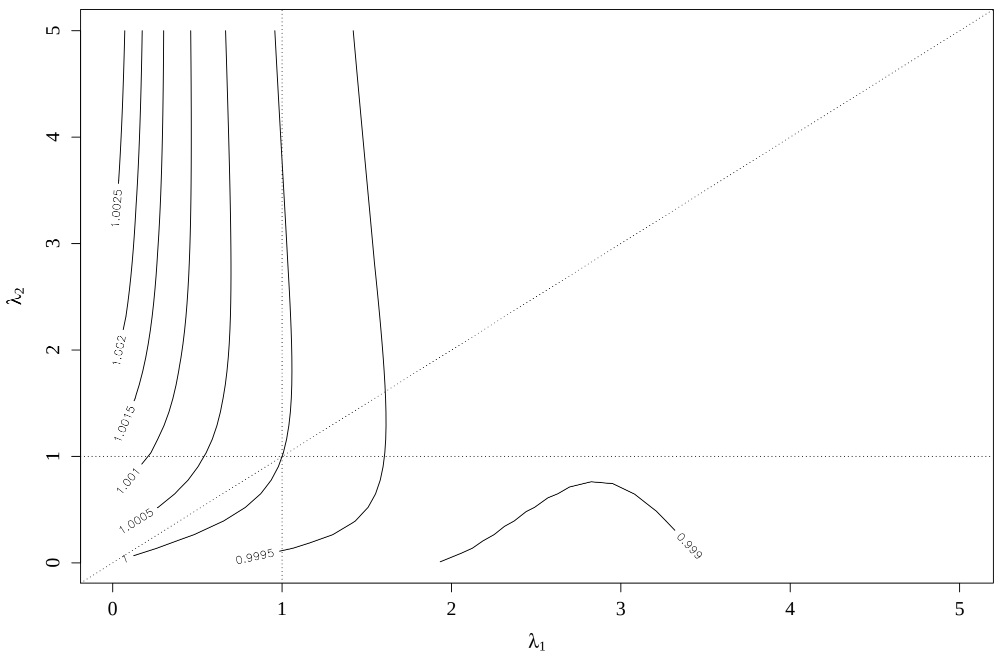

Figure 6.1: Plot of social welfare ratio $\widehat{SW}(\pi_0)$ with $\alpha_1 = 90\%, \alpha_2 = 10\%, \mu_1 = 0.01, \mu_2 = 0.04$.

We have the following results:

Result 6.2. *When both risk-groups have demand elasticities that are less than 1, and the high risks have a higher demand elasticity than the low risks, assuming insurance premium is sufficiently small compared to individuals'* *wealth, social welfare at pooled premium is greater than the social welfare at risk-differentiated premiums, i.e. the social welfare ratio* $\widehat{SW}(\pi_0)$ *is greater than 1 when* $0 < \lambda_1 < \lambda_2 < 1$.

Recall that the ordering of demand elasticities $0 < \lambda_1 < \lambda_2 < 1$ is what we might expect due to the income effect on demand which we mentioned earlier in this thesis. This result is proved in Theorem E.2.8 in Appendix E.

In Figure 6.1, Result 6.2 is represented by the fact that in the area above the $\lambda_1 = \lambda_2$ diagonal inside the unit square, all social welfare ratio contours are greater than 1.

Result 6.3. *When* $0 < \lambda_2 < \lambda_1 < 1$*, either full or no risk classification maximises social welfare* $\hat{S}(\underline{\pi})$.

In Figure 6.1, this result is represented by the fact that in the area below the $\lambda_1 = \lambda_2$ diagonal within the unit square, there are social welfare ratio contours greater or smaller than 1. This result is proved in Theorem E.2.9 in Appendix E.

Result 6.4. *When* $\lambda_2 > \lambda_1$*,* $C(\pi_0) < 1 \Rightarrow \widehat{SW}(\pi_0) < 1$*, i.e. when the high risks have higher demand elasticities than the low risks, if loss coverage under pooling is smaller than under full risk classification, then so is the social welfare.*

Figure 6.2 is the same as Figure 6.1 but with loss coverage ratio being plotted as well in dashed blue contours, using the same underlying parameters.

From this plot, we observe that above the diagonal line where $\lambda_1 = \lambda_2$, the solid contour line representing $\widehat{SW}(\pi_0) = 1$ lies entirely to the left of the dashed blue contour line representing $C(\pi_0) = 1$. It follows that $C(\pi_0) < 1$ implies that $\widehat{SW}(\pi_0) < 1$.

This result is proved in Theorem E.2.5 in Appendix E.

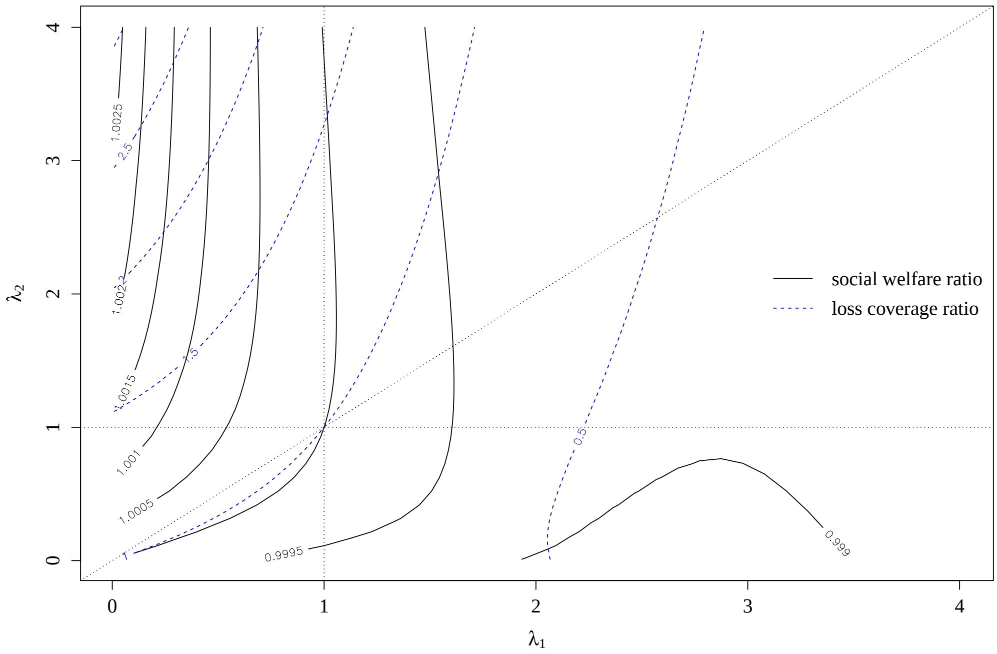

Figure 6.2: Plot of social welfare ratio $\widehat{SW}(\pi_0)$ and loss coverage ratio $C(\pi_0)$ with $\alpha_1 = 90\%, \alpha_2 = 10\%, \mu_1 = 0.01, \mu_2 = 0.04$.

Result 6.5. *When $\lambda_2 < \lambda_1$, $C(\pi_0) > 1 \Rightarrow \widehat{SW}(\pi_0) > 1$, i.e. when the low risks have higher demand elasticities than the high risks, if loss coverage under pooling exceeds that under full risk classification, then so is the social welfare.*

From Figure 6.2, we observe that below the diagonal line where $\lambda_1 = \lambda_2$, the solid contour line representing $\widehat{SW}(\pi_0) = 1$ lines entirely to the right of the dashed blue contour line representing $C(\pi_0) = 1$. It follows that $C(\pi_0) > 1$ implies that $\widehat{SW}(\pi_0) > 1$. This result is proved in Theorem E.2.5 in Appendix E.

Results in this section show consistency between the two measures: loss coverage and social welfare, when comparing the impact of risk classification regimes on social welfare.

6.3 Summary

In this chapter, we introduced social welfare, which is defined as the aggregate expected utilities of all individuals in a population with equal weight assigned to each individual. We investigate the relationship between loss coverage and social welfare. We found that, under certain assumptions, loss coverage and social welfare are directly related.

Using iso-elastic demand as an example, assuming insurance premium is sufficiently small compared to individuals' wealth, we proved that ranking risk classifications by (observable) loss coverage always gives the same ordering as ranking by (unobservable) utilitarian social welfare. In particular, maximising loss coverage is equivalent to maximising social welfare.

Chapter 7

Partial Risk Classification

Our analysis on loss coverage and social welfare so far has focused only on two risk classifications, i.e. *full risk classification* where risk-differentiated premiums are charged for different risk-groups, and *no risk classification* where all risk-groups are charged the same pooled equilibrium premium. Under this set-up, using iso-elastic demand with realistic demand elasticities, we have found that loss coverage and social welfare for pooled premium exceeds that for risk-differentiated premiums, i.e. pooling can benefit the society as a whole in terms of the proportion of loss covered by insurance and the corresponding aggregate social welfare.

In practice, there can be intermediate risk classifications, between the two extremes of *full* and *no risk classification*. We call these intermediate risk classifications, *partial risk classification*. *Partial risk classification* is possibly a more realistic strategy, because insurers can differentiate risks only to a certain extent, which may not fully reflect the differences between different risk-groups. For example, in the European Union, gender-based premiums were banned from 2012, but insures can still charge different premiums for different ages; in the U.S., the Patient Protection and Affordable Care Act allows classification only by age, location, family size and smoking status; and many countries have restricted insurers' use of genetic test results for underwriting purposes.

In this chapter, we want to analyse loss coverage and social welfare for *partial risk classification*.

Recall the following definitions:

- Loss coverage ratio C is defined as

$$C = \frac{LC(\underline{\pi})}{LC(\underline{\mu})}, \tag{7.1}$$

  i.e. the ratio of loss coverage for a premium strategy $\underline{\pi}$ (given a risk classification) to that of *full risk classification* $\underline{\mu}$.

- Social welfare ratio $\widehat{SW}$ is defined as

$$\widehat{SW} = \frac{\hat{S}(\underline{\pi})}{\hat{S}(\underline{\mu})}, \tag{7.2}$$

  i.e. the ratio of social welfare for a premium strategy $\underline{\pi}$ (given a risk classification) to that of *full risk classification* $\underline{\mu}$.

Using iso-elastic demand, we will first analyse *partial risk classification* for populations with only two risk-group: a high and a low risk-group. Then we will generalise our results to the case of more risk-groups.

7.1 Two Risk-groups

We have the following main result on maximising loss coverage in the case of two risk-groups:

Result 7.1. *Using iso-elastic demand, when there are only two risk-groups, partial risk classification can lead to higher loss coverage compared to both full and no risk classification only when the elasticities of demand of the low risks and high risks are different and both are greater than 1 with the high risks have a higher elasticity of demand up to a certain level, i.e.* $1 < \lambda_1 < \lambda_2 < L$ *for some* $L > 1$.

Figure 7.1 shows an example of the regions where different risk classifications ("No" for *no risk classification*, "Full" for *full risk classification*, and "Partial" for *partial risk classification*) maximises loss coverage ratio. λ_1, the demand elasticity for the low risk-group is on the x-axis and λ_2, the demand elasticity for the high risk-group is on the y-axis. Any combination of (λ_1, λ_2) located on the black curve gives loss coverage ratio at pooled equilibrium premium $C(\pi_0) = 1$, to the left of which leads to $C(\pi_0) > 1$ and to the right of which leads to $C(\pi_0) < 1$. The dark coloured region indicates that *full risk classification* maximises loss coverage ratio. The light coloured region indicates that certain *partial risk classification* maximises loss coverage ratio, and the region without colour indicates that *no risk classification* maximises loss coverage ratio.

The dashed blue curve has the following form:

$$\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2 - 1}{\lambda_1 - 1}\right) = \frac{\mu_2}{\mu_1} \tag{7.3}$$

where (μ_1, μ_2) are risks of the low and high risk-group. ($\frac{\mu_2}{\mu_1}$ is 4 in this example.) When both λ_1 and λ_2 are greater than 1, this curve demarcates the two areas *partial risk classification* maximises loss coverage ratio on the right of the curve while *no risk classification* maximises loss coverage ratio on the left.

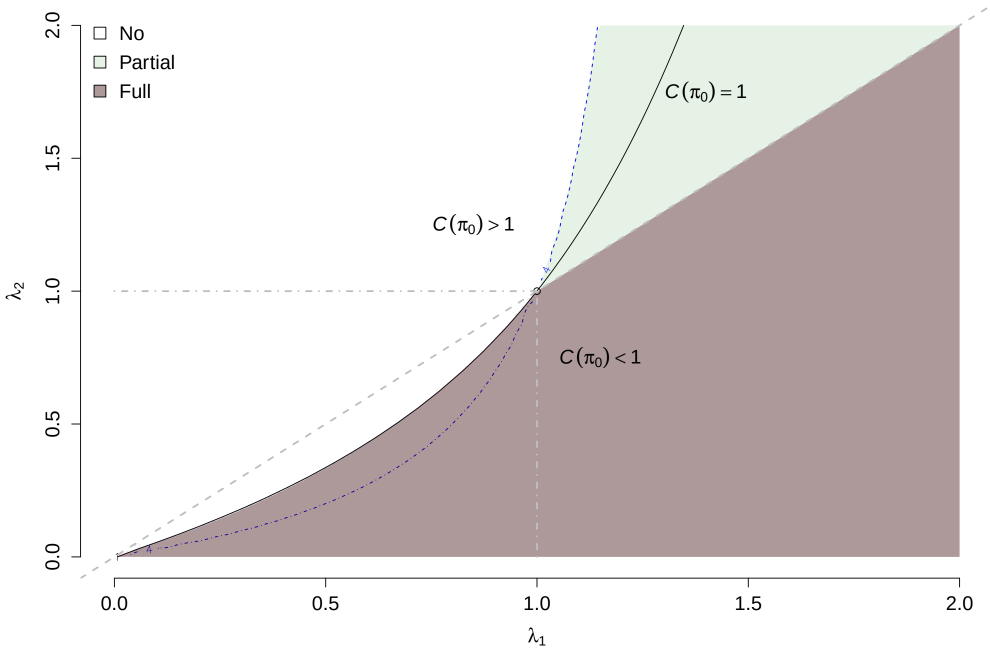

Figure 7.1: Plot of the areas where different risk classifications give the highest loss coverage ratio with $\alpha_1 = 90\%, \alpha_2 = 10\%, \mu_1 = 0.01, \mu_2 = 0.04$.

7.1.1 Equal Demand Elasticity

When both risk-groups have the same demand elasticity λ, we have already proved in Section 5.2 that loss coverage at pooled equilibrium premium is higher than loss coverage at risk-differentiated premiums if the constant demand elasticity $\lambda < 1$. Taking *partial risk classification* into account, we have the following result:

Result 7.2. *If both the low risk-group and the high risk-group have the same iso-elastic demand elasticity λ, then*

- $0 < \lambda < 1 \Rightarrow$ *No risk classification maximises loss coverage;*
- $\lambda = 1 \Rightarrow$ *All risk classifications schemes have the same loss coverage;*
- $\lambda > 1 \Rightarrow$ *Full risk classification maximises loss coverage.*

In Figure 7.1, $\lambda_1 = \lambda_2 = \lambda < 1$ falls into the area where *no risk classification* (indicated by "No") maximises loss coverage. And $\lambda_1 = \lambda_2 = \lambda > 1$ falls into the area where the *full risk classification* (indicated by "Full") maximises loss coverage. And at the special case of $\lambda_1 = \lambda_2 = \lambda = 1$, all three areas, i.e. "No", "Full" and "Partial" meet at that point. This means all three risk classification schemes lead to the same loss coverage. Hence, when the low risk-group and the high risk-group have equal demand elasticities, *partial risk classification* does not maximise loss coverage.

Result 7.2 is formally proved in Theorem F.1.3 and F.1.4 in Appendix F.

7.1.2 Different Demand Elasticities

In the case when demand elasticities for the low risk-group and high risk-group could be different, we have the following results on maximising loss coverage:

Result 7.3. *If both risk-groups have demand elasticities which are less than 1, i.e.* $0 < \lambda_1, \lambda_2 < 1$, *then*

- $0 < \lambda_1 < \lambda_2 < 1 \Rightarrow$ *No risk classification maximises loss coverage;*
- $0 < \lambda_2 < \lambda_1 < 1$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) > \frac{\mu_2}{\mu_1}$
  $\Rightarrow$ *Full risk classification maximises loss coverage;*

- $0 < \lambda_2 < \lambda_1 < 1$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) \leq \frac{\mu_2}{\mu_1}$
  $\Rightarrow$ *Either full or no risk classification maximises loss coverage.*

From Figure 7.1, we observe that $0 < \lambda_1 < \lambda_2 < 1$ falls into the region where *no risk classification* ("No") maximises loss coverage. $0 < \lambda_2 < \lambda_1 < 1$ and $\frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) > \frac{\mu_2}{\mu_1}$ (i.e. the area to the right of the dashed blue curve within the unit square) falls into the region where *full risk classification* ("Full") maximises loss coverage.

Result 7.3 is formally proved in Theorem F.1.7 and F.1.8 in Appendix F.

Result 7.4. *If both risk-groups have demand elasticities which are greater than 1, i.e.* $\lambda_1, \lambda_2 > 1$*, then*

- $\lambda_1 > \lambda_2 > 1 \Rightarrow$ *Full risk classification maximises loss coverage;*

- $\lambda_2 > \lambda_1 > 1$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2-1}{\lambda_1-1}\right) > \frac{\mu_2}{\mu_1}$
  $\Rightarrow$ *No risk classification maximises loss coverage;*

- $\lambda_2 > \lambda_1 > 1$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2-1}{\lambda_1-1}\right) \leq \frac{\mu_2}{\mu_1}$
  $\Rightarrow$ *A partial risk classification maximises loss coverage.*

This result proves that *partial risk classification* could maximise loss coverage if demand elasticities for both risk-groups are greater than 1 and the demand elasticity for the high risk-group is larger than the demand elasticity for the low risk-group up to a certain level. If the high risk-group has a much higher demand elasticity, then *no risk classification* maximises loss coverage instead. This feature is shown in Figure 7.1 where *partial risk classification* maximises loss coverage ratio in the area (i.e. the light coloured area) between the diagonal $\lambda_1 = \lambda_2 > 1$ and the dashed blue boundary curve $\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2-1}{\lambda_1-1}\right) = \frac{\mu_2}{\mu_1}$.

On the other hand, when the low risk-group has a higher demand elasticity than the high risk-group, *full risk classification* maximises loss coverage.

Result 7.4 is proved in Theorem F.1.11, F.1.12 and F.1.13 in Appendix F.

Result 7.5.

$$0 < \lambda_1 < 1 < \lambda_2 \Rightarrow \textit{ No risk classification maximises loss coverage.} \tag{7.4}$$

This result proves that when the high risk-group has a demand elasticity that is greater than 1 and the low risk-group has a demand elasticity that is less than 1, *no risk classification* maximises loss coverage.

Result 7.5 is proved in Theorem F.1.14 in Appendix F.

Result 7.6.

$$0 < \lambda_2 < 1 < \lambda_1 \Rightarrow \textit{ Full risk classification maximises loss coverage.} \tag{7.5}$$

This result proves that when the low risk-group has a demand elasticity that is greater than 1 and the high risk-group has a demand elasticity that is less than 1, *full risk classification* maximises loss coverage.

Result 7.6 is proved in Theorem F.1.15 in Appendix F.

The above results on maximising loss coverage can also be explained intuitively. Using Figure 7.1 as an example, in the uncoloured region (i.e. λ_1 'sufficiently low' compared with λ_2), *no risk classification* maximises loss coverage. This phenomenon can be explained in the following way: other things equal, lower demand elasticity for the low risk-group means demand from low risks at risk-differentiated premium is indifferent to that at the pooled premium. Higher demand elasticity for the high risk group indicates a much higher demand at pooled equilibrium compared to the risk-differentiated premium. And taking into account of the high relative risk of 4 in this example, the aggregate loss coverage at pooled premium is optimal. Conversely, in the dark coloured region, *full risk classification* maximises loss coverage.

Partial risk classification maximises loss coverage only in the light coloured middle region where both elasticities exceed 1, with λ_2 higher than λ_1 to a certain level. This is a region where both demand elasticities are 'reasonably high', i.e. demand is reasonably responsive to premiums.

7.2 Three Risk-groups

Some of the results on *partial risk classification* for two risk-groups can be further generalised for more risk-groups (e.g. m risk-groups, where $m \geq 2$). We present our findings for three risk-groups (i.e. the low risk-group, middle risk-group and high risk-group).

7.2.1 Equal Demand Elasticity

When there are three risk-groups, i.e. a low risk-group, a middle risk-group and a high risk-group, and they have the same demand elasticity, using iso-elastic demand, we have the following results:

Result 7.7. *When demand elasticity is greater than 1, full risk classification maximises loss coverage.*

This result is proved in Theorem F.2.2 in Appendix F.

Result 7.8. *When demand elasticity is less than 1 and if we assume high risks cannot be charged a premium lower than that of the low risks, i.e.* $\pi_i \geq$

π_j if $\mu_i > \mu_j$ for all i and j, then no risk classification maximises loss coverage.

This result is proved in Theorem F.2.5 in Appendix F.

In Figure 7.2 we consider an example of loss coverage ratio at different premium strategies $\underline{\pi} = (\pi_1, \pi_2, \pi_3)$ where $(\mu_1, \mu_2, \mu_3) = (0.01, 0.02, 0.04)$, $(\alpha_1, \alpha_2, \alpha_3) = (60\%, 30\%, 10\%)$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$. π_1 is on the x-axis and π_2 is on the y-axis with π_3 is the contour plot shown as dashed dark blue indifference curves. Loss coverage ratio comparing a given premium strategy (π_1, π_2, π_3) to risk-differentiated premiums (μ_1, μ_2, μ_3) are plotted as black contours. Any combination of π_1, π_2 and π_3 on this plot satisfies the equilibrium condition to ensure zero expected profits for insurers. The shaded area indicates the region: $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$. The pooled equilibrium premium π_0 for *no risk classification* is shown in Figure 7.2 as the circle. (In this example, $\pi_0 = 0.02$.)

Figure 7.2 shows that under the constraint $\pi_1 \leq \pi_2 \leq \pi_3$, *no risk classification* leads to the highest loss coverage ratio. This is the graphical representation of Result 7.8.

Figure 7.2 also shows that if the constraint that $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$ is removed, a higher loss coverage ratio (compared to $C(\pi_0)$) might be achieved by *partial risk classification*. As these specific cases may be considered to be unrealistic, we defer the discussions on theses to Appendix F.2.2.

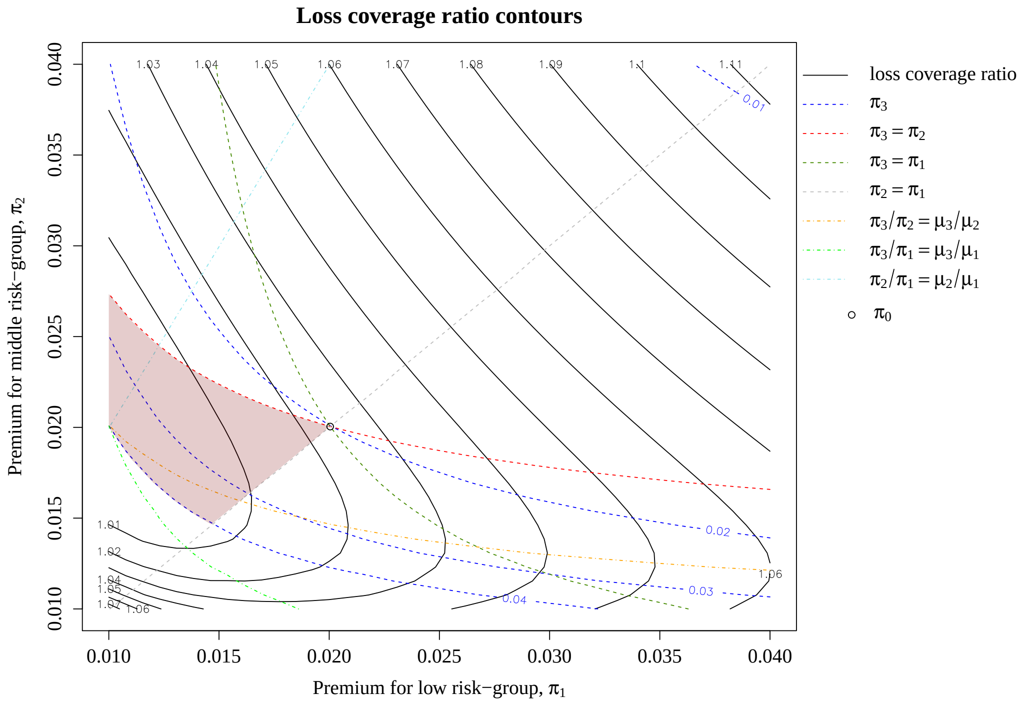

Figure 7.2: Plot of loss coverage ratio in terms of π_1, π_2 assuming $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$ with $\alpha_1 = 60\%, \alpha_2 = 30\%, \alpha_3 = 10\%, \mu_1 = 0.01, \mu_2 = 0.02, \mu_3 = 0.04$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$.

7.3 Maximising Social Welfare: Two Risk-groups

In Section 7.1, using iso-elastic demand, we have analysed loss coverage ratio in the case of two risk-groups taking *partial risk classification* into account. In this section, we explore the impact on social welfare, in the case when *partial risk classification* is allowed.

7.3.1 Equal Demand Elasticity

When both the low and the high risk-groups have the same demand elasticity λ, we have already proved in Section 6.1 that, using iso-elastic demand, social welfare at pooled equilibrium premium is higher than social welfare at risk-differentiated premiums if the constant demand elasticity $\lambda < 1$.

If *partial risk classification* is allowed, we have the following result:

Result 7.9. *When there are two risk-groups, and they have the same demand elasticity, i.e.* $\lambda_1 = \lambda_2 = \lambda$*,*

- $0 < \lambda < 1 \Rightarrow$ *No risk classification maximises social welfare;*
- $\lambda = 1 \Rightarrow$ *All risk classification schemes have the same social welfare;*
- $\lambda > 1 \Rightarrow$ *Full risk classification maximises social welfare.*

This result proves that pooling (under *no risk classification*) maximises social welfare when demand elasticity $\lambda < 1$; and risk-differentiated premiums (under *full risk classification*) maximise social welfare when demand elasticity $\lambda > 1$. Therefore, no intermediate *partial risk classification* maximises social welfare when both the low risk-group and the high risk-group have the same demand elasticity.

Recall that in Result 7.2, we have proved that *partial risk classification* does not maximise loss coverage in the case of equal demand elasticity. Hence, this result shows the consistency between social welfare and loss coverage in terms of implications from different risk classifications.

Result 7.9 is proved in Theorem E.2.3 and E.2.4 in Appendix E.

7.3.2 Different Demand Elasticities

When the low risk-group and the high risk-group could possibly have different demand elasticities, we have the following results:

Result 7.10. *When both risk-groups have demand elasticities that are less than 1,*

- $0 < \lambda_1 < \lambda_2 < 1$
  $\Rightarrow$ *No risk classification maximises social welfare;*

- $0 < \lambda_2 < \lambda_1 < 1$
  $\Rightarrow$ *Either Full or no risk classification maximises social welfare.*

This result proves that if there are two risk-groups, i.e. a low risk-group and a high risk-group, and both risk-groups have demand elasticities which are less than 1, either the full or the *no risk classification* maximises social welfare. In particular, when the high risk-group has a higher demand elasticity than the low risk-group, then *no risk classification* maximises social welfare.

Result 7.10 is proved in Theorem E.2.8 and E.2.9 in Appendix E.

When both risk-groups have demand elasticities that are greater than 1, we have the following results:

Result 7.11.

$$1 < \lambda_2 < \lambda_1 \Rightarrow \textit{ Full risk classification maximises social welfare.}$$

This result proves that when both the low risk-group and the high risk-group have demand elasticities that are greater than 1, and the low risks have a higher demand elasticity than the high risks, *full risk classification* maximises social welfare. This result is proved in Theorem E.2.12 in Appendix E.

Result 7.12. *When* $1 < \lambda_1 < \lambda_2$*:*

- $1 < \lambda_1 < \lambda_2 < K$ *for some* $K > 1$

  $\Rightarrow$ *A partial risk classification maximises social welfare.*

- *Otherwise, no risk classification maximises social welfare.*

This result proves that when both risk-groups have demand elasticities that are greater than 1, and the high risks have a higher demand elasticity than the low risks up to a certain level, *partial risk classification* maximises social welfare.

Proof of Result 7.12 is given in Theorem E.2.13 and E.2.14 in Appendix E.

Therefore, *partial risk classification* can result in higher social welfare than both *no risk classification* and *full risk classification* when demand elasticity of the high risks are higher than that of the low risks up to a certain level.

7.4 Summary

In this chapter, we examine the implication on loss coverage and social welfare when *partial risk classifications* are taken into account. *Partial risk classification* might be a more realistic strategy, because insurers usually are allowed to underwrite to some extent.

Using iso-elastic demand as an example and assume insurance premium is sufficiently small compared to individuals' wealth, we proved that for two risk-groups with demand elasticity less than 1, either *full risk classification* or *no risk classification* maximises loss coverage and social welfare.

Partial risk classifications do, in some cases, result in a higher loss coverage and/or social welfare than *full* or *no risk classification.* For two risk-groups, this happens if demand elasticities of both risk-groups are greater than 1, with high risks' demand elasticity higher (but up to a certain level) than low risks' demand elasticity.

Using iso-elastic demand, we also analysed the case of three risk-groups, i.e. the low, middle and high risk-group. When all three risk-groups have the same demand elasticity, and high risks cannot be charged a premium lower than that of the low risks, then either *full risk classification* or *no risk classification* maximises loss coverage. If high risks are charged at a lower premium than the low risks, then *partial risk classification* could maximise loss coverage ratio. This last finding is worth further researching.

Chapter 8

Conclusions

In this thesis, we analyse a generic insurance market where a pooled premium is charged across different risk-groups in the absence of insurance risk-classification. The outcome is interpreted through an equilibrium at which insurers break even. We characterise equilibrium by four quantities: equilibrium premium, adverse selection, loss coverage and social welfare. Using elasticity-driven demand functions, e.g. iso-elastic demand, negative-exponential demand and a general demand, we ask the following four fundamental questions in this thesis:

1. Is there an equilibrium premium, and is this equilibrium premium unique?

2. What is(are) the corresponding level(s) of adverse selection at this/these equilibrium premium(s)?

3. What is(are) the corresponding level(s) of loss coverage at this/these equilibrium premium(s), and whether it is a better measure compared to adverse selection?

4. What is(are) the implication(s) on social welfare and do loss coverage and social welfare lead to consistent conclusions on the impact of restricted risk classification?

8.1 Equilibrium Premium

In Chapter 3, we examined the first question. We solve for the "timeless equilibrium" premium(s) that give(s) insurers a zero expected profit for a given level of demand elasticity regardless of how equilibrium was reached, or whether profits or losses were made along the way. And we focus on the steady state where all insurers' profits and losses are competed away.

In Chapter 2, we introduced demand functions that will be used in this thesis, e.g. iso-elastic demand, negative-exponential demand and general demand, with an explanation of the heterogeneity in individuals' insurance purchasing behaviours. Then in Chapter 3, using the demand functions from Chapter 2, we explore equilibrium premium(s) in the case when all risk-groups have the same demand elasticity, and also in the case when they are not necessarily the same.

In both cases, we confirm that there is at least one equilibrium premium. In the equal-demand-elasticity case, there is a unique equilibrium premium, and the equilibrium premium increases with the common demand elasticity. This result also holds for the case when there are many risk-groups in the population, as long as all of them have the same demand elasticity.

In the different-demand-elasticity case, there might be multiple equilibria. However, multiple equilibria only arise under extreme conditions, which makes it unlikely to appear in practical situations. To have multiple equilibria, elasticity of demand from the high risk-group must be substantially lower than that from the low risk-group, and the fair premium demand share of the low risk-group must fall within a narrow interval.

Therefore, multiple equilibria are not likely to be a practical concern.

8.2 Adverse Selection

In Chapter 4, we examined the second question on adverse selection. We defined adverse selection as a ratio of the expected claim per policy to the expected loss per risk. Because when there are restrictions on risk classification, insurers are assumed to adjust the pooled premium to whatever level is necessary to ensure equilibrium, and competition between insurers in risk classification is not permitted, adverse selection does not imply insurer losses.

Our main finding is that, regardless of the relationship between demand elasticities of different risk-groups, adverse selection under pooling will always be higher than that under risk-differentiated premiums. The level of adverse selection also increases with demand elasticity. So this concept of adverse selection fails to distinguish between different scenarios where smaller or larger expected fractions of the population's losses are compensated by insurance.

8.3 Loss Coverage

In Chapter 5, we examined the third question. Loss coverage is defined as *the population's expected losses compensated by insurance* (Thomas (2008)).

Similar to the analysis on equilibrium premium, for each demand functions used in this thesis, e.g. iso-elastic demand, negative-exponential demand and general demand, we explore loss coverage in the case when all risk-groups have the same demand elasticity, and also in the case when they are not necessarily the same.

In the equal-demand-elasticity case, restricting risk classification increases loss coverage if demand elasticity is sufficiently small (e.g. less than 1 for iso-elastic demand). This result also holds for the case when there are many risk-groups in the population, as long as all of them have the same demand elasticity.

When demand elasticities are different for different risk-groups, restricting risk classification increases loss coverage if demand elasticity for low risks is sufficiently low, compared to that for high risks. This is despite the fact that restricting risk classification always increases adverse selection.

This phenomenon can be explained in the following way: adverse selection is associated with a fall in the number of insured individuals at pooled premium compared with that obtained under *full risk classification*. This reduction is usually seen as inefficient. However, adverse selection is also associated with a shift in coverage towards higher risks. If this shift is large enough, it can more than outweigh the fall in the numbers insured, so that loss coverage is increased. Since this implies that more risk is voluntarily traded and more losses are compensated, it is a counter-argument to the perception of reduced efficiency.

Therefore, the concept of loss coverage might be a better measure of the impact of restricted risk classification. We find that adverse selection is not always a bad thing, as long as loss coverage can be increased.

8.4 Social Welfare

The question on social welfare is addressed in Chapter 6. We defined social welfare as the sum of expected utilities of all individuals in a population with equal weight assigned to each individual.

We showed that for iso-elastic demand, if insurance premium is assumed to be sufficiently small compared to individuals' wealth, ranking risk classification schemes by (observable) loss coverage always gives the same ordering as ranking by (unobservable) utilitarian social welfare. In particular, if the common demand elasticity is less than 1, which is consistent with many empirical studies, then maximising loss coverage maximises social welfare as well. So loss coverage can be used as a proxy for social welfare.

When two risk-groups have different iso-elastic demand elasticities but both being less than 1, and if demand elasticity of the high risks is larger than the demand elasticity of the low risks, then pooling gives higher social welfare. Given that in practice, empirical evidence suggests that demand elasticities for insurance is always smaller than 1, and economic argument implies that higher risks should have a higher demand elasticity for insurance because of the income effect, our conditions on increasing social welfare can be satisfied in the real world.

8.5 Partial Risk Classification

In Chapter 7, we explored the impact of *partial risk classification* (or intermediate risk classification) on loss coverage and social welfare.

We found that, using iso-elastic demand, in the case of two risk-groups, *partial risk classification* can lead to higher loss coverage and/or social welfare than both *full risk classification* and *no risk classification* when demand elasticities of both risk-groups are greater than 1, and high risks have a higher (but up to a certain level) demand elasticity than the low risks. In the relatively more realistic case when demand elasticities for both risk-groups are small (e.g. less than 1), either *full* or *no risk classification* maximises loss coverage and/or social welfare. Although in the above cases *partial risk classifications* have immaterial impact on maximising loss coverage and/or social welfare, its potential implications should not be ignored, especially when there are more risk-groups.

When the above analysis was generalised using iso-elastic demand with three risk-groups, we found that *partial risk classification* could lead to a higher loss coverage when the common demand elasticity is sufficiently small (e.g. less than 1), and high risks are charged at a lower premium than the low risks, e.g. when insurers mis-classify risk-groups due to restrictions on using one risk factor, but not the other. Although we know of no empirical evidence so far that any such premium strategy is in place, this is an area that is worth further developing in the future.

8.6 Summary

Restrictions on insurance risk classification may induce adverse selection, which is usually perceived to reduce efficiency. In this thesis, we suggest a counter-argument to this perception in circumstances where modest adverse selection leads to an increase in "loss coverage", defined as the expected losses compensated by insurance for the whole population. This happens if the shift in coverage towards higher risks under adverse selection more than outweighs the fall in number of individuals insured. And this is the case when the low risks have a small insurance demand elasticity that is a non-decreasing function of premium, and the high risks have a higher demand elasticity than the low risks. Therefore, adverse selection is not always a bad thing, as long as loss coverage is increased.

We reconcile the concept of "loss coverage" and a utilitarian concept of social welfare. For iso-elastic insurance demand, if insurance premium is assumed to be sufficiently small compared to individuals' wealth, ranking risk classification schemes by (observable) loss coverage always gives the same ordering as ranking by (unobservable) utilitarian social welfare. This is a useful result from a policyholder's perspective, because maximising loss coverage does not require knowledge of individuals' (unobservable) utility functions; loss coverage is based solely on observable quantities.

Last but not the least, we also explore the implication of applying partial risk classification, in the realistic situation when insurers can differentiate risks only to a certain extent, which may not fully reflect the differences between different risk-groups. We find that partial risk classification can lead to a higher loss coverage and/or social welfare than *full risk classification* and *no risk classification* when there are more risk-groups (e.g. more than two) and there is no particular ordering in terms of the premiums charged to different risk-groups. Although this finding is lack of approval based on empirical evidence so far, we consider it as an interesting and important area for future research.

The research findings in this thesis could add to the wider public policy debate on implications of risk classification and provide necessary research insights for informed decision making by actuaries, regulators, policyholders, insurers, policy-makers, capital providers and other stakeholders.

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Appendix A

Probabilistic Model of Heterogeneous Insurance Purchasers

A.1 Model Specification

We can construct a probabilistic model by supposing that any individual sampled at random possesses two attributes, risk of suffering a loss event (or just 'risk' for short) and a utility function.

- We suppose that 'risk' is defined as the probability μ of suffering a defined loss event. For simplicity, suppose the set of possible values of μ is the finite set $M = \{\mu_1, \mu_2, \ldots, \mu_m\}$, that $\mathcal{G}$ is the power set of M and that $P[\mu = \mu_i] = p_i$.
- For simplicity, suppose that all utility functions belong to a family parameterized by a real number γ. Individuals' utility functions take values in R.

Then the idea of risk and utility being heterogeneous in a population may be modelled by the probability space $(\Omega, \mathcal{F}, P)$ where:

- The sample space is $\Omega = M \times R$.
- The sigma-algebra $\mathcal{F}$ is $\mathcal{G} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel sigma-algebra on R.
- The probability measure P is assumed to be given by a probability function $F(\mu, \gamma)$, discrete in its first component and absolutely continuous in its second component.

An individual sampled at random has the attributes μ and γ given by the probability F. We must have the marginal distribution:

$$p_i = P[\mu = \mu_i] = \int_{\{\mu_i\} \times R} dF(\mu, \gamma) = \int_R dF(\mu_i, \gamma) \tag{A.1}$$

where the first integral is Stieltjes, summing over the first component of F and integrating over the second component.

Two individuals with the same value μ_i of μ may be said to belong to the same risk group, for insurance purposes. The insurer is supposed able to observe μ and will offer the same premium π_i to everyone with risk μ_i. It is assumed that an individual with risk μ_i, offered premium π_i, will decide to buy insurance, or not, non-randomly, determined by their utility function. We suppose, however, that the insurer cannot observe γ. Since different individuals, sampled at random and allocated to the same risk-group, can have different utility functions, the insurer will observe heterogeneous behaviour within a risk-group. That is, even though all in the risk-group are offered the same premium rate, some will buy insurance and others will not. The purchasing decision, given the utility function, is non-random, but to the insurer it appears to be random because of the unobserved heterogeneity. At most, the insurer can observe the proportion of individuals in any risk-group that buy insurance. Thus the insurer may model the insurance-buying decision of an individual in a given risk-group as a Bernoulli random variable.

The insurer's premium strategy may be represented by a $\mathcal{G}-$measurable random variable on M, or by a $(\mathcal{G} \times \{\emptyset, \Omega\})-$measurable random variable on Ω. In either case, denote it by Π. The insurance purchasing decision may be represented by an indicator Q, taking the value 1 if insurance is purchased and 0 otherwise. For a given premium strategy Π on the insurer's part, Q is an $\mathcal{F}-$measurable random variable on Ω. Its restriction to a fixed value of the risk $\mu = \mu_i$ is the Bernoulli random variable that the insurer observes in that risk-group.

The proportion of risks with $\mu = \mu_i$ that buy insurance, which we may call a 'demand function' and denote by $d_i(\pi_i)$, is the conditional expected value of Q:

$$d_i(\pi_i) = P[\, Q = 1 \mid \mu_i \,] = E[\, Q \mid \mu_i \,] = \frac{\int_R Q(\mu_i, \gamma)\, dF(\mu_i, \gamma)}{\int_R dF(\mu_i, \gamma)} \tag{A.2}$$

and the expected population demand for insurance is the unconditional expected value of Q:

$$\begin{aligned} E[Q] &= \int_\Omega Q(\mu, \gamma)\, dF(\mu, \gamma) && \text{(A.3)} \\ &= \sum_{i \in M} \int_R Q(\mu_i, \gamma)\, dF(\mu_i, \gamma) && \text{(A.4)} \\ &= \sum_{i \in M} \left(\frac{\int_R Q(\mu_i, \gamma)\, dF(\mu_i, \gamma)}{\int_R dF(\mu_i, \gamma)} \times \int_R dF(\mu_i, \gamma) \right) && \text{(A.5)} \\ &= \sum_{i \in M} d_i(\pi_i)\, p_i. && \text{(A.6)} \end{aligned}$$

Define X to be a Bernoulli random variable, indicating that a loss event occurs. Given μ_i, X has parameter μ_i, and does not depend on any utility function. Observation of X is new information, not part of the model above. Then:

$$
\begin{aligned}
E[X] &= \int_{\Omega} E[\, X \mid \mu, \gamma] \, dF(\mu, \gamma) && \text{(A.7)} \\
&= \sum_{i \in M} E[\, X \mid \mu_i \,] \int_{R} dF(\mu_i, \gamma) && \text{(A.8)} \\
&= \sum_{i \in M} \mu_i p_i. && \text{(A.9)}
\end{aligned}
$$

Assume that Q and X are independent, conditional on μ_i. That is, the level of risk may influence the decision to buy insurance, but there is no moral hazard; insured individuals in any risk-group are not more likely to suffer the loss event than uninsured individuals. Then the expected claims outgo for the insurer is:

$$
\begin{aligned}
E[QX] &= \int_{\Omega} E[\, QX \mid \mu, \gamma \,] \, dF(\mu, \gamma) && \text{(A.10)} \\
&= \int_{\Omega} Q(\mu, \gamma) \, E[\, X \mid \mu, \gamma \,] \, dF(\mu, \gamma) \qquad (Q \text{ is } \mathcal{F}-\text{measurable}) && \text{(A.11)} \\
&= \sum_{i \in M} E[\, X \mid \mu_i \,] \int_{R} Q(\mu_i, \gamma) \, dF(\mu_i, \gamma) && \text{(A.12)} \\
&= \sum_{i \in M} \mu_i \, d_i(\pi_i) \, p_i \qquad \text{(following Equation (A.5))}. && \text{(A.13)}
\end{aligned}
$$

Finally, the expected premium income is:

$$\begin{aligned}
E[Q\Pi] &= \int_{\Omega} E[\,Q\Pi \mid \mu, \gamma\,]\, dF(\mu, \gamma) && \text{(A.14)}\\
&= \int_{\Omega} Q(\mu, \gamma)\, E[\,\Pi \mid \mu, \gamma\,]\, dF(\mu, \gamma) && \text{(A.15)}\\
&= \sum_{i \in M} E[\,\Pi \mid \mu_i\,] \int_{R} Q(\mu_i, \gamma)\, dF(\mu_i, \gamma) && \text{(A.16)}\\
&= \sum_{i \in M} \pi_i\, d_i(\pi_i)\, p_i \qquad \text{(following Equation (A.5)).} && \text{(A.17)}
\end{aligned}$$

Based on the formulation of expected premium income and claims outgo, the total expected profit for insurers, as a function of risk-classification scheme $\underline{\pi} = (\pi_1, \pi_2, \ldots, \pi_m)$, can be defined as:

Expected profit for insurers: $$\begin{aligned}\rho(\underline{\pi}) &= E[Q\Pi] - E[QX],\\ &= \sum_{i=1}^{m} d_i(\pi_i)\pi_i p_i - \sum_{i=1}^{m} d_i(\pi_i)\mu_i p_i.\end{aligned} \qquad \text{(A.18)}$$

A.2 General Demand: Case of Iso-elastic Demand

Theorem A.2.1. *Iso-elastic demand function is a special case of general negative-exponential demand function when $n \to 0$.*

Proof. The proof can be obtained using L'Hopital's Rule on Equation 2.30:

$$\lim_{n \to 0} \frac{\{1 - \left(\frac{\pi}{\mu}\right)^n\}\lambda}{n} = \lim_{n \to 0} \left[-\lambda \left(\frac{\pi}{\mu}\right)^n \log\left(\frac{\pi}{\mu}\right) \right] = -\lambda \log\left(\frac{\pi}{\mu}\right). \qquad \text{(A.19)}$$

$$\Rightarrow \lim_{n \to 0} d(\pi) = \tau \exp\left[-\lambda \log\left(\frac{\pi}{\mu}\right) \right] = \tau \left(\frac{\pi}{\mu}\right)^{-\lambda}. \qquad \text{(A.20)}$$

Hence proved. $\square$

A.3 Probabilistic Model of Social Welfare

Finally we define social welfare as expected utility of an individual chosen at random, i.e.

$$\begin{aligned} &\text{Social Welfare} && \text{(A.21)}\\ &= E\left[Q\,U_\Gamma(W-\pi L) + (1-Q)\left[X\,U_\Gamma(W-L) + (1-X)\,U_\Gamma(W)\right]\right]. \end{aligned}$$

as in Equation (6.1). Let us review the measurability and dependencies of the quantities we will need.

μ is $\mathcal{G}$-measureable.

Γ is $\mathcal{B}$-measureable (Borel sigma-algebra on R).

Π is $\mathcal{G}$-measureable.

Q is $\mathcal{F}$-measureable, but not independent of Π.

X is neither $\mathcal{G}$-measureable nor $\mathcal{F}$-measureable, but it is independent of Π.

Note that $E[\,X \mid \mathcal{F}\,] = E[\,X \mid \mu_i\,] = \mu_i$. Consider the right-hand side of Equation A.21 term by term.

$$E[Q\,U_\Gamma(W-\pi L)] \tag{A.22}$$

$$= E[\,E[Q\,U_\Gamma(W-\pi L) \mid \mathcal{F}\,]\,] \tag{A.23}$$

$$= \sum_{i=1}^{m} p_i \int_R Q(\mu_i,\gamma)\,U_\gamma(W-\pi_i L)\,dF(\pi_i,\gamma) \tag{A.24}$$

$$= \sum_{i=1}^{m} p_i\,d_i(\pi_i)\frac{\int_R Q(\mu_i,\gamma)\,U_\gamma(W-\pi_i L)\,dF(\pi_i,\gamma)}{d_i(\pi_i)} \tag{A.25}$$

$$= \sum_{i=1}^{m} p_i\,d_i(\pi_i)\,E\left[\int_R Q(\mu_i,\gamma)\,U_\gamma(W-\pi_i L)\,dF(\pi_i,\gamma)\,\middle|\,Q(\mu_i,\cdot)=1\right] \tag{A.26}$$

where $Q(\mu_i, \cdot)$ denotes the restriction of Q to the ith risk-group. This is equivalent to Equation (6.5) in the main text. Next:

$$E[\,(1-Q)\,X\,U_\Gamma(W-L)\,] \tag{A.27}$$

$$= E[\;E[\,(1-Q)\,X\,U_\Gamma(W-L)\mid\mathcal{F}\,]\,] \tag{A.28}$$

$$= \sum_{i=1}^{m} p_i \int_R (1-Q(\mu_i,\gamma))\,U_\gamma(W-L)\,E[\,X\mid\mathcal{F}\,]\,dF(\pi_i,\gamma) \tag{A.29}$$

$$= \sum_{i=1}^{m} p_i\,\mu_i \int_R (1-Q(\mu_i,\gamma))\,U_\gamma(W-L)\,dF(\pi_i,\gamma) \tag{A.30}$$

$$= \sum_{i=1}^{m} p_i\,\mu_i\,(1-d_i(\pi_i))\frac{\int_R (1-Q(\mu_i,\gamma))\,U_\gamma(W-L)\,dF(\pi_i,\gamma)}{1-d_i(\pi_i)} \tag{A.31}$$

$$= \sum_{i=1}^{m} p_i\,\mu_i\,(1-d_i(\pi_i)) \tag{A.32}$$

$$\times\, E\left[\int_R (1-Q)\,U_\gamma(W-L)\,dF(\mu_i,\gamma)\,\middle|\,Q(\pi_i,\cdot)=0\right]$$

$$= \sum_{i=1}^{m} p_i\,\mu_i\,(1-d_i(\pi_i))U(W-L), \tag{A.33}$$

if $U_\gamma(W-L) = U(W-L)$ for all γ.

Similarly,

$$E[\,(1-Q)\,(1-X)\,U_\Gamma(W)\,] \tag{A.34}$$

$$= \sum_{i=1}^{m} p_i\,(1-\mu_i)\,(1-d_i(\pi_i)) \tag{A.35}$$

$$\times\, E\left[\int_R (1-Q)\,U_\gamma(W)\,dF(\mu_i,\gamma)\,\middle|\,Q(\pi_i,\cdot)=0\right]$$

$$= \sum_{i=1}^{m} p_i\,(1-\mu_i)\,(1-d_i(\pi_i))U(W), \tag{A.36}$$

if $U_\gamma(W) = U(W)$ for all γ.

If we standardise the utility functions to obtain the social welfare S defined in Equation (6.2) and make the appropriate change of variables, Equations (A.33) and (A.36) simplify.

Appendix B

Equilibrium

B.1 Iso-elastic Demand

B.1.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by iso-elastic demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks for the low risk-group and the high risk-group.

- p_1, p_2 are the population proportions such that $p_1 + p_2 = 1$.

- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:
$$d_i(\pi) = \tau_i \left(\frac{\mu_i}{\pi}\right)^{\lambda_i}. \tag{B.1}$$

- The demand elasticity at premium π is given by:
$$\epsilon_i(\pi) = \lambda_i. \tag{B.2}$$

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \Rightarrow \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i. \tag{B.3}$$

Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

B.1.2 Theorems and Proofs

Theorem B.1.1. *For pooled premium, given (μ_1, μ_2) and (λ_1, λ_2), multiple equilibria exist if*

$$\lambda_2 - \lambda_1 < -\frac{\sqrt{\mu_2} + \sqrt{\mu_1}}{\sqrt{\mu_2} - \sqrt{\mu_1}}, \quad \textit{and} \tag{B.4}$$

$$\frac{a(\pi_{lo})}{1 + a(\pi_{lo})} = \alpha_{lo} < \alpha_1 < \alpha_{hi} = \frac{a(\pi_{hi})}{1 + a(\pi_{hi})}, \quad \textit{where} \tag{B.5}$$

$$a(\pi) = \left(\frac{\mu_2 - \pi}{\pi - \mu_1}\right)\left(\frac{\mu_2^{\lambda_2}}{\mu_1^{\lambda_1}}\right)\pi^{-(\lambda_2 - \lambda_1)}, \quad \textit{and} \tag{B.6}$$

$$(\pi_{lo}, \pi_{hi}) \textit{ solves: } \pi^2 - \left(\mu_1 + \mu_2 + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\right)\pi + \mu_1\mu_2 = 0. \tag{B.7}$$

Proof. Note that for the formulation of profit function in Equation 3.10, can be re-written as:

$$\rho(\pi) = (\mu_2 - \mu_1)\left[d_1(\pi)p_1 + d_2(\pi)p_2\right]\left[\alpha_1(\pi) - h(\pi)\right], \quad \text{where} \tag{B.8}$$

$$\alpha_1(\pi) = \frac{d_1(\pi)p_1}{d_1(\pi)p_1 + d_2(\pi)p_2}, \tag{B.9}$$

$$h(\pi) = \frac{\mu_2 - \pi}{\mu_2 - \mu_1}. \tag{B.10}$$

$$\text{So: } \rho(\pi) = 0 \Leftrightarrow \alpha_1(\pi) = h(\pi). \tag{B.11}$$

And, the slope of $\rho(\pi)$ is given by:

$$\begin{aligned}\rho'(\pi) =& (\mu_2 - \mu_1)\left[d_1'(\pi)p_1 + d_2'(\pi)p_2\right]\left[\alpha_1(\pi) - h(\pi)\right] \\ &+ (\mu_2 - \mu_1)\left[d_1(\pi)p_1 + d_2(\pi)p_2\right]\left[\alpha_1'(\pi) - h'(\pi)\right] \text{ where} \quad \text{(B.12)} \\ \alpha_1'(\pi) =& \alpha_1(\pi)[1 - \alpha_1(\pi)]\left(\frac{\lambda_2 - \lambda_1}{\pi}\right), \quad \text{and} \quad \text{(B.13)} \\ h'(\pi) =& -\frac{1}{\mu_2 - \mu_1}. \quad \text{(B.14)}\end{aligned}$$

As $\rho(\mu_1) < 0$ and $\rho(\mu_2) > 0$, there exists a premium π_0 for which $\rho(\pi_0) = 0$ and $\rho'(\pi_0) \geq 0$. Multiple equilibria exist if there exists a premium π_0 for which $\rho(\pi_0) = 0$ and $\rho'(\pi_0) < 0$, or equivalently:

$$\alpha_1(\pi_0) = h(\pi_0) \quad \text{(B.15)}$$

$$\alpha_1'(\pi_0) < h'(\pi_0). \quad \text{(B.16)}$$

Using the relationship in Equation B.15, the expression for $\alpha_1'(\pi_0)$ in Equation B.13 becomes:

$$\alpha_1'(\pi_0) = h(\pi_0)[1 - h(\pi_0)]\left(\frac{\lambda_2 - \lambda_1}{\pi_0}\right) = \left(\frac{\mu_2 - \pi_0}{\mu_2 - \mu_1}\right)\left(\frac{\pi_0 - \mu_1}{\mu_2 - \mu_1}\right)\left(\frac{\lambda_2 - \lambda_1}{\pi_0}\right). \quad \text{(B.17)}$$

Case 1: $\lambda_2 - \lambda_1 \geq 0$. In this case, $\alpha_1'(\pi) \geq 0$ and so condition in Equation B.16 cannot hold as $h'(\pi_0) < 0$. So, multiple equilibria is not possible in this case.

Case 2: $\lambda_2 - \lambda_1 < 0$. In this case, Condition B.16 leads to the following equivalent condition:

$$m(\pi_0) = \pi_0^2 - \left(\mu_1 + \mu_2 + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\right)\pi_0 + \mu_1\mu_2 < 0. \tag{B.18}$$

Now, $m(\pi_0) = (\pi_0 - \pi_{\text{lo}})(\pi_0 - \pi_{\text{hi}}), \quad \text{where}$ (B.19)

$$\pi_{\text{lo}} = \frac{1}{2}\left[\left(\mu_1 + \mu_2 + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\right) + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\sqrt{(\lambda_2 - \lambda_1 - c_1)(\lambda_2 - \lambda_1 - c_2)}\right], \tag{B.20}$$

$$\text{and} \quad \pi_{\text{hi}} = \frac{1}{2}\left[\left(\mu_1 + \mu_2 + \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\right) - \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}\sqrt{(\lambda_2 - \lambda_1 - c_1)(\lambda_2 - \lambda_1 - c_2)}\right], \tag{B.21}$$

$$\text{where} \quad c_1 = -\frac{\sqrt{\mu_2} + \sqrt{\mu_1}}{\sqrt{\mu_2} - \sqrt{\mu_1}} < -1, \tag{B.22}$$

$$\text{and} \quad c_2 = -\frac{\sqrt{\mu_2} - \sqrt{\mu_1}}{\sqrt{\mu_2} + \sqrt{\mu_1}} > -1. \tag{B.23}$$

Note that, as both $m(\mu_1)$ and $m(\mu_2)$ are positive, $m(\pi_0)$ can only be negative, if there are real solutions to $m(\pi_0) = 0$, which means, either $\lambda_2 - \lambda_1 > c_2$ or $\lambda_2 - \lambda_1 < c_1$.

Case 1: $\lambda_2 - \lambda_1 > c_2$. In this case:

$$\begin{aligned}\pi_{\text{hi}} - \mu_1 &= \frac{\mu_2 - \mu_1}{2(\lambda_2 - \lambda_1)}\left(1 + (\lambda_2 - \lambda_1) - \sqrt{(\lambda_2 - \lambda_1 - c_1)(\lambda_2 - \lambda_1 - c_2)}\right) \\ &< 0, \\ \Rightarrow \pi_{\text{lo}} &< \pi_{\text{hi}} < \mu_1 < \mu_2.\end{aligned} \tag{B.24}$$

As the roots are not in the range $[\mu_1, \mu_2]$ (note: $m(\mu_1) > 0$), $m(\pi_0)$ is only positive over $[\mu_1, \mu_2]$. So, multiple equilibria is not possible in this case.

Case 2: $\lambda_2 - \lambda_1 < c_1$. In this case:

$$\begin{aligned}
\pi_{\text{lo}} - \mu_1 &= \frac{\mu_2 - \mu_1}{2(\lambda_2 - \lambda_1)} \left(1 + (\lambda_2 - \lambda_1) + \sqrt{(\lambda_2 - \lambda_1 - c_1)(\lambda_2 - \lambda_1 - c_2)}\right) \\
&> 0 \\
\text{And } \pi_{\text{hi}} - \mu_2 &= \frac{\mu_2 - \mu_1}{2(\lambda_2 - \lambda_1)} \left(1 - (\lambda_2 - \lambda_1) - \sqrt{(\lambda_2 - \lambda_1 - c_1)(\lambda_2 - \lambda_1 - c_2)}\right) \\
&< 0 \\
\Rightarrow \mu_1 &< \pi_{\text{lo}} < \pi_{\text{hi}} < \mu_2 \qquad \text{(B.25)}
\end{aligned}$$

So, if $\lambda_2 - \lambda_1 < c_1$, multiple equilibria is possible, but only when an equilibrium premium π_0 falls within the interval $(\pi_{\text{lo}}, \pi_{\text{hi}})$.

Finally, to check the condition under which an equilibrium premium π_0 falls within $(\pi_{\text{lo}}, \pi_{\text{hi}})$, we rewrite Equation B.8 for iso-elastic demand, to get:

$$a(\pi_0) = \left(\frac{\mu_2 - \pi_0}{\pi_0 - \mu_1}\right) \left(\frac{\mu_2^{\lambda_2}}{\mu_1^{\lambda_1}}\right) \pi_0^{-(\lambda_2 - \lambda_1)} = \frac{\alpha_1}{1 - \alpha_1}. \qquad \text{(B.26)}$$

Now note that for $\lambda_2 - \lambda_1 < c_1 < -1$, $a(\pi_0)$ is increasing in π_0, because:

$$\frac{\partial}{\partial \pi_0} \log a(\pi_0) = \frac{(\lambda_2 - \lambda_1)\ m(\pi_0)}{\pi_0(\mu_2 - \pi_0)(\pi_0 - \mu_1)} > 0. \qquad \text{(B.27)}$$

So, multiple equilibria exists if $\lambda_2 - \lambda_1 < c_1$ and α_1 satisfies the condition:

$$\frac{a(\pi_{\text{lo}})}{1 + a(\pi_{\text{lo}})} = \alpha_{\text{lo}} < \alpha_1 < \alpha_{\text{hi}} = \frac{a(\pi_{\text{hi}})}{1 + a(\pi_{\text{hi}})}, \qquad \text{(B.28)}$$

which proves our result. $\square$

Theorem B.1.2. *For pooled premium, given (μ_1, μ_2), define $c = \frac{\sqrt{\mu_2} + \sqrt{\mu_1}}{\sqrt{\mu_2} - \sqrt{\mu_1}}$. If $\alpha_1 < \frac{\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}{1 + \left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}$ then there is a unique equilibrium.*

Proof. In the special case of $\lambda_2 - \lambda_1 = -c$, Equations B.20 and B.21 leads to:

$$\pi_{\text{lo}} = \pi_{\text{hi}} = \sqrt{\mu_1 \mu_2}. \tag{B.29}$$

$$\Rightarrow a(\pi_{\text{lo}}) = a(\pi_{\text{hi}}) = \left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{2\lambda_1 - c + 1}. \tag{B.30}$$

As in this case, $a(\pi_{\text{lo}}) = a(\pi_{\text{hi}})$ is an increasing function of λ_1, the minimum value is attained when $\lambda_1 = c$ (i.e. when $\lambda_2 = 0$). So the minimum possible value of:

$$\alpha_{\text{lo}} = \frac{a(\pi_{\text{lo}})}{1 + a(\pi_{\text{lo}})} = \frac{\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}{1 + \left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}. \tag{B.31}$$

So, if $\alpha_1 < \frac{\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}{1+\left(\sqrt{\frac{\mu_2}{\mu_1}}\right)^{c+1}}$, the conditions for multiple equilibria in Theorem B.1.1 are violated and so, in this case, there is unique equilibrium. □

Theorem B.1.3. *For pooled premium, given (μ_1, μ_2), if there is a unique equilibrium, then the equilibrium premium is an increasing function of the individual demand elasticities.*

Proof. We first keep λ_2 fixed in Equation 3.36, and differentiate with respect to λ_1, to get:

$$\frac{d}{d\lambda_1}\pi_0(\lambda_1) = \frac{\log\left(\frac{\pi_0}{\mu_1}\right)}{\frac{1}{\pi_0 - \mu_1} + \frac{1}{\mu_2 - \pi_0} + \frac{\lambda_2 - \lambda_1}{\pi_0}} = \frac{1}{-(\lambda_2 - \lambda_1)m(\pi_0)}\pi_0(\pi_0 - \mu_1)(\mu_2 - \pi_0)\log\left(\frac{\pi_0}{\mu_1}\right), \tag{B.32}$$

where $m(\pi_0)$ is defined in Equation B.18. Similarly, if we keep λ_1 fixed in

Equation 3.36, and differentiate with respect to λ_2, we get:

$$\frac{d}{d\lambda_2}\pi_0(\lambda_2) = \frac{\log\left(\frac{\mu_2}{\pi_0}\right)}{\frac{1}{\pi_0-\mu_1} + \frac{1}{\mu_2-\pi_0} + \frac{\lambda_2-\lambda_1}{\pi_0}} = \frac{1}{-(\lambda_2-\lambda_1)m(\pi_0)}\pi_0(\pi_0-\mu_1)(\mu_2-\pi_0)\log\left(\frac{\mu_2}{\pi_0}\right). \tag{B.33}$$

As $\mu_1 < \pi_0 < \mu_2$, if $\lambda_2 > \lambda_1$, then both $\pi_0'(\lambda_1)$ and $\pi_0'(\lambda_2)$ are positive. On the other hand, if $\lambda_2 < \lambda_1$, $m(\pi_0) > 0$ is sufficient for uniqueness of equilibrium (by condition B.18), which also implies that $\pi_0'(\lambda_1)$ and $\pi_0'(\lambda_2)$ are positive. In other words, uniqueness of equilibrium implies that the equilibrium premium is an increasing function of the individual demand elasticities. $\square$

B.2 Negative-exponential Demand

B.2.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by negative-exponential demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks for the low risk-group and the high risk-group.
- p_1, p_2 are the population proportions such that $p_1 + p_2 = 1$.
- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \exp\left[\left\{1 - \left(\frac{\pi}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right]. \tag{B.34}$$

- The demand elasticity at premium π is given by:

$$\epsilon_i(\pi) = \lambda_i \left(\frac{\pi}{\mu_i}\right)^n. \tag{B.35}$$

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \tag{B.36}$$

$$\Rightarrow \sum_{i=1}^{2} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_i}{\mu_i}\right)^n\right\} \frac{\lambda_i}{n}\right] \pi_i = \sum_{i=1}^{2} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_i}{\mu_i}\right)^n\right\} \frac{\lambda_i}{n}\right] \mu_i. \tag{B.37}$$

Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

For illustration purpose, our analysis on multiple equilibria is based on negative-exponential demand with $n = 1$, in which case:

- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \exp\left[\left(1 - \frac{\pi}{\mu_i}\right) \lambda_i\right]. \tag{B.38}$$

- The demand elasticity at premium π is given by:

$$\epsilon_i(\pi) = \lambda_i \frac{\pi}{\mu_i}, \tag{B.39}$$

which is a non-decreasing linear function of premium.

B.2.2 Theorems and Proofs

Theorem B.2.1. *The weight in pooled equilibrium premium for negative-exponential demand decreases with demand elasticity parameter of the low risk-group, i.e.* $u \downarrow \lambda_1$.

Proof. From Equation 3.55:

$$\frac{du}{d\lambda_1} = -\frac{\alpha_1 \alpha_2 \frac{(\beta^n - 1)}{n} e^{\lambda_1 \frac{(\beta^n-1)}{n}}}{[\alpha_1 + \alpha_2 e^{\lambda_1 \frac{(\beta^n-1)}{n}}]^2} \tag{B.40}$$

$$= -u(1-u)\Big(\frac{\beta^n - 1}{n}\Big); \tag{B.41}$$

$$\Rightarrow \frac{du}{d\lambda_1} \gtreqless 0 \Leftrightarrow \frac{\beta^n - 1}{n} \lesseqgtr 0. \tag{B.42}$$

Note:

$$\beta^n - 1 \gtreqless 0 \Leftrightarrow n \gtreqless 0, \tag{B.43}$$

$$\Rightarrow \frac{\beta^n - 1}{n} > 0 \text{ for } n \neq 0, \tag{B.44}$$

$$\text{therefore } \frac{du}{d\lambda_1} < 0 \text{ for } n \neq 0. \tag{B.45}$$

In the case when $n \to 0$, using L'Hopital's Rule,

$$\lim_{n \to 0} \frac{\beta^n - 1}{n} = \lim_{n \to 0} \frac{\beta^n \log \beta}{1} = \log \beta > 0, \tag{B.46}$$

$$\Rightarrow \frac{\beta^n - 1}{n} > 0 \text{ for } n \to 0, \tag{B.47}$$

$$\text{therefore } \frac{du}{d\lambda_1} < 0 \text{ for } n \to 0. \tag{B.48}$$

Therefore, for $n \in R$, $\frac{du}{d\lambda_1} < 0$. □

Theorem B.2.2. *The pooled equilibrium premium is an increasing function of* λ_1.

Proof.
$$\frac{\partial \pi_0}{\partial \lambda_1} = \frac{\partial \pi_0}{\partial u}\frac{du}{d\lambda_1}, \tag{B.49}$$

and
$$\frac{\partial \pi_0}{\partial u} = \mu_1 - \mu_2 < 0. \tag{B.50}$$

Thus, using Theorem B.2.1, we prove that $\frac{\partial \pi_0}{\partial \lambda_1} > 0$. □

Theorem B.2.3. *In the case when $n > 0$, demand elasticity is an increasing function of the equilibrium premium.*

Proof.

$$\lambda_0 = \lambda_1(\frac{\pi_0}{\mu_1})^n, \tag{B.51}$$

$$\Rightarrow \frac{\partial \lambda_0}{\partial \pi_0} = \frac{\partial \lambda_1}{\partial \pi_0}\left(\frac{\pi_0}{\mu_1}\right)^n + \frac{\lambda_1 n}{\mu_1^n}\pi_0^{n-1} \text{ when } n > 0, \tag{B.52}$$

using Result 3.21. Hence proved. □

Theorem B.2.4. *For $n > 0$, $\lambda_0 \uparrow \lambda_1$. i.e. When $n > 0$, the demand elasticity at pooled equilibrium premium increases as the demand elasticity parameter of low risks (and high risks) increases.*

Proof.

$$\lambda_0 = \lambda_1\left(\frac{\pi_0}{\mu_1}\right)^n \Rightarrow \frac{\partial \lambda_0}{\partial \lambda_1} = \left(\frac{\pi_0}{\mu_1}\right)^n + n\lambda_1 \frac{\pi_0^{n-1}}{\mu_1^n}\frac{\partial \pi_0}{\partial \lambda_1} > 0, \tag{B.53}$$

because $\pi_0 \uparrow \lambda_1$ from Result 3.21. □

Theorem B.2.5. *For pooled premium, given (μ_1, μ_2) and (λ_1, λ_2), multiple* *equilibria exist if*

$$\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < -\frac{4}{\mu_2 - \mu_1}, \quad \textit{and} \tag{B.54}$$

$$\frac{a(\pi_{lo})}{1 + a(\pi_{lo})} = \alpha_{lo} < \alpha_1 < \alpha_{hi} = \frac{a(\pi_{hi})}{1 + a(\pi_{hi})}, \quad \textit{where} \tag{B.55}$$

$$a(\pi) = \left(\frac{\mu_2 - \pi}{\pi - \mu_1}\right) \exp\left[\lambda_2 - \lambda_1 - (\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1})\pi\right], \quad \textit{and} \tag{B.56}$$

$$(\pi_{lo}, \pi_{hi}) \textit{ are solutions to: } \pi^2 - (\mu_1 + \mu_2)\pi + \mu_1\mu_2 - \frac{\mu_2 - \mu_1}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}} = 0. \tag{B.57}$$

Proof. Note that for the formulation of profit function in Equation 3.10 can be re-written as:

$$\rho(\pi) = (\mu_2 - \mu_1)[d_1(\pi)p_1 + d_2(\pi)p_2][\alpha_1(\pi) - h(\pi)], \tag{B.58}$$

where

$$h(\pi) = \frac{\mu_2 - \pi}{\mu_2 - \mu_1}. \tag{B.59}$$

$$\alpha_1(\pi) = \frac{d_1(\pi)p_1}{d_1(\pi)p_1 + d_2(\pi)p_2}. \tag{B.60}$$

$$\text{So } \rho(\pi_0) = 0 \Leftrightarrow \alpha_1(\pi) = h(\pi). \tag{B.61}$$

And the slope of $\rho(\pi)$ is given by:

$$\rho'(\pi) = (\mu_2 - \mu_1)\left[d_1'(\pi)p_1 + d_2'(\pi)p_2\right]\left[\alpha_1(\pi) - h(\pi)\right]$$

$$+(\mu_2 - \mu_1)\left[d_1(\pi)p_1 + d_2(\pi)p_2\right]\left[\alpha_1'(\pi) - h'(\pi)\right] \text{ where} \tag{B.62}$$

$$\alpha_1'(\pi) = \alpha_1(\pi)[1 - \alpha_1(\pi)]\left(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}\right), \quad \text{and} \tag{B.63}$$

$$h'(\pi) = -\frac{1}{\mu_2 - \mu_1}. \tag{B.64}$$

As $\rho(\mu_1) < 0$ and $\rho(\mu_2) > 0$, there exists a premium π_0 for which $\rho(\pi_0) = 0$

and $\rho'(\pi_0) \geq 0$. Multiple equilibria exist if there exists a premium π_0 for which $\rho(\pi_0) = 0$ and $\rho'(\pi_0) < 0$, or equivalently:

$$\alpha_1(\pi_0) = h(\pi_0) \tag{B.65}$$

$$\alpha_1'(\pi_0) < h'(\pi_0). \tag{B.66}$$

Using the relationship in Equation B.65, the expression for $\alpha_1'(\pi_0)$ in Equation B.63 becomes:

$$\alpha_1'(\pi_0) = h(\pi_0)[1 - h(\pi_0)]\left(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}\right) \tag{B.67}$$

$$= \left(\frac{\mu_2 - \pi_0}{\mu_2 - \mu_1}\right)\left(\frac{\pi_0 - \mu_1}{\mu_2 - \mu_1}\right)\left(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}\right). \tag{B.68}$$

Case 1: $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} \geq 0$. In this case, $\alpha_1'(\pi) \geq 0$ and so condition in Equation B.66 cannot hold as $h'(\pi_0) < 0$. So, multiple equilibria is not possible in this case.

Case 2: $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < 0$. In this case, Condition B.66 leads to the following equivalent condition:

$$m(\pi_0) = \pi_0^2 - (\mu_1 + \mu_2)\pi_0 + \mu_1\mu_2 - \frac{\mu_2 - \mu_1}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}} < 0. \tag{B.69}$$

Now, $m(\pi_0) = (\pi_0 - \pi_{\text{lo}})(\pi_0 - \pi_{\text{hi}})$, where

$$\pi_{\text{lo}} = \frac{1}{2}\left[\mu_1 + \mu_2 - \sqrt{(\mu_2 - \mu_1)(\mu_2 - \mu_1 + \frac{4}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}})}\right], \tag{B.70}$$

and

$$\pi_{\text{hi}} = \frac{1}{2}\left[\mu_1 + \mu_2 + \sqrt{(\mu_2 - \mu_1)(\mu_2 - \mu_1 + \frac{4}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}})}\right]. \tag{B.71}$$

Note that, as both $m(\mu_1)$ and $m(\mu_2)$ are positive, $m(\pi_0)$ can only be negative, if there are real solutions to $m(\pi_0) = 0$, which means,

$$\mu_2 - \mu_1 + \frac{4}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}} > 0, \tag{B.72}$$

$$\Rightarrow \frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < -\frac{4}{\mu_2 - \mu_1}. \tag{B.73}$$

In this case,

$$\pi_{\text{lo}} - \mu_1 = \frac{1}{2}\sqrt{\mu_2 - \mu_1}\left(\sqrt{\mu_2 - \mu_1} - \sqrt{\mu_2 - \mu_1 + \frac{4}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}}}\right) \tag{B.74}$$

$$> 0, \text{ because } \frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < 0. \tag{B.75}$$

Hence, $\mu_1 < \pi_{\text{lo}}$.

And

$$\pi_{\text{hi}} - \mu_2 = \frac{1}{2}\sqrt{\mu_2 - \mu_1}\left(\sqrt{\mu_2 - \mu_1 + \frac{4}{\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}}} - \sqrt{\mu_2 - \mu_1}\right) \tag{B.76}$$

$$< 0, \text{ because } \frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < 0. \tag{B.77}$$

Hence, $\pi_{\text{hi}} < \mu_2$, and consequently, $\mu_1 < \pi_{\text{lo}} < \pi_{\text{hi}} < \mu_2$. Therefore, if $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < -\frac{4}{\mu_2 - \mu_1}$, multiple equilibria is possible, but only when an equilibrium premium π_0 falls within the interval $(\pi_{\text{lo}}, \pi_{\text{hi}})$.

Finally, to check the condition under which an equilibrium premium π_0 falls within $(\pi_{\text{lo}}, \pi_{\text{hi}})$, we rewrite Equation 3.10 for negative-exponential demand, to get:

$$a(\pi_0) = \left(\frac{\mu_2 - \pi_0}{\pi_0 - \mu_1}\right) \exp[\lambda_2 - \lambda_1 - (\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1})\pi_0] = \frac{\alpha_1}{1 - \alpha_1}. \tag{B.78}$$

Now note that for $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < -\frac{4}{\mu_2 - \mu_1} < 0$, $a(\pi_0)$ is increasing in π_0, because:

$$\frac{\partial}{\partial \pi_0} \log a(\pi_0) = \frac{(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1}) m(\pi_0)}{(\mu_2 - \pi_0)(\pi_0 - \mu_1)} > 0. \tag{B.79}$$

So, multiple equilibria exists if $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} < -\frac{4}{\mu_2 - \mu_1}$ and α_1 satisfies the condition:

$$\frac{a(\pi_{lo})}{1 + a(\pi_{lo})} = \alpha_{lo} < \alpha_1 < \alpha_{hi} = \frac{a(\pi_{hi})}{1 + a(\pi_{hi})}, \tag{B.80}$$

which proves our result. Note that, if there are multiple equilibria, there can be a maximum of 3 equilibria, as $a(\pi_0)$ is monotonic in π_0 over (π_{lo}, π_{hi}).

□

Theorem B.2.6. *For pooled premium, given (μ_1, μ_2), if $\alpha_1 < \frac{e^2}{1+e^2}$, then there is a unique equilibrium.*

Proof. In the special case of $\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1} = -\frac{4}{\mu_2 - \mu_1}$,

$$\pi_{\text{lo}} = \pi_{\text{hi}} = \frac{\mu_1 + \mu_2}{2} \tag{B.81}$$

$$\Rightarrow a(\pi_{\text{lo}}) = a(\pi_{\text{hi}}) = \exp[\lambda_1(\frac{\mu_2}{\mu_1} - 1) - 2]. \tag{B.82}$$

As in this case, $a(\pi_{\text{lo}}) = a(\pi_{\text{hi}})$ is an increasing function of λ_1, the minimum value is attained when $\lambda_1 = \frac{4\mu_1}{\mu_2 - \mu_1}$(i.e. when $\lambda_2 = 0$). So the minimum possible value of:

$$\alpha_{\text{lo}} = \frac{a(\pi_{\text{lo}})}{1 + a(\pi_{\text{lo}})} = \frac{e^2}{1 + e^2}. \tag{B.83}$$

So, if $\alpha_1 < \frac{e^2}{1+e^2}$, the condition for multiple equilibria in Theorem B.2.5 are violated and so, in this case, there is a unique equilibrium. □

Theorem B.2.7. *For pooled premium, given (μ_1, μ_2), if there is a unique* *equilibrium, then the equilibrium premium is an increasing function of the individual demand elasticities.*

Proof. We first keep λ_2 fixed in Equation 3.84, and differentiate with respect to λ_1, to get:

$$\frac{d}{d\lambda_1}\pi_0(\lambda_1) = \frac{(\pi_0 - \mu_1)(\mu_2 - \pi_0)}{-(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1})m(\pi_0)}(\frac{\pi_0}{\mu_1} - 1). \tag{B.84}$$

Similarly, if we keep λ_1 fixed in Equation 3.84, and differentiate with respect to λ_2, we get:

$$\frac{d}{d\lambda_2}\pi_0(\lambda_2) = \frac{(\pi_0 - \mu_1)(\mu_2 - \pi_0)}{-(\frac{\lambda_2}{\mu_2} - \frac{\lambda_1}{\mu_1})m(\pi_0)}(1 - \frac{\pi_0}{\mu_2}). \tag{B.85}$$

As $\mu_1 < \pi_0 < \mu_2$, if $\frac{\lambda_1}{\mu_1} < \frac{\lambda_2}{\mu_2}$, then both $\pi_0'(\lambda_1)$ and $\pi_0'(\lambda_2)$ are positive. On the other hand, if $\frac{\lambda_2}{\mu_2} < \frac{\lambda_1}{\mu_1}$, $m(\pi_0) > 0$ is sufficient for uniqueness of equilibrium (by Equation B.69), which also implies that $\pi_0'(\lambda_1)$ and $\pi_0'(\lambda_2)$ are positive, i.e. uniqueness of equilibrium implies that the equilibrium premium is an increasing function of the individual demand elasticities. $\square$

Appendix C

Loss Coverage: Iso-elastic Demand

C.1 Case of Two Risk-groups

C.1.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by iso-elastic demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks for the low risk-group and the high risk-group.
- p_1, p_2 are the population proportions such that $p_1 + p_2 = 1$.
- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \left(\frac{\mu_i}{\pi} \right)^{\lambda_i}. \tag{C.1}$$

  Note: $\pi \geq 0$ *is an implicit assumption.*

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \Rightarrow \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i. \tag{C.2}$$

Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

- Loss coverage under a specific risk-classification scheme is defined as:

$$LC(\pi_1, \pi_2) = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i, \tag{C.3}$$

where (π_1, π_2) satisfy the equilibrium condition in Equation C.2.

C.1.2 Theorems and Proofs

Theorem C.1.1. *For $\lambda > 0$,*

$$\lambda \lesseqqgtr 1 \Leftrightarrow C(\lambda) \gtreqqless 1. \tag{C.4}$$

Proof. The loss coverage ratio for the case of equal demand elasticity is given in Equation 5.7 and can be expressed as follows:

$$\begin{aligned} C(\lambda) &= \frac{1}{\pi_0^\lambda}\frac{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}{\alpha_1\mu_1 + \alpha_2\mu_2}, \quad \text{where} \quad \pi_0 = \frac{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}{\alpha_1\mu_1^\lambda + \alpha_2\mu_2^\lambda}; \\ &= \left[w\mu_1^{\lambda-1} + (1-w)\mu_2^{\lambda-1}\right]^\lambda \left[w\mu_1^\lambda + (1-w)\mu_2^\lambda\right]^{1-\lambda} \qquad \text{(C.5)} \\ &= E_w\left[\mu^{\lambda-1}\right]^\lambda E_w\left[\mu^\lambda\right]^{1-\lambda}, \qquad \text{(C.6)} \end{aligned}$$

where $w = \frac{\alpha_1\mu_1}{\alpha_1\mu_1+\alpha_2\mu_2}$; and E_w denotes expectation in this context and the random variable μ takes values μ_1 and μ_2 with probabilities w and $1-w$ respectively.

Case $\lambda = 1$: It follows directly from Equation C.6 that $C(1) = 1$.

Case $0 < \lambda < 1$: Holder's inequality states that, if $1 < p, q < \infty$ where $1/p + 1/q = 1$, for positive random variables X, Y with $E[X]^p, E[Y]^q < \infty$, $E[X^p]^{1/p} E[Y^q]^{1/q} \geq E[XY]$.

Setting $1/p = \lambda$, $1/q = 1-\lambda$, $X = \mu^{\lambda(\lambda-1)}$ and $Y = 1/X$, applying Holder's inequality on Equation C.6 gives,

$$C(\lambda) \quad = E_w\left[X^{1/\lambda}\right]^{\lambda} E_w\left[Y^{1/(1-\lambda)}\right]^{1-\lambda} \geq E_w[XY] = 1. \tag{C.7}$$

Case $\lambda > 1$: Lyapunov's inequality states that, for positive random variable μ and $0 < s < t$, $E[\mu^t]^{1/t} \geq E[\mu^s]^{1/s}$.

So Equation C.6 gives:

$$C(\lambda) = \frac{E_w\left[\mu^{\lambda-1}\right]^{\lambda}}{E_w\left[\mu^{\lambda}\right]^{\lambda-1}} = \left[\frac{E_w\left[\mu^{\lambda-1}\right]^{1/(\lambda-1)}}{E_w\left[\mu^{\lambda}\right]^{1/\lambda}}\right]^{\lambda(\lambda-1)} \leq 1, \tag{C.8}$$

as $E_w\left[\mu^{\lambda-1}\right]^{1/(\lambda-1)} \leq E_w\left[\mu^{\lambda}\right]^{1/\lambda}$ for $\lambda > 1$ by Lyapunov's inequality.

□

Theorem C.1.2.

$$\max_{w,\lambda} C = \frac{1}{2}\left(\sqrt[4]{\frac{\mu_2}{\mu_1}} + \sqrt[4]{\frac{\mu_1}{\mu_2}}\right) = \frac{1}{2}\left(\sqrt[4]{\beta} + \frac{1}{\sqrt[4]{\beta}}\right). \tag{C.9}$$

Proof. Proceeding from Equation C.5, we have:

$$
\begin{aligned}
C(\lambda) &= \frac{\left[w\beta^{1-\lambda} + (1-w)\right]^{\lambda}\left[w + (1-w)\beta^{\lambda}\right]^{1-\lambda}}{\beta^{\lambda(1-\lambda)}} && \text{(C.10)}\\
\Rightarrow \frac{\partial}{\partial w}\log C(\lambda) &= \frac{\lambda(\beta^{1-\lambda}-1)}{w\beta^{1-\lambda}+(1-w)} - \frac{(1-\lambda)(\beta^{\lambda}-1)}{w+(1-w)\beta^{\lambda}} && \text{(C.11)}\\
\Rightarrow \frac{\partial^2}{\partial w^2}\log C(\lambda) &= -\frac{\lambda(\beta^{1-\lambda}-1)^2}{\left[w\beta^{1-\lambda}+(1-w)\right]^2} - \frac{(1-\lambda)^2(\beta^{\lambda}-1)^2}{\left[w+(1-w)\beta^{\lambda}\right]^2} < 0.\\
\Rightarrow \frac{\partial}{\partial w}\log C(\lambda) = 0 \;\;&\Rightarrow\;\; w = \frac{\lambda(\beta-1)-(\beta^{\lambda}-1)}{(\beta^{\lambda}-1)(\beta^{1-\lambda}-1)}, && \text{(C.12)}
\end{aligned}
$$

gives the maximum.

Inserting the value of w in Equation C.10, gives the result that: For $0 < \lambda < 1$,

$$
\max_w C(\lambda) = \frac{\beta - 1}{\beta^{\lambda(1-\lambda)}\left(\frac{\beta^{\lambda}-1}{\lambda}\right)^{\lambda}\left(\frac{\beta^{1-\lambda}-1}{1-\lambda}\right)^{1-\lambda}}, \quad \text{where} \quad \beta = \frac{\mu_2}{\mu_1} > 1. \quad \text{(C.13)}
$$

Equation C.13 can also be expressed as:

$$
\begin{aligned}
\max_w C(\lambda) &= \frac{1}{2}\left(\sqrt[4]{\beta} + \frac{1}{\sqrt[4]{\beta}}\right)\frac{2\left(\sqrt[4]{\beta} - \frac{1}{\sqrt[4]{\beta}}\right)}{\left(\frac{\left(\sqrt{\beta}\right)^{\lambda} - \frac{1}{\left(\sqrt{\beta}\right)^{\lambda}}}{\lambda}\right)^{\lambda}\left(\frac{\left(\sqrt{\beta}\right)^{1-\lambda} - \frac{1}{\left(\sqrt{\beta}\right)^{1-\lambda}}}{1-\lambda}\right)^{1-\lambda}},\\
&= \frac{1}{2}\left(\sqrt[4]{\beta} + \frac{1}{\sqrt[4]{\beta}}\right)\frac{R(½)}{R(\lambda)}, && \text{(C.14)}\\
\text{where} \quad R(\lambda) &= \left(\frac{\left(\sqrt{\beta}\right)^{\lambda} - \frac{1}{\left(\sqrt{\beta}\right)^{\lambda}}}{\lambda}\right)^{\lambda}\left(\frac{\left(\sqrt{\beta}\right)^{1-\lambda} - \frac{1}{\left(\sqrt{\beta}\right)^{1-\lambda}}}{1-\lambda}\right)^{1-\lambda}. && \text{(C.15)}
\end{aligned}
$$

The result follows from $R(\lambda) \geq R(½)$, which in turn follows from the fact that $R(\lambda)$ is symmetric and convex over $0 < \lambda < 1$. As symmetry is obvious, we only need to prove convexity of $R(\lambda)$.

Note that,

$$\log R(\lambda) = g(\lambda) + g(1-\lambda), \quad \text{where} \quad g(\lambda) = \lambda \log \left(\frac{\left(\sqrt{\beta}\right)^{\lambda} - \frac{1}{\left(\sqrt{\beta}\right)^{\lambda}}}{\lambda} \right). \tag{C.16}$$

If $g(\lambda)$ is a convex function over (0,1), then $g''(\lambda) \geq 0$ and $g''(1-\lambda) \geq 0$, so $\log R(\lambda)$ is convex, which in turn implies $R(\lambda)$ is convex. So it suffices to show that:

$$g(x) = x \log \left(\frac{a^x - a^{-x}}{x} \right) \tag{C.17}$$

is convex over (0,1), where $a = \sqrt{\beta} > 1$. Now,

$$\begin{aligned} g'(x) &= \log \left(\frac{a^x - a^{-x}}{x} \right) + \left(\frac{a^x + a^{-x}}{a^x - a^{-x}} \right) x \log a - 1. & \text{(C.18)} \\ g''(x) &= \frac{(a^x + a^{-x}) x \log a - (a^x - a^{-x})}{x(a^x - a^{-x})} + \frac{a^{2x} - a^{-2x} - 4x \log a}{(a^x - a^{-x})^2} \log a, \\ &\geq 0, & \text{(C.19)} \end{aligned}$$

as both $[(a^x + a^{-x}) x \log a - (a^x - a^{-x})]$ and $[a^{2x} - a^{-2x} - 4x \log a]$ are increasing functions starting from 0 at $x = 0$. Hence proved.

Note that given $\max_{w,\lambda} C = \frac{1}{2}\left(\beta^{1/4} + \beta^{-1/4}\right)$, $\max_{w,\lambda} C$ is also an increasing function of β, i.e. the maximised loss coverage ratio (LCR) also increases with the relative risk β.

This is because

$$\frac{d}{d\beta} \max_{w,\lambda} C \propto \beta^{-3/4} - \beta^{-5/4} > 0, \tag{C.20}$$

because $\beta > 1$. Hence proved.

Figure C.1 shows the plots of $\max_w C(\lambda)$ for $\beta = 4, 5$. It shows that LCR reaches its maximum when $\lambda = 0.5$ regardless of the value of β. And LCR is larger when β is larger.

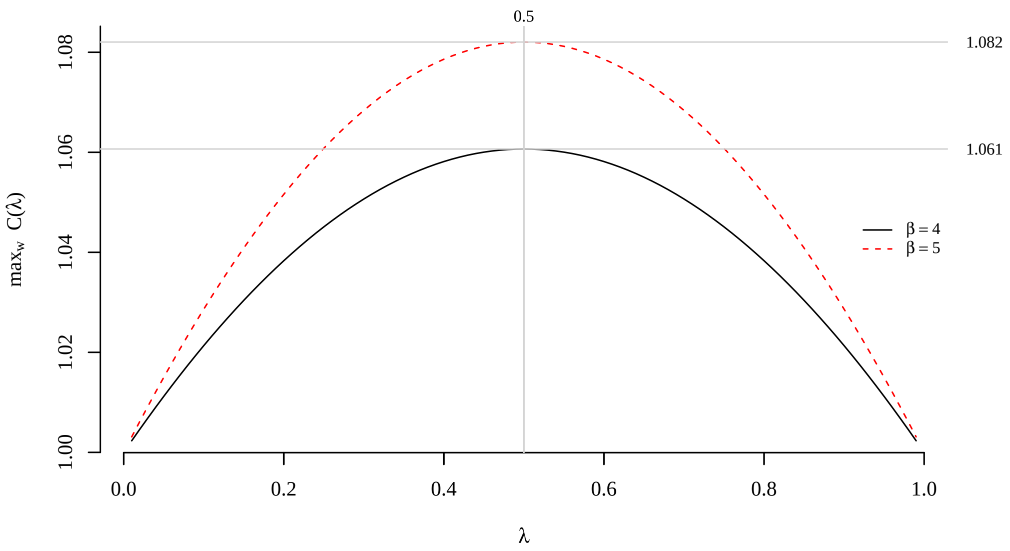

Figure C.1: Maximum loss coverage ratio as a function of λ for specific values of β.

$\square$

Theorem C.1.3. *For a given value of λ_2, the loss coverage ratio is a decreasing function of λ_1.*

Proof.

$$\frac{d}{d\lambda_1}\log C = \frac{\partial}{\partial\lambda_1}\log C + \left(\frac{\partial}{\partial\pi_0}\log C\right)\left(\frac{d\pi_0}{d\lambda_1}\right) < 0, \text{ since} \tag{C.21}$$

$$\frac{\partial}{\partial\lambda_1}\log C = -\log(\frac{\pi_0}{\mu_1}) < 0, \text{ by Equation 5.16 ,} \tag{C.22}$$

$$\frac{\partial}{\partial\pi_0}\log C = -\frac{\lambda_2}{\pi_0} - \frac{\mu_1}{\pi_0(\pi_0-\mu_1)} < 0, \text{ by Equation 5.18 ,} \tag{C.23}$$

$$\frac{d\pi_0}{d\lambda_1} > 0, \text{ by results in sub-section 3.3.3 .} \tag{C.24}$$

□

Theorem C.1.4. *For a given value of* λ_1*, loss coverage ratio is an increasing function of* λ_2 *if* $\lambda_1 \leq \frac{\mu_2}{\alpha_1(\mu_2-\mu_1)}$.

Proof. This phenomenon might be explained by the non-monotonic relationship between loss coverage ratio and equilibrium premium, π_0. Equation 5.16 gives:

$$\frac{\partial}{\partial\pi_0}\log C \propto (1-\lambda_1)\mu_2 + \lambda_1\pi_0. \tag{C.25}$$

When $0 \leq \lambda_1 \leq 1$, $\frac{\partial}{\partial\pi_0}\log C > 0$, i.e. loss coverage ratio is an increasing function of π_0. And based on results in sub-section 3.3.3, all else being fixed, including λ_1, the pooled equilibrium premium can only increase if λ_2 increases. Therefore, loss coverage ratio is an increasing function of λ_2 if $0 \leq \lambda_1 \leq 1$.

When $\lambda_1 > 1$, $\frac{\partial}{\partial\pi_0}\log C > 0$ if $\lambda_1 < \frac{\mu_2}{\mu_2-\pi_0}$. Result 3.13 shows that $\pi_0 \geq \alpha_1\mu_1 + \alpha_2\mu_2$. Therefore, as long as $\lambda_1 \leq \frac{\mu_2}{\mu_2-(\alpha_1\mu_1+\alpha_2\mu_2)} = \frac{\mu_2}{\alpha_1(\mu_2-\mu_1)}$, loss coverage ratio is an increasing function of λ_2.

As a result, there is a monotonic relationship between loss coverage ratio and λ_2 for $\lambda_1 \leq \frac{\mu_2}{\alpha_1(\mu_2-\mu_1)}$. $\square$

Theorem C.1.5. *For $0 < \lambda_1 < \lambda_2 < 1$, loss coverage ratio, $C(\lambda)$, is greater than 1.*

Proof. The equilibrium pooled premium π_0 given different demand elasticities can be rewritten as

$$\lambda_1 \log\left(\frac{\pi_0}{\mu_1}\right) + \lambda_2 \log\left(\frac{\mu_2}{\pi_0}\right) = \log\left(\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)}\right) \tag{C.26}$$

(which is given in Equation 3.36).

Now Equation C.26 implies that there exists a λ_*, where $\lambda_1 \leq \lambda_* \leq \lambda_2$, such that:

$$\lambda_* \log\left(\frac{\pi_0}{\mu_1}\right) + \lambda_* \log\left(\frac{\mu_2}{\pi_0}\right) = \log\left(\frac{\alpha_1(\pi_0-\mu_1)}{\alpha_2(\mu_2-\pi_0)}\right) \Rightarrow \lambda_* = \frac{\log\left(\frac{\mu_2}{\pi_0}\right)}{\log \beta}. \tag{C.27}$$

In other words, given a population P with $\lambda_2 \geq \lambda_1$, there exists a population T with the same constant iso-elastic demand elasticity, λ_*, for both high and low risk-groups. such that the equilibrium premium π_0 is the same for both populations, P and T. The two populations, P and T, differ only in their demand elasticities, and nothing else.

Now, as $\lambda_1 \leq \lambda_* \leq \lambda_2$, we observe:

$$\tau_1\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} \geq \tau_1\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_*} \text{ and } \tau_2\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} \geq \tau_2\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_*}. \tag{C.28}$$

In other words, for any equilibrium premium π_0, proportional demand for both high and low risk-groups are higher for population P compared to that of population T.

Consequently, the loss coverage ratio for population P with demand elasticities (λ_1, λ_2) where $\lambda_2 \geq \lambda_1$:

$$C = \frac{\alpha_1 \left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} \mu_1 + \alpha_2 \left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} \mu_2}{\alpha_1\mu_1 + \alpha_2\mu_2}, \tag{C.29}$$

$$\geq \frac{\alpha_1 \left(\frac{\mu_1}{\pi_0}\right)^{\lambda_*} \mu_1 + \alpha_2 \left(\frac{\mu_2}{\pi_0}\right)^{\lambda_*} \mu_2}{\alpha_1\mu_1 + \alpha_2\mu_2}, \tag{C.30}$$

i.e. the loss coverage ratio of population T

$$= \frac{1}{\pi_0^{\lambda_*}} \frac{\alpha_1\mu_1^{\lambda_*+1} + \alpha_2\mu_2^{\lambda_*+1}}{\alpha_1\mu_1 + \alpha_2\mu_2}, \tag{C.31}$$

$$\geq 1, \text{ if } \lambda_* \leq 1, \text{ by Theorem C.1.1 .} \tag{C.32}$$

□

C.2 Case of More Risk-groups

Theorem C.2.1. *Let V be a positive random variable and λ be a positive constant, such that $E[V^\lambda] = E[V^{\lambda+1}]$. Then:*

$$\lambda \lesseqqgtr 1 \Rightarrow E[V^\lambda] \gtreqqless E[V]. \tag{C.33}$$

Proof. **Case: $\lambda = 1$:** It follows directly from the definition.

Case: $0 < \lambda < 1$: Holder's inequality states that, if $1 < p, q < \infty$ where $1/p + 1/q = 1$, for positive random variables X, Y with $E[X^p], E[Y^q] < \infty$:

$$(E[X^p])^{1/p}(E[Y^q])^{1/q} \geq E[XY]. \tag{C.34}$$

Setting $1/p = \lambda, 1/q = 1 - \lambda, X = V^{\lambda^2}$ and $Y = V^{1-\lambda^2}$, Holder's inequality gives:

$$\left(E\left[V^{\lambda^2\frac{1}{\lambda}}\right]\right)^{\lambda}\left(E\left[V^{(1-\lambda^2)\frac{1}{1-\lambda}}\right]\right)^{1-\lambda} \geq E\left[V^{\lambda^2}V^{1-\lambda^2}\right], \tag{C.35}$$

$$\Rightarrow\left(E\left[V^{\lambda}\right]\right)^{\lambda}\left(E\left[V^{\lambda+1}\right]\right)^{1-\lambda} \geq E[V]. \tag{C.36}$$

$$\Rightarrow E\left[V^{\lambda}\right] \geq E[V], \quad \text{since } E\left[V^{\lambda}\right] = E\left[V^{\lambda+1}\right]. \tag{C.37}$$

Case: $\lambda > 1$**:** Young's inequality states that, for $a, b \geq 0$ and $p, q > 0$ such that $1/p + 1/q = 1$:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{C.38}$$

Setting $p = \lambda, q = \frac{\lambda}{\lambda-1}, a = V^{\frac{1}{\lambda}}$ and $b = V^{\lambda-\frac{1}{\lambda}}$, Young's inequality gives:

$$V^{\frac{1}{\lambda}}V^{\lambda-\frac{1}{\lambda}} \leq \frac{1}{\lambda}V^{\frac{1}{\lambda}\lambda} + \frac{\lambda-1}{\lambda}V^{(\lambda-\frac{1}{\lambda})\frac{\lambda}{\lambda-1}}, \tag{C.39}$$

$$\Rightarrow V^{\lambda} \leq \frac{1}{\lambda}V + \frac{\lambda-1}{\lambda}V^{\lambda+1}. \tag{C.40}$$

$$\Rightarrow E\left[V^{\lambda}\right] \leq \frac{1}{\lambda}E[V] + \frac{\lambda-1}{\lambda}E\left[V^{\lambda+1}\right], \tag{C.41}$$

$$\Rightarrow E\left[V^{\lambda}\right] \leq E[V], \quad \text{since } E\left[V^{\lambda}\right] = E\left[V^{\lambda+1}\right]. \tag{C.42}$$

Hence proved.

□

Theorem C.2.2. *Let V be a positive random variable and $f(v)$ be a positive function, such that $E\left[V^{f(V)}\right] = E\left[V^{f(V)+1}\right]$. Define $\lambda_{lo} = \max_{v\leq 1} f(v)$ and $\lambda_{hi} = \min_{v>1} f(v)$. Then:*

$$\lambda_{lo} < 1 \textit{ and } \lambda_{hi} \geq \lambda_{lo} \Rightarrow E\left[V^{f(V)}\right] \geq E\left[V\right]. \tag{C.43}$$

Proof. Holder's inequality states that, if $1 < p, q < \infty$ where $1/p + 1/q = 1$, for positive random variables X, Y with $E[X^p], E[Y^q] < \infty$:

$$(E[X^p])^{1/p} (E[Y^q])^{1/q} \geq E[XY]. \tag{C.44}$$

For any λ, such that $0 < \lambda < 1$, set $1/p = \lambda$, $1/q = 1 - \lambda$, $X = V^{\lambda f(V)}$ and $Y = V^{(1-\lambda)(f(V)+1)}$, Holder's inequality gives:

$$\left(E\left[V^{f(V)}\right]\right)^{\lambda} \left(E\left[V^{f(V)+1}\right]\right)^{1-\lambda} \geq E\left[V^{\lambda f(V)} V^{(1-\lambda)(f(V)+1)}\right], \tag{C.45}$$

$$\Rightarrow E\left[V^{f(V)}\right] \geq E\left[V^{f(V)+1-\lambda}\right], \quad \text{since } E\left[V^{f(V)}\right] = E\left[V^{f(V)+1}\right]. \tag{C.46}$$

The relationship in Equation C.46 holds for any positive $\lambda < 1$, Now, set $\lambda = \lambda_{lo} < 1$.

Case $V < 1$:

$$\lambda_{lo} = \max_{v \leq 1} f(v) \Rightarrow f(V) \leq \lambda_{lo} = \lambda \Rightarrow f(V)+1-\lambda \leq 1 \Rightarrow V^{f(V)+1-\lambda} \geq V. \tag{C.47}$$

Case $V = 1$:

$$V^{f(V)+1-\lambda} = V. \tag{C.48}$$

Case $V > 1$:

$$\lambda_{hi} = \min_{v > 1} f(v) \Rightarrow f(V) \geq \lambda_{hi} \geq \lambda_{lo} = \lambda \Rightarrow f(V)+1-\lambda \geq 1 \Rightarrow V^{f(V)+1-\lambda} \geq V. \tag{C.49}$$

Hence, $V^{f(V)+1-\lambda} \geq V$ for all cases, which implies

$$E\left[V^{f(V)+1-\lambda}\right] \geq E[V]. \tag{C.50}$$

Combining Equations C.46 and C.50, we have:

$$E\left[V^{f(V)}\right] \geq E\left[V\right], \tag{C.51}$$

$$\Rightarrow C(\pi_0) = E\left[V^{f(V)}\right]/E\left[V\right] \geq 1. \tag{C.52}$$

□

Appendix D

Loss Coverage: Negative-exponential Demand

D.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by negative-exponential demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks for the low risk-group and the high risk-group.
- p_1, p_2 are the population proportions such that $p_1 + p_2 = 1$.
- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \exp\left[\left\{1 - \left(\frac{\pi}{\mu_i}\right)^n\right\} \frac{\lambda_i}{n}\right]. \tag{D.1}$$

- The demand elasticity at premium π is given by:

$$\epsilon_i(\pi) = \lambda_i \left(\frac{\pi}{\mu_i}\right)^n. \tag{D.2}$$

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \tag{D.3}$$

$$\Rightarrow \sum_{i=1}^{2} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_i}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right]\pi_i = \sum_{i=1}^{2} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_i}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right]\mu_i. \tag{D.4}$$

Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

- Loss coverage under a specific risk-classification scheme is defined as:

$$LC(\pi_1, \pi_2) = \sum_{i=1}^{2} p_i \tau_i \exp\left[\left\{1 - \left(\frac{\pi_i}{\mu_i}\right)^n\right\}\frac{\lambda_i}{n}\right]\mu_i, \tag{D.5}$$

where (π_1, π_2) satisfy the equilibrium condition in Equation D.4.

D.2 Theorems and Proofs

Theorem D.2.1.

$$C(\pi_0) \leq 1, \ \textit{when } n \leq -1. \tag{D.6}$$

Proof. Equation 5.34 can be re-written as:

$$\log C(\pi_0) = \log\Big[\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}\Big] - \frac{\lambda_0}{n} + \log \pi_0 - \log\Big[\alpha_1\mu_1 + \alpha_2\mu_2\Big], \tag{D.7}$$

$$\Rightarrow \frac{\partial \log C(\pi_0)}{\partial \pi_0} = \frac{1}{n}\left[\frac{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2\beta^n e^{\frac{\lambda_2}{n}}}{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}}\right]\frac{d\lambda_1}{d\pi_0} - \frac{1}{n}\frac{d\lambda_0}{d\pi_0} + \frac{1}{\pi_0}, \tag{D.8}$$

$$= \frac{1}{n}\Big[u + (1-u)\beta^n\Big]\frac{d\lambda_1}{d\pi_0} - \frac{1}{n}\frac{d\lambda_0}{d\pi_0} + \frac{1}{\pi_0}, \tag{D.9}$$

where $u = \frac{\alpha_1}{\alpha_1 + \alpha_2 \exp[\frac{\lambda_1(\beta^n - 1)}{n}]}$.

Recall that:

$$\lambda_0 = \lambda_1\left(\frac{\pi_0}{\mu_1}\right)^n, \tag{D.10}$$

$$\Rightarrow \frac{d\lambda_0}{d\pi_0} = \frac{d\lambda_1}{d\pi_0}\left(\frac{\pi_0}{\mu_1}\right)^n + \frac{\lambda_1 n}{\pi_0}\left(\frac{\pi_0}{\mu_1}\right)^n. \tag{D.11}$$

Substitute Equation D.11 into Equation D.9 to get:

$$\frac{\partial \log C(\pi_0)}{\partial \pi_0} = \frac{1}{n}\left\{u + (1-u)\beta^n - [u + (1-u)\beta]^n\right\}\frac{d\lambda_1}{d\pi_0} - \frac{\lambda_0}{\pi_0} + \frac{1}{\pi_0}, \tag{D.12}$$

because $\pi_0 = u\mu_1 + (1-u)\mu_2$ by Equation 3.54.

According to Theorem B.2.2,

$$\frac{d\pi_0}{d\lambda_1} = (\mu_2 - \mu_1)u(1-u)\left(\frac{\beta^n - 1}{n}\right), \tag{D.13}$$

$$\Rightarrow 1 = (\mu_2 - \mu_1)u(1-u)\left(\frac{\beta^n - 1}{n}\right)\frac{d\lambda_1}{d\pi_0}, \tag{D.14}$$

$$\Rightarrow \frac{1}{\pi_0} = \frac{(\mu_2 - \mu_1)u(1-u)(\beta^n - 1)}{n\pi_0}\frac{d\lambda_1}{d\pi_0}. \tag{D.15}$$

Thus, Equation D.12 becomes,

$$\frac{\partial \log C(\pi_0)}{\partial \pi_0} = \frac{1}{n\mu_1^n \pi_0} \frac{d\lambda_1}{d\pi_0}$$
$$\left\{[u\mu_1^n + (1-u)\mu_2^n]\pi_0 - [u\mu_1 + (1-u)\mu_2]^{n+1} + (\mu_2 - \mu_1)u(1-u)(\mu_2^n - \mu_1^n)\right\}$$
$$- \frac{\lambda_0}{\pi_0}, \tag{D.16}$$
$$= \frac{1}{n\mu_1^n \pi_0} \left\{ \underbrace{\left[u\mu_1^{n+1} + (1-u)\mu_2^{n+1}\right] - \left[u\mu_1 + (1-u)\mu_2\right]^{n+1}}_{\text{A}} \right\} \frac{d\lambda_1}{d\pi_0} - \frac{\lambda_0}{\pi_0}. \tag{D.17}$$

We are able to prove that

$$\begin{cases} A > 0 & \text{if} \quad n > 0 \text{ or } n < -1; \\ A = 0 & \text{if} \quad n = 0 \text{ or } n = -1 \text{ where } n = 0 \text{ is the case of iso-elastic demand;} \\ A < 0 & \text{if} \quad -1 < n < 0, \end{cases} \tag{D.18}$$

using **Jensen's inequality**:

- For a real convex function g, numbers x_1, x_2 in its domain, and positive weights t, then:

$$tg(x_1) + (1-t)g(x_2) \geq g(tx_1 + (1-t)x_2). \tag{D.19}$$

- For a real concave function g, numbers x_1, x_2 in its domain, and positive weights t, then:

$$tg(x_1) + (1-t)g(x_2) \leq g(tx_1 + (1-t)x_2). \tag{D.20}$$

In our case,

- the function: $g = \mu^{n+1}$;
- $x_1 : \mu_1, x_2 : \mu_2$;
- $t : u$.

Note:

$$\frac{d^2 g(\mu)}{d\mu^2} = n(n+1)\mu^{n-1}, \tag{D.21}$$

$$\Rightarrow \begin{cases} g(\mu) \text{ is a convex function of } \mu & \text{if} \quad n \geq 0 \text{ or } n \leq -1; \\ g(\mu) \text{ is a concave function of } \mu & \text{if} \quad -1 < n < 0. \end{cases} \tag{D.22}$$

Therefore, in Equation D.17,

$$\begin{cases} \frac{A}{n\mu_1^n \pi_0} \frac{d\lambda_1}{d\pi_0} > 0 & \text{if} \quad n > -1; \\ \frac{A}{n\mu_1^n \pi_0} \frac{d\lambda_1}{d\pi_0} \leq 0 & \text{if} \quad n \leq -1, \end{cases} \tag{D.23}$$

because $\frac{\partial \pi_0}{\partial \lambda_1} > 0$ proved in Theorem B.2.2.

As a result, when $n \leq -1$, in Equation D.17, $\frac{\partial \log C(\pi_0)}{\partial \pi_0} \leq 0 \Leftrightarrow \frac{\partial C(\pi_0)}{\partial \pi_0} \leq 0$ for any valid π_0. This result shows that in the case of general negative-exponential demand function, when the *"the second order elasticity"* is smaller than -1, loss coverage at pooled equilibrium premium is always smaller than the loss coverage at risk-differentiated premiums.

□

Theorem D.2.2.

$$\lim_{\pi_0 \to \alpha_1\mu_1 + \alpha_2\mu_2} C(\pi_0) > 1, \ \ \textit{when } n > -1. \tag{D.24}$$

Proof. Based on the analysis in Theorem D.2.1, we have the following comments: When $n > -1$, whether loss coverage at pooled equilibrium premium is higher or lower than the loss coverage at risk-differentiated premiums depends on the size of $\frac{A}{n\mu_1^n \pi_0}\frac{d\lambda_1}{d\pi_0}$ and $\frac{\lambda_0}{\pi_0}$ in Equation D.17.

We have proved in Result 3.21 in Section 3.4.1 that $\frac{\partial \pi_0}{\partial \lambda_1} > 0$ for all $n \in R$. Therefore,

$$\lim_{\pi_0 \to \alpha_1\mu_1+\alpha_2\mu_2} \frac{\partial \log C(\pi_0)}{\partial \pi_0} = \lim_{\lambda_1 \to 0} \frac{\partial \log C(\pi_0)}{\partial \lambda_1}. \tag{D.25}$$

Using Equation 5.34, we have

$$\frac{\partial}{\partial \lambda_1}\log C = \frac{1}{n}\frac{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}\beta^n}{\alpha_1 e^{\frac{\lambda_1}{n}} + \alpha_2 e^{\frac{\lambda_2}{n}}} - \frac{1}{n}\frac{\partial \lambda_0}{\partial \lambda_1} + \frac{\partial}{\partial \lambda_1}\log \pi_0, \tag{D.26}$$

$$= \frac{1}{n}\left[u + (1-u)\beta^n\right] - \frac{1}{n}\left(\frac{\pi_0}{\mu_1}\right)^n - \lambda_0 \frac{\partial}{\partial \lambda_1}\log \pi_0 + \frac{\partial}{\partial \lambda_1}\log \pi_0, \tag{D.27}$$

$$= \frac{1}{n}\left[\{u + (1-u)\beta^n\} - \{u + (1-u)\beta\}^n\right] + (1-\lambda_0)\frac{\partial}{\partial \lambda_1}\log \pi_0, \tag{D.28}$$

where

$$u = \frac{\alpha_1}{\alpha_1 + \alpha_2 e^{\frac{\lambda_1(\beta^n - 1)}{n}}} \tag{D.29}$$

defined in Equation 3.55.

And

$$\lambda_1 \to 0 \Rightarrow \lambda_2 \to 0,\ u \to \alpha_1,\ \pi_0 \to \alpha_1\mu_1 + \alpha_2\mu_2,\ \lambda_0 = \lambda_1\pi_0/\mu_1 \to 0. \tag{D.30}$$

$$\lim_{\lambda_1\to 0}\frac{\partial u}{\partial \lambda_1}=\lim_{\lambda_1\to 0}u\frac{\partial}{\partial \lambda_1}\log u=\alpha_1\alpha_2\left(\frac{1-\beta^n}{n}\right),\ \text{by Equation B.40,} \tag{D.31}$$

$$\lim_{\lambda_1\to 0}\frac{\partial}{\partial \lambda_1}\log \pi_0=\lim_{\lambda_1\to 0}\frac{1}{\pi_0}\frac{\partial \pi_0}{\partial \lambda_1}=\lim_{\lambda_1\to 0}\frac{1}{\pi_0}(\mu_2-\mu_1)\frac{\partial u}{\partial \lambda_1}=\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2\beta}\left(\frac{\beta^n-1}{n}\right)(\beta-1), \tag{D.32}$$

$$\lim_{\lambda_1\to 0}\frac{\partial}{\partial \lambda_1}\log C=\lim_{\lambda_1\to 0}\left\{\frac{1}{n}\left[\{u+(1-u)\beta^n\}-\{u+(1-u)\beta\}^n\right]+(1-\lambda_0)\frac{\partial}{\partial \lambda_1}\log \pi_0\right\}, \tag{D.33}$$

$$=\frac{1}{n}\left[(\alpha_1+\alpha_2\beta^n)-(\alpha_1+\alpha_2\beta)^n+\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2\beta}(\beta^n-1)(\beta-1)\right], \tag{D.34}$$

$$=\frac{1}{n(\alpha_1+\alpha_2\beta)}\left[\left(\alpha_1+\alpha_2\beta^{n+1}\right)-(\alpha_1+\alpha_2\beta)^{n+1}\right], \tag{D.35}$$

$$\geq 0,\ \text{as } n>-1, \tag{D.36}$$

using similar proof as in Equations D.18, i.e. Lyapunov's inequality. That is,

$$\left(\alpha_1+\alpha_2\beta^{n+1}\right)-(\alpha_1+\alpha_2\beta)^{n+1}\begin{cases}>0 & \text{when } n>0;\\ <0 & \text{when } -1<n<0.\end{cases} \tag{D.37}$$

Hence proved. $\square$

Theorem D.2.3. *Given the "second-order elasticity" $m>n>0$, at a given equilibrium premium, π_0, the loss coverage ratio $C_m(\pi_0)>C_n(\pi_0)$.*

Proof. To analyse this situation, consider two possible negative exponential demand elasticity curves with different curvatures m and n, where $m>n>$

0. So extending the notations used in Equation 3.50, we have:

$$\epsilon_{(m)}(\pi) = k_m\,\pi^m \Rightarrow \frac{\lambda_{1m}}{\mu_1^m} = \frac{\lambda_{2m}}{\mu_2^m} = \frac{\lambda_{0m}}{\pi_{0m}^m} = k_m; \tag{D.38}$$

$$\epsilon_{(n)}(\pi) = k_n\,\pi^n \Rightarrow \frac{\lambda_{1n}}{\mu_1^n} = \frac{\lambda_{2n}}{\mu_2^n} = \frac{\lambda_{0n}}{\pi_{0n}^n} = k_n. \tag{D.39}$$

Now, suppose that the equilibrium premiums under both demand elasticities are the same, i.e. $\pi_{0m} = \pi_{0n}$, and so the weights u in Equation 3.55 have to be the same, i.e.

$$\lambda_{1m}\frac{\beta^m - 1}{m} = \lambda_{1n}\frac{\beta^n - 1}{n} \tag{D.40}$$

Using Equation 5.34, the loss coverage ratios under the two scenarios are then:

$$C_m = \frac{\alpha_1 e^{\frac{\lambda_{1m}}{m}} + \alpha_2 e^{\frac{\lambda_{2m}}{m}}}{\alpha_1\mu_1 + \alpha_2\mu_2}\, e^{-\frac{\lambda_{0m}}{m}}\, \pi_{0m}, \tag{D.41}$$

$$C_n = \frac{\alpha_1 e^{\frac{\lambda_{1n}}{n}} + \alpha_2 e^{\frac{\lambda_{2n}}{n}}}{\alpha_1\mu_1 + \alpha_2\mu_2}\, e^{-\frac{\lambda_{0n}}{n}}\, \pi_{0n}. \tag{D.42}$$

Taking the ratio, gives us:

$$\frac{C_m}{C_n} = \frac{\alpha_1 e^{\frac{\lambda_{1m}}{m}} + \alpha_2 e^{\frac{\lambda_{2m}}{m}}}{\alpha_1 e^{\frac{\lambda_{1n}}{n}} + \alpha_2 e^{\frac{\lambda_{2n}}{n}}} \exp\left[-\frac{\lambda_{0m}}{m} + \frac{\lambda_{0n}}{n}\right], \tag{D.43}$$

$$= \frac{\alpha_1 + \alpha_2 e^{\lambda_{1m}\frac{\beta^m - 1}{m}}}{\alpha_1 + \alpha_2 e^{\lambda_{1n}\frac{\beta^n - 1}{n}}} \exp\left[-\frac{\lambda_{0m} - \lambda_{1m}}{m} + \frac{\lambda_{0n} - \lambda_{1n}}{n}\right], \tag{D.44}$$

$$= \exp\left[-\frac{\lambda_{0m} - \lambda_{1m}}{m} + \frac{\lambda_{0n} - \lambda_{1n}}{n}\right], \quad \text{by Equation D.40.} \tag{D.45}$$

So,

$$
\begin{aligned}
& C_m \gtreqless C_n && \text{(D.46)}\\
\Leftrightarrow & \frac{\lambda_{0n}-\lambda_{1n}}{n} \gtreqless \frac{\lambda_{0m}-\lambda_{1m}}{m}, && \text{(D.47)}\\
\Leftrightarrow & \frac{\lambda_{1n}}{n}\left[\left(\frac{\pi_0}{\mu_1}\right)^n-1\right] \gtreqless \frac{\lambda_{1m}}{m}\left[\left(\frac{\pi_0}{\mu_1}\right)^m-1\right], \text{ by Equations D.38 and D.39,} && \text{(D.48)}\\
\Leftrightarrow & \left(\beta^m-1\right)\left[\left(\frac{\pi_0}{\mu_1}\right)^n-1\right] \gtreqless \left(\beta^n-1\right)\left[\left(\frac{\pi_0}{\mu_1}\right)^m-1\right], \text{ by Equation D.40,} && \text{(D.49)}\\
\Leftrightarrow & \frac{\beta^m-1}{\phi^m-1} \gtreqless \frac{\beta^n-1}{\phi^n-1}, \text{ by denoting } \phi=\frac{\pi_0}{\mu_1}. && \text{(D.50)}
\end{aligned}
$$

Note that $1 \leq \phi \leq \beta$ as $\mu_1 \leq \pi_0 \leq \mu_2$.

If we can now prove that the following is an increasing function of x, we would have proved $C_m > C_n$ for $m > n$:

$$Z(x) = \frac{\beta^x-1}{\phi^x-1}; \tag{D.51}$$

$$\text{or, equivalently: } \log Z(x) = \log\left(\beta^x-1\right) - \log\left(\phi^x-1\right). \tag{D.52}$$

Now:

$$\frac{\partial}{\partial x}\log Z(x) = \frac{\beta^x\log\beta}{\beta^x-1} - \frac{\phi^x\log\phi}{\phi^x-1} > 0, \tag{D.53}$$

because,

$$g(y) = \frac{y^x\log y}{y^x-1} \tag{D.54}$$

is an increasing function of y (and recalling $\beta > \phi$). This can be proved as follows:

$$\frac{\partial}{\partial y}g(y) = \frac{y^{x-1}\left(y^x-1-\log y^x\right)}{\left(y^x-1\right)^2} > 0, \text{ as } \log a < a-1 \text{ for } a > 0. \tag{D.55}$$

Hence we have proved that for a given equilibrium premium, $C_m > C_n$ for $m > n$. $\square$

Theorem D.2.4. *For any $n > 0$, the loss coverage ratio is maximised for:*

$$\lambda_0^* = 1 + \frac{[\{u + (1-u)\beta^n\} - \{u + (1-u)\beta\}^n]}{n\frac{\partial}{\partial\lambda_1}\log\pi_0}, \tag{D.56}$$

where

$$u = \frac{\alpha_1}{\alpha_1 + \alpha_2 e^{\frac{\lambda_1(\beta^n - 1)}{n}}} \tag{D.57}$$

defined in Equation 3.55. And

1. *when $0 < n < 1$, $\lambda_0^* < 1$;*

2. *when $n = 1$, $\lambda_0^* = 1$; and*

3. *when $n > 1$, $\lambda_0^* > 1$.*

Proof. This theorem follows directly from Equation D.28 in Theorem D.2.2, by setting $\frac{\partial}{\partial\lambda_1}\log C = 0$.

In particular,

1. when $0 < n < 1$, $\lambda_0^* < 1$, because $\{u + (1-u)\beta^n\} < \{u + (1-u)\beta\}^n$, using Jensen's inequality in Equation D.20 and noting $\frac{\partial}{\partial\lambda_1}\log\pi_0 > 0$;

2. when $n = 1$, $\lambda_0^* = 1$ as noted previously;

3. when $n > 1$, $\lambda_0^* > 1$, because $\{u + (1-u)\beta^n\} > \{u + (1-u)\beta\}^n$, using Jensen's inequality in Equation D.19 and noting $\frac{\partial}{\partial\lambda_1}\log\pi_0 > 0$.

$\square$

Appendix E

Social Welfare: Iso-elastic Demand

E.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by iso-elastic demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks.
- p_1, p_2 are the population proportions.
- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \left(\frac{\mu_i}{\pi} \right)^{\lambda_i}. \tag{E.1}$$

  Note: $\pi \geq 0$ *is an implicit assumption.*

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \Rightarrow \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i. \tag{E.2}$$

  Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

- Social welfare under a specific risk-classification scheme is approximated as:

$$\hat{S}(\pi_1, \pi_2) = LC(\pi_1, \pi_2) - PA(\pi_1, \pi_2) + E[1 - X], \text{ by Equation 6.17} \tag{E.3}$$

  where

$$LC(\pi_1, \pi_2) = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i; \tag{E.4}$$

$$PA(\pi_1, \pi_2) = \sum_{i=1}^{2} \frac{\lambda_i}{\lambda_i + 1} p_i \tau_i \mu_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}; \tag{E.5}$$

$$E[1 - X] = 1 - \sum_{i=1}^{2} p_i \mu_i; \tag{E.6}$$

  and (π_1, π_2) satisfy the equilibrium condition in Equation E.2.

  Equation E.3 can then be simplified as

$$\hat{S}(\pi_1, \pi_2) = \sum_{i=1}^{2} \frac{1}{\lambda_i + 1} p_i \tau_i \mu_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} + 1 - \sum_{i=1}^{2} p_i \mu_i. \tag{E.7}$$

E.2 Theorems and Proofs

Theorem E.2.1. *For the case of equal demand elasticities, i.e.* $\lambda_1 = \lambda_2 = \lambda$, *we consider the Lagrangian function:*

$$\begin{aligned} W(\pi_1, \pi_2, \Lambda) = & \frac{1}{\lambda+1} \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i + 1 - \sum_{i=1}^{2} p_i \mu_i \\ & + \Lambda \left(\sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \pi_i - \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i \right), \end{aligned} \tag{E.8}$$

i.e. social welfare is set as the objective function with the equilibrium condition as a constraint. This formulation leads to an extremum solution $(\pi_1^*, \pi_2^*) = (\mu_1, \mu_2)$.

Proof.

$$\frac{\partial W}{\partial \pi_1} = 0 \Rightarrow \pi_1 = \mu_1 \Big[\frac{\lambda}{1-\lambda}\Big] \Big[\frac{1-(\lambda+1)\Lambda}{(\lambda+1)\Lambda}\Big], \tag{E.9}$$

$$\frac{\partial W}{\partial \pi_2} = 0 \Rightarrow \pi_2 = \mu_2 \Big[\frac{\lambda}{1-\lambda}\Big] \Big[\frac{1-(\lambda+1)\Lambda}{(\lambda+1)\Lambda}\Big], \tag{E.10}$$

$$\frac{\partial W}{\partial \Lambda} = 0 \Rightarrow \Lambda = \frac{\lambda}{1+\lambda} \Rightarrow \pi_1 = \mu_1,\ \pi_2 = \mu_2. \tag{E.11}$$

So $(\pi_1^*, \pi_2^*) = (\mu_1, \mu_2)$ provides an extremum. □

Theorem E.2.2. *Given a risk classification* (π_1, π_2), *when* $0 < \lambda_1 = \lambda_2 = \lambda < 1$, $\frac{\partial \pi_2}{\partial \pi_1} < 0$, *i.e.* π_1 *and* π_2 *have a monotonic relationship. This means increasing the premium for low risks required a decrease in the premium for high risks, and vice versa, to ensure that the equilibrium condition in Equation E.2 is satisfied.*

Proof. Equation E.2 gives:

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1\tau_1\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}[\lambda + (1-\lambda)\frac{\pi_1}{\mu_1}]}{\frac{p_2\tau_2\mu_2^{\lambda+1}}{\pi_2^{\lambda+1}}[\lambda + (1-\lambda)\frac{\pi_2}{\mu_2}]} < 0, \tag{E.12}$$

when $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$.

Note that this result holds if we look at $\frac{\partial \pi_1}{\partial \pi_2}$, i.e. for $0 < \lambda < 1$, π_2 is a well-defined function of π_1 and π_1 is also a well-defined function of π_2.

□

Theorem E.2.3. *If $0 < \lambda_1 = \lambda_2 = \lambda < 1$ and we assume the premium for low risks cannot be higher than the pooled premium (and correspondingly the premium for high risks cannot be lower than the pooled premium) i.e. $\mu_1 \leq \pi_1 \leq \pi_0 \leq \pi_2 \leq \mu_2$, then social welfare $\hat{S}(\underline{\pi})$ is maximised at the pooled equilibrium premium π_0.*

Proof. When $\lambda_1 = \lambda_2 = \lambda$, Equation E.7 becomes

$$\hat{S}(\pi_1, \pi_2) = \frac{1}{\lambda+1}\sum_{i=1}^{2} p_i\tau_i\mu_i\left(\frac{\mu_i}{\pi_i}\right)^{\lambda} + 1 - \sum_{i=1}^{2} p_i\mu_i, \tag{E.13}$$

$$= \frac{1}{\lambda+1}LC(\pi_1, \pi_2) + 1 - \sum_{i=1}^{2} p_i\mu_i, \tag{E.14}$$

where $LC(\pi_1, \pi_2)$ is the corresponding loss coverage at equal demand elasticity.

Therefore,

$$\frac{\partial \hat{S}(\pi_1, \pi_2)}{\partial \pi_1} = \frac{1}{\lambda+1}\frac{\partial LC(\pi_1, \pi_2)}{\partial \pi_1}, \tag{E.15}$$

$$> 0, \tag{E.16}$$

because we have already proved $\frac{\partial LC(\pi_1,\pi_2)}{\partial \pi_1} > 0$ in Theorem F.1.3.

This implies that $\hat{S}(\pi_1, \pi_2)$ is an increasing function of π_1 (and decreasing function of π_2 by Theorem E.2.2).

In Result 5.3, we have proved that loss coverage at pooled premium under *no risk classification* is higher than or equal to loss coverage at risk-differentiated premiums under *full risk classification* when $\lambda < 1$. And the following relationship is also proved, i.e.

$$\hat{S}(\underline{\pi}_1) \geq \hat{S}(\underline{\pi}_2) \Leftrightarrow LC(\underline{\pi}_1) \geq LC(\underline{\pi}_2) \tag{E.17}$$

from Equation 6.28. Therefore, both measures, "loss coverage" and "social welfare", point to the same conclusion that pooling provides greater or equal social efficacy of insurance compared to risk-differentiated premiums when $\lambda < 1$; and vice versa.

Therefore, social welfare is maximised when $\pi_1 = \pi_2 = \pi_0$ (within the restriction that $\pi_1 \leq \pi_2$), i.e. the case of pooled equilibrium premium. □

Theorem E.2.4. *If $\lambda_1 = \lambda_2 = \lambda > 1$, then social welfare is maximised at the risk-differentiated premiums (μ_1, μ_2).*

Proof. In this case, we know that $\hat{S}(\mu_1, \mu_2) > \hat{S}(\pi_0, \pi_0)$ because $C(\mu_1, \mu_2) \geq C(\pi_0, \pi_0)$ when $\lambda > 1$ (Result 5.3) and hence the extremum $\hat{S}(\mu_1, \mu_2)$ is a maximum. So among all risk classification schemes, the *full risk classification* gives the maximum social welfare (ratio) and hence there is no meed to explore *partial risk classification* further in this case. □

Theorem E.2.5. *When the low risks and the high risks have different demand elasticities, i.e. $\lambda_1 \neq \lambda_2$, under no risk classification (with pooled equilibrium premium π_0) and full risk classification (with risk-differentiated* *premiums $\underline{\mu} = (\mu_1, \mu_2)$), we have the following relationship between social welfare ratio SW and loss coverage ratio C:*

$$\widehat{SW}(\pi_0) \gtreqless 1 \Leftrightarrow \left(\frac{\lambda_2 - \lambda_1}{1+\lambda_1}\right)\left[1 - \left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1}\right] \lesseqgtr (C(\pi_0) - 1)\left(\frac{\alpha_1 + \alpha_2\beta}{\alpha_1}\right), \tag{E.18}$$

where

$$\widehat{SW}(\pi_0) = \frac{\hat{S}(\underline{\pi_0})}{\hat{s}(\underline{\mu})}; \tag{E.19}$$

$$C(\pi_0) = \frac{LC(\pi_0)}{LC(\underline{\mu})} \text{ as defined in Equation 5.4 }; \tag{E.20}$$

$$\alpha_i = \frac{p_i\tau_i}{p_1\tau_1 + p_2\tau_2} \text{ for } i = 1, 2, \text{ and } \beta = \frac{\mu_2}{\mu_1}. \tag{E.21}$$

Proof. Based on Equation E.7, under *no risk classification* and *full risk classification*:

$$\widehat{SW}(\pi_0) = \frac{\hat{S}(\pi_0)}{\hat{S}(\underline{\mu})} = \frac{\sum_{i=1}^{2} \frac{1}{1+\lambda_i} d_i(\pi_0) p_i \mu_i + K}{\sum_{i=1}^{2} \frac{1}{1+\lambda_i} d_i(\mu_i) p_i \mu_i + K}, \tag{E.22}$$

$$\Rightarrow \widehat{SW}(\pi_0) \gtreqless 1 \Leftrightarrow \frac{p_1\tau_1\mu_1}{1+\lambda_1}\left[\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} - 1\right] + \frac{p_2\tau_2\mu_2}{1+\lambda_2}\left[\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} - 1\right] \gtreqless 0; \tag{E.23}$$

$$\Leftrightarrow \left(\frac{1+\lambda_2}{1+\lambda_1}\right)\alpha_1\left[\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} - 1\right] + \alpha_2\beta\left[\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} - 1\right] \gtreqless 0. \tag{E.24}$$

Using the definition of loss coverage ratio in Equation 5.15:

$$C(\pi_0) = \frac{LC(\pi_0)}{LC(\underline{\mu})} = \frac{\alpha_1\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} + \alpha_2\beta\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2}}{\alpha_1 + \alpha_2\beta}, \tag{E.25}$$

$$\Rightarrow \alpha_2\beta\left[\left(\frac{\mu_2}{\pi_0}\right)^{\lambda_2} - 1\right] = (C(\pi_0) - 1)(\alpha_1 + \alpha_2\beta) - \alpha_1\left[\left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1} - 1\right]. \tag{E.26}$$

Therefore, Equation E.24 becomes:

$$\widehat{SW}(\pi_0) \gtreqless 1 \Leftrightarrow \left(\frac{\lambda_2 - \lambda_1}{1+\lambda_1}\right) \underbrace{\left[1 - \left(\frac{\mu_1}{\pi_0}\right)^{\lambda_1}\right]}_{\geq 0} \lesseqgtr (C(\pi_0) - 1)\left(\frac{\alpha_1 + \alpha_2 \beta}{\alpha_1}\right). \tag{E.27}$$

Therefore,

- when $\lambda_2 < \lambda_1$, left hand side of the inequality E.27 is never greater than zero. Thus if $C(\pi_0) \geq 1$, $\widehat{SW}(\pi_0) \geq 1$;
- when $\lambda_2 > \lambda_1$, left hand side of the inequality E.27 is never smaller than zero. Thus if $C(\pi_0) \leq 1$, $\widehat{SW}(\pi_0) \leq 1$;

□

Theorem E.2.6. *For the case of different demand elasticities, the Lagrangian function takes the following form:*

$$\begin{aligned} W(\pi_1, \pi_2, \Lambda) = &\sum_{i=1}^{2} \frac{1}{\lambda_i + 1} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i + 1 - \sum_{i=1}^{2} p_i \mu_i \\ &+ \Lambda \left(\sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i - \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i\right), \end{aligned} \tag{E.28}$$

provided $\pi_i \geq 0$ *for* $i = 1, 2$.

Case 1: *When* $0 < \lambda_1, \lambda_2 < 1$, *there is a single extremum solution* $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ *where* $\min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < 1$.

Case 2: *When* $\lambda_1, \lambda_2 > 1$, *there is a single extremum solution* $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ *where* $\min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2})$.

Proof.

$$\frac{\partial W}{\partial \pi_1} = 0 \Rightarrow \pi_1 = \mu_1 \left(\frac{\lambda_1}{1-\lambda_1}\right)\left[\frac{1-(1+\lambda_1)\Lambda}{(1+\lambda_1)\Lambda}\right], \tag{E.29}$$

$$\frac{\partial W}{\partial \pi_2} = 0 \Rightarrow \pi_2 = \mu_2 \left(\frac{\lambda_2}{1-\lambda_2}\right)\left[\frac{1-(1+\lambda_2)\Lambda}{(1+\lambda_2)\Lambda}\right], \tag{E.30}$$

$$\begin{aligned}\frac{\partial W}{\partial \Lambda} = 0 \Rightarrow f(\Lambda) = p_1\tau_1\mu_1 &\left(\frac{1-\lambda_1}{\lambda_1}\right)^{\lambda_1}\left[\frac{(1+\lambda_1)\Lambda}{1-(1+\lambda_1)\Lambda}\right]^{\lambda_1}\left[\frac{\lambda_1-(1+\lambda_1)\Lambda}{(1-\lambda_1^2)\Lambda}\right] \\ + p_2\tau_2\mu_2 &\left(\frac{1-\lambda_2}{\lambda_2}\right)^{\lambda_2}\left[\frac{(1+\lambda_2)\Lambda}{1-(1+\lambda_2)\Lambda}\right]^{\lambda_2}\left[\frac{\lambda_2-(1+\lambda_2)\Lambda}{(1-\lambda_2^2)\Lambda}\right] = 0.\end{aligned} \tag{E.31}$$

Case 1: When $0 < \lambda_1, \lambda_2 < 1$,

$$\pi_1 \geq 0 \Rightarrow 0 < \Lambda \leq \frac{1}{1+\lambda_1}, \tag{E.32}$$

$$\pi_2 \geq 0 \Rightarrow 0 < \Lambda \leq \frac{1}{1+\lambda_2}. \tag{E.33}$$

For $0 < \Lambda \leq \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$,

$$f(\Lambda) > 0, \text{ for } 0 < \Lambda \leq \min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}), \tag{E.34}$$

$$f(\Lambda) < 0, \text{ for } 0 < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) \leq \Lambda \leq \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}), \tag{E.35}$$

$$f(\Lambda) \text{ is decreasing over } 0 < \Lambda \leq \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}), \tag{E.36}$$

which is proved in Theorem E.2.15.

Hence there is exactly one root, Λ^* where $0 < \min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}) < 1$ for which $f(\Lambda^*) = 0$. The risk classification scheme (π_1^*, π_2^*), which corresponds to Λ^*, gives the extremum social welfare (which can be either a minimum or a maximum).

Case 2: When $\lambda_1, \lambda_2 > 1$,

$$\pi_1 \geq 0 \Rightarrow \Lambda \geq \frac{1}{1+\lambda_1}, \tag{E.37}$$

$$\pi_2 \geq 0 \Rightarrow \Lambda \geq \frac{1}{1+\lambda_2}. \tag{E.38}$$

For $\Lambda \geq \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$,

$$f(\Lambda) < 0 \text{ for } \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}) < \Lambda < \min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}), \tag{E.39}$$

$$f(\Lambda) > 0 \text{ for } \lambda > \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}), \tag{E.40}$$

$$f(\lambda) \text{ is increasing over } \Lambda \geq \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}), \tag{E.41}$$

which is proved in Theorem E.2.16.

Hence there is exactly one root, Λ^* where $\min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2})$, for which $f(\Lambda^*) = 0$. The risk classification scheme (π_1^*, π_2^*), which corresponds to Λ^*, gives the extremum social welfare (which can be either a minimum or a maximum).

□

Theorem E.2.7. *For $0 < \lambda_1, \lambda_2 < 1$, $\hat{S}(\pi_1^*, \pi_2^*)$ is a minimum of social welfare for $\pi_1 \geq 0, \pi_2 \geq 0$.*

Proof. For the case of different demand elasticities,

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1\tau_1\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}]}{\frac{p_2\tau_2\mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]} < 0, \tag{E.42}$$

for $0 < \lambda_1, \lambda_2 < 1$.

$$\hat{S}(\pi_1, \pi_2) = \frac{p_1\tau_1}{1+\lambda_1}\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1}} + \frac{p_2\tau_2}{1+\lambda_2}\frac{\mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2}} + 1 - (p_1\mu_1 + p_2\mu_2), \tag{E.43}$$

$$\Rightarrow \frac{\partial}{\partial \pi_1}\hat{S}(\pi_1, \pi_2) = -\frac{\lambda_1}{1+\lambda_1}p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}} - \frac{\lambda_2}{1+\lambda_2}p_2\tau_2\frac{\mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}\left(\frac{\partial \pi_2}{\partial \pi_1}\right), \tag{E.44}$$

$$= \frac{-\lambda_1}{1+\lambda_1}p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}} + \frac{\lambda_2}{1+\lambda_2}p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}\left(\frac{\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}}{\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}}\right), \tag{E.45}$$

using Equation E.42,

$$= p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}} \left[\frac{\lambda_2(1+\lambda_1)[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}] - \lambda_1(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]}{(1+\lambda_1)(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]}\right], \tag{E.46}$$

$$\propto A(\pi_1, \pi_2)\frac{B(\pi_1, \pi_2)}{C(\pi_1, \pi_2)}, \tag{E.47}$$

where

$$A(\pi_1, \pi_2) = \pi_1^{-(\lambda_1+1)}, \tag{E.48}$$

$$B(\pi_1, \pi_2) = \lambda_2(1+\lambda_1)[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}] - \lambda_1(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}], \text{ and} \tag{E.49}$$

$$C(\pi_1, \pi_2) = (1+\lambda_1)(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}] > 0 \text{ for } \lambda_2 < 1. \tag{E.50}$$

Therefore,

$$\frac{\partial}{\partial \pi_1}\hat{S}(\pi_1,\pi_2) \gtreqless 0 \Leftrightarrow B(\pi_1,\pi_2) \gtreqless 0, \tag{E.51}$$

$$\Leftrightarrow \lambda_2(1+\lambda_1)[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}] \gtreqless \lambda_1(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]. \tag{E.52}$$

Note:

$$B(\pi_1^*,\pi_2^*) = 0, \text{ i.e. } \lambda_2(1+\lambda_1)[\lambda_1 + (1-\lambda_1)\frac{\pi_1^*}{\mu_1}] = \lambda_1(1+\lambda_2)[\lambda_2 + (1-\lambda_2)\frac{\pi_2^*}{\mu_2}], \tag{E.53}$$

where π_1^*, π_2^* is the extremum solution to the Lagrangian function in Theorem E.2.6.

We have proved that social welfare at (π_1^*, π_2^*) is an extremum, we can then check the second derivative of social welfare with respect to π_1 at (π_1^*, π_2^*) to find out whether the extremum is a maximum or minimum. *Note that π_1 and π_2 have a monotonic relationship (based on Equation E.42) when $0 < \lambda_1, \lambda_2 < 1$, and $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$.*

Using Equation E.47, we can get

$$\frac{\partial^2}{\partial \pi_1^2}\hat{S}(\pi_1,\pi_2) \propto \frac{\partial A}{\partial \pi_1}\frac{B}{C} + A\frac{\frac{\partial B}{\partial \pi_1}C - B\frac{\partial C}{\partial \pi_1}}{C^2}. \tag{E.54}$$

Then,

$$\frac{\partial^2}{\partial \pi_1^2}\hat{S}(\pi_1,\pi_2)|_{\pi_1^*,\pi_2^*} \propto \frac{\partial B}{\partial \pi_1}|_{\pi_1^*,\pi_2^*}, \tag{E.55}$$

because $B(\pi_1^*,\pi_2^*) = 0$ from Equation E.53 and $A(\pi_1^*,\pi_2^*) > 0, C(\pi_1^*,\pi_2^*) > 0$.

Note:

$$\frac{\partial B}{\partial \pi_1}|_{\pi_1^*,\pi_2^*} = \frac{\lambda_2(1-\lambda_1^2)}{\mu_1} - \frac{\lambda_1(1-\lambda_2^2)}{\mu_2}\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*,\pi_2^*} > 0, \tag{E.56}$$

because $\frac{\partial \pi_2}{\partial \pi_1} < 0$ for $0 < \lambda_1, \lambda_2 < 1$, and $\pi_1, \pi_2 > 0$.

Therefore,

$$\frac{\partial^2}{\partial \pi_1^2} \hat{S}(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} > 0, \tag{E.57}$$

i.e. $\hat{S}(\pi_1^*, \pi_2^*)$ is a minimum and either pooled premium under *no risk classification* or risk-differentiated premiums under *full risk classification* maximises social welfare when $0 < \lambda_1, \lambda_2 < 1$.

Note that the above analysis is still valid if we start by looking at how π_1 varies with π_2. This is because, for $0 < \lambda_1, \lambda_2 < 1$, based on Equation E.42, π_2 is a well defined function of π_1 and π_1 is also a well defined function of π_2, as long as π_1, π_2 satisfy the equilibrium premium condition in Equation E.2.

□

Theorem E.2.8. *When* $0 < \lambda_1 < \lambda_2 < 1$*, no risk classification maximises social welfare, i.e.* $\hat{S}(\pi_0, \pi_0)$ *is the maximum.*

Proof. According to Theorem E.2.6, when $0 < \lambda_1, \lambda_2 < 1$, there is a Λ^* where $\min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < 1$, such that $\hat{S}(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ is an extremum. And Theorem E.2.7 shows that this extremum is actually a minimum. We now decide whether *full* or *no risk classification* maximises social welfare by comparing the position of $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ to (μ_1, μ_2).
Using Equation E.29 and E.30, at Λ^*,

$$\pi_1^* \gtreqless \mu_1 \Leftrightarrow \Lambda^* \lesseqgtr \frac{\lambda_1}{1+\lambda_1}, \tag{E.58}$$

$$\pi_2^* \gtreqless \mu_2 \Leftrightarrow \Lambda^* \lesseqgtr \frac{\lambda_2}{1+\lambda_2}, \text{ and} \tag{E.59}$$

$$\pi_1^* \gtreqless \pi_2^* \Leftrightarrow \frac{\lambda_1[1-(1+\lambda_1)\Lambda^*]}{\lambda_2[1-(1+\lambda_2)\Lambda^*]} \left(\frac{1-\lambda_2^2}{1-\lambda_1^2}\right) \gtreqless \frac{\mu_2}{\mu_1}. \tag{E.60}$$

In this case, when $0 < \lambda_1 < \lambda_2 < 1$,

$$\frac{\lambda_1}{1+\lambda_1} < \Lambda^* < \frac{\lambda_2}{1+\lambda_2} \Rightarrow \pi_1^* < \mu_1 \text{ and } \pi_2^* > \mu_2. \tag{E.61}$$

Hence, social welfare is a monotonically increasing function of π_1 for $\mu_1 < \pi_1 < \pi_0$, because $\frac{\partial \pi_2}{\partial \pi_1} < 0$ by Equation E.42 and $\hat{S}(\pi_1^*, \pi_2^*)$ is a minimum by Theorem E.2.7.

Therefore,

$$\hat{S}(\pi_1^*, \pi_2^*) < \hat{S}(\mu_1, \mu_2) < \hat{S}(\pi_0, \pi_0), \tag{E.62}$$

i.e. *no risk classification* maximises social welfare.

□

Theorem E.2.9. *When $0 < \lambda_2 < \lambda_1 < 1$, either full or no risk classification maximises social welfare.*

Proof. In this case, according to Theorem E.2.6, there is a Λ^* such that $\frac{\lambda_2}{1+\lambda_2} < \Lambda^* < \frac{\lambda_1}{1+\lambda_1}$. And as a result, $\pi_1^* > \mu_1$ and $\pi_2^* < \mu_2$ (using Equation E.58 and E.59). We need to further check the relationship between π_1^* and π_2^*.

- When $\frac{\lambda_1[1-(1+\lambda_1)\Lambda^*]}{\lambda_2[1-(1+\lambda_2)\Lambda^*]}\left(\frac{1-\lambda^2}{1-\lambda_1^2}\right) < \frac{\mu_2}{\mu_1}$, i.e. $\pi_1^* \leq \pi_2^*$ according to Equation E.60, because π_1 and π_2 have a monotonic relationship as proved in Equation E.42, we can deduce the following relationship:

$$\mu_1 < \pi_1^* \leq \pi_0 \leq \pi_2^* < \mu_2. \tag{E.63}$$

  And because $\hat{S}(\pi_1^*, \pi_2^*)$ is a minimum by Theorem E.2.7, we reach the following result:

$$\hat{S}(\pi_1^*, \pi_2^*) \leq \min[\hat{S}(\mu_1, \mu_2), \hat{S}(\pi_0, \pi_0)], \tag{E.64}$$

i.e. whether it is *full* or *no risk classification* maximises social welfare depends on their relative value.

- When $\frac{\lambda_1[1-(1+\lambda_1)\Lambda^*]}{\lambda_2[1-(1+\lambda_2)\Lambda^*]}\left(\frac{1-\lambda^2}{1-\lambda_1^2}\right) > \frac{\mu_2}{\mu_1}$, i.e. $\pi_1^* > \pi_2^*$ according to Equation E.60, and because π_1 and π_2 have a monotonic relationship as proved in Equation E.42, we can deduce the following relationship:

$$\mu_1 < \pi_0 < \pi_1^*, \text{ and} \tag{E.65}$$

$$\pi_2^* < \pi_0 < \mu_2. \tag{E.66}$$

  Because $\hat{S}(\pi_1^*, \pi_2^*)$ is a minimum by Theorem E.2.7, hence, social welfare is a monotonically decreasing function of π_1 for $\mu_1 \leq \pi_1 \leq \pi_0$. This means

$$\hat{S}(\mu_1, \mu_2) > \hat{S}(\pi_0, \pi_0), \tag{E.67}$$

  i.e. in this case, *full risk classification* maximises social welfare.

All in all, when $0 < \lambda_2 < \lambda_1 < 1$, either *full* or *no risk classification* maximises social welfare.

□

Theorem E.2.10. *For* $\lambda_1 > 1$, $\frac{\partial \pi_2}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \pi_1 \gtreqless \mu_1 \frac{\lambda_1}{\lambda_1 - 1}$. *This result shows that whether premiums for high and low risks have a monotonic relationship depends on the value of premium for low risks,* π_1*.*

Proof. Recall that for the case of different demand elasticities,

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1 \tau_1 \mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}]}{\frac{p_2 \tau_2 \mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]}, \tag{E.68}$$

and

$$\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1} \gtreqless 0 \Leftrightarrow \pi_1 \lesseqgtr \mu_1 \frac{\lambda_1}{\lambda_1 - 1} \text{ for } \lambda_1 > 1, \tag{E.69}$$

$$\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} > 0 \text{ for } 0 < \lambda_2 < 1, \text{ and} \tag{E.70}$$

$$\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} \gtreqless 0 \Leftrightarrow \pi_2 \lesseqgtr \mu_2 \frac{\lambda_2}{\lambda_2 - 1} \text{ for } \lambda_2 > 1. \tag{E.71}$$

(*Note:* $\mu_2 \frac{\lambda_2}{\lambda_2 - 1} > \mu_2$ *when* $\lambda_2 > 1$.)

Hence, for $0 < \pi_2 \leq \mu_2$ (or $0 < \pi_2 < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$), $\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} > 0$, i.e. π_2 is a well-defined function of π_1 for $\mu_1 \leq \pi_1 \leq \mu_2$. As a result,

$$\frac{\partial \pi_2}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \pi_1 \gtreqless \mu_1 \frac{\lambda_1}{\lambda_1 - 1} \text{ for } \lambda_1 > 1. \tag{E.72}$$

Therefore,

$$\frac{\partial \pi_2}{\partial \pi_1} < 0 \tag{E.73}$$

as long as $0 < \pi_1 < \mu_1 \frac{\lambda_1}{\lambda_1 - 1}$ and $0 < \pi_2 < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$. □

Theorem E.2.11. *For* $\lambda_1, \lambda_2 > 1$, $\hat{S}(\pi_1^*, \pi_2^*)$ *is a maximum of social welfare for* $\pi_1, \pi_2 \geq 0$.

Proof. We aim to use the same approach as the one we used to prove Theorem E.2.7, i.e. find out the second derivative of social welfare with respect to π_1 at (π_1^*, π_2^*). But we need to check first whether π_1 and π_2 have a monotonic relationship at (π_1^*, π_2^*).

According to Theorem E.2.6, when $\lambda_1, \lambda_2 > 1$, there is a Λ^* where $\min(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2}) < \Lambda^* < \max(\frac{\lambda_1}{1+\lambda_1}, \frac{\lambda_2}{1+\lambda_2})$, and the corresponding $\hat{S}(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$

is an extremum. In this case,

$$\pi_1^* = \mu_1 \left(\frac{\lambda_1}{\lambda_1 - 1} \right) \left[\frac{(\lambda_1 + 1)\Lambda^* - 1}{(1 + \lambda_1)\Lambda^*} \right] < \mu_1 \frac{\lambda_1}{\lambda_1 - 1}, \tag{E.74}$$

$$\pi_2^* = \mu_2 \left(\frac{\lambda_2}{\lambda_2 - 1} \right) \left[\frac{(\lambda_2 + 1)\Lambda^* - 1}{(1 + \lambda_2)\Lambda^*} \right] < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}. \tag{E.75}$$

Hence,

$$\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} < 0, \text{ by Theorem E.2.10} \tag{E.76}$$

i.e. π_1 and π_2 still hold a monotonic relationship before reaching (π_1^*, π_2^*). Hence we can use the same approach as the one in Theorem E.2.7.

Recall that

$$\frac{\partial^2}{\partial \pi_1^2} \hat{S}(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} \propto \frac{\partial B}{\partial \pi_1}|_{\pi_1^*, \pi_2^*}, \text{ from Equation E.55} \tag{E.77}$$

where

$$B(\pi_1, \pi_2) = \lambda_2(1 + \lambda_1)[\lambda_1 + (1 - \lambda_1)\frac{\pi_1}{\mu_1}] - \lambda_1(1 + \lambda_2)[\lambda_2 + (1 - \lambda_2)\frac{\pi_2}{\mu_2}]. \tag{E.78}$$

And,

$$\frac{\partial B}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} = \frac{\lambda_2(1 - \lambda_1^2)}{\mu_1} - \frac{\lambda_1(1 - \lambda_2^2)}{\mu_2} \frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} < 0, \tag{E.79}$$

because $\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} < 0$ for $\lambda_1, \lambda_2 > 1$, and $\pi_1, \pi_2 > 0$.

Therefore,

$$\frac{\partial^2}{\partial \pi_1^2} \hat{S}(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} < 0, \tag{E.80}$$

i.e. $\hat{S}(\pi_1^*, \pi_2^*)$ is a maximum and *partial risk classification* could maximise social welfare when $\lambda_1, \lambda_2 > 1$. $\square$

Theorem E.2.12. *When $1 < \lambda_2 < \lambda_1$, full risk classification maximises social welfare for $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$.*

Proof. Using Equation E.29 and E.30, at Λ^* we get

$$\pi_1^* \gtreqless \mu_1 \Leftrightarrow \Lambda^* \gtreqless \frac{\lambda_1}{1+\lambda_1}, \text{ for } \lambda_1 > 1 \tag{E.81}$$

$$\pi_2^* \gtreqless \mu_2 \Leftrightarrow \Lambda^* \gtreqless \frac{\lambda_2}{1+\lambda_2}, \text{ for } \lambda_2 > 1, \text{ and} \tag{E.82}$$

$$\pi_1^* \gtreqless \pi_2^* \Leftrightarrow \frac{\lambda_1}{\lambda_2}\left[\frac{(1+\lambda_1)\Lambda^*-1}{(1+\lambda_2)\Lambda^*-1}\right]\left(\frac{\lambda_2^2-1}{\lambda_1^2-1}\right) \gtreqless \frac{\mu_2}{\mu_1} \tag{E.83}$$

Recalling from Theorem E.2.6, when $1 < \lambda_2 < \lambda_1$, there is a $\frac{\lambda_2}{1+\lambda_2} < \Lambda^* < \frac{\lambda_1}{1+\lambda_1}$, and as a result,

$$\pi_1^* < \mu_1 \text{ and } \pi_2^* > \mu_2. \tag{E.84}$$

Because $S(\pi_1^*, \pi_2^*)$ is a maximum by Theorem E.2.11, $\pi_2^* < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$ and $\frac{\partial \pi_2}{\partial \pi_1} < 0$, according to Equation E.73, social welfare is a monotonically decreasing function of π_1 for $\mu_1 \leq \pi_1$.

Note that in this case, there might be multiple equilibria[1].

Therefore,

- when there is a unique equilibrium premium, π_0,

  $$\hat{S}(\pi_1^*, \pi_2^*) > \hat{S}(\mu_1, \mu_2) > \hat{S}(\pi_0, \pi_0). \tag{E.85}$$

  If we focus on the case when $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$ (in which case, (π_1^*, π_2^*) is out of interest), then *full risk classification* maximises social welfare.

- When there are multiple equilibrium premiums, π_{0i} with $i \in Z, i \leq 3$ (because there could be at most 3 equilibrium premiums), if $\pi_{01} <$

[1] Results on multiple equilibria can be found in Section 3.6.

$\pi_{02} < \pi_{03}$, then

$$\hat{S}(\pi_1^*, \pi_2^*) > \hat{S}(\mu_1, \mu_2) > \hat{S}(\pi_{01}, \pi_{01}) > \hat{S}(\pi_{02}, \pi_{02}) > \hat{S}(\pi_{03}, \pi_{03}). \tag{E.86}$$

In this case, *full risk classification* still maximises social welfare.

$\square$

Theorem E.2.13. *When* $1 < \lambda_1 < \lambda_2$ *and* $\frac{\lambda_1}{\lambda_2}\left[\frac{(1+\lambda_1)\Lambda^*-1}{(1+\lambda_2)\Lambda^*-1}\right]\left(\frac{\lambda_2^2-1}{\lambda_1^2-1}\right) > \frac{\mu_2}{\mu_1}$ *(with* $\frac{\lambda_1}{1+\lambda_1} < \Lambda^* < \frac{\lambda_2}{1+\lambda_2}$*), no risk classification maximises social welfare.*

Proof. In this case, $\pi_1^* > \pi_2^*$ by Equation E.83. Because of Equation E.73, we can deduce the following result:

$$\mu_1 < \pi_0 < \pi_1^*, \text{ and} \tag{E.87}$$

$$\pi_2^* < \pi_0 < \mu_2. \tag{E.88}$$

Because $\hat{S}(\pi_1^*, \pi_2^*)$ is a maximum by Theorem E.2.11, social welfare is a monotonically increasing function of π_1 for $\mu_1 \le \pi_1 \le \pi_0$.

Therefore,

$$\hat{S}(\pi_0, \pi_0) > \hat{S}(\mu_1, \mu_2), \tag{E.89}$$

i.e. in this case, *no risk classification* maximises social welfare. $\square$

Theorem E.2.14. *When* $1 < \lambda_1 < \lambda_2$ *and* $\frac{\lambda_1}{\lambda_2}\left[\frac{(1+\lambda_1)\Lambda^*-1}{(1+\lambda_2)\Lambda^*-1}\right]\left(\frac{\lambda_2^2-1}{\lambda_1^2-1}\right) < \frac{\mu_2}{\mu_1}$ *(with* $\frac{\lambda_1}{1+\lambda_1} < \Lambda^* < \frac{\lambda_2}{1+\lambda_2}$*), partial risk classification maximises social welfare.*

Proof. In this case, $\pi_1^* \le \pi_2^*$ by Equation E.83. And we have also proved that π_1 and π_2 are still monotonically related when the extremum is reached (by Equation E.76).

We can deduce the following relationship:

$$\mu_1 < \pi_1^* \leq \pi_0 \leq \pi_2^* < \mu_2. \tag{E.90}$$

Hence,

$$\hat{S}(\pi_1^*, \pi_2^*) \geq \max[\hat{S}(\mu_1, \mu_2), \hat{S}(\pi_0, \pi_0)], \tag{E.91}$$

because $\hat{S}(\pi_1^*, \pi_2^*)$ is a maximum by Theorem E.2.11.

Therefore, *partial risk classification* with $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ maximises social welfare. $\square$

Theorem E.2.15. *When $0 < \lambda_1, \lambda_2 < 1$, the function $f(\Lambda) = \frac{\partial W}{\partial \Lambda}$ is a decreasing function of Λ where $0 < \Lambda < \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$. (W is the Lagrangian function in Theorem E.2.6.)*

Proof. Recall from Equation E.31,

$$\begin{aligned} f(\Lambda) = {} & p_1\tau_1\mu_1 \left(\frac{1-\lambda_1}{\lambda_1}\right)^{\lambda_1} \left[\frac{(1+\lambda_1)\Lambda}{1-(1+\lambda_1)\Lambda}\right]^{\lambda_1} \left[\frac{\lambda_1 - (1+\lambda_1)\Lambda}{(1-\lambda_1^2)\Lambda}\right] \\ & + p_2\tau_2\mu_2 \left(\frac{1-\lambda_2}{\lambda_2}\right)^{\lambda_2} \left[\frac{(1+\lambda_2)\Lambda}{1-(1+\lambda_2)\Lambda}\right]^{\lambda_2} \left[\frac{\lambda_2 - (1+\lambda_2)\Lambda}{(1-\lambda_2^2)\Lambda}\right], \end{aligned} \tag{E.92}$$

which can be re-written as

$$f(\Lambda) = \frac{p_1\tau_1\mu_1}{1-\lambda_1^2} \left(\frac{1-\lambda_1^2}{\lambda_1}\right)^{\lambda_1} D_1 + \frac{p_2\tau_2\mu_2}{1-\lambda_2^2} \left(\frac{1-\lambda_2^2}{\lambda_2}\right)^{\lambda_2} E_1, \text{ where} \tag{E.93}$$

$$D_1 = \left[\frac{\lambda_1 - (1+\lambda_1)\Lambda}{\Lambda}\right]\left[\frac{\Lambda}{1-(1+\lambda_1)\Lambda}\right]^{\lambda_1}, \tag{E.94}$$

$$E_1 = \left[\frac{\lambda_2 - (1+\lambda_2)\Lambda}{\Lambda}\right]\left[\frac{\Lambda}{1-(1+\lambda_2)\Lambda}\right]^{\lambda_2}. \tag{E.95}$$

Hence,

$$\frac{\partial}{\partial \Lambda} f(\Lambda) = \frac{p_1 \tau_1 \mu_1}{1-\lambda_1^2} \left(\frac{1-\lambda_1^2}{\lambda_1} \right)^{\lambda_1} \frac{\partial D_1}{\partial \Lambda} + \frac{p_2 \tau_2 \mu_2}{1-\lambda_2^2} \left(\frac{1-\lambda_2^2}{\lambda_2} \right)^{\lambda_2} \frac{\partial E_1}{\partial \Lambda}. \tag{E.96}$$

Based on Equations E.94 and E.95, we have

$$\frac{\partial D_1}{\partial \Lambda} = \left[\frac{\Lambda}{1-(1+\lambda_1)\Lambda} \right]^{\lambda_1} \frac{\lambda_1}{\Lambda^2} \left[\frac{\lambda_1 - 1}{[1-(1+\lambda_1)\Lambda]} \right] < 0, \tag{E.97}$$

$$\text{when } 0 < \lambda_1 < 1, \tag{E.98}$$

and $\Lambda < \frac{1}{1+\lambda_1}$ (to ensure $\pi_1 > 0$).

Similarly,

$$\frac{\partial E_1}{\partial \Lambda} = \left[\frac{\Lambda}{1-(1+\lambda_2)\Lambda} \right]^{\lambda_2} \frac{\lambda_2}{\Lambda^2} \left[\frac{\lambda_2 - 1}{[1-(1+\lambda_2)\Lambda]} \right] < 0, \tag{E.99}$$

$$\text{when } 0 < \lambda_2 < 1, \tag{E.100}$$

and $\Lambda < \frac{1}{1+\lambda_2}$ (to ensure $\pi_2 > 0$).

Therefore,

$$\frac{\partial}{\partial \Lambda} f(\Lambda) < 0, \tag{E.101}$$

i.e. $f(\Lambda)$ is a decreasing function of Λ for $0 < \Lambda < \min(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$. □

Theorem E.2.16. *When $\lambda_1, \lambda_2 > 1$, the function $f(\Lambda) = \frac{\partial W}{\partial \Lambda}$ is an increasing function of Λ when $\Lambda \geq \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$. ($W$ is the Lagrangian function in Theorem E.2.6.)*

Proof. From Theorem E.2.15,

$$\frac{\partial}{\partial\Lambda}f(\Lambda) = \frac{p_1\tau_1\mu_1}{1-\lambda_1^2}\left(\frac{1-\lambda_1^2}{\lambda_1}\right)^{\lambda_1}\frac{\partial D_1}{\partial\Lambda} + \frac{p_2\tau_2\mu_2}{1-\lambda_2^2}\left(\frac{1-\lambda_2^2}{\lambda_2}\right)^{\lambda_2}\frac{\partial E_1}{\partial\Lambda}, \tag{E.102}$$

$$\begin{aligned}&= p_1\tau_1\mu_1\Big[\frac{\Lambda(1-\lambda_1^2)}{\lambda_1[1-(1+\lambda_1)\Lambda]}\Big]^{\lambda_1}\frac{\lambda_1}{\Lambda^2}\Big[\frac{(1+\lambda_1)(\lambda_1-1)^2}{(1+\lambda_1)\Lambda-1}\Big]\\ &+ p_2\tau_2\mu_2\Big[\frac{\Lambda(1-\lambda_2^2)}{\lambda_2[1-(1+\lambda_2)\Lambda]}\Big]^{\lambda_2}\frac{\lambda_2}{\Lambda^2}\Big[\frac{(1+\lambda_2)(\lambda_2-1)^2}{(1+\lambda_2)\Lambda-1}\Big],\end{aligned} \tag{E.103}$$

$$> 0 \text{ when } \lambda_1, \lambda_2 > 1, \text{ and } \Lambda \geq \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2}). \tag{E.104}$$

Hence, $f(\Lambda)$ is an increasing function of Λ when $\Lambda \geq \max(\frac{1}{1+\lambda_1}, \frac{1}{1+\lambda_2})$.

$\square$

Appendix F

Partial Risk Classification on Loss Coverage

F.1 Notations and Proofs for Two Risk-groups

F.1.1 Notations and Assumptions

We assume that there are two risk-groups and demand for insurance is driven by iso-elastic demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2$ are the underlying risks for the low risk-group and the high risk-group.
- p_1, p_2 are the population proportions such that $p_1 + p_2 = 1$.
- The proportional demand for insurance for risk-group $i = 1, 2$ at premium π is given by:

$$d_i(\pi) = \tau_i \left(\frac{\mu_i}{\pi} \right)^{\lambda_i}. \tag{F.1}$$

  Note: $\pi \geq 0$ *is an implicit assumption.*

- Equilibrium is achieved when the following condition is satisfied:

$$\sum_{i=1}^{2} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{2} p_i d_i(\pi_i)\mu_i \Rightarrow \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i. \tag{F.2}$$

  Each solution (π_1, π_2) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2) = (\mu_1, \mu_2)$ represents *full risk classification* and $(\pi_1, \pi_2) = (\pi_0, \pi_0)$ represents the pooled equilibrium premium under *no risk classification*.

- Loss coverage under a specific risk-classification scheme is defined as:

$$LC(\pi_1, \pi_2) = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i, \tag{F.3}$$

  where (π_1, π_2) satisfy the equilibrium condition in Equation F.2.

F.1.2 Theorems and Proofs

Theorem F.1.1. *For the particular case of equal demand elasticities, i.e. $\lambda_1 = \lambda_2 = \lambda$, we consider the Lagrangian function:*

$$W(\pi_1, \pi_2, \Lambda) = \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i + \Lambda \left(\sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \pi_i - \sum_{i=1}^{2} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i\right), \tag{F.4}$$

i.e. loss coverage is set as the objective function with the equilibrium condition as a constraint. This formulation leads to an extremum solution $(\pi_1^*, \pi_2^*) = (\mu_1, \mu_2)$.

Proof.

$$\frac{\partial W}{\partial \pi_1} = 0 \Rightarrow \pi_1 = \mu_1 \frac{\lambda(1-\Lambda)}{\Lambda(1-\lambda)}, \tag{F.5}$$

$$\frac{\partial W}{\partial \pi_2} = 0 \Rightarrow \pi_2 = \mu_2 \frac{\lambda(1-\Lambda)}{\Lambda(1-\lambda)}, \tag{F.6}$$

$$\frac{\partial W}{\partial \Lambda} = 0 \Rightarrow \Lambda = \lambda \Rightarrow \pi_1 = \mu_1,\ \pi_2 = \mu_2. \tag{F.7}$$

So $(\pi_1^*, \pi_2^*) = (\mu_1, \mu_2)$ provides an extremum. □

Theorem F.1.2. *For a given risk classification scheme* (π_1, π_2), *when* $0 < \lambda_1 = \lambda_2 = \lambda < 1$, $\frac{\partial \pi_2}{\partial \pi_1} < 0$, *i.e.* π_1 *and* π_2 *have a monotonic relationship. This means increasing the premium for low risks required a decrease in the premium for high risks, and vice versa, to ensure that the equilibrium condition in Equation F.2 is satisfied.*

Proof. Equation F.2 gives:

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1\tau_1\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}[\lambda + (1-\lambda)\frac{\pi_1}{\mu_1}]}{\frac{p_2\tau_2\mu_2^{\lambda+1}}{\pi_2^{\lambda+1}}[\lambda + (1-\lambda)\frac{\pi_2}{\mu_2}]} < 0, \tag{F.8}$$

when $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$.

Note that this result holds if we look at $\frac{\partial \pi_1}{\partial \pi_2}$, i.e. for $0 < \lambda < 1$, π_2 is a well-defined function of π_1 and π_1 is also a well-defined function of π_2.

□

Theorem F.1.3. *If* $0 < \lambda_1 = \lambda_2 = \lambda < 1$ *and we assume the premium for low risks cannot be higher than the pooled premium (and correspondingly the premium for high risks cannot be lower than the pooled premium) i.e.* $\mu_1 \leq \pi_1 \leq \pi_0 \leq \pi_2 \leq \mu_2$, *the pooled equilibrium premium* π_0 *maximises loss coverage.*

Proof.

$$LC(\pi_1,\pi_2) = p_1\tau_1\frac{\mu_1^{\lambda+1}}{\pi_1^{\lambda}} + p_2\tau_2\frac{\mu_2^{\lambda+1}}{\pi_2^{\lambda}}, \tag{F.9}$$

$$\Rightarrow \frac{\partial}{\partial\pi_1}LC(\pi_1,\pi_2) = -\lambda p_1\tau_1\frac{\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}} - \lambda p_2\tau_2\frac{\mu_2^{\lambda+1}}{\pi_2^{\lambda+1}}\left(\frac{\partial\pi_2}{\partial\pi_1}\right), \tag{F.10}$$

$$= -\lambda p_1\tau_1\frac{\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}} + \lambda p_1\tau_1\frac{\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}\left(\frac{\lambda+(1-\lambda)\frac{\pi_1}{\mu_1}}{\lambda+(1-\lambda)\frac{\pi_2}{\mu_2}}\right), \tag{F.11}$$

using Equation F.8,

$$= -\lambda p_1\tau_1\frac{\mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}\left[\frac{(1-\lambda)\left(\frac{\pi_2}{\mu_2}-\frac{\pi_1}{\mu_1}\right)}{\lambda+(1-\lambda)\frac{\pi_2}{\mu_2}}\right], \tag{F.12}$$

$$> 0, \text{ since } \frac{\pi_2}{\mu_2} < 1 < \frac{\pi_1}{\mu_1}. \tag{F.13}$$

This implies that $LC(\pi_1,\pi_2)$ is an increasing function of π_1 (and decreasing function of π_2 by Theorem F.1.2), and is maximised when $\pi_1 = \pi_2 = \pi_0$ (within the restriction that $\pi_1 \leq \pi_2$), i.e. the case of pooled equilibrium premium. □

Theorem F.1.4. *If $\lambda_1 = \lambda_2 = \lambda > 1$, risk-differentiated premiums (μ_1,μ_2) maximise loss coverage.*

Proof. In this case, we have proved in Result 5.3 that $LC(\mu_1,\mu_2) > LC(\pi_0,\pi_0)$, and hence the extremum $LC(\mu_1,\mu_2)$ is a maximum. So among all risk classification schemes, *full risk classification* maximises loss coverage and hence there is no need to explore *partial risk classification* further in this case. □

Theorem F.1.5. *For the case of different demand elasticities, provided $\pi_i \geq 0, i = 1,2,$ the Lagrangian function takes the following form:*

$$W(\pi_1,\pi_2,\Lambda) = \sum_{i=1}^{2} p_i\tau_i\left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}\mu_i + \Lambda\left(\sum_{i=1}^{2} p_i\tau_i\left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}\pi_i - \sum_{i=1}^{2} p_i\tau_i\left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i}\mu_i\right). \tag{F.14}$$

And we have the following results on the extreme solution:

Case 1: *When* $0 < \lambda_1, \lambda_2 < 1$, *there is a single extremum solution* $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ *where:* $0 < \min(\lambda_1, \lambda_2) < \Lambda^* < \max(\lambda_1, \lambda_2) < 1$.

Case 2: *When* $\lambda_1, \lambda_2 > 1$, *there is a single extremum solution* $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ *where:* $1 < \min(\lambda_1, \lambda_2) < \Lambda^* < \max(\lambda_1, \lambda_2)$.

Case 3: *When* $0 < \lambda_1 < 1 < \lambda_2$, *loss coverage is a monotonic function of* π_1.

Case 4: *When* $0 < \lambda_2 < 1 < \lambda_1$, *loss coverage is a monotonic function of* π_1.

Proof.

$$\frac{\partial W}{\partial \pi_1} = 0 \Rightarrow \pi_1 = \mu_1 \frac{\lambda_1(1-\Lambda)}{\Lambda(1-\lambda_1)}, \tag{F.15}$$

$$\frac{\partial W}{\partial \pi_2} = 0 \Rightarrow \pi_2 = \mu_2 \frac{\lambda_2(1-\Lambda)}{\Lambda(1-\lambda_2)}, \tag{F.16}$$

$$\frac{\partial W}{\partial \Lambda} = 0 \Rightarrow$$

$$f(\Lambda) = p_1\tau_1 \left[\frac{\Lambda(1-\lambda_1)}{\lambda_1(1-\Lambda)}\right]^{\lambda_1} \left[\frac{\lambda_1-\Lambda}{1-\lambda_1}\right] \mu_1 + p_2\tau_2 \left[\frac{\Lambda(1-\lambda_2)}{\lambda_2(1-\Lambda)}\right]^{\lambda_2} \left[\frac{\lambda_2-\Lambda}{1-\lambda_2}\right] \mu_2 = 0. \tag{F.17}$$

Case 1: When $0 < \lambda_1, \lambda_2 < 1$,

$$\pi_1 \geq 0 \Rightarrow 0 < \Lambda < 1, \tag{F.18}$$

$$\pi_2 \geq 0 \Rightarrow 0 < \Lambda < 1. \tag{F.19}$$

For $0 < \Lambda < 1$,

$$f(\Lambda) > 0, \text{ for } 0 < \Lambda < \min(\lambda_1, \lambda_2) < 1, \tag{F.20}$$

$$f(\Lambda) < 0, \text{ for } 0 < \max(\lambda_1, \lambda_2) < \Lambda < 1, \tag{F.21}$$

$$f(\Lambda) \text{ is concave over } 0 < \Lambda < 1 \text{ (This result is proved Theorem F.1.16)}. \tag{F.22}$$

Hence there is exactly one root, Λ^*, $0 < \min(\lambda_1, \lambda_2) < \Lambda^* < \max(\lambda_1, \lambda_2) < 1$ for which $f(\Lambda^*) = 0$. The risk classification scheme (π_1^*, π_2^*), which corresponds to Λ^*, gives the extremum loss coverage (which can either be a minimum or a maximum).

Case 2: When $\lambda_1, \lambda_2 > 1$,

$$\pi_1 \geq 0 \Rightarrow \Lambda > 1, \tag{F.23}$$

$$\pi_2 \geq 0 \Rightarrow \Lambda > 1. \tag{F.24}$$

For $\Lambda > 1$,

$$f(\Lambda) < 0, \text{ for } 1 < \Lambda < \min(\lambda_2, \lambda_2), \tag{F.25}$$

$$f(\Lambda) > 0, \text{ for } \Lambda > \max(\lambda_1, \lambda_2), \tag{F.26}$$

$$f(\Lambda) \text{ is concave over } \Lambda > 1 \text{ (This result is proved Theorem F.1.17)}. \tag{F.27}$$

Hence there is exactly one root, Λ^*, $1 < \min(\lambda_1, \lambda_2) < \Lambda^* < \max(\lambda_1, \lambda_2)$ for which $f(\Lambda^*) = 0$. The risk classification scheme (π_1^*, π_2^*), which corresponds to Λ^*, gives the extremum loss coverage (which can be either be a minimum or a maximum).

Case 3: When $0 < \lambda_1 < 1 < \lambda_2$,

$$\pi_1 \geq 0 \Rightarrow 0 < \Lambda < 1, \text{ because } 0 < \lambda_1 < 1, \tag{F.28}$$

$$\pi_2 \geq 0 \Rightarrow \Lambda > 1, \text{ because } \lambda_2 > 1. \tag{F.29}$$

Hence, there is no feasible solution to Equation F.17, which means loss coverage is a monotonic function of π_1.

Case 4: When $0 < \lambda_2 < 1 < \lambda_1$, similar to Case 3,

$$\pi_1 \geq 0 \Rightarrow \Lambda > 1, \text{ because } \lambda_1 > 1, \tag{F.30}$$

$$\pi_2 \geq 0 \Rightarrow 0 < \Lambda < 1, \text{ because } 0 < \lambda_2 < 1. \tag{F.31}$$

Hence, there is no feasible solution to Equation F.17, which means loss coverage is a monotonic function of π_1.

□

Theorem F.1.6. *For $0 < \lambda_1, \lambda_2 < 1$, $LC(\pi_1^*, \pi_2^*)$ is a minimum of loss coverage for $\pi_1 \geq 0, \pi_2 \geq 0$.*

Proof. For the case of different demand elasticities,

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1 \tau_1 \mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}]}{\frac{p_2 \tau_2 \mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]} < 0, \tag{F.32}$$

for $0 < \lambda_1, \lambda_2 < 1$.

$$LC(\pi_1,\pi_2) = p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1}} + p_2\tau_2\frac{\mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2}}, \tag{F.33}$$

$$\Rightarrow \frac{\partial}{\partial\pi_1}LC(\pi_1,\pi_2) = -\lambda_1 p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}} - \lambda_2 p_2\tau_2\frac{\mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}\left(\frac{\partial\pi_2}{\partial\pi_1}\right), \tag{F.34}$$

$$= -\lambda_1 p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}} + \lambda_2 p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}\left(\frac{\lambda_1+(1-\lambda_1)\frac{\pi_1}{\mu_1}}{\lambda_2+(1-\lambda_2)\frac{\pi_2}{\mu_2}}\right), \tag{F.35}$$

using Equation F.32,

$$= p_1\tau_1\frac{\mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}\left[\frac{\lambda_2(1-\lambda_1)\frac{\pi_1}{\mu_1} - \lambda_1(1-\lambda_2)\frac{\pi_2}{\mu_2}}{\lambda_2+(1-\lambda_2)\frac{\pi_2}{\mu_2}}\right], \tag{F.36}$$

$$\propto A(\pi_1,\pi_2)\frac{B(\pi_1,\pi_2)}{C(\pi_1,\pi_2)}, \tag{F.37}$$

where

$$A(\pi_1,\pi_2) = \pi_1^{-(\lambda_1+1)}, \tag{F.38}$$

$$B(\pi_1,\pi_2) = \lambda_2(1-\lambda_1)\frac{\pi_1}{\mu_1} - \lambda_1(1-\lambda_2)\frac{\pi_2}{\mu_2}, \text{ and} \tag{F.39}$$

$$C(\pi_1,\pi_2) = \lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}. \tag{F.40}$$

Therefore,

$$\frac{\partial}{\partial\pi_1}LC(\pi_1,\pi_2) \gtreqless 0 \Leftrightarrow \lambda_2(1-\lambda_1)\frac{\pi_1}{\mu_1} \gtreqless \lambda_1(1-\lambda_2)\frac{\pi_2}{\mu_2}, \tag{F.41}$$

$$\Leftrightarrow B(\pi_1,\pi_2) \gtreqless 0. \tag{F.42}$$

Note:

$$B(\pi_1^*,\pi_2^*) = 0, \text{ i.e. } \lambda_2(1-\lambda_1)\frac{\pi_1^*}{\mu_1} = \lambda_1(1-\lambda_2)\frac{\pi_2^*}{\mu_2}, \tag{F.43}$$

where π_1^*, π_2^* is the extremum solution to the Lagrangian function in Theorem

F.1.5.

We have proved that loss coverage at (π_1^*, π_2^*) is an extremum, we can then check the second derivative of loss coverage with respect to π_1 at (π_1^*, π_2^*) to find out whether the extremum is a maximum or minimum. *Note that* π_1 *and* π_2 *have a monotonic relationship (based on Equation F.32) when* $0 < \lambda_1, \lambda_2 < 1$*, and* $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$*.*

Using Equation F.37, we can get

$$\frac{\partial^2}{\partial \pi_1^2} LC(\pi_1, \pi_2) \propto \frac{\partial A}{\partial \pi_1} \frac{B}{C} + A \frac{\frac{\partial B}{\partial \pi_1} C - B \frac{\partial C}{\partial \pi_1}}{C^2}. \tag{F.44}$$

Then,

$$\frac{\partial^2}{\partial \pi_1^2} LC(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} \propto \frac{\partial B}{\partial \pi_1}|_{\pi_1^*, \pi_2^*}, \tag{F.45}$$

because $B(\pi_1^*, \pi_2^*) = 0$ from Equation F.43 and $A(\pi_1^*, \pi_2^*) > 0, C(\pi_1^*, \pi_2^*) > 0$.

Note:

$$\frac{\partial B}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} = \frac{\lambda_2(1-\lambda_1)}{\mu_1} - \frac{\lambda_1(1-\lambda_2)}{\mu_2} \frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} > 0, \tag{F.46}$$

because $\frac{\partial \pi_2}{\partial \pi_1} < 0$ for $0 < \lambda_1, \lambda_2 < 1$, and $\pi_1, \pi_2 > 0$.

Therefore,

$$\frac{\partial^2}{\partial \pi_1^2} LC(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} > 0, \tag{F.47}$$

i.e. $LC(\pi_1^*, \pi_2^*)$ is a minimum and *partial risk classification* does not maximise loss coverage when $0 < \lambda_1, \lambda_2 < 1$.

Note that the above analysis is still valid if we start by looking at how π_1 varies with π_2. This is because, for $0 < \lambda_1, \lambda_2 < 1$, based on Equation F.32, π_2 is a well defined function of π_1 and π_1 is also a well defined function of π_2, as long as π_1, π_2 satisfy the equilibrium premium condition in Equation E.2.

□

Theorem F.1.7. *When* $0 < \lambda_1 < \lambda_2 < 1$*, no risk classification maximises loss coverage, i.e.* $LC(\pi_0, \pi_0)$ *is the maximum.*

Proof. According to Theorem F.1.5, when $0 < \lambda_1 < \lambda_2 < 1$, there is a Λ^* where $0 < \lambda_1 < \Lambda^* < \lambda_2 < 1$, such that $LC(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ is an extremum. Moreover, Theorem F.1.6 shows that this extremum is actually a minimum. We can now decide whether the full or *no risk classification* maximises loss coverage by comparing the position of $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ to (μ_1, μ_2).

Using Equation F.15 and F.16, at Λ^*,

$$\pi_1^* \gtreqless \mu_1 \Leftrightarrow \lambda_1 \gtreqless \Lambda^*, \tag{F.48}$$

$$\pi_2^* \gtreqless \mu_2 \Leftrightarrow \lambda_2 \gtreqless \Lambda^*, \text{ and} \tag{F.49}$$

$$\pi_1^* \gtreqless \pi_2^* \Leftrightarrow \frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) \gtreqless \frac{\mu_2}{\mu_1}. \tag{F.50}$$

In this case,

$$\lambda_1 < \Lambda^* < \lambda_2 \Rightarrow \pi_1^* < \mu_1 \text{ and } \pi_2^* > \mu_2. \tag{F.51}$$

Hence, loss coverage is a monotonically increasing function of π_1 for $\mu_1 < \pi_1 < \pi_0$, because $\frac{\partial \pi_2}{\partial \pi_1} < 0$ by Equation F.32 and $LC(\pi_1^*, \pi_2^*)$ is a minimum by Theorem F.1.6.

Therefore,

$$LC(\pi_1^*, \pi_2^*) < LC(\mu_1, \mu_2) < LC(\pi_0, \pi_0), \tag{F.52}$$

i.e. *no risk classification* maximises loss coverage. $\square$

Theorem F.1.8. *When* $0 < \lambda_2 < \lambda_1 < 1$*, either full or no risk classification maximises loss coverage.*

Proof. In this case, according to Theorem F.1.5, there is a Λ^* such that $\lambda_2 < \Lambda^* < \lambda_1$. And as a result, $\pi_1^* > \mu_1$ and $\pi_2^* < \mu_2$ (using Equation F.48 and F.49). We need to further check the value of π_1^* and π_2^*:

- When $\frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) \leq \frac{\mu_2}{\mu_1}$, i.e. $\pi_1^* \leq \pi_2^*$ according to Equation F.50, because π_1 and π_2 have a monotonic relationship as proved in Equation F.32, we can deduce the following relationship:

$$\mu_1 < \pi_1^* \leq \pi_0 \leq \pi_2^* < \mu_2. \tag{F.53}$$

  And because $LC(\pi_1^*, \pi_2^*)$ is a minimum by Theorem F.1.6, hence

$$LC(\pi_1^*, \pi_2^*) \leq \min[LC(\mu_1, \mu_2), LC(\pi_0, \pi_0)], \tag{F.54}$$

  i.e. whether it is *full* or *no risk classification* maximises loss coverage depends on their relative value.

- When $\frac{\lambda_1}{\lambda_2}\left(\frac{1-\lambda_2}{1-\lambda_1}\right) > \frac{\mu_2}{\mu_1}$, i.e. $\pi_1^* > \pi_2^*$ according to Equation F.50, and because π_1 and π_2 have a monotonic relationship as proved in Equation F.32, we can deduce the following relationship:

$$\mu_1 < \pi_0 < \pi_1^*, \text{ and} \tag{F.55}$$

$$\pi_2^* < \pi_0 < \mu_2. \tag{F.56}$$

  And because $LC(\pi_1^*, \pi_2^*)$ is a minimum by Theorem F.1.6, therefore, loss coverage is a monotonically decreasing function of π_1 for $\mu_1 \leq \pi_1 \leq \pi_0$. This means

$$LC(\mu_1, \mu_2) > LC(\pi_0, \pi_0), \tag{F.57}$$

  i.e. in this case, *full risk classification* maximises loss coverage.

Therefore, when $0 < \lambda_2 < \lambda_1 < 1$, either *full* or *no risk classification* maximises loss coverage.

□

Theorem F.1.9. *For $\lambda_1 > 1$, $\frac{\partial \pi_2}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \pi_1 \gtreqless \mu_1 \frac{\lambda_1}{\lambda_1 - 1}$. This result shows that whether premiums for high and low risks have a monotonic relationship depends on the value of premium for low risks, π_1.*

Proof. Recall that for the case of different demand elasticities,

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{p_1 \tau_1 \mu_1^{\lambda_1+1}}{\pi_1^{\lambda_1+1}}[\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1}]}{\frac{p_2 \tau_2 \mu_2^{\lambda_2+1}}{\pi_2^{\lambda_2+1}}[\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}]}, \tag{F.58}$$

and

$$\lambda_1 + (1-\lambda_1)\frac{\pi_1}{\mu_1} \gtreqless 0 \Leftrightarrow \pi_1 \lesseqgtr \mu_1 \frac{\lambda_1}{\lambda_1 - 1} \text{ for } \lambda_1 > 1, \tag{F.59}$$

$$\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} > 0 \text{ for } 0 < \lambda_2 < 1, \text{ and} \tag{F.60}$$

$$\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} \gtreqless 0 \Leftrightarrow \pi_2 \lesseqgtr \mu_2 \frac{\lambda_2}{\lambda_2 - 1} \text{ for } \lambda_2 > 1. \tag{F.61}$$

(*Note:* $\mu_2 \frac{\lambda_2}{\lambda_2 - 1} > \mu_2$ *when* $\lambda_2 > 1$.)

Hence, for $0 < \pi_2 \le \mu_2$ (or $0 < \pi_2 < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$), $\lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2} > 0$, i.e. π_2 is a well-defined function of π_1 for $\mu_1 \le \pi_1 \le \mu_2$. As a result,

$$\frac{\partial \pi_2}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \pi_1 \gtreqless \mu_1 \frac{\lambda_1}{\lambda_1 - 1} \text{ for } \lambda_1 > 1. \tag{F.62}$$

Therefore,

$$\frac{\partial \pi_2}{\partial \pi_1} < 0 \tag{F.63}$$

as long as $0 < \pi_1 < \mu_1 \frac{\lambda_1}{\lambda_1 - 1}$ and $0 < \pi_2 < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$.

There are a few remarks which worth mentioning here:

Remark 1: For $\mu_1 \le \pi_1, \pi_2 \le \mu_2$ that satisfy the equilibrium condition, although π_2 is a well defined function of π_1, π_1 might not always be a well-defined function of π_2. This is because, if we take an inverse of Equation F.58, $\frac{\partial \pi_1}{\partial \pi_2}$ becomes undefined if $\pi_1 = \frac{\lambda_1}{\lambda_1 - 1}\mu_1$.

Remark 2: The relationship between π_1, π_2 still holds in the presence of multiple equilibria. (Note that multiple equilibria is possible in the case when $\lambda_1 > 1$.[1]) Figure F.1 is an example showing that the result in Theorem F.1.9 still holds when there are multiple equilibria (three equilibrium premiums $\pi_{01}, \pi_{02}, \pi_{03}$ in this case). In this example, we observe that given any π_1 within $[\mu_1, \mu_2]$, there is a π_2 within $[\mu_1, \mu_2]$ that satisfies the equilibrium condition as in Equation F.2.

Figure F.2 is another example showing how π_2 behaves with respect to π_1 in the case of multiple equilibria (in this case, π_{01} is very close to μ_1). One point to emphasize is that, in this example, for some values of π_1, there are corresponding π_2 which can solve the equilibrium condition as in Equation E.2. But these π_2 could be beyond our interest range, e.g. $0 < \pi_2 < \mu_1$ as shown in Figure F.2.

It is difficult to find out the exact conditions which lead to this scenario analytically at this stage apart from some numerical analysis.

□

Theorem F.1.10. *For $\lambda_1, \lambda_2 > 1$, $LC(\pi_1^*, \pi_2^*)$ is a maximum of loss coverage for $\pi_1, \pi_2 \geq 0$.*

Proof. We aim to use the same approach as the one we used to prove Theorem F.1.6, i.e. find out the second derivative of loss coverage with respect to π_1 at (π_1^*, π_2^*). But we need to check whether π_1 and π_2 have a monotonic relationship at (π_1^*, π_2^*) first.

[1]Detailed results on multiple equilibria for iso-elastic demand can be found in Section 3.6.

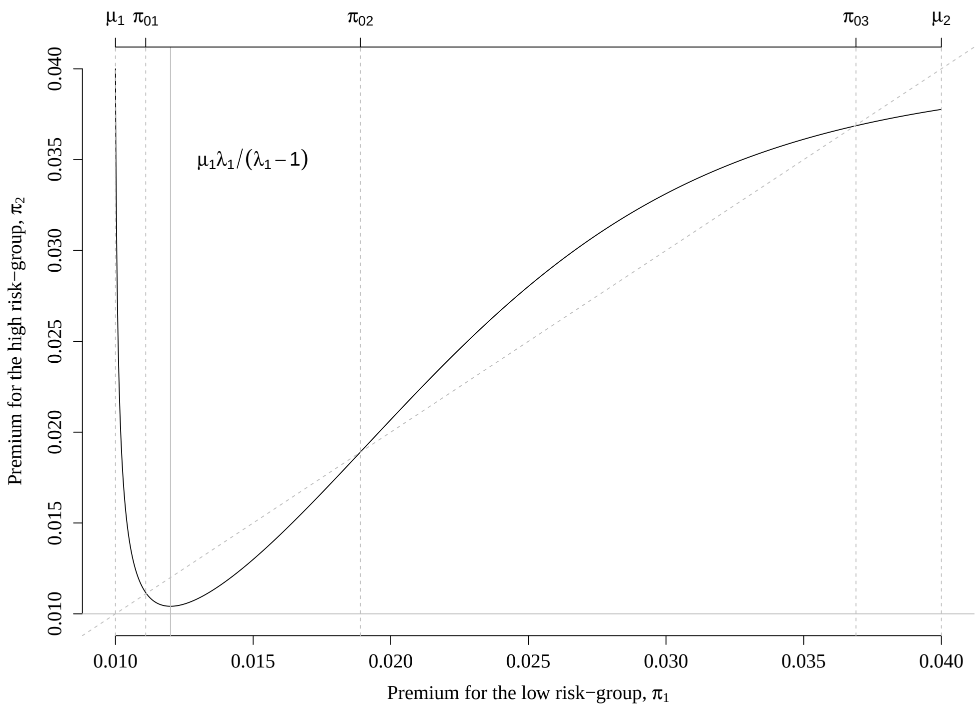

Figure F.1: Plot of premium for the high risk-group π_2 as a function of premium for the low risk-group π_1 in the case of multiple equilibria with $\alpha_1 = 99.7\%, \alpha_2 = 0.3\%, \mu_1 = 0.01, \mu_2 = 0.04$ and $\lambda_1 = 6, \lambda_2 = 1.5$.

According to Theorem F.1.5, when $\lambda_1, \lambda_2 > 1$, there is a Λ^* where $1 < \min(\lambda_1, \lambda_2) < \Lambda^* < \max(\lambda_1, \lambda_2)$, and the corresponding $LC(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ is an extremum. In this case,

$$\pi_1^* = \mu_1 \left(\frac{\lambda_1}{\lambda_1 - 1}\right)\left(\frac{\Lambda^* - 1}{\Lambda^*}\right) < \mu_1 \frac{\lambda_1}{\lambda_1 - 1}, \tag{F.64}$$

$$\pi_2^* = \mu_2 \left(\frac{\lambda_2}{\lambda_2 - 1}\right)\left(\frac{\Lambda^* - 1}{\Lambda^*}\right) < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}. \tag{F.65}$$

Hence,

$$\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*, \pi_2^*} < 0, \text{ by Theorem F.1.9} \tag{F.66}$$

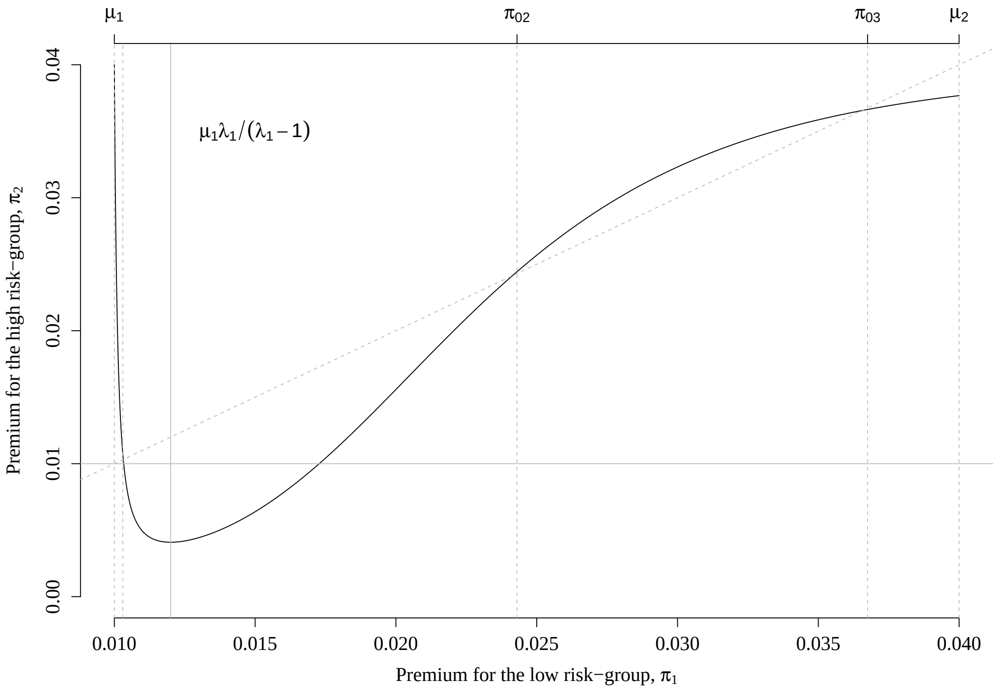

Figure F.2: Plot of premium for the high risk-group π_2 as a function of premium for the low risk-group π_1 in the case of multiple equilibria with $\alpha_1 = 99.7\%, \alpha_2 = 0.3\%, \mu_1 = 0.01, \mu_2 = 0.04$ and $\lambda_1 = 6, \lambda_2 = 0.8$.

i.e. even though π_1 and π_2 might not be monotonically related, this situation would not happen before the extremum is reached. Hence we can use the same approach as the one in Theorem F.1.6.

Recall that

$$\frac{\partial^2}{\partial \pi_1^2} LC(\pi_1, \pi_2)|_{\pi_1^*, \pi_2^*} \propto \frac{\partial B}{\partial \pi_1}|_{\pi_1^*, \pi_2^*}, \text{ from Equation F.45} \tag{F.67}$$

where

$$B(\pi_1, \pi_2) = \lambda_2(1 - \lambda_1)\frac{\pi_1}{\mu_1} - \lambda_1(1 - \lambda_2)\frac{\pi_2}{\mu_2}. \tag{F.68}$$

And,

$$\frac{\partial B}{\partial \pi_1}|_{\pi_1^*,\pi_2^*} = \frac{\lambda_2(1-\lambda_1)}{\mu_1} - \frac{\lambda_1(1-\lambda_2)}{\mu_2}\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*,\pi_2^*} < 0, \tag{F.69}$$

because $\frac{\partial \pi_2}{\partial \pi_1}|_{\pi_1^*,\pi_2^*} < 0$ for $\lambda_1, \lambda_2 > 1$, and $\pi_1, \pi_2 > 0$.

Therefore,

$$\frac{\partial^2}{\partial \pi_1^2} LC(\pi_1, \pi_2)|_{\pi_1^*,\pi_2^*} < 0, \tag{F.70}$$

i.e. $LC(\pi_1^*, \pi_2^*)$ is a maximum and *partial risk classification* could maximise loss coverage when $\lambda_1, \lambda_2 > 1$. $\square$

Theorem F.1.11. *When $1 < \lambda_2 < \lambda_1$, full risk classification maximises loss coverage for $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$.*

Proof. Using Equations F.15 and F.16, we get

$$\pi_1^* \gtreqless \mu_1 \Leftrightarrow \lambda_1 \lesseqgtr \Lambda^*, \text{ for } \lambda_1 > 1 \tag{F.71}$$

$$\pi_2^* \gtreqless \mu_2 \Leftrightarrow \lambda_2 \lesseqgtr \Lambda^*, \text{ for } \lambda_2 > 1, \text{ and} \tag{F.72}$$

$$\pi_1^* \gtreqless \pi_2^* \Leftrightarrow \frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2 - 1}{\lambda_1 - 1}\right) \gtreqless \frac{\mu_2}{\mu_1}. \tag{F.73}$$

Recalling from Theorem F.1.5, when $1 < \lambda_2 < \lambda_1$, there is a $\lambda_2 < \Lambda^* < \lambda_1$, and as a result,

$$\pi_1^* < \mu_1 \text{ and } \pi_2^* > \mu_2. \tag{F.74}$$

Hence loss coverage is a monotonically decreasing function of π_1 for $\mu_1 \leq \pi_1$ because $\pi_2^* < \mu_2 \frac{\lambda_2}{\lambda_2 - 1}$ and $\frac{\partial \pi_2}{\partial \pi_1} < 0$, according to Equation F.62 and $LC(\pi_1^*, \pi_2^*)$ is a maximum by Theorem F.1.10.

Note that in this case, there might be multiple equilibria[2].

Therefore,

[2]Results on multiple equilibria can be found in Section 3.6.

- when there is a unique equilibrium premium, π_0,

$$LC(\pi_1^*, \pi_2^*) > LC(\mu_1, \mu_2) > LC(\pi_0, \pi_0). \tag{F.75}$$

  If we focus on the case when $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$, then *full risk classification* maximises loss coverage.

- When there are multiple equilibrium premiums, π_{0i} with $i \in Z, i \leq 3$ (because there could be at most 3 equilibrium premium), if $\pi_{01} < \pi_{02} < \pi_{03}$, then

$$LC(\pi_1^*, \pi_2^*) > LC(\mu_1, \mu_2) > LC(\pi_{01}, \pi_{01}) > LC(\pi_{02}, \pi_{02}) > LC(\pi_{03}, \pi_{03}). \tag{F.76}$$

  Full risk classification still maximises loss coverage (ratio).

Figure F.3 is an example showing how loss coverage ratio behaves with respect to π_1 in the presence of multiple equilibria. Figure F.3 uses the same values of parameters as in Figure F.1 (i.e. in both figures, we have the same equilibrium premiums $\pi_{01}, \pi_{02}, \pi_{03}$).

In this example, although π_1, π_2 might not have a monotonic relationship, loss coverage ratio is a monotonically decreasing function of π_1. Therefore, *full risk classification* maximises loss coverage.

$\square$

Theorem F.1.12. *When* $1 < \lambda_1 < \lambda_2$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2 - 1}{\lambda_1 - 1}\right) > \frac{\mu_2}{\mu_1}$, *no risk classification maximises loss coverage.*

Proof. In this case, $\pi_1^* > \pi_2^*$ by Equation F.73. Because of Equation F.63,

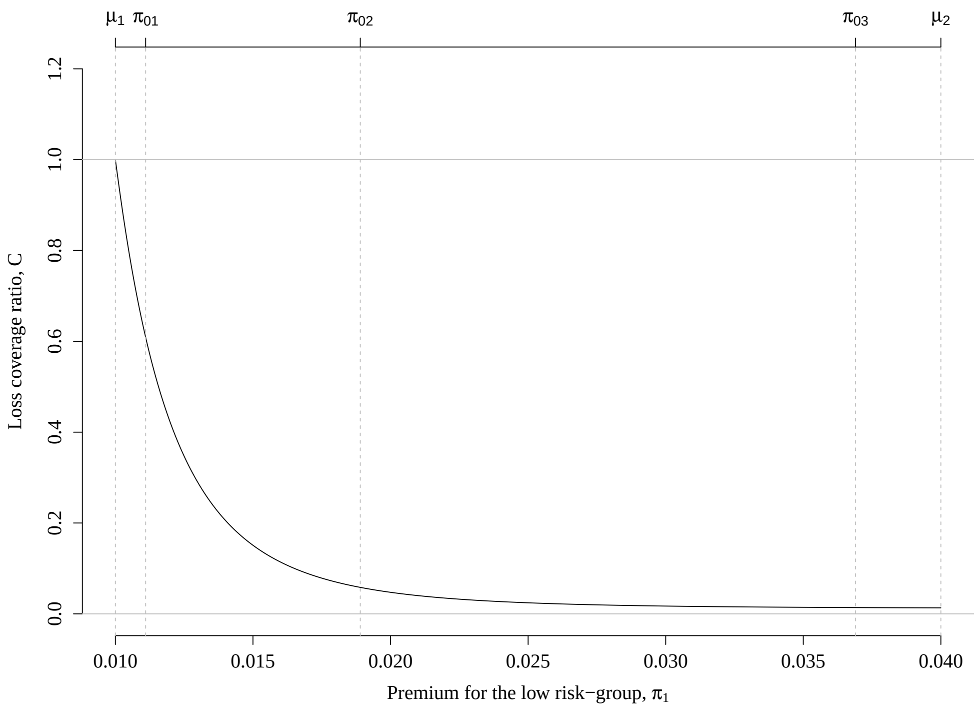

Figure F.3: Plot of loss coverage ratio C as a function of premium for the low risk-group π_1 in the case of multiple equilibria with $\alpha_1 = 99.7\%, \alpha_2 = 0.3\%, \mu_1 = 0.01, \mu_2 = 0.04$ and $\lambda_1 = 6, \lambda_2 = 1.5$.

we can deduce the following result:

$$\mu_1 < \pi_0 < \pi_1^*, \text{ and} \tag{F.77}$$

$$\pi_2^* < \pi_0 < \mu_2. \tag{F.78}$$

Hence loss coverage is a monotonically increasing function of π_1 for $\mu_1 \leq \pi_1 \leq \pi_0$ because $LC(\pi_1^*, \pi_2^*)$ is a maximum by Theorem F.1.10.

Therefore,

$$LC(\pi_0, \pi_0) > LC(\mu_1, \mu_2), \tag{F.79}$$

i.e. in this case, *no risk classification* maximises loss coverage. $\square$

Theorem F.1.13. *When* $1 < \lambda_1 < \lambda_2$ *and* $\frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2-1}{\lambda_1-1}\right) \leq \frac{\mu_2}{\mu_1}$*, a partial risk classification maximises loss coverage.*

Proof. In this case, $\pi_1^* \leq \pi_2^*$ by Equation F.73. And we have also proved that π_1 and π_2 are still monotonically related when the extremum is reached (by Equation F.66).

We can deduce the following result:

$$\mu_1 < \pi_1^* \leq \pi_0 \leq \pi_2^* < \mu_2. \tag{F.80}$$

Hence,

$$LC(\pi_1^*, \pi_2^*) \geq \max[LC(\mu_1, \mu_2), LC(\pi_0, \pi_0)], \tag{F.81}$$

because $LC(\pi_1^*, \pi_2^*)$ is a maximum by Theorem F.1.10.

Therefore, a *partial risk classification* with $(\pi_1^*(\Lambda^*), \pi_2^*(\Lambda^*))$ maximises loss coverage in this case. $\square$

Theorem F.1.14. *When* $0 < \lambda_1 < 1 < \lambda_2$*, no risk classification maximises loss coverage.*

Proof. Recall that

$$\frac{\partial}{\partial \pi_1} LC(\pi_1, \pi_2) \propto A(\pi_1, \pi_2)\frac{B(\pi_1, \pi_2)}{C(\pi_1, \pi_2)} \text{ from Equation F.37 }, \tag{F.82}$$

where

$$A(\pi_1, \pi_2) = \pi_1^{-(\lambda_1+1)}, \tag{F.83}$$

$$B(\pi_1, \pi_2) = \lambda_2(1-\lambda_1)\frac{\pi_1}{\mu_1} - \lambda_1(1-\lambda_2)\frac{\pi_2}{\mu_2}, \text{ and} \tag{F.84}$$

$$C(\pi_1, \pi_2) = \lambda_2 + (1-\lambda_2)\frac{\pi_2}{\mu_2}. \tag{F.85}$$

Moreover,

$$C(\pi_1, \pi_2) > 0 \text{ for } 0 < \pi_2 < \mu_2 < \mu_2 \frac{\lambda_2}{\lambda_2 - 1} \tag{F.86}$$

by Equation F.65.

Hence,

$$\frac{\partial}{\partial \pi_1} LC(\pi_1, \pi_2) > 0 \text{ because } B(\pi_1, \pi_2) > 0, \tag{F.87}$$

for $0 < \lambda_1 < 1 < \lambda_2$ and $\pi_1, \pi_2 > 0$, especially $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$. This result shows that loss coverage is an increasing function of π_1.

Therefore,

$$LC(\pi_0, \pi_0) > LC(\mu_1, \mu_2), \tag{F.88}$$

i.e. *no risk classification* maximises loss coverage. □

Theorem F.1.15. *When $0 < \lambda_2 < 1 < \lambda_1$, full risk classification maximises loss coverage.*

Proof. Similar to the proof for Theorem F.1.14, in this case,

$$\frac{\partial}{\partial \pi_1} LC(\pi_1, \pi_2) < 0 \text{ because } B(\pi_1, \pi_2) < 0, \tag{F.89}$$

for $0 < \lambda_2 < 1 < \lambda_1$ and $\pi_1, \pi_2 > 0$. This result shows that loss coverage is a decreasing function of π_1.

Note that in this case, there might be multiple equilibria[3].

Therefore,

- when there is a unique equilibrium premium π_0,

$$LC(\mu_1, \mu_2) > LC(\pi_0, \pi_0), \tag{F.90}$$

[3]Results on multiple equilibria can be found in Section 3.6.

i.e. *full risk classification* maximises loss coverage.

- When there are multiple equilibrium premiums, π_{0i} with $i \in Z, i \leq 3$ (because there could be at most 3 equilibrium premiums), if $\pi_{01} < \pi_{02} < \pi_{03}$, then

$$LC(\mu_1, \mu_2) > LC(\pi_{01}, \pi_{01}) > LC(\pi_{02}, \pi_{02}) > LC(\pi_{03}, \pi_{03}). \quad \text{(F.91)}$$

  In this case, even though π_1 and π_2 might not have a monotonic relationship (by Equation F.63), the loss coverage is a monotonic function of π_1.

  Figure F.4 is an example showing that loss coverage ratio monotonically decreases with respect to π_1 when there are multiple equilibria. Note that in this example, the values of all parameters are the same as those in Figure F.2.

  Note that in this example, there might not be solutions to π_2 within $[\mu_1, \mu_2]$ for some π_1. For the purpose of illustration, if we are restricting $\mu_1 \leq \pi_1, \pi_2 \leq \mu_2$, then the loss coverage ratio curve might not be continuous for some π_1.

□

Theorem F.1.16. *When* $0 < \lambda_1, \lambda_2 < 1$*, the function* $f(\Lambda) = \frac{\partial W}{\partial \Lambda}$ *is a concave function of* Λ *where* $0 < \Lambda < 1$*. (*W *is the Lagrangian function as described in Theorem F.1.5.)*

Proof. Recall from Equation F.17,

$$f(\Lambda) = p_1 \tau_1 \left[\frac{\Lambda(1-\lambda_1)}{\lambda_1(1-\Lambda)}\right]^{\lambda_1} \left[\frac{\lambda_1 - \Lambda}{1-\lambda_1}\right] \mu_1 + p_2 \tau_2 \left[\frac{\Lambda(1-\lambda_2)}{\lambda_2(1-\Lambda)}\right]^{\lambda_2} \left[\frac{\lambda_2 - \Lambda}{1-\lambda_2}\right] \mu_2, \quad \text{(F.92)}$$

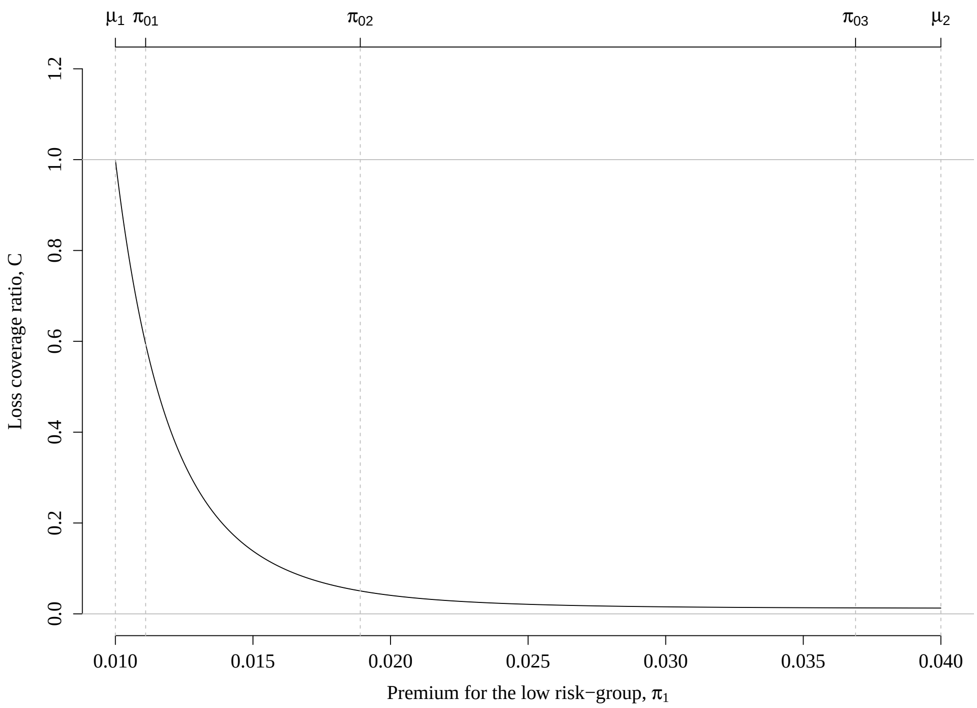

Figure F.4: Plot of loss coverage ratio C as a function of premium for the low risk-group π_1 in the case of multiple equilibria with $\alpha_1 = 99.7\%, \alpha_2 = 0.3\%, \mu_1 = 0.01, \mu_2 = 0.04$ and $\lambda_1 = 6, \lambda_2 = 0.8$.

which can be re-written as

$$f(\Lambda) = \frac{p_1 \tau_1 \mu_1}{1-\lambda_1} \left(\frac{1-\lambda_1}{\lambda_1} \right)^{\lambda_1} D_1 + \frac{p_2 \tau_2 \mu_2}{1-\lambda_2} \left(\frac{1-\lambda_2}{\lambda_2} \right)^{\lambda_2} E_1, \tag{F.93}$$

where

$$D_1 = (\lambda_1 - \Lambda) \left(\frac{\Lambda}{1-\Lambda} \right)^{\lambda_1}, \tag{F.94}$$

$$E_1 = (\lambda_2 - \Lambda) \left(\frac{\Lambda}{1-\Lambda} \right)^{\lambda_2}. \tag{F.95}$$

Hence,

$$\frac{\partial^2}{\partial \Lambda^2} f(\Lambda) = \frac{p_1 \tau_1 \mu_1}{1-\lambda_1} \left(\frac{1-\lambda_1}{\lambda_1}\right)^{\lambda_1} \frac{\partial^2 D_1}{\partial \Lambda^2} + \frac{p_2 \tau_2 \mu_2}{1-\lambda_2} \left(\frac{1-\lambda_2}{\lambda_2}\right)^{\lambda_2} \frac{\partial^2 E_1}{\partial \Lambda^2}. \quad \text{(F.96)}$$

Based on Equation F.94 and F.95, we get

$$\frac{\partial D_1}{\partial \Lambda} = \left(\frac{\Lambda}{1-\Lambda}\right)^{\lambda_1} \left[\frac{\lambda_1(\lambda_1 - \Lambda)}{\Lambda(1-\Lambda)} - 1\right], \quad \text{(F.97)}$$

$$\Rightarrow \frac{\partial^2 D_1}{\partial \Lambda^2} = \lambda_1 \left(\frac{\Lambda}{1-\Lambda}\right)^{\lambda_1} \frac{(\lambda_1 - 1)(\lambda_1 + \Lambda)}{\Lambda^2 (1-\Lambda)^2} < 0 \text{ when } 0 < \lambda_1 < 1 \text{ and } 0 < \Lambda < 1. \quad \text{(F.98)}$$

Similarly,

$$\frac{\partial E_1}{\partial \Lambda} = \left(\frac{\Lambda}{1-\Lambda}\right)^{\lambda_2} \left[\frac{\lambda_2(\lambda_2 - \Lambda)}{\Lambda(1-\Lambda)} - 1\right], \quad \text{(F.99)}$$

$$\Rightarrow \frac{\partial^2 E_1}{\partial \Lambda^2} = \lambda_2 \left(\frac{\Lambda}{1-\Lambda}\right)^{\lambda_2} \frac{(\lambda_2 - 1)(\lambda_2 + \Lambda)}{\Lambda^2 (1-\Lambda)^2} < 0 \text{ when } 0 < \lambda_2 < 1 \text{ and } 0 < \Lambda < 1. \quad \text{(F.100)}$$

Hence,

$$\frac{\partial^2 f(\Lambda)}{\partial \Lambda^2} < 0, \quad \text{(F.101)}$$

i.e. $f(\Lambda)$ is a concave function of Λ for $0 < \Lambda < 1$. □

Theorem F.1.17. *When $\lambda_1, \lambda_2 > 1$, the function $f(\Lambda) = \frac{\partial W}{\partial \Lambda}$ is a concave function of Λ where $\Lambda > 1$. (W is the Lagrangian function as described in Theorem F.1.5.)*

Proof. Similar to the proof for Theorem F.1.16, in this case, the function

$f(\Lambda)$ can be re-written as

$$f(\Lambda) = \frac{p_1\tau_1\mu_1}{\lambda_1 - 1}\left(\frac{\lambda_1 - 1}{\lambda_1}\right)^{\lambda_1} D_2 + \frac{p_2\tau_2\mu_2}{\lambda_2 - 1}\left(\frac{\lambda_2 - 1}{\lambda_2}\right)^{\lambda_2} E_2, \tag{F.102}$$

where

$$D_2 = (\Lambda - \lambda_1)\left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_1}, \tag{F.103}$$

$$E_2 = (\Lambda - \lambda_2)\left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_2}. \tag{F.104}$$

Hence,

$$\frac{\partial^2}{\partial\Lambda^2}f(\Lambda) = \frac{p_1\tau_1\mu_1}{\lambda_1 - 1}\left(\frac{\lambda_1 - 1}{\lambda_1}\right)^{\lambda_1}\frac{\partial^2 D_2}{\partial\Lambda^2} + \frac{p_2\tau_2\mu_2}{\lambda_2 - 1}\left(\frac{\lambda_2 - 1}{\lambda_2}\right)^{\lambda_2}\frac{\partial^2 E_2}{\partial\Lambda^2}. \tag{F.105}$$

Based on Equation F.103 and F.104, we get

$$\frac{\partial D_2}{\partial\Lambda} = \left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_1}\left[1 - \frac{\lambda_1(\Lambda - \lambda_1)}{\Lambda(\Lambda - 1)}\right], \tag{F.106}$$

$$\Rightarrow \frac{\partial^2 D_2}{\partial\Lambda^2} = -\lambda_1\left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_1}\frac{(\lambda_1 - 1)(\lambda_1 + \Lambda)}{\Lambda^2(\Lambda - 1)^2} < 0 \text{ when } \lambda_1 > 1 \text{ and } \Lambda > 1. \tag{F.107}$$

Similarly,

$$\frac{\partial E_2}{\partial\Lambda} = \left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_2}\left[1 - \frac{\lambda_2(\Lambda - \lambda_2)}{\Lambda(\Lambda - 1)}\right], \tag{F.108}$$

$$\Rightarrow \frac{\partial^2 E_2}{\partial\Lambda^2} = -\lambda_2\left(\frac{\Lambda}{\Lambda - 1}\right)^{\lambda_2}\frac{(\lambda_2 - 1)(\lambda_2 + \Lambda)}{\Lambda^2(\Lambda - 1)^2} < 0 \text{ when } \lambda_2 > 1 \text{ and } \Lambda > 1. \tag{F.109}$$

Hence,

$$\frac{\partial^2 f(\Lambda)}{\partial \Lambda^2} < 0, \tag{F.110}$$

i.e. $f(\Lambda)$ is a concave function of Λ for $\Lambda > 1$. $\square$

F.2 Notations and Proofs for Three Risk-groups

F.2.1 Notations and Assumptions

We assume that there are three risk-groups and demand for insurance is driven by iso-elastic demand elasticity. We use the following notations and assumptions:

- $\mu_1 < \mu_2 < \mu_3$ are the underlying risks for the low, middle and high risk-group.

- p_1, p_2, p_3 are the population proportions such that $p_1 + p_2 + p_3 = 1$.

- The proportional demand for insurance for risk-group $i = 1, 2, 3$ at premium π is given by:

  $$d_i(\pi) = \tau_i \left(\frac{\mu_i}{\pi}\right)^{\lambda_i}. \tag{F.111}$$

  Note: $\pi \geq 0$ is an implicit assumption.

- Equilibrium is achieved when the following condition is satisfied:

  $$\sum_{i=1}^{3} p_i d_i(\pi_i)\pi_i = \sum_{i=1}^{3} p_i d_i(\pi_i)\mu_i \Rightarrow \sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \pi_i = \sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i. \tag{F.112}$$

  Each solution (π_1, π_2, π_3) to the above equilibrium condition represents a specific risk-classification scheme. Special cases: $(\pi_1, \pi_2, \pi_3) =$

(μ_1, μ_2, μ_3) represents *full risk classification* and $(\pi_1, \pi_2, \pi_3) = (\pi_0, \pi_0, \pi_0)$ represents *no risk classification.*

- Loss coverage under a specific risk-classification scheme is defined as:

$$LC(\pi_1, \pi_2, \pi_3) = \sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i, \tag{F.113}$$

where (π_1, π_2, π_3) satisfy the equilibrium condition in Equation F.112.

And the loss coverage ratio is defined as

$$C(\pi_1, \pi_2, \pi_3) = \frac{LC(\pi_1, \pi_2, \pi_3)}{LC(\mu_1, \mu_2, \mu_3)}, \tag{F.114}$$

$$= \frac{\sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda_i} \mu_i}{\sum_{i=1}^{3} p_i \tau_i \mu_i}. \tag{F.115}$$

F.2.2 Additional Observations

When there are three risk-groups (i.e. a low risk-group, a middle risk-group and a high risk-group) and they have the same demand elasticity $\lambda_1 = \lambda_2 = \lambda_3 < 1$, we have some additional observations on the loss coverage ratio from the impact of *partial risk classification.* If the restriction that $\pi_1 \leq \pi_2 \leq \pi_3$ is relaxed, a higher loss coverage ratio (compared to $C(\pi_0)$) might be achieved by a *partial risk classification.*

In Figure F.5 we consider an example of loss coverage ratio at different premium strategies $\underline{\pi} = (\pi_1, \pi_2, \pi_3)$ where $(\mu_1, \mu_2, \mu_3) = (0.01, 0.02, 0.04)$, $(\alpha_1, \alpha_2, \alpha_3) = (60\%, 30\%, 10\%)$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$. π_1 is on the x-axis and π_2 is on the y-axis with π_3 being plot as the dashed dark blue indifference curves. Loss coverage ratios comparing a given premium strategy (π_1, π_2, π_3) to risk-differentiated premiums (μ_1, μ_2, μ_3) are plotted as black contours. Any combination of π_1, π_2 and π_3 on this plot satisfies the equilibrium condition to ensure zero expected profits for insurers. The pooled equilibrium premium π_0 at *no risk classification* is shown in Figure 7.2 as the circle. (In this example, $\pi_0 = 0.02$.) The other colourful contours indicating different boundary conditions which will be discussed in Theorem F.2.4.

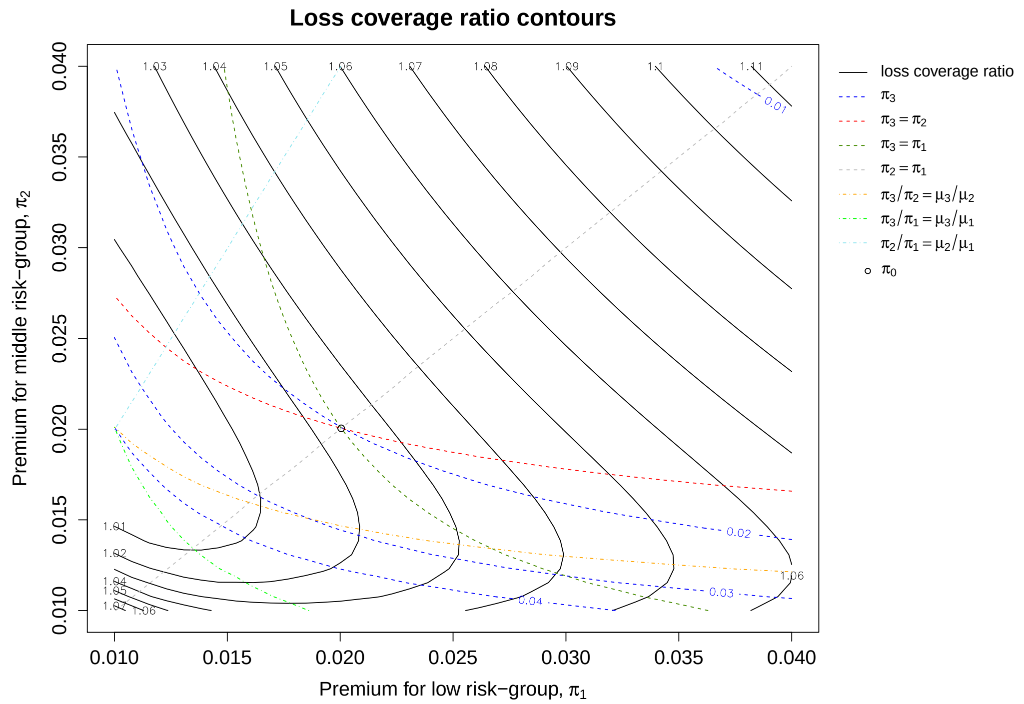

Figure F.5: Plot of loss coverage ratio in terms of π_1, π_2 with $p_1 = 60\%, p_2 = 30\%, p_3 = 10\%, \tau_1 = \tau_2 = \tau_3 = 0.5, \mu_1 = 0.01, \mu_2 = 0.02, \mu_3 = 0.04$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$.

When demand elasticity is less than 1, if we pool two risk-groups by charging them the same premium, and charge the other risk-group a different premium, without any restriction on the ordering of the premiums, in some cases, this can maximise loss coverage. In particular:

Result F.1. *If we pool the low and the middle risk-groups by charging them the same premium, and charge the high risk-group a possibly different premium, loss coverage is maximised when the premium charged to the high risk-group is minimised.*

In Figure F.5, along the dashed grey diagonal line where $\pi_1 = \pi_2$, decreasing π_3 increases loss coverage ratio, i.e. moving towards the top right corner of the plot. When π_3 reaches $\mu_1 = 0.01$ (in this case), loss coverage ratio reaches its maximum. This result is intuitive because high risks' demand for insurance at a very low premium can be very large such that the shift in risk coverage towards the high risks outweighs the reduction in demand from the low and the middle risk-groups. As a result, the aggregate loss coverage increases. This result is proved in Theorem F.2.6 with an example given in Theorem F.2.8.

Result F.2. *If we pool the middle and the high risk-groups by charging them the same premium, and charge the low risk-group a possibly different premium, loss coverage is maximised when the premium charged to the low risk-group is maximised.*

In Figure F.5, along the dashed red line where $\pi_2 = \pi_3$, increasing π_1 increases loss coverage ratio, i.e. moving from left-hand side towards right-hand side of the plot. When π_1 reaches $\mu_3 = 0.04$ (in this case), loss coverage ratio reaches its maximum. This result is proved in Theorem F.2.7 with an example given in Theorem F.2.10.

However, the result is not so straightforward when pooling the low and the high risk-groups. In particular:

Result F.3. *If $\mu_1 \leq \pi_2 \leq \pi_1 \leq \pi_3 \leq \mu_3$ is allowed, then loss coverage is maximised when charging the same premium for the low and the high risk-group and minimising the premium for the middle risk-group.*

In Figure F.5, the loss coverage (ratio) is maximised along the dark green dashed curve approaching the bottom right corner of the plot where $\pi_2 \to \mu_1 = 0.01$. This result is proved in Theorem F.2.11.

Result F.4. *If $\mu_1 \leq \pi_1 \leq \pi_3 \leq \pi_2 \leq \mu_3$ is allowed, then loss coverage is maximised when charging the same premium for the low and the high risk-group and maximising the premium for the middle risk-group.*

In Figure F.5, the loss coverage ratio is maximised along the dark green dashed curve approaching the top left corner of the plot where $\pi_2 \to \mu_3 = 0.04$. This result is proved in Theorem F.2.12.

Result F.3 and F.4 show some intuition. When pooling the low risk-group and the high risk-group and charging both risk-groups a premium (π_1 say) somewhere between (μ_1, μ_3) (i.e. the risk-differentiated premiums for the low and high risk-group respectively), demand from the middle risk-group depends on how the premium charged to this group (π_2) compared to its risk (μ_2). And because of the inverse relationship between π_1 and π_2 in terms of satisfying the equilibrium condition, there could be different premium strategies in terms of maximising loss coverage.

Partial risk classification could maximise loss coverage ratio when the common constant demand elasticity is less than 1, and high risks are also allowed to be charged at a lower premium than the low risks, e.g. when insurers cannot access all information about policyholders due to restrictions by regulators.

Note that for the above observations in this sub-section, we know of no empirical evidence that any such premium strategy is in place. All the results are for the purpose of completion and are remained to be tested.

F.2.3 Theorems and Proofs

Theorem F.2.1. *For the particular case of equal demand elasticities, i.e.* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, *we consider the Lagrangian function:*

$$W(\pi_1, \pi_2, \pi_3, \Lambda) = \sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i + \Lambda \left(\sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \pi_i - \sum_{i=1}^{3} p_i \tau_i \left(\frac{\mu_i}{\pi_i}\right)^{\lambda} \mu_i \right), \tag{F.116}$$

i.e. loss coverage is set as the objective function with the equilibrium condition as a constraint. This formulation leads to an extremum solution $(\pi_1^*, \pi_2^*, \pi_3^*) = (\mu_1, \mu_2, \mu_3)$.

Proof.

$$\frac{\partial W}{\partial \pi_1} = 0 \Rightarrow \pi_1 = \mu_1 \frac{\lambda(1-\Lambda)}{\Lambda(1-\lambda)}, \tag{F.117}$$

$$\frac{\partial W}{\partial \pi_2} = 0 \Rightarrow \pi_2 = \mu_2 \frac{\lambda(1-\Lambda)}{\Lambda(1-\lambda)}, \tag{F.118}$$

$$\frac{\partial W}{\partial \pi_3} = 0 \Rightarrow \pi_3 = \mu_3 \frac{\lambda(1-\Lambda)}{\Lambda(1-\lambda)}, \tag{F.119}$$

$$\frac{\partial W}{\partial \Lambda} = 0 \Rightarrow \Lambda = \lambda \Rightarrow \pi_1 = \mu_1,\ \pi_2 = \mu_2,\ \pi_3 = \mu_3. \tag{F.120}$$

So $(\pi_1^*, \pi_2^*, \pi_3^*) = (\mu_1, \mu_2, \mu_3)$ provides an extremum. □

Theorem F.2.2. *When* $\lambda > 1$, *full risk classification, i.e.* $(\pi_1 = \mu_1, \pi_2 = \mu_2, \pi_3 = \mu_3)$ *maximises loss coverage.*

Proof. We have proved in Theorem F.2.1 that the solution from Lagrangian function $(\pi_1^*, \pi_2^*, \pi_3^*) = (\mu_1, \mu_2, \mu_3)$ provides an extremum of loss coverage, and we have also proved in Result 5.5 that loss coverage at risk-differentiated premiums is higher than loss coverage at pooled equilibrium premium, i.e. $C(\pi_0) < 1$ when $\lambda > 1$. Therefore, $LC(\pi_1 = \mu_1, \pi_2 = \mu_2, \pi_3 = \mu_3)$ must be a maximum. $\square$

Theorem F.2.3. *When $0 < \lambda < 1$, to maintain the equilibrium condition defined in equation F.112,*

1. *given π_1, $\frac{\partial \pi_3}{\partial \pi_2} < 0$ (or $\frac{\partial \pi_2}{\partial \pi_3} < 0$), i.e. π_2, π_3 have a monotonic relationship;*

2. *given π_2, $\frac{\partial \pi_3}{\partial \pi_1} < 0$ (or $\frac{\partial \pi_1}{\partial \pi_3} < 0$), i.e. π_1, π_3 have a monotonic relationship;*

3. *given π_3, $\frac{\partial \pi_2}{\partial \pi_1} < 0$ (or $\frac{\partial \pi_1}{\partial \pi_2} < 0$), i.e. π_1, π_2 have a monotonic relationship.*

Proof. In the case of *partial risk classification*, i.e. insurers can charge different premiums to different risk-groups, equation F.112 can be rewritten as

$$\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_3}\right)^{\lambda} (\pi_3 - \mu_3) = 0, \quad \text{(F.121)}$$

with $\alpha_i = \frac{p_i \tau_i}{p_1\tau_1 + p_2\tau_2 + p_3\tau_3}$, $i = 1, 2, 3$.

Proof of result 1: Given a $\pi_1 \geq 0$, and differentiate both sides of equation F.121 with respect to π_2 to get:

$$\frac{\partial \pi_3}{\partial \pi_2} = -\frac{\frac{\alpha_2 \mu_2^{\lambda+1}}{\pi_2^{\lambda+1}} \left[\lambda + (1-\lambda)\frac{\pi_2}{\mu_2}\right]}{\frac{\alpha_3 \mu_3^{\lambda+1}}{\pi_3^{\lambda+1}} \left[\lambda + (1-\lambda)\frac{\pi_3}{\mu_3}\right]} < 0, \text{ for } 0 < \lambda < 1. \quad \text{(F.122)}$$

(And similarly, $\frac{\partial \pi_2}{\partial \pi_3} < 0$.)

Proof of result 2: Given a $\pi_2 \geq 0$, and differentiate both sides of equation F.121 with respect to π_1 to get:

$$\frac{\partial \pi_3}{\partial \pi_1} = -\frac{\frac{\alpha_1 \mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}\left[\lambda + (1-\lambda)\frac{\pi_1}{\mu_1}\right]}{\frac{\alpha_3 \mu_3^{\lambda+1}}{\pi_3^{\lambda+1}}\left[\lambda + (1-\lambda)\frac{\pi_3}{\mu_3}\right]} < 0, \text{ for } 0 < \lambda < 1. \tag{F.123}$$

(And similarly, $\frac{\partial \pi_1}{\partial \pi_3} < 0$.)

Proof of result 3: Given a $\pi_3 \geq 0$, and differentiate both sides of equation F.121 with respect to π_1 to get:

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{\alpha_1 \mu_1^{\lambda+1}}{\pi_1^{\lambda+1}}\left[\lambda + (1-\lambda)\frac{\pi_1}{\mu_1}\right]}{\frac{\alpha_2 \mu_2^{\lambda+1}}{\pi_2^{\lambda+1}}\left[\lambda + (1-\lambda)\frac{\pi_2}{\mu_2}\right]} < 0, \text{ for } 0 < \lambda < 1. \tag{F.124}$$

(And similarly, $\frac{\partial \pi_1}{\partial \pi_2} < 0$.)

Figure F.6 is an example interpreting premium for the high risk-group, π_3 in terms of π_1, π_2 using $(\mu_1, \mu_2, \mu_3) = (0.01, 0.02, 0.04)$, $(\alpha_1, \alpha_2, \alpha_3) = (60\%, 30\%, 10\%)$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$. Values of π_3 are plotted as the contours with π_1 on the x-axis, π_2 on the y-axis. In this example, we assume $\mu_1 \leq \pi_1, \pi_2, \pi_2 \leq \mu_3$, and there is no further restriction on the relationship between π_1, π_2, and π_3.

The above results are presented in this figure as:

1. To keep π_1 the same, i.e. moving vertically on the plot for any chosen π_1 value, increasing π_2 means decreasing π_3.

2. To keep π_2 the same, i.e. moving horizontally on the plot for any chosen π_2 value, increasing π_1 means decreasing π_3.

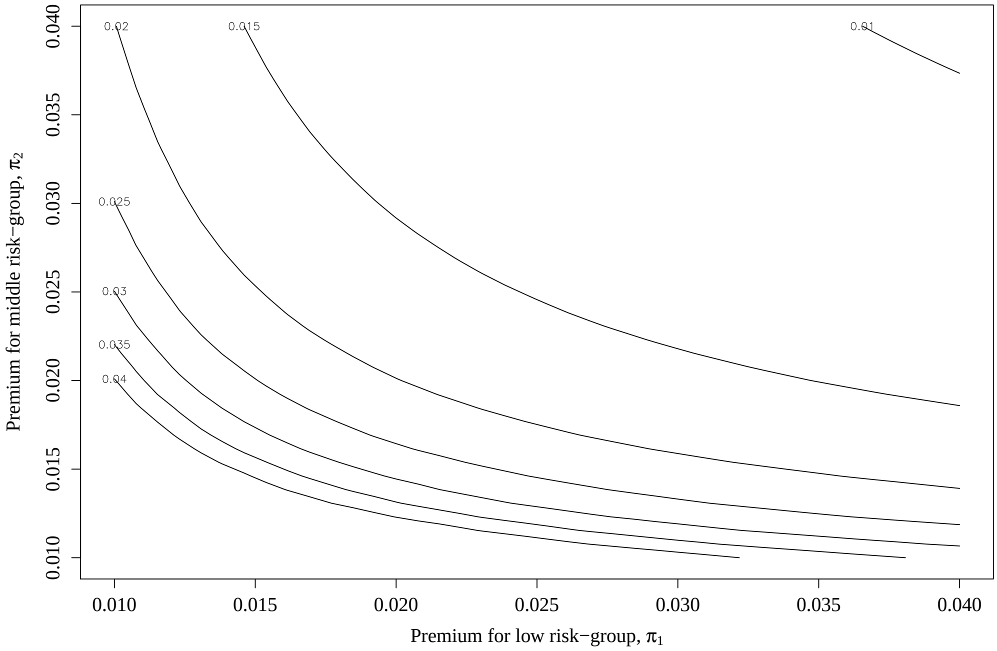

Figure F.6: Plot of premium for the high risks given π_1, π_2 with $\alpha_1 = 60\%, \alpha_2 = 30\%, \alpha_3 = 10\%, \mu_1 = 0.01, \mu_2 = 0.02, \mu_3 = 0.04$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0.8$.

3. To keep π_3 the same, i.e. moving along one of the indifference curve, increasing π_1 means decreasing π_2.

Another observation is that, because we focus on the case when $\mu_1 \leq \pi_1, \pi_2, \pi_3 \leq \mu_3$, not all combinations of π_1, π_2 will lead to a feasible $\mu_1 \leq \pi_3 \leq \mu_3$ such that the equilibrium condition in equation F.112 is satisfied. For example, when π_1, π_2 are both very small, i.e. the bottom-left corner of Figure F.6 where no π_3 within $(\mu_1, \mu_3) = (0.01, 0.04)$ is obtainable in this case. $\square$

Theorem F.2.4. *When $0 < \lambda < 1$, we get the following results on the loss* *coverage ratio, C, defined in Equation F.115:*

1. *Given $\pi_1 \geq 0$,*

$$\frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_2} \lesseqgtr \frac{\mu_3}{\mu_2} \text{ and } \frac{\partial C}{\partial \pi_3} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_2} \gtreqless \frac{\mu_3}{\mu_2}. \tag{F.125}$$

2. *Given $\pi_2 \geq 0$,*

$$\frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \lesseqgtr \frac{\mu_3}{\mu_1} \text{ and } \frac{\partial C}{\partial \pi_3} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \gtreqless \frac{\mu_3}{\mu_1}. \tag{F.126}$$

3. *Given $\pi_3 \geq 0$,*

$$\frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_2}{\pi_1} \lesseqgtr \frac{\mu_2}{\mu_1} \text{ and } \frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \frac{\pi_2}{\pi_1} \gtreqless \frac{\mu_2}{\mu_1}. \tag{F.127}$$

These results indicate the relationship between the premiums charged to any two risk-groups, when fixing the premium charged to the remaining risk-group. These relationships provide a foundation built on which loss coverage can be analysed in some later theorems.

Proof. Recalling Equation F.115:

$$C = \frac{\alpha_1\mu_1\left(\frac{\mu_1}{\pi_1}\right)^\lambda + \alpha_2\mu_2\left(\frac{\mu_2}{\pi_2}\right)^\lambda + \alpha_3\mu_3\left(\frac{\mu_3}{\pi_3}\right)^\lambda}{\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3}. \tag{F.128}$$

Proof of Result 1: Given a $\pi_1 \geq 0$, then differentiate both sides of Equation F.128 with respect to π_2, we can get:

$$\frac{\partial C}{\partial \pi_2} \propto -\lambda\Big[\alpha_2\left(\frac{\pi_2}{\mu_2}\right)^{-\lambda-1} + \alpha_3\left(\frac{\pi_3}{\mu_3}\right)^{-\lambda-1}\frac{\partial \pi_3}{\partial \pi_2}\Big], \tag{F.129}$$

$$= -\lambda\Big[\alpha_2\left(\frac{\pi_2}{\mu_2}\right)^{-\lambda-1} - \alpha_3\left(\frac{\pi_3}{\mu_3}\right)^{-\lambda-1}\frac{\frac{\alpha_2\mu_2^{\lambda+1}}{\pi_2^{\lambda+1}}\left[\lambda+(1-\lambda)\frac{\pi_2}{\mu_2}\right]}{\frac{\alpha_3\mu_3^{\lambda+1}}{\pi_3^{\lambda+1}}\left[\lambda+(1-\lambda)\frac{\pi_3}{\mu_3}\right]}\Big] \tag{F.130}$$

using Result 1 in Theorem F.2.3

$$= -\lambda\alpha_2\left(\frac{\pi_2}{\mu_2}\right)^{-\lambda-1}\Big[\frac{(1-\lambda)(\frac{\pi_3}{\mu_3}-\frac{\pi_2}{\mu_2})}{\lambda+(1-\lambda)\frac{\pi_3}{\mu_3}}\Big]. \tag{F.131}$$

Hence,

$$\frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_2} \lesseqgtr \frac{\mu_3}{\mu_2} \text{ when } 0<\lambda<1. \tag{F.132}$$

And because of the inverse relationship between π_2 and π_3 in Theorem F.2.3,

$$\frac{\partial C}{\partial \pi_3} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_2} \gtreqless \frac{\mu_3}{\mu_2}. \tag{F.133}$$

Proof of Result 2: Similar to the previous proof, given a $\pi_2 \geq 0$, then differentiate both sides of Equation F.128 with respect to π_1, we can get:

$$\frac{\partial C}{\partial \pi_1} \propto -\lambda\alpha_1\left(\frac{\pi_1}{\mu_1}\right)^{-\lambda-1}\frac{(1-\lambda)(\frac{\pi_3}{\mu_3}-\frac{\pi_1}{\mu_1})}{\lambda+(1-\lambda)\frac{\pi_3}{\mu_3}}, \text{ using result 2 in Theorem F.2.3} \tag{F.134}$$

$$\Rightarrow \frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \lesseqgtr \frac{\mu_3}{\mu_1} \text{ when } 0<\lambda<1. \tag{F.135}$$

And because of the inverse relationship between π_1 and π_3 in Theorem

F.2.3,

$$\frac{\partial C}{\partial \pi_3} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \gtreqless \frac{\mu_3}{\mu_1}. \tag{F.136}$$

Proof of Result 3: Similarly, given a $\pi_3 \geq 0$, and differentiate both sides of Equation F.128 with respect to π_1, we can get:

$$\frac{\partial C}{\partial \pi_1} \propto -\lambda\alpha_1 \left(\frac{\pi_1}{\mu_1}\right)^{-\lambda-1} \frac{(1-\lambda)(\frac{\pi_2}{\mu_2} - \frac{\pi_1}{\mu_1})}{\lambda + (1-\lambda)\frac{\pi_2}{\mu_2}}, \text{ using result 3 in Theorem F.2.3} \tag{F.137}$$

$$\Rightarrow \frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_2}{\pi_1} \lesseqgtr \frac{\mu_2}{\mu_1} \text{ when } 0 < \lambda < 1. \tag{F.138}$$

And because of the inverse relationship between π_1 and π_2 in Theorem F.2.3,

$$\frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \frac{\pi_2}{\pi_1} \gtreqless \frac{\mu_2}{\mu_1}. \tag{F.139}$$

Results in this theorem are shown in Figure F.5:

- Result 1 is shown as the orange dashed line. When $\frac{\pi_3}{\pi_2} > \frac{\mu_3}{\mu_2}$, i.e. the area below this dashed line, at any given π_1, increasing π_2 decreases loss coverage ratio, i.e. $\frac{\partial C}{\partial \pi_2} < 0$. When $\frac{\pi_3}{\pi_2} < \frac{\mu_3}{\mu_2}$, i.e. the area above this line, at any given π_1, increasing π_2 increases loss coverage ratio, i.e. $\frac{\partial C}{\partial \pi_2} > 0$. (This argument also works if we look from the perspective of π_3 instead of π_2.)

- Result 2 is shown as the light green dashed line. In this example, if we focus on cases when $\mu_1 \leq \pi_1, \pi_2, \pi_3 \leq \mu_3$, then,

$$\frac{\pi_3}{\mu_3} \leq 1 \leq \frac{\pi_1}{\mu_1}, \tag{F.140}$$

$$\Rightarrow \frac{\pi_3}{\pi_1} \leq \frac{\mu_3}{\mu_1}, \tag{F.141}$$

$$\Rightarrow \frac{\partial C}{\partial \pi_1} \geq 0, \tag{F.142}$$

i.e. at a given π_2, loss coverage ratio increases with π_1. (This argument also works if we look from the perspective of π_3 instead of π_1.) In other words, our main interest is in the area to the right of the light green dashed line.

- Result 3 is shown as the light blue dashed line. When $\frac{\pi_2}{\pi_1} > \frac{\mu_2}{\mu_1}$, i.e. the area to the left of this dashed line, along any given π_3 (i.e. the dark blue-dashed indifference curves), increasing π_1 decreases loss overage ratio, i.e. $\frac{\partial C}{\partial \pi_1} < 0$. When $\frac{\pi_2}{\pi_1} < \frac{\mu_2}{\mu_1}$, i.e. the area to the right of this dashed line, along any given π_3, increasing π_1 increases loss coverage ratio, i.e. $\frac{\partial C}{\partial \pi_1} > 0$. (This argument also works if we look from the perspective of π_2 instead of π_1.)

Note: we have not put any restrictions on the relationship between π_1, π_2 and π_3, as long as they are non-negative.

$\square$

Theorem F.2.5. *If $0 < \lambda < 1$, and $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$, no risk classification maximises loss coverage (ratio), i.e. $C(\pi_1 = \pi_2 = \pi_3 = \pi_0)$ is the maximum.*

Proof. When $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$, we have proved in Result 2 of Theorem F.2.4 that $\frac{\pi_3}{\pi_1} < \frac{\mu_3}{\mu_1}$. This indicates that at a given π_2, loss coverage ratio increases with π_1. However, $\max \pi_1 = \pi_2$. Hence, the maximised loss coverage ratio locates on the line $\pi_1 = \pi_2$ (which is the grey dashed diagonal line in Figure F.5).

Therefore, the maximisation problem becomes to find out the maximum loss coverage ratio by pooling the low and the middle risk-groups with one premium, and charging the high risk-group another premium, i.e.

$$\max C(\pi_1, \pi_2, \pi_3) \text{ becomes } \max C(\pi_1 = \pi_2, \pi_3) \tag{F.143}$$

subject to $0 < \lambda < 1$, and $\mu_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$.

In this case, the equilibrium condition in equation F.112 becomes:

$$\alpha_1 \left(\frac{\mu_1}{\pi_2}\right)^\lambda (\pi_2 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^\lambda (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_3}\right)^\lambda (\pi_3 - \mu_3) = 0. \tag{F.144}$$

Differentiate both sides of F.144 with respect to π_2, we get:

$$\begin{aligned} &\frac{\alpha_1}{\mu_1} \left(\frac{\mu_1}{\pi_2}\right)^{\lambda+1} [(1-\lambda)\pi_2 + \lambda\mu_1] + \frac{\alpha_2}{\mu_2} \left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1} [(1-\lambda)\pi_2 + \lambda\mu_2] \\ &+ \frac{\alpha_3}{\mu_3} \left(\frac{\mu_3}{\pi_3}\right)^{\lambda+1} [(1-\lambda)\pi_3 + \lambda\mu_3] \frac{\partial \pi_3}{\partial \pi_2} = 0, \end{aligned} \tag{F.145}$$

$$\Rightarrow \frac{\partial \pi_3}{\partial \pi_2} = - \frac{\frac{\alpha_1}{\mu_1} \left(\frac{\mu_1}{\pi_2}\right)^{\lambda+1} [(1-\lambda)\pi_2 + \lambda\mu_1] + \frac{\alpha_2}{\mu_2} \left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1} [(1-\lambda)\pi_2 + \lambda\mu_2]}{\frac{\alpha_3}{\mu_3} \left(\frac{\mu_3}{\pi_3}\right)^{\lambda+1} [(1-\lambda)\pi_3 + \lambda\mu_3]} < 0, \tag{F.146}$$

for $0 < \lambda < 1$, i.e. π_2 and π_3 have a monotonic relationship.

Then we look at loss coverage ratio. Equation F.128 becomes

$$C = \frac{\alpha_1\mu_1 \left(\frac{\mu_1}{\pi_2}\right)^\lambda + \alpha_2\mu_2 \left(\frac{\mu_2}{\pi_2}\right)^\lambda + \alpha_3\mu_3 \left(\frac{\mu_3}{\pi_3}\right)^\lambda}{\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3}. \tag{F.147}$$

Differentiate C in the above equation with respect to π_2 to get:

$$\frac{\partial C}{\partial \pi_2} \propto -\lambda\left[\alpha_1\left(\frac{\mu_1}{\pi_2}\right)^{\lambda+1} + \alpha_2\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1} + \alpha_3\left(\frac{\mu_3}{\pi_3}\right)^{\lambda+1}\frac{\partial \pi_3}{\partial \pi_2}\right], \quad \text{(F.148)}$$

$$\propto -\lambda(1-\lambda)\left[\alpha_1\left(\frac{\mu_1}{\pi_2}\right)^{\lambda+1}\left(\frac{\pi_3}{\mu_3}-\frac{\pi_2}{\mu_1}\right) + \alpha_2\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1}\left(\frac{\pi_3}{\mu_3}-\frac{\pi_2}{\mu_2}\right)\right] \quad \text{(F.149)}$$

using equation F.146

$$\Rightarrow \frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \alpha_1\mu_1^{\lambda+1}\left(\frac{\pi_3}{\mu_3}-\frac{\pi_2}{\mu_1}\right) + \alpha_2\mu_2^{\lambda+1}\left(\frac{\pi_3}{\mu_3}-\frac{\pi_2}{\mu_2}\right) \lesseqgtr 0, \text{ becuase } 0<\lambda<1 \quad \text{(F.150)}$$

$$\Leftrightarrow \frac{\pi_3}{\pi_2} \lesseqgtr \frac{\alpha_1\mu_1^{\lambda}+\alpha_2\mu_2^{\lambda}}{\alpha_1\mu_1^{\lambda+1}+\alpha_2\mu_2^{\lambda+1}}\mu_3. \quad \text{(F.151)}$$

Note: because we focus on the case when $\pi_3 \leq \mu_3$, in equation F.144:

$$\alpha_1\left(\frac{\mu_1}{\pi_2}\right)^{\lambda}(\pi_2-\mu_1) + \alpha_2\left(\frac{\mu_2}{\pi_2}\right)^{\lambda}(\pi_2-\mu_2) \geq 0, \quad \text{(F.152)}$$

$$\Rightarrow \pi_2 \geq \frac{\alpha_1\mu_1^{\lambda+1}+\alpha_2\mu_2^{\lambda+1}}{\alpha_1\mu_1^{\lambda}+\alpha_2\mu_2^{\lambda}}. \quad \text{(F.153)}$$

Therefore,

$$\frac{\pi_3}{\pi_2} \leq \frac{\alpha_1\mu_1^{\lambda}+\alpha_2\mu_2^{\lambda}}{\alpha_1\mu_1^{\lambda+1}+\alpha_2\mu_2^{\lambda+1}}\mu_3, \quad \text{(F.154)}$$

$$\Leftrightarrow \frac{\partial C}{\partial \pi_2} \geq 0, \quad \text{(F.155)}$$

i.e. loss coverage ratio is a non-decreasing function of π_2 for $0<\lambda<1$, and $\mu_1 \leq \pi_2, \pi_3 \leq \mu_3$.

Remind that in our assumptions: $\mu_1 \leq \pi_1 \leq \pi_2 \leq \pi_3 \leq \mu_3$. Hence $\max \pi_2 = \pi_3$, and therefore $C(\pi_1 = \pi_2 = \pi_3 = \pi_0)$ is the maximum.

□

Theorem F.2.6. *When $0 < \lambda < 1$, if insurers pool the low and the middle risk-groups by charging the same premium (say π_1) and charge another premium (say π_3) to the high risk-group, then loss coverage ratio C is maximised by minimising π_3.*

Proof. The premium π_1 charged to the low and the middle risk-groups and the premium π_3 charged to the high risk-group should satisfy the equilibrium condition given in Equation F.112, i.e.

$$\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_3}\right)^{\lambda} (\pi_3 - \mu_3) = 0, \quad \text{(F.156)}$$

where $\alpha_i = \frac{p_i \tau_i}{p_1\tau_1 + p_2\tau_2 + p_3\tau_3}$, $i = 1, 2, 3$.

Differentiate both sides of the above equation with respect to π_1 gives:

$$\frac{\partial \pi_3}{\partial \pi_1} = -\frac{\frac{\alpha_1}{\mu_1}\left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1}\Big[(1-\lambda)\pi_1 + \lambda\mu_1\Big] + \frac{\alpha_2}{\mu_2}\left(\frac{\mu_2}{\pi_1}\right)^{\lambda+1}\Big[(1-\lambda)\pi_1 + \lambda\mu_2\Big]}{\frac{\alpha_3}{\mu_3}\left(\frac{\mu_3}{\pi_3}\right)^{\lambda+1}\Big[(1-\lambda)\pi_3 + \lambda\mu_3\Big]} < 0, \quad \text{(F.157)}$$

for $0 < \lambda < 1$. This result shows that, to maintain the equilibrium position, increasing π_1 means π_3 has to be reduced.

Loss coverage ratio can be written as:

$$
\begin{aligned}
C &= \frac{\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^\lambda \mu_1 + \alpha_2 \left(\frac{\mu_2}{\pi_1}\right)^\lambda \mu_2 + \alpha_3 \left(\frac{\mu_3}{\pi_3}\right)^\lambda \mu_3}{\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3}, && \text{(F.158)}\\
\Rightarrow \frac{\partial C}{\partial \pi_1} &\propto -\lambda\left[\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1} + \alpha_2 \left(\frac{\mu_2}{\pi_1}\right)^{\lambda+1} + \alpha_3 \left(\frac{\mu_3}{\pi_3}\right)^{\lambda+1} \frac{\partial \pi_3}{\partial \pi_1}\right], && \text{(F.159)}\\
&\propto \frac{-\lambda(1-\lambda)}{(1-\lambda)\pi_3 + \lambda\mu_3}\left[\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1} \left(\frac{\pi_3}{\pi_1} - \frac{\mu_3}{\mu_1}\right) + \alpha_2 \left(\frac{\mu_2}{\pi_1}\right)^{\lambda+1} \left(\frac{\pi_3}{\pi_1} - \frac{\mu_3}{\mu_2}\right)\right] && \text{(F.160)}
\end{aligned}
$$

using equation F.157 ,

$$
\Rightarrow \frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \lesseqgtr \frac{\alpha_1\mu_1^\lambda + \alpha_2\mu_2^\lambda}{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}\mu_3. \quad \text{(F.161)}
$$

Because in Equation F.157, we have proved that $\frac{\partial \pi_3}{\partial \pi_1} < 0$, therefore,

$$
\frac{\partial C}{\partial \pi_3} \gtreqless 0 \Leftrightarrow \frac{\pi_3}{\pi_1} \gtreqless \frac{\alpha_1\mu_1^\lambda + \alpha_2\mu_2^\lambda}{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}\mu_3. \quad \text{(F.162)}
$$

Note: if we restrict $\mu_1 \leq \pi_1, \pi_2, \pi_3 \leq \mu_3$, then in Equation F.156:

$$
\begin{aligned}
&\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^\lambda (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_1}\right)^\lambda (\pi_1 - \mu_2) \geq 0, && \text{(F.163)}\\
&\Rightarrow \pi_1 \geq \frac{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}{\alpha_1\mu_1^\lambda + \alpha_2\mu_2^\lambda}. && \text{(F.164)}
\end{aligned}
$$

Therefore,

$$
\frac{\pi_3}{\pi_1} \leq \frac{\alpha_1\mu_1^\lambda + \alpha_2\mu_2^\lambda}{\alpha_1\mu_1^{\lambda+1} + \alpha_2\mu_2^{\lambda+1}}\mu_3 \Rightarrow \frac{\partial C}{\partial \pi_3} \leq 0, \quad \text{(F.165)}
$$

when pooling the low and the middle risk-groups, loss coverage ratio is a decreasing function of the premium charged to the high risk-group.

Therefore, when the low and the middle risk-groups are pooled together, loss coverage ratio is maximised when the premium charged to the high risk-groups is minimised.

$\square$

Theorem F.2.7. *When $0 < \lambda < 1$, if insurers pool the middle and the high risk-groups by charging the same premium (say π_2) and charge another premium (say π_1) to the low risk-group, the loss coverage ratio C is maximised by maximising π_1.*

Proof. The premium π_2 charged to the middle and the high risk-groups and the premium π_1 charged to the low risk-group should satisfy the equilibrium condition given in Equation F.112, i.e.

$$\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_3) = 0, \quad \text{(F.166)}$$

where $\alpha_i = \frac{p_i \tau_i}{p_1\tau_1 + p_2\tau_2 + p_3\tau_3}, i = 1, 2, 3.$

Differentiate both sides of the above equation with respect to π_2 gives:

$$\frac{\partial \pi_1}{\partial \pi_2} = -\frac{\frac{\alpha_2}{\mu_2}\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1}\Big[(1-\lambda)\pi_2 + \lambda\mu_2\Big] + \frac{\alpha_3}{\mu_3}\left(\frac{\mu_3}{\pi_2}\right)^{\lambda+1}\Big[(1-\lambda)\pi_2 + \lambda\mu_3\Big]}{\frac{\alpha_1}{\mu_1}\left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1}\Big[(1-\lambda)\pi_1 + \lambda\mu_1\Big]} < 0, \quad \text{(F.167)}$$

for $0 < \lambda < 1$. This result shows that, to maintain the equilibrium position, increasing π_1 means π_2 has to be reduced.

Loss coverage ratio can be written as:

$$C = \frac{\alpha_1\left(\frac{\mu_1}{\pi_1}\right)^{\lambda}\mu_1 + \alpha_2\left(\frac{\mu_2}{\pi_2}\right)^{\lambda}\mu_2 + \alpha_3\left(\frac{\mu_3}{\pi_2}\right)^{\lambda}\mu_3}{\alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3}, \quad \text{(F.168)}$$

$$\Rightarrow \frac{\partial C}{\partial \pi_2} \propto \frac{-\lambda(1-\lambda)}{(1-\lambda)\pi_1 + \lambda\mu_1}\left[\alpha_2\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1}\left(\frac{\pi_1}{\pi_2} - \frac{\mu_1}{\mu_2}\right) + \alpha_3\left(\frac{\mu_3}{\pi_2}\right)^{\lambda+1}\left(\frac{\pi_1}{\pi_2} - \frac{\mu_1}{\mu_3}\right)\right] \quad \text{(F.169)}$$

using equation F.167 ,

$$\Rightarrow \frac{\partial C}{\partial \pi_2} \gtreqless 0 \Leftrightarrow \frac{\pi_1}{\pi_2} \lesseqgtr \frac{\alpha_2\mu_2^{\lambda} + \alpha_3\mu_3^{\lambda}}{\alpha_2\mu_2^{\lambda+1} + \alpha_3\mu_3^{\lambda+1}}\mu_1. \quad \text{(F.170)}$$

Because in Equation F.167, we have proved that $\frac{\partial \pi_1}{\partial \pi_2} < 0$, therefore,

$$\frac{\partial C}{\partial \pi_1} \gtreqless 0 \Leftrightarrow \frac{\pi_1}{\pi_2} \gtrless \frac{\alpha_2 \mu_2^{\lambda} + \alpha_3 \mu_3^{\lambda}}{\alpha_2 \mu_2^{\lambda+1} + \alpha_3 \mu_3^{\lambda+1}} \mu_1. \tag{F.171}$$

Note: if we restrict $\mu_1 \leq \pi_1, \pi_2, \pi_3 \leq \mu_3$, then in Equation F.156:

$$\alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_3) \leq 0, \tag{F.172}$$

$$\Rightarrow \pi_2 \leq \frac{\alpha_2 \mu_2^{\lambda+1} + \alpha_3 \mu_3^{\lambda+1}}{\alpha_2 \mu_2^{\lambda} + \alpha_3 \mu_3^{\lambda}}. \tag{F.173}$$

Therefore,

$$\frac{\pi_1}{\pi_2} \geq \frac{\alpha_2 \mu_2^{\lambda} + \alpha_3 \mu_3^{\lambda}}{\alpha_2 \mu_2^{\lambda+1} + \alpha_3 \mu_3^{\lambda+1}} \mu_1 \Rightarrow \frac{\partial C}{\partial \pi_1} \geq 0, \tag{F.174}$$

when pooling the middle and the high risk-groups, loss coverage ratio is an increasing function of the premium charged to the low risk-group.

Therefore, when the middle and the high risk-groups are pooled together, loss coverage ratio is maximised when the premium charged to the low risk-groups is maximised.

□

Theorem F.2.8. *If* $0 < \lambda < 1$, *and* $\mu_1 \leq \pi_3 \leq \pi_1 \leq \pi_2 \leq \mu_3$ *is allowed, a premium strategy that pools the low and the middle risk-groups, and minimises the premium for the high risk-group maximises loss coverage ratio.*

Proof. Given π_2,

$$\pi_3 \leq \pi_1 \Rightarrow \frac{\pi_3}{\pi_1} \leq 1 < \frac{\mu_3}{\mu_1} \Rightarrow \frac{\partial C}{\partial \pi_1} > 0 \text{ by Equation F.126 }. \tag{F.175}$$

Thus, given a π_2, loss coverage ratio is an increasing function of π_1.

Because $\max \pi_1 = \pi_2$, maximising $C(\underline{\pi})$ becomes maximising $C(\pi_1 = \pi_2, \pi_3)$, which means the maximum loss coverage ratio locates on the diagonal line $\pi_1 = \pi_2$. Thus we are pooling the low and the middle risk-groups. Using the result in Theorem F.2.6, loss coverage ratio is maximised by minimising π_3. □

Theorem F.2.9. *If* $0 < \lambda < 1$, *and* $\mu_1 \leq \pi_3 \leq \pi_2 \leq \pi_1 \leq \mu_3$ *is allowed, a premium strategy that pools the low and the middle risk-groups, and minimises the premium for the high risk-group maximises loss coverage ratio.*

Proof. Given a π_1,

$$\pi_3 \leq \pi_2 \Rightarrow \frac{\pi_3}{\pi_2} \leq 1 < \frac{\mu_3}{\mu_2} \Rightarrow \frac{\partial C}{\partial \pi_2} > 0 \text{ by Equation F.125 .} \tag{F.176}$$

Thus, given a π_1, loss coverage ratio is an increasing function of π_2.

Because $\max \pi_2 = \pi_1$, maximising $C(\underline{\pi})$ becomes maximising $C(\pi_1 = \pi_2, \pi_3)$, i.e. the maximised loss coverage ratio locates on the diagonal line $\pi_1 = \pi_2$. Thus we are pooling the low and the middle risk-groups. Using the result in Theorem F.2.6, loss coverage ratio is maximised by minimising π_3.

Both Theorem F.2.8 and F.2.9 leads to the same results in terms of maximising loss coverage ratio. In Figure F.5, the maximised loss coverage ratio is achieved towards the top right corner of the plot where π_3 is minimised towards $\mu_1 = 0.01$, and π_1, π_2 are towards $\mu_3 = 0.04$. □

Theorem F.2.10. *If* $0 < \lambda < 1$, *and* $\mu_1 \leq \pi_2 \leq \pi_3 \leq \pi_1 \leq \mu_3$ *is allowed, a premium strategy that pools the middle and the high risk-groups, and maximises the premium for the low risk-group maximises loss coverage ratio.*

Proof.

$$\pi_3 \leq \pi_1 \Rightarrow \frac{\pi_3}{\pi_1} \leq 1 < \frac{\mu_3}{\mu_1} \Rightarrow \frac{\partial C}{\partial \pi_3} < 0 \text{ given a } \pi_2 \text{, by Equation F.126 .} \tag{F.177}$$

In this case, $\min \pi_3 = \pi_2$. So the maximised loss coverage ratio locates on the line $\pi_2 = \pi_3$. Therefore, the maximisation problem becomes to find out the maximum loss coverage ratio by pooling the middle and the high risk-groups with one premium, and charging the low risk-group another premium. This is saying that,

$$\max C(\underline{\pi}) \text{ becomes } \max C(\pi_1, \pi_2 = \pi_3), \tag{F.178}$$

subject to $0 < \lambda < 1$. Using the result in Theorem F.2.7, loss coverage ratio is maximised by maximising π_1. In Figure F.5, the maximised loss coverage ratio locates on the far-right end of the red dashed curve (where $\pi_2 = \pi_3$) in which case, $\pi_1 \to \mu_3 = 0.04$. □

Theorem F.2.11. *If $0 < \lambda < 1$, and $\mu_1 \leq \pi_2 \leq \pi_1 \leq \pi_3 \leq \mu_3$ is allowed, a premium strategy that pools the low and the high risk-groups, and minimises the premium for the middle risk-group maximises loss coverage ratio.*

Proof. At a given π_3,

$$\pi_2 \leq \pi_1 \Rightarrow \frac{\pi_2}{\pi_1} \leq 1 < \frac{\mu_2}{\mu_1} \Rightarrow \frac{\partial C}{\partial \pi_1} > 0, \text{ by equation F.127 }, \tag{F.179}$$

i.e. at a given π_3, C is an increasing function of π_1.

Because $\max \pi_1 = \pi_3$, maximising $C(\underline{\pi})$ becomes maximising $C(\pi_1 = \pi_3, \pi_2)$, i.e. the maximised loss coverage ratio locates on the dark green dashed curve $\pi_1 = \pi_3$ in Figure F.5.

The equilibrium condition in Equation F.112 becomes:

$$\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^\lambda (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^\lambda (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_1}\right)^\lambda (\pi_1 - \mu_3) = 0. \tag{F.180}$$

Differentiate both sides of the above equation with respect to π_1, gives:

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{\alpha_1}{\mu_1}\left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1}[(1-\lambda)\pi_1 + \lambda\mu_1] + \frac{\alpha_3}{\mu_3}\left(\frac{\mu_3}{\pi_1}\right)^{\lambda+1}[(1-\lambda)\pi_1 + \lambda\mu_3]}{\frac{\alpha_2}{\mu_2}\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1}[(1-\lambda)\pi_2 + \lambda\mu_2]} < 0, \tag{F.181}$$

when $0 < \lambda < 1$. This result indicates that, to maintain the equilibrium position, increasing π_1 means π_2 has to be reduced.

Note that at a given π_3 (which equals π_1 in this scenario), $\frac{\pi_2}{\pi_1} < \frac{\mu_2}{\mu_1}$ also indicates that $\frac{\partial C}{\partial \pi_2} < 0$ (by Equation F.127). Therefore, the maximisation problem turns out to be to find out the maximum loss coverage ratio by pooling the low and the high risk-groups with one premium, and minimising the premium charged to the middle risk-group; while at the meantime, the equilibrium position is maintained (because decreasing π_2 will automatically increase π_1).

Note that in this case, $\pi_2 \le \pi_1 = \pi_3$, thus loss coverage (ratio) is maximised by minimising π_2, i.e. when $\pi_2 \to \mu_1$. In Figure F.5, the maximised loss coverage ratio locates on the far-right end of the dark green dashed curve (where $\pi_1 = \pi_3$) in which case, $\pi_2 \to \mu_1 = 0.01$.

$\square$

Theorem F.2.12. *If $0 < \lambda < 1$, and $\mu_1 \le \pi_1 \le \pi_3 \le \pi_2 \le \mu_3$ is allowed, a premium strategy that pools the low and the high risk-groups, and maximises the premium for the middle risk-group maximises loss coverage ratio.*

Proof.

$$\pi_3 \le \pi_2 \Rightarrow \frac{\pi_3}{\pi_2} \le 1 < \frac{\mu_3}{\mu_2} \Rightarrow \frac{\partial C}{\partial \pi_3} < 0 \text{ by Equation F.125 .} \tag{F.182}$$

So, given a π_1, loss coverage ratio C is a decreasing function of π_3.

Because $\min \pi_3 = \pi_1$, maximising $C(\underline{\pi})$ becomes maximising $C(\pi_1 = \pi_3, \pi_2)$, i.e. the maximised loss coverage ratio locates on the dark green dashed curve $\pi_1 = \pi_3$ in Figure F.5.

The equilibrium condition in Equation F.112 becomes:

$$\alpha_1 \left(\frac{\mu_1}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_1) + \alpha_2 \left(\frac{\mu_2}{\pi_2}\right)^{\lambda} (\pi_2 - \mu_2) + \alpha_3 \left(\frac{\mu_3}{\pi_1}\right)^{\lambda} (\pi_1 - \mu_3) = 0. \quad \text{(F.183)}$$

Differentiate both sides of the above equation with respect to π_1, gives:

$$\frac{\partial \pi_2}{\partial \pi_1} = -\frac{\frac{\alpha_1}{\mu_1}\left(\frac{\mu_1}{\pi_1}\right)^{\lambda+1}[(1-\lambda)\pi_1 + \lambda\mu_1] + \frac{\alpha_3}{\mu_3}\left(\frac{\mu_3}{\pi_1}\right)^{\lambda+1}[(1-\lambda)\pi_1 + \lambda\mu_3]}{\frac{\alpha_2}{\mu_2}\left(\frac{\mu_2}{\pi_2}\right)^{\lambda+1}[(1-\lambda)\pi_2 + \lambda\mu_2]} < 0, \quad \text{(F.184)}$$

when $0 < \lambda < 1$. This result indicates that, to maintain the equilibrium position, increasing π_1 means π_2 has to be reduced.

Note that at a given π_1 (which equals π_3 in this scenario), $\frac{\pi_3}{\pi_2} < \frac{\mu_3}{\mu_2}$ also indicates that $\frac{\partial C}{\partial \pi_2} > 0$ (by Equation F.127). Therefore, the maximisation problem turns out to be to find out the maximum loss coverage ratio by pooling the low and the high risk-groups with one premium, and maximising the premium charged to the middle risk-group; while at the meantime, the equilibrium position is maintained (because decreasing π_2 will automatically increase π_1).

Note that in this case, $\pi_2 \geq \pi_1 = \pi_3$, thus loss coverage (ratio) is maximised by maximising π_2, i.e. when $\pi_2 \to \mu_3$. In Figure F.5, the maximised loss coverage ratio locates on the far-left end of the dark green dashed curve (where $\pi_1 = \pi_3$) in which case, $\pi_2 \to \mu_3 = 0.04$.

$\square$