Monoidality of Franke's Exotic Model

David Barnes[*] [1]
The University of Western Ontario
Department of Mathematics
London, Ontario N6A 5B7
Canada
d.j.barnes@shef.ac.uk

Constanze Roitzheim[†]
The University of Glasgow
Department of Mathematics
University Gardens, Glasgow G12 8QW
United Kingdom
c.roitzheim@maths.gla.ac.uk

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Abstract

We discuss the monoidal structure on Franke's algebraic model for the $K_{(p)}$-local stable homotopy category at odd primes and show that its Picard group is isomorphic to the integers.

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Introduction

A well-established method to study the structure of the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ is chromatic filtration. This concerns Bousfield localisation with respect to homology theories $E(n)$, $n \in \mathbb{N}$ for a fixed prime p absent from notation. The resulting "chromatic layers" $\mathrm{Ho}(L_n\mathcal{S})$ provide a better and better approximation of $\mathrm{Ho}(\mathcal{S})$ at p as n increases.

For $n = 1$, $E(1)$ is the Adams summand of p-local complex K-theory, so $\mathrm{Ho}(L_1\mathcal{S})$ is the $K_{(p)}$-local stable homotopy category. Since K-theory has been studied extensively, we can make use of a wealth of tools to study this first chromatic layer. Startlingly, the behaviour of $\mathrm{Ho}(L_1\mathcal{S})$ at odd primes differs significantly from $p = 2$. For $p = 2$ all higher homotopy structure of $\mathrm{Ho}(L_1\mathcal{S})$ is encoded in its triangulated structure, meaning that $\mathrm{Ho}(L_1\mathcal{S})$ is *rigid* at $p = 2$, see [1]. As a consequence, $\mathrm{Ho}(L_1\mathcal{S})$ at $p = 2$ cannot be described by an algebraic model. For odd primes however, the situation is completely different.

[*]Partially supported by NSERC
[†]Supported by EPSRC grant EP/G051348/1, corresponding author
[1]Current Address: The University of Sheffield, Hicks Building, Sheffield S3 7RH, United Kingdom

For odd primes Jens Franke constructed an algebraic model category $C^{(T,N)}(\mathcal{A})$ whose homotopy category $D^{(T,N)}(\mathcal{A})$ is equivalent to $\mathrm{Ho}(L_1\mathcal{S})$, [2]. But the underlying models $L_1\mathcal{S}$ and $C^{(T,N)}(\mathcal{A})$ are not Quillen equivalent by [2] and [3]. As $C^{(T,N)}(\mathcal{A})$ has such a different homotopical behaviour from the standard model $L_1\mathcal{S}$, it is called an *exotic model*.

Although some general differences between $p = 2$ and $p > 2$ have been studied in [1, Section 6], it is still mysterious how this exotic model came along. Furthermore, it is still almost entirely unknown for which n and p, $\mathrm{Ho}(L_n\mathcal{S})$ possesses exotic models and if it does, how many. It is not even known if Franke's model is the only algebraic model in its range.

One structural tool that has not yet been made use of is monoidality. We are going to focus on it in this paper. After defining a monoidal product on $C^{(T,N)}(\mathcal{A})$, we find that Franke's model structure on this category is not monoidal. This means that it does not induce a monoidal structure on $D^{(T,N)}(\mathcal{A})$. So our goal is to construct a new model structure on $C^{(T,N)}(\mathcal{A})$ that is Quillen equivalent to Franke's model while also being compatible with the monoidal product.

Furthermore, we want the newly defined derived product $\wedge^L_{\mathbb{P}\mathcal{I}}$ on $D^{(T,N)}(\mathcal{A})$ to interact reasonably with the smash product $\wedge^L$ on $\mathrm{Ho}(L_1\mathcal{S})$. Let

$$\mathcal{R} : D^{(T,N)}(\mathcal{A}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

denote Franke's triangulated equivalence. In [4], Nora Ganter constructed a natural isomorphism

$$\mathcal{R}(C_* \otimes^L_{E(1)_*} D_*) \cong \mathcal{R}(C_*) \wedge^L \mathcal{R}(D_*).$$

However, that paper does not accurately define the non-derived tensor product of quasi-periodic cochain complexes so the definition of the derived tensor product is not complete.

Our construction is compatible with the assumptions needed for Ganter's result. Hence, it closes a gap in [4] and so allows us to use the isomorphism

$$\mathcal{R}(C_* \wedge^L_{\mathbb{P}\mathcal{I}} D_*) \cong \mathcal{R}(C_*) \wedge^L \mathcal{R}(D_*)$$

to relate our new monoidal product $\wedge^L_{\mathbb{P}\mathcal{I}}$ to $\wedge^L$. It should be noted that it is unknown if $\mathcal{R}$ is associative for $p > 5$ and it is definitely not associative for $p = 3$.

Even further, combining this with work of Hovey and Sadofsky [5], we can read off the Picard group of $D^{(T,N)}(\mathcal{A})$. We hope that our results contribute to understanding the concept of exotic models in the future.

Organisation

In Section 1 we revise the concept of quasi-periodic chain complexes $C^{(T,N)}(\mathcal{G})$. Here, $\mathcal{G}$ denotes a Grothendieck abelian category, T a self-equivalence of $\mathcal{G}$ and $N \geq 0$ the periodicity index. Quasi-periodic chain complexes form the basis of Franke's construction. In particular, they are chain complexes in $\mathcal{G}$. We recall how to create model structures on $C^{(T,N)}(\mathcal{G})$ using the forgetful functor to chain complexes $C(\mathcal{G})$.

In Section 2, we explain how $C^{(T,N)}(\mathcal{G})$ can be described as a category of modules over a ring object in $C(\mathcal{G})$. The ring object will be the "periodified" unit $\mathbb{P}\mathcal{I}$.

Section 3 recalls some definitions and properties about comodules over Hopf algebroids. They are used in Section 4, which concerns Franke's category. Here, we specify the Grothendieck abelian category $\mathcal{A}$, the self-equivalence T and period $N = 1$. The abelian category $\mathcal{A}$ is equivalent to $E(1)_*E(1)$-comodules. We then describe the resulting model structure and some of its properties.

In Franke's case, $\mathcal{A}$ does not have enough projectives, only enough injectives. But the injective model structure is not monoidal. A step towards a solution is the *relative projective model structure* described in Section 5. This was first introduced by Christensen and Hovey in [6]. The induced model structure on $C^{(T,1)}(\mathcal{A})$ is monoidal but is not Quillen equivalent to Franke's model as it does not have enough weak equivalences.

We use the above to construct a *quasi-projective model structure* in Section 6. For the construction, we formally add weak equivalences to the relative projective model structure. Eventually we arrive at a model category that is Quillen equivalent to Franke's model and is a monoidal model category.

Finally, in Section 7, we relate our result to Ganter's theorem and compute the Picard group of the exotic model, $\mathrm{Pic}(D^{(T,1)}(\mathcal{A}))$. We further place it in context with other results about the $E(n)$-local stable homotopy category, hopefully shedding some light on the existence of exotic models versus rigidity.

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1 Quasi-periodic chain complexes

We use $\mathcal{G}$ for a general abelian category and reserve $\mathcal{A}$ for Franke's category of Section 4. For model structure purposes we will usually assume that $\mathcal{G}$ is a Grothendieck abelian category, which is our reason for choosing this letter. We will always assume that we have a self-equivalence $T\colon \mathcal{G} \to \mathcal{G}$ and we further assume that $\mathcal{G}$ has all small limits and colimits.

In this section, we introduce the category $C^{(T,N)}(\mathcal{G})$ of quasi-periodic chain complexes of period N, i.e. chain complexes with values in $\mathcal{G}$ that are periodic up to a "twist" by T. Given a model structure on chain complexes $C(\mathcal{G})$, we are then going to discuss how $C^{(T,N)}(\mathcal{G})$ inherits a model structure from $C(\mathcal{G})$.

Definition 1.1 *The category, $C^{(T,N)}(\mathcal{G})$, of **quasi-periodic chain complexes** (or **twisted chain complexes**) in $\mathcal{G}$, has objects the class of chain complexes $C(\mathcal{G})$ in $\mathcal{G}$ which have a specified isomorphism $\alpha_C : T(C_*) \longrightarrow C[N]_*$. A morphism is then a chain map which commutes with the given isomorphisms as above.*

Here, $C[N]_* = C_{*-N}$, the differential on $C[N]$ is $d_{C[N]} = (-1)^N d_C$. For further details on this category see [2, Example 1.3.3] or [3, Subsection 2.2].

The forgetful functor $U : C^{(T,N)}(\mathcal{G}) \longrightarrow C(\mathcal{G})$ from quasi-periodic chain complexes to chain complexes on $\mathcal{G}$ has both a left and a right adjoint. We are most interested in the left adjoint, which we call **periodification**. Given a chain complex M, we define

$$\mathbb{P}M = \bigoplus_{k\in\mathbb{Z}} T^k M[-kN].$$

Thus $\mathbb{P}M_n = \bigoplus_{k\in\mathbb{Z}} T^k M_{n+kN}$. The differential on summand $T^k M_{n+kN}$ is given by

$$(-1)^{kN} T^k d_{n+kN} : T^k M_{n+kN} \to T^k M_{n+kN-1}.$$

This is a functor, the action on maps being to send g to that map which on level n and summand k is given by $T^k g_{n+kN}$. Furthermore $\mathbb{P}M$ is a quasi-periodic chain complex, the quasi-periodicity isomorphism is the following composite.

$$\begin{aligned} T\mathbb{P}M_n = T\bigoplus_{k\in\mathbb{Z}} T^k M_{n+kN} &\cong \bigoplus_{k\in\mathbb{Z}} T^{k+1} M_{n+kN} \\ &= \bigoplus_{k\in\mathbb{Z}} T^{k+1} M_{n+(k+1)N-N} \\ &= \bigoplus_{k\in\mathbb{Z}} T^{k+1} M[N]_{n+(k+1)N} \\ &= \mathbb{P}M[N]_n \end{aligned}$$

Lemma 1.2 *The functor $\mathbb{P}$ is the left adjoint to the forgetful functor U from quasi-periodic chains on $\mathcal{G}$ to chains on $\mathcal{G}$.*

Proof Let $f : \mathbb{P}M \to X$ be a quasi-periodic chain map. Let f^n_k be the map from the k-summand of $\mathbb{P}M_n$ to X_n, so $f^n_k : T^k M_{n+kN} \to X_n$. The collection $f^n_0 : M_n \to X_n$ defines a chain map $\hat{f} : M \to X$.

For the inverse, let $g : M \to X$ be a chain map. Define a collection g^n_k by the following composite, where the second map is coming from the quasi-periodic structure of X.

$$T^k M_{n+kN} \xrightarrow{T^k g_{n+kN}} T^k X_{n+kN} \xrightarrow{(\alpha_X)^k} X_n$$

We then define a map $\tilde{g} : \mathbb{P}M \to X$, which on summand k is g^n_k. To see that $\tilde{g}$ is a quasi-periodic map, we note that the following diagram commutes

$$\begin{array}{ccc} T\mathbb{P}M & \xrightarrow{T\tilde{g}} & TX \\ \downarrow{\scriptstyle \alpha_{\mathbb{P}M}} & & \downarrow{\scriptstyle \alpha_X} \\ \mathbb{P}M[N] & \xrightarrow{\tilde{g}[N]} & X[N] \end{array}$$

because the diagram below commutes.

$$\begin{array}{ccccc} TT^k M_{n+kN} & \xrightarrow{TT^k g_{n+kN}} & TT^k X_{n+kN} & \xrightarrow{(\alpha_X)^k} & TX^n \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \alpha_X} \\ T^{k+1} M_{n+kN} & \xrightarrow{T^{k+1} g_{n+kN}} & T^{k+1} X_{n+kN} & \xrightarrow{(\alpha_X)^{k+1}} & X_{n-N} \end{array}$$

It is simple to check that $\tilde{g}$ is compatible with the differentials. That $\hat{\tilde{g}} = g$ is immediate. One can also check that $\tilde{\hat{f}} = f$, by noting that the diagram below commutes. This holds since f is quasi-periodic and the top path is $\tilde{\hat{f}}^n_k$ whereas the lower path is f^n_k.

$$\begin{array}{ccc} T^k M_{n+kN} & \xrightarrow{T^k f_0^{n-kN}} & T^k X_{n+kN} \\ \downarrow{=} & & \downarrow{(\alpha_X)^k} \\ T^k M_{n+kN} & \xrightarrow{f^n_k} & X^n \end{array}$$

■

A dual argument will show that the forgetful functor has a right adjoint $\mathbb{R}$, which sends a chain complex M to the quasi-periodic chain complex $\mathbb{R}M = \prod_{k\in\mathbb{Z}} T^k M[-kN]$.

Proposition 1.3 *Assume that there is a cofibrantly generated model structure on $C(\mathcal{G})$ and that T is a left Quillen functor. Then the forgetful functor*

$$U : C^{(T,N)}(\mathcal{G}) \longrightarrow C(\mathcal{G})$$

creates a model structure on $C^{(T,N)}(\mathcal{G})$. That is, there is a model structure on the category of quasi-periodic chain complexes, $C^{(T,N)}(\mathcal{G})$, where a map f is a weak equivalence or a fibration if and only if Uf is so in $C(\mathcal{G})$.

Proof Let I be the generating cofibrations and J the generating trivial cofibrations of $C(\mathcal{G})$. Using the lifting criterion of [7, Theorem 11.3.2], one only needs to show that $\mathbb{P}I$ and $\mathbb{P}J$ satisfy the small-object argument and that a $\mathbb{P}J$-cell complex is a weak equivalence. Since the forgetful functor is both a left and a right adjoint, it preserves all colimits, thus $\mathbb{P}I$ and $\mathbb{P}J$ satisfy the small-object argument. Since $\mathbb{P}$ preserves acyclic cofibrations, it follows that applying U to a $\mathbb{P}J$-cell complex gives an acyclic cofibration in $C(\mathcal{G})$. Thus the forgetful functor takes $\mathbb{P}J$-cell complexes to weak equivalences. ■

The above proposition implies that the adjoint functor pair $(\mathbb{P}, U)$ is a Quillen adjunction with $\mathbb{P}$ being the left and U being the right Quillen functor. Note that this model structure on $C^{(T,N)}(\mathcal{G})$ is also cofibrantly generated with the generating cofibrations being the image of the generating cofibrations in $C(\mathcal{G})$ under the functor $\mathbb{P}$, see [8, Appendix 1].

Lemma 1.4 *The functor $\mathbb{P}$ preserves quasi-isomorphisms.*

Proof The functor T is an equivalence, hence it preserves quasi-isomorphisms, as does the shift functor. Since $\mathbb{P}$ is just an infinite direct sum of shifts and applications of T, it preserves homology isomorphisms. ■

One particular model structure that will be of interest is the model structure created in [2]. It is used on the algebraic model for the K-local stable homotopy category. We will discuss this category in more detail in Section 4. We add the assumption that $\mathcal{G}$ is a Grothendieck abelian category so that we have cofibrant generation. Note that a Grothendieck abelian category always has enough injectives [9, Corollary X.4.3] and every object is small [10, Proposition A.2.].

Definition 1.5 *A **Grothendieck abelian category** $\mathcal{G}$ is a cocomplete abelian category, where filtered colimits commute with finite limits. Further, there is a **generator** G, that is, $\mathcal{G}(G,-)$ is faithful.*

Lemma 1.6 *If $\mathcal{G}$ is a Grothendieck abelian category, then there is a cofibrantly generated model structure on $C^{(T,N)}(\mathcal{G})$ with the quasi-isomorphisms as the weak equivalences and the monomorphisms as the cofibrations. We call this the **injective model structure**.*

Proof For $C(\mathcal{G})$ this is [11, Proposition 3.13], which uses Theorem 6.1. We can lift this model structure to $C^{(T,N)}(\mathcal{G})$ using the above results. ∎

At this point we would like to mention some parallels to topology. The main example of interest is the category $C^1(\mathcal{A})$, a special case of $C^{(T,N)}(\mathcal{G})$. We introduce it in Section 4. It comes with an equivalence of triangulated categories

$$\mathcal{R} : D^1(\mathcal{A}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S}).$$

Here, $D^1(\mathcal{A})$ is the homotopy category of $C^1(\mathcal{A})$ with the injective model structure. Further, $\mathrm{Ho}(L_1\mathcal{S})$ denotes the K-local stable homotopy category at $p > 2$. The chain complex corresponding to the sphere L_1S^0 under $\mathcal{R}$ is exactly $\mathbb{P}\mathcal{I}$. So in our construction, we would like the periodified unit $\mathbb{P}\mathcal{I}$ to play the role of a localised sphere.

The stable homotopy category itself is not equivalent to a category of chain complexes. However, the reader might find the constructions in this section similar to the following adjunction

$$L_1 = - \wedge^L L_1S^0 : \mathrm{Ho}(\mathcal{S}) \rightleftarrows \mathrm{Ho}(L_1\mathcal{S}) : U.$$

Localisation with respect to $K_{(p)}$ equals smashing with L_1S^0, which is left adjoint to the forgetful functor.

2 Quasi-periodic chain complexes as modules

We describe the category of quasi-periodic chain complexes as a category of modules over a specifically chosen monoid. It gives a nice description of the monoidal structure of $C^{(T,N)}(\mathcal{G})$.

We now assume that $\mathcal{G}$ is a closed symmetric monoidal category with tensor product $\otimes$ and unit $\mathcal{I}$. Here $\mathcal{G}$ has both a tensor product and an internal homomorphism object $F(-,-)$, which are related by the usual adjunction. In this case we must make further assumptions on T. We want T to behave like N-fold suspension, in particular we do not require T to be a monoidal functor.

Definition 2.1 *We say that T is **compatible** with the monoidal structure if there is a natural isomorphism of functors*

$$m : T \to T\mathcal{I} \otimes (-).$$

If T is compatible with the monoidal structure then for any X and Y there is a natural isomorphism $T(X \otimes Y) \to TX \otimes Y$.

Lemma 2.2 *If T is compatible with the monoidal structure, so is T^n, for any $n \in \mathbb{Z}$. There are natural isomorphisms*

$$TF(X,Y) \cong F(T^{-1}X, Y) \cong F(X, TY).$$

From now on we assume that T is compatible with the monoidal structure. We can think of $\mathcal{I}$ as an object of $C(\mathcal{G})$ concentrated in degree zero. We show that $\mathbb{P}\mathcal{I}$ is a monoid in $C(\mathcal{G})$ and that the category of quasi-periodic chain complexes is isomorphic to the category of $\mathbb{P}\mathcal{I}$-modules.

Proposition 2.3 *The category of quasi-periodic chain complexes, $C^{(T,N)}(\mathcal{G})$, is isomorphic to the category of $\mathbb{P}\mathcal{I}$-modules in $C(\mathcal{G})$.*

Proof First of all, we prove that if X is a quasi-periodic cochain complex, then X has a natural action of $\mathbb{P}\mathcal{I}$. We start by writing out level n of $\mathbb{P}\mathcal{I} \otimes X$, the tensor product in the category of chain complexes:

$$(\mathbb{P}\mathcal{I} \otimes X)_n = \bigoplus_{k \in \mathbb{Z}} T^k \mathcal{I}[-kN] \otimes X_{n+kN} \cong \bigoplus_{k \in \mathbb{Z}} T^k X_{n+kN} \cong \mathbb{P}X_n$$

the action map is then induced by the structure map of X and the fold map

$$\bigoplus_{k \in \mathbb{Z}} T^k X_{n+kN} \to \bigoplus_{k \in \mathbb{Z}} X_n \to X_n.$$

It is easy to check that this map is associative and unital, the unit of $\mathbb{P}\mathcal{I}$ being the obvious inclusion $\mathcal{I} \to \mathbb{P}\mathcal{I}$. We note that the differential of

$$(\mathbb{P}\mathcal{I} \otimes X)_n \cong \bigoplus_{k \in \mathbb{Z}} T^k X_{n+kN}$$

is given by $(-1)^{kN} T^k d_{n+kN}$, which tells us that the differentials are compatible with the action $\mathbb{P}\mathcal{I} \otimes X \to X$.

For the converse we prove that if Y is a $\mathbb{P}\mathcal{I}$-module, then Y is a quasi-periodic chain complex. The action map of Y takes the following form:

$$(\mathbb{P}\mathcal{I} \otimes X)_n \cong \bigoplus_{k \in \mathbb{Z}} T^k Y_{n+kN} \to Y_n.$$

Let

$$\phi(k)_n : T^k Y_{n+kN} \to Y_n$$

be the k component of the above map. We can assemble these to obtain a map

$$\phi(k) : T^k Y \to Y[kN].$$

We need to see that $\phi(1)$ is an isomorphism, it will be our periodicity map. Since the action map is unital, $\phi(0)$ is the identity. Associativity of the action shows that

$$\phi(l+k) = \phi(k)\phi(l),$$

in particular

$$\phi(1)\phi(-1) = \phi(0) = \phi(-1)\phi(1),$$

so $\phi(1)$ is an isomorphism. ∎

Another consequence of our computations is the following, which says that $\mathbb{P}$ is compatible with the monoidal structure.

Corollary 2.4 *There is an isomorphism of quasi-periodic chain complexes $\mathbb{P}\mathcal{I} \otimes X \cong \mathbb{P}X$, which is natural in X.*

Thus we have shown that the category of quasi-periodic chain complexes and $\mathbb{P}\mathcal{I}$-modules are isomorphic. We can also think of $\mathbb{P}$ as an monad on the category of chain complexes of objects of $\mathcal{G}$, we can then describe $C^{(T,N)}(\mathcal{G})$ as the category of modules over this monad. We make no use of this monad description.

A result by Schwede and Shipley states that if a monoidal model category satisfies the **monoid axiom** [12, Definition 3.3], then there is an induced model structure on the category of modules over a fixed commutative monoid. We apply this result to our case and arrive at the following.

Proposition 2.5 *Assume that there is a model structure on $C(\mathcal{G})$ which is cofibrantly generated, monoidal and satisfies the monoid axiom. Then the category of $\mathbb{P}\mathcal{I}$-modules has a cofibrantly generated model structure where the weak equivalences and fibrations are the underlying weak equivalences and fibrations. The generating cofibrations and acyclic cofibrations are given by applying $\mathbb{P}\mathcal{I} \otimes (-)$ to the generating cofibrations and acyclic cofibrations of $C(\mathcal{G})$.*

Corollary 2.6 *This model category on $C^{(T,N)}(\mathcal{G})$ is precisely the model category of quasi-periodic chain complexes with the lifted model structure. Furthermore, since $\mathbb{P}\mathcal{I}$ is a commutative monoid, this model category is monoidal and satisfies the monoid axiom, with monoidal product given as the tensor over $\mathbb{P}\mathcal{I}$: $X \otimes_{\mathbb{P}\mathcal{I}} Y$.*

This follows from [12, Theorem 4.1].

3 Comodules over Hopf algebroids

We are going to recall some definitions, conventions and basic properties about comodules over a Hopf algebroid as this is the abelian category we are most interested in. We refer to [13] and [14, Appendix B.3] for more details.

Let k be a commutative ring. Then a **Hopf algebroid** is a pair (A, Γ) of commutative k-algebras such that for every k-algebra B, the pair

$$(\mathrm{Hom}_{k-alg}(A, B), \mathrm{Hom}_{k-alg}(\Gamma, B))$$

forms a groupoid (i.e. a small category where every morphism is an isomorphism) with $\mathrm{Hom}_{k-alg}(A,B)$ being the objects and $\mathrm{Hom}_{k-alg}(\Gamma,B)$ being the morphisms. This means that there are structure maps

- $\Delta : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$ (coproduct, inducing composition of morphisms)
- $c : \Gamma \longrightarrow \Gamma$ (conjugation, inducing inverses)
- $\epsilon : \Gamma \longrightarrow A$ (augmentation, inducing identity morphisms)
- $\eta_R : A \longrightarrow \Gamma$ (right unit, inducing target)
- $\eta_L : A \longrightarrow \Gamma$ (left unit, inducing source)

satisfying certain conditions. Note that η_R and η_L make Γ into an A-bimodule. By $\otimes_A$ we mean the tensor product of A-bimodules.

For technical reasons we are going to consider only **flat Adams Hopf algebroids**, i.e. Hopf algebroids (A,Γ) where Γ is a filtered colimit of finitely generated projectives over A. The main topological example we have in mind are Hopf algebroids of the form (R_*, R_*R) where R is a topologically flat commutative ring spectrum.

Definition 3.1 *A (A,Γ)-**comodule** is a left A-module M together with a map*

$$\psi_M : M \longrightarrow \Gamma \otimes_A M$$

satisfying a coassociativity and counit condition.

The category (A,Γ)-comod is a cocomplete abelian category [13, Lemma 1.1.1]. It is also a closed monoidal category with symmetric monoidal product $\wedge$ with unit A. For two (A,Γ)-comodules M and N, $M \wedge N$ denotes the tensor product of M and N as left A-modules (A is assumed to be commutative). The comodule structure map is then given by

$$M \otimes N \xrightarrow{\psi_M \otimes \psi_N} (\Gamma \otimes_A M) \otimes (\Gamma \otimes_A N) \xrightarrow{\gamma} \Gamma \otimes_A (M \otimes N)$$

where

$$\gamma((x \otimes m) \otimes_A (y \otimes n)) = xy \otimes_A (m \otimes n),$$

see [13, Lemma 1.1.2]. As for the closed structure, the right adjoint of the monoidal product $\wedge$ is denoted by $F(-,-)$. It is left exact in the first variable and right exact in the second one. If M is finitely presented over A, then

$$F(M,N) \cong A\text{-mod}(M,N)$$

as A-modules, but $F(M,N)$ is generally not isomorphic to (A,Γ)-comod(M,N). For more properties of F see [13, Subsection 1.3].

When one has a flat Adams Hopf algebroid, the category of comodules is a Grothendieck abelian category by [13, Propositions 1.4.1 and 1.4.4]. This property will be important for Sections 5 and 6.

By $C(A,\Gamma)$ we mean the category of chain complexes in (A,Γ)-comodules. There are two model structures on $C(A,\Gamma)$ as described in [13], namely the relative projective model which we consider in Section 5 and the homotopy model structure. The homotopy model structure is a Bousfield localisation of the relative model structure and has various technical advantages over the latter. However, for our purposes it is more appropriate to consider the relative model structure. Let us summarise a few properties.

Theorem 3.2 (Hovey) *The relative projective model structure on $C(A,\Gamma)$ for a flat Adams Hopf algebroid (A,Γ) is cofibrantly (and finitely) generated, proper, stable and monoidal. The cofibrations are precisely the degreewise split monomorphisms whose cokernel is a complex of relative projectives with no differential. It satisfies the monoid axiom and if X is cofibrant and f is a projective equivalence, then $X \wedge f$ is a projective equivalence.*

4 Franke's exotic model

For a spectrum E, the E-local stable homotopy category is obtained from the stable homotopy category by formally inverting those maps that induce isomorphisms in E_*-homology. The resulting category is especially sensitive towards phenomena related to E_*. For certain special homology theories this is an important structural tool for studying the stable homotopy category itself.

Jens Franke used quasi-periodic chain complexes to give an algebraic description of $\mathrm{Ho}(L_1\mathcal{S})$, the K-local stable homotopy category at an odd prime p. Equivalently, one can consider the $E(1)$-local stable homotopy category for the p-local Adams summand $E(1)$. We briefly recall Franke's result, the abelian categories and the self-equivalences used in it.

Theorem 4.1 (Franke) *There is an equivalence of categories*

$$\mathcal{R} : D^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

where $D^{2p-2}(\mathcal{B})$ denotes the derived category of quasi-periodic chain complexes over the abelian category $\mathcal{B}$ and $\mathrm{Ho}(L_1\mathcal{S})$ the $E(1)$-local stable homotopy category. Further, there is a natural isomorphism

$$E(1)_*(\mathcal{R}(C)) \cong \bigoplus_{i=0}^{2p-3} H_i(C)[i].$$

We would like to remark that Franke's theorem also holds for $\mathrm{Ho}(L_n\mathcal{S})$ whenever $n^2+n < 2p-2$. However, the description of the abelian category is less explicit. This is why we only formulate it for $n=1$ and $p>2$, although our main results will also hold in the whole of Franke's range.

Let us recall the ingredients of this theorem. We will first describe a category $\mathcal{A}$ which is equivalent to the category of $E(1)_*E(1)$-comodules as introduced by Bousfield in [15], see also [16].

Definition 4.2 *Let p be an odd prime, set $\mathcal{B}$ to be the category of modules over $\mathbb{Z}_{(p)}$ (the p-local integers), with Adams operations ψ^k, $k \in \mathbb{Z}_{(p)}^*$, such that, for each $M \in \mathcal{B}$:*

- *There is an eigenspace decomposition*

$$M \otimes \mathbb{Q} \cong \bigoplus_{j \in \mathbb{Z}} W_{j(p-1)}$$

  such that for all $w \in W_{j(p-1)}$, and $k \in \mathbb{Z}_{(p)}^*$, $(\psi^k \otimes \mathrm{Id})w = k^{j(p-1)}w$.

- *For each $x \in M$ there is a finitely generated submodule $C(x)$, which contains x, such that for all $m \geqslant 1$, there is an n such that the action of $\mathbb{Z}_{(p)}^*$ on $C(x)/p^m(x)$ factors through the quotient of $(\mathbb{Z}/p^{n+1})^*$ by its subgroup of order $p-1$.*

There is a self-equivalence $T^{j(p-1)} \colon \mathcal{B} \to \mathcal{B}$, for each $j \in \mathbb{Z}$. It leaves the underlying $\mathbb{Z}_{(p)}$-module unchanged but ψ^k acts on this as $k^{j(p-1)}\psi^k$ $(k \in \mathbb{Z}_{(p)}^*)$.

Definition 4.3 *The objects of the category $\mathcal{A}$ are collections $(M_n)_{n \in \mathbb{Z}}$, with $M_n \in \mathcal{B}$, with specified isomorphisms $T^{p-1}(M_n) \to M_{n+2p-2}$, for each $n \in \mathbb{Z}$.*

The category $\mathcal{B}$ is then a subcategory of $\mathcal{A}$, an object $M \in \mathcal{B}$ can be viewed as a collection (M_n), where $M_n = M$ whenever $n \equiv 0 \mod 2p-2$ and is zero elsewhere. The isomorphisms are then the identity on objects. So the category $\mathcal{A}$ is isomorphic to the sum of $2p-2$ shifted copies of $\mathcal{B}$.

Theorem 4.4 (Bousfield, Clarke-Crossley-Whitehouse) *The abelian category $\mathcal{A}$ is isomorphic to the category of $(E(1)_*, E(1)_*E(1))$-comodules.*

We note here that the Hopf algebroid $(E(1)_*, E(1)_*E(1))$ is a flat Adams Hopf algebroid [13, Theorem 1.4.9]. Hence we are consistent with the technical assumptions needed for talking about the relative projective model structure on $C^{(T,N)}(\mathcal{A})$ later.

Definition 4.5 *We define $C^1(\mathcal{A})$ to be $C^{(T^{p-1},1)}(\mathcal{A})$. Similarly, we rename the category $C^{(T^{(2p-2)(p-1)},2p-2)}(\mathcal{B})$ as $C^{2p-2}(\mathcal{B})$. We also rename both T^{p-1} and $T^{(2p-2)(p-1)}$ as T. Note that the two categories are isomorphic.*

Lemma 4.6 *The self-equivalence T is compatible with the monoidal structure on $\mathcal{A}$ in the sense of Definition 2.1.*

Proof The monoidal product on $\mathcal{B}$ is given by tensoring over $\mathbb{Z}_{(p)}$ and allowing $\mathbb{Z}_{(p)}^*$ to act diagonally. So on $X \otimes_{\mathbb{Z}_{(p)}} Y$,

$$\psi^k(a \otimes b) = \psi^k a \otimes_{\mathbb{Z}_{(p)}} \psi^k b.$$

That this product structure satisfies the compatibility conditions is easy to check. The natural isomorphism

$$T(X \otimes_{\mathbb{Z}_{(p)}} Y) \to TX \otimes_{\mathbb{Z}_{(p)}} Y$$

is the identity map on underlying sets. To see that this morphism is a map of $\mathcal{B}$, we note that $T(X \otimes_{\mathbb{Z}_{(p)}} Y)$, ψ^k acts as $k^{p-1}(\psi^k \otimes \psi^k)$, whereas on $TX \otimes_{\mathbb{Z}_{(p)}} Y$, ψ^k acts as $(k^{p-1}\psi^k) \otimes \psi^k$. Since we have tensored over $\mathbb{Z}_{(p)}$, these are the same. We extend this to a monoidal product on $\mathcal{A}$ in the standard manner:

$$(M \wedge N)_n = \bigoplus_{a+b=n} M_a \otimes_{\mathbb{Z}_{(p)}} N_b.$$

The unit is best described as $E(1)_*$. It follows immediately that this tensor product is compatible with T. One could see this equally well by considering the monoidal structure on $E(1)_*E(1)$-comodules, see [16]. ∎

Following Section 2, the monoidal product defined in the above proof induces a closed monoidal structure on the category $C^1(\mathcal{A})$.

Franke constructs a model structure on quasi-periodic chain complexes as follows, see also [2, Example 1.3.3]. A quasi-periodic chain map $f : X \longrightarrow Y$ is:

- a weak equivalence if it is a quasi-isomorphism
- a fibration if it is a degreewise split epimorphism with strictly injective kernel
- a cofibration if it is a monomorphism.

We call this the **injective model structure**, we briefly mentioned this model structure in Lemma 1.6. A quasi-periodic chain complex is C is said to be **strictly injective** if it is degreewise injective and every morphism from C into an acyclic complex K is nullhomotopic via a quasi-periodic homotopy.

Definition 4.7 *In the above special case we denote the respective derived categories of* $C^1(\mathcal{A})$ *and* $C^{2p-2}(\mathcal{B})$ *by* $D^1(\mathcal{A})$ *and* $D^{2p-2}(\mathcal{B})$.

The main defect of the injective model structure is that it is not monoidal (the pushout-product axiom fails). The counterexample is analogous to the one in the injective model structure on chain complexes of R-modules for a ring R [17, Subsection 4.2]. The pushout-product axiom states (in part) that in a monoidal model category with product $\otimes$, if one takes two cofibrations $f : U \longrightarrow V$ and $g : W \longrightarrow X$, then the induced map

$$f \square g : (V \otimes W) \coprod_{U \otimes W} (U \otimes X) \longrightarrow V \otimes X$$

is again a cofibration. To see that this is not the case for $C^1(\mathcal{A})$ with the injective model structure we take

$$U = \mathbb{P}\mathcal{I}, \;\; V = \mathbb{P}(\mathcal{I} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}), \;\; W = 0 \;\text{ and }\; X = \mathbb{P}(\mathcal{I} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}/p)$$

with f and g being the obvious inclusions. Remembering that

$$\mathbb{P}C \wedge_{\mathbb{P}\mathcal{I}} \mathbb{P}D = \mathbb{P}(C \wedge_{\mathcal{I}} D),$$

we see that the induced pushout-product map is $X \longrightarrow 0$, which is clearly not a monomorphism and hence not a cofibration.

5 The relative projective model structure

In this section we are going to summarise the relative projective model structure on $C(\mathcal{G})$. It is a generalisation of the projective model structure on $C(R\text{-mod})$ where R is a commutative ring. It was introduced by Christensen and Hovey in [6]. Assuming that the relative projective model structure exists on $C(\mathcal{G})$ for some $\mathcal{G}$, we are going to discuss the model structure it creates on the quasi-periodic chains $C^{(T,N)}(\mathcal{G})$. At the end of this section we specialise to the case of $C^1(\mathcal{A})$.

One begins by specifying the objects playing the role of the "projective" objects. This class of chosen objects is called a **projective class** $\mathcal{P}$, see [6, Def. 1.1]. The objects $P \in \mathcal{P}$ are called **relative projectives**. A morphism $f : A \longrightarrow B$ in $\mathcal{G}$ is called **$\mathcal{P}$-epimorphism** if it induces an epimorphism $\mathcal{G}(P, A) \longrightarrow \mathcal{G}(P, B)$ for all P in $\mathcal{P}$. Assuming that $\mathcal{G}$ is cocomplete, one way to obtain a projective class is to take any set $\mathcal{S}$ and define $\mathcal{P}$ to be the collection of retracts of coproducts of objects in $\mathcal{S}$, [6, Lemma 1.5].

We use the projective class to define a model structure on $C(\mathcal{G})$. We say that a chain map $f : X \longrightarrow Y$ is:

- a **$\mathcal{P}$-equivalence** if $f_* : \mathcal{G}(P, X) \longrightarrow \mathcal{G}(P, Y)$ is a quasi-isomorphism in $C(\mathbb{Z})$ for all $P \in \mathcal{P}$ (note that $\mathcal{G}(P, X)$ is a chain complex in the usual way with differential $(d_X)_* : \mathcal{G}(P, X_n) \longrightarrow \mathcal{G}(P, X_{n-1})$),
- a **$\mathcal{P}$-fibration** if $\mathcal{G}(P, f)$ is a degreewise surjection for all $P \in \mathcal{P}$,
- a **$\mathcal{P}$-cofibration** if it has the left lifting property with respect to all $\mathcal{P}$-fibrations that are also $\mathcal{P}$-equivalences.

Theorem 5.1 (Christensen-Hovey) *The above three classes form a model structure on the category of chain complexes $C(\mathcal{G})$ if and only if cofibrant replacements exist.*

This model structure is called the **relative projective model structure**, it is proper whenever it exists. Christensen and Hovey also characterise the cofibrant objects in this model structure.

Proposition 5.2 *A chain map $i : A \longrightarrow B$ is a $\mathcal{P}$-cofibration in $C(\mathcal{G})$ if and only if it is a degreewise split monomorphism with $\mathcal{P}$-cofibrant cokernel.*

A chain complex C is cofibrant in $C(\mathcal{G})$ if and only if it is degreewise relative projective and every map from C to a weakly $\mathcal{P}$-contractible chain complex K is nullhomotopic.

Here, **weakly $\mathcal{P}$-contractible** means $\mathcal{P}$-equivalent to 0.

Theorem 5.3 (Hovey) *If $\mathcal{G}$ is a Grothendieck abelian category and the projective class is constructed from a set $\mathcal{S}$ (using retracts and coproducts), then the relative projective model structure on $C(\mathcal{G})$ exists and is cofibrantly generated.*

The statement regarding cofibrant generation is [6, Theorem 5.7], the generating sets are below.

$$I = \{S^{n-1}P \to D^nP | P \in \mathcal{S},\ n \in \mathbb{Z}\} \quad J = \{0 \to D^nP | P \in \mathcal{S},\ n \in \mathbb{Z}\}$$

As usual, $S^{n-1}P$ denotes the chain complex that only consists of P concentrated in degree $n-1$ and D^nP is the chain complex with P in degrees $n-1$ and n (and zeroes elsewhere) with the identity as the only non-trivial differential.

We further assume that $T(\mathcal{P}) = \mathcal{P}$. This implies that T is left Quillen functor, as it preserves the generating sets above.

Proposition 5.4 *Say that a map of $C^{(T,N)}(\mathcal{G})$ is a $\mathcal{P}$-**equivalence** or a $\mathcal{P}$-**fibration** if it is so as a map of $C(\mathcal{G})$. Then these classes of maps define a cofibrantly generated model structure on quasi-periodic chains $C^{(T,N)}(\mathcal{G})$.*

Before we prove the proposition, it is worth mentioning that quasi-isomorphisms are not necessarily $\mathcal{P}$-equivalences. But in Franke's model, the weak equivalences are exactly the quasi-isomorphisms. We will see in Proposition 5.14 that consequently the relative projective model structure and Franke's injective model structure are not Quillen equivalent. However, the relative projective model structure provides a vital intermediate step towards a model structure Quillen equivalent to Franke's model with better properties than the injective model.

We continue with the proof of Proposition 5.4

Proof It is immediate that this model structure is precisely that created by the forgetful functor

$$U : C^{(T,N)}(\mathcal{G}) \longrightarrow C(\mathcal{G})$$

as discussed in Proposition 1.3. The generating cofibrations and acyclic cofibrations are given by the sets

$$\mathbb{P}I = \{\mathbb{P}S^{n-1}P \to \mathbb{P}D^nP | P \in \mathcal{S},\ n \in \mathbb{Z}\} \quad \mathbb{P}J = \{0 \to \mathbb{P}D^nP | P \in \mathcal{S},\ n \in \mathbb{Z}\}$$

where $\mathbb{P}$ is the periodification functor defined in Section 1. ∎

Corollary 5.5 *A cofibration of $C^{(T,N)}(\mathcal{G})$ is a cofibration of $C(\mathcal{G})$. An acyclic cofibration of $C^{(T,N)}(\mathcal{G})$ is an acyclic cofibration of $C(\mathcal{G})$. Thus the functor $\mathbb{R}$ defined in Section 1 is a right Quillen functor, with left adjoint U.*

Proof We prove the first of these statements, the proof of the second is identical (it also follows from the fact that U preserves cofibrations and weak equivalences). The third follows immediately.

Since T is a left Quillen functor, the periodification $\mathbb{P}\colon C(\mathcal{G}) \to C(\mathcal{G})$ is also a left Quillen functor. Hence the set $\mathbb{P}I$ above consists of cofibrations of $C(\mathcal{G})$. It follows immediately that $\mathbb{P}I$-cof (as constructed in the category $C(\mathcal{G})$) is contained in the class of cofibrations of $C(\mathcal{G})$. In turn, applying U to an element of the class $\mathbb{P}I$-cof (as constructed in the category $C^1(\mathcal{G})$) gives a cofibration of $C(\mathcal{G})$. ∎

Let us now give a characterisation of the cofibrant objects in $C^{(T,N)}(\mathcal{G})$ and the cofibrations. These results follow a well-known standard argument (for an example see [6, Section 2]) but we include them for completeness' sake. Of course, since $C^{(T,N)}(\mathcal{G})$ is cofibrantly generated, we can use the usual description of cofibrant objects as retracts of relative cell complexes.

Lemma 5.6 *A quasi-periodic chain complex C is cofibrant in $C^{(T,N)}(\mathcal{G})$ if and only if it is degreewise relative projective and every map from C to a weakly $\mathcal{P}$-contractible quasi-periodic chain complex K is nullhomotopic with quasi-periodic homotopy.*

Proof Let C be degreewise relative projective and assume that every map from C to a weakly $\mathcal{P}$-contractible quasi-periodic chain complex K is nullhomotopic with quasi-periodic homotopy. We are going to show that the inclusion $0 \longrightarrow C$ has the left lifting property with respect to all acyclic fibrations $f : X \longrightarrow Y$, that is, there is a lift in the diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle f}\,\sim \\ C & \xrightarrow[g]{} & Y. \end{array}$$

As C is degreewise relative projective and $f : X \longrightarrow Y$ is a $\mathcal{P}$-epimorphism, there are degreewise lifts $\gamma_n : C_n \longrightarrow X_n$. We can choose those lifts to be quasi-periodic, so

$$\gamma_{n-N} = \alpha_X \circ T\gamma_n \circ \alpha_C^{-1},$$

by simply choosing lifts in degrees 0 to $N-1$ and extending. However, these maps γ_n do not necessarily form a chain map, so we are going to add an extra term to obtain a chain map.

Consider the degree-wise defined map

$$\partial := d_X \circ \gamma - \gamma \circ d_C$$

from C to X. Then $f \circ \partial = 0$, so there is a lift $F : C \longrightarrow K[1]$ where K is the kernel of the acyclic fibration f, thus $F \circ j = \partial$, where $j : K \longrightarrow X$ is the inclusion. One can check that F is not just a degreewise map in $\mathcal{G}$ but a chain map. (Note that $d_{K[1]} = -d_K$.) This map F can also be chosen to be a quasi-periodic map.

The kernel K is weakly contractible, so by assumption F is nullhomotopic with quasi-periodic nullhomotopy, i.e. there is a family of maps

$$h_n : C_n \longrightarrow K_n$$

such that

$$F_n = h_{n-1} \circ d_C + d_{K[1]} \circ h_n$$

and

$$h_{n-N} = \alpha_K \circ Th_n \circ \alpha_C^{-1}.$$

Now define the desired lift $\tilde{g}$ as $\tilde{g} := \gamma + j \circ h$. This is a quasi-periodic chain map by construction and satisfies $f \circ \tilde{g} = g$.

Conversely, let $C \in C^{(T,N)}(\mathcal{G})$ be cofibrant. Because C is also cofibrant in $C(\mathcal{G})$ by Corollary 5.6, we know that C is degreewise relative projective. Further, let $f : C \longrightarrow K$ be a morphism with $K \in C^1(\mathcal{G})$ be weakly contractible. Consider the quasi-periodic chain complex $PK := K \oplus K[-1]$ with differential $d(x, y) = (dx, x - dy)$. The projection $p : PK \longrightarrow K$ is an acyclic fibration, so f factors over p because C is cofibrant. So there is a lift

$$\begin{array}{ccc} 0 & \longrightarrow & PK \\ \downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle p}\,\sim \\ C & \xrightarrow[f]{} & K. \end{array}$$

with $\tilde{f} = (f, h)$ where $h : C \longrightarrow K[-1]$ is a quasi-periodic chain map. Because $\tilde{f}$ is also a quasi-periodic chain map, we have

$$f = d \circ h + h \circ d,$$

so h also serves as a quasi-periodic nullhomotopy of f, which is what we wanted to prove. ∎

Lemma 5.7 *A quasi-periodic chain map $i : A \longrightarrow B$ is a $\mathcal{P}$-cofibration in $C^{(T,N)}(\mathcal{G})$ if and only if it is a degreewise split monomorphism with $\mathcal{P}$-cofibrant cokernel.*

Proof The cokernel of a cofibration is cofibrant as it is the pushout of i along the zero map, and cofibrations are invariant under pushouts. It is a split monomorphism, because by Corollary 5.6, it is a cofibration in $C(\mathcal{G})$.

Now let $i : A \longrightarrow B$ be a degreewise split monomorphism with cofibrant cokernel C. We would like to show that i has the LLP with respect to an acyclic fibration $p : X \longrightarrow Y$ as in the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle p}\,\sim \\ B & \xrightarrow[g]{} & Y. \end{array}$$

Because i is a split monomorphism, we can write $B = A \oplus C$ and $g = (g_A, g_C)$. Since C is cofibrant, there is a lift $\tilde{g}_C$ similarly to the previous lemma. Hence $\tilde{g} := (f, \tilde{g}_C)$ is the desired lift in the diagram, so i is a cofibration. ∎

Using the results of Section 2 we can consider monoidal structures on $C^{(T,N)}(\mathcal{G})$, Corollary 2.6 gives the following result.

Proposition 5.8 *Assume that the relative projective model structure on $C(\mathcal{G})$ is a monoidal model category that satisfies the monoid axiom. Assume further that $T(\mathcal{P}) = \mathcal{P}$ and that T is compatible with the monoidal structure of $\mathcal{G}$ in the sense of Definition 2.1. Then the induced model structure on $C^{(T,N)}(\mathcal{G})$ is monoidal and satisfies the monoid axiom.*

We now turn to our motivating example: the category of (A,Γ)-comodules for (A,Γ) a flat Adams Hopf algebroid, see Section 3. We assume that this category has a self-equivalence T that is compatible with the monoidal product. We introduce a particular projective class, we will use it to construct the quasi-projective model structure and see that it has some useful properties.

Remember that F denotes the function object of a closed monoidal category and $\mathcal{I}$ denotes the unit.

Definition 5.9 *The **dual** of an (A,Γ)-comodule M is $DM = F(M,\mathcal{I})$. A comodule is called **dualisable** if the natural map $DM \wedge N \longrightarrow F(M,N)$ is an isomorphism for all N.*

In the case of flat Adams Hopf algebroid, a comodule is dualisable if and only if it is finitely generated and projective as an underlying A-module [13, Proposition 1.3.4]. As a consequence, the collection of isomorphism classes of dualisable comodules is a set $\mathcal{S}$. The projective class associated to this set [6, Lemma 1.5] gives the the relative projective model structure on chain complexes of comodules via Theorem 5.3. We now exploit the good monoidal properties obtained by choosing the projective class to come from the dualisable objects. In particular, we now longer have to worry about asking for $T(\mathcal{P}) = \mathcal{P}$.

Lemma 5.10 *If T is compatible with the monoidal structure on the category of (A,Γ)-comodules, then P is dualisable if and only if T^nP is dualisable for each $n \in \mathbb{Z}$.*

Proof We have the following commutative diagram (see Lemma 2.2), from which the result follows.

$$\begin{array}{ccc} F(TP,X) & \xrightarrow{\cong} & T^{-1}F(P,X) \\ \downarrow & & \\ F(TP,\mathcal{I}) \wedge X & & \downarrow \cong \\ \downarrow \cong & & \\ T^{-1}F(P,\mathcal{I}) \wedge X & \xrightarrow{\cong} & T^{-1}(F(P,\mathcal{I}) \wedge X) \end{array}$$

■

Lemma 5.11 *The monoidal product of two dualisable comodules is dualisable. The dual of a dualisable comodule is dualisable. There is a natural map $X \to DDX$ that is an isomorphism if X is dualisable.*

All of these results on dualisable objects hold in a general closed monoidal category, see [18, Chapter III] for a detailed description. Together with Proposition 5.8 and [13, Proposition 2.1.4] we obtain the following corollary.

Corollary 5.12 *In the context of Section 4, $C^1(\mathcal{A})$ with the relative projective model structure is monoidal and satisfies the monoid axiom.*

The monoidal product of two objects M and N of $C^{(T,N)}((A,\Gamma)\text{-comod})$ is given by $M \wedge_{\mathbb{P}\mathcal{I}} N$. This product is particularly well behaved, as well as satisfying the pushout product axiom, we have the lemma below, which will make it easier to calculate the derived monoidal product on $D^{(T,N)}(\mathcal{G})$.

Lemma 5.13 *If X is a $\mathcal{P}$-cofibrant quasi-periodic chain complex, then the functor $X \wedge_{\mathbb{P}\mathcal{I}}(-)$ preserves $\mathcal{P}$-equivalences. Furthermore every $\mathcal{P}$-cofibrant object X of $C^1(\mathcal{G})$ is a retract of some $\mathbb{P}Y$, where Y is an I-cell complex of $C((A,\Gamma)\text{-comod})$.*

Proof The quasi-periodic chain complex X is a retract of some $\mathbb{P}I$-cell complex Z. The quasi-periodic chain complex Z is constructed as a colimit of pushouts of coproducts of maps in $\mathbb{P}I$, where $Z_0 = 0$. Since $0 = \mathbb{P}0$, it follows that $Z = \mathbb{P}Y$, where Y is a colimit of pushouts of coproducts of maps in I. This proves the second statement. For the first, take X, Y and Z as above, then

$$\mathbb{P}Y \wedge_{\mathbb{P}\mathcal{I}} (-) \cong Y \wedge_{\mathcal{I}} (-).$$

We know that Y is cofibrant in $C((A,\Gamma)\text{-comod})$, hence by [13, Proposition 2.1.4] this functor preserves $\mathcal{P}$-equivalences. The functor $X \wedge_{\mathbb{P}\mathcal{I}} (-)$ is a retract of this functor and hence also preserves $\mathcal{P}$-equivalences. ∎

Focusing upon Franke's exotic model we compare the relative projective model structure with the injective model structure on Franke's model.

Proposition 5.14 *The identity functor provides a Quillen adjoint pair between $C^1(\mathcal{A})$ with the injective model structure and the relative projective model structure. However, the two model structures are not Quillen equivalent.*

Proof The identity is a left Quillen functor from $C(\mathcal{A})$ with the relative projective model structure to $C(\mathcal{A})$ with the injective model structure: a $\mathcal{P}$-cofibration is an injective cofibration as it is in particular a monomorphism by Lemma 5.7. By [13, 2.1.5] a $\mathcal{P}$-equivalence is also a H_*-isomorphism. It follows that we have a Quillen pair between the lifted model structures on $C^1(\mathcal{A})$.

However, to obtain a Quillen equivalence the weak equivalences have to agree between $\mathcal{P}$-cofibrant X and injectively fibrant Y. We first show that this is not true for $C(\mathcal{A})$. Consider the chain complex X that consists of $E(1)_*$ concentrated in degree zero. Then we take an injectively fibrant replacement Y of X. The map $X \to Y$ is a quasi-isomorphism by definition, X is $\mathcal{P}$-cofibrant and Y injectively fibrant. This map is not a $\mathcal{P}$-equivalence. To see this, just take $P = E(1)_*$ itself. Then

$$\mathrm{H}_*(\mathcal{A}(P,X)) = \mathcal{A}(P, E(1)_*)$$

concentrated in degree 0. But

$$\mathrm{H}_*(\mathcal{A}(P,Y)) = \mathrm{Ext}^*_{\mathcal{A}}(E(1)_*, E(1)_*).$$

There are non-trivial higher Ext groups on the right side, so the two homologies are not isomorphic. One can periodify the above to get the desired counterexample. ∎

6 The quasi-projective model structure

We saw at the end of Section 5 that the relative projective model structure has fewer weak equivalences than the injective model structure — too few for the model categories to be Quillen equivalent. To fix this deficit we can add weak equivalences to the relative projective model structure via Bousfield localisation. As a result we will obtain a model structure for $D^1(\mathcal{A})$ that still has the nice monoidal properties of the relative projective model structure. For clarity, we restrict ourselves to the case of a flat Adams Hopf algebroid (A, Γ), with a self equivalence T on the category of comodules which is compatible with the monoidal product. We will comment on the general case at the end of this section.

We make use of the paper [11] to show that there is a model structure on the category of chain complexes of comodules where the cofibrations are the $\mathcal{P}$-cofibrations and the weak equivalences are the quasi-isomorphisms. We state the theorem we will use later, it is a theorem of Smith, but appears as [11, Theorem 1.7].

The key is using the notions of a class of maps having the "solution set condition" or being "accessible". It is technically awkward to perform constructions such as Bousfield localisation using a class of maps rather than a set of maps. However, if the class of maps satisfies the solution set condition, then it contains a set such that localising with respect to this set gives the Bousfield localisation with respect to the whole class. So the solution set condition can be used to avoid this awkwardness, for the full definition see [11, Definition 1.5]. Accessibility is a another technical condition [11, Definition 1.14], but in particular, an accessible class of maps in a locally presentable category has the solution set condition [11, Proposition 1.15].

Theorem 6.1 *Let $\mathcal{C}$ be a locally presentable category, $\mathcal{W}$ a subcategory and I a set of morphisms of $\mathcal{C}$. Suppose they satisfy the criteria:*

- *c0: $\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property*
- *c1: I-inj is contained in $\mathcal{W}$*
- *c2: I-cof $\cap \mathcal{W}$ is closed under taking cell complexes*
- *c3: $\mathcal{W}$ satisfies the solution set condition at I.*

Setting the weak equivalences to be $\mathcal{W}$, the cofibrations to be I-cof and the fibrations to be $(I\text{-cof} \cap W)$-inj, one obtains a cofibrantly generated model structure on $\mathcal{C}$.

The notations I-cof and I-inj are technical but standard, so we refer the reader to [17, Subsection 2.1] rather than recall them here.

We use this theorem to obtain a model structure on quasi-periodic chain complexes whose cofibrations are the $\mathcal{P}$-cofibrations as introduced in Section 5 and whose weak equivalences are the quasi-isomorphisms. Remember that our class of relative projectives $\mathcal{P}$ is constructed from the set of isomorphism classes of dualisable objects $\mathcal{S}$.

Proposition 6.2 *Let $\mathcal{W}$ be the set of quasi-isomorphisms and let I be the set*

$$I = \{S^{n-1}P \to D^nP | P \in \mathcal{S},\ n \in \mathbb{Z}\}.$$

Then the above result gives a cofibrantly generated model structure on $C((A,\Gamma)$-comod$)$, which we call the quasi-projective model structure:

- *the weak equivalences are the quasi-isomorphisms,*
- *the cofibrations are the $\mathcal{P}$-cofibrations,*
- *the fibrations are those maps that have the left-lifting-property with respect to the acyclic cofibrations.*

Proof Condition c0 is obvious. The set I is the set of generating cofibrations of the relative projective model structure on $C((A,\Gamma)$-comod$)$, so I-inj is the class of acyclic $\mathcal{P}$-fibrations. Hence condition c1 is contained within [13, Proposition 2.1.5] which states that every projective equivalence is a homology isomorphism.

For condition c2, we know that I-cof is closed under transfinite composition and under pushouts. We know that the class of monomorphisms that are quasi-isomorphisms is closed, this class is the class of acyclic cofibrations of the injective model structure. Hence their intersection is also closed. By the proof of [11, Proposition 3.13], the quasi-isomorphisms are accessible, thus by Proposition 1.15 of the same paper, the solution set condition holds and we see that c3 holds. ∎

By Proposition 1.3, we also obtain a model structure on the category of quasi-periodic chain complexes since we assumed that $T(\mathcal{P}) = \mathcal{P}$ for our chosen class of relative projectives.

Corollary 6.3 *There is a model structure on the category of quasi-periodic chain complexes $C^{(T,N)}((A,\Gamma)$-comod$)$ where the weak equivalences are the quasi-isomorphisms and the cofibrations are degreewise split monomorphisms with $\mathcal{P}$-cofibrant cokernel. We call this the* ***quasi-projective model structure****.*

Corollary 6.4 *The quasi-projective model structure is the Bousfield localisation of the relative projective model structure with respect to the class of quasi-isomorphisms.*

It should be remarked that a simpler way to construct the quasi-projective model structure can be found in [19]. Here, Cole discusses how to construct a model structure on a category from "mixing" two existing ones. However, this does not examine whether the resulting model structure is cofibrantly generated, which is what we need to discuss its monoidal properties.

Theorem 6.5 *We have the following diagram of Quillen adjunctions, where all vertical arrows are identity functors and the horizontal arrows are periodification and the forgetful functor as introduced in Proposition 1.3.*

$$\begin{array}{ccc}
(A,\Gamma)\text{-}comod_{rel\ proj} & \underset{U}{\overset{\mathbb{P}}{\rightleftarrows}} & C^{(T,N)}((A,\Gamma)\text{-}comod)_{rel\ proj} \\
\downarrow\uparrow & & \downarrow\uparrow \\
(A,\Gamma)\text{-}comod_{quasi\ proj} & \underset{U}{\overset{\mathbb{P}}{\rightleftarrows}} & C^{(T,N)}((A,\Gamma)\text{-}comod)_{quasi\ proj} \\
\downarrow\uparrow & & \downarrow\uparrow \\
(A,\Gamma)\text{-}comod_{inj} & \underset{U}{\overset{\mathbb{P}}{\rightleftarrows}} & C^{(T,N)}((A,\Gamma)\text{-}comod)_{inj}
\end{array}$$

Furthermore the injective and quasi-projective model structures are Quillen equivalent.

Proof The upper vertical pairs are Quillen pairs as the cofibrations are the same and a weak equivalence in the relative projective model structure is a quasi-isomorphism. For the lower vertical pairs, a cofibration in the quasi-projective model structure is a $\mathcal{P}$-cofibration, hence a monomorphism. The weak equivalences in both are the quasi-isomorphisms. Thus the identity functor from the quasi-projective model structure to the injective model structure preserves cofibrations and weak equivalences, hence it is a left Quillen functor. This also shows that the quasi-projective and injective model structures must be Quillen equivalent as they have the same weak equivalences. ∎

We note that one could have constructed the quasi-projective model structure on $C^{(T,N)}((A,\Gamma)\text{-comod})$ directly, taking care to show the category-theoretic conditions of Theorem 6.1. We are now going to exploit the monoidal properties of this model structure. We would like to make use of Proposition 2.5, but while the pushout product axiom holds, it is not obvious to us why the monoid axiom would hold for this model structure on $C((A,\Gamma)\text{-comod})$ or $C^{(T,N)}((A,\Gamma)\text{-comod})$. Thus we prove that we have a monoidal model structure directly.

Lemma 6.6 *The quasi-projective model structure on $C((A,\Gamma)\text{-}comod)$ is monoidal.*

Proof We first note that the unit is cofibrant. We know that the pushout of two cofibrations is a cofibration, because the relative projective model structure satisfies the pushout product axiom. Now consider the pushout product of a generating cofibration with a generating acyclic cofibration. Let k be the inclusion $S^{n-1}\mathcal{I} \to D^n\mathcal{I}$, then $P \wedge k$ is the general form of any generating cofibration (where P is a dualisable object). Let $f\colon X \to Y$ be a generating acyclic cofibration. Then the pushout product of $P \wedge k$ and f is simply P smashed with the pushout product of k and f. We must check that this map is a quasi-isomorphism. Consider the following pushout diagram

$$\begin{array}{ccc}
S^{n-1}\mathcal{I} \wedge X & \longrightarrow & D^n\mathcal{I} \wedge X \\
\downarrow & & \downarrow \\
S^{n-1}\mathcal{I} \wedge Y & \longrightarrow & Q
\end{array}$$

the left hand vertical map is a monomorphism and a homology isomorphism (modulo signs for the differential, it is just a suspension of f). It follows that the right hand vertical map is also a monomorphism and a homology isomorphism as acyclic cofibrations are preserved by pushouts.

It is easy to see that $D^n\mathcal{I} \wedge X$ has trivial homology, as does $D^n\mathcal{I} \wedge Y$, whence the pushout product of k and f (see page 12),

$$k\Box f : Q \to D^n\mathcal{I} \wedge Y$$

must be a homology isomorphism. We now need to see that $(k\Box f) \wedge P$ is a homology isomorphism. But this is a statement about underlying A-modules, where P is finitely generated and projective, hence $(k\Box f) \wedge P$ is a homology isomorphism. ∎

Corollary 6.7 *The pushout product axiom holds for $C^{(T,N)}((A,\Gamma)$-comod$)$ with the quasi-projective model structure and monoidal product $\wedge_{\mathbb{P}\mathcal{I}}$.*

Proof We copy the proof of the above lemma. We know that the relative projective model structure is monoidal, hence the pushout product of two cofibrations in the quasi-projective model structure is a cofibration. A generating acylic cofibration for this model structure on $C^1(\mathcal{A})$ has form $\mathbb{P}f : \mathbb{P}X \to \mathbb{P}Y$, where f is a generating acyclic cofibration for $C^1(\mathcal{A})$. Similarly, a generating cofibration looks like $\mathbb{P}(P \wedge k)$, for k and P as in the previous proof. Recall that now we are taking the product over $\mathbb{P}\mathcal{I}$. Drawing the relevant diagram it is easy to see that we need $\mathbb{P}((k\Box f) \wedge P)$ to be a homology isomorphism. We know that $(k\Box f) \wedge P$ is a homology isomorphism and $\mathbb{P}$ preserves homology isomorphisms, since it is just an infinite direct sum of shifts and applications of T, so we are done. ∎

Recall that a fibration in the relative projective model structure is, in particular, a surjection [13, Proposition 2.1.5.]. It follows that any quasi-projective fibration is a surjection and similarly that any quasi-projective cofibration is a monomorphism.

Lemma 6.8 *The quasi-projective model structure is proper.*

Proof The long exact sequence in homology implies that any model structure on $C((A,\Gamma)$-comod$)$ will be proper as long as weak equivalences coincide with quasi-isomorphisms, every cofibration is an injection, and every fibration is a surjection. It follows immediately that the quasi-projective model structure on $C^{(T,N)}((A,\Gamma)$-comod$)$ is proper. ∎

We summarise our work in the following theorem.

Theorem 6.9 *The quasi-projective model structure on $C^{(T,N)}((A,\Gamma)$-comod$)$ is cofibrantly generated, proper and monoidal. In the special case of T, $\mathcal{A}$ and N as in Section 4 the homotopy category of this model category is precisely $D^1(\mathcal{A})$.*

Corollary 6.10 *Franke's model $D^1(\mathcal{A})$ is a symmetric monoidal category with tensor product $\wedge^L_{\mathbb{P}\mathcal{I}}$.*

Remark 6.11 *Consider a general Grothendieck category that is closed monoidal. Assume first that that the collection of isomorphism classes of dualisable objects forms a set $\mathscr{S}$. Secondly, assume that this set generates the category, that is, the coproduct of all elements of $\mathscr{S}$ is a generator. Then the relative projective, quasi-projective and injective model structures all exist for $C^{(T,N)}(\mathcal{G})$. The first two are monoidal model categories, all three are proper and the obvious analogue of Theorem 6.5 holds.*

The extra work required is reproving [13, Theorem 2.1.5] in this situation, this is quite straightforward. The fibrations in the relative projective model structure are all epimorphisms, so both model structures are proper. Since dualisables are flat the pushout product axioms hold.

7 Picard groups

In this section we are going to compare the Picard group of $D^1(\mathcal{A})$ to the Picard group of the $K_{(p)}$-local stable homotopy category $\mathrm{Ho}(L_1\mathcal{S})$. Let $\mathcal{M}$ be a monoidal category with unit $\mathcal{I}$ and product $\wedge$. The **Picard group** $\mathrm{Pic}(\mathcal{M})$ is the group of invertible objects in this category: its objects are the isomorphism classes of $X \in \mathcal{M}$ such that there is an object $Y \in \mathcal{M}$ with $X \wedge Y \cong \mathcal{I}$. The group multiplication is induced by $\wedge$.

Picard groups have their origin in algebraic geometry but have increasingly been studied in stable homotopy theory. Of particular interest are the Picard groups of Bousfield localisations of the stable homotopy category or the homotopy category of R-modules for an E_∞-ring spectrum R. For example, it is well-known that the Picard group of the stable homotopy category is $\mathbb{Z}$, generated by the 1-sphere. This result was later reproved by Baker and Richter in [20] who also gave computations of $\mathrm{Ho}(R-\mathrm{mod})$ for some connective E_∞ ring spectra R.

Let $\mathrm{Ho}(L_n\mathcal{S})$ denote the $E(n)$-local stable homotopy category where $E(n)$ is the n^{th} Johnson-Wilson spectrum. Hovey and Sadofsky showed that for $n^2+n < 2p-2$,

$$\mathrm{Pic}(\mathrm{Ho}(L_n\mathcal{S})) \cong \mathbb{Z},$$

consisting of shifts of the sphere spectrum, see [5]. Georg Biedermann, in [21], later extended the computation to p and n with $p > n+1$ and $4p-3 > n^2+n$. This means we know that for p odd, $\mathrm{Pic}(\mathrm{Ho}(L_1\mathcal{S})) \cong \mathbb{Z}$, consisting of the spheres.

The previous section shows that that $D^1(\mathcal{A})$ is a symmetric monoidal category, so it makes sense to consider its Picard group and compare it to $\mathrm{Pic}(\mathrm{Ho}(L_1\mathcal{S}))$.

Let us remember that Franke's model $C^1(\mathcal{A})$ does not only work for the $E(1)$-local stable homotopy category and $p > 2$. An analogous construction works for all n and p with $n^2+n < 2p-2$. Hence in this range, the $E(n)$-local stable homotopy category possesses an exotic algebraic model. Although not obviously related, it is no coincidence that this range for n and p agrees with the range of Hovey's and Sadofsky's Picard group computation.

Both results use the fact that the $E(n)$-based Adams spectral sequence collapses for those n and p. In Franke's proof, the collapsing is used rather indirectly for an algebraic

description of some morphisms in $\mathrm{Ho}(L_n\mathcal{S})$. Hovey and Sadofsky show that an element X of $\mathrm{Pic}(\mathrm{Ho}(L_n\mathcal{S}))$ satisfies

$$E(n)_* \cong E(n)_*(X) \quad \text{in} \quad E(n)_*E(n)\text{-comod.}$$

Sparseness of the $E(n)$-Adams spectral sequence is a key ingredient for constructing a weak equivalence $L_nS^0 \longrightarrow X$.

While in the "exotic range" n, p with $n^2 + n < 2p - 2$ the $E(n)$-local Picard group is trivial, this is not the case for $n = 1$ and $p = 2$. For $p = 2$,

$$\mathrm{Pic}(\mathrm{Ho}(L_1\mathcal{S})) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

The $\mathbb{Z}/2$-summand is generated by the so-called question mark complex [5, Theorem 6.1]. Also, we know that for $p = 2$, $\mathrm{Ho}(L_1\mathcal{S})$ is rigid and hence has no exotic models. It would be an interesting topic to relate Picard groups to the existence of exotic models.

Lemma 7.1 $\mathrm{Pic}(D^1(\mathcal{A}))$ *is a set.*

Proof Our category $D^1(\mathcal{A})$ is triangulated, so we apply [22, A.2.8, 2.1.3 and 2.3.6]. ■

Franke's theorem tells us that $D^1(\mathcal{A})$ and $\mathrm{Ho}(L_1\mathcal{S})$ are equivalent as triangulated categories via the functor $\mathcal{R}$. However, $\mathcal{R} : D^1(\mathcal{A}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$ is not monoidal as it is not associative [4, Remark 1.4.2]. Hence it does not automatically induce a group homomorphism between the respective Picard groups. Extra work is needed to see that $\mathcal{R}$ preserves just enough structure to use it for comparing these Picard groups.

Theorem 7.2 (Ganter) *There is a natural isomorphism*

$$\mathcal{R}(C \wedge^L_{\mathbb{P}\mathcal{I}} D) \cong \mathcal{R}(C) \wedge^L \mathcal{R}(D)$$

where $\wedge^L$ *denotes the smash product in* $\mathrm{Ho}(L_1\mathcal{S})$, $\mathcal{I} = E(1)_*$ *and* $\wedge^L_{\mathbb{P}\mathcal{I}}$ *is the monoidal product of* $D^1(\mathcal{A})$.

Note that in her theorem, Ganter denotes the derived tensor product of quasi-periodic chain complexes by $\otimes^L_{E(1)_*}$. This is not to be confused with the tensor product in $D(\mathcal{A})$.

She defines the monoidal product on $D^1(\mathcal{A})$ as the tensor product of underlying degreewise flat replacements, i.e. flat as $E(1)_*$-modules. Ganter's proof that this gives a monoidal structure uses homological algebra following [**?**]. While Ganter mentions the concept of monoidal model categories, she does not address the question of whether $C^1(\mathcal{A})$ is such a category.

Theorem 6.9 answers this question. For a monoidal model category $\mathcal{M}$ with product $\otimes$, the derived product $\otimes^L$ on $\mathrm{Ho}(\mathcal{M})$ is defined as

$$X \otimes^L Y = QX \otimes QY$$

where QX and QY are cofibrant replacements of $X, Y \in \mathcal{M}$ [17, 4.3.2]. Since in our case the cofibrant objects are also degreewise flat, we are consistent with Ganter's result and can write down the above theorem. Further, we can use it to compute $\mathcal{R}(C \wedge^L_{\mathbb{P}\mathcal{I}} D)$.

Theorem 7.3 $\mathrm{Pic}(D^1(\mathcal{A})) \cong \mathbb{Z}$

Proof Let C and D be an inverse pair of quasi-periodic chain complexes in $D^1(\mathcal{A})$, so

$$C \wedge^L_{\mathbb{P}\mathcal{I}} D \cong \mathbb{P}\mathcal{I}.$$

Applying $\mathcal{R}$ to this equation and using Ganter's theorem gives

$$\mathcal{R}(C \wedge^L_{\mathbb{P}\mathcal{I}} D) \cong \mathcal{R}(\mathbb{P}\mathcal{I}) \cong \mathcal{R}(C) \wedge^L \mathcal{R}(D).$$

Furthermore, we know that $\mathcal{R}$ sends the unit $\mathbb{P}\mathcal{I}$ to the $E(1)$-local sphere $L_1 S^0$. This can be read off the natural isomorphism given in Franke's theorem and the fact that $E(1)_*$ reflects isomorphisms. So we arrive at the statement

$$L_1 S^0 \cong \mathcal{R}(C) \wedge^L \mathcal{R}(D).$$

This means that $\mathcal{R}(C)$ and $\mathcal{R}(D)$ are in the Picard group of $\mathrm{Ho}(L_1\mathcal{S})$ and hence must be suspensions of the $E(1)$-local sphere. Being an equivalence of triangulated categories $\mathcal{R}$ reflects isomorphisms, so C and D are shifts of $\mathbb{P}\mathcal{I}$. Since

$$\mathbb{P}\mathcal{I}[i] \wedge^L_{\mathbb{P}\mathcal{I}} \mathbb{P}\mathcal{I}[j] \cong \mathbb{P}\mathcal{I}[i+j]$$

in $D^1(\mathcal{A})$, this completes the proof of our theorem. ∎

As mentioned, the construction of Franke's functor $\mathcal{R}$ also extends to the $E(n)$-local stable homotopy category for $n^2+n \leq 2p-2$, i.e. there is an equivalence of triangulated categories

$$\mathcal{R} : D^1(\mathcal{A}) \longrightarrow \mathrm{Ho}(L_n\mathcal{S})$$

for some abelian category $\mathcal{A}$ that is equivalent to the category of $E(n)_*E(n)$-comodules. But the monoidal behaviour of $\mathcal{R}$ in this general case is not yet known.

In particular, Ganter's construction only works for $n = 1$. It will be worth investigating whether our results about the properties of $C^1(\mathcal{A})$ give a more straightforward analogue of Theorem 7.2.

Ganter's isomorphism for $n = 1$ allows us to read off the Picard group of Franke's model in a simple way. However, it would be interesting to know if, particularly for higher n, this Picard group can be calculated more directly. We hope that considering these questions will lead to more insight into the existence of exotic models and hence understanding the structure of the $E(n)$-local stable homotopy categories.

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