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Order structures, Jordan algebras, and geometry

In this article Bas Lemmens and Onno van Gaans discuss one of the central themes of the Lorentz Center workshop 'Order Structures, Jordan Algebras, and Geometry' held in May 2017 at the Lorentz Center in Leiden, which was organised by the authors in collaboration with Cho-Ho Chu from Queen Mary University of London.

The notion of a Jordan algebra has a long and rich history in mathematics. It was originally introduced by Pascual Jordan in a quest to find alternative algebraic settings for quantum mechanics. Although this program failed, Jordan algebras turned out to have deep connections with diverse areas of mathematics including Lie theory, differential geometry and mathematical analysis.

A *real Jordan algebra* is a real vector space $\mathcal{A}$ with a bilinear product $(a,b) \in \mathcal{A} \times \mathcal{A} \mapsto a \bullet b \in \mathcal{A}$ satisfying

1. $a \bullet b = b \bullet a$,
2. $a^2 \bullet (a \bullet b) = a \bullet (a^2 \bullet b)$ (Jordan Identity).

So, Jordan algebras are commutative, but in general fail to be associative — the Jordan Identity only gives power associativity.

Throughout this article we will assume that the Jordan algebra has a *unit*, denoted e. A Jordan algebra is said to be *formally real* if $a^2 + b^2 = 0$ implies $a = 0$ and $b = 0$. A prime example of a formally real Jordan algebra is the space of $n \times n$ Hermitian matrices, $\mathcal{H}_n(\mathbb{C})$, with Jordan product

$$A \bullet B = \frac{AB + BA}{2} \tag{1}$$

for $A, B \in \mathcal{H}_n(\mathbb{C})$.

More generally one can consider the space of bounded self-adjoint operators on a Hilbert space H, denoted $B(H)_{\mathrm{sa}}$, and define a Jordan product $A \bullet B$ as in (1). Another interesting class of examples are the so called *spin factors* which are defined as follows. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $V = H \oplus \mathbb{R}$. For (x, α) and (y, β) in V (so $x, y \in H$ and $\alpha, \beta \in \mathbb{R}$) one defines a Jordan product by

$$(x, \alpha) \bullet (y, \beta) = (\beta x + \alpha y, \alpha\beta + \langle x, y \rangle).$$

Finally we should also mention the space $C(K)$ of continuous functions on a compact Hausdorff space K with Jordan product $f \bullet g = fg$, which is an associative Jordan algebra.

In a famous paper Jordan, von Neumann and Wigner classified the finite dimensional formally real Jordan algebras [6]. They showed, in finite dimensions, that every formally real Jordan algebra can be written as a direct sum of simple ones of which there are only five types: the space of symmetric $n \times n$ real matrices, $\mathcal{S}_n(\mathbb{R})$, with $n \geq 3$, the space of $n \times n$ Hermitian matrices, $\mathcal{H}_n(\mathbb{F})$, over the fields $\mathbb{C}$ and $\mathbb{H}$ with $n \geq 3$, the spin factors with H an n-dimensional real inner-product space with $n \geq 0$, and an exceptional one $\mathcal{H}_3(\mathbb{O})$, where $\mathbb{O}$ are the octonians. This is a 27-dimensional formally real Jordan algebra which is also known as the Albert algebra. In all but the spin factors the Jordan product is as in (1).

Symmetric cones

A deep connection between finite dimensional formally real Jordan algebras and the geometry of cones was independently discovered by Koecher [8] and Vinberg [16].

Here a *cone* C is a convex subset of a real vector space V such that $C \cap (-C) = \{0\}$

Photo: Konrad Jacobs, Mathematisches Forschungsinstitut Oberwolfach

Max Koecher

Ernest Vinberg

and $\lambda C \subseteq C$ for all $\lambda \geq 0$. Koecher and Vinberg showed that the interior, $\mathcal{A}_+^\circ$, of the cone of squares $\mathcal{A}_+ = \{x^2 : x \in \mathcal{A}\}$ in a finite-dimensional formally real Jordan algebra is a *symmetric cone*. Recall that the interior, C°, of a cone C in a finite dimensional vector space V is a *symmetric cone* if

1. there exists an inner product $(\cdot|\cdot)$ on V such that C is *self-dual*, i.e.,
$$\begin{aligned} C &= C^* \\ &= \{y \in V : (y \mid x) \geq 0 \text{ for all } x \in C\}, \end{aligned}$$
2. C° is *homogeneous*, that is to say, the group of (linear) automorphisms of C,
$$\mathrm{Aut}(C) = \{A \in \mathrm{GL}(V) : A(C) = C\},$$
acts transitively on C°.

Conversely, any symmetric cone in a finite-dimensional vector space V can be realised as the interior of the cone of squares of a formally real Jordan algebra on V. For example, the *Lorentz cone*,

$$\begin{aligned} \Lambda = \{(x_1,\ldots,x_n,x_{n+1}) \in \mathbb{R}^{n+1} &: \\ \sqrt{x_1^2 + \cdots + x_n^2} \leq x_{n+1}\}&, \end{aligned}$$

is the cone of squares in the spin factor $\mathbb{R}^n \oplus \mathbb{R}$.

This characterisation of finite-dimensional Euclidean Jordan algebras provides a connection with the geometry of real manifolds. Indeed, symmetric cones are prime examples of Riemannian symmetric spaces. To explain this connection in more detail we need to recall some basic results from the Jordan theory. Let $\mathcal{A}$ be a real Jordan algebra with unit e. The *spectrum* of $a \in \mathcal{A}$ is given by

$$\sigma(a) = \{\lambda \in \mathbb{R} : a - \lambda e \text{ is not invertible}\}.$$

An element c of $\mathcal{A}$ is called an *idempotent* if $c^2 = c$, and it is said to be *primitive* idempotent if it cannot be written as the sum of two non-zero idempotents.

Now suppose that $\mathcal{A}$ is a finite-dimensional Euclidean Jordan algebra. A set $\{c_1,\ldots,c_k\} \subseteq \mathcal{A}$ of primitive idempotents is called a *complete system of orthogonal primitive idempotents*, or, a *Jordan frame* if

1. $c_i \bullet c_j = 0$ for all $i \neq j$,
2. $c_1 + \cdots + c_k = e$.

The Spectral Theorem [5, Theorem III.1.2] says that for each a in a finite-dimensional Euclidean Jordan algebra $\mathcal{A}$ there exists a Jordan frame $\{c_1,\ldots,c_k\}$ and unique real numbers $\lambda_1 \leq \cdots \leq \lambda_k$ such that $a = \lambda_1 c_1 + \cdots + \lambda_k c_k$. In fact, $\sigma(a) = \{\lambda_1,\ldots,\lambda_k\}$. Note that some of the λ_i may be equal. Thus, any element has a spectral decomposition in terms of orthogonal primitive idempotents. Using this fact it can be shown that the interior, $\mathcal{A}_+^\circ$, of the cone of squares satisfies

$$\begin{aligned} \mathcal{A}_+^\circ &= \{x \in \mathcal{A} : \sigma(x) \subseteq (0,\infty)\} \\ &= \{x^2 : x \in \mathcal{A} \text{ invertible}\}. \end{aligned}$$

We also have a functional calculus. For example, for $a = \lambda_1 c_1 + \cdots + \lambda_k c_k$ we can define

$$\log a = (\log \lambda_1) c_1 + \cdots + (\log \lambda_k) c_k$$

and

$$a^{-1/2} = \lambda_1^{-1/2} c_1 + \cdots + \lambda_k^{-1/2} c_k.$$

Given $a \in \mathcal{A}$, the linear map $Q_a : \mathcal{A} \to \mathcal{A}$ given by

$$Q_a b = 2(a \bullet (a \bullet b)) - a^2 \bullet b$$

for $b \in V$ is called the *quadratic representation of* a. In case of $\mathcal{H}_n(\mathbb{C})$ it is easy to check that $Q_A B = ABA$ for all B.

It is known that if $a \in \mathcal{A}$ is invertible, then $Q_a \in \mathrm{Aut}(\mathcal{A}_+)$, see [5, Proposition III.2.2].

Using these results we can now make the connection with Riemannian symmetric spaces. Indeed, if $\mathcal{A}$ is a finite-dimensional formally real Jordan algebra, then the symmetric cone $\mathcal{A}_+^\circ$ can be equipped with a Riemannian metric,

$$\begin{aligned} \delta(a,b) &= \| \log Q_{a^{-1/2}} b \|_2 \\ &= \sqrt{\textstyle\sum_{i=1}^k \log^2 \lambda_i (Q_{a^{-1/2}} b)}, \end{aligned}$$

where the $\lambda_i(Q_{a^{-1/2}} b)$ are the eigenvalues of $Q_{a^{-1/2}} b$ including multiplicities. Indeed, δ is a length metric, i.e.,

$$\delta(a,b) = \inf_\gamma L(\gamma),$$

where the infimum is taken over all (piecewise) smooth paths $\gamma : [\alpha,\beta] \to \mathcal{A}_+^\circ$ from a to b, and the length of γ is given by

$$L(\gamma) = \int_\alpha^\beta \| Q_{\gamma(t)^{-1/2}} \gamma'(t) \|_2 \, \mathrm{d}t.$$

The Riemannian manifold $(\mathcal{A}_+^\circ, \delta)$ is a *symmetric space*. In fact, at each $a \in C^\circ$ the map $S_a : \mathcal{A}_+^\circ \to \mathcal{A}_+^\circ$ given by

$$S_a(b) = Q_a b^{-1} \text{ for } b \in \mathcal{A}_+^\circ,$$

is a *symmetry* at a, i.e., a δ-isometry that has a as an isolated fixed point and satisfies $S_a^2 = \mathrm{Id}$ on $\mathcal{A}_+^\circ$.

Infinite-dimensional symmetric cones

There exists no analogue of the Koecher–Vinberg characterisation of formally real Jordan algebras in terms of the geometry of cones in infinite dimensions. One obvious obstruction is the fact that most infinite-dimensional formally real Jordan algebras are not realised in an inner-product space, so there is no natural notion of self-duality, nor, can one define a Riemannian metric on the interior of the cone of squares. Recent works [3,4,10,17], however, indicate that there may exist alternative notions of 'symmetric' cones that would allow one to characterise the formally real Jordan algebras in arbitrary dimensions and thereby extending the Koecher–Vinberg result. The main purpose of the workshop was to explore these possibilities. In the remainder of this article we will outline some of the promising approaches that were discussed.

To set up the problem in infinite dimensions it is natural to consider a beautiful infinite-dimensional generalisation of the formally real Jordan algebras due to Alfsen, Schultz and Stormer [1] which are called

JB-algebras. A *JB-algebra* is a real Jordan algebra $\mathcal{A}$, which is equipped with a norm $\|\cdot\|$ making it a Banach space, where the norm satisfies:

1. $\|a \bullet b\| \leq \|a\|\|b\|$ (Banach algebra condition),
2. $\|a^2\| = \|a\|^2$ (C^*-algebra condition),
3. $\|a^2\| \leq \|a^2 + b^2\|$ (positivity condition),

for all $a, b \in \mathcal{A}$. Note that condition 3 ensures that the Jordan algebra is formally real. Moreover, the JB-algebra norm satisfies $\|e\| = \|e^2\| = \|e\|^2$, so that $\|e\| = 1$.

The JB-algebras naturally belong to category of so called complete order-unit spaces. Recall that a cone C in a real vector space V induces a partial ordering $\leq$ on V by $v \leq w$ if $w - v \in C$. An element $u \in C$ is called an *order-unit* if for each $v \in V$ there exists $\lambda \in [0, \infty)$ such that

$$-\lambda u \leq v \leq \lambda u.$$

The triple (V, C, u) is called an *order-unit space* if, in addition, C is *Archimedean*, meaning that if $v \in V$ and $w \in C$ are such that $nv \leq w$ for all $n = 1, 2, \ldots$, then $v \leq 0$. An order-unit space has a natural norm,

$$\|v\|_u = \inf\{\lambda > 0 : -\lambda u \leq v \leq \lambda u\},$$

which is called the *order-unit norm*. With respect to this norm the cone C is closed and has a non-empty interior. In fact, $u \in C^\circ$. We call the order-unit space *complete* if it is complete with respect to the order-unit norm.

If $\mathcal{A}$ is JB-algebra with cone of squares $\mathcal{A}_+$ and unit e, then the triple $(\mathcal{A}, \mathcal{A}_+, e)$ is a complete order-unit space and the JB-algebra norm $\|\cdot\|$ coincides with the order unit norm $\|\cdot\|_e$. We should mention that in finite dimensions the JB-algebra norm is different from the norm induced by the inner-product under which the cone of squares is self-dual.

As each finite-dimensional vector space V with a closed cone C that has a non-empty interior, is a complete order-unit space, the Koecher–Vinberg characterisation can be recast as saying that a finite dimensional order-unit space (V, C, u) is a JB-algebra if and only if the interior of the cone is symmetric. So, it is natural to ask the following question.

Question 1. *Can we characterise the JB-algebras among the complete order-unit spaces in terms of the geometry of the cone in arbitrary dimensions*?

There appear to be several natural approaches to establish such a characterisation.

A Finsler geometric approach

The first one takes a Finsler geometric point of view. It relies on the fact that the interior of the cone in an order-unit space (V, C, u) can be equipped with a natural Finsler metric, namely Thompson's metric. This metric connects the order structure of the cone with its metric geometry in the following way. On C°, *Thompson's metric* is defined by

$$d_T(v, w) = \max\{\log M(v/w), \log M(w/v)\}$$

for $v, w \in C^\circ$,where

$$M(x/y) = \inf\{\beta > 0 : x \leq \beta y\}$$

for $x, y \in C^\circ$. For example, in the case of the JB-algebra $B(H)_{\mathrm{sa}}$ we have for A and B in the interior of the cone $B(H)^+_{\mathrm{sa}}$ that $A \leq \beta B$ if and only if

$$B^{-1/2} A B^{-1/2} = Q_{B^{-1/2}} A \leq \beta I,$$

as $Q_{B^{-1/2}}$ is an automorphism of the cone. Thus,

$$\begin{aligned} &\log M(A/B) \\ &\quad = \sup\{\log \lambda : \lambda \in \sigma(B^{-1/2} A B^{-1/2})\} \\ &\quad = \sup \sigma(\log B^{-1/2} A B^{-1/2}). \end{aligned}$$

On the other hand, $B \leq \alpha A$ if and only if

$$\alpha^{-1} I \leq B^{-1/2} A B^{-1/2} = Q_{B^{-1/2}} A,$$

as $Q_{B^{-1/2}}$ is in automorphism of the cone. Thus,

$$\begin{aligned} &\log M(B/A) \\ &\quad = \sup\{-\log \lambda : \lambda \in \sigma(B^{-1/2} A B^{-1/2})\} \\ &\quad = -\inf \sigma(\log B^{-1/2} A B^{-1/2}). \end{aligned}$$

So, we get that

$$\begin{aligned} &d_T(A, B) \\ &\quad = \max\{|\mu| : \mu \in \sigma(\log B^{-1/2} A B^{-1/2})\} \\ &\quad = \left\|\log B^{-1/2} A B^{-1/2}\right\|. \end{aligned}$$

In fact, on the interior of the cone in a JB-algebra $\mathcal{A}$, Thompson's metric satisfies

$$\begin{aligned} d_T(a, b) &= \max\{|\mu| : \mu \in \sigma(\log Q_{b^{-1/2}} a)\} \\ &= \|\log Q_{b^{-1/2}} a\| \end{aligned}$$

for all $a, b \in \mathcal{A}^\circ_+$. In an order-unit space Thompson's metric is a length metric on C° with a Finsler structure [14]. Indeed, if v and w are points in C°, then

$$d_T(v, w) = \inf_\gamma \mathrm{Length}(\gamma),$$

where the infimum is taken over all piecewise smooth paths $\gamma : [\alpha, \beta] \to C^\circ$ from v to w, and

$$\mathrm{Length}(\gamma) = \int_\alpha^\beta \|\gamma'(t)\|_{\gamma(t)} \, \mathrm{d}t.$$

Here $\|\cdot\|_{\gamma(t)}$ is the order-unit norm with respect to the order-unit $\gamma(t) \in C^\circ$. So, (C°, d) is a Finsler manifold.

As in finite-dimensional Euclidean Jordan algebras the group $\mathrm{Aut}(\mathcal{A}_+)$ acts transitively on the interior of the cone of squares in a JB-algebra. Indeed, given $a, b \in \mathcal{A}^\circ_+$, the automorphism $Q_{b^{1/2}} \circ Q_{a^{-1/2}}$ maps a to b. So, $(\mathcal{A}^\circ_+, d_T)$ is a homogeneous Finsler manifold.

In analogy to the Riemannian case we call a Finsler manifold (M, d) *symmetric* if for each $x \in M$ there exists a d-isometry $S_x : M \to M$ which has x as an isolated fixed point and satisfies $S_x^2 = \mathrm{Id}$ on M.

This definition is motivated by the fact that if $\mathcal{A}$ is a JB-algebra, the Finsler manifold $(\mathcal{A}^\circ_+, d_T)$ is symmetric. In fact, in that case, the symmetries coincide with the Riemannian symmetries in finite dimensions. So for each $a \in C^\circ$ the symmetry at a is given by

$$S_a(b) = Q_a b^{-1} \quad \text{for } b \in C^\circ.$$

To see that S_a is indeed a d_T-isometry we first note that if $T : C \to C$ is an automorphism of the cone C in an order-unit space, then for each $x, y \in C^\circ$ we have that $x \leq \beta y$ if and only if $Tx \leq \beta Ty$, and hence $M(x/y) = M(Tx/Ty)$. Thus, every automorphism of $\mathcal{A}_+$ is a d_T-isometry. The symmetry S_a is the composition of the automorphism Q_a of $\mathcal{A}$ and the map $\iota : b \mapsto b^{-1}$.

Now note that $b^{-1} \leq \beta a^{-1}$ is equivalent to $e \leq \beta Q_{b^{1/2}} a^{-1}$. As $(Q_x y)^{-1} = Q_{x^{-1}} y^{-1}$ for all $x, y \in C^\circ$, we deduce that $b^{-1} \leq \beta a^{-1}$ if and only if $e \leq \beta (Q_{b^{-1/2}} a)^{-1}$, which is equivalent to

$$Q_{b^{-1/2}} a = Q_{(Q_{b^{-1/2}} a)^{1/2}} e \leq \beta e.$$

This implies that $b^{-1} \leq \beta a^{-1}$ is equivalent to $a \leq \beta b$, and hence $M(\iota(b)/\iota(a)) = M(b^{-1}/a^{-1}) = M(a/b)$ for all $a, b \in \mathcal{A}^\circ_+$. This proves that ι is a d_T-isometry as well.

Using the notion of a symmetric Finsler manifold a natural way to answer Question 1 would be by establishing the following conjecture.

Conjecture 1. *If (V,C,u) is a complete order-unit space, then (C°,d_T) is a symmetric Finsler manifold if and only if V is a JB-algebra with unit u, cone of squares C, and JB-algebra norm $\|\cdot\|_u$.*

A significant complication to solve Conjecture 1 arises through the fact that geodesics are in general not unique for Thompson's metric [9], which is a key difference with the finite-dimensional Riemannian case. However, if the order-unit space is a JB-algebra $\mathcal{A}$, then there are *distinguished geodesics* between points $a,b \in \mathcal{A}_+^\circ$ for Thompson's metric, which are given by $\gamma_{ab}(t) := Q_{a^{1/2}}(Q_{a^{-1/2}}b)^t$, see [12]. In finite-dimensional formally real Jordan algebras these distinguished geodesics are precisely the geodesics for the Riemannian metric. It turns out that in a JB-algebra, the symmetries S_a map distinguished geodesics to distinguished geodesics [11]. Thus, in the JB-algebra setting the Finsler geometry of Thompson's metric shares certain geometric features with the Riemannian geometry in finite dimensions.

One way to establishing Conjecture 1 would be by connecting it with existing results for complex Jordan algebras, known as JB*-algebras, see [15]. If (V,C,u) is an order unit space, then we can consider its complexification $V_{\mathbb{C}} = V \oplus \mathrm{i}V$. The set

$$T_C = \{z \in V_{\mathbb{C}} : \operatorname{Im} z \in C^\circ\}$$

is called a *tube domain* if it is biholomorphic to a bounded domain in $V_{\mathbb{C}}$. A tube domain T_C is called *symmetric* if at each $z \in T_C$ there exists a holomorphic involution $S_z : T_C \to T_C$ which has z as an isolated fixed point. In [7] Braun, Kaup and Upmeier showed that the symmetric tube domains are in one-one correspondence with JB*-algebras. Since JB*-algebras are exactly the complexification of JB-algebras, the one-one correspondence would provide a way to establish Conjecture 2.

Alternatively, one could start by making additional assumptions on the symmetries in the Finsler manifold (C°,d_T). Inspired by Loos's definition of symmetric spaces [13] one could for example assume in addition that the symmetries S_x are smooth and satisfy

$$S_x(S_y(S_x(z))) = S_{S_x(y)}(z) \text{ for all } x,y,z \in C^\circ.$$

Both these assumptions hold in the case of a JB-algebra.

An order theoretic approach

The second approach to answering Question 1 takes a purely order theoretic point of view and is more ambitious. If we consider the symmetry S_e at the unit e in a JB-algebra $\mathcal{A}$, we get the inverse map $\iota : a \mapsto a^{-1}$ on $\mathcal{A}_+^\circ$. This map has a special order theoretic property; namely, it is an *order-antimorphism*, i.e.,

$$a \le b \text{ if and only if } \iota(b) \le \iota(a).$$

Moreover, ι is *anti-homogeneous* in the sense that $\iota(\lambda a) = \lambda^{-1}\iota(a)$ for all $\lambda > 0$ and $a \in \mathcal{A}_+^\circ$.

There is some evidence indicating that the following striking order theoretic characterisation of JB-algebras holds.

Conjecture 2. *If (V,C,u) is a complete order-unit space, then there exists a bijective anti-homogeneous order-antimorphism $g : C^\circ \to C^\circ$ if and only if V is a JB-algebra with unit u, cone of squares C, and JB-algebra norm $\|\cdot\|_u$.*

In fact, this conjecture is known to hold for finite-dimensional order-unit spaces by recent work of Walsh [17]. Additional supporting evidence was provided by Lemmens, Roelands and van Imhoff in [10], where it was shown that if (V,C,u) is a complete order-unit space with a strictly convex cone, then there exists a bijective anti-homogeneous order-antimorphism $g : C^\circ \to C^\circ$ if and only if V is a spin factor.

During the workshop various related problems and approaches were discussed. Overall it was a very fruitful meeting, which provided an ideal opportunity for researchers with diverse mathematical backgrounds to meet. The format of the workshop worked well with four hours of lectures each day: a two-hour introductory talk introducing the main theme of the day and two one-hour talks that were more specialised. There was plenty of time for discussion and small group collaborations. We also had two stand-up sessions for which people could sign up to present further thoughts, or lead a discussion, on the problems and results. These sessions worked well and quickly became very lively with a lot of audience participation. The workshop has already stimulated new work in this area, see for example the recent paper by Bertram [2], and will undoubtedly have further impact in years to come.

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