THE INTEGRAL ISOMORPHISM BEHIND ROW REMOVAL PHENOMENA FOR SCHUR ALGEBRAS

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Abstract. We explain and generalise row and column removal phenomena for Schur algebras via integral isomorphisms between subquotients of these algebras. In particular, we prove new reduction formulae for p-Kostka numbers.

1. Introduction

This paper is concerned with the study of the representation theory of the symmetric and general linear groups over $\Bbbk$ a field of characteristic $p \geq 0$ or (more generally) a commutative noetherian ring.

Given a partition λ of n into at most d non-zero parts, we have associated GL_d-modules: $L(\lambda)$ the simple module of highest weight λ; $\Delta(\lambda)$ (respectively $\nabla(\lambda)$) the Weyl (respectively dual Weyl) module of highest weight λ; and $I(\lambda)$ the injective cover of $L(\lambda)$. Applying the Schur functor to these modules, we obtain the simple modules $D(\lambda)$ (or zero); the Specht (and dual Specht) modules S^λ (and S_λ); and the Young modules $Y(\lambda)$ for the symmetric group $\mathfrak{S}_n$.

One of the main open problems in the representation theory of general linear and symmetric groups is the following.

Problem A: *Given λ and μ partitions of n, provide a combinatorial interpretation of the decomposition numbers, $d_{\lambda\mu} = [\nabla(\lambda) : L(\mu)]$.*

It is well-known that Problem A is equivalent to the following (see for instance [Jam83, Theorem 3.1] and [Erd96]).

Problem B: *Given λ and μ partitions of n, provide a combinatorial interpretation of the p-Kostka numbers, $[\mathrm{Sym}^\lambda(\Bbbk^d) : I(\mu)] = K_{\lambda\mu} = [\mathrm{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\Bbbk) : Y(\mu)]$.*

Young modules, and p-Kostka numbers in particular, have been extensively studied; see for example [Erd93, Erd01, EH02, FHK08, Gil14, Gra85, Hen05, Jam83, Kl83]. Of similar interest are questions concerning the cohomological structure of the general linear and symmetric groups.

Problem C: *Given λ and μ partitions of n, calculate the homomorphisms and* Ext-*groups* $\mathrm{Ext}^i_{\mathrm{GL}_d}(\Delta(\lambda), \Delta(\mu))$, $\mathrm{Ext}^i_{\mathrm{GL}_d}(\Delta(\lambda), L(\mu))$ *and* $\mathrm{Ext}^i_{\mathrm{GL}_d}(L(\lambda), L(\mu))$ *for* $i \geq 0$.

Let $\lambda = (\lambda_1, \dots, \lambda_d)$ and $\mu = (\mu_1, \dots, \mu_d)$ be partitions of n. For any fixed $1 \leq r \leq d$, we define partitions

$$\lambda^T = (\lambda_1, \lambda_2, \dots, \lambda_r), \quad \lambda^B = (\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_d).$$

We say that (λ, μ) admits a horizontal cut (after the rth row) if $|\lambda^T| = m = |\mu^T|$ for some $m \in \mathbb{N}$. Vertical column cuts are defined similarly.

Assuming that $\Bbbk$ is a (often algebraically closed) field, important reduction formulas for Problems A, B, and C exist across the literature. The first result in this direction, due to James [Jam81], reduced Problem A in the case of a first-row-removal cut (the $r = 1$ case of the above). This was later extended to arbitrary horizontal cuts by Donkin [Don85] and was later graded by Chuang–Miyachi–Tan [CMT02]. In the case of Problem B, the first-row-removal result was conjectured in [Hen05] and later proven by Fang–Henke–Koenig [FHK08]. The reduction theorem for the homomorphism and extension groups between two Weyl modules (or between two Specht modules) by Donkin, Fayers–Lyle, Lyle–Mathas, and Parshall–Scott [Don98, Don07, FL03, LM05, PS08].

The main result of this paper is Theorem 4.13. There we explicitly construct a family of isomorphisms between products of subquotient algebras of the integral forms of the Schur algebras of the general linear groups. As this isomorphism is on the level of the integral forms of these Schur algebras, it allows us to relate the representation theories of these algebras over any integral domain (this should be of particular interest over commutative noetherian rings, when the Schur algebras are quasi-hereditary). For the reader whose primary interest is representation theory over a field, we remark that our isomorphism explains the fact that Problems A, B, and C are 'independent of the characteristic' of the underlying field.

The isomorphisms of Theorem 4.13 allow us to explain and generalise all the aforementioned horizontal row (and vertical columns) removal phenomena for the general linear and symmetric groups and to (where relevant) extend them to arbitrary fields. For example, we obtain new reductions for p-Kostka numbers (generalising the $r = 1$ case proven in [FHK08]) and for extension groups between Weyl and simple modules.

Corollary 1.1. *Let (λ, μ) be a pair of partitions of n that admits a horizontal row cut. Then we have equalities*

$$K_{\lambda\mu} = K_{\lambda^T\mu^T} \cdot K_{\lambda^B\mu^B} \qquad d_{\lambda\mu} = d_{\lambda^T\mu^T} \cdot d_{\lambda^B\mu^B}$$

and isomorphisms between the Ext groups of a Weyl and simple module

$$\mathrm{Ext}^k_{S^{\Bbbk}_{n,d}}(\Delta(\lambda), L(\mu)) = \bigoplus_{i+j=k} \mathrm{Ext}^i_{S^{\Bbbk}_{m,r}}(\Delta(\lambda^T), L(\mu^T)) \otimes \mathrm{Ext}^j_{S^{\Bbbk}_{n-m,d-r}}(\Delta(\lambda^B), L(\mu^B))$$

and similar isomorphisms between the Ext groups of two Weyl modules

$$\mathrm{Ext}^k_{S^{\Bbbk}_{n,d}}(\Delta(\lambda), \Delta(\mu)) = \bigoplus_{i+j=k} \mathrm{Ext}^i_{S^{\Bbbk}_{m,r}}(\Delta(\lambda^T), \Delta(\mu^T)) \otimes \mathrm{Ext}^j_{S^{\Bbbk}_{n-m,d-r}}(\Delta(\lambda^B), \Delta(\mu^B)).$$

and similar results hold for pairs of partitions admitting a vertical column cut.

The paper is structured as follows. In the first two sections we give a review of the construction of the Schur algebra and tensor space. The exposition here does not follow the chronological development of the theory, but is cherry-picked to be as simple and combinatorial as possible. We follow Doty–Giaquinto [DG02] for the definition of the Schur algebra via generators and relations. We also recall J. A. Green's construction of the co-determinant basis of the Schur algebra and Murphy's construction of an analogous basis of tensor space. We stress that Sections 2 and 3 of this paper do not contain any new results — however they do provide a new and slick presentation to this material which unifies various ideas and approaches taken in the literature.

In Section 4 we construct explicit isomorphisms between subquotients of the Schur algebra and tensor space (Theorem 4.13). As a consequence, we prove Corollary 1.1. In Section 5 we recall standard facts concerning the Schur functor and hence restate the results of Section 4 in the setting of the symmetric group. Finally for those in-the-know, we remark that all of our methods and results can be easily quantised to the q-Schur and Hecke algebras; we have refrained from doing this (although it is routine) as we believe one of the merits of our approach is its stark simplicity.

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2. The combinatorics of tensor space

We let $\Lambda_{n,d}$ denote the set of compositions of n into at most d non-zero parts. That is, the set of sequences, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_d$ equals n. We let $\Lambda^+_{n,d} \subseteq \Lambda_{n,d}$ denote the subset consisting of the sequences $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ and refer to such sequences as partitions. With a partition, λ, is associated its Young diagram, which is the set of nodes

$$[\lambda] = \left\{ (i,j) \in \mathbb{Z}^2_{>0} \ \mid j \leq \lambda_i \right\}.$$

We let λ' denote the conjugate partition obtained by flipping the Young diagram $[\lambda]$ through the north-west to south-easterly diagonal. Given $\lambda, \mu \in \Lambda^+_{n,d}$ we say that λ dominates μ, and write $\lambda \trianglerighteq \mu$ if

$$\textstyle\sum_{1 \leq i \leq r} \lambda_i \geq \sum_{1 \leq i \leq r} \mu_i$$

for all $1 \leq r \leq d$. There is a surjective map $\Lambda_{n,d} \to \Lambda^+_{n,d}$ given by rearranging the rows of a composition to obtain a partition in the obvious fashion (for example if $n = 9$ and $d = 4$, then $(5,0,1,3) \mapsto (5,3,1,0)$). Under the pullback of this map we obtain the dominance ordering on the set of compositions, $\Lambda_{n,d}$, and we extend the notation in the obvious fashion.

Given $\lambda \in \Lambda^+_{n,d}$ and $\mu \in \Lambda_{n,d}$, we define a λ-tableau of weight μ to be a map $\mathsf{T} : [\lambda] \to \{1, \dots, d\}$ such that $\mu_i = |\{x \in [\lambda] : \mathsf{T}(x) = i\}|$ for $i \geq 1$. If T is a λ-tableau of weight μ, we say that T is semistandard if the rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. We let T^λ denote the unique element of $\mathrm{SStd}(\lambda, \lambda)$.

The set of all semistandard tableaux of shape λ and weight μ is denoted $\mathrm{SStd}(\lambda,\mu)$ and we let $\mathrm{SStd}(\lambda,-) := \cup_{\mu\in\Lambda_{n,d}}\mathrm{SStd}(\lambda,\mu)$. For $d \geq n$, we have that $\omega = (1^n, 0^{d-n})$ belongs to $\Lambda^+_{n,d}$. We refer to the tableaux of weight ω as the set of standard tableaux; we let $\mathrm{Std}(\lambda) := \mathrm{SStd}(\lambda,\omega)$. We let t^λ denote the element of $\mathrm{Std}(\lambda)$ in which the first row contains the entries $1, 2, \dots, \lambda_1$ the second row contains entries $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_2$ etc.

2.1. **Symmetric groups and tensor space.** Let $\Bbbk$ be an integral domain. Fix a pair n, d of positive integers and let $\Bbbk^d$ be the $\Bbbk$-module of rank d, spanned by the column vectors, $v_1, \dots, v_d$, over $\Bbbk$ and let $\mathbb{T} = (\Bbbk^d)^{\otimes n}$, denote the the nth tensor power of $\Bbbk^d$. The module $\mathbb{T}$ is called tensor space. Tensor space has a natural basis given by the elementary tensors of the form

$$v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_n},$$

for some $(i_1, i_2, \dots, i_n) \in \{1, \dots, d\}^n$. We let $\mathfrak{S}_{\{1,2,\dots,n\}}$ (or simply $\mathfrak{S}_n$) denote the symmetric group of permutations of the set $\{1, 2, \dots, n\}$. The symmetric group $\mathfrak{S}_n$ acts naturally on the right of $\mathbb{T}$. This action is given by the place permutation of the subscripts of the elementary tensors,

$$(v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_n}) \cdot s = v_{i_{s^{-1}(1)}} \otimes v_{i_{s^{-1}(2)}} \cdots \otimes v_{i_{s^{-1}(n)}}.$$

and extending $\Bbbk$-linearly.

Given $\mu \in \Lambda_{n,d}$ and w an elementary tensor in $\mathbb{T}$, we say that the vector w has weight μ if $|\{i_x \mid 1 \leq x \leq n, i_x = j\}| = \mu_j$, for all $j \in \{1, \dots, d\}$. We define the μ-weight space to be the subspace $\mathbb{T}_\mu$ of $\mathbb{T}$ spanned by the set of elementary tensors of weight μ.

It is clear that the symmetric group acts by transitively permuting the set of elementary vectors of a given weight, $\mu \in \Lambda_{n,d}$. In particular, the elementary tensor

$$\underbrace{e_1 \otimes \cdots \otimes e_1}_{\mu_1} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{\mu_2} \otimes \dots \otimes \underbrace{e_d \otimes \cdots \otimes e_d}_{\mu_d}$$

is a generator of the $\mathfrak{S}_n$-module $\mathbb{T}_\mu$ and the stabiliser subgroup, denoted $\mathfrak{S}_\mu$, is equal to the subgroup

$$\mathfrak{S}_{\{1,2,\dots,\mu_1\}} \times \mathfrak{S}_{\{\mu_1+1,\mu_1+2,\dots,\mu_2\}} \times \cdots \times \mathfrak{S}_{\{n-\mu_d+1,n-\mu_d+2,\dots,n\}}.$$

2.2. **Murphy's basis of tensor space.** We shall now define Murphy's basis of tensor space over several steps

- Let $\lambda \in \Lambda^+_{n,d}$ and $\mu \in \Lambda_{n,d}$. Given $\mathsf{S} \in \mathrm{SStd}(\lambda,\mu)$ we define the row-reading element $e_\mathsf{S} \in \mathbb{T}$ by recording the entries of S, as read from left to right along successive rows, as the subscripts in the tensor power. For example, if

$$\mathsf{S} = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}$$

then

$$e_\mathsf{S} = v_1 \otimes v_1 \otimes v_3 \otimes v_2 \otimes v_2.$$

- For $\lambda \in \Lambda^+_{n,d}$, we have a corresponding Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ given by the stabiliser of e_{T^λ}. We let $\mathcal{O}_\lambda(e_{\mathsf{S}})$ denote the orbit sum of vectors conjugate to e_{S} under the natural right action of $\mathfrak{S}_\lambda$.
- For $\mathsf{t} \in \mathrm{Std}(\lambda)$ we let d_{t} denote the permutation on n letters such that $(\mathsf{t}^\lambda)d_{\mathsf{t}} = \mathsf{t}$.
- Given $\mathsf{S} \in \mathrm{SStd}(\lambda, \mu)$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We define

$$\rho_{\mathsf{St}} = (\mathcal{O}_\lambda(e_{\mathsf{S}}))d_{\mathsf{t}}$$

Theorem 2.1 (Murphy [Mur95])**.** *Tensor space $\mathbb{T} = (\Bbbk^d)^{\otimes n}$ is free as a $\mathbb{Z}$-module with basis given by*

$$\{\rho_{\mathsf{Tt}} \mid \mathsf{T} \in \mathrm{SStd}(\lambda, \mu), \mathsf{t} \in \mathrm{Std}(\lambda), \lambda \in \Lambda^+_{n,d}, \mu \in \Lambda_{n,d}\}.$$

Example 2.2. Given $\lambda = (3, 2)$, $\mu = (2, 2, 1)$ and S as above, we have that

$$\rho_{\mathsf{St}^\lambda} = v_1 \otimes v_1 \otimes v_3 \otimes v_2 \otimes v_2 + v_1 \otimes v_3 \otimes v_1 \otimes v_2 \otimes v_2 + v_3 \otimes v_1 \otimes v_1 \otimes v_2 \otimes v_2.$$

Example 2.3. Tensor space $\mathbb{T} = (\Bbbk^2)^{\otimes 4}$ is 16 dimensional. We have that $\Lambda_{4,2} = \{(2, 2), (3, 1), (1, 3), (4, 0), (0, 4)\}$. The semistandard tableaux, S, T, and U of weight $(2, 2)$ are as follows

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline\end{array}\ .$$

The standard tableaux $\mathsf{s}_1, \mathsf{s}_2, \mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3$ and u are as follows

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}\ .$$

The space of vectors of weight $(2, 2)$ is 6-dimensional with basis

$$\begin{aligned}
\rho_{\mathsf{Ss}_1} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 \\
\rho_{\mathsf{Ss}_2} &= v_1 \otimes v_2 \otimes v_1 \otimes v_2 \\
\rho_{\mathsf{Tt}_1} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_1 \otimes v_2 \\
\rho_{\mathsf{Tt}_2} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_2 \otimes v_1 \\
\rho_{\mathsf{Tt}_3} &= v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 + v_2 \otimes v_2 \otimes v_1 \otimes v_1 \\
\rho_{\mathsf{Uu}} &= v_1 \otimes v_1 \otimes v_2 \otimes v_2 + v_1 \otimes v_2 \otimes v_1 \otimes v_2 + v_1 \otimes v_2 \otimes v_2 \otimes v_1 \\
&\quad + v_2 \otimes v_1 \otimes v_1 \otimes v_2 + v_2 \otimes v_1 \otimes v_2 \otimes v_1 + v_2 \otimes v_2 \otimes v_1 \otimes v_1.
\end{aligned}$$

3. The Schur algebra and the co-determinant basis

Let Φ be the root system of type A_{d-1}: $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq d\}$. Here the ε_is form the standard orthonormal basis of the euclidean space $\mathbb{R}^d$. Let $(\ ,\ )$ denote the inner product on this space and define $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then $\{\alpha_1, \dots, \alpha_{d-1}\}$ is a base of simple roots and $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$ is the corresponding set of positive roots.

The following definition of the Schur algebra over $\mathbb{Q}$ is due to Doty and Giaquinto [DG02, Theorem 2.4] and is very much inspired by Lusztig's modified form of the quantum universal enveloping algebra.

Definition 3.1. The $\mathbb{Q}$-algebra $S^{\mathbb{Q}}_{n,d}$ is the associative algebra (with 1) given by generators 1_λ ($\lambda \in \Lambda_{n,d}$), $e_{i,i+1}$, $f_{i,i+1}$ ($1 \le i \le d-1$) subject to the relations

$$1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda, \quad \sum_{\lambda \in \Lambda_{n,d}} 1_\lambda = 1 \tag{R1}$$

$$e_{i,i+1} f_{j,j+1} - f_{j,j+1} e_{i,i+1} = \delta_{ij} \sum_{\lambda \in \Lambda_{n,d}} (\alpha_i, \lambda)\, 1_\lambda \tag{R2}$$

$$e_{i,i+1} 1_\lambda = \begin{cases} 1_{\lambda+\alpha_i} e_{i,i+1} & \text{if } \lambda + \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases} \tag{R3}$$

$$f_{i,i+1} 1_\lambda = \begin{cases} 1_{\lambda-\alpha_i} f_{i,i+1} & \text{if } \lambda - \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases} \tag{R4}$$

$$1_\lambda e_{i,i+1} = \begin{cases} e_{i,i+1} 1_{\lambda-\alpha_i} & \text{if } \lambda - \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases} \tag{R5}$$

$$1_\lambda f_{i,i+1} = \begin{cases} f_{i,i+1} 1_{\lambda+\alpha_i} & \text{if } \lambda + \alpha_i \in \Lambda_{n,d} \\ 0 & \text{otherwise} \end{cases} \tag{R6}$$

Here, for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda_{n,d}$ we identify $\lambda + \alpha_i$ with the composition $(\lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1} - 1, \lambda_{i+2}, \dots, \lambda_n)$.

Remark 3.2. It was pointed out by Rouquier (see [DG, Introduction]) that the Serre relations ($R7$) and ($R8$) as stated in [DG02, Theorem 1.4] follow from ($R1$) to ($R6$) and hence may be omitted.

Definition 3.3. For $1 \le i < j \le d$, we inductively define elements

$$e_{i,j} = e_{i,j-1} e_{j-1,j} - e_{j-1,j} e_{i,j-1} \quad f_{i,j} = f_{j-1,j} f_{i,j-1} - f_{i,j-1} f_{j-1,j}.$$

We define the divided powers

$$e^{[m]}_{i,j} = \frac{e^m_{i,j}}{m!} \quad f^{[m]}_{i,j} = \frac{f^m_{i,j}}{m!}$$

We formally set $e_{i,i} = 1 = f_{i,i}$. The integral Schur algebra $S^{\mathbb{Z}}_{n,d}$ is the subring of $S^{\mathbb{Q}}_{n,d}$ generated by all divided powers and the idempotents 1_λ for $\lambda \in \Lambda_{n,d}$.

Moreover, $S^{\mathbb{Q}}_{n,d}$ acts naturally on $\mathbb{T}$ as follows,

$$e_{i,i+1}(v_{j_1} \otimes \dots \otimes v_{j_n}) = \sum_{\substack{1 \le a \le n \\ j_a = i+1}} (v_{j_1} \otimes \dots \otimes v_{j_a - 1} \otimes \dots \ \otimes \dots v_{j_n})$$

$$f_{i,i+1}(v_{j_1} \otimes \dots \otimes v_{j_n}) = \sum_{\substack{1 \le a \le n \\ j_a = i}} (v_{j_1} \otimes \dots \otimes v_{j_a + 1} \otimes \dots \ \otimes \dots v_{j_n})$$

$$1_\lambda(v_{j_1} \otimes v_{j_2} \otimes \dots v_{j_n}) = \begin{cases} (v_{j_1} \otimes v_{j_2} \otimes \dots v_{j_n}) & \text{if the vector is of weight } \lambda \\ 0 & \text{otherwise} \end{cases}$$

The action of the divided powers is easily deduced from the above. Let $v \in \mathbb{T}$ be a tensor with p occurrences of the tensorand v_j for $1 \leq j \leq d$. The divided power $e_{i,j}^{[m]}$ sends v to the sum over the $\binom{p}{m}$ vectors obtainable from v by swapping a total of m of the v_j's for v_i's. The other divided powers, $f_{i,j}^{[m]}$, act similarly.

Definition 3.4. Given $1 \leq i, j \leq d$ and $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$, we let $\mathsf{T}(i,j)$ denote the number of entries equal to j lying in the ith row of T. Since T is semistandard we have that $\mathsf{T}(i,j) = 0$ for $i > j$ and $\sum_{1 \leq i \leq d} \mathsf{T}(i,j) = \mu_j$.

Definition 3.5. Given $\mathsf{S}, \mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$ we let

$$\xi_{\mathsf{S}\lambda} = \prod_{i=1}^{d} \left(\prod_{j=1}^{d} f_{i,j}^{[\mathsf{S}(i,j)]} \right) \qquad \xi_{\lambda\mathsf{T}} = \prod_{i=d}^{1} \left(\prod_{j=d}^{1} e_{i,j}^{[\mathsf{T}(i,j)]} \right)$$

(notice the ordering on these products) and we define

$$\xi_{\mathsf{ST}} = \xi_{\mathsf{S}\lambda} 1_\lambda \xi_{\lambda\mathsf{T}}$$

Example 3.6. Let $\lambda = (3,3)$, $\mu = (2,2,1,1)$, and $\nu = (2,1,2,1)$. We let S and T denote the tableaux

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 3 & 4 \\ \hline \end{array},$$

respectively. We have that $\mathsf{S} \in \mathrm{SStd}(\lambda, \mu)$, $\mathsf{T} \in \mathrm{SStd}(\lambda, \nu)$. We have that $\mathsf{S}(2,4) = 1$, $\mathsf{S}(2,2) = 2$, $\mathsf{S}(1,3) = 1$, $\mathsf{S}(1,1) = 2$, and all other $\mathsf{S}(i,j)$ are equal to zero. Similarly, $\mathsf{T}(1,2) = 1$, $\mathsf{T}(2,3) = 2$, $\mathsf{T}(2,4) = 1$ and all other $\mathsf{T}(i,j) = 0$. Therefore,

$$\xi_{\mathsf{ST}} = f_{1,3}^{[1]} f_{2,4}^{[1]} 1_\lambda e_{2,4}^{[1]} e_{2,3}^{[2]} e_{1,2}^{[1]}.$$

We now reconstruct Green's co-determinant of the Schur algebra [Gre93]. While the basis itself is well-known, this formulation in terms of Doty and Guiaquinto's presentation is new and is necessary for our proof of the main theorem.

Theorem 3.7. *The Schur algebra $S_{n,d}^{\mathbb{Z}}$ is free as a $\mathbb{Z}$-module with basis*

$$\{\xi_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu) \text{ for } \lambda \in \Lambda_{n,d}^+, \mu, \nu \in \Lambda_{n,d}\}.$$

If $\mathsf{S} \in \mathrm{SStd}(\lambda, -)$, $\mathsf{T} \in \mathrm{SStd}(\lambda, -)$ for some $\lambda \in \Lambda_{n,d}^+$, and $a \in S_{n,d}^{\mathbb{Z}}$ then there exist scalars $r(a; \mathsf{S}, \mathsf{U}) \in \mathbb{Z}$, which do not depend on T, such that

$$a\xi_{\mathsf{ST}} = \sum_{\mathsf{U} \in \mathrm{SStd}(\lambda, -)} r(a; \mathsf{S}, \mathsf{U}) \xi_{\mathsf{UT}} \mod (S_{n,d}^{\mathbb{Z}})^{\rhd\lambda}$$

where $(S_{n,d}^{\mathbb{Z}})^{\rhd\lambda}$ is two-sided ideal generated by the idempotent

$$\sum_{\{\mu \in \Lambda_{n,d} \mid \mu \rhd \lambda\}} 1_\mu.$$

The ideal $(S_{n,d}^{\mathbb{Z}})^{\rhd\lambda}$ is spanned by

$$\{\xi_{\mathsf{QR}} \mid \mathsf{Q}, \mathsf{R} \in \mathrm{SStd}(\mu, -), \mu \in \Lambda_{n,d}^+, \mu \rhd \lambda\}.$$

Moreover, the $\mathbb{Z}$*-linear map* $* : S^{\mathbb{Z}}_{n,d} \to S^{\mathbb{Z}}_{n,d}$ *determined by* $(\xi_{\mathsf{ST}})^* = \xi_{\mathsf{TS}}$*, for all* $\lambda \in \Lambda^+_{n,d}$ *and all* $\mathsf{S}, \mathsf{T} \in \mathrm{SStd}(\lambda, -)$*, is an anti-isomorphism of* $S^{\mathbb{Z}}_{n,d}$*. Therefore the Schur algebra is a cellular algebra in the sense of* [GL96].

Proof. In [DJM98] the authors constructed a cellular basis of $\mathrm{End}_{\Bbbk\mathfrak{S}_n}(\mathbb{T})$. Having established the action of Doty and Guiaquinto's presentation on tensor space, it is easy to see that the $\xi_{\mathsf{ST}} \in \mathrm{End}_{\Bbbk}(\mathbb{T})$ defined above coincide precisely with the construction of the corresponding homomorphisms (denoted by φ_{ST}) defined in [DJM98] as endomorphisms of tensor space. Thus the statement above is merely a re-imagining of [DJM98, The semistandard basis theorem]. □

Definition 3.8. Given $\Bbbk$ a commutative noetherian ring, we define the Schur algebra to be $S^{\Bbbk}_{n,d} := S^{\mathbb{Z}}_{n,d} \otimes \Bbbk$.

We remark that the algebra $S^{\Bbbk}_{n,d}$ is quasi-hereditary (for $\Bbbk$ any commutative noetherian ring) by [Gre93, Corollary 7.2].

Definition 3.9. Given $\lambda \in \Lambda^+_{n,d}$, we define the Weyl module $\Delta^{\mathbb{Z}}(\lambda)$ to be the left $S^{\mathbb{Z}}_{n,d}$–module with basis

$$\{\xi_{\mathsf{ST}^\lambda} + (S^{\mathbb{Z}}_{n,d})^{\rhd\lambda} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, -)\}$$

and the dual Weyl module $\nabla^{\mathbb{Z}}(\lambda)$ to be the left $S^{\mathbb{Z}}_{n,d}$–module with basis

$$\{\rho_{\mathsf{St}^\lambda} + \mathbb{T}^{\rhd\lambda} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, -)\},$$

where $\mathbb{T}^{\rhd\lambda}$ is the left $S^{\mathbb{Z}}_{n,d}$–module of $\mathbb{T}$ with basis $\{\rho_{\mathsf{St}} \mid \mathsf{S} \in \mathrm{SStd}(\mu, -), \mathsf{t} \in \mathrm{Std}(\mu), \mu \rhd \lambda\}$. We let $\Delta^{\Bbbk}(\lambda)$ (respectively $\nabla^{\Bbbk}(\lambda)$) denote the module $\Delta^{\mathbb{Z}}(\lambda) \otimes_R \Bbbk$ (respectively $\nabla^{\mathbb{Z}}(\lambda) \otimes_R \Bbbk$). When the context is clear, we drop the ring over which the module is defined.

Definition 3.10. If a module, M, has a filtration of the form

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$$

where each M_{i+1}/M_i for $1 \leq i < k$ is isomorphic to some $\Delta(\lambda^{(i)})$ (respectively $\nabla(\lambda^{(i)})$) for some $\lambda^{(i)} \in \Lambda^+_{n,d}$, then we say that M has a Δ- (respectively ∇-) filtration and write $M \in \mathcal{F}(\Delta)$ (respectively $M \in \mathcal{F}(\nabla)$).

Definition 3.11. For any $\lambda \in \Lambda^+_{n,d}$ there exists a unique tilting module $T(\lambda)$ for $S^{\Bbbk}_{n,d}$ of highest weight λ with both a Δ- and ∇-filtration.

Given any $\lambda \in \Lambda^+_{n,d}$ the Weyl module, $\Delta(\lambda)$, is equipped with a bilinear form $\langle\ ,\ \rangle_\lambda$ determined by

$$\xi_{\mathsf{US}}\xi_{\mathsf{TV}} \equiv \langle \xi_{\mathsf{ST}^\lambda}, \xi_{\mathsf{TT}^\lambda} \rangle_\lambda \xi_{\mathsf{U},\mathsf{V}}$$

modulo $\mathrm{Span}\{\xi_{\mathsf{ST}} \mid \mathsf{S}, \mathsf{T} \in \mathrm{SStd}(\mu, -) \text{ and } \mu \rhd \lambda\}$ for $\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V} \in \mathrm{SStd}(\lambda, -)$. We define $L(\lambda)$ to be the quotient of the corresponding Weyl module $\Delta(\lambda)$ by the radical of the bilinear form $\langle\ ,\ \rangle_\lambda$. Finally we denote by $I(\lambda)$ the injective envelope of $L(\lambda)$ as an $S^{\Bbbk}_{n,d}$-module.

3.1. **Generalised symmetric powers.** For $\lambda \in \Lambda^+_{n,d}$, $\mu \in \Lambda_{n,d}$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$ let $\mu(\mathsf{t})$ be the λ-tableau of weight μ obtained from t by replacing each entry i in t by r if i appears in row r of t^μ. Given $\mathsf{t} \in \mathrm{Std}(\lambda)$, we let $[\mathsf{t}]_\mu$ denote the set $\{\mathsf{s} \in \mathrm{Std}(\lambda) \mid \mu(\mathsf{s}) = \mu(\mathsf{t})\}$. If $\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$, we write $\mathsf{t} \in \mathsf{T}$ if $\mu(\mathsf{t}) = \mathsf{T}$. On the other hand, it will be convenient to say that $\mu(\mathsf{t}) = 0$, whenever $\mu(\mathsf{t})$ is not semistandard. Finally, for $\mathsf{S},\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$ we set

$$\rho_{\mathsf{ST}} := \sum_{\mathsf{t}\in\mathsf{T}} \rho_{\mathsf{St}}. \tag{7}$$

Remark 3.12. In the case that $\mu = \omega$, the map $\omega : \mathrm{Std}(\lambda) \to \mathrm{SStd}(\lambda,\omega)$ is the bijective map which identifies standard tableaux with semistandard tableaux of weight ω.

Example 3.13. Let $n = 4$ and $d = 2$. Adopting the same notation as in Example 2.3 it is easy to observe that there is a unique element of $\mathrm{SStd}(\lambda,(2,2))$ for each $\lambda \in \Lambda^+_{4,2}$. These are the tableaux S, T and U of Example 2.3. The pullback under $\mathrm{Std}(\lambda) \to \mathrm{SStd}(\lambda,(2,2))$ is given by

$$[\mathsf{s}_1]_{(2,2)} = \{\mathsf{s}_1\} \quad [\mathsf{t}_1]_{(2,2)} = \{\mathsf{t}_1,\mathsf{t}_2\} \quad [\mathsf{u}]_{(2,2)} = \{\mathsf{u}\},$$

for λ equal to $(2,2),(3,1)$ and (4), respectively. Therefore

$$\begin{aligned}\rho_{\mathsf{TT}} =& e_1\otimes e_2\otimes e_1\otimes e_2 + e_2\otimes e_1\otimes e_1\otimes e_2\\ &+ e_1\otimes e_2\otimes e_2\otimes e_1 + e_2\otimes e_1\otimes e_2\otimes e_1 + 2e_1\otimes e_1\otimes e_2\otimes e_2.\end{aligned}$$

Definition 3.14. Given $\mu \in \Lambda_{n,d}$, we let

$$\mathrm{Sym}^\mu(\Bbbk^d) = \mathrm{Sym}^{\mu_1}(\Bbbk^d)\otimes\cdots\otimes\mathrm{Sym}^{\mu_d}(\Bbbk^d)$$

denote the generalised symmetric tensor of the natural $S^{\Bbbk}_{n,d}$-module, $\Bbbk^d$.

Proposition 3.15. *Let $\Bbbk$ be a field. The module $\mathrm{Sym}^\mu(\Bbbk^d)$ has a basis given by sums of elements in the Murphy basis of tensor space of Theorem 2.1, as follows*

$$\{\rho_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\nu), \mathsf{T} \in \mathrm{SStd}(\lambda,\mu), \lambda \in \Lambda^+_{n,d}, \nu \in \Lambda_{n,d}\}.$$

Proof. For each $\mathsf{S} \in \mathrm{SStd}(\lambda,\nu)$ and $\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$ we have that $\mathfrak{S}_\mu$ acts transitively on the set $\{\rho_{\mathsf{St}} \mid \mathsf{t} \in \mathsf{T}\}$. Moreover, the stabiliser of any element ρ_{St} is $\mathfrak{S}_\mu \cap d_{\mathsf{t}}^{-1}\mathfrak{S}_\lambda d_{\mathsf{t}}$ (see for example [Mat99, Proposition 4.4]). Therefore the element ρ_{ST} is fixed by the action of $\mathfrak{S}_\mu$. Hence, for every $\mathsf{S} \in \mathrm{SStd}(\lambda,\nu)$ and $\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$ we have that $\rho_{\mathsf{ST}} \in \mathrm{Sym}^\mu(\Bbbk^d)$.

The elements ρ_{St} are linearly independent and the orbits $\{\mathsf{t} \mid \mu(\mathsf{t}) = \mathsf{T}\}$ for $\mathsf{T} \in \mathrm{SStd}(\lambda,\mu)$ are disjoint. Therefore the elements ρ_{ST} are linearly independent (over any field, as their coefficients in the sum in equation (7) are all 0 or 1). The result now follows from a dimension count using the formula

$$\dim_\Bbbk(\mathrm{Sym}^\mu(\Bbbk^d)) = \sum_\lambda [\mathrm{Sym}^\mu(\Bbbk^d) : \nabla(\lambda)] \dim_\Bbbk(\nabla(\lambda)) = \sum_\lambda |\mathrm{SStd}(\lambda,\mu)||\mathrm{SStd}(\lambda,\nu)|$$

where the second equality is simply Young's rule [Jam78, Section 14] together with the cellular basis of Definition 3.9. □

Proposition 3.16 (Lemma 3.4 [Don93]). *Let $\Bbbk$ be a field. The injective indecomposable $S^{\Bbbk}_{n,d}$-modules, $I(\lambda)$, are precisely the indecomposable summands of* $\mathrm{Sym}^{\mu}(\Bbbk^d)$ *for $\lambda, \mu \in \Lambda^+_{n,d}$. For $\mu, \lambda \in \Lambda^+_{n,d}$, we have*

$$[\mathrm{Sym}^{\mu}(\Bbbk^d) : I(\lambda)] = K_{\mu\lambda} = \dim 1_{\lambda} L(\mu)$$

where the coefficients, $K_{\mu\lambda}$, are known as the p-Kostka numbers. In particular, $[\mathrm{Sym}^{\mu}(\Bbbk^d) : I(\lambda)] = 1$ *for $\lambda = \mu$ and 0 unless $\mu \trianglelefteq \lambda$.*

4. Isomorphisms between subquotients of integral Schur algebras

In this section, we prove the main results of this paper, culminating in Theorem 4.13 below. In Subsection 4.1 we consider certain subsets, $\Lambda^+_{n,d}(r, c, m) \subseteq \Lambda^+_{n,d}$. We recall the definition of generalised row cuts on pairs of partitions and show that if (λ, μ) admit such a cut and $\lambda \rhd \mu$, then λ and μ both belong to one our subsets $\Lambda^+_{n,d}(r, c, m)$. In Subsections 4.2 and 4.3 we construct explicit isomorphisms between certain subquotients of the Schur algebras corresponding to the sets $\Lambda^+_{n,d}(r, c, m)$; all of these isomorphisms are given simply on the level of the tableaux bases.

The subquotients in which we are interested are of the following form.

Definition 4.1. Let P denote a partially ordered set and Q denote a subset of P. We say that Q is saturated if for any $\alpha \in Q$ and $\beta \in P$ with $\beta \lhd \alpha$, we have that $\beta \in Q$. We say that Q is co-saturated if its complement in P is saturated. If a set is saturated, co-saturated, or the intersection of a saturated and a co-saturated set, we shall say that it is closed under the dominance order.

Definition 4.2. Let M be a $S^{\Bbbk}_{n,d}$-module, and $\pi \subseteq \Lambda^+_{n,d}$ denote some closed subset under the dominance order. We say that M belongs to π if the simple composition factors of M are labelled by weights from π. We write $M \in \mathcal{F}_{\pi}(\Delta)$ (respectively $M \in \mathcal{F}_{\pi}(\nabla)$) if M has a Δ-filtration (respectively ∇-filtration) in which the Δ (respectively ∇) factors are labelled by weights from π.

We shall use standard facts about saturated and co-saturated sets in what follows, referring to [CPS88] (or [Don98, Appendix]) for more details. Much of the representation theoretic information is preserved under taking such subquotients. In particular, this allows us to simplify the proof and slightly extend the (unquantised) results of [Jam81, LM05] and [Don98, 4.2(17)]. In Subsection 4.3 we then deduce that higher extension groups and decomposition numbers are preserved under taking generalised row cuts. In Subsection 4.4, we consider the image of the generalised symmetric powers under these functors and hence prove Corollary 1.1.

4.1. Combinatorics of partitions and generalised row cuts.

We now recall the combinatorics of generalised row cuts. Given a partition $\lambda \in \Lambda_{n,d}$ and $1 \leq r \leq d$, we let

$$\lambda^T = (\lambda_1, \lambda_2, \ldots, \lambda_r), \quad \lambda^B = (\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_d).$$

Definition 4.3. Given $r, c, m \in \mathbb{N}$, we let $\Lambda_{n,d}(r,c,m) \subseteq \Lambda_{n,d}$ denote the set

$$\{\lambda \in \Lambda_{n,d} \mid \lambda_j \le c \le \lambda_i, \text{ for } 1 \le i \le r \text{ and } r+1 \le j \le d, |\lambda^T| = m\}$$

and we let

$$\Lambda^+_{n,d}(r,c,m) = \{\lambda \in \Lambda^+_{n,d} \mid \lambda_r \ge c \ge \lambda_{r+1}, |\lambda^T| = m\}$$

in other words, $\Lambda^+_{n,d}(r,c,m) = \Lambda^+_{n,d} \cap \Lambda_{n,d}(r,c,m)$. Extending the above notation we denote by $\Lambda^+_{n,d}(0,c,0)$ the subset of $\Lambda^+_{n,d}$ consisting of all the partitions λ such that $\lambda_1 \le c$.

Remark 4.4. The subset $\Lambda^+_{n,d}(r,c,m) \subseteq \Lambda^+_{n,d}$ can be thought of diagrammatically as in Figure 1.

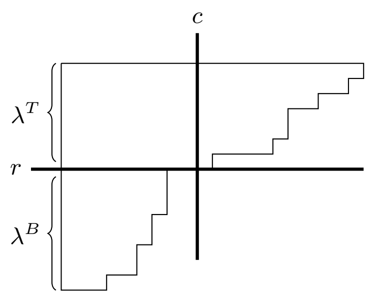

FIGURE 1. A partition λ such that $\lambda_r \ge c \ge \lambda_{r+1}$.

Proposition 4.5. *The map $\lambda \mapsto \lambda^T \times \lambda^B$ is a bijection between $\Lambda_{n,d}(r,c,m)$ and $\Lambda_{m,r}(r,c,m) \times \Lambda_{n-m,d-r}(0,c,0)$. Moreover, for $\lambda, \mu \in \Lambda_{n,d}(r,c,m)$, we have that $\lambda \trianglerighteq \mu$ if and only if $\lambda^T \trianglerighteq \mu^T$ and $\lambda^B \trianglerighteq \mu^B$. This restricts in the obvious way to the sets of partitions as the subsets of compositions.*

Proof. Clear from the definitions. □

Example 4.6. For example, the map in Proposition 4.5 takes the element in Figure 1 to the pair of elements in Figure 2.

Let $r, c, m \in \mathbb{N}$ be such that $\Lambda^+_{n,d}(r,c,m) \ne \emptyset$. We note that the set $\Lambda^+_{n,d}(r,c,m)$ has a unique maximal and a unique minimal element (under the dominance ordering on partitions). One can describe these partitions directly, however we use Proposition 4.5 to make the statements simpler. The unique maximal and minimal elements of any non-empty $\Lambda^+_{z,r}(0,c,0)$ are equal to

$$\alpha(r,c,z) = (c^{\lfloor \frac{z}{c} \rfloor}, z - c\lfloor \tfrac{z}{c} \rfloor) \quad \text{and} \quad \zeta(r,c,z) = (r^{\lfloor \frac{z}{r} \rfloor}, z - r\lfloor \tfrac{z}{r} \rfloor)'$$

respectively. For $r, c \ge z$ we have that $\alpha(r,c,z) = (z)$ and $\zeta(r,c,z) = (1^z)$. Finally, given two partitions $\lambda, \mu \in \Lambda(n,d)$, we set

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots, \lambda_d + \mu_d).$$

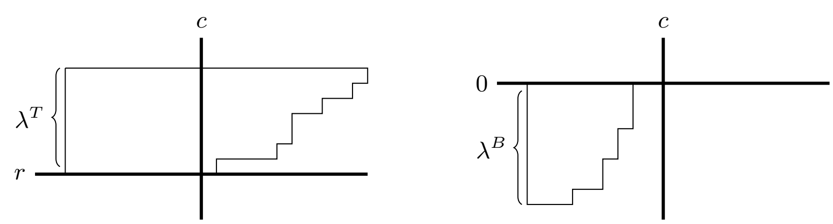

FIGURE 2. The element of $\Lambda^+_{m,r}(r,c,m) \times \Lambda^+_{n-m,d-r}(0,c,0)$ obtained from the element in Figure 1 under the map in Proposition 4.5.

Proposition 4.7. *If $\Lambda^+_{n,d}(r,c,m) \neq \emptyset$, then it has a unique maximal element*

$$\sigma := \sigma(r,c,m) = (c^r, \alpha(d-r,c,n-m)) + (m-cr)$$

and a unique minimal element

$$\gamma := \gamma(r,c,m) = (c^r, \zeta(d-r,c,n-m)) + \zeta(r,m-cr,m-cr).$$

Proof. This follows from Proposition 4.5. □

Having defined the maximal and minimal elements of $\Lambda^+_{n,d}(r,c,m)$ we now define

$$\Sigma^+_{n,d}(r,c,m) = \{\mu \in \Lambda^+_{n,d} \mid \mu \trianglelefteq \sigma\}$$
$$\Gamma^+_{n,d}(r,c,m) = \{\mu \in \Lambda^+_{n,d} \mid \mu \trianglerighteq \gamma\}.$$

The set $\Sigma^+_{n,d}(r,c,m)$ (respectively $\Gamma^+_{n,d}(r,c,m)$) is clearly a saturated (respectively co-saturated) subset of $\Lambda^+_{n,d}$ in the sense of [Don98, Appendix].

We let $\Gamma_{n,d}(r,c,m)$ (respectively $\Sigma_{n,d}(r,c,m)$) denote the sets of compositions which can be obtained from a partition in $\Gamma^+_{n,d}(r,c,m)$ (respectively $\Sigma^+_{n,d}(r,c,m)$) by permutation of the rows $\{1,\dots,r\}$ and the rows $\{r+1,\dots,d\}$. The sets of minimal and maximal elements of $\Lambda_{n,d}(r,c,m)$ are those which are mapped to γ and σ respectively under the map $\Lambda_{n,d}(r,c,m) \to \Lambda^+_{n,d}(r,c,m)$.

Example 4.8. The set $\Lambda^+_{11,5}(3,2,9)$ consists of six elements

$$(5,2^3) \quad (4,3,2^2) \quad (3^3,2) \quad (5,2^2,1^2) \quad (4,3,2,1^2) \quad (3^3,1^2)$$

and here we have $\sigma = (5,2^3)$ and $\gamma = (3^3,1^2)$.

Proposition 4.9. *We have that*

$$\Lambda^+_{n,d}(r,c,m) = \Sigma^+_{n,d}(r,c,m) \cap \Gamma^+_{n,d}(r,c,m)$$

Proof. It is clear that $\Lambda^+_{n,d}(r,c,m) \subseteq \Sigma^+_{n,d}(r,c,m) \cap \Gamma^+_{n,d}(r,c,m)$. We now prove the reverse containment. Suppose that $\mu \in \Lambda^+_{n,d}$ is such that $\gamma \trianglelefteq \mu \trianglelefteq \sigma$. We have that $\sum_{1\le i\le r} \gamma_i = m = \sum_{1\le i\le r} \sigma_i$ and therefore

$$m \le \textstyle\sum_{1\le i\le r} \mu_i \le m.$$

Therefore $\sum_{1\leq i\leq r}\mu_i = m$; putting this together with $\mu \trianglelefteq \sigma$ and $\sigma_r \geq c$, we deduce that $\mu_r \geq c$. Similarly, we have that $\mu \trianglelefteq \sigma$ and $\sigma_{r+1} \leq c$; therefore $\mu_{r+1} \leq c$. Therefore $\mu \in \Lambda^+_{n,d}(r, c, m)$, as required. □

Definition 4.10. Given $\lambda, \mu \in \Lambda_{n,d}$ and $1 \leq r \leq d$, we say that λ and μ admit a horizontal cut after the rth row if

$$\sum_{1\leq i\leq r}\lambda_i = \sum_{1\leq i\leq r}\mu_i.$$

Proposition 4.11. *Let $\lambda, \mu \in \Lambda^+_{n,d}$ be a pair of partitions that admits a horizontal cut after the rth row. If $\lambda \trianglerighteq \mu$, then $\mu \in \Lambda^+_{n,d}(r, \lambda_r, |\lambda^T|)$. Moreover $\lambda \trianglerighteq \mu$ if and only if $\lambda^T \trianglerighteq \mu^T$ and $\lambda^B \trianglerighteq \mu^B$.*

Proof. Let $\lambda, \mu \in \Lambda^+_{n,d}$ and suppose that λ and μ admit a horizontal cut after the rth row and $\lambda \trianglerighteq \mu$. In which case,

$$\mu_{r+1} \leq \lambda_{r+1} \leq \lambda_r \leq \mu_r$$

and so $\mu \in \Lambda^+_{n,d}(r, \lambda_r, |\lambda^T|)$. The second statement is clear. □

4.2. **Subquotient algebras of Schur algebras.** Let Ω be any subset of $\Lambda_{n,d}$. We denote by 1_Ω the idempotent defined by

$$1_\Omega = \sum_{\mu\in\Omega} 1_\mu.$$

Given $r, c, m \in \mathbb{N}$, it will be convenient to denote $1_{\Lambda^+_{n,d}(r,c,m)}$ by $1^{n,d}_{r,c,m}$. Similarly we will denote by $S^{\mathbb{Z}}(\Lambda^+_{n,d}(r, c, m))$ the associated subquotient algebra

$$S^{\mathbb{Z}}(\Lambda^+_{n,d}(r, c, m)) = 1_{\Gamma^+_{n,d}(r,c,m)}(S^{\mathbb{Z}}_{n,d}/S^{\mathbb{Z}}_{n,d}1_{\Lambda_{n,d}\setminus\Sigma_{n,d}(r,c,m)}S^{\mathbb{Z}}_{n,d})1_{\Gamma^+_{n,d}(r,c,m)}.$$

We have a functor $h_{r,c,m} : S^{\mathbb{Z}}_{n,d}\text{-mod} \to S^{\mathbb{Z}}_{n,d}(\Lambda^+_{n,d}(r, c, m))\text{-mod}$ given by

$$h_{r,c,m}(M) = 1_{\Gamma^+_{n,d}(r,c,m)}(M/\langle 1_{\Lambda_{n,d}\setminus\Sigma_{n,d}(r,c,m)}M\rangle).$$

We let $S^{\Bbbk}_{n,d}(\Lambda^+_{n,d}(r, c, m)) = S^{\mathbb{Z}}_{n,d}(\Lambda^+_{n,d}(r, c, m)) \otimes \Bbbk$.

Proposition 4.12. *The algebra $S^{\mathbb{Z}}_{n,d}(\Lambda^+_{n,d}(r, c, m))$ has identity $1^{n,d}_{r,c,m}$. The algebra is free as a $\mathbb{Z}$-module with cellular basis*

$$\{\xi_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu) \text{ for } \lambda, \mu, \nu \in \Lambda^+_{n,d}(r, c, m)\}.$$

The algebra $S^{\Bbbk}_{n,d}(\Lambda^+_{n,d}(r, c, m))$ is quasi-hereditary with a full set of non-isomorphic simple, standard, injective, and tilting $S^{\Bbbk}_{n,d}(\Lambda^+_{n,d}(r, c, m))$-modules given by

$$h_{r,c,m}(L(\lambda)) \quad h_{r,c,m}(\Delta(\lambda)) \quad h_{r,c,m}(I(\lambda)) \quad h_{r,c,m}(T(\lambda))$$

respectively, for $\lambda \in \Lambda^+_{n,d}(r, c, m)$. We have that

$$[\Delta(\lambda) : L(\mu)]_{S^{\Bbbk}_{n,d}} = [h_{r,c,m}(\Delta(\lambda)) : h_{r,c,m}(L(\mu))]_{S^{\Bbbk}(\Lambda^+_{n,d}(r,c,m))}.$$

Let N be any module belonging to $\Sigma^+_{n,d}(r,c,m)$ and let $M \in \mathcal{F}_{\Lambda^+_{n,d}(r,c,m)}(\Delta)$, we have

$$\mathrm{Ext}^j_{S^{\Bbbk}_{n,d}}(M,N) \cong \mathrm{Ext}^j_{S^{\Bbbk}(\Lambda^+_{n,d}(r,c,m))}(h_{r,c,m}(M), h_{r,c,m}(N)).$$

Proof. Given λ,μ,ν and $\mathsf{S} \in \mathrm{SStd}(\lambda,\mu)$, $\mathsf{T} \in \mathrm{SStd}(\lambda,\nu)$ we have that

$$\xi_{\mathsf{ST}} = \xi_{\mathsf{S}\lambda} 1_\lambda \xi_{\lambda\mathsf{T}} = 1_\mu \xi_{\mathsf{S}\lambda}\xi_{\lambda\mathsf{T}} = \xi_{\mathsf{S}\lambda}\xi_{\lambda\mathsf{T}} 1_\nu,$$

and so the algebra $S^{\mathbb{Z}}_{n,d}(\Lambda^+_{n,d}(r,c,m))$ has the stated integral cellular basis. We have $\Gamma^+_{n,d}(r,c,m)$ and $\Sigma^+_{n,d}(r,c,m)$ are saturated and co-saturated sets and therefore the result follows by first applying [Don98, Propositions A3.11 and A3.13] followed by [Don98, Propositions A3.3 and A3.4]. □

4.3. **Isomorphisms between subquotients of integral Schur algebras.** We now construct the isomorphism between the subquotient algebras in which we are interested. We first extend the combinatorics of cuts to semistandard tableaux. Given $\lambda,\mu \in \Lambda^+_{n,d}(r,c,m)$, let

$$\psi : \mathrm{SStd}(\lambda,\mu) \longrightarrow \mathrm{SStd}(\lambda^T,\mu^T) \times \mathrm{SStd}(\lambda^B,\mu^B),$$

be the map defined as follows. For $\mathsf{S} \in \mathrm{SStd}(\lambda,\mu)$, let $\psi(\mathsf{S}) = \mathsf{S}^T \times \mathsf{S}^B$, where S^T is obtained from S by deleting the $(r+1)$th, $(r+2)$th, … rows; and S^B is obtained from S by deleting the first r rows and replacing each entry i with the entry $i-r$.

We briefly show that ψ is a well defined bijection. Since S is semistandard, for all $j \in \{1,\dots,r\}$ we have that j appears only in the first r rows of S. Since $|\mu^T| = |\lambda^T| = m$, we deduce that no number greater than r appears in the first r rows of S. Hence $\mathsf{S}^T \in \mathrm{SStd}(\lambda^T,\mu^T)$. From this it clearly follows that $\mathsf{S}^B \in \mathrm{SStd}(\lambda^B,\mu^B)$. Hence ψ is well defined. Let now η be the map from $\mathrm{SStd}(\lambda^T,\mu^T) \times \mathrm{SStd}(\lambda^B,\mu^B)$ to $\mathrm{SStd}(\lambda,\mu)$, defined as follows. Let $\mathsf{S} \times \mathsf{T} \in \mathrm{SStd}(\lambda^T,\mu^T) \times \mathrm{SStd}(\lambda^B,\mu^B)$. We denote by $\widetilde{\mathsf{T}}$ the tableau obtained by replacing each entry i of T by $i+r$. For all $j \in \{1,2,\dots,r\}$ let the jth row of $\eta(\mathsf{S}\times\mathsf{T})$ coincide with the jth row of S. For all $j \in \{r+1,\dots,n\}$ let the jth row of $\eta(\mathsf{S}\times\mathsf{T})$ coincide with the $(j-r)$th row of $\widetilde{\mathsf{T}}$. The map η is well defined, since $\eta(\mathsf{S}\times\mathsf{T})$ is clearly an element of $\mathrm{SStd}(\lambda,\mu)$. Moreover, $\psi\cdot\eta$ is the identity map of $\mathrm{SStd}(\lambda^T,\mu^T) \times \mathrm{SStd}(\lambda^B,\mu^B)$ and $\eta\cdot\psi$ is the identity map of $\mathrm{SStd}(\lambda,\mu)$. This shows that ψ is a bijection.

Theorem 4.13. *The map*

$$\varphi : S^{\mathbb{Z}}(\Lambda^+_{n,d}(r,c,m)) \to S^{\mathbb{Z}}(\Lambda^+_{m,r}(r,c,m)) \times S^{\mathbb{Z}}(\Lambda^+_{n-m,d-r}(0,c,0))$$

given by

$$\varphi(\xi_{\mathsf{ST}}) = \xi_{\mathsf{S}^T\mathsf{T}^T} \times \xi_{\mathsf{S}^B\mathsf{T}^B}$$

is an isomorphism of $\mathbb{Z}$-algebras.

Proof. The map φ is an isomorphism of $\mathbb{Z}$-modules by definition. If $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$ and $\mathsf{T}(i,j) \neq 0$, then this implies $i, j \in \{1, \dots, r\}$ or $i, j \in \{r+1, \dots, d\}$. Therefore we can factorise the elements $\xi_{\lambda \mathsf{T}}$ as follows,

$$\xi_{\lambda\mathsf{T}} = \prod_{i=1}^{d}\prod_{j=1}^{d} e_{i,j}^{[\mathsf{T}(i,j)]} = \left(\prod_{i=r+1}^{d}\prod_{j=r+1}^{d} e_{i,j}^{[\mathsf{T}(i,j)]}\right) \times \left(\prod_{i=1}^{r}\prod_{j=1}^{r} e_{i,j}^{[\mathsf{T}(i,j)]}\right) =: \xi_{\lambda\mathsf{T}^B} \times \xi_{\lambda\mathsf{T}^T}$$

and similarly for the elements $\xi_{\mathsf{S}\lambda}$. We remark that the symbols $\xi_{\lambda\mathsf{T}^B}$ and $\xi_{\lambda\mathsf{T}^T}$ are naturally defined by the equation above. Therefore given $\mathsf{S} \in \mathrm{SStd}(\lambda, \alpha)$, $\mathsf{T} \in \mathrm{SStd}(\lambda, \beta)$ and $\mathsf{U} \in \mathrm{SStd}(\mu, \gamma)$ and $\mathsf{V} \in \mathrm{SStd}(\mu, \delta)$ we have that

$$\begin{aligned}
\xi_{\mathsf{ST}}\xi_{\mathsf{UV}} &= (\xi_{\mathsf{S}^T\lambda}1_\lambda\xi_{\lambda\mathsf{T}^T}\xi_{\mathsf{S}^B\lambda}1_\lambda\xi_{\lambda\mathsf{T}^B})(\xi_{\mathsf{U}^T\mu}1_\mu\xi_{\mu\mathsf{V}^T}\xi_{\mathsf{U}^B\mu}1_\mu\xi_{\mu\mathsf{V}^B}) \\
&= 1_\alpha(\xi_{\mathsf{S}^T\lambda}\xi_{\lambda\mathsf{T}^T}\xi_{\mathsf{S}^B\lambda}\xi_{\lambda\mathsf{T}^B})(\xi_{\mathsf{U}^T\mu}\xi_{\mu\mathsf{V}^T}\xi_{\mathsf{U}^B\mu}\xi_{\mu\mathsf{V}^B})1_\delta \\
&= 1_\alpha(\xi_{\mathsf{S}^T\lambda}\xi_{\lambda\mathsf{T}^T}\xi_{\mathsf{U}^T\mu}\xi_{\mu\mathsf{V}^T})(\xi_{\mathsf{S}^B\lambda}\xi_{\lambda\mathsf{T}^B}\xi_{\mathsf{U}^B\mu}\xi_{\mu\mathsf{V}^B})1_\delta \\
&= 1_\alpha\left(r(\mathsf{S}^T,\mathsf{T}^T,\mathsf{U}^T,\mathsf{V}^T,\mathsf{W}^T,\mathsf{X}^T)\xi_{\mathsf{W}^T\mathsf{X}^T} + S^{\mathbb{Z}}(\Sigma^+_{n,d}(r,c,m))\right) \\
&\quad \times \left(r(\mathsf{S}^B,\mathsf{T}^B,\mathsf{U}^B,\mathsf{V}^B,\mathsf{W}^B,\mathsf{X}^B)\xi_{\mathsf{W}^B\mathsf{X}^B} + S^{\mathbb{Z}}(\Sigma^+_{n,d}(r,c,m))\right)1_\delta
\end{aligned}$$

where $\mathsf{W}^T \in \mathrm{SStd}(-,\alpha)$, $\mathsf{X}^T \in \mathrm{SStd}(-,\beta)$ and $\mathsf{W}^B \in \mathrm{SStd}(-,\gamma)$, $\mathsf{X}^B \in \mathrm{SStd}(-,\delta)$ and where $r(\mathsf{S}^T,\mathsf{T}^T,\mathsf{U}^T,\mathsf{V}^T,\mathsf{W}^T,\mathsf{X}^T), r(\mathsf{S}^B,\mathsf{T}^B,\mathsf{U}^B,\mathsf{V}^B,\mathsf{W}^B,\mathsf{X}^B) \in \mathbb{Z}$. This is because the product $\xi_{\mathsf{S}^T\lambda}\xi_{\lambda\mathsf{T}^T}\xi_{\mathsf{S}^B\lambda}\xi_{\lambda\mathsf{T}^B}$ (respectively $\xi_{\mathsf{U}^T\mu}\xi_{\mu\mathsf{V}^T}\xi_{\mathsf{U}^B\mu}\xi_{\mu\mathsf{V}^B}$) can be written entirely in divided powers of the $e_{i,i+1}$ and $f_{i,i+1}$ for $1 \leq i \leq r-1$ (respectively $r+1 \leq i \leq d-1$).

Therefore, the map φ can be seen to be given by taking the products of generators on the left-hand side to those of the right-hand side as follows:

$$\varphi(1_\lambda) = \begin{cases} 1_{\lambda^T} \times 1_{\lambda^B} & \text{if } \lambda \in \Lambda_{n,d}(r,c,m) \\ 0 & \text{otherwise} \end{cases}$$

$$\varphi(e_{i,i+1}^{[m]}) = \begin{cases} e_{i,i+1}^{[m]} \times 1_{0,c,0}^{n-m,d-r} & \text{if } 1 \leq i \leq r-1 \\ 1_{r,c,m}^{m,r} \times e_{i-r,i-r+1}^{[m]} & \text{if } r+1 \leq i \leq d \end{cases}$$

$$\varphi(f_{i,i+1}^{[m]}) = \begin{cases} f_{i,i+1}^{[m]} \times 1_{0,c,0}^{n-m,d-r} & \text{if } 1 \leq i \leq r-1 \\ 1_{r,c,m}^{m,r} \times f_{i-r,i-r+1}^{[m]} & \text{if } r+1 \leq i \leq d \end{cases}$$

and the products can easily be seen to agree modulo the respective ideals. ☐

We immediately obtain a new reduction theorem concerning extension groups between a Weyl module and a simple module and reprove reduction theorems concerning extension groups between a pair of Weyl modules [LM05] and [Don98, 4.2(17)].

Corollary 4.14. *Let $\Bbbk$ be a commutative noetherian ring. If λ, μ admit a horizontal cut after the rth row, then we have the following isomorphisms between* Ext*-groups*

$$\mathrm{Ext}^k_{S^{\Bbbk}_{n,d}}(\Delta(\lambda), L(\mu)) \cong \bigoplus_{i+j=k} \mathrm{Ext}^i_{S^{\Bbbk}_{m,r}}(\Delta(\lambda^T), L(\mu^T)) \otimes \mathrm{Ext}^j_{S^{\Bbbk}_{n-m,d-r}}(\Delta(\lambda^B), L(\mu^B)).$$

and

$$\mathrm{Ext}^k_{S^{\Bbbk}_{n,d}}(\Delta(\lambda),\Delta(\mu)) \cong \bigoplus_{i+j=k} \mathrm{Ext}^i_{S^{\Bbbk}_{m,r}}(\Delta(\lambda^T),\Delta(\mu^T)) \otimes \mathrm{Ext}^j_{S^{\Bbbk}_{n-m,d-r}}(\Delta(\lambda^B),\Delta(\mu^B)).$$

Moreover, we have the following equality of decomposition numbers

$$[\Delta(\lambda):L(\mu)] = [\Delta(\lambda^T):L(\mu^T)] \times [\Delta(\lambda^B):L(\mu^B)].$$

Proof. This is immediate from Proposition 4.11 and Theorem 4.13. □

Corollary 4.15. *Let $\Bbbk$ be a commutative noetherian ring. Let $T(\lambda)$ be a tilting module whose Δ-factors belong to $\Lambda^+_{n,d}(r,c,m)$. For $\mu \in \Lambda^+_{n,d}(r,c,m)$ we have that*

$$\mathrm{Ext}^k_{S^{\Bbbk}_{n,d}}(T(\lambda),L(\mu)) \cong \bigoplus_{i+j=k} \mathrm{Ext}^i_{S^{\Bbbk}_{m,r}}(T(\lambda^T),L(\mu^T)) \otimes \mathrm{Ext}^j_{S^{\Bbbk}_{n-m,d-r}}(T(\lambda^B),L(\mu^B)).$$

Proof. This is immediate from Proposition 4.11 and Theorem 4.13. □

Remark 4.16 (Removing a single row)**.** We now consider the example of row cuts for $r=1$. In this case, the isomorphisms above (and implications for decomposition numbers and extension groups) were proven in [FHK08]. In this case, the results can also be seen to follow by tensoring with the determinant representation and applying a duality (as noted by Donkin in [FHK08, Appendix]).

4.4. **Generalised symmetric powers and p-Kostka numbers.** By Proposition 4.12, we know that injective, standard, and simple modules are all preserved under the functors $h_{r,c,m}$ and the isomorphism φ. It remains to check that the generalised symmetric powers are also preserved.

Theorem 4.17. *Let $\Bbbk$ be an arbitrary field. Given $\lambda,\mu \in \Lambda^+_{n,d}(r,c,m)$, we have that*

$$h_{r,c,m}(\mathrm{Sym}^{\mu}(\Bbbk^d)) \cong h_{r,c,m}(\mathrm{Sym}^{\mu^T}(\Bbbk^r)) \otimes h_{0,c,0}(\mathrm{Sym}^{\mu^B}(\Bbbk^{d-r}))$$

and

$$h_{r,c,m}(I(\lambda)) \cong h_{r,c,m}(I(\lambda^T)) \otimes h_{0,c,0}(I(\lambda^B)).$$

In particular, the p-Kostka numbers are preserved under generalised row cuts.

Proof. First, we note that $K_{\mu\lambda} \neq 0$ implies $\lambda \trianglerighteq \mu$. The isomorphism of injective modules is clear from Proposition 4.12 and Theorem 4.13. The result will therefore follow once we prove the isomorphism between the images of the generalised symmetric powers. Recall that the module $\mathrm{Sym}^{\mu}(\Bbbk^d)$ has basis

$$\{\rho_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\nu), \mathsf{T} \in \mathrm{SStd}(\lambda,\mu), \lambda \in \Lambda^+_{n,d}, \nu \in \Lambda_{n,d}\}.$$

Therefore $h_{r,c,m}(\mathrm{Sym}^{\mu}(\Bbbk^d))$ is the module with basis

$$\{\rho_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\nu), \mathsf{T} \in \mathrm{SStd}(\lambda,\mu), \lambda,\nu \in \Lambda^+_{n,d}(r,c,m)\}$$

and, of course, one obtains similar bases for both of the modules $h_{r,c,m}(\mathrm{Sym}^{\mu^T}(\Bbbk^r))$ and $h_{0,c,0}(\mathrm{Sym}^{\mu^B}(\Bbbk^{d-r}))$.

Any $\mathsf{T} \in \mathrm{SStd}(\lambda, \mu)$ has the entry s in each of the first c columns of the sth row for each $1 \leq s \leq r$. Therefore, any tableau t such that $\mu(\mathsf{t}) \neq 0$ must necessarily have entries $1, \dots, m$ in the first r rows and the entries $m+1, \dots, n$ in the final $d-r$ rows. Therefore, for $\lambda, \mu \in \Lambda^+_{n,d}(r, c, m)$, we have that the set

$$\{\mathsf{s} \in \mathrm{Std}(\lambda) \mid \mu(\mathsf{s}) \neq 0\}$$

is naturally in bijection with the set

$$\{\mathsf{t} \in \mathrm{Std}(\lambda^T) \mid \mu^T(\mathsf{t}) \neq 0\} \times \{\mathsf{u} \in \mathrm{Std}(\lambda^B) \mid \mu^B(\mathsf{u}) \neq 0\},$$

via the map $\varphi(s) = \mathsf{s}^T \times \mathsf{s}^B$, where

- s^T is obtained from s by deleting the $(r+1)$th, $(r+2)$th, ... rows;
- s^B is obtained from s by deleting the first r rows and replacing each entry i with the entry entry $i - m$.

Therefore, the map : $\mathsf{T} \mapsto \mathsf{T}^T \times \mathsf{T}^B$ lifts to an isomorphism

$$\psi : h_{r,c,m}(\mathrm{Sym}^\lambda(\Bbbk^d)) \longrightarrow h_{r,c,m}(\mathrm{Sym}^{\lambda^T}(\Bbbk^r)) \otimes h_{0,c,0}(\mathrm{Sym}^{\lambda^B}(\Bbbk^{d-r}))$$

given by

$$\psi\left(\rho_{\mathsf{ST}}\right) = \psi\left(\sum_{\mathsf{t} \in \mathsf{T}} \rho_{\mathsf{St}}\right) = \left(\sum_{\mathsf{t}^T \in \mathsf{T}^T} \rho_{\mathsf{S}^T\mathsf{t}^T}\right) \times \left(\sum_{\mathsf{t}^B \in \mathsf{T}^B} \rho_{\mathsf{S}^B\mathsf{t}^B}\right) = \rho_{\mathsf{S}^T\mathsf{T}^T} \times \rho_{\mathsf{S}^B\mathsf{T}^B}.$$

To complete the proof, it is enough to observe that if λ does not dominate μ then either λ^T does not dominate μ^T or λ^B does not dominate μ^B, by Proposition 4.11. Hence, by Proposition 3.16, we have that $K_{\mu\lambda} = 0 = K_{\mu^T\lambda^T} \cdot K_{\mu^B\lambda^B}$. □

4.5. **Generalised column cuts.** Given $\lambda \in \Lambda^+_{n,d}$ and $1 \leq c \leq n$, we define partitions

$$\lambda^L = (\lambda'_1, \lambda'_2, \dots \lambda'_c)' \quad \lambda^R = (\lambda'_{c+1}, \dots \lambda'_n)'.$$

We say that a pair of partitions λ and μ admit a generalised column cut after the cth column if

$$\sum_{1 \leq i \leq c} \lambda'_i = \sum_{1 \leq i \leq c} \mu'_i$$

for some $1 \leq c \leq n$. One can define similar subsets of $\Lambda^+_{n,d}(r, c, m)$ and generalise all the arguments and isomorphisms of the previous sections to cover these reduction theorems for generalised column cuts. However, it is also easy to deduce these results in two steps as follows. If $c = 1$, the isomorphisms are easily deduced by tensoring with the determinant representation (plus the use of an idempotent truncation if $d < n$, see [FHK08] for more details). The result now follows by applying this isomorphism along with the isomorphisms of Proposition 4.12. These arguments are standard for such results, see [FL03, Proof of Proposition 2.4]. We go through this argument more explicitly for p-Kostka numbers below.

Corollary 4.18. *The p-Kostka numbers are preserved under generalised column cuts. In other words, $K_{\lambda\mu} = K_{\lambda^L\mu^L} K_{\lambda^R\mu^R}$.*

Proof. Suppose that $\lambda, \mu \in \Lambda^+_{n,d}$ are such that $\lambda \trianglerighteq \mu$ and (λ, μ) admits a vertical cut after the cth column; we let $r = \lambda'_c$. It is easy to see that (λ, μ) admits a horizontal cut after the rth row. Therefore,

$$\begin{aligned} K_{\lambda\mu} &= K_{\lambda^T\mu^T} K_{\lambda^B\mu^B} \\ &= K_{(\lambda^T_1 - c, \dots \lambda^T_r - c)(\mu^T_1 - c, \dots \mu^T_r - c)} K_{\lambda^B\mu^B} \\ &= K_{\lambda^R\mu^R} K_{\lambda^B\mu^B} \\ &= K_{\lambda^R\mu^R} K_{(c^r, \lambda^B)(c^r, \mu^B)} \\ &= K_{\lambda^R\mu^R} K_{\lambda^L\mu^L} \end{aligned}$$

where the first equality follows from Theorem 4.17; the second (respectively fourth) equality follows from a total of c applications of first column removal [FHK08, Corollary 9.1] (respectively r applications of first row addition Theorem 4.17); and the third and fifth equalities follows by definition and our choice of $r = \lambda'_c$. □

Remark 4.19. In the case $r = 1$ or $c = 1$, the above reduction theorems for p-Kostka numbers were first proven in [FHK08].

5. The Schur functor

When $d \geq n$, the symmetric group acts faithfully on $1_\omega \mathbb{T}$ and we obtain an isomorphic copy of $\Bbbk\mathfrak{S}_n$ as the idempotent subalgebra $1_\omega S^\Bbbk_{n,d} 1_\omega$ of $S^\Bbbk_{n,d}$. In this section, we recall how one can use this idempotent truncation map (the Schur functor) to the study of the representation theory of $\Bbbk\mathfrak{S}_n$.

5.1. The Murphy basis of the symmetric group.

Given $\mathsf{t} \in \mathrm{Std}(\lambda)$, recall that d_t is the element of $\mathfrak{S}_n$ such that $(\mathsf{t}^\lambda) d_\mathsf{t} = \mathsf{t}$. For $\lambda \in \Lambda^+_{n,d}$ we denote by x_λ the element of the group algebra of the symmetric group defined by

$$x_\lambda = \sum_{x \in \mathfrak{S}_\lambda} x.$$

Theorem 5.1 (Murphy)**.** *The group algebra of the symmetric group is free as a $\mathbb{Z}$-module with basis*

$$\{x_{\mathsf{st}} \mid x_{\mathsf{st}} := d_\mathsf{s} x_\lambda d_\mathsf{t}^{-1}, \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \textit{ for } \lambda \in \Lambda^+_{n,d}\}.$$

Recall our bijective map $\omega : \mathrm{Std}(\lambda) \to \mathrm{SStd}(\lambda, \omega)$. Suppose that $\omega(\mathsf{s}) = \mathsf{S}$ and $\omega(\mathsf{t}) = \mathsf{T}$. Under this identification we obtain an isomorphism $1_\omega S^\Bbbk_{n,d} 1_\omega \cong \Bbbk\mathfrak{S}_n$ given by : $\xi_{\mathsf{ST}} \mapsto x_{\mathsf{st}}$. Therefore the basis in Theorem 5.1 is a cellular basis (in the sense of [GL96]) under the inherited cell structure (in other words, it satisfies the properties detailed in Theorem 3.7). In particular, we have the following.

Definition 5.2. Given $\lambda \in \Lambda^+_{n,d}$, we define the Specht module S^λ to be the left $\Bbbk\mathfrak{S}_n$–module with basis

$$\{x_{\mathsf{st}^\lambda} + \Bbbk\mathfrak{S}_n^{\rhd\lambda} \mid \mathsf{s} \in \mathrm{Std}(\lambda)\}$$

where $\Bbbk\mathfrak{S}_n^{\rhd\lambda}$ is the $\Bbbk$-module with basis $\{x_{\mathsf{uv}} \mid \mathsf{u},\mathsf{v} \in \mathrm{Std}(\mu), \mu \rhd \lambda\}$. Similarly, we define the dual Specht module S_λ to be the left $\Bbbk\mathfrak{S}_n$–module with basis

$$\{\rho_{\mathsf{ST}^\lambda} + 1_\omega \mathbb{T}^{\rhd\lambda} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\omega)\}$$

where $1_\omega\mathbb{T}^{\rhd\lambda}$ is the $\Bbbk$-module with basis $\{\rho_{\mathsf{Uv}} \mid \mathsf{U} \in \mathrm{SStd}(\mu,\omega), \mathsf{v} \in \mathrm{Std}(\mu), \mu \rhd \lambda\}$.

Definition 5.3. We say that $\lambda \in \Lambda^+_{n,d}$ is p-restricted if $\lambda_i - \lambda_{i+1} < p$ for all $1 \leq i < d$. If $\lambda \in \Lambda^+_{n,d}$ is not p-restricted, we say that it is p-singular.

Each Specht module S^λ is equipped with the bilinear form $\langle\ ,\ \rangle_\lambda$, inherited from the idempotent truncation. This form is degenerate if and only if λ is p-singular. Given a p-restricted $\lambda \in \Lambda^+_{n,d}$, we define the the simple module $D(\lambda)$ to be the quotient of the Specht module S^λ by the radical of the the bilinear form $\langle\ ,\ \rangle_\lambda$. By elementary properties of idempotent truncation functors, we have that

$$[S^\lambda : D(\mu)] = [\Delta(\lambda) : L(\mu)]$$

for all $\lambda \in \Lambda^+_{n,d}$ and all p-restricted $\mu \in \Lambda^+_{n,d}$. By Corollary 4.14, we have the following corollary (see also [Don85]).

Corollary 5.4. *Let λ denote a partition of n and let μ denote a p-regular partition of n. If (λ,μ) admit a horizontal cut after the rth row, then*

$$[S^\lambda : D(\mu)] = [S^{\lambda^T} : D(\mu^T)] \times [S^{\lambda^B} : D(\mu^B)].$$

Proof. This follows immediately from Corollary 4.14 and the above. □

5.2. **Young permutation modules.** Given $\mu \in \Lambda_{n,d}$, we let $M(\mu)$ denote the image of the generalised symmetric power under the Schur functor,

$$M(\mu) = 1_\omega(\mathrm{Sym}^\mu(\Bbbk^d)).$$

We refer to these modules as the Young permutation modules. By definition, the module $M(\mu)$ has basis given by the subset of all the vectors of weight ω in Proposition 3.15 as follows

$$\{\rho_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda,\omega), \mathsf{T} \in \mathrm{SStd}(\lambda,\mu), \lambda \in \Lambda^+_{n,d}\}.$$

Under the identification of $\mathrm{SStd}(\lambda,\omega)$ and $\mathrm{Std}(\lambda)$, we recover Murphy's basis of these permutation modules [Mur95].

Proposition 5.5 (J. A. Green [Gre80])**.** *For $\lambda,\mu \in \Lambda^+_{n,d}$, the module $M(\mu)$ decomposes as a direct sum as follows*

$$M(\mu) = \bigoplus_{\lambda \vdash n} K_{\mu\lambda} Y(\lambda)$$

where $Y(\lambda) = 1_\omega(I(\lambda))$; we refer to the module $Y(\lambda)$ as the indecomposable Young module of weight *λ.*

Corollary 5.6. *Let $\Bbbk$ be a field. If (λ,μ) admit a horizontal cut after the rth row, then*

$$[M(\lambda) : Y(\mu)] = [M(\lambda^T) : Y(\mu^T)] \times [M(\lambda^B) : Y(\mu^B)].$$

Proof. This follows immediately from Theorem 4.17 and the above. □

5.3. **The faithfulness of the Schur functor.** The following theorem, first proved in [KN01, Section 6.4] and [Don07, Proposition 10.5], states the degree to which cohomological information is preserved under the Schur functor.

Theorem 5.7. *Let $\Bbbk$ denote an algebraically closed field of characteristic $p \geq 3$. The Schur algebra $S^{\Bbbk}_{n,d}$ is a $(p-3)$-faithful cover (in the sense of* [Rou08]*) of the symmetric group, $\Bbbk\mathfrak{S}_n$. That is,*

$$\mathrm{Ext}^i_{S^{\Bbbk}_{n,d}}(\Delta(\lambda), \Delta(\mu)) \cong \mathrm{Ext}^i_{\Bbbk\mathfrak{S}_n}(S^\lambda, S^\mu)$$

for all $\lambda, \mu \in \Lambda^+_{n,d}$ and all $0 \leq i \leq p-3$.

Corollary 5.8. *Let $\Bbbk$ denote an algebraically closed field of characteristic $p \geq 3$. If (λ, μ) admit a horizontal cut after the rth row, then*

$$\mathrm{Ext}^i_{\Bbbk\mathfrak{S}_n}(S^\lambda, S^\mu) \cong \bigoplus_{i+j=k} \mathrm{Ext}^i_{\Bbbk\mathfrak{S}_m}(S^{\lambda^T}, S^{\mu^T}) \otimes \mathrm{Ext}^j_{\Bbbk\mathfrak{S}_{n-m}}(S^{\lambda^B}, S^{\mu^B})$$

for all $\lambda, \mu \in \Lambda^+_{n,d}$ and all $0 \leq i \leq p-3$.

Proof. This follows immediately from Corollary 4.14 and the above. □

Remark 5.9. This result can be partially extended to $p = 2$ [LM05, Theorem 1.1].

Remark 5.10. These results can be extended to cyclotomic Hecke algebras [FS].

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