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The modular Weyl–Kac character formula

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Abstract

We classify and explicitly construct the irreducible graded representations of anti-spherical Hecke categories which are concentrated one degree. Each of these homogeneous representations is one-dimensional and can be cohomologically constructed via a BGG resolution involving every (infinite dimensional) standard representation of the category. We hence determine the complete first row of the inverse parabolic p -Kazhdan–Lusztig matrix for an arbitrary Coxeter group and an arbitrary parabolic subgroup. This generalises the Weyl–Kac character formula to all Coxeter systems (and their parabolics) and proves that this generalised formula is rigid with respect to base change to fields of arbitrary characteristic.

Keywords Hecke categories · p -Kazhdan-Lusztig polynomials · Weyl-Kac character formula

Introduction

The discovery of counterexamples to the expected bounds of Lusztig’s conjecture was an earthquake in representation theory. It marked the beginning of a new era of Lie theory, in which diagrammatic Hecke categories play centre stage in our attempts to understand the structure of algebraic groups in terms of parabolic “ p -Kazhdan–Lusztig polynomials”. There are precious few general results concerning either the simple representations of these diagrammatic Hecke categories, or the underlying combinatorics of parabolic p -Kazhdan–Lusztig polynomials.

We let \mathbb{k} denote an algebraically closed field of characteristic $p > 2$. Given W an arbitrary Coxeter group and P an arbitrary parabolic subgroup, we classify and construct the homoge-

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neous (simple) representations of the anti-spherical Hecke category $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ (note that $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ is a \mathbb{k} -linear graded category and so this notion makes sense) constructed from the “geometric realisation” of W . We prove that a $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module is homogeneous if and only if it is one-dimensional if and only if it is the simple $L(1_{P \setminus W})$ labelled by the identity coset $1_{P \setminus W} \in {}^P W$ (and we provide a basis of $L(1_{P \setminus W})$ by way of Libedinsky’s light leaves construction).

Theorem A *The anti-spherical Hecke category $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ admits a unique homogeneous simple module, $L(1_{P \setminus W})$, labelled by the identity coset $1_{P \setminus W}$. This simple module is a one-dimensional quotient of the infinite-dimensional standard $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module $\Delta(1_{P \setminus W})$.*

Concurrently, we provide a cohomological construction of the unique homogeneous $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module by way of a BGG resolution. Within this BGG resolution, every one of the (infinite-dimensional) standard $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -representations $\Delta(w)$ for $w \in {}^P W$ appears in degree as dictated by the length function on the underlying Hecke algebra. Our BGG resolutions allow us to calculate the complete first row of the inverse parabolic p -Kazhdan–Lusztig matrix for W an arbitrary Coxeter group and P an arbitrary parabolic subgroup. This provides the first family of explicit (inverse) p -Kazhdan–Lusztig polynomials to admit a uniform description across arbitrary Coxeter groups and their parabolic subgroups. In the case that W is an affine Weyl group and P is the maximal finite parabolic subgroup this gives new character formulas for representations of the corresponding algebraic groups through [1, 2].

Theorem B *Associated to the unique homogeneous simple $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module, $L(1_{P \setminus W})$, we have a complex $C_{\bullet}(1_{P \setminus W}) = \bigoplus_{w \in {}^P W} \Delta(w)\langle \ell(w) \rangle$ with differential given by an alternating sum over all “simple reflection homomorphisms”. This complex is exact except in degree zero, where $H_0(C_{\bullet}(1_{P \setminus W})) = L(1_{P \setminus W})$. We hence conclusively generalise the Weyl–Kac character formula to all (parabolic) Coxeter systems via the formula*

$$[L(1_{P \setminus W})] = \sum_{w \in {}^P W} (-v)^{\ell(w)} [\Delta(w)]$$

and thus conclude that the first row of the inverse parabolic p -Kazhdan–Lusztig matrix has entries $(-v)^{\ell(w)}$ regardless of the characteristic $p \neq 2$.

Specialising to the case of (affine) Weyl groups, our character formulas and resolutions have a long history. For finite Weyl groups, Bernstein–Gelfand–Gelfand constructed their eponymous resolutions in the context of finite dimensional Lie algebras [8]. For Kac–Moody Lie algebras these were the subject of Kac–Kazhdan’s conjecture [34] (over \mathbb{C}) which was verified by Wakimoto (for $W = \widehat{\mathfrak{S}}_2$ [46]), Hayashi (for classical type [25]) and Feigin, Frenkel, and Ku (in full generality [22, 36]) and was extended to arbitrary fields by Mathieu [40] and subsequently reproven by Arakawa using W -algebras [3, 4]. For parabolic subgroups of finite Weyl groups, our resolutions were first constructed in [37] and went on to have spectacular applications in the study of the Laplacian on Euclidean space [15]. For the infinite dihedral Weyl group with two generators, these resolutions were generalised to the Virasoro and blob algebras of algebraic statistical mechanics [21, 23, 41]. For W the finite symmetric group and P a maximal parabolic, these resolutions were one of the highlights of Brundan–Stroppel’s founding work on categorical representation theory [10, 11]. For W the affine symmetric group and P the maximal finite parabolic and $\mathbb{k} = \mathbb{C}$, Theorem B proves a recent conjecture of Berkesch–Griffeth–Sam [5] concerning BGG resolutions of unitary modules for Cherednik algebras.

Kazhdan–Lusztig conjectured that much of combinatorial Lie theory should generalise beyond the realm of Weyl groups (where our resolutions admit the geometric realisations

discussed above) to arbitrary Coxeter groups. Hecke categories provide the structural perspective in which the Kazhdan–Lusztig conjecture was finally proven [19] and serve as the archetypal setting for studying all Lie theoretic objects. In this light, Theorem B provides the prototype for all the aforementioned BGG resolutions and vastly generalises their construction to all parabolic Coxeter systems (which are poorly understood in general, but include for example the finite, affine, compact, paracompact, hypercompact, and Lorentzian Coxeter groups) and to fields of positive characteristic — the instances for Lie groups, Kac–Moody Lie algebras, and their parabolic analogues are merely the examples for which a classical geometric structure exists.

Our proof is surprisingly elementary, the key idea is to exploit the inductive nature of the light leaves construction. This plays off both the monoidal structure and the highest-weight structure of the Hecke category, which are compatible with certain truncated restriction functors. Our proof is more compact than many of the proofs for (affine) Weyl groups cited above and should be accessible to those who are new to the research area.

p -Kazhdan–Lusztig polynomials

The anti-spherical Kazhdan–Lusztig polynomials for crystallographic Coxeter systems were first studied in the language of Hecke categories by Libedinsky–Williamson over the complex numbers [38]. The authors remark that their localisation methods do not carry over to fields of positive characteristic and so they do not define p -Kazhdan–Lusztig polynomials (or light leaves bases) in their paper. In this paper we make the (rather trivial) observation that one does not need such localisation methods in order to define p -Kazhdan–Lusztig polynomials (or light leaves bases). In Sect. 1.6 we discuss under what circumstances one can restrict to $\mathbb{Z} \subset \mathbb{C}$ and hence obtain light leaves bases by “reduction modulo p ”. In this manner, we define the p -Kazhdan–Lusztig polynomials for arbitrary (parabolic) Coxeter systems.

With this machinery in place, we note that Theorem B provides the first instance of a complete row/column of the (inverse) (p -)Kazhdan–Lusztig matrix to be calculated for all Coxeter groups (and their parabolics). This is especially noteworthy considering just how difficult it is to calculate p -Kazhdan–Lusztig polynomials — for example, the current state-of-the-art “billiards conjecture” of Jensen–Lusztig–Williamson describes an infinitesimally small region of the p -Kazhdan–Lusztig matrix in type $A_2 \setminus \widehat{A}_2$ [31, 39]. The only other examples of p -Kazhdan–Lusztig polynomials which are explicitly known are the famous “torsion explosion” counter examples to the Lusztig conjecture [47]. Therefore we expect our examples of (characteristic-free) p -Kazhdan–Lusztig polynomials to be of wide interest in their community, as they provide the most general family of these polynomials calculated to date.

1 The diagrammatic Hecke categories

We begin by recalling the basics of diagrammatic Hecke categories. Almost everything from this section is lifted from Elias–Williamson’s original paper [20] or is an extension of their results to the parabolic setting [38].

Remark 1.1 The cyclotomic quotients of (anti-spherical) Hecke categories are small categories with finite-dimensional morphism spaces given by the light leaves basis of [20, 38]. Working with such a category is equivalent working to with a locally unital algebra, as defined in [12, Section 2.2], see [12, Remark 2.3]. Throughout this paper we will work in the latter

setting. The reader who prefers to think of categories can equivalently phrase everything in the this paper in terms of categories and representations of categories.

1.1 Coxeter systems

Let (W, S) be a Coxeter system: W is the group generated by the finite set S subject to the relations $(\sigma\tau)^{m_{\sigma\tau}} = 1$ for $m_{\sigma\tau} \in \mathbb{N} \cup \{\infty\}$ satisfying $m_{\sigma\tau} = m_{\tau\sigma}$ and that $m_{\sigma\tau} = 1$ if and only if $\sigma = \tau \in S$. Let $\ell : W \rightarrow \mathbb{N}$ be the corresponding length function. Let $\mathcal{L} = \mathbb{Z}[v, v^{-1}]$ be the ring of Laurent polynomials with integer coefficients in one variable v .

Consider $S_P \subseteq S$ an arbitrary subset and (P, S_P) its corresponding Coxeter group. We say that (P, S_P) is the parabolic subgroup corresponding to the set $S_P \subseteq S$. We denote by ${}^P W \subseteq W$ the set of minimal coset representatives in $P \backslash W$.

Given an expression $\underline{w} = \sigma_1\sigma_2 \cdots \sigma_\ell$ for $\sigma_i \in W$ for $1 \leq i \leq \ell$, we let w be the corresponding element of W . We define a subword of \underline{w} to be a sequence $\mathbf{t} = (t_1, t_2, \dots, t_\ell) \in \{0, 1\}^\ell$ and we set $\underline{w}^{\mathbf{t}} := \sigma_1^{t_1}\sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}$ and we emphasise that $s_i^0 = 1_W \in W$. We write $y \leq w$ if for some (or equivalently, every) reduced expression \underline{w} there exists a subword \mathbf{t} and a reduced expression \underline{y} such that $\underline{w}^{\mathbf{t}} = \underline{y}$. We let $\text{exp}_P^\ell(w)$ denote the set of all expressions $\underline{w} = \sigma_1\sigma_2 \cdots \sigma_\ell$ of w of length ℓ such that $\sigma_1 \dots \sigma_k \in {}^P W$ for each $1 \leq k \leq \ell$, we let $\text{exp}_P(w) = \cup_{\ell \geq 0} \text{exp}_P^\ell(w)$, and $\text{exp}_P^\ell = \cup_{w \in W} \text{exp}_P^\ell(w)$. We set $\text{rexp}_P(w) := \text{exp}_P^{\ell(w)}(w)$.

1.2 Bi-coloured quantum numbers and Cartan matrices

We define the x - and y - bicoloured quantum numbers as follows. First set

$$[0]_x = [0]_y = 0 \quad [1]_x = [1]_y = 1 \quad [2]_x = x \quad [2]_y = y \tag{1.1}$$

and then inductively define

$$[2]_x[k]_y = [k + 1]_x + [k - 1]_x \quad [2]_y[k]_x = [k + 1]_y + [k - 1]_y. \tag{1.2}$$

When k is odd, $[k]_x = [k]_y$. The following definition allows one to speak of Cartan matrices of Coxeter groups.

Definition 1.2 Let \mathbb{k} be a complete local ring in which 2 is invertible. A balanced Cartan matrix of (W, S) over \mathbb{k} is an $|S| \times |S|$ -matrix $(a_{\sigma\tau})_{\sigma, \tau \in S}$ such that

- (1) for all $\sigma \in S$ we have $a_{\sigma\sigma} = 2$;
- (2) for all distinct $\sigma, \tau \in S$ such that $m_{\sigma\tau} < \infty$, set $x = -a_{\sigma\tau}$ and $y = -a_{\tau\sigma}$. We require that

$$[m_{\sigma\tau}]_x = [m_{\sigma\tau}]_y = 0 \quad [m_{\sigma\tau} - 1]_x = [m_{\sigma\tau} - 1]_y = 1 \tag{1.3}$$

1.3 Soergel graphs

Let (W, S) denote an arbitrary Coxeter system with S finite. Given $\sigma \in S \cup \{\emptyset\}$ we define the monochrome Soergel generators to be the framed graphs



and given any $\sigma, \tau \in S$ with $m_{\sigma\tau} = m < \infty$ we have the bi-chrome generator

$$\text{braid}_{\sigma\tau}^{\tau\sigma}(m) = \text{diagram} \quad \text{braid}_{\sigma\tau}^{\tau\sigma}(m) = \text{diagram}$$

for m odd, or even respectively. Here the northern edges are coloured with the sequence

$$\underbrace{\tau\sigma\tau\sigma \dots \sigma\tau}_{m \text{ times}} \quad \underbrace{\tau\sigma\tau\sigma \dots \tau\sigma}_{m \text{ times}}$$

for m odd or even respectively and the southern edges are coloured

$$\underbrace{\sigma\tau\sigma\tau \dots \tau\sigma}_{m \text{ times}} \quad \underbrace{\sigma\tau\sigma\tau \dots \sigma\tau}_{m \text{ times}}$$

for m odd or even respectively. We define the northern/southern reading word of a Soergel generator (or its dual) to be word in the alphabet S obtained by reading the colours of the northern/southern edge of the frame respectively (ignoring any \emptyset symbols). Pictorially, we define the duals of these generators to be the graphs obtained by reflection through their horizontal axes. Non-pictorially, we simply swap the sub- and superscripts. We denote this duality by $*$. For example, the dual of the fork generator is pictured as follows

$$\text{fork}_{\sigma\sigma}^{\sigma\sigma} = \text{diagram} = (\text{fork}_{\sigma\sigma}^{\sigma\sigma})^* \tag{1.4}$$

Given any two (dual) Soergel generators D and D' we define $D \otimes D'$ to be the diagram obtained by horizontal concatenation (and we extend this linearly). The northern/southern colour sequence of $D \otimes D'$ is the concatenation of those of D and D' ordered from left to right. Given any two (dual) Soergel generators, we define their product $D \circ D'$ to be the vertical concatenation of D on top of D' if the southern reading word of D is equal to the northern reading word of D' and to be zero otherwise. Finally, we define a Soergel graph to be any graph obtained by repeated horizontal and vertical concatenation of the Soergel generators and their duals.

1.4 Some specific graphs

For $\underline{w} = \sigma_1 \dots \sigma_\ell$ an expression, we define

$$1_{\underline{w}} = 1_{\sigma_1} \otimes 1_{\sigma_2} \otimes \dots \otimes 1_{\sigma_\ell} \tag{1.5}$$

and given $k > 1$ and $\sigma, \tau \in S$ we set

$$1_{\sigma\tau}^k = 1_\sigma \otimes 1_\tau \otimes 1_\sigma \otimes 1_\tau \dots \tag{1.6}$$

to be the alternately coloured idempotent on k strands (so that the final strand is σ - or τ -coloured if k is odd or even respectively). Given $\sigma, \tau \in S$ with $m_{\sigma\tau} = m$ even, let

$$\underline{w} = \rho_1 \dots \rho_k(\sigma\tau \dots \sigma\tau)\rho_{m+k+1} \dots \rho_\ell \quad \underline{\underline{w}} = \rho_1 \dots \rho_k(\tau\sigma \dots \tau\sigma)\rho_{m+k+1} \dots \rho_\ell \tag{1.7}$$

be two reduced expressions for $w \in W$. We say that \underline{w} and $\underline{\underline{w}}$ are adjacent and we set

$$\text{braid}_{\underline{\underline{w}}}^{\underline{w}} = 1_{\rho_1} \otimes \dots \otimes 1_{\rho_k} \otimes \text{braid}_{\tau\sigma}^{\sigma\tau}(m) \otimes 1_{\rho_{m+k+1}} \otimes \dots \otimes 1_{\rho_\ell} \tag{1.8}$$



Fig. 1 The fork-braid and Jones–Wenzl relations for $m_{\sigma\tau} = 3$

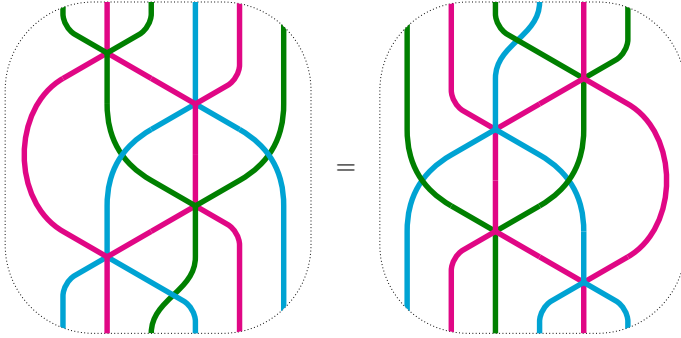


Fig. 2 The Zamolodchikov relation for A_3

(similarly for m odd). Now, given a sequence of adjacent reduced expressions,

$$\underline{w} = \underline{w}^{(1)}, \underline{w}^{(2)}, \dots, \underline{w}^{(q)} = \underline{w} \tag{1.9}$$

and the value q is minimal such that this sequence exists, then we set

$$\text{braid}_{\underline{w}}^{\underline{w}} = \prod_{1 \leq p < q} \text{braid}_{\underline{w}^{(p+1)}}^{\underline{w}^{(p)}} \in \mathcal{H}_W^k. \tag{1.10}$$

While this element is not uniquely defined, only the minimality will matter for our purposes (Fig. 1).

Example 1.3 The left and righthand diagrams depicted in Fig. 2 are both of the form $\text{braid}_{\underline{w}}^{\underline{w}}$ for

$$\underline{w} = \rho\sigma\rho\tau\sigma\tau\rho\sigma\tau \quad \text{and} \quad \underline{w} = \tau\sigma\rho\tau\sigma\tau. \tag{1.11}$$

The corresponding sequences of adjacent reduced expressions are recorded in the Zamolodchikov relation pictured in Fig. 2.

1.5 The diagrammatic Hecke categories

Let (W, S) be a Coxeter system with a balanced Cartan matrix $(a_{\sigma\tau})_{\sigma, \tau \in S}$. Suppose $\sigma, \tau \in S$ with $m = m_{\sigma\tau} < \infty$. In order to save space, we set

$$\text{jw}_{\sigma\sigma\sigma}^{\sigma\tau\sigma} = (1_\sigma \otimes \text{spot}_{\mathfrak{H}}^{\sigma} \otimes 1_\sigma)(\text{fork}_{\sigma\sigma}^{\sigma} \otimes 1_\sigma) \quad \text{wj}_{\sigma\tau\sigma}^{\sigma\sigma\sigma} = (1_\sigma \otimes \text{fork}_{\sigma\sigma}^{\sigma\sigma})(1_\sigma \otimes \text{spot}_{\tau}^{\theta} \otimes 1_\sigma) \tag{1.12}$$

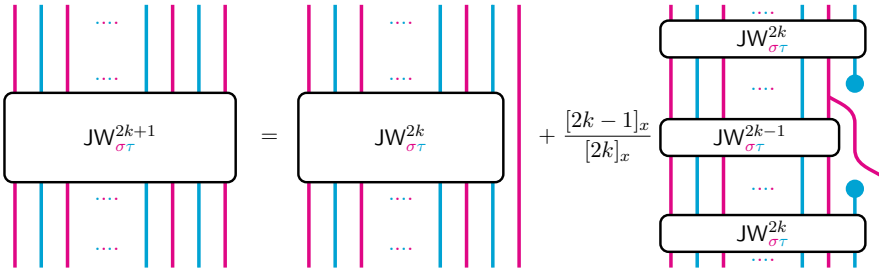
We are now ready to inductively define the Jones–Wenzl projector $\text{JW}_{\sigma\tau}^{2k+1}$ to be the element

$$\text{JW}_{\sigma\tau}^{2k} \otimes 1_\sigma + \frac{[2k-1]_x}{[2k]_x} (\text{JW}_{\sigma\tau}^{2k} \otimes 1_\sigma)(1_{\sigma\tau}^{2k-2} \otimes \text{jw}_{\sigma\sigma\sigma}^{\sigma\tau\sigma})(\text{JW}_{\sigma\tau}^{2k-1} \otimes 1_{\sigma\sigma})(1_{\sigma\tau}^{2k-2} \otimes \text{wj}_{\sigma\tau\sigma}^{\sigma\sigma\sigma})(\text{JW}_{\sigma\tau}^{2k} \otimes 1_\sigma)$$

and the Jones–Wenzl projector $JW_{\sigma\tau}^{2k}$ to be the element

$$JW_{\sigma\tau}^{2k-1} \otimes 1_\tau + \frac{[2k-2]_x}{[2k-1]_x} (JW_{\sigma\tau}^{2k-1} \otimes 1_\tau) (1_{\sigma\tau}^{2k-3} \otimes jw_{\tau\tau\tau}^{\tau\sigma\tau}) (JW_{\sigma\tau}^{2k-2} \otimes 1_{\tau\tau}) (1_{\sigma\tau}^{2k-3} \otimes wj_{\tau\sigma\tau}^{\tau\tau\tau}) (JW_{\sigma\tau}^{2k-1} \otimes 1_\tau).$$

We remark that in each case the leftmost strand is coloured with σ and the second term has coefficient equal to a ratio of x -bicoloured quantum integers. The pictorial version of the first recursion (for $2k + 1$ odd) is as follows



The elements $JW_{\tau\sigma}^{2k}$ and $JW_{\tau\sigma}^{2k+1}$ are the same as the above except with the inverted colour pattern and coefficients equal to y -bicoloured quantum integers. Specifically, we set $JW_{\tau\sigma}^{2k+1}$ to be the element

$$JW_{\tau\sigma}^{2k} \otimes 1_\tau + \frac{[2k-1]_y}{[2k]_y} (JW_{\tau\sigma}^{2k} \otimes 1_\tau) (1_{\tau\sigma}^{2k-2} \otimes jw_{\tau\tau\tau}^{\tau\sigma\tau}) (JW_{\tau\sigma}^{2k-1} \otimes 1_{\tau\tau}) (1_{\tau\sigma}^{2k-2} \otimes wj_{\tau\sigma\tau}^{\tau\tau\tau}) (JW_{\tau\sigma}^{2k} \otimes 1_\tau)$$

and we set $JW_{\tau\sigma}^{2k}$ to be the element

$$JW_{\tau\sigma}^{2k-1} \otimes 1_\sigma + \frac{[2k-2]_y}{[2k-1]_y} (JW_{\tau\sigma}^{2k-1} \otimes 1_\sigma) (1_{\tau\sigma}^{2k-3} \otimes jw_{\sigma\sigma\sigma}^{\sigma\tau\sigma}) (JW_{\tau\sigma}^{2k-2} \otimes 1_{\sigma\sigma}) (1_{\tau\sigma}^{2k-3} \otimes wj_{\sigma\sigma\sigma}^{\sigma\sigma\sigma}) (JW_{\tau\sigma}^{2k-1} \otimes 1_\sigma).$$

Finally, we define $JW_{\sigma\tau}$ and $JW_{\tau\sigma}$ to be the evaluation of the diagrams $JW_{\sigma\tau}^m$ and $JW_{\tau\sigma}^m$ respectively at $x = -a_{\sigma\tau}$ and $y = -a_{\sigma\tau}$.¹ The Jones–Wenzl relation for $m = 3$ is pictured in Fig. 2.

Definition 1.4 Let \mathbb{k} be an arbitrary commutative ring. Let (W, S) be a Coxeter system with a balanced Cartan matrix $(a_{\sigma\tau})_{\sigma, \tau \in S}$ over a commutative ring \mathbb{k} . We define $\mathcal{H}_W^{\mathbb{k}}$ to be the locally-unital associative \mathbb{Z} -graded \mathbb{k} -algebra spanned by all Soergel-graphs, with duality $*$, and multiplication given by vertical concatenation of diagrams modulo the following local relations and their duals.

$$\begin{aligned} 1_\sigma 1_\tau &= \delta_{\sigma, \tau} 1_\sigma & 1_\emptyset 1_\sigma &= 0 & 1_\emptyset^2 &= 1_\emptyset \\ 1_\emptyset \text{spot}_\sigma^\emptyset 1_\sigma &= \text{spot}_\sigma^\emptyset & 1_\sigma \text{fork}_{\sigma\sigma}^\sigma 1_\sigma &= \text{fork}_{\sigma\sigma}^\sigma & 1_\sigma^m \text{braid}_{\sigma\tau}^{\tau\sigma}(m) &= \text{braid}_{\sigma\tau}^{\tau\sigma}(m) \end{aligned}$$

For each $\sigma \in S$ we have monochrome relations

$$\begin{aligned} (\text{spot}_\sigma^\emptyset \otimes 1_\sigma) \text{fork}_{\sigma\sigma}^{\sigma\sigma} &= 1_\sigma & (1_\sigma \otimes \text{fork}_{\sigma\sigma}^\sigma) (\text{fork}_{\sigma\sigma}^{\sigma\sigma} \otimes 1_\sigma) &= \text{fork}_{\sigma\sigma}^{\sigma\sigma} \text{fork}_{\sigma\sigma}^\sigma \\ \text{fork}_{\sigma\sigma}^\sigma \text{fork}_{\sigma\sigma}^{\sigma\sigma} &= 0 & (\text{spot}_\sigma^\emptyset \text{spot}_\sigma^\emptyset) \otimes 1_\sigma + 1_\sigma \otimes (\text{spot}_\sigma^\emptyset \text{spot}_\sigma^\emptyset) &= 2(\text{spot}_\sigma^\emptyset \text{spot}_\sigma^\emptyset) \end{aligned}$$

For every ordered pair $(\sigma, \tau) \in S^2$ with $\sigma \neq \tau$, the bi-chrome relations: The two-colour barbell,

$$(\text{spot}_\tau^\emptyset \text{spot}_\tau^\emptyset) \otimes 1_\sigma - 1_\sigma \otimes (\text{spot}_\tau^\emptyset \text{spot}_\tau^\emptyset) = a_{\sigma\tau} ((\text{spot}_\tau^\emptyset \text{spot}_\sigma^\emptyset) - 1_\sigma \otimes (\text{spot}_\sigma^\emptyset \text{spot}_\tau^\emptyset)).$$

¹ Here we are using the fact that Jones–Wenzl projectors can be computed “generically” [16, Theorem 6.13].

If $m = m_{\sigma\tau} < \infty$ we also have the fork-braid relations

$$\begin{aligned} \text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau}(\text{fork}_{\sigma\sigma}^{\sigma} \otimes 1_{\tau\sigma}^{m-1})(1_{\sigma} \otimes \text{braid}_{\tau\sigma\cdots\tau\sigma}^{\sigma\tau\cdots\tau\sigma}) &= (1_{\tau\sigma}^{m-1} \otimes \text{fork}_{\tau\tau}^{\tau})(\text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau} \otimes 1_{\tau}) \\ \text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau}(\text{fork}_{\sigma\sigma}^{\sigma} \otimes 1_{\tau\sigma}^{m-1})(1_{\sigma} \otimes \text{braid}_{\tau\sigma\cdots\tau\sigma}^{\sigma\tau\cdots\tau\sigma}) &= (1_{\tau\sigma}^{m-1} \otimes \text{fork}_{\sigma\sigma}^{\sigma})(\text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau} \otimes 1_{\sigma}) \end{aligned}$$

for m odd and even, respectively. We require the cyclicity relation,

$$\begin{aligned} (1_{\tau\sigma}^m \otimes (\text{spot}_{\sigma}^{\emptyset} \text{fork}_{\sigma\sigma}^{\sigma})) (1_{\tau} \otimes \text{braid}_{\tau\sigma}^{\sigma\tau}(m) \otimes 1_{\sigma}) ((\text{fork}_{\tau\tau}^{\tau\tau} \text{spot}_{\emptyset}^{\tau}) \otimes 1_{\sigma\tau}^m) &= \text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau} \\ (1_{\tau\sigma}^m \otimes (\text{spot}_{\tau}^{\emptyset} \text{fork}_{\tau\tau}^{\tau})) (1_{\tau} \otimes \text{braid}_{\tau\sigma}^{\sigma\tau}(m) \otimes 1_{\tau}) ((\text{fork}_{\tau\tau}^{\tau\tau} \text{spot}_{\emptyset}^{\tau}) \otimes 1_{\sigma\tau}^m) &= \text{braid}_{\sigma\tau\cdots\tau\sigma}^{\tau\cdots\sigma\tau}. \end{aligned}$$

for m odd or even, respectively. We have the Jones–Wenzl relations

$$\begin{aligned} (1_{\tau\sigma}^{m-1} \otimes \text{spot}_{\tau}^{\emptyset}) \text{braid}_{\sigma\tau}^{\tau\sigma}(m) &= (\text{spot}_{\sigma}^{\emptyset} \otimes 1_{\tau\sigma}^{m-1}) \text{JW}_{\sigma\tau} \\ (1_{\tau\sigma}^{m-1} \otimes \text{spot}_{\sigma}^{\emptyset}) \text{braid}_{\sigma\tau}^{\tau\sigma}(m) &= (\text{spot}_{\sigma}^{\emptyset} \otimes 1_{\tau\sigma}^{m-1}) \text{JW}_{\sigma\tau} \end{aligned}$$

for m odd or even, respectively. For $(\sigma, \tau, \rho) \in S^3$ with $m_{\sigma\rho} = m_{\rho\tau} = 2$ and $m_{\sigma\tau} = m$, we have

$$(\text{braid}_{\sigma\tau}^{\tau\sigma}(m) \otimes 1_{\rho}) \text{braid}_{\rho\sigma\tau\cdots\sigma}^{\sigma\tau\cdots\sigma\rho} = \text{braid}_{\rho\tau\sigma\cdots\tau}^{\tau\sigma\cdots\tau\rho} (1_{\rho} \otimes \text{braid}_{\sigma\tau}^{\tau\sigma}(m)).$$

We have the three Zamolodchikov relations: for a type A_3 triple $\sigma, \tau, \rho \in S$ with $m_{\sigma\tau} = 3 = m_{\sigma\rho}$ and $m_{\tau\rho} = 2$ we have that

$$\begin{aligned} &\text{braid}_{\sigma\rho\tau\sigma\rho}^{\rho\sigma\rho\tau\sigma\rho} \text{braid}_{\sigma\rho\tau\sigma\rho}^{\sigma\rho\tau\sigma\rho} \text{braid}_{\sigma\tau\rho\sigma\tau\rho}^{\sigma\rho\tau\sigma\tau\rho} \text{braid}_{\sigma\tau\rho\sigma\tau\rho}^{\sigma\tau\rho\sigma\rho\tau} \text{braid}_{\tau\sigma\tau\rho\sigma\tau}^{\sigma\tau\sigma\rho\sigma\tau} \text{braid}_{\tau\sigma\rho\tau\sigma\tau}^{\tau\sigma\rho\tau\sigma\tau} \\ &= \text{braid}_{\rho\sigma\tau\rho\sigma\rho}^{\rho\sigma\tau\rho\sigma\rho} \text{braid}_{\rho\sigma\tau\rho\sigma\rho}^{\rho\sigma\tau\rho\sigma\rho} \text{braid}_{\rho\sigma\tau\rho\sigma\rho}^{\rho\sigma\tau\rho\sigma\rho} \text{braid}_{\rho\sigma\tau\rho\sigma\rho}^{\rho\sigma\tau\rho\sigma\rho} \text{braid}_{\tau\rho\sigma\tau\rho\sigma}^{\tau\rho\sigma\tau\rho\sigma} \text{braid}_{\tau\rho\sigma\tau\rho\sigma}^{\tau\rho\sigma\tau\rho\sigma} \text{braid}_{\tau\rho\sigma\tau\rho\sigma}^{\tau\rho\sigma\tau\rho\sigma}. \end{aligned}$$

For a type B_3 triple $\sigma, \tau, \rho \in S$ such that $m_{\sigma\rho} = 4, m_{\tau\rho} = 2, m_{\sigma\tau} = 3$, we have that

$$\begin{aligned} &\text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\sigma\rho\sigma\tau\sigma\rho\sigma\tau}^{\sigma\rho\sigma\tau\sigma\rho\sigma\tau} \text{braid}_{\sigma\rho\sigma\tau\sigma\rho\sigma\tau}^{\sigma\rho\sigma\tau\sigma\rho\sigma\tau} \text{braid}_{\sigma\rho\sigma\tau\sigma\rho\sigma\tau}^{\sigma\rho\sigma\tau\sigma\rho\sigma\tau} \times \\ &\text{braid}_{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\sigma\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\tau\sigma\tau\rho\sigma\tau\rho\sigma}^{\tau\sigma\tau\rho\sigma\tau\rho\sigma} \\ &= \text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\rho\sigma\rho\tau\sigma\rho\sigma\tau}^{\rho\sigma\rho\tau\sigma\rho\sigma\tau} \text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \times \\ &\text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \text{braid}_{\tau\rho\sigma\tau\rho\sigma\tau\rho}^{\tau\rho\sigma\tau\rho\sigma\tau\rho} \end{aligned}$$

and for a type H_3 triple $\sigma, \tau, \rho \in S$ such that $m_{\sigma\rho} = 2, m_{\tau\rho} = 5, m_{\sigma\tau} = 3$, we have a final H_3 relation², for which we refer to [20, Definition 5.2]. Further, we require the bifactoriality relation

$$((D_1 \circ 1_{\underline{w}}) \otimes (D_2 \circ 1_{\underline{x}}))((1_{\underline{w}} \circ D_3) \otimes (1_{\underline{x}} \circ D_4)) = (D_1 \circ 1_{\underline{w}} \circ D_3) \otimes (D_2 \circ 1_{\underline{x}} \circ D_4) \tag{1.13}$$

and the monoidal unit relation

$$1_{\emptyset} \otimes D_1 = D_1 = D_1 \otimes 1_{\emptyset}$$

for all diagrams D_1, D_2, D_3, D_4 and all words $\underline{w}, \underline{x}$. Finally, we require the (non-local) cyclo-tomic relation

$$\text{spot}_{\sigma}^{\emptyset} \text{spot}_{\emptyset}^{\sigma} \otimes 1_{\underline{w}} = 0 \quad \text{for all } \underline{w} \in \text{exp}(w), w \in W, \text{ and all } \sigma \in S.$$

The \mathbb{Z} -grading on the algebra \mathcal{H}_W^k is defined on the generators (and their duals) as follows:

$$\text{deg}(1_{\emptyset}) = 0 \quad \text{deg}(1_{\sigma}) = 0 \quad \text{deg}(\text{spot}_{\sigma}^{\emptyset}) = 1 \quad \text{deg}(\text{fork}_{\sigma\sigma}^{\sigma}) = -1 \quad \text{deg}(\text{braid}_{\tau\sigma}^{\sigma\tau}(m)) = 0$$

² To the authors’ knowledge, this relation has not been explicitly determined (but can be given more computing power). We invite the reader to either believe that this can be written down (as is now standard in this area) or to read all results in this paper “modulo” any Coxeter group W with a parabolic subgroup of type H_3 .

for $\sigma, \tau \in S$ arbitrary and $m \geq 2$.

Remark 1.5 The cyclotomic relation amounts to considering diagrammatic Soergel modules instead of diagrammatic Soergel bimodules, or equivalently, to considering finite dimensional \mathbb{k} -modules rather than modules of finite rank over the polynomial ring, R , generated by the “barbells”, $\text{spot}_\sigma^\emptyset \text{spot}_\emptyset^\sigma$, for $\sigma \in S$. If we omit the cyclotomic relation in the above definition of \mathcal{H}_W we obtain a diagrammatic Bott–Samelson category \mathcal{H}_{BS} for W (viewed as a locally unital associative \mathbb{k} -algebra).

Diagrammatic Bott–Samelson categories are normally defined using a reflection representation $\mathfrak{h} = (V, \{\alpha_\sigma^\vee : \sigma \in S\}, \{\alpha_\sigma : \sigma \in S\})$ of the Coxeter group W called a realisation. Our construction of \mathcal{H}_{BS} implicitly uses the universal realisation of W with respect to the balanced Cartan matrix $(a_{\sigma\tau})_{\sigma, \tau \in S}$, defined as follows. Abusing notation slightly, let V^* be a free \mathbb{k} -module with basis $\{\alpha_\sigma : \sigma \in S\}$, and let $V = (V^*)^*$. For each $\sigma \in S$ define $\alpha_\sigma^\vee \in V$ by setting $\langle \alpha_\sigma^\vee, \alpha_\tau \rangle = a_{\sigma\tau}$ for all $\tau \in S$. The Coxeter group W acts on V^* via $\sigma(\beta) = \beta - \langle \alpha_\sigma^\vee, \beta \rangle \alpha_\sigma$ for all $\sigma \in S$ and $\beta \in V^*$. If \mathcal{H}'_{BS} is the Bott–Samelson category for another realisation of W with the same Cartan matrix, then there is a unique monoidal functor $\mathcal{H}_{\text{BS}} \rightarrow \mathcal{H}'_{\text{BS}}$, which descends to an isomorphism after taking cyclotomic quotients (cf. [45, Lemma 11.2]).

Definition 1.6 Given $S_P \subseteq S$ we define the anti-spherical Hecke category $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ to be the quotient of $\mathcal{H}_W^{\mathbb{k}}$ by the homogeneous (non-local) relation

$$1_\sigma \otimes 1_{\underline{w}} = 0$$

for all $\sigma \in S_P \subseteq S$ and $\underline{w} \in \text{exp}(w)$ for $w \in W$.

1.6 Parabolic light leaves tableaux and cellular bases

We now recall the combinatorics of cellular bases for diagrammatic Hecke categories. This is well known in the non-parabolic setting (see e.g. [20, §6.1] or [17, Chapter 10.4]); a good reference for this material in the parabolic setting is [38, §5]. Our notation is closely analogous to that in previous work of the first and second authors ([7] and [6]) and Ryom–Hansen [44]. In particular, we will use the language of tableaux (rather than words in the Coxeter generators) to describe the indexing sets for our cellular bases. We provide extended examples after the definitions see Examples 1.16 and 1.19 (in particular, we highlight the role played by the parabolic in these examples).

We will consider certain truncations of $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$, and to that end we define, for any $w \in {}^P W$, a poset

$$\mathcal{P}_{\leq w} = \{x \mid x \in {}^P W \text{ and } x \leq w\} \tag{1.14}$$

partially ordered by the Bruhat order. Fix $\underline{w} = \sigma_1 \dots \sigma_\ell \in \text{exp}_P^\ell(w)$ (not necessarily reduced). Given \mathfrak{t} a subword of $\underline{w} \in \text{exp}_P^\ell(w)$, we define $\text{Shape}_k(\mathfrak{t}) = \sigma_1^{t_1} \sigma_2^{t_2} \dots \sigma_k^{t_k} \in W$ for $1 \leq k \leq \ell$. In the non-parabolic case, the set of tableaux of shape x and weight \underline{w} will then be given by

$$\text{Std}_{\leq \underline{w}}(x) = \{\mathfrak{t} \mid \text{Shape}_\ell(\mathfrak{t}) = x\}$$

and we define the set of parabolic tableaux of shape x and weight \underline{w} to be

$$\text{Std}_{\leq \underline{w}}^P(x) = \{\mathfrak{t} \mid \text{Shape}_k(\mathfrak{t})\sigma_{k+1} \in {}^P W \text{ for } 0 \leq k < \ell \text{ and } \text{Shape}_\ell(\mathfrak{t}) = x\} \subseteq \text{Std}_{\leq \underline{w}}(x).$$

Finally, we take the union over all possible weights to obtain the set of all parabolic tableaux

$$\text{Std}_{\leq \ell}^P(x) = \bigcup_{\substack{w \in {}^P W \\ \underline{w} \in \text{exp}_P^\ell(w)}} \text{Std}_{\leq \underline{w}}^P(x), \quad \text{Std}^P(x) = \bigcup_{\ell \geq 0} \text{Std}_{\leq \ell}^P(x).$$

Given $x < x\tau$ and $\mathbf{t} \in \text{Std}_{\leq \underline{w}}^P(x)$, we define

$$\mathbf{t}^+ = (t_1, \dots, t_q, 1) \in \text{Std}_{\leq \underline{w}\tau}(x\tau), \quad \mathbf{t}^- = (t_1, \dots, t_q, 0) \in \text{Std}_{\leq \underline{w}\tau}(x)$$

and this will be the backbone of how we grow the cellular bases. We can decompose the diagrammatic anti-spherical Hecke category in the following manner,

$$\mathcal{H}_{P \setminus W}^{\mathbb{k}} = \bigoplus_{\substack{v \in \text{exp}_P(v) \\ \underline{w} \in \text{exp}_P(w) \\ v, w \in W}} 1_{\underline{v}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\underline{w}}$$

and hence regard this algebra as a locally unital associative algebra in the sense of [12, Section 2.2]. Each one of these finite-dimensional pieces $1_{\underline{v}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\underline{w}}$ is the space of morphisms between the Bott–Samelson objects labelled by \underline{v} and \underline{w} .

Recall that given $\underline{w} = \sigma_1 \sigma_2 \cdots \sigma_\ell$ and $\mathbf{t} = (t_1, t_2, \dots, t_\ell) \in \{0, 1\}^\ell$, we set $\underline{w}^{\mathbf{t}} := \sigma_1^{t_1} \sigma_2^{t_2} \cdots \sigma_\ell^{t_\ell}$. We define

$$1_{\leq \underline{w}} = \sum_{\mathbf{t} \in \{0, 1\}^\ell} 1_{\underline{w}^{\mathbf{t}}}.$$

In particular, we will wish to consider the subalgebras

$$1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}$$

in order to understand $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$. We will construct a spanning set of $1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}$ in an iterative fashion. For any fixed expression $\underline{w} \in \text{exp}_P(w)$, we have an embedding

$$1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}} \hookrightarrow (1_{\leq \underline{w}} \otimes 1_\tau) \mathcal{H}_{P \setminus W}^{\mathbb{k}} (1_{\leq \underline{w}} \otimes 1_\tau) \tag{1.15}$$

given by $D \mapsto D \otimes 1_\tau$. Note that the image of this embedding lies inside an idempotent truncation of $1_{\leq \underline{w}\tau} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}\tau}$. We consider each of these embeddings in turn (for all $\tau \in S$) in order to provide the iterative construction of the ‘‘anti-spherical light leaves’’ elements of $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$.

We now inductively construct the light leaves basis. For $\mathbf{t} \in \text{Std}_{\leq 1_W}(1_W)$, we formally set $c_{\mathbf{t}} = 1_\emptyset$ and we recall that \mathbf{t}^+ and \mathbf{t}^- are defined in equation(1.36). If $y\tau > y$, then for any $\underline{y} \in \text{rexp}_P(y)$, $\underline{y}^+ \in \text{rexp}_P(y\tau)$, $\underline{y}^- \in \text{rexp}_P(y)$ and $\mathbf{t} \in \text{Std}_{\leq \underline{w}}(y)$ we define

$$c_{\mathbf{t}^+} = \text{braid}_{\underline{y}^+}^{\mathbf{t}^+}(c_{\mathbf{t}} \otimes 1_\tau) \quad c_{\mathbf{t}^-} = \text{braid}_{\underline{y}^-}^{\mathbf{t}^-}(c_{\mathbf{t}} \otimes \text{spot}_\tau^\emptyset). \tag{1.16}$$

If $y\tau = y' < y$, then we let $\underline{y}'\tau$ be a reduced expression for y . For any $\underline{y}^+ \in \text{rexp}_P(y)$, $\underline{y}^- \in \text{rexp}_P(y\tau)$ and $\mathbf{t} \in \text{Std}_{\leq \underline{w}}(y'\tau)$ we define

$$c_{\mathbf{t}^+} = \text{braid}_{\underline{y}'\tau}^{\mathbf{t}^+}(1_{\underline{y}'} \otimes \text{fork}_{\tau}^{\mathbf{t}})(c_{\mathbf{t}} \otimes 1_\tau) \quad c_{\mathbf{t}^-} = \text{braid}_{\underline{y}^-}^{\mathbf{t}^-}(1_{\underline{y}^-} \otimes \text{spot}_\tau^\emptyset \text{fork}_{\tau}^{\mathbf{t}})(c_{\mathbf{t}} \otimes 1_\tau). \tag{1.17}$$

Fix a choice of reduced expression \underline{x} for $x \in \mathcal{P}_{\leq w}$ and construct elements c_s, c_t for $s, t \in \text{Std}_{\leq w}^P(x)$. We set $c_{st} := (c_s^*)c_t$.

The definition of the anti-spherical Hecke category $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ is extremely general, making sense over arbitrary rings and Coxeter systems. In order for it to be well behaved we will make the following (very mild) assumption.

Assumption 1.7 The anti-spherical light leaves elements

$$\{c_{st} \mid s, t \in \text{Std}_{\leq w}^P(x), x \in \mathcal{P}_{\leq w}\} \tag{1.18}$$

are \mathbb{k} -linearly independent in $1_{\leq w} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq w}$.

Theorem 1.8 ([20, Section 6.4] and [38, Theorem 5.3]) *If Assumption 1.7 holds, then the algebra $1_{\leq w} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq w}$ is finite-dimensional with graded cellular basis (1.18) with respect to the Bruhat ordering on $\mathcal{P}_{\leq w}$ and anti-involution $*$. For \mathbb{k} a field, we have that $1_{\leq w} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq w}$ is quasi-hereditary.*

Proof In the proof of [38, Theorem 5.3] it is proven that (1.18) always spans, so Assumption 1.7 implies that it is in fact a basis. Cellularity is not mentioned explicitly, but follows in a completely analogous fashion to [20, Section 6.4]. The only point of the theorem which is not explicitly stated in [20, Section 6.4] and [38, Theorem 5.3] is that the algebra is quasi-hereditary. However, this is immediate from the fact that each layer of the cellular basis contains (at least one) idempotent $c_{ss} = 1_{\underline{x}}$ for s the unique tableau in $\text{Std}_{\leq \underline{x}}(x) \subseteq \text{Std}_{\leq w}(x)$. \square

Remark 1.9 We note that when Assumption 1.7 does not hold, the analogue of Soergel’s categorification theorem for $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ is false [38, Theorem 6.2]. In this instance it is debatable whether $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ should even be called the “anti-spherical Hecke category”! The following result is our attempt to give a reasonably general condition for when Assumption 1.7 holds.

Theorem 1.10 ([38, Theorem 5.3]) Let \mathcal{O} be a complete local ring in which 2 is invertible, and let $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$ be a balanced Cartan matrix for W over \mathcal{O} . If the universal realisation for W with respect to $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$ is faithful, then Assumption 1.7 holds for $\mathcal{H}_{P \setminus W}^{\mathcal{O}}$, the anti-spherical Hecke category defined over \mathcal{O} with respect to $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$. Moreover, if there is a ring homomorphism $\mathcal{O} \rightarrow \mathbb{k}$ such that $(a_{\sigma\tau})_{\sigma, \tau \in S}$ is the image in \mathbb{k} of $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$, then Assumption 1.7 holds for $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$.

Proof Assumption 1.7 is preserved by base change, so the second statement follows immediately from the first. The special case where $\mathcal{O} = \mathbb{R}$ and $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$ is the “geometric” Cartan matrix for W over \mathbb{R} is proved in [38, Proposition 5.5]. In fact this proof is valid for any universal realisation over a complete local ring in which the “parabolic property” [38, 2.3] holds, including faithful realisations. \square

There are two incredibly important realisations upon which we now focus our attention.

Example 1.11 Let p be an odd prime.

- (1) Let \mathbb{k} be any field of characteristic p and let W be a crystallographic Coxeter group. Let A be a generalised Cartan matrix for W , and let $(a_{\sigma\tau})_{\sigma, \tau \in S}$ be the image of A over \mathbb{k} . Set $\mathcal{O} = \mathbb{Z}_p$ take $(a_{\sigma\tau}^{\mathcal{O}})_{\sigma, \tau \in S}$ to be the image of A over \mathcal{O} . The universal realisation for W with respect to A is faithful (see e.g. [33, Chapter 3]), so the same holds true over \mathcal{O} . Thus Assumption 1.7 holds for $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ by Theorem 1.10.

(2) Let \mathbb{k} be a field of characteristic p which contains the algebraic integers $2 \cos(\pi/m_{\sigma\tau})$ for all $\sigma, \tau \in S$, and define the “geometric” Cartan matrix by setting $a_{\sigma\tau} = -2 \cos(\pi/m_{\sigma\tau})$. As mentioned above, it is known that the universal realisation with respect to this Cartan matrix over \mathbb{R} is faithful, so the same holds true over $\mathcal{O} = \mathbb{Z}_p[2 \cos(\pi/m_{\sigma\tau}) \mid \sigma, \tau \in S]$. Thus Assumption 1.7 holds for $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ by Theorem 1.10.

Remark 1.12 Typically the realisations in Example 1.11 do *not* satisfy the parabolic property over \mathbb{k} . For example, suppose W is an affine Weyl group, $(a_{\sigma\tau})_{\sigma, \tau \in S}$ is the (image in \mathbb{k} of the) associated affine Cartan matrix, and P is the finite Weyl subgroup. Then the corresponding realisation from Example 1.11(1) does not satisfy the parabolic property [26, Lemma 1.11]. This is also discussed in [38, Sections 2.3 and 3.1].

Remark 1.13 In particular, we note that if \mathbb{k} is algebraically closed and of odd (or infinite) characteristic then Assumption 1.7 holds for the geometric Cartan matrix (by Example 1.11(2)). Thus the reader unfamiliar with realisations can focus on just this case.

Remark 1.14 We have assumed that $p \neq 2$ as we do not wish to discuss the technicalities of Demazure surjectivity. Demazure surjectivity sometimes fails for the natural and geometric realisations in characteristic 2 (even for crystallographic types). One can fix this technicality, but the details are tedious and are often glossed over entirely in the literature (see for example the B_2 and C_2 examples for $p = 2$ [32]).

Fix $x \in {}^P W$ and \underline{x} a reduced word. We define right “cell” ideals

$$\mathcal{H}_{P \setminus W}^{\leq \underline{x}} = 1_{\leq \underline{x}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} \quad \mathcal{H}_{P \setminus W}^{< \underline{x}} = \mathcal{H}_{P \setminus W}^{\leq \underline{x}} \cap \mathbb{k}\{c_{st} \mid s, t \in \text{Std}^P(y), y < x\}. \tag{1.19}$$

We define the standard $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module, $\Delta_{\mathcal{H}_{P \setminus W}}(x)$, as follows:

$$\Delta_{\mathcal{H}_{P \setminus W}}(x) = \mathcal{H}_{P \setminus W}^{\leq \underline{x}} / \mathcal{H}_{P \setminus W}^{< \underline{x}} = \{c_s + \mathcal{H}_{P \setminus W}^{< \underline{x}} \mid s \in \text{Std}^P(x)\}. \tag{1.20}$$

We will almost always drop the subscript and simply write $\Delta(x) := \Delta_{\mathcal{H}_{P \setminus W}}(x)$. We recall that the cellular structure allows us to define, for each $x \in {}^P W$, a bilinear form $\langle \cdot, \cdot \rangle^x$ on $\Delta(x)$ which is determined by

$$c_{st}c_{uv} \equiv \langle c_t, c_u \rangle^x c_{sv} \pmod{\mathcal{H}_{P \setminus W}^{< \underline{x}}} \tag{1.21}$$

for any $s, t, u, v \in \text{Std}^P(x)$. When \mathbb{k} is a field, we obtain a complete set of non-isomorphic simple $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -modules $L(x)$ for $x \in {}^P W$ via exact sequences as follows:

$$0 \rightarrow \text{rad}(\langle \cdot, \cdot \rangle^x) \rightarrow \Delta(x) \rightarrow L(x) \rightarrow 0. \tag{1.22}$$

We will not discuss projective indecomposable $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -modules³. The right $1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}$ -modules

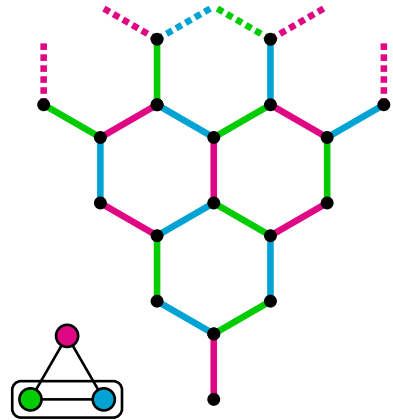
$$\Delta_{\leq \underline{w}}(x) := \Delta(x)1_{\leq \underline{w}} \quad L_{\leq \underline{w}}(x) := L(x)1_{\leq \underline{w}} \tag{1.23}$$

for $x \in \mathcal{P}_{\leq \underline{w}}$ provide complete sets of non-isomorphic standard and simple modules respectively. The projective indecomposable $1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}$ -modules $P_{\leq \underline{w}}(x)$ are the direct summands

$$1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}} = \bigoplus_{x \leq \underline{w}} \dim_v(L_{\leq \underline{w}}(x)) P_{\leq \underline{w}}(x). \tag{1.24}$$

³ “That whereof we cannot speak, thereof we must remain silent”, Wittgenstein.

Fig. 3 The Coxeter and weak Bruhat graphs for $\mathfrak{S}_3 \leq \widehat{\mathfrak{S}}_3$. The weak Bruhat graph can be thought of as a hexagonal-tiling of a sixth of the plane



Assumption 1.15 For the remainder of the paper, we will assume that k is a field of characteristic $p \neq 2$.

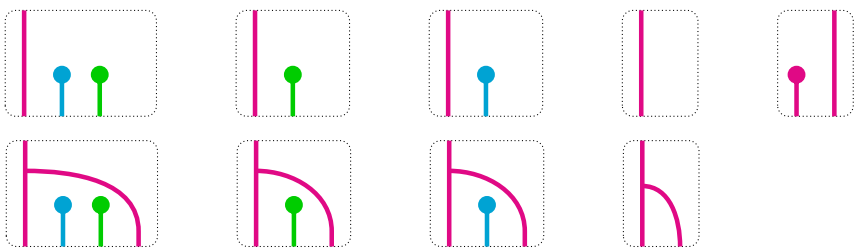
Example 1.16 Let W be the affine symmetric group $\widehat{\mathfrak{S}}_3$ with generators s_1, s_2, s_3 and let P be the maximal finite parabolic generated by s_2 and s_3 . The Coxeter graph and Bruhat graphs are depicted in Fig. 3. For $\underline{w} = s_1 s_2 s_3 s_1$, the algebra $1_{\leq \underline{w}} \mathcal{H}_P^k \setminus W 1_{\leq \underline{w}}$ has graded dimension

$$1^2 + (v + 1)^2 + (v + 1)^2 + (v + 1)^2 + (v^2 + 4v + 3 + v^{-1})^2 + (2v^2 + 3v + 1)^2$$

where the sum is over the squares of the graded dimensions of the standard modules, which have labels

$$s_1 s_2 s_3 s_1 \quad s_1 s_2 s_3 \quad s_1 s_2 \quad s_1 s_3 \quad s_1 \quad 1_{P \setminus W}$$

respectively. For example, the basis of the 9-dimensional standard module has basis



(1.25)

We have that $\mathcal{H}_{P \setminus W}$ is the quotient of \mathcal{H}_W by the 2-sided ideal

$$J = \langle 1_\sigma \otimes 1_{\underline{w}} \mid \sigma \in S_P, \underline{w} \in \exp(w) \text{ for } w \in W \rangle. \tag{1.26}$$

We have a functor

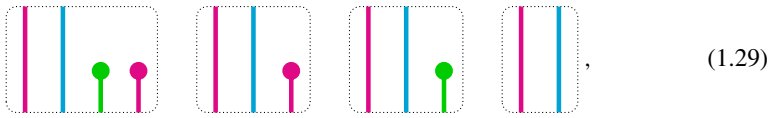
$$\begin{aligned} \pi : \mathcal{H}_W\text{-mod} &\rightarrow \mathcal{H}_{P \setminus W}\text{-mod} \\ \pi : M &\mapsto M/JM. \end{aligned} \tag{1.27}$$

We have that

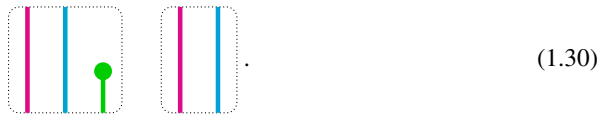
$$\pi(\Delta_{\mathcal{H}_W}(x)) = \begin{cases} \Delta_{\mathcal{H}_{P \setminus W}}(x) & \text{for } x \in {}^P W \\ 0 & \text{otherwise} \end{cases} \quad \pi(L_{\mathcal{H}_W}(x)) = \begin{cases} L_{\mathcal{H}_{P \setminus W}}(x) & \text{for } x \in {}^P W \\ 0 & \text{otherwise} \end{cases} \tag{1.28}$$

This is simply because every basis element of the standard module is killed by taking this quotient (this is mentioned explicitly in the proof of [38, Theorem 5.3]).

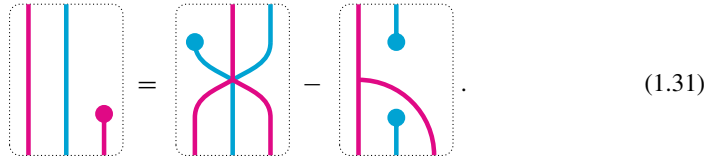
Example 1.17 We continue with Example 1.16, we note that $s_1 s_2 \in {}^P W$ and that $s_1 s_2 s_1 \in P$. For those unfamiliar with anti-spherical light leaves, we now illustrate the manner in which a basis element of \mathcal{H}_W^k will die when we take the quotient $\mathcal{H}_{P \setminus W}^k$. We have that the $1_{\leq s_1 s_2 s_3 s_1} \mathcal{H}_W^k 1_{\leq s_1 s_2 s_3 s_1}$ -module $\Delta_{\leq s_1 s_2 s_3 s_1}(s_1 s_2)$ is $(v^2 + 2v + 1)$ -dimensional and spanned by the diagrams



however the $1_{\leq s_1 s_2 s_3 s_1} \mathcal{H}_{P \setminus W}^k 1_{\leq s_1 s_2 s_3 s_1}$ -module $\Delta_{\leq s_1 s_2 s_3 s_1}(s_1 s_2)$ is $(v + 1)$ -dimensional module with basis



We illustrate how this works with the second diagram in (1.29). We rewrite the second diagram using the Jones–Wenzl relation (depicted explicitly in Fig. 1) as follows



We recall that $s_2 \in P$; therefore the first diagram is now zero by the $1_{s_2} \otimes D = 0$ relation and the second diagram belongs to the 2-sided cell-ideal generated by 1_{s_1} .

In fact, *no* diagram in (1.30) has a rightmost s_1 -strand. In more detail, we have $s_1 s_2 < s_1 s_2 s_1 \notin {}^P W$, so for any tableau t of shape $s_1 s_2$ and weight ending in s_1 , the light leaf c_t factors through $1_{s_1 s_2} \otimes \text{spot}_{s_1}$. By the above reasoning c_t must therefore vanish in $\mathcal{H}_{P \setminus W}$. More generally, if $y < y\tau \notin {}^P W$ then there are no parabolic tableaux of shape y and weight ending in τ . This illustrates an important aspect of the restriction functors we will define in the next section, and we will refer back to this example in the proof of Corollary 1.18.

1.7 Branching rules for standard modules

We define the τ -restriction functor

$$\begin{aligned} \text{Res}_{\leq w}^{\leq w\tau} : 1_{\leq w\tau} \mathcal{H}_{P \setminus W}^k 1_{\leq w\tau} \text{-mod} &\rightarrow 1_{\leq w} \mathcal{H}_{P \setminus W}^k 1_{\leq w} \text{-mod} \\ : M &\mapsto \text{res}(M(1_{\leq w} \otimes 1_\tau)) \end{aligned} \tag{1.32}$$

where

$$\text{res} : 1_{\leq w\tau} \mathcal{H}_{P \setminus W}^k 1_{\leq w\tau} \text{-mod} \longrightarrow 1_{\leq w} \mathcal{H}_{P \setminus W}^k 1_{\leq w} \text{-mod} \tag{1.33}$$

denotes the ordinary restriction functor defined by the embedding of equation (1.15). While we use the term restriction for $\text{Res}_{\leq w}^{\leq w\tau}$, we emphasise that this functor does not preserve the underlying vector space. This is because the inclusion 1.15 is the non-unital inclusion of an idempotent subalgebra, so restriction kills the complementary idempotent.

Theorem 1.8 has the following immediate corollary.

Corollary 1.18 *Let $x = y\tau > y$, with $x, y \in {}^P W$. We have that*

$$0 \rightarrow \mathbb{k}\{c_{t^+} \mid t \in \text{Std}_{\leq w}(y)\} \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(x)) \rightarrow \mathbb{k}\{c_{t^+} \mid t \in \text{Std}_{\leq w}(x)\} \rightarrow 0 \tag{1.34}$$

and

$$0 \rightarrow \mathbb{k}\{c_{t^-} \mid t \in \text{Std}_{\leq w}(y)\} \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y)) \rightarrow \mathbb{k}\{c_{t^-} \mid t \in \text{Std}_{\leq w}(x)\} \rightarrow 0 \tag{1.35}$$

where the submodule is isomorphic to $\Delta_{\leq w}(y)$ (respectively $\Delta_{\leq w}(y)\langle 1 \rangle$) and the quotient module is isomorphic to $\Delta_{\leq w}(x)\langle -1 \rangle$ (respectively $\Delta_{\leq w}(x)$). Finally, if $y < y\tau = x \notin {}^P W$, then

$$\text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y)) = 0. \tag{1.36}$$

Proof We have that the maps

$$\Delta_{\leq w}(y) \hookrightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(x)) \quad \phi : c_t \mapsto c_t \otimes 1_\tau \tag{1.37}$$

$$\Delta_{\leq w}(y) \hookrightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y)) \quad \phi : c_t \mapsto c_t \otimes \text{spot}_\tau^\emptyset \tag{1.38}$$

for $t \in \text{Std}_{\leq w}(y)$ are injective $(1_{\leq w} \otimes 1_\tau) \mathcal{H}_{P \setminus W}^k (1_{\leq w} \otimes 1_\tau)$ -homomorphisms by construction. Similarly, the maps

$$\Delta_{\leq w}(x) \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(x))/\phi(\Delta_{\leq w}(y)) \quad : c_t \mapsto (c_t \otimes 1_\tau)(1_y \otimes \text{fork}_\tau^{\tau\tau}) \tag{1.39}$$

$$\Delta_{\leq w}(x) \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y))/\phi(\Delta_{\leq w}(y)) \quad : c_t \mapsto (c_t \otimes 1_\tau)(1_y \otimes (\text{fork}_\tau^{\tau\tau} \text{spot}_\tau^\emptyset)) \tag{1.40}$$

for $t \in \text{Std}_{\leq w}(x)$ are $(1_{\leq w} \otimes 1_\tau) \mathcal{H}_{P \setminus W}^k (1_{\leq w} \otimes 1_\tau)$ -isomorphisms by construction (as we have simply multiplied on the right by an element of the algebra). On the other hand, if $y < y\tau = x \notin {}^P W$, then $\Delta_{\leq w\tau}(y)(1_{\leq w} \otimes 1_\tau) = 0$ by the light leaves construction, so $\text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y)) = 0$. We refer back to Example 1.17 for an illustrative example. \square

Example 1.19 We continue with the notation of Example 1.16. We consider the restriction functor $\text{Res}_{\leq x}^{\leq x\sigma}$ for $\sigma = s_1 \in W$ and $x = s_1 s_2 s_3$ (note that $x\sigma = w$ for w as in Example 1.16). The southern reading words of the first three diagrams in Eq. 1.25 are not of the form $y\sigma$ for some $y \leq x$ and therefore these diagrams are sent to zero by the restriction functor. The remaining diagrams in the first row of Eq. 1.25 form a submodule, isomorphic to $\Delta_{\leq s_1 s_2 s_3}(1_{P \setminus W})$, the isomorphism is depicted on basis elements in Fig. 4. The diagrams from the second row of Eq. 1.25 form a quotient module, isomorphic to $\Delta_{\leq s_1 s_2 s_3}(s_1)$, the isomorphism is depicted on basis elements in Fig. 5.

The reader will notice that this sequence is non-split. To see this, note that the monochrome diagrams from the first row of Eq. 1.25 are both obtained from the monochrome diagram in the second row as follows:

$$1_\sigma = \text{fork}_{\sigma\sigma}^\sigma(\text{spot}_{\emptyset}^\sigma \otimes 1_\sigma) \quad \text{and} \quad (\text{spot}_{\emptyset}^\sigma \otimes 1_\sigma) = \text{fork}_{\sigma\sigma}^\sigma(\text{spot}_{\emptyset}^\sigma \text{spot}_{\emptyset}^\sigma \otimes 1_\sigma).$$

Fig. 4 The isomorphism $\Delta_{\leq s_1 s_2 s_3}(1_{P \setminus W}) \cong \{c_{t^+} \mid t \in \text{Std}_{\leq s_1 s_2 s_3}(1_{P \setminus W})\}$ as in Example 1.19. The righthand-side forms the submodule of $\text{Res}_{\leq s_1 s_2 s_3}^{\leq s_1 s_2 s_3 s_1}(\Delta_{\leq s_1 s_2 s_3 s_1}(s_1))$ in the short exact sequence of 1.34

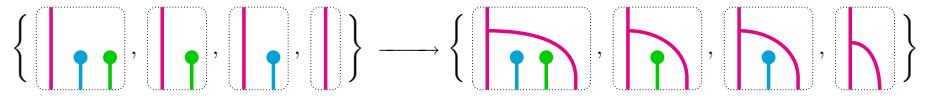
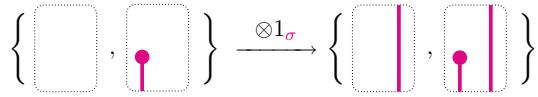


Fig. 5 The isomorphism $\Delta_{\leq s_1 s_2 s_3}(s_1) \cong \{c_{t^+} \mid t \in \text{Std}_{\leq s_1 s_2 s_3}(s_1)\}$ as in Example 1.19. The righthand-side forms the quotient module of $\text{Res}_{\leq s_1 s_2 s_3}^{\leq s_1 s_2 s_3 s_1}(\Delta_{\leq s_1 s_2 s_3 s_1}(s_1))$ in the short exact sequence of 1.34

Whereas, there does not exist any $D \in 1_{\leq s_1 s_2 s_3} \mathcal{H}_{P \setminus W}^k 1_{\leq s_1 s_2 s_3}$ such that either

$$1_\sigma(D \otimes 1_\sigma) = \text{fork}_{\sigma\sigma}^\sigma \quad \text{or} \quad (\text{spot}_{\emptyset}^\sigma \otimes 1_\sigma)(D \otimes 1_\sigma) = \text{fork}_{\sigma\sigma}^\sigma.$$

1.8 *p*-Kazhdan–Lusztig polynomials

The categorical (rather than geometric) definition of the *p*-Kazhdan–Lusztig polynomials is given via the *diagrammatic character* of [20, Definition 6.23]. In the language of this paper, the definition of the anti-spherical *p*-Kazhdan–Lusztig polynomial, ${}^p n_{x,y}(v)$ for $x, y \in {}^P W$, is as follows,

$${}^p n_{x,y}(v) := \sum_{k \in \mathbb{Z}} \dim_v(\text{Hom}_{1_{\leq w} \mathcal{H}_{P \setminus W}^k 1_{\leq w}}(P_{\leq w}(x), \Delta_{\leq w}(y))) = \sum_{k \in \mathbb{Z}} [\Delta_{\leq w}(y) : L_{\leq w}(x)(k)] v^k \tag{1.41}$$

for any $x, y \leq w$ and $\underline{x} \in \text{rex}_P(x), \underline{y} \in \text{rex}_P(y), \underline{w} \in \text{rex}_P(w)$ are arbitrary (note that the definition of these polynomials is independent of w). The anti-spherical *p*-Kazhdan–Lusztig polynomials are recorded in the $|{}^P W| \times |{}^P W|$ -matrix

$$\mathbf{D}_{P \setminus W} = ({}^p n_{x,y}(v))_{x,y \in {}^P W} \tag{1.42}$$

and we set

$$\mathbf{D}_{P \setminus W}^{-1} = ({}^p n_{x,y}^{-1}(v))_{x,y \in {}^P W} \tag{1.43}$$

to be the inverse of this matrix (which exists, as $\mathbf{D}_{P \setminus W}$ is lower uni-triangular). The non-parabolic (*p*-)Kazhdan–Lusztig polynomials are obtained by setting $P = \{1_W\} \leq W$.

2 The classification and construction of homogeneous $\mathcal{H}_{P \setminus W}^k$ -modules

It is, in general, a hopeless task to attempt to understand all *p*-Kazhdan–Lusztig polynomials or to understand all simple $\mathcal{H}_{P \setminus W}^k$ -modules. In particular, it was shown in [47] that one can embed certain number-theoretic questions (for which no combinatorial solution could possibly be hoped to exist) into the *p*-Kazhdan–Lusztig matrices of affine symmetric groups.

Thus we restrict our attention to classes of modules which we can hope to understand. Over the complex numbers, the first port of call would be to attempt to understand the unitary modules; for Lie groups this ongoing project is Vogan’s famous Atlas of Lie groups. Over arbitrary fields, the notion of unitary no longer makes sense; however, for graded algebras the homogeneous representations seem to provide a suitable replacement. For quiver Hecke algebras, the homogeneous representations were classified and constructed by Kleshchev–Ram [35]. For (quiver) Hecke algebras of symmetric groups, the notions of unitary and homogeneous representations coincide over the complex field [9, Theorem 8.1] and the beautiful cohomological and structural properties of these (homogeneous) representations are entirely independent of the field [9, 35].

In this section, we fix W an arbitrary Coxeter group and fix P an arbitrary parabolic subgroup and we classify and construct the homogeneous representations of the diagrammatic Hecke category $\mathcal{H}_{P \setminus W}$. We first provide a cohomological construction of the module $L(1_{P \setminus W})$ via a BGG resolution. This cohomological construction allows us to immediately deduce a basis-theoretic construction of $L(1_{P \setminus W})$, from which we easily read-off the fact that $L(1_{P \setminus W})$ is homogeneous. We then prove that $L(w)$ is inhomogeneous for any $1 \neq w \in {}^P W$.

Definition 2.1 Given $w, y \in {}^P W$, we say that (w, y) is a Carter–Payne pair if $y \leq w$ and $\ell(y) = \ell(w) - 1$. We let CP_ℓ denote the set of Carter–Payne pairs (w, y) with $\ell(w) = \ell \in \mathbb{N}$.

For $P \subseteq W$ an affine Weyl group and its maximal finite parabolic subgroup, the following family of homomorphisms were first considered (in the context of algebraic groups) by Carter–Payne in [13].

Theorem 2.2 For $(w, y) \in CP_\ell$, pick an arbitrary reduced expression $\underline{w} = \sigma_1 \dots \sigma_\ell$ and suppose that $\underline{y} = \sigma_1 \dots \sigma_{p-1} \widehat{\sigma}_p \sigma_{p+1} \dots \sigma_\ell$ is the subexpression of y obtained by deleting precisely one element $\sigma_p \in S$. We have that

$$\text{Hom}_{\mathcal{H}_{P \setminus W}^{\mathbb{k}}}(\Delta(w), \Delta(y))$$

is v^1 -dimensional. Given choices of \underline{w} and \underline{y} as above this homomorphism space is spanned by the map

$$\varphi_y^w(c_t) = (1_{\sigma_1 \dots \sigma_{p-1}} \otimes \text{spot}_{\sigma_p}^\emptyset \otimes 1_{\sigma_{p+1} \dots \sigma_\ell})c_t$$

for $t \in \text{Std}(w)$.

Proof Since $\mathcal{P}_{\leq w}$ is a co-saturated subset of ${}^P W$, we can truncate to the algebra $1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}$ by [14, Appendix A3.13]. We have that

$$\Delta(y)1_{\underline{w}} = \text{Span}_{\mathbb{k}}\{1_{\sigma_1 \dots \sigma_{p-1}} \otimes \text{spot}_{\sigma_p}^\emptyset \otimes 1_{\sigma_{p+1} \dots \sigma_\ell}\}$$

by Theorem 1.8, as there is a unique tableau t with shape y [28, Theorem 5.8]. Moreover this space is of strictly positive degree, namely v^1 . Whereas, the character of the simple head, $L(y)$ of $\Delta(y)$, is invariant under swapping v and v^{-1} by [27, Proposition 2.18]. Therefore

$$\Delta(y)1_{\underline{w}} = \text{rad}(\Delta(y)1_{\underline{w}}) \quad \text{and} \quad L(y)1_{\underline{w}} = 0. \tag{2.1}$$

By our assumption that (w, y) is a Carter–Payne pair, there does not exist an $x \in W$ such that $y < x < w$. We now apply this assumption twice. Firstly, we note that $[\Delta_{\leq \underline{w}}(y) : L_{\leq \underline{w}}(x)] \neq 0$ implies that $y \leq x \leq w$. Putting this together with Eq. 2.1 we have that $\text{rad}(\Delta_{\leq \underline{w}}(y)) = L_{\leq \underline{w}}(w)\langle 1 \rangle$ and the graded decomposition number is equal to

$$\dim_v(\text{Hom}_{1_{\leq \underline{w}} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \underline{w}}} (P_{\leq w}(w), \Delta_{\leq \underline{w}}(y))) = \sum_{k \in \mathbb{Z}} [\Delta_{\leq \underline{w}}(y) : L_{\leq \underline{w}}(w)\langle k \rangle] = v^1.$$

Now applying our assumption again, we conclude that this homomorphism factors through the projection $P_{\leq w}(w) \rightarrow \Delta_{\leq w}(w)$ by highest weight theory and so we have

$$\dim_v(\text{Hom}_{1_{\leq w}\mathcal{H}_{P \setminus W}^k 1_{\leq w}}(P_{\leq w}(w), \Delta_{\leq w}(y))) = \dim_v(\text{Hom}_{1_{\leq w}\mathcal{H}_{P \setminus W}^k 1_{\leq w}}(\Delta_{\leq w}(w), \Delta_{\leq w}(y))),$$

and thus the result follows. □

We set $\mathcal{P}_\ell = \{w \in {}^P W \mid \ell(w) = \ell\}$ for each $\ell \in \mathbb{N}$. Following a construction going back to work of Bernstein–Gelfand–Gelfand and Lepowsky [8, 24], we are going to define a complex of graded $\mathcal{H}_{P \setminus W}^k$ -modules

$$\cdots \longrightarrow \Delta_2 \xrightarrow{\delta_2} \Delta_1 \xrightarrow{\delta_1} \Delta_0 \xrightarrow{\delta_0} 0, \tag{2.2}$$

where

$$\Delta_\ell := \bigoplus_{w \in \mathcal{P}_\ell} \Delta(w)(\ell(w)). \tag{2.3}$$

We will refer to this as the BGG complex. We momentarily assume that $P = 1 \leq W$ is the trivial parabolic (so that ${}^P W = W$). Suppose $w, z \in W$ such that $\ell(w) = \ell(z) + 2$ and $w > z$ in the Bruhat ordering. By [8, Lemma 10.3] there exists a unique pair of distinct elements $x, y \in W$ such that $w > x, y > z$. We refer to the quadruple $w, x, y, z \in W$ as a diamond and we have homomorphisms of \mathcal{H}_W^k -modules

$$\begin{array}{ccc}
 & \Delta(x) & \\
 \varphi_x^w \nearrow & & \searrow \varphi_z^x \\
 \Delta(w) & & \Delta(z) \\
 \varphi_y^w \searrow & & \nearrow \varphi_z^y \\
 & \Delta(y) &
 \end{array} \tag{2.4}$$

given by our Carter–Payne homomorphisms. By a generalization of [8, Lemma 10.4] to arbitrary Coxeter groups (see [28, §6.17]), it is possible to pick a sign $\varepsilon(\alpha, \beta)$ for each Carter–Payne pair (α, β) such that for every diamond the product of the signs associated to its four arrows is equal to -1 . We can now define the \mathcal{H}_W^k -differential $\delta_\ell : \Delta_\ell \rightarrow \Delta_{\ell-1}$ for $\ell \geq 1$ to be the sum of the maps

$$\varepsilon(\alpha, \beta)\varphi_\beta^\alpha : \Delta(\alpha)(\ell) \rightarrow \Delta(\beta)(\ell - 1) \tag{2.5}$$

over all Carter–Payne pairs $(\alpha, \beta) \in \text{CP}_\ell$. We let $C_\bullet(1_W) = \bigoplus_{\ell \geq 0} \Delta_\ell(\ell)$ together with the differential $(\delta_\ell)_{\ell \geq 0}$.

Lemma 2.3 *We have that $\varphi_x^w \varphi_z^x = \varphi_y^w \varphi_z^y$.*

Proof We truncate to consider a subalgebra $1_{\leq w}\mathcal{H}_W^k 1_{\leq w}$ for $w = \sigma_1 \sigma_2 \cdots \sigma_\ell$ a fixed reduced expression of $w \in W$. Since $\ell(x) = \ell(y) = \ell - 1$ there are unique subexpressions $\underline{x}, \underline{y}$ for x, y respectively inside \underline{w} . Similarly there are unique subexpressions for z inside $\underline{x}, \underline{y}$, which induce subexpressions $\underline{z}_x, \underline{z}_y$ for z inside \underline{w} . On the other hand, there is a unique diagram in $\Delta_{\leq w}(z)1_{\underline{w}}$ of maximal degree equal to $\ell(w) - \ell(z) = 2$, this diagram is equal to

$$(1_{\sigma_1 \cdots \sigma_{t-1}} \otimes \text{spot}_{\sigma_t}^\emptyset \otimes 1_{\sigma_{t+1} \cdots \sigma_{u-1}} \otimes \text{spot}_{\sigma_u}^\emptyset \otimes 1_{\sigma_{u+1} \cdots \sigma_\ell})$$

for some $1 \leq t < u \leq \ell$. The corresponding reduced expression

$$\sigma_1 \cdots \sigma_{t-1} \widehat{\sigma}_t \sigma_{t+1} \cdots \sigma_{u-1} \widehat{\sigma}_u \sigma_{u+1} \cdots \sigma_\ell$$

is equal to (at least) one of the expressions \underline{z}_x or \underline{z}_y . Without loss of generality, we suppose it is \underline{z}_y ; this implies that

$$\varphi_y^w \varphi_z^y(c_t) = (1_{\sigma_1 \dots \sigma_{t-1}} \otimes \text{spot}_{\sigma_t}^\emptyset \otimes 1_{\sigma_{t+1} \dots \sigma_{u-1}} \otimes \text{spot}_{\sigma_u}^\emptyset \otimes 1_{\sigma_{u+1} \dots \sigma_\ell})c_t.$$

We now consider the other composition $\varphi_x^w \varphi_z^x$. Let $1 \leq p < q \leq \ell$ denote the indices of the terms σ_p and σ_q which are deleted from \underline{w} to obtain \underline{z}_x . If $\underline{z}_x \neq \underline{z}_y$, then $\underline{x} = \sigma_1 \dots \sigma_{q-1} \widehat{\sigma}_q \sigma_{q+1} \dots \sigma_\ell$, whereas $\sigma_1 \dots \sigma_{p-1} \widehat{\sigma}_p \sigma_{p+1} \dots \sigma_\ell$ is a non-reduced expression. It follows from Humphrey’s Deletion condition [28, Section 5.8] that there exists some $q' < p < q$ and that the subexpression $\underline{z}' = \sigma_1 \dots \sigma_{q'-1} \widehat{\sigma}_{q'} \sigma_{q'+1} \dots \sigma_{p-1} \widehat{\sigma}_p \sigma_{p+1} \dots \sigma_\ell$ is also a reduced expression for z . By applying a sequence of braids we may assume that $q' = p - 1$ and $q = p + 1$ (so that the cancellation in the non-reduced expression involves two adjacent reflections); thus we can apply the (local) relation

(2.6)

to see that

$$\begin{aligned} & (1_{\sigma_1 \dots \sigma_{p-1}} \otimes \text{spot}_{\sigma_p}^\emptyset \otimes 1_{\sigma_{p+1} \dots \sigma_{q-1}} \otimes \text{spot}_{\sigma_q}^\emptyset \otimes 1_{\sigma_{q+1} \dots \sigma_\ell}) \\ &= (1_{\sigma_1 \dots \sigma_{q'}} \otimes \text{spot}_{\sigma_{q'}}^\emptyset \otimes 1_{\sigma_{q'+1} \dots \sigma_{p-1}} \otimes \text{spot}_{\sigma_p}^\emptyset \otimes 1_{\sigma_{p+1} \dots \sigma_\ell}) \pmod{\mathcal{H}_W^{<z}} \end{aligned}$$

(Any diagram with less than $\ell(z)$ propagating strands is zero; thus the second term in Eq. 2.6 is immediately zero and the third term is zero by the barbell and cyclotomic relations.) Now we consider the expression obtained from \underline{w} by deleting $\sigma_{q'}$. If it is reduced, then by uniqueness of maximal degree basis elements, the expression obtained by deleting $\sigma_{q'}$ and σ_p must be \underline{z}_y , and we are done. Otherwise, we can repeat the above process. This must eventually terminate, as the indices of the deleted generators get smaller with each step at which point, we deduce that

$$\begin{aligned} & (1_{\sigma_1 \dots \sigma_{p-1}} \otimes \text{spot}_{\sigma_p}^\emptyset \otimes 1_{\sigma_{p+1} \dots \sigma_{q-1}} \otimes \text{spot}_{\sigma_q}^\emptyset \otimes 1_{\sigma_{q+1} \dots \sigma_\ell}) \\ &= (1_{\sigma_1 \dots \sigma_{t-1}} \otimes \text{spot}_{\sigma_t}^\emptyset \otimes 1_{\sigma_{t+1} \dots \sigma_{u-1}} \otimes \text{spot}_{\sigma_u}^\emptyset \otimes 1_{\sigma_{u+1} \dots \sigma_\ell}) \pmod{\mathcal{H}_W^{<z}} \end{aligned}$$

and the result follows, as we are working in the standard module $\Delta(z) = \mathcal{H}_W^{<z} / \mathcal{H}_W^{<z}$. □

Corollary 2.4 *We have that $\text{Im}(\delta_{\ell+1}) \subseteq \ker(\delta_\ell)$, in other words $C_\bullet(1_W)$ is a complex.*

Proof We have defined the differential (via the scalars $\varepsilon(w, z)$) so that the composition $\delta_\ell \delta_{\ell-1}$ restricted to a given diamond is equal to $\varphi_x^w \varphi_z^x - \varphi_y^w \varphi_z^y$ and so the result follows from Lemma 2.3. □

Now, we apply the quotient functor $: \mathcal{H}_W\text{-mod} \rightarrow \mathcal{H}_{P \setminus W}\text{-mod}$ to $C_\bullet(1_W)$ and hence obtain $C_\bullet(1_{P \setminus W}) = \bigoplus_{\ell \geq 0} \Delta_\ell(\ell)$ together with the differential $(\delta_\ell)_{\ell \geq 0}$. (We have abused notation slightly by identifying the differentials for $\mathcal{H}_{P \setminus W}$ - and \mathcal{H}_W -modules.)

Proposition 2.5 *We have that $C_\bullet(1_{P \setminus W})$ is a complex.*

Proof For arbitrary $P \leq W$, we note that $\mathcal{H}_{P \setminus W}$ is the quotient of \mathcal{H}_W by the parabolic annihilation relation (1.14). Taking quotients preserves complexes and so the result follows from Corollary 2.4. □

Remark 2.6 In the quotient, diamonds can “collapse”. For example, if $y \notin {}^P W$ then we obtain

$$\begin{array}{ccc}
 & \Delta(x) & \\
 \varphi_x^w \nearrow & & \searrow \varphi_z^x \\
 \Delta(w) & & \Delta(z) \\
 \searrow & & \nearrow \\
 & 0 &
 \end{array} \tag{2.7}$$

in which case, we have that $\varphi_x^w \varphi_z^x = 0$. (To see this, simply note that the equality $\varphi_x^w \varphi_z^x = \varphi_y^w \varphi_z^y$ continues to hold, but that the righthand-side of the equality factors through the zero module). For example, this happens in Example 1.16 with $w = s_1 s_2$, $x = s_1$, $y = s_2$, and $z = 1_W$.

We have already encountered one drawback of the τ -restriction functors from the previous section: they kill any standard module $\Delta_{\leq w}(x)$ such that $x < x\tau \notin {}^P W$ (and therefore the simple head is also killed). To remedy this, we define slightly larger algebras

$$1_{\leq \ell} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \ell} \text{ for } 1_{\leq \ell} = \sum_{\substack{c \underline{w} \in \exp_P^{\mathbb{k}}(w) \\ 0 \leq k \leq \ell \\ w \in W}} 1_{\underline{w}} \tag{2.8}$$

and we define $\text{Res}_{\ell}^{\ell+1} : 1_{\leq \ell+1} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \ell+1} \rightarrow 1_{\leq \ell} \mathcal{H}_{P \setminus W}^{\mathbb{k}} 1_{\leq \ell}$ to be the functor

$$\text{Res}_{\ell}^{\ell+1} = \bigoplus_{\substack{\underline{w} \in \exp_P^{\ell} \\ \tau \in S}} \text{Res}_{\leq \underline{w}}^{\leq w\tau}. \tag{2.9}$$

Lemma 2.7 *Let $1_{P \setminus W} \neq x$, $w \in {}^P W$ and suppose that $x \leq w$. We have that*

$$\text{Res}_{\leq \underline{w}}^{\leq w\tau} (L_{\leq w\tau}(x)) \neq 0$$

for some $\tau \in S$ and $\underline{w} \in \exp_P(w)$. Therefore $\text{Res}_{\ell}^{\ell+1}(L(x)1_{\leq \ell+1}) = 0$ implies $x = 1_{P \setminus W}$.

Proof For $1_{P \setminus W} \neq x \in {}^P W$, there exists some $\tau \in S$, $x' \in {}^P W$ such that $x\tau = x' < x$. In which case, $\underline{x} = \underline{x'}\tau$ is a reduced expression for x and $1_{\underline{x'}\tau} \in L(x)$. Our assumption that $x\tau = x' \leq x \leq w \leq w\tau$ implies that the preimage of $1_{\underline{x'}\tau} \in 1_{\leq w\tau} \mathcal{H}_{P \setminus W} 1_{\leq w\tau}$ under the map of (1.15) is equal to $1_{\underline{x'}} \in 1_{\leq \underline{w}} \mathcal{H}_{P \setminus W} 1_{\leq \underline{w}}$ and so the result follows. \square

We are now ready to prove that $C_{\bullet}(1_{P \setminus W})$ is a BGG resolution of the $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -module $L(1_{P \setminus W})$. For W the affine symmetric group, P the maximal finite parabolic and $\mathbb{k} = \mathbb{C}$, the existence of these BGG resolutions was conjectured by Berkesch–Griffeth–Sam in [5]. This conjecture was proven by way of the KZ-functor in the context of the quiver Hecke algebras of type A (by the first and third authors with José Simental, [9]). In type A, the diagrammatic Hecke categories and (truncations of) quiver Hecke algebras were recently shown to be isomorphic in [6]. Thus the following theorem generalises the BGG resolutions [5, 9] to all Coxeter groups, W , and all parabolic subgroups, P , and arbitrary fields, \mathbb{k} .

Theorem 2.8 *Fix W an arbitrary Coxeter group and fix P an arbitrary parabolic subgroup. The $\mathcal{H}_{P \setminus W}^{\mathbb{k}}$ -complex $C_{\bullet}(1_{P \setminus W})$ is exact except in degree zero, where $H_0(C_{\bullet}(1_{P \setminus W})) = L(1_{P \setminus W})$. Moreover, we have that*

$$L(1_{P \setminus W}) = \mathbb{k}\{c_s \mid \text{Shape}_k(s) = 1_{P \setminus W} \text{ for all } k \geq 1\}.$$

Proof By applying the restriction functor to Proposition 2.5, we have that

$$\text{Res}_\ell^{\ell+1}(C_\bullet(1_{P \setminus W}) 1_{\leq \ell+1}) \tag{2.10}$$

forms a complex of $1_{\leq \ell} \mathcal{H}_{P \setminus W}^{\text{kk}} 1_{\leq \ell}$ -modules. Moreover, we can idempotent-truncate

$$D_{\bullet}^{\tau, w}(1_{P \setminus W}) = (\text{Res}_\ell^{\ell+1}(1_{\leq \ell+1} C_\bullet(1_{P \setminus W}))) (1_{\leq w} \otimes 1_\tau) \tag{2.11}$$

and hence obtain a complex of $1_{\leq w} \mathcal{H}_{P \setminus W}^{\text{kk}} 1_{\leq w}$ -modules (through the identification of $1_{\leq w} \mathcal{H}_{P \setminus W} 1_{\leq w} \hookrightarrow 1_{\leq w\tau} \mathcal{H}_{P \setminus W} 1_{\leq w\tau}$). Let $x, y \in {}^P W$ with $x = y\tau > y$. For $y \in {}^P W$, we have that already seen that

$$0 \rightarrow \{c_{t^+} \mid t \in \text{Std}_{\leq w}(y)\} \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(x)) \rightarrow \{c_{t^+} \mid t \in \text{Std}_{\leq w}(x)\} \rightarrow 0 \tag{2.12}$$

and

$$0 \rightarrow \{c_{t^-} \mid t \in \text{Std}_{\leq w}(y)\} \rightarrow \text{Res}_{\leq w}^{\leq w\tau}(\Delta_{\leq w\tau}(y)) \rightarrow \{c_{t^-} \mid t \in \text{Std}_{\leq w}(x)\} \rightarrow 0 \tag{2.13}$$

where in both cases the submodule is isomorphic to $\Delta_{\leq w}(y)$ and the quotient module is isomorphic to $\Delta_{\leq w}(x)$. Since $\underline{x} = y\tau$, we have that

$$\varphi_y^x(c_{t^+}) = (1_{\underline{y}} \otimes \text{spot}_{\vartheta}^{\tau})c_{t^+} = c_{t^-} \tag{2.14}$$

for any $t \in \text{Std}_{\leq w}(x)$ or $t \in \text{Std}_{\leq w}(y)$ by definition. Therefore, we have that

$$(\text{Res}_{\leq w}^{\leq w\tau} \circ \varphi_y^x) = \text{id}_{\underline{x}} \langle 1 \rangle + \text{id}_{\underline{y}} \langle 1 \rangle \tag{2.15}$$

for $x, y \in {}^P W$ where

$$\text{id}_{\underline{z}} \langle 1 \rangle \in \text{Hom}_{1_{\leq w} \mathcal{H}_{P \setminus W}^{\text{kk}} 1_{\leq w}}(\Delta_{\leq w}(z) \langle \ell(z) \rangle, \Delta_{\leq w}(z) \langle \ell(z) + 1 \rangle) \tag{2.16}$$

is simply the graded shift of the identity map for $\underline{z} = \underline{x}, \underline{y}$ for $x, y \in W$. This implies that

$$D_{\bullet}^{\tau, w}(1_{P \setminus W}) = \bigoplus_{\substack{c y \tau > y \\ y \leq w}} (\Delta(y) \langle \ell(y) \rangle \oplus \Delta(y) \langle \ell(y) + 1 \rangle) \tag{2.17}$$

with differential

$$\text{Res}_{\leq w}^{\leq w\tau} \circ \delta_\ell = \sum_{\substack{(x, y) \in \text{CP}_\ell \\ \underline{x} = y\tau}} (\text{id}_{\underline{x}} \langle 1 \rangle + \text{id}_{\underline{y}} \langle 1 \rangle) + \sum_{\substack{(s, t) \in \text{CP}_\ell \\ s \neq t\tau}} (\text{Res}_{\leq w}^{\leq w\tau} \circ \varphi_t^s). \tag{2.18}$$

Thus we have that

$$H_j((\text{Res}_{\leq w}^{\leq w\tau}((C_\bullet(1_{P \setminus W}))(1_{\leq w} \otimes 1_\tau))) 1_{\leq \ell+1}) = 0 \tag{2.19}$$

for all $j \geq 0$. Now, summing over all $\tau \in S, w \in W$, and $\underline{w} \in \exp_P^\ell(w)$ we deduce that

$$\text{Res}_\ell^{\ell+1}(C_\bullet(1_{P \setminus W}) 1_{\leq \ell+1}) \tag{2.20}$$

forms a complex with zero homology in every degree. By Lemma 2.7, we have that restriction kills no simple $\mathcal{H}_{P \setminus W}^{\text{kk}}$ -module $L(w)$ for $1 \neq w \in {}^P W$. Moreover,

$$\text{Head}(\Delta(1_{P \setminus W}) 1_{\leq \ell+1}) = L(1_{P \setminus W}) 1_{\leq \ell+1} \not\subset \text{Im}(\delta_1) \tag{2.21}$$

and $[1_{\ell+1}\Delta(w) : 1_{\ell+1}L(1_{P\setminus W})] = 0$ for $1_{P\setminus W} \neq w \in {}^P W$ simply because the highest weight structure on $\mathcal{H}_{P\setminus W}$ is given by the Bruhat order. Therefore

$$H_j(C_\bullet(1_{P\setminus W})1_{\leq \ell+1}) = \begin{cases} L(1_{P\setminus W})1_{\leq \ell+1} & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2.22}$$

Finally, we have proven that $L(1_{P\setminus W})$ is killed by multiplication by the idempotent 1_τ at the ℓ th point for any $\ell \geq 1$ and for any $\tau \in S$. Thus $L(1_{P\setminus W})$ is spanned by c_s for s the empty tableau, as required. \square

We immediately deduce the following corollary, which is new even for the classical (inverse) parabolic and non-parabolic Kazhdan–Lusztig polynomials (in other words, for \mathbb{k} the complex field). Indeed, this seems to be the first non-trivial family of parabolic (p -)Kazhdan–Lusztig polynomials which admits a uniform construction across all Coxeter groups and all parabolic subgroups.

Corollary 2.9 (The Weyl–Kac character formula for Coxeter groups) *In the graded Grothendieck group of $\mathcal{H}_{P\setminus W}$, we have that*

$$[L(1_{P\setminus W})] = \sum_{w \in {}^P W} (-v)^{\ell(w)} [\Delta(w)]$$

Thus the complete first row of the inverse p -Kazhdan–Lusztig matrix is given by

$${}^p n_{1,w}^{-1} = (-v)^{\ell(w)}$$

for all $w \in {}^P W$.

Theorem 2.10 *The module $L(1_{P\setminus W})$ is both the unique homogeneous $\mathcal{H}_{P\setminus W}^{\mathbb{k}}$ -module and the unique 1-dimensional $\mathcal{H}_{P\setminus W}^{\mathbb{k}}$ -module.*

Proof That the module $L(1_{P\setminus W})$ is homogeneous is clear (as it is 1-dimensional). We now prove the converse, namely for any $1 \neq w \in {}^P W$ we show that $L(w)$ is inhomogeneous and of dimension strictly greater than 1. Let $1 \neq w \in {}^P W$ and choose τ such that $w\tau = w' < w$. By Theorem 1.8, the elements

$$1_{\underline{w}'} \otimes \text{spot}_{\tau}^{\emptyset} \otimes 1_{\tau} \quad 1_{\underline{w}'} \otimes \text{fork}_{\tau\tau}^{\tau} \tag{2.23}$$

span $\Delta_{\leq w\tau\tau}(w)$. The former is homogeneous of degree $+1$ and the latter is homogeneous of degree -1 . We have that

$$(1_{\underline{w}'} \otimes \text{fork}_{\tau\tau}^{\tau})(1_{\underline{w}'} \otimes \text{spot}_{\emptyset}^{\tau} \otimes 1_{\tau}) = 1_{\underline{w}'} \tag{2.24}$$

and

$$(1_{\underline{w}'} \otimes \text{spot}_{\tau}^{\emptyset} \otimes 1_{\tau})(1_{\underline{w}'} \otimes \text{spot}_{\emptyset}^{\tau} \otimes 1_{\tau}) = 1_{\underline{w}'} \otimes \text{spot}_{\tau}^{\emptyset} \text{spot}_{\emptyset}^{\tau} \otimes 1_{\tau} = 0 \pmod{\mathcal{H}_{P\setminus W}^{<w} 1_{\leq w\tau\tau}} \tag{2.25}$$

since the degree of this element is $+2$ (whereas the degree of the idempotent spanning the weight space of the cell module is, of course, of degree 0) and

$$(1_{\underline{w}'} \otimes \text{fork}_{\tau\tau}^{\tau})(1_{\underline{w}'} \otimes \text{fork}_{\tau\tau}^{\tau}) = 0.$$

Therefore the Gram matrix for this weight space of the cellular form is the 2×2 -matrix with 0s on the diagonal and 1s off the diagonal. This matrix has rank 2 and so neither element in (2.23) belongs to the radical of the cellular form. Therefore both elements in (2.23) belong to $L_{\leq w\tau\tau}(w)$ and the result follows. \square

Remark 2.11 We recall from the introduction that the conjecture of Berkesch–Griffeth–Sam (or rather, its equivalent formulation for homogeneous representations of quiver Hecke algebras) follows immediately from Theorem B. This might be surprising to the reader familiar with the homogeneous representations of quiver Hecke algebras. In [35] it is shown that there are up to $e - 1$ distinct homogeneous representations of any block of the quiver Hecke algebras (and for sufficiently large rank, there are precisely $e - 1$ such representations for a “regular block”). Whereas, in this paper we have seen that there is precisely one homogeneous representation of $\mathcal{H}_{P \setminus W}^k$ for $\mathfrak{S}_h = P \subset W = \widehat{\mathfrak{S}}_h$ for $h \in \mathbb{N}$. Therefore, one might think that there are “more” homogeneous representations of the quiver Hecke algebra. However, for each $1 \leq h < e$ there is an isomorphism between a finite truncation of $\mathcal{H}_{P \setminus W}^k$ and the Serre quotient of the quiver Hecke algebra corresponding to the set of partitions with at most h columns [6]. Through these isomorphisms, one can obtain the $e - 1$ distinct BGG resolutions of the $e - 1$ distinct homogeneous simple modules of the quiver Hecke algebra predicted by Berkesch–Griffeth–Sam [5].

We now provide an elementary infinite family of simple modules which do not admit BGG resolutions, in order to justify our claim in the introduction that such resolutions are “rare”. In [8] an example of such a simple for $W = \mathfrak{S}_4$ is given. We focus on the simplest case, namely that of the anti-spherical category controlling the algebraic group $SL_2(\mathbb{k})$.

Proposition 2.12 *Let \mathbb{k} be a field of finite characteristic $p > 0$. There exist infinitely many simple $\mathcal{H}_{\mathfrak{S}_2 \setminus \widehat{\mathfrak{S}}_2}^k$ -modules which do not admit BGG resolutions.*

Proof This is a standard Temperley–Lieb type result, we provide references in Remark 2.14 below but we include a proof for the sake of completeness. The Coxeter presentation of \mathfrak{S}_2 is $\langle \sigma, \tau \mid \tau^2 = \sigma^2 = 1 \rangle$ and we let P denote the finite parabolic generated by the reflection τ . We will provide an infinite family of examples of $x \in W$ such that $\text{rad}(\Delta(x))$ is not generated by the homomorphic images of standard modules, thus showing that each such $L(x)$ does not admit a BGG resolution. For $n \in \mathbb{N}$, we set

$$\underline{x} = \underbrace{\sigma\tau\sigma\tau\sigma \dots}_{np-1} \quad \underline{z} = \underbrace{\sigma\tau\sigma\tau\sigma \dots}_{np} \quad \underline{y} = \underbrace{\sigma\tau\sigma\tau\sigma \dots}_{np+1} \tag{2.26}$$

so that, in the notation of Eq. 1.5, we have

$$1_{\underline{x}} = 1_{\sigma\tau}^{np-1} \quad 1_{\underline{z}} = 1_{\sigma\tau}^{np} \quad 1_{\underline{y}} = 1_{\sigma\tau}^{np+1}. \tag{2.27}$$

Suppose that $L(y)$ is a subquotient of $\Delta(x)$ and that $L(y)$ belongs to the submodule generated by the homomorphic images of standard modules. Then $L(y)$ must be in the image of a homomorphism from $\Delta(y)$ or $\Delta(z)$ by highest weight theory. (In more detail, we simply note that $[\Delta(w) : L(y)] \neq 0$ implies that $w \leq y$ and that $\text{Hom}_{\mathcal{H}_{P \setminus W}^k}(\Delta(w), \Delta(x)) \neq 0$ implies that $x \leq w$ and therefore $x \leq w \leq y$; thus $w \in \{x, y, z\}$.) The module $\Delta(x)1_{\underline{y}}$ is $(np - 1)$ -dimensional and spanned by the light leaves basis elements

$$f_k := \begin{cases} (1_{\sigma\tau}^{k-1} \otimes \text{fork}_{\sigma\sigma}^{\sigma} \otimes 1_{\tau\sigma}^{np-1-k})(1_{\sigma\tau}^k \otimes \text{spot}_{\tau}^{\emptyset} \otimes 1_{\sigma\tau}^{np-k}) & \text{if } k \text{ is odd,} \\ (1_{\sigma\tau}^{k-1} \otimes \text{fork}_{\tau\tau}^{\tau} \otimes 1_{\sigma\tau}^{np-1-k})(1_{\sigma\tau}^k \otimes \text{spot}_{\sigma}^{\emptyset} \otimes 1_{\tau\sigma}^{np-k}) & \text{if } k \text{ is even,} \end{cases} \tag{2.28}$$

for $1 \leq k < np$. We now calculate the full submodule structures of standard modules and hence verify that $L(y)$ is not in the image of any homomorphism $\text{Hom}_{\mathcal{H}_{P \setminus W}^k}(\Delta(w), \Delta(x))$ for $w \in \{x, y, z\}$.

Decomposition numbers. We will prove that the Gram matrix of $\Delta(x)1_{\underline{y}}$ has rank $\dim(\Delta_{\leq y}(x)) - \dim(L_{\leq y}(y)) = np - 1 - 1 = np - 2$. Thus proving that $[\Delta(x) : L(y)] = 1$

using Eq. 1.22 (this is a standard cellular argument). The Gram-matrix of the cell-form of $\Delta(x)1_{\underline{y}}$ has -2 for each of the diagonal entries and 1 for each of the super and subdiagonals (in other words, it is equal to -1 times the Cartan matrix of type A_{np-1}). The determinant of this matrix is np which is zero in \mathbb{k} . Over \mathbb{k} , the rank of this matrix is $np - 2$ and so $[\Delta(x) : L(y)] = 1$ as required.

Submodule structures and homomorphisms. By considering the light leaves basis, one deduces that $\Delta_{\leq \underline{y}}(y) = L_{\leq \underline{y}}(y)$ is 1-dimensional and that $\Delta_{\leq \underline{y}}(z)$ is 2-dimensional with simple head $L_{\leq \underline{y}}(z)$ and simple socle $L_{\leq \underline{y}}(y)$. Suppose that the socle of $\Delta_{\leq \underline{y}}(x)$ contains a submodule isomorphic to $L_{\leq \underline{y}}(y)$. This submodule must be the span of some element $g = \sum \alpha_k f_k$ for $\alpha_k \in \mathbb{k}$ with gf_k^* vanishing in $\Delta_{\leq \underline{y}}(x)$ for all $1 \leq k < np$, i.e.

$$gf_k^* \in \mathcal{H}_{\mathfrak{S}_2 \setminus \hat{\mathfrak{S}}_2}^{<x}$$

for all $1 \leq k < np$. We have that $f_{k\pm 1}f_k^* = 1_{\underline{x}}$ and $f_k f_k^* = -2 \cdot 1_{\underline{x}}$ modulo $\mathcal{H}_{\mathfrak{S}_2 \setminus \hat{\mathfrak{S}}_2}^{<x}$. Thus

$$g = \alpha(f_{np-1} + 2f_{np-2} + 3f_{np-3} + 4f_{np-4} + 5f_{np-5} + \dots + (np - 1)f_1),$$

for some $\alpha \in \mathbb{k} \setminus \{0\}$. However, we notice that for g as above,

$$\begin{aligned} \alpha 1_{\sigma_{\tau}}^{np-1} \otimes \text{spot}_{\tau}^{\theta} &= g(1_{\sigma_{\tau}}^{np-2} \otimes \text{spot}_{\theta}^{\sigma}) && \text{if } np \text{ is even,} \\ \alpha 1_{\sigma_{\tau}}^{np-1} \otimes \text{spot}_{\sigma}^{\theta} &= g(1_{\sigma_{\tau}}^{np-2} \otimes \text{spot}_{\theta}^{\tau}) && \text{if } np \text{ is odd,} \end{aligned}$$

and so the submodule generated by g contains $L_{\leq \underline{y}}(z)$ as a submodule. Thus $\Delta_{\leq \underline{y}}(x)$ is uniserial with simple head $L_{\leq \underline{y}}(x)$, simple socle $L_{\leq \underline{y}}(z)$, and the middle composition factor $L_{\leq \underline{y}}(y)$. Thus $L_{\leq \underline{y}}(x)$ not the image of a homomorphism from $\Delta_{\leq \underline{y}}(z)$ or $\Delta_{\leq \underline{y}}(y)$, as required. □

Remark 2.13 There is a theory of “ $SL_2(\mathbb{k})$ -strings” for algebraic groups. This theory allows to inflate decomposition numbers and extension groups between standard modules for $SL_2(\mathbb{k})$ to calculate certain decomposition numbers and homomorphisms between standard modules for arbitrary algebraic groups (see [30, Part II, 5.21 (2)] and [18, Introduction] for decomposition numbers and extension groups respectively). One can use the equivalence between algebraic groups and anti-spherical Hecke categories from [42] in order to translate these statements to the setting of $\mathcal{H}_{P \setminus W}$ -modules (in the case that P is the maximal parabolic of an affine Weyl group W). Thus one can use Proposition 2.12 to provide many examples of simple $\mathcal{H}_{P \setminus W}$ -modules which do not admit BGG resolutions. We do not go into further details here.

Remark 2.14 Through the isomorphism of [6], we can rephrase the above as a question concerning decomposition numbers and homomorphisms for the symmetric group \mathfrak{S}_{np^2+p} in characteristic $p > 2$. We let $S(\lambda)$ denote the Specht module and $D(\mu)$ is the simple head for λ, μ partitions (the latter p -regular). We have that $[S(np^2, p) : D(np^2 + p)] = 1$, however $D(np^2 + p)$ is not in the image of any homomorphism $\text{Hom}_{\mathbb{k}\mathfrak{S}_{np^2+p}}(S(\lambda), S(np^2, p))$. This example was already known to Gordon James in [29, 24.5 Examples] for $p = 2$ and the general case is similar, see [29, 24.4 Theorem] and [29, 24.15 Theorem]. For p arbitrary, the full submodule structures of Specht modules labelled by 2-part partitions were determined in [43].

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