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KAON DECAY PHENOMENOLOGY IN A D.K.P. FRAMEWORK

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by

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I declare the material in the following dissertation to be original except in so far as explicit reference is made.

CONTENTS

INTRODUCTION

The main purpose of this dissertation, is to describe a new attempt to determine some characteristics of the form factors describing K_{ℓ_3} decays, using the D.K.P. formalism. Within the framework of V-A theory, K_{ℓ_3} decays are pure vector transitions and characterized by two form factors f_+ and f_- which are functions only of $t \equiv (p_K - p_\pi)^2$, the four momentum transfer squared between the K and π meson. The study of the form factors in K_{ℓ_3} decay is interesting both because of the relative simplicity of the theory and the relative accessibility of the effects, induced by strong interactions, to experimental measurement. If there were no strong interactions, only f_+ would be present, and it would be a constant. The strong interaction effects can be gauged through the variation, of f_+ and f_-, with the momentum transfer t.

Most of the studies dealing with K_{ℓ_3} decays, assume that SU(3) is a good symmetry, and only later incorporate the breaking of SU(3); usually through the empirical parameter θ_c ; the Cabibbo angle.

A different approach in which symmetry breaking can be incorporated is in the use of the D.K.P. formalism to describe the fields of the pion and the kaon. One is led right from the start to a decomposition of the hadronic matrix element which is different from that given in the Klein-Gordon formalism. A way to see the difference in the description of the mesons is to say that because of the different dimension of the D.K.P. field, various mass factors must be separated out in order that the associated form factors be dimensionless.

We shall show that if we assume the simultaneous participation of three interrelated (only two are independent) D.K.P. currents, then a specific value for $\xi(0)$ and λ_- can be obtained. The value of $\xi(0)$ is large and negative and equal to -1.6 . The value of λ_- is - 0.018 . These values agree well with the over all (quadratic) fit made by Chounet, et al [10] in 1972, but do not seem to agree well with the more recent experiments. The assumption that $\lambda_- \neq 0$ implies that the form factors $f_+(t)$ and $f_0(t)$ should have a quadratic t-dependence. Since most of the experimental data is given under the assumption that $\lambda_- = 0$ it is not possible to compare our results with those experiments. One may expect experimental difficulties in the determination of f_- . This happens because in expressions for observed quantities one finds the factor $[m_\ell/m_K][f_-/f_+]$. Thus, the hadronic effects expected due to the presence of f_- are expected to be small. The fit for the experimental determination of $\xi(0)$ will depend on wether one assumes a constant or linear t-dependence behaviour for the f_- form factor.

The D.K.P. analysis of $K_{\ell 3}$ decays, incorporates from the begining an SU(3) symmetry breaking term by requiring that $m_\pi \neq m_K$. The particular model we use, illustrates a mechanism through which two of the $K_{\ell 3}$ decay parameters can be understood.

A very important problem which is left unanswered is the normalization values one is to attribute to $f_+(0)$, $(f_0(0))$ (or more precisely $f_+ \sin\theta_c$) and f_-. However, we discuss the possibility of determining $f_+(0)\sin\theta_c$ in terms of the $K_{\ell 3}$ decay masses ($m_\pi, m_K ; m_\ell$). In an informal way we discuss and make some observations concerning $f_+(0)\sin\theta_c$ concluding that it could be possible that the D.K.P. formalism would help to understand this problem of normalization.

Comparing our results with other models used in the study of $K_{\ell 3}$ decay we can say the following: Pole dominance, gives slopes (λ_+, λ_0) which are positive and determined by the masses of the exchanged particles, i.e. $\lambda_+ = m_\pi^2/m_{K^*}$ and $\lambda_0 = m_\pi^2/m_K^2$. Thus if $\lambda_0 < 0$ (as in our case), then the pole dominance model cannot be valid. On the other hand the Callan-Treiman result which, in a certain sense, reflects how badly SU(3) symmetry is broken, agrees very poorly with our result that requires a rather more radical departure from SU(3) conservation.

I THE D.K.P. FORMALISM

Introduction.-

To begin the present chapter, we want to describe the D.K.P. formalism[1], for non-interacting spin-0 fields. As was stressed by Kemmer[1], the meson equations will appear as equations of the Dirac type, but will involve matrices obeying a different scheme of commutation rules than those corresponding to the Dirac matrices. The equations of motion are first order matrix-differential equations resembling the Dirac equation.

The β matrices appearing in the first-order linear differential equation are four 16X16 matrices. The algebra of the β matrices has three irreducible representations and these are one, five, and ten-dimensional representations respectively. To each representation there corresponds a field determined as usual by their spin. It turns out that the D.K.P. field $\psi(x)$ consist of three irreducible fields, the first is the trivial $\psi = 0$ field, the second one represents spin-0 mesons and the third one describes spin-1 mesons. In general the D.K.P. formalism describes spin-0 and spin-1 mesons, having a non-vanishing rest-mass.

After it was shown that the simpler Klein-Gordon formalism for spin-0 particles was equivalent to the D.K.P. formalism for free fields, as well as for the field interacting with electromagnetism [2], and for the field interacting with the Dirac field [3], interest in the D.K.P. formalism has not been very great in the past. It was not until it was suggested [4], that this equivalence between the two formalism might not hold for certain particular cases, that interest in the D.K.P. formalism arose again. As was pointed out then, it is with the introduction of interactions that differences between the two formalism can appear.

For K_{ℓ_3} decays, the mesons involved, are the kaon and the pion. Experimentally, their masses have beeen found to be quite different, i.e.; $m_K - m_\pi \sim 2.7\ m_\pi$. This mass difference is considered to be a measure of the SU(3) symmetry breaking. Since the D.K.P. formalism mixes the meson masses in a way, that does not occur with the Klein-Gordon formalism, we naturally consider the former method to be the correct way to incorporate directly the symmetry breaking, in the study of meson decay processes. We will take advantage of this inequivalence in this Thesis to study meson $\rightarrow$ meson + $\ell\nu$ processes.

1-1) D.K.P. Theoretical Models.

We will study the case of spin-0 mesons, but much of the formalism is similar to the case of spin-1 mesons.

The Lagrangian density for the free D.K.P. field is

$$\mathcal{L}(x) = \bar{\psi}(x) \left[i \partial_\mu \beta^\mu - m \right] \psi(x) \tag{1}$$

where β -matrices satisfy the relation

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu \tag{2}$$

and

$$\bar{\psi} \equiv \psi^+ \eta \tag{3}$$

where

$$\eta = 2 \beta^{o2} - 1 \tag{4}$$

satisfies

$$\eta = \eta^+ \; ; \; \eta \beta^{\mu +} \eta^{-1} = \beta^\mu$$

From this Lagrangian we can write the equations of motion

$$i \partial_\mu \beta^\mu \psi(x) - m \psi(x) = 0 \tag{5-a}$$

$$i \partial_\mu \bar{\psi}(x) \beta^\mu + m \bar{\psi}(x) = 0 \tag{5-b}$$

and we can also write the vector current:

$$J^\mu(x) = i \bar{\psi}(x) \beta^\mu \psi(x) , \tag{6}$$

an immediate consequence is that this current is conserved

$$\partial_\mu J^\mu = 0 \tag{7}$$

However, if we are interested in two different D.K.P. fields, $\psi_1(x)$ and $\psi_2(x)$, we can define[5] the expression

$$J^\mu(x) \equiv i \bar{\psi}_2(x) \beta^\mu \psi_1(x) \tag{8}$$

and we see that the divergence of this current

$$\partial_\mu J^\mu = i\, \partial_\mu \bar{\psi}_2 \beta^\mu \psi_1 + i \bar{\psi}_2 \beta^\mu \partial_\mu \psi_1$$

$$= (m_1 - m_2)\, \bar{\psi}_2 \psi_1 \qquad (9)$$

is not conserved if $m_1 \neq m_2$.

We note that for the Klein-Gordon fields φ_1 and φ_2 , we can also define

$$j^\mu = i\, \varphi_2^* \overset{\leftrightarrow}{\partial^\mu} \varphi_1 \qquad (10)$$

the divergence of this current is given by

$$\partial_\mu j^\mu = (m_1^2 - m_2^2)\, i\, \varphi_2^* \varphi_1 \qquad (11)$$

where we have used the Klein-Gordon equation.

The above considerations are an indication that the D.K.P. formalism may yield in some cases different results than those obtained with the Klein-Gordon formalism.

It is known [6] that in the presence of conserved currents, the D.K.P. formalism, for spin-0 and spin-1 particles, is equivalent to the Klein-Gordon and Proca formulations respectively. It was pointed out recently [4], that when there is a broken symmetry relating the fields of a meson of one mass to a meson of another mass, this no longer holds in general. Therefore the difference in the results may be compared with experiment.

The matrix element of the Klein-Gordon current above, (10) can be expressed in momentum space as:

$$\langle P_2 | \mathcal{J}^{\mu} | P_1 \rangle = \frac{1}{2(E_1E_2)^{1/2}} \left[P_2^{\mu} + P_1^{\mu} \right] \tag{12}$$

It will be useful to define the matrix element of an antisymmetrical current with respect to the interchange $P_1 \leftrightarrow P_2$ as:

$$\langle P_2 | \mathcal{J}^{-\mu} | P_1 \rangle \equiv \frac{1}{2(E_1E_2)^{1/2}} \left[P_2^{\mu} - P_1^{\mu} \right] \tag{13}$$

In momentum space the matrix element of the D.K.P. current J_1^{μ} (8) is given by:

$$\langle p_2 | J_a^{\mu} | p_1 \rangle = \frac{1}{2}\left[\frac{m_1 m_2}{E_1 E_2}\right]^{1/2}\left[\frac{p_2^{\mu}}{m_2} + \frac{p_1^{\mu}}{m_1}\right] \tag{14}$$

where m_1 and m_2 are masses of initial and final mesons.

In analogy with (13), let us define the matrix element of the antisymmetrical D.K.P. current.

$$\langle p_2 | J_b^{\mu} | p_1 \rangle = \frac{1}{2}\left[\frac{m_1 m_2}{E_1 E_2}\right]^{1/2}\left[\frac{p_2^{\mu}}{m_2} - \frac{p_1^{\mu}}{m_1}\right] \tag{15}$$

This current, can be expressed in terms of the β^{μ} as (51)

$$J_b^{\mu}(x) = i\,\bar{\psi}_2(x)\,\tau\,\beta^{\mu}\,\psi_1(x) \tag{16}$$

where τ is the charge-conjugation matrix. In the particular representation we will choose below for the β-matrices, the matrix τ satisfies

$$\tau\beta^{\mu} = \beta^{\mu *}\tau \quad ; \quad \tau^{*}\tau = 1 \tag{17}$$

if we choose for the β^{μ} the particular representation

$$\beta^1 = \frac{1}{i}\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{bmatrix} \qquad \beta^2 = \frac{1}{i}\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{bmatrix}$$

$$\beta^3 = \frac{1}{i}\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{bmatrix} \qquad \beta^0 = \frac{1}{i}\begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -1&0&0&0&0 \end{bmatrix} \tag{18}$$

this leads to

$$\tau = \begin{bmatrix} -1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&-1&0&0\\ 0&0&0&-1&0\\ 0&0&0&0&1 \end{bmatrix} = \tau^* \tag{19}$$

and we have

$$\eta = 2\beta^0\beta^0 - 1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

If we let the five component D.K.P wave function be given by

$$\psi_p(x) = \left(\frac{1}{2\pi}\right)^{3/2} \sqrt{\frac{m}{p^0}}\; u(p)\, e^{-ipx} \tag{21}$$

where

$$u(p) = \frac{1}{\sqrt{2m^2}} \begin{bmatrix} -ip^0 \\ ip^1 \\ ip^2 \\ ip^3 \\ m \end{bmatrix} \tag{22}$$

and

$$\bar{u}(p) = \frac{1}{\sqrt{2m^2}} \left[ip^0, ip^1, ip^2, ip^3, m \right] \tag{23}$$

The following relations can be satisfied for particular representations of the Dirac γ-matrices, with choice (18) above and with γ_μ and γ_5 replaced by β_μ and η respectively:

$$\beta_\mu^* = -\beta_\mu \, , \quad \beta_\mu^T = -\beta^\mu \, , \quad \beta_\mu^+ = \beta^\mu$$

$$\eta \beta_\mu = \beta^\mu \eta \, , \quad \eta^2 = 1 \quad , \eta^+ = \eta^* = \eta^T = \eta \tag{24}$$

where :

$\beta^K = -\beta_K$ for $K = 1, 2, 3$ and $\beta^0 = \beta_0$, and where T means the transpose matrix, $+$ means the complex conjugate transpose matrix, i.e. $A^+ = (A^T)^*$.

The particular relationships that follow with our choice (22) and (23) are

$$\bar{u}(p)\, u(p) = 1$$

$$\bar{u}(p)\, u(-p) = 0 \tag{25}$$

The D.K.P. equations can be expressed as

$$(p^{\mu}\beta_{\mu} - m)\, u(p) = 0 \qquad (26\text{-a})$$

and

$$\bar{u}(p)\,(p^{\mu}\beta_{\mu} - m) = 0 \qquad (26\text{-b})$$

When $p_1 \neq p_2$, that is when the mesons have different masses, we obtain:

$$\bar{u}'(p_2)\, u(p_1) = \frac{1}{2 m_1 m_2}\left[m_1 m_2 + p_1 p_2 \right] \qquad (27)$$

for the vector current matrix element, we have:

$$\bar{u}'(p_2)\, \beta^{\mu}\, u(p_1) = \frac{1}{2}\left[\frac{p_2^{\ \mu}}{m_2} + \frac{p_1^{\ \mu}}{m_1} \right] \qquad (28)$$

and for the antisymmetric vector current matrix element

$$\bar{u}(p_2)\, \beta^{\mu}\tau\, u(p_1) = \frac{1}{2}\left[\frac{p_2^{\ \mu}}{m_2} - \frac{p_1^{\ \mu}}{m_1} \right] \qquad (29)$$

We can now define the phenomenological D.K.P. current as

$$\mathcal{J}^{\mu} \equiv h_a J_a^{c} + h_b J_b^{\mu} + h_c J_c^{\mu} \tag{30}$$

where the matrix element for J_c^{μ} is defined to be proportional to:

$$(P_2^{\mu} - P_1^{\mu})\, \bar{u}(P_2)\, u(P_1)$$

and $h_{a,b,c}$ are the corresponding coupling constants. $\langle P_2 | J_c^{\mu} | P_1 \rangle$ is not independent of (14) and (15), and can be expressed as a linear combination of those two currents. If we notice that (27) can be expressed in the form

$$\frac{t_0 - t}{4 m_1 m_2}$$

where $t_0 \equiv (m_1 + m_2)^2$ and $t \equiv (P_1 - P_2)^2$, we can define $\langle P_2 | J_c^{\mu} | P_1 \rangle$ to be given by

$$\langle P_2 | J_c^{\mu} | P_1 \rangle \equiv \frac{1}{2} \left[\frac{m_1 m_2}{E_1 E_2}\right]^{1/2} \frac{(t_0 - t)}{2 m_1 m_2 (\omega_1 + \omega_2)} (P_2 - P_1)^{\mu} \tag{31}$$

where we have chosen the same normalization constant as we had for J_a^μ and J_b^μ in (14) and (15).

We shall assume throughout that only these three vector-like currents contribute to the decay processes we are going to consider. The choice of what currents to consider has been the subject of much discussion for and against the argument that the D.K.P. formalism, when used to describe K_{ℓ_3} decays, leads to a more satisfactory theory for the K_{ℓ_3} form factors than does the conventional Klein-Gordon formalism. There is the group [4] that only considers J_a^μ and J_c^μ and who take the stand that the D.K.P. formalism yields qualitatively different results than those obtained by the traditional Klein-Gordon theory. On the other hand there is the opposite view [8] held by those who, essentially view the D.K.P. formalism as equivalent to the Klein-Gordon formalism this group considers J_a^μ and J_b^μ only.

As long as there is no convincing way of showing what currents to use, we consider that the proper "phenomenological attitude" is to take into account all three currents. In what follows we are going to choose particular values of the coupling constants $h_{a,b,c}$ to analyse the semi-leptonic kaon decay process (see Appendix for details and conventions), $K \to \pi \ell \nu_\ell$.

1-2) D.K.P. Phenomenological Analysis of $K\ell_3$ Form Factors.-

From the phenomenological D.K.P. expression for the meson current (30) we are going to analyse the particular values we obtain for the relevant (see Appendix) quantities appearing in the expression for the form factors f_+ , f_- and ξ (Appendix equations (139),(141)) by choosing particular values for the coupling constants $h_{a,b,c}$ appearing in (30).

For the process $K \to \pi \ell \nu$ it is customary to parametrize (see Appendix p., 84) the matrix element of the vector meson current and to assume it to be proportional to.

$$(P_K + P_\pi)^\mu f_+(t) + (P_K - P_\pi) f_-(t) \tag{32}$$

where f_+ and f_- depend only on t, in a way that has to do with the (virtual) strong interactions. If we identify $\langle P_2 | \mathcal{J}^\mu | P_1 \rangle$ in (30) with

$$\frac{1}{2(E_1E_2)^{1/2}} \left\{ (P_1+P_2)^\mu f_+(t) + (P_1 - P_2)^\mu f_-(t) \right\} \tag{33}$$

and $\quad P_1 \equiv P_K \ , \ P_2 \equiv P_\pi \ ;$

we obtain the following relationships for the form factors:

$$f_+(t) = \frac{1}{2(m_K m_\pi)^{1/2}} \left[(m_K + m_\pi) h_a(t) \; m_K - m_\pi) h_b(t) \right] \tag{34}$$

and

$$f_-(t) = -\frac{1}{2(m_K m_\pi)^{1/2}} \left\{ (m_K - m_\pi) h_a(t) + (m_K + m_\pi) h_b(t) + \left(\frac{t_0 - t}{m_K + m_\pi} \right) h_c(t) \right\} \tag{35}$$

These relations mean that the ratio of form factors is:

$$\xi(t) \equiv \frac{f_-(t)}{f_+(t)} = \frac{-(m_K^2 - m_\pi^2) h_a - t_0 (h_b + h_c) + t h_c}{t_0 h_a + (m_K^2 - m_\pi^2) h_b} \tag{36}$$

It will be useful to have an expression for the divergence of $\mathcal{J}^\mu$ in terms of the $h_i \; ; i = a, b, c$. Using (30) we can express the matrix element of $\partial_\mu \mathcal{J}^\mu$ as:

$$\langle P_2 | \partial_\mu J^\mu | P_1 \rangle = \left[\frac{m_\pi m_K}{E_1 E_2}\right]^{1/2} \left(\frac{1}{4 m_\pi m_K}\right) \Big\{ (m_K - m_\pi)(t_0 - t) h_a +$$

$$+ \sqrt{t_0} \left[(m_K - m_\pi)^2 - t \right] h_b + \tag{37}$$

$$+ \frac{t}{\sqrt{t_0}} \left[t - t_0 \right] h_c \Big\}$$

We see that there are three interesting values for t, namely

$$t = t_0 \quad , \quad t = (m_K - m_\pi)^2 \ , \qquad t = 0 \tag{38}$$

In the work of Fischbach et, al [4] where they do not consider J_b^μ, it is argued that the value of t where the divergence vanishes; i.e. $t = t_0$ indicates the possibility of a meson resonance with a mass value around $\sqrt{t_0} \simeq 628.7$ mev. On the other hand Willey et, al [8] do not see the reason to consider J_c^μ and as a consequence, the divergence does not vanish at any particular value for t. In reality, the argument between Fischbach et, al [4] and Willey et, al [8] can be traced down to their different assumptions.

Note that J_c^μ can not be constructed out of general invariance considerations. It has a phenomenological origin. In terms of the D.K.P. fields,

ψ_1 and ψ_2 , it can be expressed as:

$$J_c^{\mu} = i\,\partial^{\mu}\,(\bar{\psi}_2\,\psi_1) \tag{39}$$

and has the form of a derivative current. On the other hand it can be argued phenomenologicaly that there is no strong a priori reason to exclude either J_b or J_c . We shall now make some particular assumptions concerning the values of h_a , h_b and h_c .

1-3) Particular Values of the h_i Coupling Constants and the Corresponding Values for the K_{ℓ_3} Parameters.-

In order to study what values to choose for h_a , h_b and h_c we shall consider various possibilities. To begin with, the simplest assumption to make is that only one of the three currents making up $\mathcal{J}^{\mu}$ in expression (30), is relevant, in the description of K_{ℓ_3} decay. If this is so, and we consider that the K_{ℓ_3} functions $(f_{\pm}, \xi)$ are to be calculated when the momentum transfer vanishes; i.e. t = 0 , we obtain, the values shown in TABLE 1 at the end of this section where, in each case we assumed that only one coupling constant, h_i , did not vanish; in order to consider only one current at a time. Present experimental estimates seem to favor the choice $h_a \neq 0$; $h_b = h_c = 0$. In this case, when h_a is taken to be unity, we obtain for the ratio of the form factors at t = 0 , the value $\xi(0) = -0.6$ which is in good agreement with some experiments performed after 1971 (see Appendix pp., 95, 98, and 100)

If we consider next that only two of the three currents defining the D.K.P. currents; $\mathcal{J}^{\mu}$, are meaningful, we obtain for the K_{ℓ_3} parameters the values shown in TABLE-II at the end of this section.

Another case is when the three currents are considered simultaneously. It can be seen, when all the coupling constants are equal that one obtains:

$$f_+ = \left[\frac{m_K}{m_\pi}\right]^{1/2} \sim 1.9 \tag{40}$$

$$f_- = -\frac{3\,m_K + m_\pi}{2\,(m_K m_\pi)^{1/2}} \sim -3.1 \tag{41}$$

and

$$\xi = -\frac{3\,m_K + m_\pi}{2\,m_K} \sim -1.6 \tag{42}$$

For the sake of completeness we show in table III at the end of this section, the values of the K_{ℓ_3} parameters when the meson masses are equal; i.e., $m_K = m_\pi$.

The above values for the K_{ℓ_3} parameters are very rough estimates and are only meant to help to find the best particular combination of the h_i coupling constants. Nevertheless it may be useful at this stage to look (cf TABLES I, II, and III pp., 26 and 27) at the experimental values of these parameters (see the Appendix p.p. 93-100).

If the concept of SU(3) symmetry is meaningful for the analysis of the K_{ℓ_3} parameters, then their true values are considered to be close to the numbers one calculates when one considers SU(3) to be an exact ~~conser-ved~~ symmetry. If this is the case, f_- vanishes (see Appendix p., 86 , and it is assumed that the true value for f_- should be a small number compared with f_+, which takes a value in SU(3) theory of order ~~compatible with~~ unity.

If one assumes SU(3) conservation it can be seen from TABLE I that the acceptable choice is when only J_a^μ is present. Nevertheless, since symmetry breaking is large, we shall take the point of view that the three currents should be considered simultaneously using a particular linear combination determined by the h_i parameters. The concrete way we shall do this is going to be treated in the next section, but before we do this, we would like to end this section with an illustration. If we consider that the form factors f_+ and f_- are t-independent and are determined by relations (34) and (35) it can be seen from the Appendix, that the branching ratio

$$\frac{\Gamma\,(K \to \pi \mu \nu)}{\Gamma\,(K \to \pi e \nu)} \qquad (43)$$

can be expressed (from Appendix, eq (54)) by the relation

$$\frac{\Gamma(K_{\mu 3})}{\Gamma(K_{e3})} = 0.64 + 0.12\,\xi(0) + 0.019\,\xi^2(0) \qquad (44)$$

where we have neglected terms in λ_+ .

Subtituting the ξ values in TABLE-I we find

	$h_a \neq 0$	$h_b \neq 0$
$\frac{\Gamma(K_{\mu 3})}{\Gamma(K_{e3})}$	0.57	0.49

Performing the same operation with the ξ values in TABLE-II we obtain, using (36);

	$h_b, h_a \neq 0$	$h_c, h_a \neq 0$	$h_c, h_b \neq 0$
$\frac{\Gamma(K_{\mu 3})}{\Gamma(K_{e3})}$	0.54	0.50	0.45

where we have used in all the calculations, the masses of $m_{K^\pm}$ and of m_{π^0} .

The values shown above should be compared with the experimental $\Gamma(K_{\mu 3})/\Gamma(K_{e3})$ value [9] for $K^\pm$ i.e.;

$$\frac{\Gamma(K_{\mu 3})}{\Gamma(K_{e3})} = 0.663 \pm .018 \qquad (45)$$

TABLE-I

$h_a \neq 0$	$h_b \neq 0$	$h_c \neq 0$
$f_+ = \frac{m_K + m_\pi}{\Delta}$	$f_+ = \frac{m_K - m_\pi}{\Delta}$	$f_+ = 0$
~ 1.2	~ 0.67	
$f_- = -\frac{m_K - m_\pi}{\Delta}$	$f_- = \frac{-m_K - m_\pi}{\Delta}$	$f_- = -\frac{m_K + m_\pi}{\Delta}$
~ -0.67	~ -1.2	~ -1.2
$\xi = -\frac{m_K - m_\pi}{m_K + m_\pi}$	$\xi = -\frac{m_K + m_\pi}{m_K - m_\pi}$	$\xi = -\infty$
~ -0.6	~ -1.7	

where

$$\Delta \equiv 2\,(m_K m_\pi)^{1/2}$$

TABLE-II

$h_a = h_b \neq 0$	$h_a = h_c \neq 0$	$h_b = h_c \neq 0$
$f_+ = \left[\frac{m_K}{m_\pi}\right]^{1/2}$	$f_+ = \frac{m_K + m_\pi}{\Delta}$	$f_+ = \frac{m_K - m_\pi}{\Delta}$
~ 1.91	~ 1.22	~ 0.69
$f_- = -\left[\frac{m_K}{m_\pi}\right]^{1/2}$	$f_- = -\left[\frac{m_K}{m_\pi}\right]^{1/2}$	$f_- = -\frac{2(m_K + m_\pi)}{\Delta}$
~ -1.91	~ -1.91	~ -2.43

where $\Delta \equiv 2\,(m_K m_\pi)^{1/2}$

TABLE-III

	$h_a \neq 0$	$h_b \neq 0$	$h_c \neq 0$	$h_a = h_b \neq 0$	$h_a = h_c \neq 0$	$h_b = h_c \neq 0$
f_+	1	0	0	1	1	0
$\xi(0)$	0	$-\infty$	$-\infty$	-1	-1	$-\infty$

where $m_K^2 \simeq m_\pi^2$

II A TWO CURRENT ANALYSIS IN THE D.K.P. FORMALISM.-

One may adopt the point of view that there are really two fundamental currents (12) and (13). Thus there will only be two coupling constants, and from (30), we can express the phenomenological D.K.P. current as:

$$\mathcal{J}^\mu = \frac{1}{2(m_1 m_2)^{1/2}} \left\{ 2m_1 h J^\mu + \left[2m_1 + \frac{t_0 - t}{m_1 + m_2} \right] g \mathcal{J}^\mu \right\} \tag{46}$$

where

$$h = \frac{1}{2}\left[h_a + h_b + (h_a - h_b)\frac{m_2}{m_1} \right] \tag{47}$$

and

$$g = \frac{(h_a + h_b) m_1 + (h_b - h_a) m_2 + h_c \dfrac{t_0 - t}{m_1 + m_2}}{2m_1 + \dfrac{t_0 - t}{m_1 + m_2}} \tag{48}$$

In what follows we will analyse the consequences of the above scheme, comparing the phenomenological parameters used to analyse all the K_{ℓ_3} processes, assuming that $h(t)$ and $g(t)$ are smooth functions of t.

The particular parametrization we will choose for the D.K.P. vector currents, will have a specific effect on the dynamics of the processes we shall study. The decision of which parametrization to use is a phenomenological one, which will have to be confronted with the data. We will see that our parametrization is compatible with a quadratic fit of both polarization experiments and Dalitz plot data, obtained by Chounet, et al [10].

If we define

$$I^{\mu}_{(+)} \equiv \frac{1}{2(m_1 m_2)^{1/2}} \left[2m_1 \, j^{\mu} \right] \tag{49}$$

and

$$I^{\mu}_{(-)} \equiv \frac{1}{2(m_1 m_2)^{1/2}} \left[2m_1 + \frac{t_0 - t}{m_1 + m_2} \right] j^{\mu} \tag{50}$$

we can express $\mathcal{J}^{\mu}$ in (46) as follows:

$$\mathcal{J}^{\mu} = h \, I^{\mu}_{(+)} + g \, I^{\mu}_{(-)} \tag{51}$$

In the analysis of K_{ℓ_3} decays;

$$K \rightarrow \pi + \ell + \nu$$
$$(1) \qquad (2) \qquad (3) \qquad (4)$$

the matrix element of the vector current defines the form factors f_+ and f_- (see Appendix Eq(107));

$$\langle \pi | V^\mu | K \rangle = \frac{1}{2}\left[(P_2 + P_1)^\mu f_+(t) + (P_2 - P_1)^\mu f_-(t) \right]$$

Comparing the above expression with (51) it follows that

$$f_+(t) = \left(\frac{m_K}{m_\pi} \right)^{1/2} h(t) \tag{52}$$

and

$$f_-(t) = - \left(\frac{1}{2 m_\pi m_K} \right)^{1/2} \left[2 m_K + \frac{t_0 - t}{m_K + m_\pi} \right] g(t) \tag{53}$$

In what follows we shall determine+ λ_- and $\xi(t)$ in terms of the meson masses m_K and m_π. These will turn out to be very important quantities. These results will be obtained phenomenologically from the specific parametrization of the D.K.P. currents that

+N.B. for the definition of the parameter λ_-, and $\xi(t)$ see the Appendix. p., 90 .

we have chosen and some normalization assumptions.

If we consider g(t) to be fairly constant with respect to t, and we assume that f_- varies linearly with t, i.e. ;

$$f_-(t) = f_-(0)\left[1 + \lambda_- \frac{t}{m_\pi^2}\right] \tag{54}$$

we can compare this with (53). It follows then that we can express λ_- in terms of m_K and m_π, as:

$$\begin{aligned} \lambda_- &= \frac{-m_\pi^2}{(3m_K+m_\pi)(m_K+m_\pi)} \\ &= -0.0179 \end{aligned} \tag{55}$$

This value should be compared with the experimental (over all fit) value [10]:

$$\lambda_- = -0.03 \begin{array}{l} +0.08 \\ -0.05 \end{array} \tag{56}$$

obtained from polarization and Dalitz plot experi_ ments.

On the other hand, from the above expresions for f_+ and f_-, we obtain for $\xi = f_- / f_+$ the relation

$$\xi(t) = -\frac{\left[2m_K + \frac{t_0 - t}{m_K + m_\pi}\right]}{2m_K}\left[\frac{g(t)}{h(t)}\right] \tag{57}$$

If $g(t) \sim h(t)$, it can be seen, that inside the physical region, i.e. $m_\ell^2 \leq t \leq (m_K - m_\pi)^2$ we have

$$\xi(0) \sim -1.64 \leq \xi(t) \leq -1.43 \sim \xi\left[(m_K - m_\pi)^2\right] \qquad (58)$$

Thus the variation of ξ is about 15% within the decay region. This crude estimate is inside the limits within wich $\xi(t)$ has varied in most of the measurements performed before 1970 [10], and resulting in

$$-2 \leq \xi(0) \leq 0 \qquad (59)$$

but is not compatible with the majority of values found for ξ after 1972^{+}, for which

$$-1.5 < \xi < 0 \qquad (60)$$

Assuming linearity, the fits [9] (1975) to all the experimental values for λ_+, λ_0 and $\xi(0)$ are shown in the next table:

+N.B. see Tables B-2, C-1 and D-1 in the Appendix.

TABLE OF OVER ALL LINEAR FITS TO K_{ℓ_3} EXPERIMENTS

	λ_+	λ_0	$\xi(0)$
$K^+_{e_3}$	0.0285 ± .0043		
$K^0_{e_3}$	0.0288 ± .0028		
$K^+_{\mu_3}$	0.027 ± .008	-0.009 ± .007	-0.45 ± .14
$K^0_{\mu_3}$	0.034 ± .006	0.021 ± .006	-0.17 ± .10

Data obtained from:

Rev Mod Phys. Vol. 48, No. 2 Part II (1976).

2-1) Quadratic Variation of the f_+ and f_0 Form Factors.-

It is conventional to expand the form factors in low order polynomials over the decay region:

$$m_\ell^2 \leq t \leq (m_K - m_\pi)^2$$

Let the form factors be expanded as

$$f_\pm(t) = f_\pm(0)\left[1 + \lambda_\pm \frac{t}{m_\pi^2} + \lambda'_\pm \frac{t^2}{m_\pi^4} + \cdots\right] \qquad (61)$$

and+

$$f_0(t) = f_0(0)\left[1 + \lambda_0 \frac{t}{m_\pi^2} + \lambda'_0 \frac{t^2}{m_\pi^4} + \cdots\right] \qquad (62)$$

These parameters are related by

$$f_0(0) = f_+(0) \qquad (63)$$

+ N.B. for the definition of $f_0(t)$ see Appendix p., 86 or see below.

$$\lambda_0 = \lambda_+ + \xi(0) \frac{m_\pi^2}{m_K^2 - m_\pi^2} \tag{64}$$

where

$$\xi(0) \equiv \frac{f_-(0)}{f_+(0)}$$

and

$$\lambda_0' = \lambda_+' + \lambda_- \xi(0) \frac{m_\pi^2}{m_K^2 - m_\pi^2} \tag{65}$$

etc...

Since $\lambda_+(\lambda_0)$, $\lambda_+'(\lambda_0')$,... are the physically relevant+ parameters an analysis will be biased if terms in f_- are retained to the same order as terms in f_+ . It turns out that when one determines a term in $f_0(t)$ to a given order, the contribution from f_+ is in each case more accurately determined experimentally than that from f_- [10)]. This means that to second order, an analysis which retains λ_- must also retain λ_+' .

+ N.B. see the note in the Appendix concerning the di-lepton amplitudes 0^+ and 1^- to which f_0 and f_+ are directly proportional.(see p., 87)

As can be seen from the Appendix the f_0 form factor is defined to be given by the matrix element of the current divergence:

$$\langle \pi | \partial^{\mu} V_{\mu} | K \rangle = (m_K^2 - m_\pi^2)\, f_0(t) \tag{66}$$

From the theoretical point of view the f_+ and f_0 form factors can be seen[+] to correspond to two dynamically independent amplitudes (0^+ and 1^-).

From the experimental point of view f_+ and f_0 are generally less correlated in a Dalitz plot analysis than are f_+ and f_- .

The $\Delta I = 1/2$ rule requires[+] the equality of the form factors in K^+ and K^0 decay.

μ-e universality implies[+] that the form factors involving $\pi\mu\nu$ are the same as those involving $\pi e\nu$.

The polarization experiments determine $\xi(t)$ directly whereas the branching ratio experiments are mainly sensitive to the slope of $f_0(t)$:

$$\lambda_0 = \lambda_+ + \frac{m_\pi^2}{m_K^2 - m_\pi^2}\, \xi(0)$$

+ N.B. for more details see Appendix. p., 87

The value that we obtained in (57) for $\xi(0)$, i.e. ; - 1.64 , leads to a negative value for the slope (λ_0) of f_0 , if $\lambda_+ \lesssim 0.132$. Fixing λ_+ to the value

$$\lambda_+ = 0.045 \tag{67}$$

using (64), we obtain:

$$\lambda_0 = - 0.087 \tag{68}$$

which is not at all negligeable. This value for λ_0 can be compared with the following values+, obtained in 1972 by Chounet, assuming a linear variation of f_0 .

Branching Ratio	$\lambda_0 = - 0.015 \pm 0.01$	
Fit to Dalitz Plot Data	$\lambda_0 = - 0.038 \pm 0.020$	(69)
Fit to Polarization Data	$\lambda_0 = - 0.03^{+0.09}_{-0.04}$	

As we can see from the table on p., 28 taking into account all the measurements performed after 1972, the λ_0 value seems to be positive now. Nevertheless, the value for λ_0 is still not definitive despite the numerous experiments performed to deter_ mine it.

When an analysis is made assuming a quadratic variation of the form factors, λ'_+ has been found to be positive, in the three instances it has been deter_ mined.

+ N.B. These are the values given by M.K. Gaillard in Proceedings of the XI Internationale Universitätswochen, in Schladming ed. by P. Urban (pag. 283 Springer-Verlag 1972).

The values found for λ_+ and λ'_+ in quadratic fits over K_{e_3} data have been:

a) Chounet, et al (ii)+; the results obtained from the data of 6++ high statistics experiments were

$$\begin{aligned} \lambda_+ &= 0.012 \pm 0.005 \\ \lambda'_+ &= 0.0052 \pm 0.0013 \end{aligned} \tag{70}$$

b) Chien, et al (xxiii); a fit obtained from a high statistics (16000) experiment, gave

$$\begin{aligned} \lambda_+ &= 0.026 \pm 0.006 \\ \lambda'_+ &= 0.0045 \pm 0.0015 \end{aligned} \tag{71}$$

c) Gjesdal, et al (xvi); a fit obtained from a very high statistics (500,000) experiment gave

$$\begin{aligned} \lambda_+ &= 0.0246 \pm 0.0043 \\ \lambda'_+ &= 0.0014 \pm 0.0008 \end{aligned} \tag{72}$$

On the other hand the values found for λ_0 and λ'_0 in K_{μ_3} fits were:

+ N.B. references given in Roman numbers are listed in the Appendix p., 101

++ N.B. the experiments were : (1458)-(iii), (2707)-(iv), (16000)-(viii), (42000)-(ix) and Basile, et al (4800), in Phys. Lett. 26B, 542, 1968.

a) Chounet, et al (ii); to limit the number of free parameters they used their λ_+ and λ'_+ values (see (70)) as input to extract λ_0 and λ'_0 ; they obtained for a combination of Dalitz plot and polarization experiments

$$\begin{aligned} \lambda_0 &= -0.11 \pm 0.03 \\ \lambda'_0 &= 0.0085 \pm 0.0065 \end{aligned} \tag{73}$$

b) Dally et al (xxv) obtain in a high statistics (16000) fit to their experiment

$$\begin{aligned} \lambda_0 &= -0.080 \pm 0.272 \\ \lambda'_0 &= -0.006 \pm 0.045 \end{aligned} \tag{74}$$

Those values should be compared with our values shown below, for λ_0 and λ'_0 , for the three different values of λ_+ and λ'_+ shown in (70), (71) and (72); using (64) and (65) we obtain:

$$\begin{array}{llll} \text{with eq. (70);} & \lambda_0 = -0.12 & \text{and} & \lambda'_0 = 0.0076 \\ \text{with eq. (71);} & \lambda_0 = -0.106 & \text{and} & \lambda'_0 = 0.0069 \\ \text{with eq. (72);} & \lambda_0 = -0.107 & \text{and} & \lambda'_0 = 0.0038 \end{array} \tag{75}$$

where we have used our value for λ_- in (55) and for

$\xi(0)$, the expression in (57) (with the assumption that $g(t) \sim h(t)$), therefore, it follows that (64) and (65) can be written as:

$$\lambda_+ - \lambda_0 = \left[\frac{3m_K + m_\pi}{2\, m_K}\right]\left[\frac{m_\pi^2}{m_K^2 - m_\pi^2}\right] \simeq 0.132 \tag{76}$$

and

$$\lambda_0' - \lambda_+' = \frac{m_\pi^4}{2m_K\,(m_K + m_\pi)(m_K^2 - m_\pi^2)} \simeq 0.00237 \tag{77}$$

respectively. These are the relation we have used.

A quadratic fit (determination of $\lambda_+, \lambda_+', \lambda_0$ and λ_0') has not been carried out with all the present data. The last one, was worked out by Chounet[ii], taking into account all the relevant data available before 1972.

We are next going to compare our results with each one of the different types of measurements available that can determine $\xi(0)$ and λ_- , or λ_0 and λ_0'. For this purpose, due to the reason just mentioned, we shall use (except for the branching ratio results) the experimental values given in reference (ii)

2-2) D.K.P. Determination of the Branching Ratio for the Charged K_{ℓ_3} Decay Mode for a Quadratic Variation of the Form Factors.-

The experimental measurements of the branching ratios for the charged kaons, gives the (fitted) value[i]

$$\frac{\Gamma(K^+_{\mu_3})}{\Gamma(K^+_{e_3})} = 0.663 \pm .018 \tag{78}$$

In a linear approximation for the form factors, with the assumption that $\lambda_+ = 0.030$, the over-all value found[i] for $\xi(0)$ is $-0.20 \pm .15$. This means that our value of -1.64 for $\xi(0)$ will not give (78), assuming the same value for λ_+. The same situation will repeat itself when we take a quadratic variation for the form factors. To see this, let us write the branching ratio for charged kaons in terms of λ_+, λ'_+, λ_0 and λ'_0, as follows

$$\frac{\Gamma(K^+_{\mu_3})}{\Gamma(K^+_{e_3})} = 0.6457 + 2.236\,\lambda_+ - 2.021\,\lambda_+^2 + 7.734\,\lambda'_+$$

$$+ 30.53\,\lambda_+\lambda'_+ + 66.85\,\lambda'^2_+ + 1.565\,\lambda_0 + 5.890\,\lambda'_0 + 0.0032\,\lambda_0\lambda_+$$

$$+ 0.098\,(\lambda'_+\lambda_0 + \lambda_+\lambda'_0) + 0.27\,\lambda'_+\lambda'_0 + 25.75\,\lambda_0\lambda'_0 + 2.94\,\lambda_0^2$$

$$+ 62.85\,\lambda'^2_0 \;/\; 1 + 3700\,\lambda_+ + 5.478\,\lambda_+^2 + 10.956\,\lambda'_+ + 40.48\,\lambda_+\lambda'_+$$

$$+ 85.29\,\lambda'^2_+ \tag{79}$$

Our calculations for different values of λ_+ , λ'_+ λ_0 and λ'_0 are given in the following table:

λ_+	0.012	0.0246
λ'_+	0.0052	0.0014
λ_0	-0.11	-0.107
λ'_0	0.0085	0.0038
$\frac{\Gamma(K^+_{\mu 3})}{\Gamma(K^+_{e3})}$	0.551	0.539

The first column corresponds to the values given in ref(ii), the the second column to the values shown in (72) with the corresponding value for λ_0 and λ'_0 found in (75). Our value for the branching ratio of the charged kaons, i.e. 0.539, does not compare well with the experimental value in (78). There has always been a discrepancy between the values found for $\xi(0)$ and λ_+ (linear fit) in branching ratio measurements and those values found for the same parameters in the Dalitz plot density and polarization measurements.

In general terms the experimental results for each kind of measurement (Dalitz plot density, branching ratio and polarization measurements) has always been so inconsistent (see relevant section of the Appendix) to render very suspicious the world average values[ii] for $\xi(0)$ and λ_- or, λ_0 and λ'_0 . Nevertheless the world average values for λ_+ and $\xi(0)$ or λ_0 have been obtained in ref(i)$^{+}$, but in view of the poor agreement of many of the experiments included in the world average, this world average value should not be taken too seriously. We consider also that the experiments on K_{ℓ_3} decays have not ruled out a significant departure from strict linearity.

We are now going to compare our results for $\xi(0)$ with the Dalitz plot and the polarization measurements.

Comparison between the K_{μ_3} Dalitz plot and polarization measurements with the D.K.P. values for $\xi(0)$ (λ_0) and λ_- (λ_0') .-

Polarization measurements have in the recent past (before 1973) consistently given large negative values for $\xi(0)$. Only recently (see Appendix p., 100) not so large negative values have been published.

Notice that for $\xi(0)$, around -1, it generally implies that $\lambda_0 < 0$ for $\lambda_+ < 0.045$.

For $K^+_{\mu_3}$ polarization measurements the over-all value (before 1972), considering a linear fit, was[ii)]

$\xi(0) = -1.45 \pm .70$; a combination of $K^+_{\mu_3}$ and $K^0_{\mu_3}$ polarization experiments the values obtained[ii)] were

a) linear fit

$$\xi(0) = -2.0 \pm 0.7$$

$$\Lambda = 0.18 \pm 0.15$$

+ N.B. see p., 33

b) linear expension of f_0 and f_+

$\xi(0) = -2.2 \pm 0.80$

$\lambda_+ = 0.19 \pm 0.16$

$\lambda_0 = 0.01 \pm 0.09$.

with $\lambda_- \equiv 0$.

c) quadratic fit, with λ_+ and λ_+' fixed as in (70)

$\xi(0) = -1.9 \pm 0.6$

$\lambda_- = -0.06 \begin{smallmatrix} + 0.09 \\ - 0.05 \end{smallmatrix}$

For K_{μ_3} Dalitz plot measurements the values obtained[ii)] for K^+ together with K^0 were:

a) linear fit

$\xi(0) = -1.5 \pm 0.5$

$\lambda = 0.10 \pm 0.13$

b) linear expantion of f_0 and f_+ :

$\xi(0) = -1.6 \pm 0.6$

$\lambda_+ = 0.10 \pm 0.12$

$\lambda_0 = -0.03 \pm 0.08$

with $\lambda_- \equiv 0$.

c) quadratic fit; for λ_+ and λ_+' fixed as in (70)

$\xi(0) = -1.3 \pm 0.5$

$\lambda_- = 0.0 \begin{smallmatrix} + 0.13 \\ - 0.07 \end{smallmatrix}$

For a combination of Dalitz plot and polarization measurements the values obtained in an over-all fit[ii) was:

a) linear fit

$\xi(0) = -1.6 \pm 0.4$

$\Lambda = 0.11 \pm 0.10$

b) linear expansion of f_0 and f_+

$\xi(0) = -1.7 \begin{smallmatrix} + 0.35 \\ - 0.7 \end{smallmatrix}$

$\lambda_+ = 0.11 \begin{smallmatrix} + 0.014 \\ - 0.07 \end{smallmatrix}$

$\lambda_0 = -0.03 \begin{smallmatrix} + 0.09 \\ - 0.04 \end{smallmatrix}$

with $\lambda_- \equiv 0$.

c) quadratic fit, for λ_+ and λ'_+ fixed as in (70)

$\xi(0) = -1.50 \pm 0.40$

$\lambda_- = -0.03 \begin{smallmatrix} + 0.08 \\ - 0.05 \end{smallmatrix}$

or

$\lambda_0 = -0.11 \pm 0.03$

$\lambda'_0 = 0.0085 \pm 0.0065$

All the values given above show an excellent agreement with our values for $\xi(0)$ and λ_- ; $\xi(0) = -1.64$ and $\lambda_- = -0.018$

If we consider all the K^+_μ measurements performed up to the present[i) we can notice that of a total of eight experiments quoted in ref (i) only one[+] has obtained a positive $\xi(0)$.

+ N.B. Borreani, et al Phys. Rev 140 B,1686 (1965) gives $\xi(0) = 1.2 \pm 2.4$

On the other hand of the eight experiments three are not compatible with our value for $\xi(0)$: the latest experiment in the Data Card Listings[i)] (page 72), has such a big error band that it is also compatible.

For the $K^0_{\mu 3}$ polarization experiments we notice[i)] that there are five experiments on record (see page 80 of ref(i)).

$\xi(0)$	
-1.2 $\pm$ 0.5	Auerbach (1966)
-1.6 $\pm$ 0.5	Abrams (1968)
-1.81 $\pm$ 0.5	Longo (1969)
-0.385 $\pm$ .105	Sandweiss (1973)
0.178 $\pm$ .105	Shen (1975)

It can be seen that the two most recent (but they are also the ones that have the highest statistics) experiments, are far from a value of -1.6 for $\xi(0)$, while the other three are in good agreement with that value.

The individual Dalitz plot density measurements performed after 1972, are not in good agreement with our value of $\xi(0)$ = -1.6

To summarize:

The result given here for $\xi(0)$ (-1.64) and for λ_- (-0.0179) are in very good agreement with the values given in ref (ii). In other words with experiments performed before 1972.

This agreement does not continue for most of the (high statistics) experiments made after 1972, where one finds that λ_0 is mostly positive. Since almost all of the experimental fits assume that $\lambda_- = 0$; and consecuently, with only one exception+, all assume that f_+ and f_0 have a linear t-dependence, this means that we can not compare our value for λ_- (λ_- =-0.018) with experiment.

+ N.B. Dally et al see page 39

2-3) Discussion on Alternatives to Broken SU(3) Symmetry.-

The Cabibbo theory is supposed to be valid in the limit of exact SU(3)-symmetry. Since $m_K \neq m_\pi$, the symmetry is badly broken in K_{ℓ_3} decays. Unfortunately it is not known how the symmetry breaking should be taken into account.

The empirical supression factor for strangeness-changing weak amplitudes is the Cabibbo angle, which is considered to be related in an unknown way to symmetry-breaking effects in the hadronic weak current.

From the way the D.K.P. formalism leads to the mixing of the meson masses, we may observe that in K_{ℓ_3} decays, $f_+(0) \sin \theta_C$ could be connected with the meson masses. Strictly speaking one could say that the lepton masses are also involved since the minimum t value is $t = m_\ell^2$ and therefore, one is dealing with a factor $f_+(m_\ell^2)$. To disregard this effect in K_{e_3} may be justified on account of the smallness of the electron mass, but may not be such a good approximation for K_{μ_3} because $m_\mu \sim m_\pi$.

$f_+(0) \sin \theta_C$ is connected with K_{ℓ_3} decays, and $f_+(0) \cos \theta_C$ with π_{e_3}, i.e. $\pi^+ \to \pi^0 e^+ \nu$. In the D.K.P. formalism $f_+(0) = h(0) \sqrt{\frac{m_K}{m_\pi}}$ (see(52)).

Hence

$$h(0)\left[\frac{m_{K^+}}{m_{\pi^0}}\right]^{1/2} \rightarrow f_+(0)\sin\theta_c \qquad (K^+\rightarrow\pi^0 e^+\nu)$$

and

$$h(0)\left[\frac{m_{\pi^+}}{m_{\pi^0}}\right]^{1/2} \rightarrow f_+(0)\cos\theta_c \qquad (\pi^+\rightarrow\pi^0 e^+\nu)$$

where we assumed the same $h(0)$ for both reactions.

Let

$$h(m_\ell^2)\left(\frac{m_1}{m_2}\right)^{1/2} \equiv \Omega\,(m_1^2, m_2^2; m_\ell^2)$$

$$(m_1 \rightarrow m_2 + \ell\nu)$$

where m_1 and m_2 are the initial an final mesons respectively.

Hence+ $\Omega\,(m_{K^+}, m_{\pi^0})$ is associated with K_{e_3} decays and $\Omega\,(m_{\pi^+}, m_{\pi^0})$ with $\pi^+_{e_3}$ decays. We have the same function associated with each process. In an effort to determine Ω phenomenologically, we could (SU(3) symmetry considerations should not be used here) set

$$\Omega(m_{K^+}; m_{\pi^0}) \text{ and } \Omega\,(m_{\pi^+}, m_{\pi^0}) = \text{constant} \qquad (80)$$

+ N.B. we have neglected m_e^2 .

If by some means we could set

$$\Omega^2(m_K, m_{\pi^0}) + \Omega^2(m_{\pi^+}, m_{\pi^0}) = 1 \tag{81}$$

from (80) we obtain

$$\begin{aligned} h(0) &= \left[\frac{m_2}{m_1}\right]^{1/2} \Omega(m_1, m_2) \\ &= \left[\frac{m_2}{m_1}\right]^{1/2} \text{const.} \end{aligned} \tag{82-a}$$

Relation (81) is suggested from the good agreement with experiment of the Cabibbo relation.

$$\cos^2\theta_{exp} + \sin^2\theta_{exp} \sim 1$$

it is possible to find simple forms for the Ω' s such that (81) is obeyed, the simplest is

$$\Omega(m_1, m_2) \equiv \left(\frac{m_2}{m_1}\right) = \text{const.} \tag{82-b}$$

we have

$$\left(\frac{m_{\pi^0}}{m_{K^+}}\right)^2 + \left(\frac{m_{\pi^0}}{m_{\pi^+}}\right)^2 = 1.0098$$

which is not far from 1. This suggests that it may be possible to replace the Cabibbo assumption that requires the same parameter θ_c for strangeness-changing ($\sin\theta_c$) and strangeness-conserving ($\cos\theta_c$) processes, by the assumption that requires the same Ω function for both processes.

In general, different decay processes would in general lead to different Ω functions. However one could introduce the hypothesis that decay processes with the same number of meson states, irrespective of the electric charge of the mesons, would have the same Ω function. For example $K^0_{e_3}$ would lead to the same Ω function as $K^+_{e_3}$

The relation $m_K = m_\pi$ valid when SU(3) is a conserved symmetry, could be replaced , in the presence of symmetry breaking, by the weaker requirement that Ω for K decays is the same as for π decays.

We have seen that if one uses the D.K.P. formalism, mass factors appear, which were not present in the Klein-Gordon approach. If we adopt the point of view that their appearence is a reflection of a SU(3) symmetry breaking effect, an alternative view to SU(3) symmetry maybe possible in which all masses take their experimentally observed values, and such that the choice of the D.K.P. currents, automatically lead to the observed parameters for the K_{ℓ_3} form factors

This may lead to interesting future investigations concerning partial conservation of currents in the sense that matrix elements of the divergence of the D.K.P. current may have a "dip" or perhaps exact zeros.

With respect to this, we would like to mention an interesting point connected with the zero's of the divergence form factor. If $f_0(t)$ is given by the quadratic approximation

$$f_0(0)\left(1+\lambda_0 \frac{t}{m_\pi^2}+\lambda_0' \frac{t^2}{m_\pi^4}\right)$$

and we ask at what value it vanishes, then it is clear that this happens when t is given by

$$t=\left[-\frac{\lambda_0}{2\lambda_0'} \pm \frac{\sqrt{\lambda_0^2-4\lambda_0'}}{2\lambda_0'}\right] m_\pi^2 \tag{83}$$

This gives two values where the matrix element of the divergence of the vector current (involved in $K_{\ell 3}$ decays) vanishes. It may happen that these zeros are meaningful inside or very near the physical t-region: for example, if $f_0(t)$ has a double root, then (83) reduces to

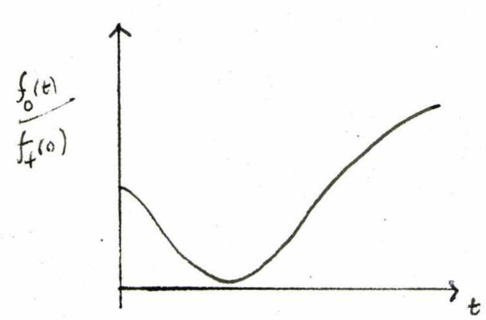
$$t=-m_\pi^2\left(\frac{\lambda_0}{2\lambda_0'}\right)=\left(-\frac{2}{\lambda_0}\right) m_\pi^2$$

if
$$\lambda_0^2 = 4\lambda_0'$$

This restriction is compatible with the experimental values for λ_0 and λ_0' (see p., 39). On the other hand this suggests a reason for λ_0' to be an order of magnitude smaller than $|\lambda_0|$ and on the other hand gives a reason why λ_0' has always been found (experimentally) to be positive.

The pole dominance model requires that λ_0 and λ_0' be given by
$$\lambda_0 = m_\pi^2 / m_K^2 \qquad (\lambda_0 > 0)$$
and
$$\lambda_0' = 2\left\{ m_\pi^2 / m_K^2 \right\}^2 = 2\lambda_0^2$$
also offers an explanation for the positivity of λ_0' and the different order of magnitude between λ_0' and λ_0 but is a relation that is considered to be valid outside, and not near the physical region of t. Actually if $m_K \sim 1$ Gev, $t \sim 8(m_K - m_\pi)^2$.

It may be possible to consistently unify this two points of view. After all, both are gross approximations dealing with two different t-regions. A behaviour of f_0 depicted in the following figure



indicates how this might arise.

This may open up a field of research to help us understand the peculiar behaviour of kaon decay processes through a study of the divergence form factor for small as well as relatively large values of t.

APPENDIX

A-1) Notation, Conventions and Formulae.-

a) Metric, γ- Matrices, Related Properties and Definitions.-

i) The metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} \equiv g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{84}$$

$$\left(g_{\mu\nu} g^{\mu\nu} = 1 \right)$$

This corresponds to

$$a \cdot b \equiv a^\mu b_\mu = a^0 b_0 - \vec{a} \cdot \vec{b}$$

$$p^\mu p_\mu = p^2 = m^2$$

$$p_1 \cdot p_2 = E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2$$

$$\partial_\mu A^\mu = \frac{\partial}{\partial t} A^0 - \vec{\nabla} \cdot \vec{A}$$

$$= \frac{\partial}{\partial t} A_0 + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$x^\mu \equiv (x^0, \vec{x}) = (t, \vec{x})$$

$$x_\mu = g_{\mu\nu} x^\nu = (t, -\vec{x})$$

$$\vec{x} \equiv (x, y, z) \equiv (x_1, x_2, x_3)$$

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

ii) The γ^μ matrices in the Dirac equation

$$i \gamma^\mu \partial_\mu \psi(x) - m \psi(x) = 0 \tag{85}$$

satisfy the anti-commutation relations

$$\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \tag{86}$$

A convenient representation of the Dirac matrices is +

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

where $\vec{\sigma}$ denotes the 2x2 Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In this realization for the γ matrices we have

$$\gamma^{0+} = \gamma^0 \quad , \quad \vec{\gamma}^{+} = -\vec{\gamma}$$

thus

$$\gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0 \tag{87}$$

The matrix γ_5 is defined as

$$\gamma_5 \equiv \gamma^5 \quad , \quad \gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma_5 = -i\,\gamma_0\,\gamma_1\gamma_2\gamma_3$$

+ N.B. In this section, we shall follow very closely the book:Elementary Particle Physics by S. Gasiorowicz J. Wiley & Sons. Inc. 1966.

iii) Traces of the γ matrices.-

Some traces that shall use and which can be verified from the explicit representation for the γ matrices above (they are true for any representation) are:

$$Tr\,\{\gamma_\mu\} = 0$$

$$Tr\,\{\gamma_\mu \gamma_\nu\} = 4\, g_{\mu\nu}$$

$$Tr\,\{\gamma_\mu \gamma_\nu \gamma_\alpha\} = 0$$

valid for $\mu, \nu, \alpha = 0, 1, 2, 3, 5$

and

$$Tr \left\{ \text{odd number of } \gamma\text{'s} \right\} = 0$$

$$Tr \left\{ \gamma_5 \quad \text{with less than four other } \gamma\text{'s} \right\} = 0$$

excluding γ_5 and where all indices are different.

Hence the trace of the product of 4 γ matrices differs from zero only if the indices are equal 2 by 2 :

$$Tr\,\{\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta\} = 4\left[g_{\mu\nu}\, g_{\alpha\beta} - g_{\mu\alpha}\, g_{\nu\beta} + g_{\mu\beta}\, g_{\nu\alpha}\right] \tag{88}$$

it follows that:

$$T_r\{\not a \not b \not c \not d\} = 4\left[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right]$$

$$(\not a \equiv a\cdot\gamma \equiv a^\mu \gamma_\mu) \tag{89}$$

Let $\varepsilon_{\mu\nu\alpha\beta}$ be the completly antisymmetric tensor with 4 indices. It is defined to have the properties

$$\varepsilon_{0123} = +1$$

$$\varepsilon_{\mu\nu\alpha\beta} = (-1)^n , \qquad \{\mu\nu\alpha\beta\} = \mathrm{Perm}\{0,1,2,3\}$$

where n is the number of permutations of the indices, therefore

$$\gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta = i\,\varepsilon_{\mu\nu\alpha\beta}\,\gamma_5$$

and

$$T_r\{\gamma_5 \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta\} = i\,\varepsilon_{\mu\nu\alpha\beta}\, T_r\{\gamma_5^2\}$$

$$= 4i\,\varepsilon_{\mu\nu\alpha\beta}$$

Hence

$$T_r\{\gamma_5 \not a \not b \not c \not d\} = 4i\,\varepsilon_{\mu\nu\alpha\beta}\, a_\mu b_\nu c_\alpha d_\beta \tag{90}$$

b) Normalizations and Projection Operators.-

i) Normalization of plane wave states

In calculating $|T|^2$ (transition matrix squared) for non-relativistic potential scattering, one may use plane waves described by

$$\psi = \frac{1}{\sqrt{V}} e^{i\vec{p}\cdot\vec{r}} \qquad \text{(non-rel)}$$

which are normalized to 1/V particles per unit volume. In the relativistic case, the wave functions are normalized to 1/V per unit volume and they are proportional to

$$\frac{1}{\sqrt{2EV}} e^{-ip\cdot x} \longrightarrow \begin{cases} \psi_{\text{boson}} \propto \left(\frac{1}{2EV}\right)^{1/2} e^{-ipx} \\ \psi_{\text{fermion}} \propto \left(\frac{1}{2EV}\right)^{1/2} e^{-ipx} \end{cases}$$

$$\frac{1}{\sqrt{2EV}} \longrightarrow \frac{1}{\sqrt{2E}} \;;\; (V=1)$$

the proportionality factor may be different for fermions and bosons or may be the same depending on the choice for the normalization of the spinors

ii) Spinor normalizations, completeness relation, and the energy projection operator.

The Dirac equation may be written for spinors $u\,(E>0)$ and $\vartheta(E<0)$ as:

$$(\not p - m)\, u(p) = 0 \tag{91-a}$$

$$(\not p + m)\, \vartheta(p) = 0 \tag{91-b}$$

where

$$\psi_{\text{Dirac}}(x) = \frac{1}{(2\pi)^{3/2}} \left(u(p)\, e^{-ipx} + \vartheta(p)\, e^{ipx} \right)$$

on the other hand the spinors are normalized to read

$$\bar{u}\, u = 1 \qquad\qquad u^{+} u = \frac{E}{m} \tag{92-a}$$

$$\bar{\vartheta}\, \vartheta = -1 \qquad\qquad \vartheta^{+} \vartheta = \frac{E}{m} \tag{92-b}$$

This choice leads to the completeness relation

$$\Lambda_{+} + \Lambda_{-} = \sum_{r} \left[u^{(r)}(p)\, \bar{u}^{(r)}(p) - \vartheta^{(r)}(p)\, \bar{\vartheta}^{(r)}(p) \right] = 1$$

where

$$(\Lambda_{+})_{\alpha\beta} \equiv \sum_{r} u^{(r)}_{\alpha}(p)\, \bar{u}^{(r)}_{\beta}(p)$$

$$(\Lambda_{-})_{\alpha\beta} \equiv -\sum_{r} \vartheta^{(r)}_{\alpha}(p)\, \bar{\vartheta}^{(r)}_{\beta}(p) \,,$$

and

$$(\Lambda_-)_{\alpha\beta} \equiv - \sum_r v_\alpha^{(r)}(p)\, \bar{v}_\beta^{(r)}(p)$$

are the projection operators Λ_+ and Λ_- and may be written in the form

$$\Lambda_+ = \frac{1}{2m}(m + \not{p}) \qquad (93\text{-a})$$

and

$$\Lambda_- = \frac{1}{2m}(m - \not{p}) \qquad (93\text{-b})$$

and obey the relations

$$\Lambda_\pm^2 = \Lambda_\pm \qquad \Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0$$

$$\Lambda_+ u = u \qquad \Lambda_+ v = 0$$

$$\Lambda_- u = 0 \qquad \Lambda_- v = v$$

iii) Trace relationship for the Dirac spinors.

We shall be interested in the calculation of the differential probability $d\Gamma(E_2, E_3)$ for the semi-leptonic decay $\underset{(1)}{K^+} \to \underset{(2)}{\pi^0}\, \underset{(3)}{\ell^+}\, \underset{(4)}{\nu}$, it will be necessary then to sum $|\langle f|T|i\rangle|^2$ (the square of the probability amplitude) over all polarization states of the leptons. In cases such as this one we can replace the sum over the polarization states of the lepton by a sum over the 4 basis states by introducing the projector on the positive (negative) energy which cancels the contribution of the negative (positive) energy states (see p.,59). Calculations are greatly simplified by this procedure.

Let us consider a complete set of spinor states constructed out of the four basis spinors describing states with positive and negative energy and spin up and down, and denote them by the symbol u'; then let

$$u' = |u'\rangle \quad \text{and} \quad \bar{u}' = \langle u'|$$

In the basis of the vectors $|u'\rangle$, a linear operator ϕ is represented by a matrix

$$\phi_{ij} \equiv \langle i|\phi|j\rangle$$

we assume the closing relation

$$\sum_{all} |i\rangle\langle i| = 1$$

where the sum is over all basis vectors. We have then

$$\sum_i \langle i | \not{\phi} | i \rangle = \sum_i \phi_{ii} \equiv T_r \{ \not{\phi} \}$$

in particular this implies that

$$\sum_{all} \langle u' | \not{\phi} | u' \rangle = T_r \{ \not{\phi} \} \tag{94}$$

in terms of the projection ($|i\rangle\langle i|$) operator for negative energy states Λ_- (we will need this since we consider the current $\ell^+\nu$):

$$\sum{}' \bar{\nu}_\ell \not{\phi} \nu_\ell = \sum_{all} u' \not{\phi} [-\Lambda_-] u' \tag{95}$$

$$= T_R \{ \not{\phi} (-1) \Lambda_- \}$$

the prime on the $\sum$ symbol means summation over polarization states. where the minus sign in front of Λ_- takes into account the normalization $\bar{\nu}\nu = -1$

c) Transition Matrix Element and the Differential Decay Probability.-

We have mentioned before that the weak Hamiltonian is assumed to be of the current X current form;

$$H_W = \frac{G}{\sqrt{2}} \int d^4x \; J_\lambda^{+}(x) \, J^\lambda(x) \;, \tag{96}$$

where $J_\lambda(x)$ is the total weak current. It is connected to the S-operator by the usual perturbation expansion

$$S = 1 - i \, H_W + O(G^2) \tag{97}$$

In addition to the matrix element of this S-operator we may also define matrix elements of the T-operator (to form the transition matrix element $\langle f | T | i \rangle$) by

$$\langle f | S | i \rangle = \delta_{fi} + i \, (2\pi)^4 \, \delta^{(4)}(P_f - P_i) \mathcal{N}_{ToT} \langle f | T | i \rangle \tag{98}$$

where $\mathcal{N}_{ToT}(E_i, E_f)$ is the following normalization factor

$$\mathcal{N}_{ToT} = \prod_{in} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_i}} \prod_{fin} \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_f}} \tag{99}$$

we shall be interested in dealing with the reaction $K^+ \to \pi^0 \ell^+ \nu$ we have :

$$\underset{(1)}{K^+} \to \underset{(2)}{\pi^0} \underset{(3)}{\ell^+} \underset{(4)}{\nu}$$

$$\mathcal{N}^{(K^+\to\pi^0\ell^+\nu)} = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_1}} \frac{1}{(2\pi)^{9/2}} \left(\frac{1}{\sqrt{2E_2}\sqrt{2E_3}\sqrt{2E_4}}\right)$$

in the K^+ center of mass system $(E_1 = m_K)$;

$$\mathcal{N}^{(K^+\to\pi^0 e^+\nu)} = \frac{1}{\sqrt{2m_K}} \left(\frac{1}{2\pi}\right)^6 \frac{1}{\sqrt{2E_2^*}\sqrt{2E_3^*}\sqrt{2E_4^*}} \qquad (100)$$

where the star in E_j^* means the energy of the j particle with respect to the kaon rest mass system.

The relation between the T-matrix element and the matrix element of the Hamiltonian (using (97) and (98)) is :

$$\delta_{fi} + i(2\pi)^4 \delta^{(4)}(P_f - P_i)\mathcal{N}\langle f|T|i\rangle = \delta_{fi} - i\int d^4x \langle f| H_W(x)|i\rangle$$

$$H_W(x) = \frac{G}{\sqrt{2}} J_\lambda^+(x) J^\lambda(x)$$

neglecting terms of order G^2 . This implies that

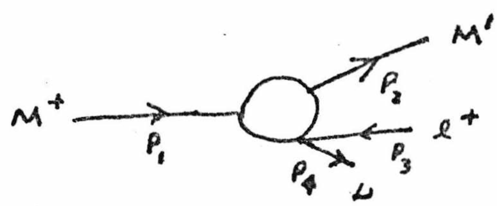
$$-i(2\pi)^4 \delta^{(4)}(P_f - P_i) \langle f | H_W^{(0)} | i \rangle = i(2\pi)^4 \delta^{(4)}(P_f - P_i) \mathcal{N} \langle f | T | i \rangle \tag{101}$$

The matrix element of the weak interaction Hamiltonian density can be written down in the most general form, following invariance arguments. If all the normalization factors $\left(\frac{m}{E}\right)$ for the fermions $\left(\frac{1}{2E}\right)$ for the bosons, are taken out, the rest is a Lorentz invariant and may involve (when fermions are involved) the spinors of the fermions (baryons and/or leptons).

In what follows we shall set up the formalism to decribe a general semi-leptonic meson decay of the form

$$M^+ \rightarrow M' + \ell^+ + \nu \tag{102}$$

it is depicted in the following figure



and we shall assume it is a $0^- \rightarrow 0^-$ mesonic transition.

The S-matrix element describing the particular kinematical configuration of process (102) is:

$$\langle P_2, P_3, P_4 | S_{eff} | P_1 \rangle = - \frac{iG}{\sqrt{2}} \int d^4x \, \langle P_2 | h^+_\lambda(x) | P_1 \rangle \langle P_3, P_4 | \ell^\lambda(x) | 0 \rangle$$

$$J^\lambda(x) = \underbrace{\ell^\lambda(x)}_{lep.} + \underbrace{h^\lambda(x)}_{had.} \tag{105}$$

The matrix element of the lepton current can directly be found to be

$$\langle P_3, P_4 | \ell^\lambda(x) | 0 \rangle = \frac{1}{(2\pi)^3} \left[\frac{m_\ell m_\nu}{E_3 E_4} \right]^{1/2} \bar{u}_\nu(P_4) \gamma^\lambda (1+\gamma_5) v_\ell(P_3) \, e^{ix(P_3+P_4)} \tag{106}$$

In order to use a common normalization for the leptons one assumes $m_\nu \neq 0$; a term with the form $\left[1/m_\nu \right]$ will appear in such a way that the neutrino mass will be canceled out.

The most general form of the mesonic matrix element is given by

$$\langle P_2 | h_\lambda(x) | P_1 \rangle = \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_1 \cdot 2E_2}} \phi_\lambda(P_1, P_2) \, e^{ix(P_2 - P_1)}$$

where $\phi_\lambda(P_1, P_2)$ is as yet an unspecified vertex function.+ All we know about ϕ_λ is that is a Lorentz vector. This matrix element of the hadron current can be parametrized in terms of two form factors (one for each momentum vector available).

A-2)Transition Matrix Element for the reaction $K^+ \to \pi^0 \ell^+ \nu$

We shall be interested in the Decay $K^+ \to \pi^0 \ell^+ \nu$, when we calculate (in the next section) the Dalitz plot density $\Gamma(E_2, E_3)$.

The interaction is a purely vector one. The matrix element of the hadronic current can be parametrized such that

$$\phi^\lambda(P_1, P_2) = P_1^\lambda f_1(t) + P_2^\lambda f_2(t)$$

$$= \frac{1}{2}\left[(P_1+P_2)^\lambda f_+(t) + (P_1-P_2)^\lambda f_-(t)\right] \tag{107}$$

where $$t \equiv (P_1-P_2)^\lambda (P_1-P_2)_\lambda = P_1^2 + P_2^2 - 2P_1 \cdot P_2$$

$$= (P_3+P_4)^2$$

$$= P_3^2 + P_4^2 + 2P_3 \cdot P_4$$

$$= m_\ell^2 + 2E_3 |\vec{P}_4| - 2\vec{P}_3 \cdot \vec{P}_4 \qquad (m_\nu = 0)$$

+ N.B. As it is known in this dissertation we are dealing precisely with the study of this function. We are of the opinion that the D.K.P. formalism (through the inclusion of the derivative current J_c^μ) is able, in a natural way, to yield more information on ϕ_λ.

and+ :

$$f_+ = f_1 + f_2$$

$$f_- = f_1 - f_2 \tag{108}$$

Therefore using (107) we can write the hadronic strangeness-changing vector current matrix element as

$$\langle \pi^0 (P_2) | V_\lambda^{(1)} (x) | K (P_1) \rangle =$$

$$\left(\frac{1}{2\pi}\right)^3 \frac{1}{\sqrt{4E_1E_2}} \left\{ (P_1+P_2)_\lambda f_+(t) + (P_1-P_2)_\lambda f_-(t) \right\} e^{ix(P_2-P_1)} \tag{109}$$

where $V_\lambda^{(1)}(0)$ symbolizes the strangeness changing vector current.

Subtituting eq.,(106) and eq., (109) in eq.,(101). we can write the invariant T-matrix element as

$$T \equiv \langle f | T | i \rangle = - \frac{G \sin\theta_c}{\sqrt{2}} (2\pi)^3 \sqrt{4E_1E_2} \langle \pi^0 | V_{(0)}^{(1)\lambda} | K^+ \rangle \bar{u}_\nu(P_4) \gamma_\lambda (1+\gamma_5) v_\ell(P_3)$$

$$\bar{u}_\nu (P_4) \equiv \bar{u}_\nu (P_4, \lambda_4) \; ; \quad v_\ell (P_3) \equiv v_\ell (P_3, \lambda_3) \tag{110}$$

where we have introduced the Cabibbo assumption i.e. $\sin\theta$.

+ N.B. Historically f_1 and f_2 were used first, but more recently f_+ and f_- where introduced.

Finally, taking into account that $p_2 = p_1 - p_3 - p_4$ and using the Dirac equation the invariant t-matrix element can be written in the form

$$T = \frac{G \sin\theta}{\sqrt{2}} \left(2 f_+(t)\right) \bar{u}_\nu(p_4)\,(1-\gamma_5)\,(\beta + \not{p}_1)\, v_\ell(p_3) \tag{111}$$

where

$$\beta = \frac{m_\ell}{2} \left(1 - \xi(t)\right)$$

and

$$\xi(t) \equiv \frac{f_-(t)}{f_+(t)}$$

A-3) Basic Theoretical Framework.-

The current-current theory of weak interactions[12)] is consistent with most of the present knowledge of weak interaction phenomena. The underlying assumption is that a perturbation expansion in terms of the weak coupling constant is possible due to the smallness of G.[+] Since a theory of higher order weak interactions is not available, it is not possible to write down an interaction Lagrangian (in the sense of electromagnetic interactions, for example, where this can be done) and the best one can do, is to write down an "effective" Lagrangian; the corresponding matrix elements are then assumed to describe lowest order weak processes. It is in this spirit that one may write the effective weak S-operator in the conventional way.

$$S = 1 - \frac{iG}{\sqrt{2}} \int d^4x\, d^4x'\, I^{+}_{\lambda}(x)\, \rho^{\lambda\lambda'}(x-x')\, I_{\lambda'}(x') \tag{112}$$

+ N.B. Here G (has dimensions, (energy)x(volume)) is the Fermi constant. The values given by K. Kleinknecht in the 17$^{\text{th}}$ International conference in H.E. Physics (London 1974) was for the Fermi coupling constant derived from the μ-decay
G_μ = $(1.43583 \pm .00003)\times 10^{-49}$ erg-cm^3
and from β-decays
G_β = $(1.413 \pm .002)\times 10^{-49}$ erg-cm^3

corresponding to an intrinsic four-fermion interaction. In principle mesons are taken into account if we assume that they can, in some way, be thought of as bound states of fermion-antifermion systems, like for example, the formation of a quark-antiquark state.

If we assume that intermediate boson do not exist one can approximate the weak interaction processes via the direct coupling of the weak current with it self. Usually it is assumed that the interactions involving the weak processes are local in nature.[+] For small momentum transfers the available empirical information has been in good accord with the assumption that the structure tensor has the most simple form:

$$\rho^{\lambda\lambda'}(x-x') = g^{\lambda\lambda'}\,\delta^{(4)}(x-x') \tag{113}$$

+ N.B. A fundamental problem of weak interactions is the discovery of a complete theory for which the Fermi theory (which in a certain sense, we are presently discussing) is the low energy limit. In the theory of weak interactions the Lagrangian is a phenomenological Lagrangian, a low energy description of weak interactions. The intermediate vector boson theory is an alternative low energy description of the weak interactions.

which leads to the four fermion Fermi model and which has been found to be a good first approximation for low energy processes. That is one way to couple directly the weak currents to each other. The self-coupled weak current $I_\lambda(x)$ is assumed to be separable into two parts and to be of the form

$$I_\lambda(x) = h_\lambda^{(hadrons)}(x) + \ell_\lambda^{(leptons)}(x) \tag{114}$$

and where it is assumed also, that to lowest order in the weak coupling G, the leptons may be treated as free particles, if we disregard the electromagnetic interactions. Which can be calculated to any order.

Usually the following is assumed: [12)]

(a) only vector and axial-vector currents contribute.

(b) phenomenological current-current V-A Lagrangian.

i) existence of a phenomenological lepton weak current.

ii) locality of lepton production; the fermions are produced at the same vertex.

iii) local non-derivative coupling.

iv) first order terms of weak interactions dominate.

(c) final state interactions are ignored.

(d) violation of C and P, but CP invariance.

(e) translation invariance and (restricted) Lorentz covariance.

In general terms it is expected that strong interaction renormalization effects will modify not only the effective coupling constants but the space-time structure as well of the weak processes. It has become standard practice to express the strong interaction renormalization effects, for hadronic currents, by form factors, which in general, depend on Lorentz invariant combination of four-momentum variables that characterize the particles participating in a given process.

If we agree to write the (bare) interaction Lagrangian in the current-current form, then the weak interaction process is usually classified into four different categories. They are, the purely-leptonic processes, the strangeness-conserving semi-leptonic processes, the strangeness-violating semi-leptonic processes, and the non-leptonic weak interactions. The two currents assumed to describe all the possible interactions are: the leptonic weak current

$$l_\lambda = \bar{\psi}_{\nu_e} \gamma_\lambda (1+\gamma_5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma_\lambda (1+\gamma_5) \psi_\mu \qquad (115)$$

where ψ_μ and $\psi_{\bar\nu_\mu}$ are the fields of muon and its anti-neutrino (and similarly for ψ_e and $\psi_{\bar\nu_e}$), which has been written in the usual maximum parity violating V-A form, with left handed neutrinos, and right handed anti-neutrinos. The other current contains the hadronic part of the weak interactions. With regard to semi-leptonic processes, in which we are really going to be interested here, there is evidence for the following isospin selection rules: $|\Delta\vec{I}| = 1$, $\Delta Y = 0$ and $|\Delta\vec{I}| = 1/2$, $\Delta Y \neq 0$ where Y is the hypercharge. We can take account of them by assuming that the hadronic weak current is the sum of two currents; an isovector current (I = 1), which is hypercharge-conserving $\Delta Y = 0$ and an isospinor current ($I = \frac{1}{2}$), that is not hypercharge-changing $\Delta Y \neq 0$. The first of these two currents transforms like an I-spin raising (or lowering) operator, i.e.; containing, for example, the algebraic properties of the charged pions, for instance, it can take an initial neutron state into a final proton state. The second current in the total hadronic current transforms as a v-spin raising operator, i.e.; containing the algebraic properties of the charged Kaons; as an example, this operator can take an initial lambda state into a final proton state.

All that has been said above, is the basis of the theory of Cabibbo[11)] which we shall briefly describe. It will be assumed that h_λ is formed out of a vector current and an axial-vector current, i.e.;

$$h_\lambda \equiv V_\lambda + A_\lambda \tag{116}$$

The basic Cabibbo postulates are:

a) The weak currents are members of a single self-conjugate octet. This characteristic has to do with the transformation properties of the current-operator under rotations in SU(3)-space. This means that the vector and axial-vactor currents transform like the infinitesimal generators of SU(3). The octet of vector currents consist of eight operators $V^{(j)}$ $(j = 1,\ldots,8)$, where

$$V_\lambda^{(1 \pm i2)} \equiv V_\lambda^{(1)} \pm i\, V_\lambda^{(2)} \tag{117}$$

carry the quantum numbers $S = 0$, $I = 1$, and $I_3 = \pm 1$, where S is the strangeness quantum number. They may be thought of as the $\pi^\pm$ members of the vector currents. On the other hand

$$V_\lambda^{(4\pm i5)} \equiv V_\lambda^{(4)} \pm i\, V_\lambda^{(5)} \tag{118}$$

carry the quantum numbers S=1, I=$\frac{1}{2}$, and $I_3 = \pm\frac{1}{2}$. They correspond to the $K^\pm$ members of the same octet. Similarly, $V_\lambda^{(6\pm i7)}$, $V_\lambda^{(3)}$ and $V_\lambda^{(8)}$ correspond respectively to the $K^0(\bar{K}^0)$, π^0 and η^0 members of the octet. The vector currents with upper indexes 4,5,6,7 charge strangeness, and are assumed to be conserved only when SU(3) is conserved. Furtheremore, the axial octet operators A_λ^j (j=i,....8) are assumed to exist. Again $A_\lambda^{(1)} \pm i\, A_\lambda^{(2)}$ are the $\pi^\pm$ members of the octet of axial currents, etc...

One usually assumes that the hadronic weak current h_λ is made out of two pieces, one corresponding to $\Delta S = 0$ and the other to $\Delta S = \pm 1$ one writes

$$V_\lambda = a\, V_\lambda(\Delta S = 0) + b\, V_\lambda(\Delta S = \pm 1) \tag{119}$$

and

$$A_\lambda = a'\, A_\lambda(\Delta S = 0) + b'\, A_\lambda(\Delta S = \pm 1) \tag{120}$$

where

$$V_\lambda(\Delta S=0) \cdot \quad V_\lambda^{(3)} \; ; \; V_\lambda^{(8)} \; ; \qquad V_\lambda^{(1)} \pm i V_\lambda^{(2)}$$

$$V_\lambda(\Delta S=\pm 1) \cdot \quad V_\lambda^{(6)} \pm i V_\lambda^{(7)} \quad ; \qquad V_\lambda^{(4)} \pm i V_\lambda^{(5)}$$

and similarly for A_λ . The following selection rules have been taked into account:

$$\Delta Q = \pm 1 \quad ; \qquad \Delta I_3 = \pm 1 \qquad \text{for} \quad \Delta S = 0$$

and

$$\Delta Q = \pm 1 \quad ; \qquad \Delta I_3 = \pm \tfrac{1}{2} \qquad \text{for} \quad \Delta S = \pm 1$$

b) Also Cabibbo makes the assumption that a = a' and b = b' .

c) It is required that the electromagnetic weak decay coupling G_μ has the same value as the complete hadronic weak coupling "strength". This is assumed to be expressed by $a^2+b^2=1$. This relation can be used to define the so called Cabibbo angle, i.e.; a = $\cos\theta$ and b = $\sin\theta$. Historically it was introduced to take into account the differences in "strength" between the $\Delta S=0$ processes (like $\pi^+ \to \pi^0 e^+ \nu_e$) and the $\Delta S = \pm 1$ processes (like $K^+ \to \pi^0 e^+ \nu_e$). Experimentally, the Cabibbo angle varies between 0.27 and 0.21 depending on the type of process observed. The 1974 world average fit to all the experiments gave a value centered around 0.230 (quoted by K. Kleinknecht, see foot note page 72.

To summarize, let us say that the neutrino fields (to first order in G) are considered to be free fields, on the other hand, the charged leptons are assumed to be interacting fields but only with respect to the electromagnetic interactions where perturbation theory is applicable. The hadron current cannot be specified in detail except in models (like the quark model): however, an important hypotesis is that it belongs to a multiplet of local currents fulfilling a strict group algebra at equal times. The group algebra of the total weak current reflects the universality of lepton and hadron couplings. Universality is interpreted to mean that the vector part of h_λ is coupled to l_λ with the same strenght as l_λ is coupled to itself.

Since the hadrons appear to have a non-trivial extension in space, this makes the current operator h_λ non-local. This may be expressed by the form factors. The very structure of the hadrons must be accounted for in terms of these form factors.

The main objective of this dissertation will be to determine part of the structure of the form factors appearing in the parametrization of the matrix element of the weak hadronic current that is responsable for the decay process $K \rightarrow \pi \ell \nu$. In the method we employ, SU(3) symmetry breaking is assumed from the start.

A-4) Semi-Leptonic Meson Decays.-

In this thesis we are mainly interested in two meson decays: $K \to \pi \ell \nu$ and $\pi \to \pi \ell \nu$. As we have seen the first involves a factor G sinθ the latter the factor G cosθ . In the SU(3) symmetry limit ($m_\pi = m_K$) the matrix elements for both decay processes must be essentially the same.

A-4a) Definition of the form factors for K_{ℓ_3} decays.-

The invariant transition matrix element for K_{ℓ_3} decay is given by (110).

$$-\frac{G}{\sqrt{2}} \sin\theta (2\pi)^3 \sqrt{4E_\pi E_K} \langle \pi | V_{\lambda(0)}^{4 \pm i5} | K \rangle \bar{u} \gamma^\lambda (1+\gamma_5) \nu$$

where we have a $0^- \to 0^-$ transition. It follows that only the vector part, of the hadronic strangeness-changing weak current is involved.

For K_{ℓ_3} decay, we have

$$\begin{array}{ccccccc} K & \to & \pi & + & \ell & + & \nu_\ell \\ (P_1) & & (P_2) & & (P_3) & & (P_4) \end{array}$$

where ℓ is an electron or muon, ν_ℓ their correspon-
ding neutrino, and K and π are charged or neutral kaons and pions respectively. The assumption that the lepton current enters the matrix element in the form

$$\bar{\psi}_\nu(p_4)\gamma^\lambda(1+\gamma_5)\psi_\ell(p_3)$$

sugests that a possible form for the hadronic vertex function is the vector function

$$<\pi(p_2)|\ V_\lambda\ |K(p_1)> = \phi_\lambda(p_1,p_2)$$

so that it can be contracted with the former current. The matrix element of the hadronic vector current is assumed to be a function of only p_1 and p_2, the kaon and the pion 4-momenta. One can parametrize the matrix element as follows (see (107)):

$$(2\pi)^3\sqrt{E_1E_2}<\pi(p_2)|V^\lambda|K> = f_1(t)\,p_1^\lambda + f_2(t)\,p_2^\lambda \tag{121}$$

where $t \equiv (p_1-p_2)^2$, and the form factors are functions of t only. This follows from Lorentz covariance.

The principal aim of experiments on K_{ℓ_3} has been to determine the form factors as functions of the invariant momentum transfer :

$$t \equiv (P_1 - P_2)^\lambda (P_1 - P_2)_\lambda \quad ; \qquad (122\text{-a})$$

$$t = m_K^2 + m_\pi^2 - 2 m_K E_2 \qquad (122\text{-b})$$

where t has been calculated in the coordinate system where the kaon is at rest, i.e. $P_1 = (\vec{0}, m_K)$. The range of t in the physical region is

$$m_\ell^2 \leq t \leq (m_K - m_\pi)^2 \qquad (123)$$

The form factors are assumed to be smooth functions of t and to have no singularities within the decay region.

The aspects of theoretical interest in form factors are related to the difficulty in the task of calculating their mathematical form from relativistic field theory.

In a phenomenological model one tries to formulate a technique of parametrizing the experimental data in the most economical way. Progress in the knowledge of the behaviour of the form factors with t is correlated with the progress of our theoretical methods.

The decomposition of the matrix element in two linear combinations of p_1^λ and p_2^λ is absolutely arbitrary. One usually introduces the form factors $\tilde{f}_+$ and $\tilde{f}_-$ defined by the hadron transition matrix element

$$(2\pi)^3 \sqrt{4E_1E_2} \langle \pi(p_2) | V_\lambda(x) | K(p_1) \rangle = e^{-i(p_1-p_2)x}$$
$$\times f_{4+i5,\,jK} \left[(p_1+p_2)_\lambda \tilde{f}_+(t) + (p_1-p_2)_\lambda \tilde{f}_-(t) \right] \tag{124}$$

where $f_{4+i5,\,jK}$ is the appropiate SU(3) structure constant. In the limit of exact SU(3) where the masses are degenerate and the current is conserved one finds

$$\tilde{f}_+(0) = \text{constant} \tag{125-a}$$

$$\tilde{f}_-(0) = 0 \tag{125-b}$$

More often one expresses the matrix element (124) in terms of the form factors $f_\pm(t)$ where

$$f_{\pm}(t) \equiv f_{4+i5,\,jk}\,\tilde{f}_{\pm}(t) \tag{126}$$

A convenient normalization for $\tilde{f}_{+}(0)$ is

$$\tilde{f}_{+}(0) = 1$$

exact SU(3) symmetry then implies

$$f_{+}(0) = \frac{1}{\sqrt{2}} \tag{127-a}$$

for $K^{\mp} \to \pi^{0}\, \ell^{\mp}\, \nu$

and

$$f_{+}(0) = 1 \tag{127-b}$$

for $K^{0} \to \pi^{\mp}\, \ell^{\pm}\, \nu$

In the limit of perfect SU(3) symmetry, V_{λ} is considered to be a conserved current, and therefore we expect that in (124), the following relations hold:

$$f_{+}(t) \neq 0 \quad , \quad f_{-}(t) = 0 \qquad t \neq 0$$

if we have a perfect SU(3) symmetry. This follows from the divergence of V_λ which can be seen to be proportional to

$$(p_1^2 - p_2^2)\, f_+(t) + t\, f_-(t)\,, \tag{128}$$

if the divergence vanishes then

$$f_-(t) = 0 \qquad \text{when} \qquad p_1^2 = p_2^2$$

The matrix element of the divergence of the current V_λ is given by

$$(2\pi)^3 \sqrt{4E_1E_2}\, \langle \pi(p_2) | \partial^\lambda V_\lambda(x) | K(p_1) \rangle = -i \left\{ (\omega_K^2 - \omega_\pi^2)\, f_+(t) + \right.$$

$$\left. + \; t\, f_-(t) \right\} e^{-ix\cdot(p_1 - p_2)}\,, \tag{129}$$

and in a sense, it can be seen to be of the order of the SU(3) breaking. A useful parametrization is defined by the, so called, divergence or scalar form factor

$$f_0(t) \equiv f_+(t) + \frac{t}{m_K^2 - m_\pi^2}\, f_-(t) \tag{130}$$

The matrix element of the divergence of the current can be parametrized as

$$i(2\pi)^3\sqrt{4E_1E_2}\,\langle \pi(P_2)|\partial^\lambda V_\lambda|K(P_1)\rangle = (m_K^2 - m_\pi^2)\, f_0(t) \tag{131}$$

It is interesting to note that f_+ and f_0 are connected with the 1^- and 0^+ transition amplitudes.

$f_+(t)$ is the form factor corresponding to the transverse current (spin one exchange), $f_0(t)$ is the form factor corresponding to the longitudinal part of the current (spin zero exchange). This can be seen as follows:

Expanding (124) in the centre of mass of the leptons (ℓ, ν), we have:

$$(2\pi)^3\sqrt{4E_1E_2}\,\langle \pi(P_2)|V_0(0)|K(P_1)\rangle = \frac{m_K^2 - m_\pi^2}{\sqrt{t}}\, f_0(t) \tag{132}$$

$$(2\pi)^3\sqrt{4E_1E_2}\,\langle \pi(P_2)|\vec{V}(0)|K(P_1)\rangle = 2\,\vec{P}_2\, f_+(t) \tag{133}$$

We can see that the first relation above is valid in the following way:

In the center of mass of the leptons

$$P_1 = P_2 + P_3 + P_4 \quad ; \quad \vec{P}_1 - \vec{P}_2 = 0 = \vec{P}_3 + \vec{P}_4$$

so

$$t = P_1^2 + P_2^2 - 2P_1P_2 = (P_1 - P_2)_0^2 - (\vec{P}_1 - \vec{P}_2)^2$$
$$= (P_1 - P_2)_0^2$$
$$= (E_1 - E_2)^2$$

therefore

$$\sqrt{t} = E_1 - E_2 \quad , \tag{134}$$

we have also

$$(E_1 + E_2)(E_1 - E_2) = (E_1 + E_2)\sqrt{t} = (m_K^2 - m_\pi^2)$$

hence

$$E_1 + E_2 = \frac{m_K^2 - m_\pi^2}{\sqrt{t}} \quad , \tag{135}$$

using now

$$f_-(t) = \frac{m_K^2 - m_\pi^2}{t}\left[f_0(t) - f_+(t)\right] \tag{136}$$

we obtain $$(p_1+p_2)_\lambda\, f_+(t) + (p_1-p_2)_\lambda\, f_-(t) =$$

$$= \left\{ (p_1+p_2)_\lambda + \frac{\omega_K^2-\omega_\pi^2}{t}(p_2-p_1)_\lambda \right\} f_+(t) + \frac{\omega_K^2-\omega_\pi^2}{t}(p_1-p_2)_\lambda\, f_0(t)$$

it can be seen that when $\lambda=0$, the above relation reduces to

$$\left\{ (E_1+E_2) + \frac{\omega_K^2-\omega_\pi^2}{t}(E_2-E_1) \right\} f_+(t) + \frac{\omega_K^2-\omega_\pi^2}{t}(E_1-E_2)\, f_0(t) \tag{137}$$

using eq (134)and eq (135), the term multiplying f_+ vanishes and we obtain (132).

To verify eq (133), all we have to do is note that

$$(2\pi)^3\sqrt{4E_1E_2}\,\langle \pi(p_2)|\vec{V}(0)|K(p_1)\rangle = \left\{ (\vec{p}_1+\vec{p}_2) + \frac{\omega_K^2-\omega_\pi^2}{t}(\vec{p}_2-\vec{p}_1) \right\} f_+(t) +$$

$$+ \frac{\omega_K^2-\omega_\pi^2}{t}(\vec{p}_1-\vec{p}_2)\, f_0(t)$$

will reduce to eq (133) if we notice that in our reference system

$$\vec{P}_1 - \vec{P}_2 = 0 \qquad \text{and} \qquad \vec{P}_1 + \vec{P}_2 = 2\vec{P}_2 + \vec{P}_3 + \vec{P}_4 = 2\vec{P}_2$$

In K_{ℓ_3} decays, one usually introduces the ratio of f_- and f_+ and define the parameter

$$\xi(t) \equiv \frac{f_-(t)}{f_+(t)} = \frac{m_K^2 - m_\pi^2}{t}\left\{\frac{f_0(t) - f_+(t)}{f_+(t)}\right\} \tag{138}$$

Due to the limited experimental statistics for K_{ℓ_3} events, one usually assumes some simple model for the variation of f_+ and f_- with respect to t. Since the range of t covered in K_{ℓ_3} decays is rather small, one may hope that the behaviour of the form factors in the physical region is relatively smooth. Historically, f_+ and f_- were assumed to have a linear t dependence. It is possible, however, to analyse the experiments in terms of f_+ and f_0 which are associated to the amplitudes that have definite spin and parity. We can start by assuming that f_+ and f_- have a linear t dependence and write

$$f_\pm(t) = f_\pm(0)\left(1 + \lambda_\pm \frac{t}{m_\pi^2}\right) \tag{139}$$

hence

$$\xi(t) = \xi(0) + \Lambda \frac{t}{m_\pi^2} \qquad (140)$$

with the following relations between the six parameters

$$\xi(0) = f_-(0) / f_+(0) \qquad (141)$$

and $\Lambda = \lambda_- - \lambda_+$ when $\lambda_+ << 1$.

On the other hand, if we assume that f_+ and f_0 have a linear t dependence, f_0 will have the form

$$f_0(t) = f_0(0) \left(1 + \lambda_0 \frac{t}{m_\pi^2}\right) \qquad (142)$$

and we can notice that these assumptions are inconsistent with a linear expansion for f_-.

Assuming the above parametrization for f_+ and f_- to analyse the data, it seems, that the analysis depends on λ_+ and ξ in a much more sensitive way that on λ_-, i.e. λ_- is determined with much less precision. Nevertheless the variation of f_- with t in $K_{\ell 3}$ decay cannot be ignored since the energy release is of the same order of magnitude as the hadron masses.

In this thesis, we take the point of view that the apparent discrepancy between the polarization, and branching ratio and Dalitz analysis, reflects insufficient consideration of a posible quadratic variation with t of the form factors. In this thesis we adopt the point of view that $\lambda_- \neq 0$. In our model, it turns out to be completely determined by the hadronic masses.+

The parameters $\lambda_\pm$ and $\xi(0)$, can be measured in different independent ways.

A-4b) Experimental determination of the K_{l_3} form factors

There are four types of measurements one can perform to obtain information.

(1) The measurement of Dalitz plot density for K_{e_3} decay allows one to determine $f_+(t)$. More specifically one can measure λ_+ and λ'_+ . Most experiments are concerned only with a linear dependence of $f_+(t)$ and measure λ_+ , see table (A-1) below.

In general, the overall agreement between various experiments has been rather poor, and thus it is not clear how meaningful is the world average. The inclusion of a quadratic term in the expansion of $f_+(t)$, i. e.

+ N.B. see main text eq., (55).

λ_+' , reduced the discrepancy between various experiments (see ref. (ii)).

(2) An analysis of the Dalitz plot+ in $K_{\mu 3}$ decay can yield information on both f_+ and ξ (or f_+ and f_0). Unfortunately, these two parameters seem to be strongly correlated, and the answers that one obtains depend on the parametrization used.

The most recent $K_{\mu 3}$ experiments have had a sufficiently large number of events to do a parameter independent fit, i.e. analysis of the distribution in a small enough band of E_2, so that the variation of the form factors in this band can be ignored. The fitted value of f_+^2 and ξ (or f_0) in individual bands can then be used to extract the t dependence of these form factors.

The recent K^+ and K^0 Dalitz plot analysis experiments are summarized in Table (B1,2) below. Special attention needs to be paid to the various assumptions on the dependence of the form factors for the selection of the experimental values in table (B-2) because of the previously mentioned correlation.

(3) For the branching ratio $\mathbb{R}_{(\mu/e)}$, defined by

$$\mathbb{R}_{(\mu/e)} \equiv \frac{\Gamma(K_{\mu 3})}{\Gamma(K_{e 3})} \tag{143}$$

+N.B. For the theoretical method see p., 104

the experimental data is summarized in table (C-1).

There are some inconsistencies between various measurements of $\Gamma(K^+ \to \pi^0 \mu^+ \nu)$. For example, the value of $\mathbb{R}_{(\mu,e)}$ for K^0 seems to be greater than the value of $\mathbb{R}_{(\mu,e)}$ for $K^\pm$. One can probably summarize by saying that the experimental situation on branching ratios is still far from settled, and is probably too early to conclude that the difference between $\mathbb{R}_{(\mu,e)}$ for K^0 and K^+ has been definitely established.

(4) The measurement of the muon polarization in $K \to \pi \mu \nu$ decay tests the extent to which time reversal invariance is a good symmetry principle, by requiring the polarization to be in the $\mu \pi$ plane. Furthermore, for every point of the Dalitz plot, there exists a direction along which the muon is totally polarized, and this direction is completely specified by the value of $\xi(t)$. Thus measurement of this direction corresponds to a measurement of $\xi(t)$. Several recent measurements of muon polarization for both K^0 and K^+; $\pi \mu \nu$ decays are summarized in table (D-1) below.

A-4c) Experimental data

a) K_{e_3} Dalitz plot and pion spectrum data.-

The f_- form factor can be neglected for K_{e_3} because it appears in the calculations multiplied by the kinematical factor $\left[\frac{m_e}{m_K}\right]^2$.

The f_+ term is usually assumed to be linear in t. The linear parametrization has usually the form

$$f_+(t) = f_+(0)\left[1 + \lambda_+ \frac{t}{m_\pi^2}\right]$$

The λ_+ values seem to be consistent and the average over all the spectra experiments is i)

$$K^+_{e_3} : \qquad \lambda_+ = 0.0285 \pm 0.0043 \qquad (144\text{-a})$$

$$K^0_{e_3} : \qquad \lambda_+ = 0.0288 \pm 0.0028 \qquad (144\text{-b})$$

Next we show a list of the recent data for $\pi e \nu$ decays involving K^+ and K^0 respectively.

TABLE A-1

$K^+_{e_3}$	λ_+ Data	$K^o_{e_3}$	λ_+ Data
(ii)	0.026 ± .008	(ii)	0.017 ± .007
(iii)	0.045 ± .015	(viii)	0.050 ± .010
(iv)	0.027 ± .010	(ix)	0.023 ± .005
(v)	0.029 ± .011	(x)	0.022 ± .014
(vi)	0.027 ± .008	(xi)	0.055 ± .010
(vii)	0.025 ± .007	(xii)	0.019 ± .013
		(xiii)	0.040 ± .012
		(xiv)	0.0270 ± .0028
		(xv)	0.044 ± .006
		(xvi)	0.0312 ± .0025

b) K_{μ_3} Dalitz plot and pion spectrum data.-

In this case the f_- form factor can not be neglected. This is usually taken into account through the ξ parameter defined as

$$\xi \equiv \frac{f_-}{f_+} = \xi(0)\left[\,1 + \Lambda\, t/m_\pi^2\,\right]$$

where $\xi(0) = f_-(0)\,/f_+(0)$ and $\Lambda \equiv \lambda_- - \lambda_+$ for $\lambda_+ \ll 1$

In most experiments it is assumed that f_+ depends linearly in t and that f_- is constant.

On the other hand it has been found that λ_+ and $\xi(0)$ are strongly correlated.

TABLE B-1

$K^+_{\mu_3}$	λ_+ Data
(ii)	0.043 ± .017
(xvii)	0.050 ± .018
(xviii)	0.024 ± .022
(xix)	0.006 ± .015
(xx)	0.025 ± .017
(xxi)	0.027 ± .019
(xxii)	0.025 ± .030

$K^0_{\mu_3}$	λ_+ Data
(ii)	0.030
(xxiii)	0.08 ± .01
(xxiv)	0.085 ± .015
(xxv)	0.11 ± .04
(xxvi)	0.046 ± .008
(xxvi)	0.076 ± .004
(xxvii)	0.030 ± .003
(xxviii)	0.046 ± .030

TABLE B-2

$K^+_{\mu_3}$	$\xi(0)$ Data
(ii)	
(xvii)	-1.1 ± .56
(xviii)	-0.62 ± .28
(xix)	0.45 ± .28
(xx)	-0.36 ± .40
(xxi)	-0.8 ± .8
(xxii)	-0.57 ± .24

$K^0_{\mu_3}$	$\xi(0)$ Data
(ii)	
(xxix)	-3.9 ± .4
(xxiii)	-0.68 ± .12
(xiv)	-1.5 ± .7
(xxv)	0.50 ± .61
(xxx)	1.00 ± .45
(xxvi)	-0.26 ± .21
(xxvi)	-2.41 ± .17
(xxvii)	-0.11 ± .07
(xxviii)	-0.25 ± .22

c) Branching ratio experiments.-

Here one determines either λ_+ or $\xi(0)$, since they cannot be known at the same time. The mathematical relationship between these parameters as given by Chounet, et al [ii)] is; for $K^{\pm}$:

$$\mathbb{R}_{(\mu,e)} = 0.646 + 3.801\ \lambda_+ + 6.812\ \lambda_+^2 + 0.127\ \xi(0) + 0.476\ \xi(0)\ \lambda_+ + 0.019\ \xi^2(0) \ / \ 1 + 3.700\ \lambda_+ + 5.478\ \lambda_+^2 \tag{145-a}$$

and for K^0:

$$\mathbb{R}_{(\mu,e)} = 0.645 + 3.546\ \lambda_+ + 5.932\ \lambda_+^2 + 0.125\ \xi(0) + 0.437\ \xi(0)\ \lambda_+ + 0.019\ \xi^2(0) \ / \ 1 + 3.457\ \lambda_+ + 4.779\ \lambda_+^2 \tag{145-b}$$

If we make the approximation that λ_+ is very small and neglect λ_+^2, the Particle Data Group [i)] give the following relationship for $K^{\pm}$:

$$\mathbb{R}_{(\mu,e)} = 0.6457 + 1.4115\ \lambda_+ + 0.1264\ \xi(0) + 0.01912\ \xi^2(0) + 0.0080\ \lambda_+\ \xi(0) \tag{146-a}$$

and for K^0:

$$\mathbb{R}_{(\mu,e)} = 0.6452 + 1.3162\ \lambda_+ + 0.1246\ \xi(0) + 0.0186\ \xi^2(0) + 0.0064\ \lambda_+\ \xi(0) \tag{146-b}$$

From this relationships we can determine λ_+ ($\xi(0)$) in terms of $\xi(0)$ (λ_+). Results are usually quoted as values of $\xi(0)$ at fixed λ_+. We list these results in the table below. The method used to evaluate these parameters, is to substitute the measured experimental number for $\mathbb{R}_{(\mu,e)}$ in (146-a) or (146-b). depending on wether one is dealing with $K^{\pm}$ or with K^{0}. In this compilation of data μ-e universality is assumed.

In a fit over the available data [i] (up to 1975), and assuming that λ_+ is fixed to the value, $\lambda_+ = 0.030$, the values of $\mathbb{R}_{(\mu,e)}$ and $\xi(0)$; λ_0 are:

TABLE-C

	$K^{\pm}$	K^{0}
$\mathbb{R}_{(\mu,e)}$	0.663 ± 0.18	0.696 ± 0.017
$\xi(0)$	-0.20 ± 0.15	0.09 ± 0.13
λ_0	0.014 ± 0.012	0.038 ± 0.011

where the λ_0 and $\xi(0)$ parameters are related as follows

$$\lambda_0 - \lambda_+ = \frac{m_\pi^2}{m_K^2 - m_\pi^2}\,\xi(0) \qquad (147)$$

TABLE C-1

$K^{\pm}$	$\xi(0)$		λ_+
(iii)	$-0.35 \pm .22$	for	$0.45 \pm .015$
(xvii)	$-0.81 \pm .27$	for	0.028
(xix)	$0.0 \pm .15$	for	0.03

K^{0}	$\xi(0)$		λ_+
(xxix)	$-0.5 \pm .5$	for	0.02
(xii)	$0.5 \pm .4$	for	$0.19 \pm .013$
(xxx)	$-0.08 \pm .25$	for	0.02

In an analysis made in 1970, ii) in a linear fit over the available $\mathbb{R}_{(\mu,e)}$ data they found for $K^{\pm}$;

$$\mathbb{R}^{(K^+)}_{(\mu,e)} = 0.626 \pm 0.019$$

that

$$\lambda_+ = 0.000 \quad ; \quad \xi(0) = -0.17 \pm 0.15$$

$$\lambda_+ = 0.030 \quad ; \quad \xi(0) = -0.53 \pm 0.18$$

$$\lambda_+ = 0.045 \quad ; \quad \xi(0) = -0.71 \pm 0.20$$

and for K^0

$$\mathbb{R}^{(K^0)}_{(\mu e)} = 0.684 \pm 0.018$$

they found that

$$\lambda_+ = 0 \qquad ; \qquad \xi(0) = 0.30 \pm 0.15$$

where the $\mathbb{R}_{(\mu e)}$ results reffer to mean values and $\lambda_- \equiv 0$

d) $K_{\mu 3}$ polarization experiments.-

The polarization measurements deal with the average of $\xi(t)$ over the t range of the experiment. They measure ξ directly.

The over all value given in ref. (ii) for experiments performed before 1972, was:

$$\xi(0) = -2.0 \pm .7$$
$$\Lambda = 0.18 \pm .15$$

where

$$\xi(t) = \xi(0)\,(1 + \Lambda t)$$

TABLE D-1

$K^{\pm}$	$\xi(0)$	K^{0}	$\xi(0)$
(ii)	$-1.45 \pm .70$	(xxix)	$-0.385 \pm .105$
(xxi)	$-0.64 \pm .27$	(xxx)	$0.178 \pm .105$
(vii)	-0.25 ± 1.20		

e) Over all fits to the K_{μ_3} experiments.-

The fits to all the different kinds of K_{μ_3} experiments are given in ref. (i). The values for λ_+, λ_0 and $\xi(0)$ are:

TABLE E

	$K^{\pm}_{\mu_3}$	$K^{0}_{\mu_3}$
λ_+	$0.027 \pm .008$	$0.034 \pm .006$
λ_0	$-0.009 \pm .007$	$0.021 \pm .006$
$\xi(0)$	$-0.45 \pm .14$	$-0.17 \pm .10$

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A-5) Dalitz Plot Density.-

The three body decay K_{ℓ_3} can be charactrized by only two variables: there are three final particles which means we have to consider 9 components of momenta but there are four conservation equations (three-momentum and energy), and the decay configuration is defined to within a rotation in space so this eliminates another three variables (the Euler angles), and consequently we are left with two independent variables. The variables used for the Dalitz plot are E_2 and E_3 . One obtains the Dalitz plot density; $\Gamma(E_2, E_3)$ by summing over the polarization states of the leptons and by integrating with respect to the neutrino momentum and the angular variables. A convenient way of expressing the matrix element of the hadronic current will be to write it in terms of P_1 , P_3 and P_4 instead of in terms of P_1 and P_2 . Thus we change the parametrization of the matrix element of the hadron current in (128) from

$$(P_1+P_2)^\lambda f_+(t) + (P_1-P_2)^\lambda f_-(t)$$

to

$$f_+(t)\left\{2P_1^\lambda - [1-\xi(t)](P_3+P_4)^\lambda\right\} \qquad (148)$$
$$(\xi \equiv f_-/f_+)$$

Then the transition amplitude can be written as fo - llows (see (110)):

$$-\frac{G\sin\theta_c}{\sqrt{2}}\left[2f_+\right]\bar{u}_\nu(p_4)\left\{\not{p}_1-\tfrac{1}{2}(1-\xi)(\not{p}_3+\not{p}_4)\right\}(1+\gamma_5)\,v_\ell(p_3) \tag{149}$$

We apply the Dirac equation to the term in $\not{p}_3+\not{p}_4$:

$$\bar{u}_\nu(1-\gamma_5)\underbrace{\not{p}_3\,v_\ell}_{-m_\ell v_\ell}+\underbrace{\bar{u}_\nu\not{p}_4}_{m_\nu\bar{u}_\nu\to 0}(1+\gamma_5)\,v$$

: γ_5 anticommutes with γ_μ .

hence, setting

$$\beta=\frac{m_\ell}{2}(1-\xi)$$

the amplitude can be written as

$$T=\frac{-G\sin\theta_c}{\sqrt{2}}(2f_+(t))\,\bar{u}_\nu(p_4)(1-\gamma_5)(\not{p}_1+\beta)\,v_\ell(p_3) \tag{150}$$

We have to calculate $|T|^2\equiv\langle f|T|i\rangle^{+}\langle f|T|i\rangle$.

$$T^{+} = \langle f | T | i \rangle^{+} = \langle i | T^{+} | f \rangle$$

$$= \frac{-G \sin\theta_c}{\sqrt{2}} (2 f_{+}^{*})\, v_{\ell}^{+} \underbrace{\{\beta^{*} + \not{p}_1^{+}\}}_{\gamma_0 \not{p}_1 \gamma_0} \underbrace{\{1-\gamma_5\}^{+}}_{1-\gamma_5} \underbrace{\gamma_0^{+}}_{\gamma_0} u_{\nu}$$

$$v_{\ell}^{+} \{\beta^{*} + \gamma_0 \not{p}_1 \gamma_0\}\{1-\gamma_5\}\, \gamma_0\, u_{\nu}$$

$$\bar{v}_{\ell} \{\beta^{*} + \not{p}_1\}\{1+\gamma_5\}\, u_{\nu}$$

$$= \frac{-G \sin\theta_c}{\sqrt{2}} (2 f_{+}^{*})\, \bar{v}_{\ell} (\beta^{*} + \not{p}_1)(1+\gamma_5)\, u_{\nu} \tag{151}$$

hence

$$|T|^2 = T^{+} T$$

$$= \frac{G^2 \sin^2\theta_c}{2} |2 f_{+}|^2\, \bar{v}_{\ell} (\beta^{*} + \not{p}_1)(1+\gamma_5)\, u_{\nu} \bar{u}_{\nu} (1-\gamma_5)$$

$$\times (\beta + \not{p}_1)\, v_{\ell}$$

$$= G'^{2}\, \bar{v}_{\ell} (\beta^{*} + \not{p}_1)(1+\gamma_5)\, u_{\nu} \bar{u}_{\nu} (1-\gamma_5)(\beta + \not{p}_1)\, v_{\ell} \tag{152}$$

where

$$G'^{2} \equiv \frac{G^2}{2} \sin^2\theta_c\, |2 f_{+}(t)|^2 \tag{153}$$

We seek the differential decay probability, $d\Gamma(E_1, E_2)$; the Dalitz plot density, so we have to sum $|T|^2$ over all polarization states of the leptons: for each lepton we have seen in (93) that.

$$\sum_r u_\alpha^{(r)}(p)\, \bar{u}_\beta^{(r)}(p) = \frac{1}{2m}(m + \not{p})_{\alpha\beta}$$

hence, for the neutrino we have that: $\dfrac{\not{p}_4}{2m_\nu}$

On the other hand, we replace the sum over the polarization states of the lepton by a sum over the 4 basis (spinors) states by the introduction of the projection on the positive energy states, which cancels the contribution of the negative energy states. This artifice makes it possible to considerably simplify the calculations.

We recall that in the basis of the vectors $|u'\rangle$ a linear operator ϕ is represented by a matrix

$$\phi_{ij} = \langle i | \phi | j \rangle$$

and

$$\sum_i \langle i | \phi | i \rangle = \sum_i \phi_{ii} = T_r \{\phi\}$$

We have to calculate $\sum |T|^2$ to obtain the Dalitz plot density. The sum extends over all the spin directions of the leptons. Hence

$$\sum |T|^2 = G' \sum \bar{v}_\ell (\beta^* + \not{p}_1)(1+\gamma_5) \underbrace{u_\nu \bar{u}_\nu}_{\not{p}_4/2m_\nu} (1-\gamma_5)(\beta + \not{p}_1) v_\ell$$

$$= -G' \, T_r \left\{ (\beta^* + \not{p}_1)(1+\gamma_5) \frac{\not{p}_4}{2m_\nu} (1-\gamma_5)(\beta+\not{p}_1) \left[\frac{m_\ell - \not{p}_3}{2m_\ell} \right] \right\}$$

where we used (95). It follows that

$$\sum |T|^2 = \frac{-G'}{4m_\nu m_\ell} T_r \left\{ (\beta^* + \not{p}_1) \not{p}_4 \underbrace{(1-\gamma_5)^2}_{2(1-\gamma_5)} (\beta + \not{p}_1)(m_\ell - \not{p}_3) \right\}$$

$$= \frac{-G'}{2m_\nu m_\ell} T_r \left\{ (\beta^* + \not{p}_1) \not{p}_4 (1-\gamma_5)(\beta + \not{p}_1)(m_\ell - \not{p}_3) \right\} \tag{154}$$

To calculate this, the traces of the products of γ matrices are involved. The properties and relations that we need are in pp., 55-58

The only non-zero terms are those containing 2 or 4γ– matrices, or the product of 4γ– matrices by γ^5. The latter term has the form

$$T_r \{ \not{p}_1 \not{p}_4 \gamma_5 \not{p}_1 \not{p}_3 \} = T_r \{ \gamma_5 \not{p}_1 \not{p}_4 \not{p}_1 \not{p}_3 \}$$

the trace of which is

$$4 i \varepsilon_{\mu\nu\alpha\beta} \, p_1^\mu p_4^\nu p_1^\alpha p_3^\beta = 0$$

There remains the terms

$$T_r \{ m_\ell \not{p}_1 \not{p}_4 \beta \} = 4 m_\ell \beta \; p_1 \cdot p_4$$

$$T_r \{ \beta^* \not{p}_4 \not{p}_1 m_\ell \} = 4 m_\ell \beta^* \; p_4 \cdot p_1$$

which on adding gives

$$8 m_\ell \, Re \beta \;\; p_1 \cdot p_4$$

and

$$- t_r \{ |\beta|^2 \not{p}_4 \not{p}_3 \} = - 4 |\beta|^2 \, p_4 \cdot p_3$$

$$- T_r \{ \not{p}_1 \not{p}_4 \not{p}_1 \not{p}_3 \} = - 4 \left[2 (p_1 \cdot p_4)(p_1 \cdot p_3) - m_K^2 (p_4 \cdot p_3) \right)$$

hence

$$\sum |T|^2 = -\left[\frac{2G'}{m_\ell m_\nu}\right] \left[2\,(m_\ell \, R_e \beta - P_1\cdot P_3)\, P_1\cdot P_4 + (m_\ell^2 - |\beta|^2)\, P_4\cdot P_3\right] \tag{155}$$

We evaluate $\sum |T|^2$ in the K^+ rest system, all the energies evaluated here, except for $E_1 = m_K$, will be designated as E^* namely: $P_1\cdot P_3 = m_K E_3^*$, $P_1\cdot P_4 = m_K E_4^*$

We have the following relations

$$P_3^2 + P_4^2 + 2P_3\cdot P_4 = m_K^2 + m_\pi^2 - 2m_K E_2^* \Rightarrow P_3\cdot P_4 = m_K \left[E_2^{max} - E_2^*\right]$$

where

$$E_2^{max} = \frac{m_K^2 + m_\pi^2 - m_\ell^2}{2\, m_K} \tag{156}$$

is the maximum energy of the pion. The minimum energy is

$$E_2^{min} = m_\pi^2 \tag{157}$$

therefore

$$\sum |T|^2 = \frac{-2G'}{m_\ell m_\nu} \left\{ 2\,(m_\ell \, R_e \beta - m_K E_3^*)\, m_K E_4^* + (m_\ell^2 - |\beta|^2)\, E_2' m_K \right\} \tag{158}$$

where

$$E_2' = E_2^{max} - E_2^*$$

and $\sum|T|^2$ can be written in the following form

$$\sum|T|^2 = \frac{2G' m_K}{m_\nu m_\ell}\Big\{[2 E_3^* E_4^* - m_K E_2'] + m_\ell^2\Big[\frac{E_2'}{4} - E_4^*\Big]$$

$$+ R_e \xi\Big[m_\ell^2\Big(E_4^* - \frac{E_2'}{2}\Big)\Big] + |\xi|^2 \frac{m_\ell^2 E_2'}{4}\Big\} \qquad (159)$$

$$\beta = \frac{m_\ell}{2}(1-\xi) \qquad\qquad |\beta|^2 = \frac{m_\ell^2}{4}|1-\xi|^2$$

$$R_e\beta = \frac{m_\ell}{2}(1 - R_e\xi) \qquad\qquad = \frac{m_\ell^2}{4}\big[1+|\xi|^2 - 2R_e\xi\big]$$

hence we arrive to the expression

$$\sum|T|^2 = \frac{G^2 \sin^2\theta_c |2f_+|^2}{m_\ell\, m_\nu}\Big\{A + B R_e\xi + C|\xi|^2\Big\} m_K \qquad (160)$$

$$A = m_K[2E_3^* E_4^* - m_K E_2'] + m_\ell^2\Big(\frac{E_2'}{4} - E_4^*\Big)$$

$$B = m_\ell^2\Big[E_4^* - \frac{E_2'}{2}\Big]$$

$$C = \frac{m_\ell^2 E_2'}{4}$$

We can now obtain the expression for the differential decay probability: the Dalitz plot density

$$d\Gamma(E_2^*, E_3^*) = \frac{G^2 \sin\theta_c |2f_+|^2}{16\pi^3}\Big[A + B R_e\xi + C|\xi|^2\Big] dE_2^* dE_3^* \qquad (161)$$

It follows that when $m_\ell = m_{electron}$ B and C, in the above expression are small and thus ξ plays no role in K_{ℓ_3} decays. This is as long as f_- is small with respect to $m^2_{electron}$

Since the form factors are functions of $t =$ $m_K^2 + m_\pi^2 - 2\, m_K E_2^*$, and therefore of E_2^*, it is also possible to determine their t dependence without prior parametrization. Thus, one can do a fit through the events in a band of constant E_2^* on the Dalitz plot.

A-6) The Pion Energy Spectrum

We would like to calculate the π^0 energy spectrum. It is not necessary to assume any particular t-dependence for the form factors, because f_+ and f_- are functions only of, t , the π- energy, and the integration is with respect to E_3^* the electron (muon) energy.

We calculated (161)

$$d\Gamma = \frac{-G^2 \sin\theta |2f_+|^2}{16\pi^3} \left[2 \left(m_\ell \, \mathrm{Re}\,\beta - m_K E_3^* \right) m_K E_4^* + \left(m_K^2 - |\beta|^2 \right) m_K E_2' \right] dE_2^* \, dE_3^* \tag{162}$$

since

$$E_4^* = m_K - E_2^* - E_3^*$$

we obtain

$$\frac{d\Gamma(E_2^*, E_3^*)}{dE_2^* \, dE_3^*} = \frac{-G^2 \sin^2\theta |2f_+|^2}{16\pi^3} \Big[2 m_K^2 \, E_3^{*2} - E_3^* \left\{ 2 m_K^2 (m_K - E_2^*) + 2 m_\ell \, m_K \, \mathrm{Re}\,\beta \right\}$$
$$+ 2 m_\ell \, \mathrm{Re}\,\beta (m_K - E_2^*) m_K + (m_K^2 - |\beta|^2) m_K E_2' \Big] \tag{163}$$

$$\frac{d\Gamma(E_2^*)}{dE_2^*} = \frac{-G^2 \sin^2\theta |2f_+|^2}{16\pi^3} \int \left[a E_3^{*2} + b E_3^* + c \right] dE_3^* \tag{164}$$

it follows that

where

$$a = 2\, m_K^2$$

$$b = -2 m_K \left\{ m_K (m_K - E_2^*) + m_\ell m_K \, R_e \beta \right\} \tag{165}$$

$$c = m_K \left\{ 2 m_\ell \, R_e \beta \, (m_K - E_2^*) + (m_K^2 - |\beta|^2) E_2' \right\}$$

if we now set

$$E_3^{*\,max} = \frac{(m_K - E_2^* + |\vec{P}_2^*|)^2 + m_\ell^2}{2(m_K - E_2^* + |\vec{P}_2^*|)} \equiv \Delta_1 \tag{166-a}$$

$$E_3^{*\,min} = \frac{(m_K - E_2^* - |\vec{P}_2^*|)^2 + m_\ell^2}{2(m_K - E_2^* - |\vec{P}_2^*|)} \equiv \Delta_2 \tag{166-b}$$

and calculate the following integral

$$\int_{\Delta_2}^{\Delta_1} (a\, E_3^{*2} + b\, E_3^* + c)\, dE_3^* = \frac{a}{3}\left[\Delta_1^3 - \Delta_2^3\right] + $$

$$+ \frac{b}{2}\left[\Delta_1^2 - \Delta_2^2\right] + c\left[\Delta_1 - \Delta_2\right] \tag{167}$$

$$= (\Delta_1 - \Delta_2)\left[\frac{a}{3}\left\{(\Delta_1 + \Delta_2)^2 - \Delta_1\Delta_2\right\} + \frac{b}{2}(\Delta_1 + \Delta_2) + c \right]$$

using the following relations

$$\Delta' \equiv E_2' + \frac{m_\ell^2}{2m_K} \tag{168-a}$$

$$\Delta_1 - \Delta_2 = |\vec{P}_2^*| \frac{E_2'}{\Delta'} \tag{168-b}$$

$$\Delta_1 + \Delta_2 = (m_K - E_2^*)\left(2 - \frac{E_2'}{\Delta'}\right) \tag{168-c}$$

$$(\Delta_1+\Delta_2)^2 - \Delta_1\Delta_2 = \frac{3}{4}(m_K - E_2^*)^2\left(2 - \frac{E_2'}{\Delta'}\right)^2 + \frac{|\vec{P}_2^*|^2}{4}\left(\frac{E_2'}{\Delta'}\right)^2 \tag{168-d}$$

we can write the integral in the form

$$|\vec{P}_2^*|\left[\frac{E_2'}{\Delta'}\right]^2\left[\frac{a}{3}\left\{\left(\frac{3}{4}(m_K - E_2^*)^2\right)\left(\frac{4\Delta'}{E_2'} + \frac{E_2'}{\Delta'} - 4\right) + \frac{|\vec{P}_2^*|^2}{4}\left(\frac{E_2'}{\Delta'}\right)\right\} + \frac{b}{2}\left\{(m_K - E_2)\left(2\frac{\Delta'}{E_2'} - 1\right)\right\} + c\frac{\Delta'}{E_2'}\right] \tag{169}$$

substituting this expression in (164), gives for the pion energy spectrum

$$\frac{d\Gamma(E_2^*)}{dE_2^*} = \frac{G^2\sin^2\theta|2f_+|^2}{16\pi^3}|\vec{P}_2^*|\left(\frac{E_2'}{\Delta'}\right)^2\left[\frac{a}{3}\left\{\left(\frac{3}{4}(m_K-E_2^*)^2\left[\frac{4\Delta'}{E_2'}+\frac{E_2'}{\Delta'}-4\right]\right.\right.\right. + \frac{|\vec{P}_2^*|^2}{4}\left(\frac{E_2'}{\Delta'}\right)\Big\} + \frac{b}{2}\left\{(m_K-E_2^*)\left(2\frac{\Delta'}{E_2'}-1\right)\right\} + c\frac{\Delta'}{E_2'}\Big] \tag{170}$$

For $K^+_{e_3}$ decay, the mass of the lepton can be neglected, thus (168) reduces to

$$\Delta' = E_2' \qquad (171\text{-a})$$

$$\Delta_1 - \Delta_2 = |\vec{p}_2^{\,*}| \qquad (171\text{-b})$$

$$\Delta_1 + \Delta_2 = (m_K - E_2^*) \qquad (171\text{-c})$$

$$(\Delta_1 + \Delta_2)^2 - \Delta_1 \Delta_2 = 3/4\,(m_K - E_2^*)^2 + \frac{|\vec{p}_2^{\,*}|^2}{4} \qquad (171\text{-d})$$

On the other hand (165) becomes

$$a = 2m_K^2$$

$$b = -2m_K^2\,(m_K - E_2^*) \qquad (172)$$

$$c = m_K^3\,E_2'$$

where

$$E_2' = \frac{m_K^2 + m_\pi^2}{2m_K} - E_2^* \qquad (173)$$

All these expressions can be used to write the pion energy spectrum for $K^+_{e_3}$ decay

$$\frac{d\Gamma}{dE_2^*} = \frac{G^2 \sin^2\theta\,|2f_+|^2\,m_K\,|\vec{p}_2^{\,*}|^3}{48\,\pi^3} \qquad (174)$$

From this expression we can obtain information about $f_+(t) = f_+(E_2^*)$; even without any assumption on the form of the t-dependence. All one has to do is to fix E_2^* . This observation is usually put into effective use at the time of the experimental measurements. This has been discussed earlier in the section dealing with the experimental determination of the K_{ℓ_3} form factors.

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