Correlation and Inequality in Weighted Majority Voting Games

Sanjay Bhattacherjee · Palash Sarkar

Abstract In a weighted majority voting game, the weights of the players are determined based on some socio-economic parameter. A number of measures have been proposed to measure the voting powers of the different players. A basic question in this area is to what extent does the variation in the voting powers reflect the variation in the weights? The voting powers depend on the winning threshold. So, a second question is what is the appropriate value of the winning threshold? In this work, we propose two simple ideas to address these and related questions in a quantifiable manner. The first idea is to use Pearson's Correlation Coefficient between the weight vector and the power profile to measure the similarity between weight and power. The second idea is to use standard inequality measures to quantify the inequality in the weight vector as well as in the power profile. These two ideas answer the first question. Both the weight-power similarity and inequality scores of voting power profiles depend on the value of the winning threshold. For situations of practical interest, it turns out that it is possible to choose a value of the winning threshold which maximises the similarity score and also minimises the difference in the inequality scores of the weight vector and the power profile. This provides an answer to the second question. Using the above formalisation, we are able to quantitatively argue that it is sufficient to consider only the vector of swings for the players as the power measure. We apply our methodology to the voting games arising in the decision making processes of the International Monetary

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Fund (IMF) and the European Union (EU). In the case of IMF, we provide quantitative evidence that the actual winning threshold that is currently used is sub-optimal and instead propose a winning threshold which has a firm analytical backing. On the other hand, in the case of EU, we provide quantitative evidence that the presently used threshold is very close to the optimal.

Keywords Voting games · Weighted majority · Power measure · Correlation · Inequality · Gini index · Coefficient of variation.

1 Introduction

Voting is arguably the most important aspect of decision making in a democratic set up. A committee settles an issue by accepting or rejecting some resolution related to the issue. While unanimity or consensus is desirable, this may not always be possible due to the conflicting interests of the different committee members. In such a situation, a voting procedure among the members is used to either accept or reject a resolution. A resolution is accepted or passed, if a certain number of persons vote in its favour, else it fails and is rejected.

In its basic form, each committee member has a single vote. Many scenarios of practical interest, on the other hand, assign weights to the committee members. These weights need not be the same for all the members. In the context of weighted voting, a resolution is accepted, if the sum total of the weights of the members who vote in its favour cross a previously decided upon threshold. A common example of weighted voting is a company boardroom, where the members have weights in proportion to the shares that they hold in the company. Important examples of weighted voting in the context of public policy are the European Union (EU) and the International Monetary Fund (IMF).

Voting procedures have been formally studied in the game theory literature under the name of voting games. Due to its real life importance, weighted majority voting games have received a lot of attention. In the literature on voting games, the members are called players. One of the basic questions is how much influence does a player have in determining the outcome of a voting procedure? In other words, what is the power of a player in a voting game? In quantitative terms, it is desirable to measure the power of a player in a voting game by assigning a non-negative real number to the player. A power measure assigns such a number to each player in the game. This leads to the basic question of what constitutes a good measure of power of a player in a voting game. The literature contains a number of power measures. Each one of these measures aim to capture certain aspects of the informal notion of power in a voting game. We refer to Felsenthal and Machover (1998) for a comprehensive discussion to voting games and the various power measures. An introduction to the area can be found in Chakravarty et al. (2015).

Consider the setting of weighted majority voting games. For any such game, the players are assigned weights based on socio-economic parameters. As a

result, there is a variation in the weights of the players. Further, given any power measure, we obtain a variation in the powers of the different players. It is well known that the variation in the voting powers does not necessarily reflect the variation in the weights. In this context, the following three questions can be formulated.

1. To what extent does the variation in the voting powers reflect the variation in the weights?
2. Is the inequality present in the weights preserved in the voting powers?
3. How does the value of the winning threshold (i.e., the threshold which is required to be crossed for a motion to be passed) affect the above two questions?

This work addresses the above questions. The questions posed are not merely theoretical. Similar questions have been posed in Leech (2002b) in the context of measurement of voting power in IMF. For example, the following text fragments appear in Leech (2002b).

> "... weighted voting raises the important question of whether the resulting inequality of power over actual decisions is precisely what was intended for the relationship between power and contribution."
>
> "How does the voting power of individual countries compare with their nominal votes? To what extent is the degree of inequality in the distribution of votes reflected in the distribution of voting power?"
>
> "Different types of decisions use different decision rules, some requiring a special supermajority. What effect do different decision rules have on the distribution of power and also on the power of the voting body itself to act?"

The work Leech (2002b) makes a qualitative analysis of the above issues. Our work allows a quantitative analysis of these issues. In more details, our work makes the following contributions.

Measurement of similarity between weights and voting powers. We propose the use of Pearson's correlation coefficient as a measure of similarity between the weight vector of the players and the vector of voting powers of the players.

Measurement of inequality in weights and voting powers. There is a large literature on the measurement of inequality in a vector of values obtained from measurement of various social parameters. A survey on measurement of inequality appears in Cowell (2016). The variation of the values in such a vector is captured by an inequality index. A number of inequality indices have been proposed in the literature. We propose the use of such inequality indices to measure the inequality in the weights and also in the voting powers. This allows the comparison of the inequality present in the weights to that present in the voting powers.

Winning threshold as a controllable parameter. Our formalisations of both the similarity between the weights and the voting powers as well as the measurement of inequality in the voting powers have the winning threshold as a parameter. By varying this parameter, both the weight-power similarity and the voting power inequality can be controlled. So, given a vector of weights, the winning threshold can be set to a certain value to maximise the weight-power similarity or, to minimise the difference between the inequality in the weights and the inequality in the voting powers.

In this context, we would like to discuss the broader issue of designing games to achieve certain desirable power profiles. This is often called the inverse problem for voting games. Usually the goal is to determine a set of weights which result in the target powers. For example, in the context of the IMF voting game, an iterative algorithm to determine weights has been proposed in Leech (2002b). There is one major drawback of this approach. As mentioned earlier, in a weighted majority voting game, the weights often represent a socio-economic parameter. When the weights are artificially obtained (say, using an iterative algorithm), their interpretation in the socio-economic context is lost. It then becomes hard to provide a natural justification of the weights.

Our approach of having the winning threshold as a controllable parameter provides an alternative method of designing games. For the complete specification of a game, both the weights and the winning threshold need to be specified. In our approach, the weights do not change and hence they retain their original interpretation arising from the background socio-economic application. We only suggest tuning the winning threshold so that the resulting power profile is "imbued" with the intuitive natural justification of the weights. Games designed using such an approach can be much better explained to the general public than games where the weights are artificially obtained.

Detailed study. We consider seven different voting power measures and two different inequality indices. We show that the scaling invariance property of an inequality measure as well as that of the Pearson's correlation coefficient divides the voting power measures into three groups. The non-normalised Banzhaf measure, the normalised Banzhaf index and the two Coleman measures fall into one group; the public good measure and public good index defined by Holler fall into a second group and the Deegan-Packel measure is in the third group. We show that any two power measures in the same group have the same behaviour with respect to both the similarity index and the inequality index. This brings down the complexity of the analysis.

There has been a lot of discussion in the literature on the comparative suitabilities of the Banzhaf and the Coleman indices Banzhaf (1965); Brink and Laan (1998); Coleman (1971); Dubey and Shapley (1979); Laruelle and Valenciano (2001); Lehrer (1998); Laruelle and Valenciano (2011); Barua et al. (2009). This discussion has both been qualitative and also formal in the sense of axiomatically deriving the indices Brink and Laan (1998); Lehrer (1998). Our work provides a new perspective to this discussion. The stand-alone values

of the powers of the players as measured by any power measure are perhaps not of much interest by themselves. It is only in a relative sense that they acquire relevance. There are two ways to consider this relative sense, in comparison to the weights and in comparison among themselves. We propose to quantify the relative notion in comparison to the weights by the correlation between the weight vector and the power profile and to quantify the relative values of the powers among themselves by an appropriate inequality score. Under both of these quantifications, we prove that the Banzhaf and the Coleman power measures turn out to be the same. Based on this result, we put forward the suggestion that there is perhaps no *essential* difference between these power measures. It is sufficient to consider only the swings for the different players as was originally proposed by Banzhaf (and sometimes called the raw Banzhaf measure). While this may sound a bit radical, our analysis based on correlation and inequality does not leave scope for any other consideration. It is of course possible that there is some other quantifiable way of distinguishing between the relative spreads of the Banzhaf and the Coleman measures. This can be a possible future research question.

The literature contains a number of voting power measures which have been proposed as fundamentally different from the swing based Banzhaf measure. Intuitive arguments have been forwarded as to why these measures are appropriate for certain applications. In our opinion, a basic requirement for any power measure is to reflect the "content" of weights. In addition to the Banzhaf measure, we have also considered the Holler measures and the Deegan-Packel measure. Our simulation experiments as well as computations with real-life data shows that the "content" of the weights is best captured by the Banzhaf measure and neither the Holler measure nor the Deegan-Packel measure are good indicators of this "content". Based on this evidence, we put forward the suggestion that it is sufficient to consider the swings as the only measure of power in voting games.

Applications. IMF decision making procedures have been modelled as voting games Leech (2002b). Decision making in the EU has also been discussed in the context of voting games Leech (2002a).

The notions of similarity between the variations in the weights and the voting powers as well as the relation between the inequality in the weights and that in the voting powers have been informally discussed. Our proposals for measuring weight-power similarity and the voting power inequality formalises this intuition. We compute the various measures for the IMF game and (a simplified version of) the EU voting game and suggest that the winning threshold can be used as a parameter in achieving target values of similarity or inequality.

In both the IMF and the EU voting games, there is a "natural" justification for assigning weights to the different players. In the context of IMF, the weights reflect the proportion of financial contribution made by the different countries while in the case of EU, the weights reflect the population of the different countries. This is reasonable, since the IMF is a financial organisation while

the EU is essentially a political organisation. In both cases, however, the choice of the winning threshold is not backed by any quantifiable parameter.

Our work provides methods for choosing a winning threshold which has a *quantifiable* justification. There are two options. In the first option, one should choose a value of the winning threshold which maximises the correlation between the weight vector and an appropriate power profile. In the second option, one should choose a value of the winning threshold which yields an inequality score for an appropriate power profile which is closest to the inequality score of the weight vector. In both the cases of IMF and EU, both the options lead to similar values of the winning threshold. Based on this analysis, we put forward the suggestion that the voting rule for IMF should be modified to reflect the optimal value of the winning threshold. In the case of EU, our results provide evidence that the presently used winning threshold is close to the optimal value.

Previous and Related Works

The Shapley-Shubik power index was introduced in Shapley (1953); Shapley and Shubik (1954), Banzhaf index was introduced in Banzhaf (1965) while Coleman indices were introduced in Coleman (1971). Later work by Holler (1982) and Holler and Packel (1983) introduced the public good measure/index. Deegan and Packel introduced another power measure in Deegan and Packel (1978). There are other known measures/indices and we refer to Felsenthal and Machover (1998); Chakravarty et al. (2015) for further details.

The first work to address the problem of inequality in voting games is Einy and Peleg (1991). They provided an axiomatic deduction of an inequality index for the Shapley-Shubik power measure. A more general axiomatic treatment of inequality for power measures appears in a paper by Laruelle and Valenciano (2004). This work postulates axioms and deduces an inequality measure for a class of power indices which includes the Banzhaf index. A more recent work by Weber (2016) suggests the use of the Coefficient of Variation as an inequality index for measuring inequality arising from the Banzhaf index. Later we provide a more detailed discussion of the relationship of these prior works to our contribution.

2 Preliminaries

2.1 Voting Games

We provide some standard definitions arising in the context of voting games. For details the reader may consult Felsenthal and Machover (1998); Chakravarty et al. (2015). In the following, the cardinality of a finite set S will be denoted by $\#S$ and the absolute value of a real number x will be denoted by $|x|$.

Let $N = \{A_1, A_2, \ldots, A_n\}$ be a set of n players. A subset of N is called a *voting coalition*. The set of all voting coalitions is denoted by 2^N. A voting

game G is given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$ where a winning coalition is assigned the value 1 and a losing coalition is assigned the value 0. Below we recall some basic notions about voting games.

1. For any $S \subseteq N$ and player $A_i \in N$, A_i is said to be a *swing* in S if $A_i \in S$, $\widehat{G}(S) = 1$ but $\widehat{G}(S \setminus \{A_i\}) = 0$.
2. For a voting game G, the number of swings for A_i will be denoted by $m_G(A_i)$.
3. A player $A_i \in N$ is called a *dummy player* if A_i is not a swing in any coalition, i.e., if $m_G(A_i) = 0$.
4. For a voting game G, the set of all winning coalitions will be denoted by $W(G)$ and the set of all losing coalitions will be denoted by $L(G)$.
5. A coalition $S \subseteq N$ is called a *minimal winning coalition* if $\widehat{G}(S) = 1$ and there is no $T \subset S$ for which $\widehat{G}(T) = 1$.
6. The set of all minimal winning coalitions in G will be denoted by $\mathsf{MW}(G)$ and the set of minimal winning coalitions containing the player A_i will be denoted as $\mathsf{MW}_G(A_i)$.
7. A voting game G is said to be *proper* if for any coalition $S \subseteq N$, $\widehat{G}(S) = 1$ implies that $\widehat{G}(N \setminus S) = 0$. In other words, in a proper game it is not allowed for both S and its complement to be winning.

Definition 1 Consider a triplet $(N, \mathbf{w}, q)$, where $N = \{A_1, \ldots, A_n\}$ is a set of players; $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is a vector of non-negative weights with w_i being the weight of A_i; and q is a real number in $(0, 1)$. Let $\omega = \sum_{i=1}^n w_i$. The triplet $(N, \mathbf{w}, q)$ defines a *weighted majority voting game* G given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$ in the following manner. Let $w_S = \sum_{A_i \in S} w_i$ denote the sum of the weights of all the players in the coalition $S \subseteq N$. Then

$$\widehat{G}(S) = \begin{cases} 1 & \text{if } w_S/\omega \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $G = (N, \mathbf{w}, q)$ to denote the weighted majority voting game arising from the triplet $(N, \mathbf{w}, q)$.

For a weighted majority voting game $G = (N, \mathbf{w}, q)$ to be proper, it is necessary that $q > 0.5$. For the technical analysis of weighted majority voting games, we do not restrict to proper games. When considering applications, as is conventional, one should consider only proper games.

2.2 Voting Power

The notion of power is an important concept in a voting system. A *power measure* captures the capability of a player to influence the outcome of a vote.

Given a game G on a set of players N and a player A_i in N, a power measure $\mathcal{P}$ associates a non-negative real number $v_i = \mathcal{P}_G(A_i)$ to the player A_i. The number v_i captures the power that A_i has in the game G. If $\sum_{A_i \in G} \mathcal{P}_G(A_i) = 1$

for all games G, then $\mathcal{P}$ is called a *power index*. In other words, for a power index the powers of the individual players sum to 1.

A widely studied index of voting power is the Shapley-Shubik index. This index, however, is defined for a voting game where the order in which the players cast their votes is important. In our application of voting power to the voting games arising in the IMF and EU decision making processes, the order of casting votes is not important. So, we do not consider the Shapley-Shubik index in this work. Below we provide the definitions of some of the previously proposed power measures. See Felsenthal and Machover (1998); Chakravarty et al. (2015) for further details.

Banzhaf Power Measures. The *raw Banzhaf power measure* $\mathsf{BR}_G(A_i)$ for a player A_i in the game G is defined as the number of distinct coalitions in which A_i is a swing. Hence,

$$\mathsf{BR}_G(A_i) = m_G(A_i).$$

The *non-normalized Banzhaf power measure* $\mathsf{BZN}_G(A_i)$ is defined as follows.

$$\mathsf{BZN}_G(A_i) = \frac{\mathsf{BR}_G(A_i)}{2^{n-1}} = \frac{m_G(A_i)}{2^{n-1}}.$$

The *Banzhaf normalized power index* $\mathsf{BZ}_G(A_i)$ is defined as follows.

$$\mathsf{BZ}_G(A_i) = \frac{\mathsf{BR}_G(A_i)}{\sum_{j=1}^{n} \mathsf{BR}_G(A_j)} = \frac{m_G(A_i)}{\sum_{j=1}^{n} m_G(A_j)}.$$

Coleman Power Measures. The *Coleman preventive power measure* $\mathsf{CP}_G(A_i)$ for a player A_i in the game G is a measure of its ability to stop a coalition S from achieving $w_S \geq q$. It is defined as follows.

$$\mathsf{CP}_G(A_i) = \frac{m_G(A_i)}{\#W(G)}.$$

The *Coleman initiative power measure* $\mathsf{CI}_G(A_i)$ for a player A_i in the game G is a measure of its ability to turn an otherwise losing coalition S with $w_S < q$ into a winning coalition with $w_{S \cup \{A_i\}} \geq q$. It is defined as follows.

$$\mathsf{CI}_G(A_i) = \frac{m_G(A_i)}{\#L(G)} = \frac{m_G(A_i)}{2^n - \#W(G)} = \frac{\mathsf{CP}_G(A_i)}{\frac{2^n}{\#W(G)} - 1}.$$

Holler Public Good Index. Holler proposed the *public good index* $\mathsf{PGI}_{A_i}(G)$ as follows.

$$\mathsf{PGI}_{A_i}(G) = \frac{\#\mathsf{MW}_G(A_i)}{\sum_{A_j \in N} \#\mathsf{MW}_G(A_j)}.$$

The non-normalized version of $\mathsf{PGI}_{A_i}(G)$ is called the *absolute public good measure*. It is defined as

$$\mathsf{PGM}_{A_i}(G) = \frac{\#\mathsf{MW}_G(A_i)}{\#\mathsf{MW}(G)}.$$

Deegan-Packel Power Measure. The *Deegan-Packel power measure* $\mathsf{DP}_G(A_i)$ for a player A_i in the game G is defined to be

$$\mathsf{DP}_G(A_i) = \frac{1}{\#\mathsf{MW}(G)} \sum_{S \in \mathsf{MW}_G(A_i)} \frac{1}{\#S}.$$

Power Profile. Suppose $\mathcal{P}$ is a measure of voting power. Then $\mathcal{P}$ assigns a non-negative real number to each of the n players in the game. So, $\mathcal{P}$ is given by a vector of non-negative real numbers. This vector is called the $\mathcal{P}$-power profile of the game.

Computing Voting Powers. A weighted majority voting game $G = (N, \mathbf{w}, q)$ is completely specified by the set of players N, a weight vector $\mathbf{w}$ and the threshold q. Given this data, it is of interest to be able to compute the $\mathcal{P}$-power profile for any power measure $\mathcal{P}$. There are known dynamic programming based algorithms for computing the values of the different voting power indices. We refer to Matsui and Matsui (2000); Chakravarty et al. (2015) for an introduction to algorithms for computing voting powers. In our work, we have implemented the algorithms for computing the $\mathcal{P}$-power profiles where $\mathcal{P}$ is any of the power measures defined above.

There is a large literature on voting powers. The various indices mentioned above have been introduced to model certain aspects of voting games which are not adequately covered by the other indices. There have been axiomatic characterisations of these indices. A detailed discussion of the relevant literature is not really within the focus of the present work. Instead we refer to the highly respected monograph Felsenthal and Machover (1998) and the more recent textbook Chakravarty et al. (2015) for such details. *Our concern in this work is how to quantify the efficacy of any particular voting power measure that one may choose for a particular application.*

2.3 Pearson's Correlation Coefficient

Given vectors $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, Pearson's correlation coefficient is the standard measure of linear correlation between these two vectors. It is defined as follows.

$$\mathsf{PCC}(\mathbf{w}, \mathbf{v}) = \begin{cases} 0 & \text{if } w_1 = \cdots = w_n \text{ or } v_1 = \cdots = v_n; \\ \frac{\sum_{i=1}^{n}(w_i - \mu_{\mathbf{w}})(v_i - \mu_{\mathbf{v}})}{\sqrt{\sum_{i=1}^{n}(w_i - \mu_{\mathbf{w}})^2}\sqrt{\sum_{i=1}^{n}(v_i - \mu_{\mathbf{v}})^2}} & \text{otherwise.} \end{cases} \tag{1}$$

Here $\mu_{\mathbf{w}}$ and $\mu_{\mathbf{v}}$ are the means of $\mathbf{w}$ and $\mathbf{v}$ respectively.

From (1), it follows that for any two positive real numbers γ and δ,

$$\mathsf{PCC}(\mathbf{w}, \mathbf{v}) = \mathsf{PCC}(\gamma\mathbf{w}, \delta\mathbf{v}). \tag{2}$$

The relation captured in (2) can be considered to be a scale invariance property of the Pearson's Correlation Coefficient.

2.4 Inequality Indices

The notion of inequality has been considered for social parameters including income, skills, education, health and wealth Cowell (2016). There are several methods for measuring inequality. At a basic level, the idea of an inequality index $\mathcal{I}$ is the following. Given a vector $\mathbf{a}$ whose components are real numbers, $\mathcal{I}(\mathbf{a})$ produces a non-negative real number r. In other words, the index $\mathcal{I}$ assigns an inequality score of r to the vector $\mathbf{a}$. There is a large literature on inequality indices including the measurement of multidimensional inequality Chakravarty and Lugo (2016); Chakravarty (2017). In this work, we will consider only basic inequality indices. Some of the most commonly used inequality indices are mentioned below.

Given a vector $\mathbf{a}$ of real numbers, let $\mu_{\mathbf{a}}$ and $\sigma_{\mathbf{a}}$ denote the mean and standard deviation of $\mathbf{a}$. *In the definition of the inequality indices below, we will assume that the entries of $\mathbf{a}$ are non-negative and $\mu_{\mathbf{a}}$ is positive.*

Gini Index. The value of the Gini index of a vector $\mathbf{a} = (a_1, \ldots, a_n)$ is given by

$$\mathsf{GI}(\mathbf{a}) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_i - a_j|}{2n \sum_{i=1}^{n} a_i}. \tag{3}$$

Coefficient of Variation. For a vector $\mathbf{a} = (a_1, \ldots, a_n)$, the Coefficient of Variation is computed as the ratio of the standard deviation $\sigma_{\mathbf{a}}$ to the mean $\mu_{\mathbf{a}}$ of $\mathbf{a}$.

$$\mathsf{CoV}(\mathbf{a}) = \frac{\sigma_{\mathbf{a}}}{\mu_{\mathbf{a}}} = \frac{\sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)^2}}{\frac{1}{n} \sum_{i=1}^{n} a_i}. \tag{4}$$

Generalised Entropy Index. The generalised entropy index is a measure of inequality based on information theory. For a real number α, the generalized entropy index $\mathsf{GEI}_\alpha(\mathbf{a})$ is defined in the following manner.

$$\mathsf{GEI}_\alpha(\mathbf{a}) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left(\frac{1}{n} \sum_{A_i \in N} \left(\frac{a_i}{\mu_{\mathbf{a}}} \right)^\alpha - 1 \right) & \text{if } \alpha \neq 0, 1; \\ \frac{1}{n} \sum_{A_i \in N, a_i > 0} \left(\frac{a_i}{\mu_{\mathbf{a}}^+} \right) \ln \left(\frac{a_i}{\mu_{\mathbf{a}}^+} \right) & \text{if } \alpha = 1; \\ -\frac{1}{n} \sum_{A_i \in N, a_i > 0} \ln \left(\frac{a_i}{\mu_{\mathbf{a}}^+} \right) & \text{if } \alpha = 0. \end{cases} \tag{5}$$

Here ln denotes the natural logarithm and $\mu_{\mathbf{a}}^+$ denotes the mean of the positive entries in $\mathbf{a}$. Also, note that for $\alpha = 0, 1$ the sum is over positive values of a_i as otherwise the ln function gets applied to 0. In other words, for $\alpha = 0$ and 1, the computation of inequality considers only the positive entries of $\mathbf{a}$. GEI_1 is called the *Theil Index* and GEI_2 is half the square of CoV.

Remark: For application to the context of voting powers, a power of zero implies that the player is a dummy. If GEI_0 or GEI_1 is used to measure inequality, then such dummy players will get ignored. As a result, the inequality in the power profile will not be adequately captured by these two measures. Due to this reason, GEI_0 and GEI_1 are not suitable for measuring inequality in voting powers. GEI_2 is half the square of CoV and will essentially spread out the value of CoV. The relevance of GEI_k for $k > 2$ to the context of voting power is not clear. So, though we have computed, we do not report the values of GEI in this work.

Computing Inequality Indices. It is quite routine to implement an algorithm, which given a vector of non-negative quantities computes the values of the various inequality indices. In our work, we have implemented algorithms to compute the Gini Index and the Coefficient of Variation.

Desirable Properties of an Inequality Index. A few basic and natural properties have been postulated which any reasonable inequality measure should satisfy. Below we mention these properties. See Cowell (2016) for more details. Let $\mathcal{I}$ be a postulated inequality index and $\mathbf{a} = (a_1, \ldots, a_n)$ be a vector of non-negative real numbers.

Let π be a bijection from $\{1, \ldots, n\}$ to itself, i.e., π is a permutation of $\{1, \ldots, n\}$. Define $\mathbf{a}_\pi$ to be the vector $(a_{\pi(1)}, \ldots, a_{\pi(n)})$, i.e., $\mathbf{a}_\pi$ is a reordering of the components of $\mathbf{a}$.

Anonymity (ANON): $\mathcal{I}$ is said to satisfy anonymity if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\mathbf{a}_\pi)$ for all permutations π of $\{1, \ldots, n\}$. Anonymity captures the property that inequality depends only on the (multi-)set of values $\{a_1, \ldots, a_n\}$. Information related to ordering or labelling of these values using names are irrelevant for the measurement of inequality.

Egalitarian Principle (EP): $\mathcal{I}$ is said to satisfy the egalitarian principle if $\mathcal{I}(\mathbf{a}) = 0$ for all $\mathbf{a}$ such that $a_1 = \cdots = a_n$. EP captures the property that the inequality is the minimum possible when all components of the vector $\mathbf{a}$ have the same value.

Scale Invariance (ScI). $\mathcal{I}$ is said to satisfy scale invariance if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\gamma \mathbf{a})$ for all real $\gamma > 0$. The idea behind scale invariance is that if all the values are scaled by the same factor then the inequality remains unchanged.

Let $\mathbf{a}^{[k]}$ denote the vector

$$\left(\underbrace{a_1, \ldots, a_1}_{k}, \underbrace{a_2, \ldots, a_2}_{k} \ldots, \underbrace{a_n, \ldots, a_n}_{k} \right).$$

Population Principle (PP). $\mathcal{I}$ is said to satisfy the population principle if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\mathbf{a}^{[k]})$ for any integer $k \geq 1$. The vector $a^{[k]}$ contains k copies of each of the values $a_1, \ldots, a_n$. PP says that the inequality in such a vector remains the same as in the original vector, i.e., by replicating each of the components of the original vector the same number of times does not change the inequality.

For $1 \leq i < j \leq n$, let $\mathbf{a}_{i,j,\delta}$ be the vector

$$(a_1, \ldots, a_{i-1}, a_i + \delta, \ldots, a_j - \delta, a_{j+1}, \ldots, a_n).$$

Transfer Principle (TP). $\mathcal{I}$ is said to satisfy the transfer principle if $\mathcal{I}(\mathbf{a}) \geq \mathcal{I}(\mathbf{a}_{i,j,\delta})$ for any $1 \leq i < j \leq n$ and $\delta > 0$ such that $a_i < a_j$ and $a_i + \delta \leq a_j - \delta$. The transfer principle says that if δ units are transferred from a richer person to a poorer person without changing their relative ordering, then inequality cannot increase.

Suppose $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are vectors of dimensions $n_1, \ldots, n_k$ respectively with non-negative real entries. Let μ_i be the mean of $\mathbf{a}_i$ and define $\mu = (\mu_1, \ldots, \mu_k)$. Let $\mathbf{a}$ be the vector formed by concatenating the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$.
Decomposability (Decom). $\mathcal{I}$ is said to satisfy decomposability if

$$\mathcal{I}(\mathbf{a}) = \sum_{i=1}^{k} n_i \mathcal{I}(\mathbf{a}_i) + \mathcal{I}(\mu).$$

Decomposibility captures the following idea. The vector $\mathbf{a}$ is divided into k groups and inequality is measured for each of the groups. Further, the mean of each group is computed and inequality is computed for the vector composing of the means. The inequality for each group is 'within group inequality' whereas the inequality in the vector of means is some kind of 'across group inequality'. The index $\mathcal{I}$ satisfies decomposability if the overall inequality in the vector can be decomposed into a sum of 'within group inequality' and 'across group inequality'.

The Gini Index, the Coefficient of Variation and the Generalised Entropy Indices satisfy ANON, EP, ScI, PP and TP. It has been shown Shorrocks (1980) that any index which satisfies ANON, ScI, PP, TP and Decom must necessarily have the form of a generalised entropy index for some value of α.

ANON, EP, ScI and TP are natural properties that any inequality index should satisfy irrespective of the domain to which it is applied. PP becomes relevant in the context of variable population size. For voting games, the players constitute the population which is fixed. So, the application of PP to voting games is vacuous. On the other hand, it is not clear that Decom is necessarily a desirable property for *all* applications of inequality. In particular, it is not clear that Decom is relevant in the context of voting powers which is the focus of the present work.

3 Weight-Power Similarity

Let $\mathcal{P}$ be a measure of voting power. Suppose this is applied to a weighted majority voting game $G = (N, \mathbf{w}, q)$. Let $\mathbf{v}$ be the resulting power profile. It is of interest to know how similar the power profile vector $\mathbf{v}$ is to the weight vector $\mathbf{w}$. Note that the power profile vector $\mathbf{v}$ depends on the winning threshold q.

Based on the Pearson's correlation coefficient, we define the similarity index $\mathcal{P}\text{-SI}_{\mathbf{w}}(q)$ as follows.

$$\mathcal{P}\text{-SI}_{\mathbf{w}}(q) = \text{PCC}(\mathbf{w}, \mathbf{v}) \tag{6}$$

where $\mathbf{v}$ is the power profile vector generated by the voting power measure $\mathcal{P}$ applied to the weighted majority voting game $G = (N, \mathbf{w}, q)$.

So, for a fixed q, $\mathcal{P}\text{-SI}_{\mathbf{w}}(q)$ measures the similarity of the power profile vector to the weight vector by the correlation between these two vectors. Note that $\mathcal{P}\text{-SI}_{\mathbf{w}}(q)$ is a function of q. As, q changes, the power profile vector $\mathbf{v}$ will also change, though the weight vector $\mathbf{w}$ will not change. So, with change in q, the correlation between $\mathbf{w}$ and $\mathbf{v}$ changes. By varying q, it is possible to study the change in the correlation between $\mathbf{w}$ and $\mathbf{v}$.

Theorem 1 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game such that $0 < \#W(G) < 2^n$. Then for any $q \in (0, 1)$ the following holds.*

1. $\text{BZN-SI}_{\mathbf{w}}(q) = \text{BZ-SI}_{\mathbf{w}}(q) = \text{CP-SI}_{\mathbf{w}}(q) = \text{CI-SI}_{\mathbf{w}}(q)$.
2. $\text{PGI-SI}_{\mathbf{w}}(q) = \text{PGM-SI}_{\mathbf{w}}(q)$.

Proof Let $\mathbf{v} = (m_G(A_1), \ldots, m_G(A_n))$ be the vector of swings for the players $A_1, \ldots, A_n$ in the game G. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ are the power profiles for G corresponding to BZN, BZ, CP and CI respectively. Then

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 = \alpha_3 \mathbf{v}_3 = \alpha_4 \mathbf{v}_4$$

where

$$\alpha_1 = 2^{n-1}, \ \alpha_2 = \sum_{j \in N} m_G(A_j), \ \alpha_3 = \#W(G) \text{ and } \alpha_4 = \#L(G).$$

In G, the values $2^n, \sum_{j \in N} m_G(A_j), \#W(G)$ and $\#L(G)$ are fixed. So, $\alpha_1, \alpha_2, \alpha_3$ and α_4 are constants. Further, since $0 < \#W(G) < 2^n$, it follows that $0 < \#L(G) < 2^n$ and so $\sum_{j \in N} m_G(A_j) > 0$. This in particular means that $\alpha_2, \alpha_3, \alpha_4 > 0$ and clearly $\alpha_1 > 0$. So, using (2) we have

$$\text{BZN-SI}_{\mathbf{w}}(q) = \text{PCC}(\mathbf{w}, \mathbf{v}_1) = \text{PCC}(\mathbf{w}, \mathbf{v}/\alpha_1) = \text{PCC}(\mathbf{w}, \mathbf{v}).$$

In a similar manner, it follows that $\text{BZ-SI}_{\mathbf{w}}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$, $\text{CP-SI}_{\mathbf{w}}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$ and $\text{CI-SI}_{\mathbf{w}}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$.

The argument for $\text{PGI-SI}_{\mathbf{w}}(q) = \text{PGM-SI}_{\mathbf{w}}(q)$ is similar. $\qquad\square$

From the viewpoint of weight-power similarity, Theorem 1 shows that it is sufficient to consider only $\text{BZ-SI}_{\mathbf{w}}(q)$, $\text{PGI-SI}_{\mathbf{w}}(q)$ and $\text{DP-SI}_{\mathbf{w}}(q)$.

4 Measuring Inequality of Voting Powers

Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game. The weights of all the players are not equal. In fact, in several important practical situations, the voting game is designed in a manner such that the weights are indeed unequal. The inequality in the weights can be captured by applying an appropriate inequality measure. Suppose $\mathcal{I}$ is an inequality measure. Then $\mathcal{I}(\mathbf{w})$ is the inequality present in the weights.

Let $\mathcal{P}$ be a measure of voting power. Suppose $\mathcal{P}$ is applied to G to obtain the power profile vector $\mathbf{v}$. Then $\mathbf{v}$ is a vector consisting of non-negative real numbers. The inequality in the vector $\mathbf{v}$ can be measured by the inequality index $\mathcal{I}$ as $\mathcal{I}(\mathbf{v})$. The value of $\mathcal{I}(\mathbf{v})$ depends on the winning threshold q whereas the value of $\mathcal{I}(\mathbf{w})$ does not depend on q. So, by varying q, it is possible to vary $\mathcal{I}(\mathbf{v})$ with the goal of making it as close to $\mathcal{I}(\mathbf{w})$ as possible. Then one can say that the inequality present in the weights is more or less reflected in the inequality that arises in the voting powers.

Given a weighted majority voting game $G = (N, \mathbf{w}, q)$, we define the *weight inequality* of G with respect to an inequality measure $\mathcal{I}$ as

$$\mathcal{I}\text{-WI}(\mathbf{w}) = \mathcal{I}(\mathbf{w}). \tag{7}$$

We consider two different options for $\mathcal{I}$, namely, GI and CoV. This gives rise to two different measures of weight inequality, which are GI-WI and CoV-WI.

Given a weighted majority voting game $G = (N, \mathbf{w}, q)$, a voting power measure $\mathcal{P}$ and an inequality measure $\mathcal{I}$, the *power inequality* of $\mathcal{P}$ as determined by $\mathcal{I}$ is denoted by $(\mathcal{P}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q)$ and is defined in the following manner.

$$(\mathcal{P}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q) = \mathcal{I}(\mathbf{v}) \tag{8}$$

where $\mathbf{v}$ is the power profile vector generated by the power measure $\mathcal{P}$ applied to the weighted majority voting game $G = (N, \mathbf{w}, q)$.

We have considered seven options for $\mathcal{P}$, namely, BZN, BZ, CP, CI, PGI, PGM and DP. The following results shows that under a simple and reasonable condition on an inequality measure $\mathcal{I}$, it is sufficient to consider only three of these.

Theorem 2 *Let $\mathcal{I}$ be an inequality index satisfying scale invariance. Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game such that $0 < \#W(G) < 2^n$. Then for any $q \in (0, 1)$ the following holds.*

1. $(\text{BZ}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q) = (\text{BZ}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q) = (\text{CP}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q) = (\text{CI}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q).$
2. $(\text{PGI}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q) = (\text{PGM}, \mathcal{I})\text{-PI}_{\mathbf{w}}(q).$

Proof As in the proof of Theorem 1, let $\mathbf{v} = (m_G(A_1), \ldots, m_G(A_n))$ be the vector of swings and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ be the power profiles corresponding to BZN, BZ, CP and CI respectively so that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 = \alpha_3 \mathbf{v}_3 = \alpha_4 \mathbf{v}_4 \tag{9}$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 are the positive constants defined in the proof of Theorem 1.

Using the scale invariance of $\mathcal{I}$ we have

$$(\mathsf{BZN}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q) = \mathcal{I}(\mathbf{v}_1) = \mathcal{I}(\mathbf{v}/\alpha_1) = \mathcal{I}(\mathbf{v}).$$

In a similar manner, it follows that $(\mathsf{BZ}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q) = \mathcal{I}(\mathbf{v})$ $(\mathsf{CP}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q) = \mathcal{I}(\mathbf{v})$ and $(\mathsf{CI}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q) = \mathcal{I}(\mathbf{v})$.

The argument for $(\mathsf{PGI}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q) = (\mathsf{PGM}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$ is similar.   □

The Gini Index, the Coefficient of Variation and the Generalised Entropy Index satisfy the scale invariance property. Based on Theorem 2, if $\mathcal{I}$ is any of these indices, then from the viewpoint of power inequality, it is sufficient to consider $(\mathsf{BZ}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$, $(\mathsf{PGI}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$ and $(\mathsf{DP}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$. For $\mathcal{I}$, we will consider the Gini Index and the Coefficient of Variation. This means that we need to consider 6 possibilities.

Remark: Theorem 2 has been stated for weighted majority voting games. The crux of the argument is based on (9). This relation does require G to be a weighted majority voting game. So, it is possible to rewrite the proof to show that for general voting games (which are not necessarily weighted majority voting games), the inequalities of the different Banzhaf power profiles and the Coleman power profiles are all equal and also the inequality of the Holler public good index is equal to that of the Holler public good measure.

Comparison to previous works. The work Einy and Peleg (1991) on inequality in voting system is concerned with measuring inequality arising in the Shapley-Shubik power index. Since we do not consider this index in our work, we do not comment any further on the work in Einy and Peleg (1991). Instead, we simply remark that our approach can also be applied to the Shapley-Shubik power index.

Laruelle and Valenciano (2004) axiomatically derive an inequality index for a class of power indices which includes the normalised Banzhaf index. We note the following points about the work in Laruelle and Valenciano (2004).

1. The approach works only for an index, i.e., the sum of the powers must sum to one. So, for example it cannot be applied to measure inequality arising in either of the Coleman power measures.
2. The notion of power considered in the work is based on swings. So, the power measures given by PGI, PGM and DP are not covered by their work.
3. Among the axioms, ANON and EP are assumed and it is shown that the obtained measure satisfies Scl. On the other hand, PP and TP are not mentioned in the paper and it is not clear whether these two properties hold for the obtained measure.
4. Justification for one of the axioms (namely, Constant Sensitivity to Null Players) is not clear. In the discussion leading up to this axiom, the authors remark: "Thus, at this point any further step is questionable, though necessary to specify an index."

In view of the above, we feel that it might be preferable to study the behaviour of standard inequality indices on both power measures and power indices rather than relying on an axiomatically derived inequality index where at least one of the axioms does not necessarily have a natural justification.

Weber (2016) considered the application [1] of the Coefficient of Variation to the measurement of inequality for essentially the normalised Banzhaf index. In contrast, we consider all the standard inequality indices and a much larger class of power measures/indices. Even for the Coefficient of Variation, the result that the scale invariance property implies that all the swing based measures have the same inequality is not present in Weber (2016).

5 Variation of Similarity and Inequality with Winning Threshold

We have conducted some experiments to understand the dependence of the similarity and inequality indices of power profiles on the winning threshold.

In the first experiment, N was taken to be $\{A_1, \ldots, A_{30}\}$ and one hundred weight vectors were generated where the individual weights were chosen to be integers independently and uniformly in the range $[1, \ldots, 100]$. For each of the 100 weight vectors $\mathbf{w}$, the value of the winning threshold q was varied from 0.01 to 0.99 in steps of 0.01. For the game defined by the triplet $(N, \mathbf{w}, q)$, the power profiles for the different power measures were computed. From this, the similarity indices $\mathsf{BZ\text{-}SI}_{\mathbf{w}}(q)$, $\mathsf{PGI\text{-}SI}_{\mathbf{w}}(q)$ and $\mathsf{DP\text{-}SI}_{\mathbf{w}}(q)$ were computed and the inequality indices $\mathcal{I}\text{-}\mathsf{WI}(\mathbf{w})$, $(\mathsf{BZN}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$, $(\mathsf{PGI}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$ and $(\mathsf{DP}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$ were computed where $\mathcal{I}$ was taken to be GI and CoV. All the obtained results show a definite pattern.

A second experiment was conducted with $n = 30$ and non-random weights. In particular, two distinct values of weights were used, namely, n_1 of the weights were taken to be 100 and n_2 of the weights were taken to be 1 with $n_1 + n_2 = 30$. The value of q was varied as mentioned above and the corresponding similarity and inequality indices were computed. In this case, no definite pattern was observed and there was a rich variation in the behaviour.

To further explain the above experiments, we report three particular cases with $n = 30$.

Case-I: A random weight vector. The actual value of $\mathbf{w}$ (after sorting into descending order) came out to be
$$\{93, 92, 90, 86, 86, 83, 82, 77, 72, 68, 67, 67, 63, 62, 62, 59,$$
$$49, 40, 36, 35, 35, 30, 29, 27, 26, 26, 23, 21, 15, 11\}.$$
The plots of similarity, Gini Index and the Coefficient of Variation are shown in Figures 1, 2 and 3 respectively.

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[1] The author remarks: "To the best of my knowledge, I am the first to propose a measure of inequality of voting systems that can be used across different constituencies." This is an oversight since the work by Laruelle and Valenciano Laruelle and Valenciano (2004) is much earlier.

Case-II: 15 of the weights were taken to be 100 and the other 15 of the weights were taken to be 1. The plots of similarity, Gini Index and the Coefficient of Variation are shown in Figures 4, 5 and 6 respectively.

Case-III: 29 of the weights were taken to be 100 and the other weight was taken to be 1. The plots of similarity, Gini Index and the Coefficient of Variation are shown in Figures 7, 8 and 9 respectively.

Based on the plots, we make the following observations.

1. For the random case, compared to the Holler index and the Deegan-Packel measure, the Banzhaf index is a much better marker of similarity to the weight vector and it also much better at capturing the inequality present in the weights. For the two non-random cases, there is not much difference between the three power measures.

2. For Case-II, there are sharp spikes in the similarity and inequality plots. These correspond to choices of q for which the low weight players achieve power similar to the high weight players.

3. For Case-III, there is one player with low weight. Apart from a small range of q where this player gains significant power, for other values of q this player becomes a dummy. The inequality indices, however, do not reflect this. The inequality scores are generally quite low indicating that there is small inequality in the system. This is a feature of the inequality indices which are not sensitive to low scores of a small number of players.

6 Applications

6.1 IMF Voting Games

The IMF has two decision making bodies, namely, the Board of Governors (BoG) and the Executive Board (EB).

A total of 189 member countries make up the BoG. Each country has a specified voting share. The voting share or weight of a country is calculated as the sum of a basic weight plus an amount which is proportional to the special drawing rights of the country. The EB consists of 24 Executive Directors (EDs) representing all the 189 member countries. Eight of these directors are nominated by eight member countries while each of the other directors are elected by a group of countries. Each ED has a voting weight which is the sum of the voting weights of the countries that he or she represents.

The BoG is the highest decision making body of the IMF and is officially responsible for all major decisions. In practice, however, the BoG has delegated

most of its powers to the EB[2]. Accordingly, in this work, we will consider only
the voting game arising from the EB weights.

Actual weights of the members of the EB are available from the IMF web-
site[3]. These weights range from the minimum of 80157 to the maximum of
831407. Since these values are rather large, for the purposes of computation of
voting powers, we have divided these voting weights by 1000 and then rounded
to the nearest integer. While this is an approximation, it does not significantly
affect the voting powers. In particular, we have checked that no two members
with originally unequal weights get the same weight after this rounding off
process. The actual weight vector that has been used to compute the voting
powers is the following.

$$\mathbf{w}_{\mathsf{imf}} = \quad \{831, 310, 306, 273, 268, 267, 219, 208, 203, 203, 196, 170,$$
$$165, 162, 155, 154, 150, 149, 138, 131, 111, 101, 82, 80\}.$$

The rules specify several winning thresholds[4]. We mention these below.

> "Except as otherwise specifically provided, all decisions of the Fund
> shall be made by a majority of the votes cast."
>
> "The Fund, by a seventy percent majority of the total voting power,
> may decide at any time to distribute any part of the general reserve."
>
> "The Fund may use a member's currency held in the Investment
> Account for investment as it may determine, in accordance with rules
> and regulations adopted by the Fund by a seventy percent majority of
> the total voting power."
>
> "An eighty-five percent majority of the total voting power shall be
> required for any change in quotas."

So, three possible values of q are used: $q = 0.5$, $q = 0.7$ and $q = 0.85$.

We have computed the similarity and inequality indices for the IMF EB
voting game with q varying from 0.01 to 0.99 in steps of 0.01. The plots
of $\mathsf{BZ\text{-}SI}_{\mathbf{w}_{\mathsf{imf}}}(q)$, $\mathsf{PGI\text{-}SI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ and $\mathsf{DP\text{-}SI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ are shown in Figure 10; the
plots of $(\mathsf{BZ}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$, $(\mathsf{PGI}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ and $(\mathsf{DP}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ are shown
in Figure 11; and the plots of $(\mathsf{BZ}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$, $(\mathsf{PGI}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ and
$(\mathsf{DP}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathsf{imf}}}(q)$ are shown in Figure 12. The actual values of these indices
for the range $[0.5, 0.65]$ along with the values for $q = 0.70$ and $q = 0.85$
are shown in Tables 1 and 2. Based on these data, we have the following
observations.

1. The Holler and the Deegan-Packel indices are not good indicators of either
   the similarity to or the inequality present in the weights. So, we focus only
   on the Banzhaf index.
2. The following holds for the Banzhaf index:
   – The plots of the two inequality indices have bell curve shapes. To a
     lesser extent, the same is also true of the similarity index.
   – The maximum similarity is achieved for $q = 0.60$.

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[2] `https://en.wikipedia.org/wiki/International_Monetary_Fund`

[3] `https://www.imf.org/external/np/sec/memdir/eds.aspx`

[4] `https://www.imf.org/external/pubs/ft/aa/index.htm`

Table 1 IMF EB similarity indices.

q	BZ-SI$_{\mathbf{w}_{imf}}(q)$	PGI-SI$_{\mathbf{w}_{imf}}(q)$	DP-SI$_{\mathbf{w}_{imf}}(q)$
0.50	0.9938690	0.7743330	0.9330398
0.51	0.9939952	0.8965981	0.9531445
0.52	0.9943706	0.9363455	0.9618091
0.53	0.9949599	0.9522328	0.9669979
0.54	0.9957522	0.9600696	0.9708993
0.55	0.9966513	0.9649342	0.9740920
0.56	0.9975953	0.9688469	0.9771681
0.57	0.9985210	0.9730949	0.9806975
0.58	0.9992881	0.9769905	0.9839294
0.59	0.9998168	0.9806887	0.9869004
0.60	**0.9999911**	0.9846956	0.9899614
0.61	0.9996918	0.9882609	0.9924378
0.62	0.9988216	0.9912144	0.9942195
0.63	0.9972257	0.9936152	0.9953372
0.64	0.9948817	0.9951439	0.9954396
0.65	0.9916875	0.9956169	0.9945365
0.70	0.9609858	0.9788570	0.9715726
0.85	0.7634234	0.7787653	0.7721551

- For the Gini Index, the inequality in the power profile is closest to the inequality in the weights for $q = 0.61$.
- For the Coefficient of Variation, the inequality in the power profile is closest to the inequality in the weights for $q = 0.60$.

From Tables 1 and 2 we note that in comparison to $q = 0.60$, the choices $q = 0.50$, $q = 0.70$ and $q = 0.85$ are sub-optimal.

Based on the above analysis, we put forward the suggestion that the winning threshold of $q = 0.60$ be seriously considered for any future possible change in voting rule. For $q = 0.60$, the actual values of the different power measures are shown in Table 3.

6.2 EU Voting Games

Until Brexit is effective, the European Union Council has 28 members. It votes on different types of matters in three different ways[5]. The first is the unanimity voting where all members have to vote in favour or against for the motion to be passed or dismissed. In non-legislative issues, a simple majority voting is done where at least 15 out of the 28 members have to vote in favour. For most (80%) of the issues that are voted upon in the EU Council, the "qualified majority" method is used. This is stated as follows[6].

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[5] http://www.consilium.europa.eu/en/council-eu/voting-system/

[6] http://www.consilium.europa.eu/en/council-eu/voting-system/qualified-majority/

Table 2 Inequality as measured by GI and CoV in the IMF Game. In the table, $\mathfrak{f}_1(q) = (\mathsf{BZ}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$, $\mathfrak{f}_2(q) = (\mathsf{PGI}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$, $\mathfrak{f}_3(q) = (\mathsf{DP}, \mathsf{GI})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$, $\mathfrak{g}_1(q) = (\mathsf{BZ}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$, $\mathfrak{g}_2(q) = (\mathsf{PGI}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$, $\mathfrak{g}_3(q) = (\mathsf{DP}, \mathsf{CoV})\text{-}\mathsf{PI}_{\mathbf{w}_{\mathrm{imf}}}(q)$. Note GI-WI$(\mathbf{w}_{\mathrm{imf}}) = \mathbf{0.285042}$ and CoV-WI$(\mathbf{w}_{\mathrm{imf}}) = \mathbf{0.689527}$.

q	$\mathfrak{f}_1(q)$	$\mathfrak{f}_2(q)$	$\mathfrak{f}_3(q)$	$\mathfrak{g}_1(q)$	$\mathfrak{g}_2(q)$	$\mathfrak{g}_3(q)$
0.50	0.316535	0.013684	0.023564	0.859515	0.033982	0.061049
0.51	0.316212	0.017718	0.027366	0.857502	0.046205	0.074206
0.52	0.315214	0.021654	0.031084	0.851387	0.059712	0.086957
0.53	0.313531	0.025291	0.034457	0.841354	0.072727	0.098447
0.54	0.311215	0.028989	0.037852	0.827460	0.085147	0.108920
0.55	0.308401	0.032545	0.041122	0.810689	0.096331	0.118073
0.56	0.305067	0.035747	0.044020	0.791192	0.105924	0.125614
0.57	0.301236	0.038982	0.046930	0.769085	0.114385	0.132059
0.58	0.297133	0.042029	0.049670	0.745825	0.121321	0.137161
0.59	0.292743	0.044794	0.052123	0.721395	0.126885	0.141047
0.60	0.288019	0.047504	0.054517	**0.695812**	0.131366	0.144021
0.61	**0.283235**	0.050075	0.056795	0.670527	0.134712	0.131366
0.62	0.278285	0.052419	0.058857	0.645342	0.137090	0.147386
0.63	0.273201	0.054555	0.060691	0.620092	0.138635	0.147978
0.64	0.268155	0.056714	0.062592	0.596040	0.139694	0.148302
0.65	0.263088	0.058572	0.064200	0.572826	0.140068	0.148049
0.66	0.257927	0.060254	0.065613	0.550134	0.139974	0.147422
0.67	0.252875	0.061895	0.067025	0.528952	0.139751	0.146796
0.68	0.247854	0.063419	0.068331	0.508843	0.139248	0.145961
0.69	0.242726	0.064596	0.069269	0.489429	0.138348	0.144745
0.70	0.237753	0.065816	0.070276	0.471442	0.137616	0.143776
0.85	0.158497	0.066284	0.068153	0.278829	0.116985	0.120198

Table 3 Voting powers of the players under various power measures for the IMF-EB game with $q = 0.60$. Instead of the players, the voting powers are shown against the weights of the players.

Wts	BZN	BZ	CP	CI	PGI	PGM	DP
831	0.4033509	0.1661335	0.8949349	0.2603447	0.0662802	0.8888586	0.0682873
310	0.1508728	0.0621420	0.3347491	0.0973816	0.0435712	0.5843174	0.0442821
306	0.1488262	0.0612990	0.3302082	0.0960606	0.0434846	0.5831563	0.0441693
273	0.1322311	0.0544638	0.2933878	0.0853492	0.0427837	0.5737562	0.0432420
268	0.1297327	0.0534347	0.2878445	0.0837366	0.0426849	0.5724312	0.0431133
267	0.1292485	0.0532353	0.2867701	0.0834240	0.0426597	0.5720929	0.0430820
219	0.1055294	0.0434658	0.2341434	0.0681144	0.0416246	0.5582125	0.0417465
208	0.1001409	0.0412464	0.2221877	0.0646364	0.0414027	0.5552361	0.0414634
203	0.0976979	0.0402401	0.2167671	0.0630595	0.0413069	0.5539514	0.0413394
203	0.0976979	0.0402401	0.2167671	0.0630595	0.0413069	0.5539514	0.0413394
196	0.0942787	0.0388318	0.2091809	0.0608526	0.0411606	0.5519899	0.0411522
170	0.0816418	0.0336269	0.1811427	0.0526961	0.0406361	0.5449551	0.0404690
165	0.0792183	0.0326287	0.1757655	0.0511318	0.0405488	0.5437851	0.0403568
162	0.0777606	0.0320283	0.1725313	0.0501909	0.0404977	0.5430995	0.0402881
155	0.0743695	0.0306316	0.1650074	0.0480022	0.0403450	0.5410517	0.0400980
154	0.0738934	0.0304355	0.1639510	0.0476948	0.0403227	0.5407521	0.0400702
150	0.0719494	0.0296347	0.1596377	0.0464400	0.0402490	0.5397639	0.0399749
149	0.0714759	0.0294397	0.1585871	0.0461344	0.0402247	0.5394390	0.0399449
138	0.0661672	0.0272532	0.1468086	0.0427079	0.0399681	0.5359976	0.0396272
131	0.0627846	0.0258599	0.1393032	0.0405246	0.0397938	0.5336591	0.0394117
111	0.0531555	0.0218939	0.1179389	0.0343095	0.0389966	0.5229683	0.0384651
101	0.0483502	0.0199146	0.1072770	0.0312079	0.0383484	0.5142761	0.0377462
82	0.0392314	0.0161588	0.0870447	0.0253221	0.0361261	0.4844734	0.0353936
80	0.0382680	0.0157619	0.0849071	0.0247002	0.0356770	0.4784506	0.0349375

Table 4 Population percentages of the countries in the European Union.

Country	Pop %	$\mathbf{w}_{eu}$
Germany	16.06 %	1606
France	13.05 %	1305
United Kingdom	12.79 %	1279
Italy	12.00 %	1200
Spain	9.09 %	909
Poland	7.43 %	743
Romania	3.87 %	387
Netherlands	3.37 %	337
Belgium	2.21 %	221
Greece	2.11 %	211
Czech Republic	2.04 %	204
Portugal	2.02 %	202
Sweden	1.96 %	196
Hungary	1.92 %	192
Austria	1.71 %	172
Bulgaria	1.40 %	140
Denmark	1.12 %	112
Finland	1.07 %	107
Slovakia	1.06 %	106
Ireland	0.91 %	91
Croatia	0.82 %	82
Lithuania	0.57 %	57
Slovenia	0.40 %	40
Latvia	0.39 %	39
Estonia	0.26 %	26
Cyprus	0.17 %	17
Luxembourg	0.11 %	11
Malta	0.09 %	9

"A qualified majority needs 55% of member states, representing at
least 65% of the EU population."

It is the qualified majority voting rule that we consider in the context of
weighted majority voting games. The population percentages of the individual
countries are available[7] and are reproduced in Table 4. These percentages are
the weights of the individual countries. For computation of the voting powers,
the percentage values are multiplied by 100 to convert these into integers which
are then used as the weights. This scaling does not affect the decision making
process.

In the qualified majority voting, the passing rule is a joint condition, one
on the number of member states which are involved and the other on the
population percentage. While analysing the joint condition would be more
accurate, for the purpose of this work, we have worked with the simpler setting
where only the winning condition on the population percentage is considered.
This leads to a weighted majority voting game where the weight vector $\mathbf{w}_{eu}$ is
specified in Table 4 and the winning threshold is $q = 0.65$.

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[7] http://www.consilium.europa.eu/en/council-eu/voting-system/voting-calculator/

For the weight vector $\mathbf{w_{eu}}$ given in Table 4, we have computed the similarity and inequality indices with q varying from 0.01 to 0.99 in steps of 0.01. The plots of BZ-SI$_{\mathbf{w_{eu}}}(q)$, PGI-SI$_{\mathbf{w_{eu}}}(q)$ and DP-SI$_{\mathbf{w_{eu}}}(q)$ are shown in Figure 13; the plots of (BZ, GI)-PI$_{\mathbf{w_{eu}}}(q)$, (PGI, GI)-PI$_{\mathbf{w_{eu}}}(q)$ and (DP, GI)-PI$_{\mathbf{w_{eu}}}(q)$ are shown in Figure 14; and the plots of (BZ, CoV)-PI$_{\mathbf{w_{eu}}}(q)$, (PGI, CoV)-PI$_{\mathbf{w_{eu}}}(q)$ and (DP, CoV)-PI$_{\mathbf{w_{eu}}}(q)$ are shown in Figure 15. The actual values of these indices for the range $[0.5, 0.7]$ are shown in Tables 5 and 6.

Based on these data, we have the following observations.

1. As in the case of the IMF-EB voting game, the Holler and the Deegan-Packel indices are not good indicators of either the similarity to or the inequality present in the weights. So, again we focus only on the Banzhaf index.
2. The following holds for the Banzhaf index:
   – The plots of the two inequality indices as well as the similarity index have a somewhat bell shape nature.
   – The maximum similarity is achieved for $q = 0.66$.
   – For the Gini Index, the inequality in the power profile is closest to the inequality in the weights for $q = 0.66$.
   – For the Coefficient of Variation, the inequality in the power profile is closest to the inequality in the weights for $q = 0.66$.

   So, the value of $q = 0.66$ is the best in terms of similarity and also for inequality measured by either the Gini Index or the Coefficient of Variation.

The value of q actually used in the EU voting games is $q = 0.65$. The corresponding similarity value and the values for Gini Index and Coefficient of Variation are shown in Tables 1 and 2 respectively. These values show that in comparison to $q = 0.66$, the choice $q = 0.65$ is sub-optimal but, very close. For $q = 0.66$, the actual values of the different power measures are shown in Table 7.

Unlike the case of the IMF voting game, our analysis shows that for the weighted majority game arising in the context of EU, the winning threshold is very close to the optimal value. So, our analysis provides some quantitative backing to the actual winning threshold used in the EU game.

7 Conclusion

In this paper, we have addressed the problem of quantifying whether a power profile adequately captures the natural variation in the weights of a weighted majority voting game. Ideas based on Pearson's Correlation Coefficient and standard inequality measures such as the Gini Index and the Coefficient of Variation have been used in the formalisation. These ideas have been applied to the voting games arising in the context of the IMF and the EU. We provide concrete quantitative evidence that the actual winning threshold used in the

Table 5 EU game similarity indices.

q	BZ-SI$_{\mathbf{w}_{eu}}(q)$	PGI-SI$_{\mathbf{w}_{eu}}(q)$	DP-SI$_{\mathbf{w}_{eu}}(q)$
0.50	0.9988826	0.3320853	0.5051382
0.51	0.9989581	0.4181647	0.5640333
0.52	0.9991501	0.4929785	0.6151950
0.53	0.9993725	0.5596608	0.6631804
0.54	0.9995411	0.6230357	0.7099130
0.55	0.9996231	0.6830571	0.7540455
0.56	0.9996430	0.7350513	0.7910700
0.57	0.9996501	0.7765455	0.8199512
0.58	0.9996763	0.8063575	0.8409968
0.59	0.9997123	0.8289370	0.8576475
0.60	0.9997249	0.8477416	0.8722319
0.61	0.9996959	0.8642583	0.8846472
0.62	0.9996486	0.8777664	0.8944232
0.63	0.9996354	0.8870017	0.9006101
0.64	0.9996914	0.8932824	0.9048869
0.65	0.9997886	0.8976327	0.9086210
0.66	**0.9998332**	0.9039397	0.9150038
0.67	0.9997214	0.9135211	0.9242301
0.68	0.9994175	0.9258465	0.9351369
0.69	0.9989968	0.9381400	0.9448867
0.70	0.9986065	0.9465521	0.9507084

Table 6 Inequality as measured by GI and CoV in the EU Game. In the table $\mathfrak{h}_1(q) = (\text{BZ}, \text{GI})\text{-PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{h}_2(q) = (\text{PGI}, \text{GI})\text{-PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{h}_3(q) = (\text{DP}, \text{GI})\text{-PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_1(q) = (\text{BZ}, \text{CoV})\text{-PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_2(q) = (\text{PGI}, \text{CoV})\text{-PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_3(q) = (\text{DP}, \text{CoV})\text{-PI}_{\mathbf{w}_{eu}}(q)$. Note $\text{GI-WI}(\mathbf{w}_{eu}) = \mathbf{0.605671}$ and $\text{CoV-WI}(\mathbf{w}_{eu}) = \mathbf{1.272825}$.

q	$\mathfrak{h}_1(q)$	$\mathfrak{h}_2(q)$	$\mathfrak{h}_3(q)$	$\mathfrak{k}_1(q)$	$\mathfrak{k}_2(q)$	$\mathfrak{k}_3(q)$
0.50	0.614849	0.033960	0.046614	1.309745	0.091907	0.107286
0.51	0.614993	0.039425	0.051749	1.309901	0.095374	0.112346
0.52	0.615391	0.044681	0.056573	1.310338	0.100199	0.118139
0.53	0.615945	0.049083	0.060556	1.310962	0.105176	0.123691
0.54	0.616483	0.052608	0.063782	1.311545	0.109910	0.128951
0.55	0.616834	0.055576	0.067481	1.311731	0.114808	0.134668
0.56	0.616822	0.059365	0.071789	1.311070	0.120243	0.140936
0.57	0.616302	0.063896	0.076268	1.309147	0.126922	0.148409
0.58	0.615243	0.068081	0.080370	1.305796	0.134171	0.156036
0.59	0.613694	0.072437	0.084545	1.301119	0.142004	0.163861
0.60	0.611813	0.076339	0.088256	1.295536	0.149118	0.170797
0.61	0.609851	0.079853	0.091521	1.289678	0.155445	0.176837
0.62	0.608078	0.083431	0.094969	1.284198	0.161705	0.183060
0.63	0.606708	0.086467	0.097851	1.279626	0.167582	0.188834
0.64	0.605893	0.089760	0.101116	1.276296	0.174214	0.195451
0.65	0.605616	0.093045	0.104334	1.274202	0.181046	0.202085
0.66	**0.605689**	0.096528	0.107787	**1.272960**	0.188107	0.209023
0.67	0.605797	0.100171	0.111427	1.271778	0.195381	0.216243
0.68	0.605497	0.103949	0.115322	1.269457	0.202594	0.223577
0.69	0.604333	0.107898	0.119249	1.264638	0.210268	0.231500
0.70	0.601994	0.111935	0.123198	1.256239	0.218490	0.239777

Table 7 Voting powers of the players under various power measures for the EU game with $q = 0.66$. Instead of the players, the voting powers are shown against the weights of the players.

Wts	BZN	BZ	CP	CI	PGI	PGM	DP
1606	0.2459032	0.1582166	0.7950917	0.1454426	0.0537148	0.8173260	0.0554496
1305	0.2044021	0.1315144	0.6609039	0.1208962	0.0479912	0.7302363	0.0496147
1279	0.2003657	0.1289173	0.6478528	0.1185089	0.0475521	0.7235548	0.0491300
1200	0.1881286	0.1210439	0.6082861	0.1112711	0.0463214	0.7048273	0.0477734
909	0.1444251	0.0929245	0.4669771	0.0854221	0.0410962	0.6253212	0.0423199
743	0.1115583	0.0717777	0.3607073	0.0659826	0.0368236	0.5603092	0.0372597
387	0.0600651	0.0386465	0.1942116	0.0355263	0.0379704	0.5777581	0.0380942
337	0.0523199	0.0336631	0.1691686	0.0309453	0.0371804	0.5657380	0.0372361
221	0.0343284	0.0220872	0.1109957	0.0203039	0.0355657	0.5411688	0.0354550
211	0.0327751	0.0210878	0.1059733	0.0193852	0.0354097	0.5387953	0.0352811
204	0.0316896	0.0203894	0.1024638	0.0187432	0.0353493	0.5378758	0.0352135
202	0.0313774	0.0201886	0.1014543	0.0185586	0.0353118	0.5373050	0.0351703
196	0.0304473	0.0195901	0.0984469	0.0180085	0.0352300	0.5360610	0.0350762
192	0.0298266	0.0191907	0.0964398	0.0176413	0.0352006	0.5356133	0.0350399
171	0.0265634	0.0170911	0.0858887	0.0157112	0.0349022	0.5310729	0.0347002
140	0.0217525	0.0139958	0.0703336	0.0128658	0.0345472	0.5256718	0.0343204
112	0.0173997	0.0111951	0.0562592	0.0102913	0.0341762	0.5200262	0.0338917
107	0.0166246	0.0106964	0.0537533	0.0098328	0.0341039	0.5189262	0.0338115
106	0.0164687	0.0105961	0.0532489	0.0097406	0.0341017	0.5188932	0.0338069
91	0.0141386	0.0090969	0.0457151	0.0083625	0.0338729	0.5154111	0.0335409
82	0.0127383	0.0081959	0.0411873	0.0075342	0.0336632	0.5122204	0.0332955
57	0.0088505	0.0056945	0.0286168	0.0052347	0.0329459	0.5013056	0.0324475
40	0.0062171	0.0040004	0.0201021	0.0036772	0.0321913	0.4898234	0.0315918
39	0.0060615	0.0039000	0.0195989	0.0035851	0.0321094	0.4885777	0.0314986
26	0.0040399	0.0025993	0.0130624	0.0023894	0.0300802	0.4577017	0.0292896
17	0.0026410	0.0016992	0.0085393	0.0015621	0.0273734	0.4165147	0.0264409
11	0.0017116	0.0011013	0.0055342	0.0010123	0.0238031	0.3621886	0.0228048
9	0.0013991	0.0009002	0.0045238	0.0008275	0.0214120	0.3258058	0.0204462

IMF games is sub-optimal and instead propose a new value of the winning threshold which has firm theoretical justification. In the case of the EU game, the actual value of the winning threshold used is close to the optimal value. So, in this case, our analysis provides some quantitative backing to the value of the winning threshold that is actually used.

References

J. F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

R. Barua, S. R. Chakravarty, and P. Sarkar. Minimal-axiom characterizations of the coleman and banzhaf indices of voting power. *Mathematical Social Sciences*, 58:367–375, 2009.

R. Brink and G. Laan. Axiomatizations of the normalized banzhaf value and the shapley value. *Social Choice and Welfare*, 15(4):567–582, 1998.

S. R. Chakravarty. *Analyzing Multidimensional Well-Being: A Quantitative Approach*. John Wiley, New Jersey, 2017. in press.

S. R. Chakravarty and M. A. Lugo. Multidimensional indicators of inequality and poverty. In M. D. Adler and M. Fleurbaey, editors, *Oxford Handbook of Well-Being and Public Policy*, pages 246–285. Oxford University Press, New York, 2016.

S. R. Chakravarty, M. Mitra, and P. Sarkar. *A Course on Cooperative Game Theory.* Cambridge University Press, 2015.

J. S. Coleman. Control of collectives and the power of a collectivity to act. In B. Lieberma, editor, *Social Choice*, pages 269–298. Gordon and Breach, New York, 1971.

F. A. Cowell. Inequality and poverty measures. In M. D. Adler and M. Fleurbaey, editors, *Oxford Handbook of Well-Being and Public Policy*, pages 82–125. Oxford University Press, New York, 2016.

J. Deegan and E. W. Packel. A new index of power for simple n-person games. *International Journal of Game Theory*, 7(2):113–123, 1978.

P. Dubey and L. S. Shapley. Mathematical properties of the banzhaf power index. *Mathematics of Operations Research*, 4(2):99–131, 1979.

E. Einy and B. Peleg. Linear measures of inequality for cooperative games. *Journal of Economic Theory*, 53(2):328–344, 1991.

D. S. Felsenthal and M. Machover. *The Measurement of Voting Power*. Edward Elgar, Cheltenham, 1998.

M. J. Holler. Forming coalitions and measuring voting power. *Political Studies*, 30(2):262–271, 1982.

M. J. Holler and E. W. Packel. Power, luck and the right index. *Journal of Economics*, 43(1):21–29, 1983.

A. Laruelle and F. Valenciano. Shapley-shubik and banzhaf indices revisited. *Mathematics of Operations Research*, 26(1):89–104, 2001.

A. Laruelle and F. Valenciano. Inequality in voting power. *Social Choice and Welfare*, 22(2):413–431, 2004.

A. Laruelle and F. Valenciano. *Voting and collective decision-making.* Cambridge University Press, Cambridge, 2011.

D. Leech. Designing the voting system for the council of the european union. *Public Choice*, 113:437–464, 2002a.

D. Leech. Power in the governance of the international monetary fund. *Annals of Operations Research*, 109(1-4):375—397, 2002b.

E. Lehrer. An axiomatization of the banzhaf value. *International Journal of Game Theory*, 17(2):89–99, 1998.

T. Matsui and Y. Matsui. A survey of algorithms for calculating power indices of weighted voting games. *Journal of Operations Research Society of Japan*, 43:71–86, 2000.

L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games II (Annals of Mathematics Studies)*, pages 307–317. Princeton University Press, 1953.

L. S. Shapley and M. J. Shubik. A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48: 787–792, 1954.

A. F. Shorrocks. The class of additively decomposable inequality measures. *Econometrica*, 48(3):613–625, 1980.

M. Weber. Two-tier voting: Measuring inequality and specifying the inverse power problem. *Mathematical Social Sciences*, 79:40–45, 2016.
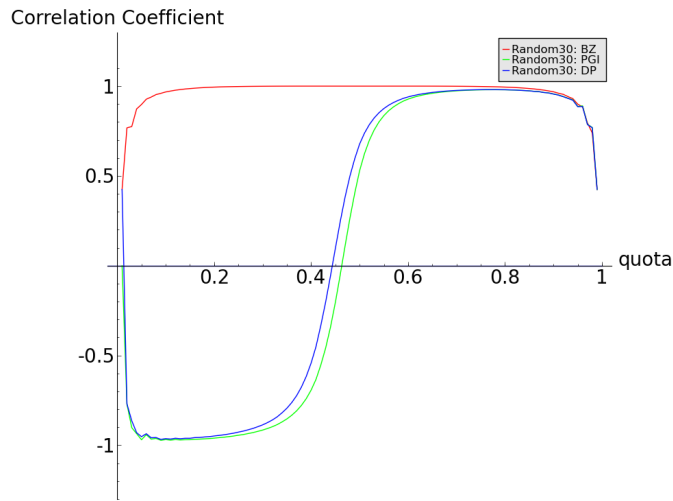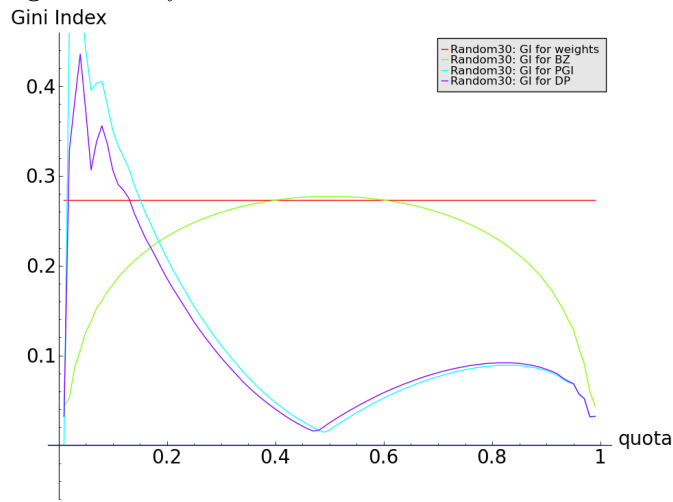
Correlation Coefficient

Fig. 1 Similarity indices for Case-I.

Gini Index

Fig. 2 Gini index for Case-I.

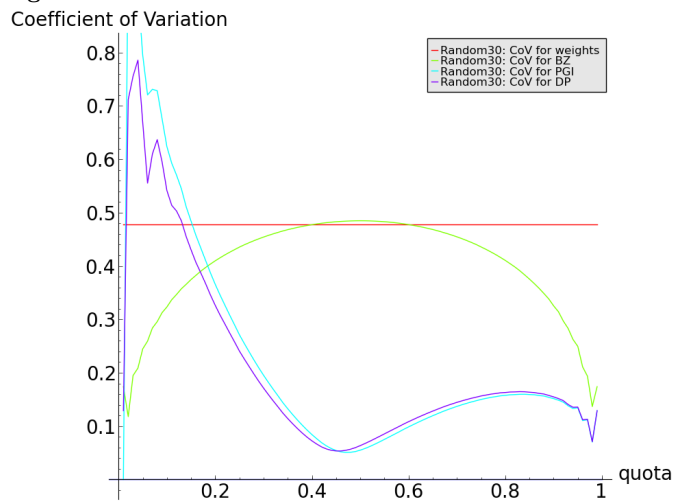Coefficient of Variation

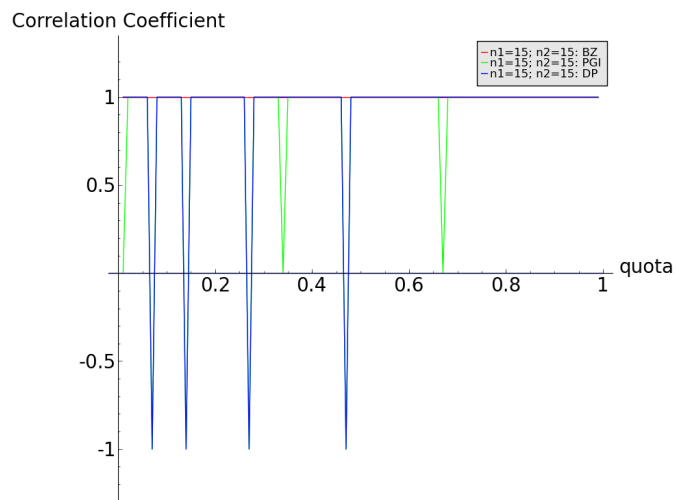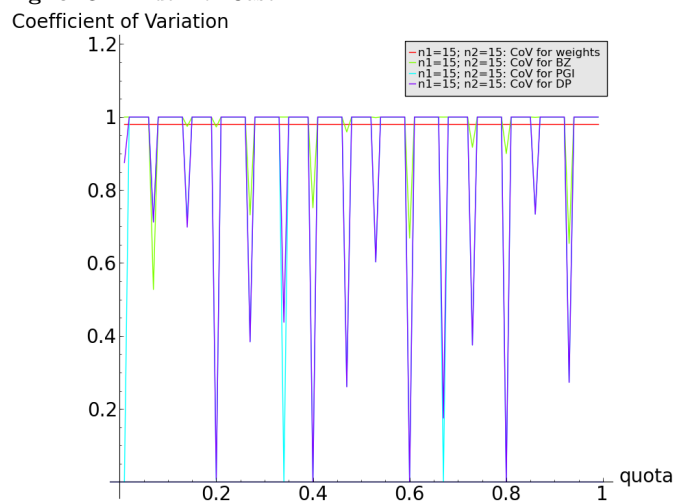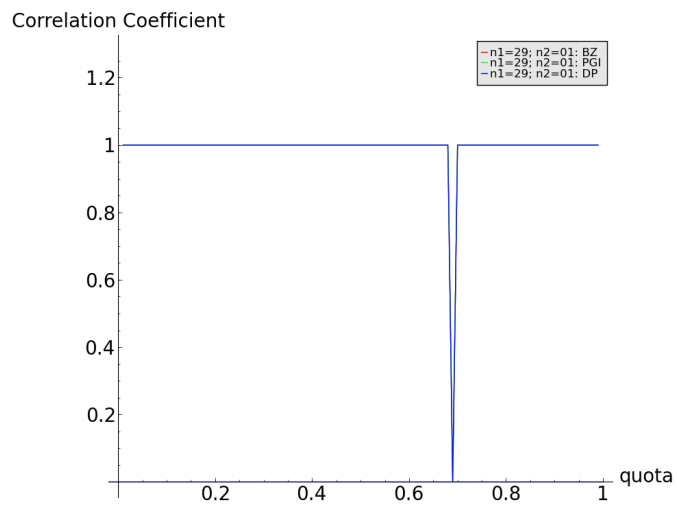**Fig. 3** Coefficient of Variation for Case-I.

Fig. 4 Similarity indices for Case-II.
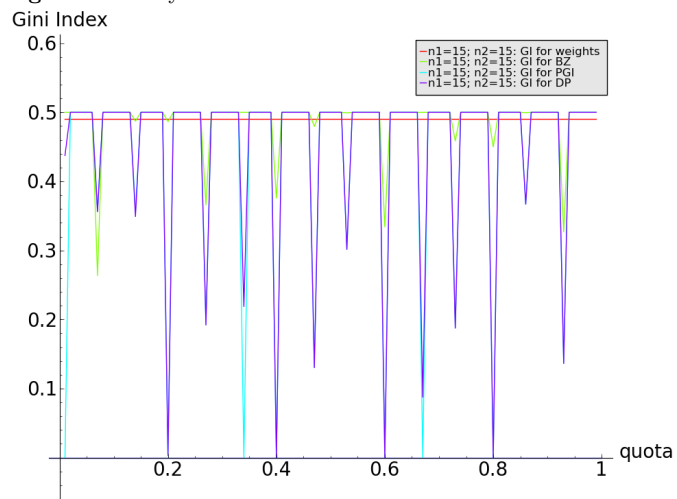
Fig. 5 Gini index for Case-II.

Fig. 6 Coefficient of Variation for Case-II.

Correlation Coefficient

Fig. 7 Similarity indices for Case-III.

Gini Index

Fig. 8 Gini index for Case-III.
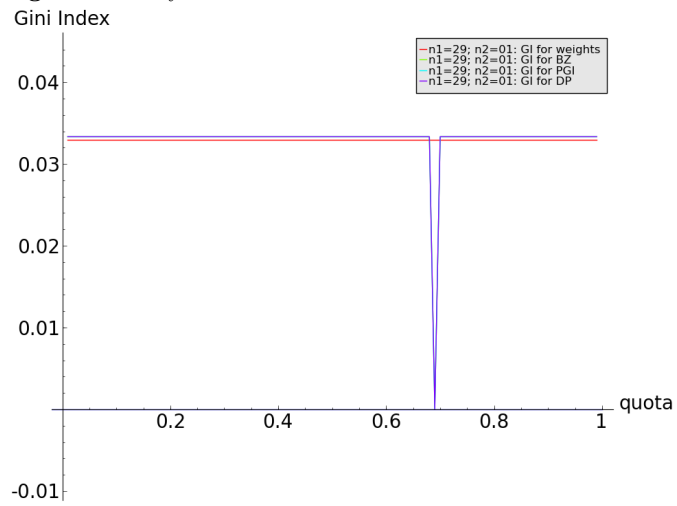
Coefficient of Variation
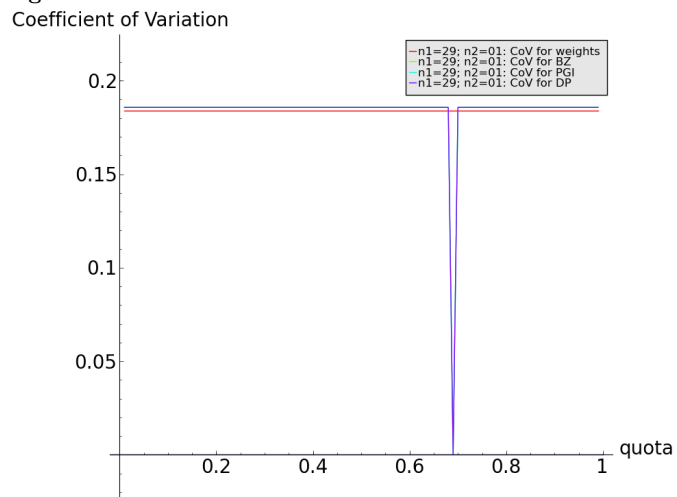
Fig. 9 Coefficient of Variation for Case-III.
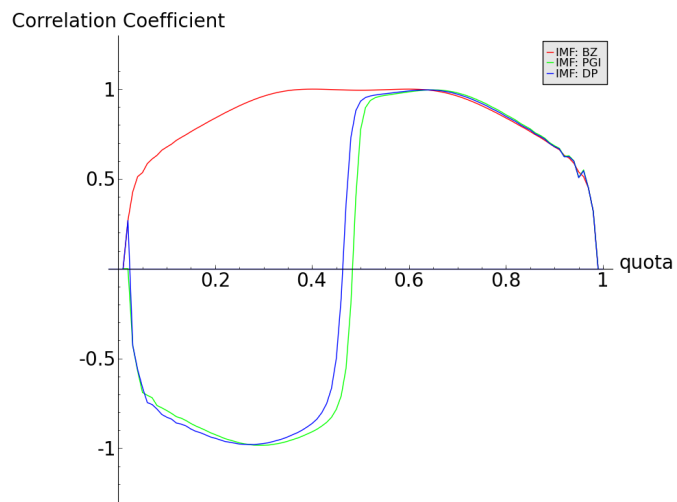
Fig. 10 Similarity indices for IMF EB weights.
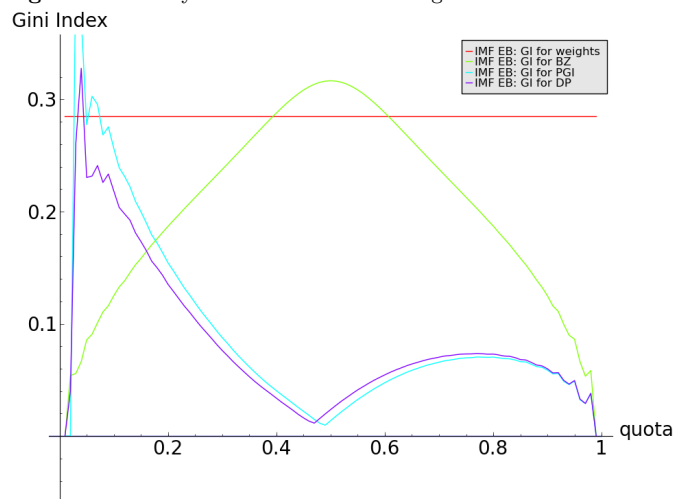
Fig. 11 Gini index for IMF EB weights.
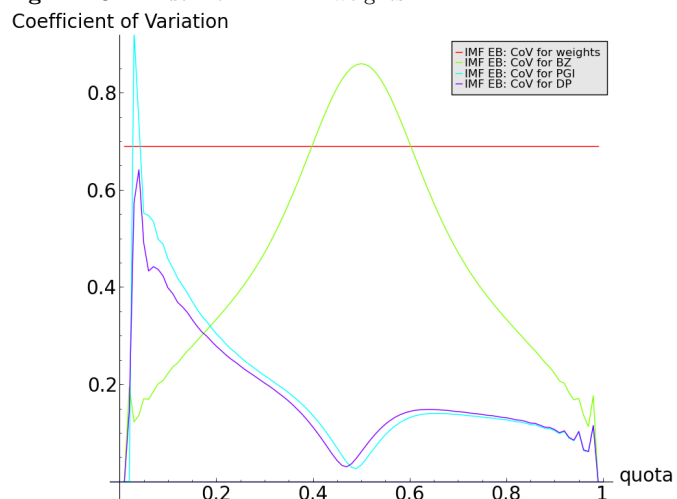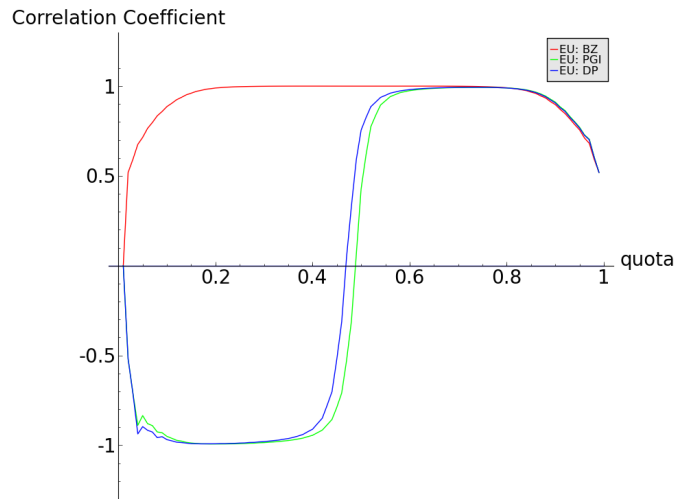

Fig. 12 Coefficient of Variation for IMF EB weights.

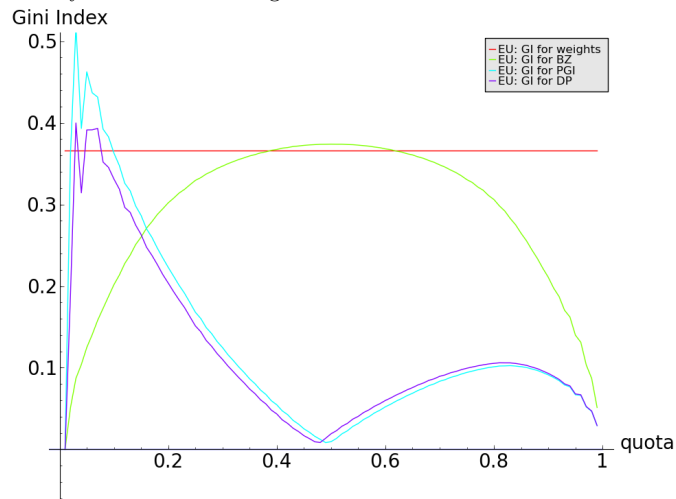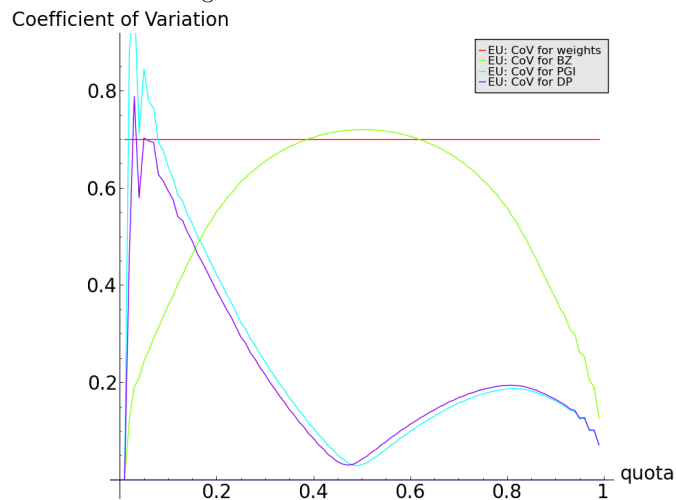**Fig. 13** Similarity indices for EU weights.

Fig. 14 Gini index for EU weights.

Fig. 15 Coefficient of Variation for EU weights.