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# Decentralized Sliding Mode Control for Output Tracking of Large-Scale Interconnected Systems*

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***Abstract*— In this paper, a class of nonlinear interconnected systems with matched and unmatched uncertainties is considered. The isolated subsystem dynamics are described by linear systems and nonlinear part. The matched uncertainties and unmatched unknown interconnection terms are assumed to be bounded by known functions. Based on sliding mode techniques, a state feedback decentralized control scheme is proposed such that the outputs of controlled interconnected systems track the given desired signals uniformly ultimately. The desired reference signals are allowed to be time-varying. Using multiple transformations, the considered system is transferred to a new interconnected system with an appropriate structure to facilitate the design of sliding surface and decentralized controller. A set of conditions is proposed to guarantee that the designed controller drives the tracking errors onto the sliding surface and the sliding mode of the error dynamics are uniformly ultimately bounded. The developed results are applied to river quality control. Simulation results show that the proposed decentralized control strategies are effective and feasible.**

## I. INTRODUCTION

Large-scale systems are often mathematically modelled by interconnections of a set of lower-dimensional subsystems. One of the characteristics of such systems is that each subsystem is usually affected by the others due to the interreactions between these subsystems. It should be noted that large-scale systems are usually distributed in space widely. Thus the designed systems should have a higher tolerance for the data-loss during data transformation, broken/unknown interconnections as well as poor communications to guarantee that the controlled large-scale systems have higher robustness. It is full of challenges to deal with large-scale interconnected systems. However, compared with centralised control, decentralized control needs local information only, and thus the information or data transfer between subsystems are not required. Specifically, when the network between different subsystems are broken, or the data transformation between subsystems are poor or unstable, the centralised control scheme cannot be implemented. Therefore, decentralized control has more advantages than centralized control and is the most popular choice in the control of large-scale interconnected systems [17].

Recently, the study on large-scale systems with the interconnected terms has made great progress, and many interesting results have been obtained. In [6], a large-scale fuzzy system with unknown interconnections was considered by Kim, Park and Joo, where matched uncertainties or disturbances are not included. There are also some results for interconnected systems (see, e.g. [7], [5], [11]) which require that interconnections are matched while unmatched interconnections and/or uncertainties are not involved in the systems. Moreover, some large-scale systems are considered just in a simple or ideal dynamic model (see, e.g. [14], [13], [4]). The structure of these considered systems lacks generality because the input only exists in one of the first-order dynamic equations. Decentralized sliding mode control is developed in [16] where the considered system is fully nonlinear with more general structure, but stabilization problem is considered while tracking problem is not involved.

Trajectory tracking and output tracking are a very important topic in both control theory and control engineering. Some tracking control results have been obtained in (see. [1], [7]). But most of the research objects are restricted to systems with a special structure (see [13], [4]). Decentralized tracking control for large-scale systems is considered in [9] where model reference control is investigated. Tracking control for the interconnected system is considered based on adaptive fuzzy techniques in [10]. It should be noted that in both [9] and [10], it is required that the isolated subsystems are linear.

Sliding mode control is very popular in dealing with complex systems with uncertainties due to its unique control characteristics ([18], [19]). On one hand, the sliding mode dynamics are a reduced-order system when compared with the original system ([17], [2]), which may simplify the corresponding system analysis and design. On the other hand, sliding mode control is totally robust to matched disturbances. Therefore, the sliding mode control method has been widely applied to deal with tracking problems, and many results have been achieved. Trajectory tracking control schemes based on sliding mode techniques are proposed for specific practical vehicles in (see. [20], [15]). The output tracking sliding mode control is designed in [12] where the considered system is linear. Although tracking control for nonlinear systems with uncertainties is considered in [3] where the event-triggered tracking is considered but only matched disturbances are considered. In [21], tracking problem for a class of large-scale systems with interconnections is considered using sliding mode control. However, it is

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required that the reference signals are constant. It should be emphasised that the results about output tracking for large-scale nonlinear interconnected systems with unknown interconnections are very few, specifically when the ideal reference signals are time-varying.

In this paper, a class of nonlinear interconnected systems is considered where both the unknown matched uncertainty and the unmatched nonlinear interconnections are considered. Suitable coordinate transformations are introduced to transfer the nominal subsystems in the interconnected system to systems with special structure. This makes each subsystem of the transferred system to separate into two parts to facilitate the system analysis and control design for output tracking. Then the tracking error dynamic systems are developed, and the sliding surface based on the tracking error system is designed. A set of conditions is proposed to guarantee the uniformly ultimately boundedness of the corresponding sliding motion. A decentralized sliding mode control scheme is proposed to drive the nonlinear interconnected systems to the designed sliding surface. Finally, the obtained results are applied to a river quality control to show the practicability and feasibility of the proposed approach.

## II. System description and basic assumptions

Consider a nonlinear large-scale system formed by $N$ interconnected subsystems as follows:

$$\begin{aligned} \dot{x}_i &= A_i x_i + f_i(x_i) + B_i(u_i + \Delta g_i(x_i)) + h_i(x) \\ y_i &= C_i x_i \qquad i = 1,2,...,N \end{aligned} \tag{1}$$

where $x = col(x_1, x_2, ..., x_N)$, $x_i \in R^{n_i}$, $u_i \in R^{m_i}$ and $y_i \in R^{m_i}$ represent the states, inputs and outputs of the $i$th subsystem respectively and $m_i < n_i$. The triple $(A_i, B_i, C_i)$ represents constant matrices of appropriate dimensions with $B_i$ and $C_i$ of full rank. The function $f_i(x_i)$ represents known nonlinear term in the $i$th subsystem, and the matched uncertainty of the $i$th isolated subsystem is denoted by $\Delta g_i(x_i)$. The terms $h_i(x)$ represent system interconnections including all unmatched uncertainties. All the nonlinear functions are assumed to be continuous in their arguments to guarantee the existence of solutions of the controlled system (1).

The object of this paper is, for a given desired signal $y_{id}(t)$, to design a decentralized sliding mode control

$$u_i = u_i(t, x_i(t), y_{id}(t))$$

such that the system output $y_i(t)$ of controlled system (1) can track the desired signal $y_{id}(t)$, i.e. the tracking errors $y_i(t) - y_{id}(t)$ are uniformly ultimately bounded for $i = 1,2,...,N$, while all the state variables of system (1) are bounded.

**Remark 1.** It should be noted that in this paper, it is required that system (1) is square, that is, the dimension of each subsystem output is equal to the dimension of the corresponding subsystem input. However, the developed results can be easily extended to the case when the dimension of subsystem output is bigger than the dimension of the subsystem input by slightly modification.

In order to deal with the tracking problem stated above, some assumptions are required to impose on the considered interconnected system (1).

**Assumption 1.** The pair $(A_i, B_i)$ is controllable and $\mathrm{rank}(C_i B_i) = m_i$ for $i = 1,2,...,N$.

It follows from the works in [17], [2]. Under Assumption 1, there exists a coordinate transformation $z_i = T_i x_i$ such that the triple $(A_i, B_i, C_i)$ with respect to the new coordinates $z_i$ has the following structure

$$\begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix}, \quad \begin{bmatrix} 0 & C_{i2} \end{bmatrix}$$

where $A_{i11} \in R^{(n_i - m_i)\times(n_i - m_i)}$, the square matrices $B_{i2} \in R^{m_i \times m_i}$ and $C_{i2} \in R^{m_i \times m_i}$ are nonsingular for $i = 1,2,...,N$.

**Assumption 2.** Suppose that $f_i(x_i)$ has the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$, where $\Gamma_i \in R^{n_i \times n_i}$ is a continuous function matrix for $i = 1,2,...,N$.

**Remark 2.** If $f_i(0) = 0$ and $f_i$ is sufficiently smooth, then the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$ is guaranteed. Therefore, the limitation to $f_i(x_i)$ in Assumption 2 is not strict.

**Assumption 3.**There exist known continuous functions $\rho_i(\cdot)$ such that

$\| \Delta g_i(x_i) \| \leq \rho_i(x_i)$ for $i = 1,2,...,N$.

**Assumption 4.** The desired output signal $y_{id}(t)$ is differentiable and satisfies

(i). $\| y_{id}(t) \| \leq L_{i1}$

(ii). $\| \dot{y}_{id}(t) \| \leq L_{i2}$

for $t \in [0, \infty)$, where $L_{i1}$ and $L_{i2}$ are known constants for $i = 1,2,...,N$.

**Remark 3.** Assumption 4 is the limitation to the desired output signals $y_{id}(t)$. It is required that the desired output signal $y_{id}(t)$ and its derivative $\dot{y}_{id}(t)$ are bounded. This assumption is quite standard and can be satisfied in most cases of reality.

## III. System structure analysis

Consider the nonlinear interconnected system in (1). Under Assumption 1, there exists a nonsingular coordinate transformation $z_i = T_i x_i$ such that in the new coordinate $z = col(z_1, z_2, ..., z_N)$, system (1) has the following form

$$\begin{aligned} \dot{z}_i &= \begin{bmatrix} A_{i11} & A_{i12} \\ A_{i21} & A_{i22} \end{bmatrix} z_i + \begin{bmatrix} F_{i1}(z_i) \\ F_{i2}(z_i) \end{bmatrix} + \begin{bmatrix} 0 \\ B_{i2} \end{bmatrix} (u_i + \Delta G_i(z_i)) \\ &\quad + \begin{bmatrix} H_{i1}(z) \\ H_{i2}(z) \end{bmatrix} \\ y_i &= \begin{bmatrix} 0 & I_{i2} \end{bmatrix} z_i, \qquad i = 1,2,...,N \end{aligned} \tag{2}$$

where $A_{i11}$ is stable, the square sub-matrices $B_{i2} \in R^{m_i \times m_i}$ are nonsingular. $I_{i2} \in R^{m_i \times m_i}$ is an identity matrix, $col(F_{i1}, F_{i2}) = T_i f_i(x_i)|_{x_i = T_i^{-1} z_i}$ and $F_{i1}(z_i) \in R^{n_i - m_i}$, $F_{i2}(z_i) \in R^{m_i}$. $\Delta G_i(z_i) = T_i \Delta g_i(x_i)|_{x_i = T_i^{-1} z_i}$, $col(H_{i1}(z), H_{i2}(z)) = T_i h_i(x)|_{x = T^{-1} z}$ and $H_{i1}(z) \in R^{n_i - m_i}$, $H_{i2}(z) \in R^{m_i}$. The coordinate transformation $T := col(T_1, T_2, ..., T_N)$.

Since $A_{i11}$ is stable for $i = 1,2,...,N$, for any $Q_i > 0$, the following *Lyapunov* equation has a unique solution $P_i > 0$ such that

$$A_{i11}^{\top} P_i + P_i A_{i11} = -Q_i, \qquad i = 1,2,...,N. \tag{3}$$

Now, in order to fully use the available structure characteristics, partition $z_i = col(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - m_i}$ and $z_{i2} \in R^{m_i}$. It follows that in the new coordinate $z$, system (2) has the following form

$$\begin{aligned} \dot{z}_{i1} &= A_{i11} z_{i1} + A_{i12} y_i + F_{i1}(z_{i1}, y_i) \\ &\quad + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \qquad (4) \\ \dot{y}_i &= A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i + \Delta G_i(z_i)) \\ &\quad + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) \qquad i = 1, 2, ..., N \qquad (5) \end{aligned}$$

From system (2) and Assumption 2,

$$\begin{aligned} col(F_{i1}, F_{i2}) &= T_i f_i(x_i)|_{x_i = T_i^{-1} z_i} \\ &= T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1} col(z_{1i}, y_i) \end{aligned} \tag{6}$$

It follows from (6) that the functions $F_{i1}(z_{i1}, y_i)$ in system (4)-(5) can be described by

$$\Gamma_{i11}(z_{i1}, y_i) z_{i1} + \Gamma_{i12}(z_{i1}, y_i) y_i = F_{i1}(z_{i1}, y_i) \tag{7}$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ are defined by

$$\begin{bmatrix} \Gamma_{i11}(\cdot) & \Gamma_{i12}(\cdot) \\ \star & \star \end{bmatrix} = T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1}$$

and the $\star$s are the function matrices which do not need to specify. Therefore, (4) can be described by

$$\begin{aligned} \dot{z}_{i1} = & A_{i11} z_{i1} + A_{i12} y_i + \Gamma_{i11}(z_{i1}, y_i) z_{i1} + \Gamma_{i12}(z_{i1}, y_i) y_i \\ & + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \end{aligned} \tag{8}$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ satisfy (7).

## IV. SLIDING MODE TRACKING CONTROL DESIGN

### A. Sliding Surface Design

Now, consider the desired output signal $y_{id}(t)$ satisfying Assumption 4. Then for system (1), the output tracking errors $e_i$ are defined by:

$$e_i(t) = y_i(t) - y_{id}(t) \qquad i = 1, 2, ..., N \tag{9}$$

and their first-time derivative is:

$$\dot{e}_i(t) = \dot{y}_i(t) - \dot{y}_{id}(t) \qquad i = 1, 2, ..., N \tag{10}$$

Combining with (4)-(5), a new system which consists of $z_{i1}$ and $e_i$ can be considered:

$$\begin{aligned} \dot{z}_{i1} &= A_{i11} z_{i1} + A_{i12} y_i + \Gamma_{i11}(z_{i1}, y_i) z_{i1} + \Gamma_{i12}(z_{i1}, y_i) y_i \\ &\quad + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \qquad (11) \\ \dot{e}_i &= A_{i21} z_{i1} + A_{i22}(e_i + y_{id}) + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i + \\ &\quad \Delta G_i(z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id}(t) \quad (12) \end{aligned}$$

for $i = 1, 2, ..., N$.

**Assumption 5.** It is easy to find a function $\gamma_i(\cdot)$ such that the following inequalities

$$\begin{aligned} & \|H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N)\| \\ & \leq \gamma_{i1}(T^{-1}(z_{11}, y_1, ..., z_{N1}, y_N))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_j \|) \end{aligned} \tag{13}$$

$$\begin{aligned} & \|H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N)\| \\ & \leq \gamma_{i2}(T^{-1}(z_{11}, y_1, ..., z_{N1}, y_N))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_j \|) \end{aligned} \tag{14}$$

hold for $i = 1, 2, ..., N$.

For systems (11)-(12), define the following sliding surface

$$col(e_1, e_2, ..., e_N) = 0 \tag{15}$$

Then, the sliding mode dynamics have the following form according to the structure of (11)-(12):

$$\begin{aligned} \dot{z}_{i1} = & A_{i11} z_{i1} + A_{i12} y_{id} + \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} \\ & + \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}) \end{aligned} \tag{16}$$

for $i = 1, 2, ..., N$.

**Remark 4.** From (13) in Assumption 5, when the states reached on the sliding surface, we could get

$$\begin{aligned} & \|H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})\| \\ & \leq \gamma_{i1}(T^{-1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_{jd} \|) \end{aligned} \tag{17}$$

hold for $i = 1, 2, ..., N$.

Obviously, the sliding surface (16) is a reduced-order interconnected system composed of $N$ subsystems whose dimension is $n_i - m_i$.

*Theorem 1:* Consider the sliding mode dynamics given in (16). Under Assumptions 1-5, the sliding mode is uniformly ultimately bounded if there exists a domain

$$\Omega = \{(z_{11}, z_{21}, ..., z_{N1}) | \, \| z_{i1} \| \leq c_i, \; i = 1, 2, ..., N\}$$

for some constants $c_i > 0$ such that $M^\top + M > 0$ in $\Omega \backslash \{0\}$ where $M := (m_{ij})_{N \times N}$ and

$$m_{ij} = \begin{cases} \lambda_{min}(Q_i) - \| R_i(\cdot) \| - 2 \| P_i \| \gamma_{i1}(\cdot), & i = j \\ -2 \| P_i \| \gamma_{i1}(\cdot), & i \neq j \end{cases} \tag{18}$$

with $P_i$ and $Q_i$ satisfying (3), and

$$R_i(\cdot) := \Gamma_{i11}(z_{i1}, y_{id})^\top P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})$$

where $\Gamma_{i11}(z_{i1}, y_i)$ is given by (2) and $\gamma_{i1}(\cdot)$ is determined by (17).

*Proof:* From the analysis above, what needs to be proved is that system (16) is uniformly ultimately bounded. For system (16), consider the following *Lyapunov* function candidate

$$V(z_{11}, z_{21}, ..., z_{N1}) = \sum_{i=1}^{N} (z_{i1})^\top P_i z_{i1} \tag{19}$$

where $P_i$ satisfies (3).

Then, the time derivative of $V(z_{11}, z_{21}, ..., z_{N1})$ along the trajectories of system (16) is given by

$$
\begin{aligned}
&\dot{V}(z_{11}, z_{21}, ..., z_{N1}) \\
&= \sum_{i=1}^{N} [(\dot{z}_{i1})^{\top} P_i z_{i1} + z_{i1}^{\top} P_i \dot{z}_{i1}] \\
&= \sum_{i=1}^{N} [z_{i1}^{\top} A_{i11}^{\top} P_i z_{i1} + y_{id}^{\top} A_{i12}^{\top} P_i z_{i1} + z_{i1}^{\top} \Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i z_{i1} \\
&\quad + y_{id}^{\top} \Gamma_{i12}(z_{i1}, y_{id})^{\top} P_i z_{i1} + H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})^{\top} P_i z_{i1} \\
&\quad + z_{i1}^{\top} P_i A_{i11} z_{i1} + z_{i1}^{\top} P_i A_{i12} y_{id} + z_{i1}^{\top} P_i \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} \\
&\quad + z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd})] \\
&= \sum_{i=1}^{N} \{ -z_{i1}^{\top} Q_i z_{i1} + z_{i1}^{\top} [\Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})] z_{i1} \\
&\quad + 2 z_{i1}^{\top} P_i A_{i12} y_{id} + 2 z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \\
&\quad + 2 z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}) \}
\end{aligned}
\tag{20}
$$

where (3) is used to establish the above. By (17) and (i) in Assumption 4, it follows that

$$
\begin{aligned}
&\dot{V}(z_{11}, z_{21}, ..., z_{N1}) \\
&\leq \sum_{i=1}^{N} \{ -\lambda_{min}(Q_i) \| z_{i1} \|^2 + \| \Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i \\
&\quad + P_i \Gamma_{i11}(z_{i1}, y_{id}) \| \| z_{i1} \|^2 + 2 \| z_{i1} \| \| P_i \| \| A_{i12} y_{id} \| \\
&\quad + 2 \| z_{i1} \| \| P_i \| \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| \\
&\quad + 2 \| z_{i1} \| \| P_i \| \| H_{i1}(z_{11}, y_{1d}, ..., z_{N1}, y_{Nd}) \| \} \\
&= -\sum_{i=1}^{N} \left\{ \lambda_{min}(Q_i) - \| R_i(\cdot) \| - 2 \| P_i \| \gamma_{i1}(\cdot) \right\} \| z_{i1} \|^2 \\
&\quad + 2 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \| P_i \| \| z_{i1} \| \gamma_{i1}(\cdot) (\| z_{j1} \| + L_{i1}) \\
&\quad + 2 \sum_{i=1}^{N} (\| A_{i12} y_{id} \| + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \|) \cdot \| P_i \| \| z_{i1} \| \\
&\leq -\frac{1}{2} \lambda_{min}(M^{\top} + M) \sum_{i=1}^{N} \| z_{i1} \|^2 + 2 \sum_{i=1}^{N} (\| A_{i12} y_{id} \| \\
&\quad + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| + \gamma_{i1}(\cdot) L_{i1}) \cdot \| P_i \| \| z_{i1} \| \\
&= -\frac{1}{2} \sum_{i=1}^{N} \{ \lambda_{min}(M^{\top} + M) \| z_{i1} \| - 4 (\| A_{i12} y_{id} \| \\
&\quad + \| \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \| + \gamma_{i1}(\cdot) L_{i1}) \| P_i \| \} \| z_{i1} \|
\end{aligned}
\tag{21}
$$

where the matrix $M$ is defined in (18). Under Assumption 4, $\| y_{id}(t) \| \leq L_{i1}$. It is clear to check $\dot{V} \leq 0$, if

$$
\| z_{i1} \| \geq \frac{4(\| A_{i12} L_{i1} \| + \| \Gamma_{i12}(z_{i1}, y_i) L_{i1} \| + \gamma_{i1}(\cdot) L_{i1}) \| P_i \|}{\lambda_{min}(M^{\top} + M)}
$$

for $i = 1, 2, .., N$. Hence, the conclusion follows. ∎

## V. Decentralized sliding mode control

For the interconnected system (1), the reachability condition [17], [16] is described by

$$
\sum_{i=1}^{N} \frac{e_i^{\top}(t) \dot{e}_i(t)}{\| e_i(t) \|} < 0 \tag{22}
$$

Then, the following control law is proposed

$$
\begin{aligned}
u_i \;=\; & -B_{i2}^{-1} \frac{e_i}{\| e_i \|} \{ \| A_{i21} z_{i1} \| + \| A_{i22} y_i \| + \| F_{i2}(z_{i1}, y_i) \| \\
& + \| B_{i2} \| \rho_i(z_{i1}, y_i) + k_i(z_{i1}, y_i) + L_{i2} \}
\end{aligned}
\tag{23}
$$

for $i = 1, 2, ..., N$, where $e_i$ and $L_{i2}$ are defined by (9) and Assumption 4, respectively. $k_i(z_{i1}, y_i)$ is the control gain to be designed later.

*Theorem 2:* Consider the nonlinear interconnected system (11)–(12). Under Assumptions 2-5, the controller (23) drives the system (11)–(12) to the composite sliding surface (16) and maintains a sliding motion on it if the controller gains $k_i(z_{i1}, y_i)$ satisfy

$$
\sum_{i=1}^{N} k_i(z_{i1}, y_i) > \sum_{i=1}^{N} \gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|) \tag{24}
$$

where $\gamma_{i2}$ are defined by Assumption 5.

*Proof:* From the analysis above, all that needs to be proved is the reachability condition (22) is satisfied. From (12) and Assumption 2,

$$
\begin{aligned}
\dot{e}_i = & A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) + B_{i2}(u_i \\
& + \Delta G_i(z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id}
\end{aligned}
\tag{25}
$$

for $i = 1, 2, ..., N$. From (23)-(25), it follows

$$
\begin{aligned}
\frac{e_i^{\top} \dot{e}_i}{\| e_i \|} = & \frac{e_i^{\top}}{\| e_i \|} \{ A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) \\
& + B_{i2} \Delta G_i((z_{i1}, y_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) - \dot{y}_{id} \} \\
& - \| A_{i21} z_{i1} \| - \| A_{i22} y_i \| - \| F_{i2}(z_{i1}, y_i) \| \\
& - \| B_{i2} \| \rho_i(z_{i1}, y_i) - k_i(z_{i1}, y_i) - L_{i2}
\end{aligned}
\tag{26}
$$

It is clear to see

$$
\begin{aligned}
& \| A_{i21} z_{i1} + A_{i22} y_i + F_{i2}(z_{i1}, y_i) \| \\
& \qquad \leq \| A_{i21} z_{i1} \| + \| A_{i22} y_i \| + \| F_{i2}(z_{i1}, y_i) \|
\end{aligned}
\tag{27}
$$

From Assumptions 3-5,

$$
\| B_{i2} \Delta G_i(z_{i1}, y_i) \| \leq \| B_{i2} \| \rho_i(z_{i1}, y_i) \tag{28}
$$

$$
\| H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N) \| \leq \gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|) \tag{29}
$$

$$
\| \dot{y}_{id} \| \leq L_{i2} \tag{30}
$$

Substituting the above four inequalities (27)-(30) into (26), it follows

$$
\sum_{i=1}^{N} \frac{e_i^{\top}(t) \dot{e}_i(t)}{\| e_i(t) \|} < -\sum_{i=1}^{N} k_i(z_{i1}, y_i) + \sum_{i=1}^{N} \gamma_{i2}(\cdot) \sum_{j=1}^{N} \| z_{j1} \|
$$

If $k_i(z_{i1}, y_i)$ is chosen to satisfy (24), then the reachability condition (22) is satisfied.

Hence, the result follows. ∎

**Remark 5.** Theorem 1 shows that the sliding mode (16) which is an interconnected system, is uniformly ultimately bounded. Theorem 2 shows that the designed control (23) can drive the considered system (11)–(12) to the sliding surface (15). According to the sliding mode theory, Theorems 1 and 2 show that the controlled systems (11)–(12) are uniformly ultimately bounded.

From Remark 7, it follows that the closed-loop systems formed by applying the control (24) to the systems (11)–(12) are uniformly ultimately bounded, which implies that the variables $\|z_{i1}(t)\|$ and $\|e_i(t)\|$ are bounded for $i = 1, 2, \ldots, N$. Further, from $e_i(t) = y_i(t) - y_{id}(t)$ and the Assumption 4 that $y_{id}(t)$ is bounded, it is straightforward to see that $y_i(t)$ are bounded due to $y_i(t) = e_i(t) + y_{id}(t)$, for $i = 1, 2, \ldots, N$. Therefore, all the state variables of the system (4)–(5) are bounded. This shows that the designed decentralized control (24) can not only make the system outputs track the desired reference signals but also keep all the state variables of the system (4)–(5) bounded.

## VI. Application to Water Quality Control

In this section, the decentralized control scheme developed in this paper will be applied to a river pollution problem [8]. The water quality of a river is mainly described by the concentrations of oxygen and pollutants. In a simplified way, this problem can be stated as the task to control the sewage discharge at different places along the river in such a way that the river pollution remains within a given band of tolerance.

Assume that the river has two regions and each region has a sewage station. Then, the river pollution system can be described by a nonlinear interconnected system as follows (see, [17])

$$\begin{aligned} \dot{x}_1 &= \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_1} (u_1 + \Delta g_1(\cdot)) \\ &\quad + \underbrace{\begin{bmatrix} sin(x_{21}) \\ 0 \end{bmatrix}}_{h_1} \quad y_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_1} x_1 \end{aligned} \tag{31}$$

$$\begin{aligned} \dot{x}_2 &= \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_2} (u_2 + \Delta g_2(\cdot)) \\ &\quad + \underbrace{\begin{bmatrix} -0.9x_{11} \\ 0 \end{bmatrix}}_{h_2} \quad y_2 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_2} x_2 \end{aligned} \tag{32}$$

where $x_1 := col(x_{11}, x_{12})$ and $x_2 := col(x_{21}, x_{22})$. The variables $x_{i1}$ and $x_{i2}$ for $i = 1, 2$. represent the concentration of biochemical oxygen demand (BOD) and the concentration of dissolved oxygen respectively, the controllers $u_i$ are the BOD of the effluent discharge into the river, $\Delta g_i$ represent the matched uncertainties added by us which is not the inherent property of the system, $h_i$ represent interconnections respectively for $i = 1, 2$. It is assumed that the concentrations of BOD for the two regions are measurable.

In this example, according to (1) the nonlinear term $f_i(x_1) = 0$, so the Assumption 2 can be ignored here. The matched uncertainties $\Delta g_1(\cdot)$ and $\Delta g_2(\cdot)$ are assumed to satisfy

$$\Delta g_1(\cdot) = -13.2x_{11} \quad \Delta g_2(\cdot) = \cos^2(x_{21}) \tag{33}$$

According to (31)-(32), the interconnections satisfy

$$\| h_1 \| \leq 1 \cdot |x_{21}| \quad \| h_2 \| \leq |0.9 \cdot x_{11}| \tag{34}$$

Combining with (33)-(34), it is clear that the Assumption 3 is satisfied.

Moreover, it can be verified that $rank(C_i B_i) = 1 = m_i$ for $i = 1, 2$. So the Assumption 1 is satisfied.

Some suitable coordinate transformation matrices $T_i$ are introduced as below: ($z_i = T_i x_i$)

$$T_1 = T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, the system (31)-(32) in $z$ coordinates can be given by

$$\begin{aligned} \dot{z}_1 &= \underbrace{\begin{bmatrix} -1.2 & -0.32 \\ 0 & -1.32 \end{bmatrix}}_{\hat{A}_1} z_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}_1} (u_1 - 13.2z_{12}) \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ \sin(z_{22}) \end{bmatrix}}_{H_1} \quad y_1 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\hat{C}_1} z_1 \end{aligned} \tag{35}$$

$$\begin{aligned} \dot{z}_2 &= \underbrace{\begin{bmatrix} -1.2 & -0.32 \\ 0 & -1.32 \end{bmatrix}}_{\hat{A}_2} z_2 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}_2} (u_2 + \cos^2(z_{22})) \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ -0.9z_{12} \end{bmatrix}}_{H_2} \quad y_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\hat{C}_2} z_2 \end{aligned} \tag{36}$$

and the sliding surfaces $S_i$ are: $\dot{z}_{i1} = -1.2z_{i1} - 0.32z_{i2}$, $i = 1, 2$.

For simulation purposes, the initial states are chosen as $z_1(0) = col(0, 1)$ and $z_2(0) = col(0, 0)$, and the desired output signals $y_{id}$ are chosen as: $y_{1d} = 2 \cdot e^{-t}$, $y_{2d} = \sin(0.5t) + 1$.

It is clear that the Assumption 4 is satisfied. Let $L_{12} = 2$, $L_{22} = 0.5$.

From (23), the proposed sliding mode controllers are as follows:

$$u_1 = -\frac{y_1 - y_{1d}}{|y_1 - y_{1d}|} (|1.32z_{12}| + |13.2z_{12}| + 3) \tag{37}$$

$$u_2 = -\frac{y_2 - y_{2d}}{|y_2 - y_{2d}|} (|1.32z_{22}| + |\cos^2(z_{22})| + 2.3) \tag{38}$$

According to (3), choose $Q_1 = Q_2 = 1$. Combining with (31)-(32), $A_{i11} = -1.2$ for $i = 1, 2$. Then $P_1 = P_2 = 0.416$. And by direct calculation, it follows from (18) that

$$M^\top + M = \begin{bmatrix} -1.664\gamma_{11} + 2 & -0.832(\gamma_{11} + \gamma_{21}) \\ -0.832(\gamma_{11} + \gamma_{21}) & -1.664\gamma_{21} + 2 \end{bmatrix}$$

According to (17), (35) and (36), $\gamma_{11} = 6 \cdot \sin(z_{11})$, $\gamma_{21} = 3 \cdot \cos(z_{21}) - 2$. And by direct verification, it is easy to check that $M^\top + M > 0$, if $|z_{11}| \leq d_1 = 5.2$, $|z_{21}| \leq d_2 = 3.9$.

According to (21) for this example: $\dot{V}(z_{11}, z_{21}) \leq 0$, if $|z_{11}| \geq 0.3$ and $|z_{21}| \geq 0.25$. Therefore, the system (31)-(32) is uniformly ultimately bounded.

The tracking results are shown in Fig. 1. The concentration of biochemical oxygen demand (BOD) of each subsystem $y_i$ can track the ideal reference $y_{id}$ with the inputs of the designed controller in (37)-(38), even in the presence of uncertainties. The time responses of the states of the system (31)-(32) are shown in Fig. 2 which indicates that the system states are bounded. Simulation results demonstrate that the developed method in this paper is effective.

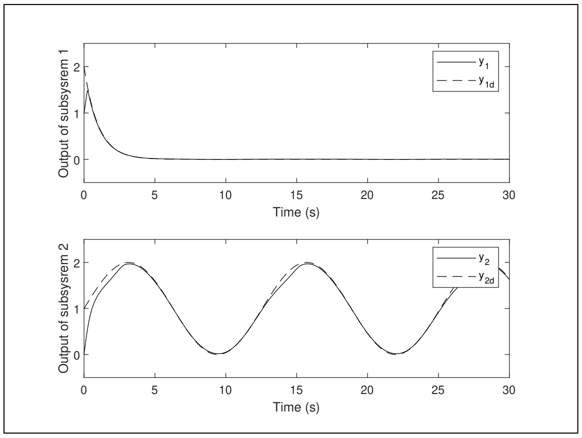


Fig. 1. Time responses of system outputs and desired outputs.

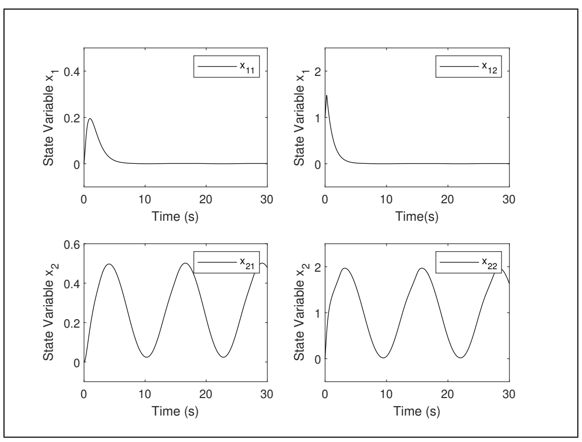


Fig. 2. Time responses of system state variables.

## VII. CONCLUSIONS

This paper has presented a sliding mode control strategy to deal with the output tracking problem of a class of large-scale systems with unmatched unknown nonlinear interconnections. The desired reference signals are allowed to be time-varying. A decentralized sliding mode control scheme has been proposed to satisfy the reachability condition driving the interconnected system onto the pre-designed sliding surface. A set of conditions is developed to guarantee that the output tracking errors are uniformly ultimately bounded while all the state variables of the interconnected system are bounded. The application of the developed result to the river pollution control system have demonstrated that the proposed approaches are effective and practicable.

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