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Relevance Vector Regression for Remaining Useful Life Prediction

Prof. Shaomin Wu

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- This presentation is based on the paper: Wang, X., Jiang, B., Wu, S., Lu, N. and Ding, S., 2021. Multivariate Relevance Vector Regression based Degradation Modeling and Remaining Useful Life Prediction. *IEEE Transactions on Industrial Electronics*, doi: 10.1109/TIE.2021.3114724.
- I would like to thank my co-authors of the paper;
- I would also like to thank the organisers of *the International Conference in Mathematical Modeling in Physical Sciences, Social Sciences and Technology (icmm-21)* for their invitations.

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Importance of RUL Prediction and our methods

- **Remaining Useful Life (RUL)** RUL is the length of an industrial item from its current time to the end of its useful life.
- **Importance of RUL estimation** It is needed in condition based maintenance, prognostics and health management.
- **Some methods for RUL estimation**
  - With lifetime data and other data, one can build a Cox regression model and then derive the probability distribution of the RUL,
  - One may also estimate the distribution of the first hitting time based on a degradation path, which will be adopted in this talk.

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Regression with OLS, Ridge, LASSO, ElasticNet

- **Ordinary least squares (OLS):** Given a set of data points, $\{(\boldsymbol{x}_1, y_1), \ldots, (\boldsymbol{x}_n, y_n)\}$, such that $\boldsymbol{x}_i (= (1, x_{i1}, \ldots, , x_{ip})^T) \in R^{n+1}$ is a feature vector for the ith case and $y_i \in R^1$ is a target output, one can build a linear regression model

$$y_n = \mathbf{w}^{\mathrm{T}} \mathbf{x} + \epsilon_n, ,$$

with the aim to minimise the error on the training datasets

$$\min \sum_{i=1}^{n} (y_i - \boldsymbol{x}_i^T \boldsymbol{w}) \tag{1}$$

where $\boldsymbol{w} = (w_0, w_1, \ldots, w_p)^T$ is the weight vector.

- **Extensions:** Ridge, LASSO (least absolute shrinkage and selection operator), and ElasticNet are extensions of OLS, with an additional penalty on the weights that aims to maximise the generalisation.

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Support Vector Regression (SVR)

- **SVR** aims to minimize the weights,

$$\min \ \frac{1}{2}\boldsymbol{w}^T\boldsymbol{w} \tag{2}$$

$$|y_i - \boldsymbol{x}_i^T\boldsymbol{w}| \leq \epsilon \tag{3}$$

where $\boldsymbol{w}$ is the weight vector and ϵ is the error term.

- ϵ-**SVR** is given by

$$\min_{w,b,\xi,\xi^*} \quad \frac{1}{2}w^T w + C\left(\sum_{i=1}^{l}\xi_i + \sum_{i=1}^{l}\xi_i^*\right)$$

$$y_i - w^T\phi(x_i) - b \leq \epsilon + \xi_i,$$

$$w^T\phi(x_i) + b - y_i \leq \epsilon + \xi_i^*,$$

$$\xi_i, \xi_i^* \geq 0, i = 1, \ldots, l.$$

where C is the regularization parameter and balances the trade-off between the model complexity and empirical error, ξ_i and ξ_i^* are slack variables.

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Limitations of SVR

- SVR has been used in many applications Nevertheless, they suffer some limitations:
  - **non-probabilistic:** SVR does not output probabilistic predictions;
  - C **and** ϵ**:** parameters of C and ϵ must be determined by cross-validation; and
  - **Mercer's condition:** the kernels must satisfy Mercer's condition. Relevance vector regression To overcome the above limitations, Tipping introduces a novel model: Relevance Vector Regression

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Relevance Vector Regression (RVR)

- Assume the relationship between $\{\boldsymbol{x}_n\}$ and $\{y_n\}_{n=1}^{N}$ is:

$$y_n = \mathbf{w}^{\mathrm{T}}\phi(\mathbf{x}) + \epsilon_n,$$

  where ϵ_n are i.i.d with the Gaussian distribution having mean-zero and variance σ^2.

- Due to the assumption of independence of the t_n, the likelihood of the complete data set can be written as

$$p(\mathbf{y}|\mathbf{w},\sigma^2) = (2\pi\sigma^2)^{-N/2}\exp\left\{-\frac{1}{2\sigma^2}\|\mathbf{y}-\mathbf{\Phi}\mathbf{w}\|^2\right\}, \quad (4)$$

  where $\mathbf{y} = (y_1 \dots y_N)^{\mathrm{T}}$, $\mathbf{w} = (w_0 \dots w_N)^{\mathrm{T}}$ and $\mathbf{\Phi}$ is the $N \times (N+1)$ 'design' matrix with $\mathbf{\Phi} = [\phi(\mathbf{x}_1), \phi(\mathbf{x}_2), \dots, \phi(\mathbf{x}_N)]^{\mathrm{T}}$, wherein $\phi(\mathbf{x}_n) = [1, K(\mathbf{x}_n,\mathbf{x}_1), K(\mathbf{x}_n,\mathbf{x}_2), \dots, K(\mathbf{x}_n,\mathbf{x}_N)]^{\mathrm{T}}$ and $K(.,.)$ is a kernel function.

RVR (cont'd)

- Tipping assumes a zero-mean Gaussian prior distribution over **w**:

$$p(\mathbf{w}|\boldsymbol{\alpha}) = \prod_{i=0}^{N} \mathcal{N}(w_i|0, \alpha_i^{-1}), \tag{5}$$

  with $\boldsymbol{\alpha}$ a vector of $N+1$ *hyperparameters*.

- The suitable priors for α and α are Gamma distributions:

$$p(\boldsymbol{\alpha}) = \prod_{i=0}^{N} \text{Gamma}(\alpha_i|a, b),$$

$$p(\beta) = \text{Gamma}(\beta|c, d),$$

  with $\beta \equiv \sigma^{-2}$ and where $\text{Gamma}(\alpha|a, b) = \Gamma(a)^{-1} b^a \alpha^{a-1} e^{-b\alpha}$, with $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$, the 'gamma function'.

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Pros and Cons of RVR

- **Pros**
  - The relevance vector machine or RVM (Tipping, 2001) is a probabilistic regression model under the Bayesian framework;
  - compared to the SVR, the RVR results in a sparser model;
  - The kernels do not need to satisfy Mercer's condition;
- **Con** RVR only allows regression from multivariate inputs to a univariate output variable, but not to multiple variables. The operation of many engineering systems is influenced by multiple variables. For example, current and voltage are indispensable for electrical systems.

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Literature review and our methods

RVR has been used to predict the remaining useful life.

- **Literature review**
  - Some authors combine a set of univariate RVR models to output multiple variables, and
  - Some authors regard that the weight matrix is inducted by separating into a vector distribution;
- **Our methods:**
  - We propose a multivatiate RVR model (MRVR), in which the weight matrix is a matrix Gaussian distribution;
  - The hyperparameters of the MRVR model are estimated by Nesterov's Accelerated Gradient (NAG) method to obtain numerical solutions

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Multivariate RVR

- An MRVR is proposed as following

$$\boldsymbol{x}_{n+l} = \boldsymbol{W}\phi(\boldsymbol{x}_n) + \epsilon, \tag{6}$$

where

- $\boldsymbol{x}_{n+l} = [x_{1,n+l}, \cdots, x_{M,n+l}]^{\mathrm{T}} \in \mathbb{R}^M$ is the $l-$step forward prediction vector, and $1 < n + l \leq N$;
- $\phi(\boldsymbol{x}_n) = [\mathbf{1}, \mathcal{K}(\boldsymbol{x}_n, \boldsymbol{x}_1), \cdots, \mathcal{K}(\boldsymbol{x}_n, \boldsymbol{x}_N)] \in \mathbb{R}^{N+1}$ denotes a design vector, in which $\mathcal{K}(\boldsymbol{x}_n, \boldsymbol{x}_j) \in \mathbb{R}$ is a kernel function,
- $\boldsymbol{W} \in \mathbb{R}^{M \times (N+1)}$ is a weight matrix of the design vector $\phi(\boldsymbol{x}_n) \triangleq \phi$, and
- ϵ is assumed to be a Gaussian distributed random error vector with the zero mean and a diagonal covariance matrix $\Sigma_0 = diag\{\sigma_1^2, \sigma_2^2, \cdots, \sigma_M^2\} \in \mathbb{R}^{M \times M}$, and $diag(\cdot)$ denotes a diagonal matrix.

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PDF of x_{n+1}

Probability Density Function (PDF) of $\boldsymbol{x}_{n+l}$ conditioned on $\boldsymbol{W}$ and Σ_0 can be written by

$$p(\boldsymbol{x}_{n+l}|\boldsymbol{W}, \Sigma_0) = (2\pi)^{-\frac{M}{2}}|\Sigma_0|^{-\frac{1}{2}} \times \exp\left(-\frac{1}{2}(\boldsymbol{x}_{n+l} - \boldsymbol{W}\phi)^{\mathrm{T}}\Sigma_0^{-1}(\boldsymbol{x}_{n+l} - \boldsymbol{W}\phi)\right), \quad (7)$$

here $|\cdot|$ is the determinant of a square matrix.

Prior and Posterior distributions of W

- **Prior distributions of W** To avoid the over-fitting problem of model (6), a prior matrix Gaussian distribution is assigned on the $M \times (N+1)$ dimension weight matrix $\boldsymbol{W}$, which is denoted as $\boldsymbol{W} \sim \mathcal{MN}_{M,N+1}(\mathbf{0}, \boldsymbol{\Psi}, \boldsymbol{\Gamma})$, which gives

$$p(\boldsymbol{W}|\boldsymbol{\Psi},\boldsymbol{\Gamma}) = (2\pi)^{-\frac{M(N+1)}{2}}|\boldsymbol{\Psi}|^{-\frac{N+1}{2}}|\boldsymbol{\Gamma}|^{-\frac{M}{2}} \times \operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Gamma}^{-1}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{\Psi}^{-1}\boldsymbol{W}\right), \tag{8}$$

  where $\operatorname{etr}(\cdot)$ is the exponential of trace of a function of the trace of the matrix.

- **Posterior distributions of W** The posterior distribution of weight matrix $\boldsymbol{W}$ is matrix Gaussian, and its PDF is formulated in the following form.

$$p(\boldsymbol{W}|\boldsymbol{x}_{n+l},\Psi,\Gamma,\Sigma_0) = (2\pi)^{-\frac{M(N+1)}{2}}|\tilde{\Psi}|^{-\frac{N+1}{2}} \times|\tilde{\Gamma}|^{-\frac{M}{2}} \operatorname{etr}\left(-\frac{1}{2}\tilde{\Gamma}^{-1}(\boldsymbol{W}-\tilde{\mu})^{\mathrm{T}}\tilde{\Psi}^{-1}(\boldsymbol{W}-\tilde{\mu})\right). \tag{9}$$

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Parameter Estimation

- **Prediction distribution** The distribution of the predicted $\mathbf{x}_{k+l}$, based on the former prediction $\mathbf{x}_{n+l}$, can be obtained by

$$p\left(\mathbf{x}_{k+l} | \mathbf{x}_{n+l}\right) = \iiiint p\left(\mathbf{x}_{k+l} | \boldsymbol{W}, \boldsymbol{\Sigma}_0\right) p\left(\boldsymbol{W} | \mathbf{x}_{n+l}, \boldsymbol{\Psi}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}_0\right) \times p\left(\boldsymbol{\Psi}, \boldsymbol{\Gamma}, \boldsymbol{\Sigma}_0 | \mathbf{x}_{n+l}\right) \mathrm{d}\boldsymbol{W} \mathrm{d}\boldsymbol{\Psi} \mathrm{d}\boldsymbol{\Gamma} \mathrm{d}\boldsymbol{\Sigma}_0. \tag{10}$$

- **Marginal likelihood function** The marginal likelihood function $p\left(x_{n+l} | \Psi, \Gamma, \Sigma_0\right)$ can be obtained by integrating over the weight parameters W as

$$p\left(x_{n+l} | \Psi, \Gamma, \Sigma_0\right) = \int p\left(x_{n+l} | W, \Sigma_0\right) p(W | \Psi, \Gamma) \mathrm{d}W = \int p\left(x_{n+l} | \mathrm{vec}\left(W^{\mathrm{T}}\right), \Sigma_0\right) p(\mathrm{vec}\left(W^{\mathrm{T}}\right) | \Psi, \Gamma) \mathrm{d}\,\mathrm{vec}\left(W^{\mathrm{T}}\right). \tag{11}$$

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Parameter Estimation (cont'd)

- The negative log of the marginal likelihood is acquired

$$\mathcal{L}\left(\boldsymbol{x}_{n+l}|\boldsymbol{\Psi},\boldsymbol{\Gamma},\boldsymbol{\Sigma}_0\right) = \frac{1}{2}\ln|\boldsymbol{\Sigma}_0| + \frac{N+1}{2}\ln|\boldsymbol{\Psi}| + \frac{M}{2}\ln|\boldsymbol{\Gamma}| + \frac{1}{2}\ln|\boldsymbol{A}| + E(\boldsymbol{\mu}) + \frac{M}{2}\ln(2\pi), \tag{12}$$

- set $\frac{\partial \mathcal{L}}{\partial \Psi_i} = 0$, $\frac{\partial \mathcal{L}}{\partial \Gamma_j} = 0$, and $\frac{\partial \mathcal{L}}{\partial \sigma_i^2} = 0$, it is difficult to obtain explicit solutions of the hyperparameters Ψ_i, Γ_j and σ_i^2.
- The NAG (Nesterov's Accelerated Gradient) method is used to obtain numerical solutions of the hyperparameters

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$$\mathcal{L}\left(\boldsymbol{x}_{n+l}|\boldsymbol{\Psi},\boldsymbol{\Gamma},\boldsymbol{\Sigma}_0\right) = \frac{1}{2}\ln|\boldsymbol{\Sigma}_0| + \frac{N+1}{2}\ln|\boldsymbol{\Psi}| + \frac{M}{2}\ln|\boldsymbol{\Gamma}| + \frac{1}{2}\ln|\boldsymbol{A}| + E(\boldsymbol{\mu}) + \frac{M}{2}\ln(2\pi), \tag{12}$$

- set $\frac{\partial\mathcal{L}}{\partial\Psi_i} = 0$, $\frac{\partial\mathcal{L}}{\partial\Gamma_j} = 0$, and $\frac{\partial\mathcal{L}}{\partial\sigma_i^2} = 0$, it is difficult to obtain explicit solutions of the hyperparameters Ψ_i, Γ_j and σ_i^2.
- The NAG (Nesterov's Accelerated Gradient) method is used to obtain numerical solutions of the hyperparameters

Parameter Estimation (cont'd)

- The negative log of the marginal likelihood is acquired

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First hitting time

- The PDF of the degradation prediction takes the form

$$p\left(\boldsymbol{x}_{k+l}|\boldsymbol{x}_{n+l}\right) = \mathcal{N}\left(\boldsymbol{x}_{k+l}|\boldsymbol{\Sigma}_0(\boldsymbol{\Sigma}_0^{-1} \otimes \boldsymbol{\phi}^{\mathrm{T}}(\boldsymbol{x}_k))\boldsymbol{\Lambda}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \boldsymbol{\Sigma}_k\right), \tag{13}$$

- Given the observed measurements $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_k$, the RUL of each variable at time t_k is defined by

$$L_{ik} \triangleq L_i(t_k) = \inf\{t_l : x_i(t_k + t_l) \in \mathcal{B}_i|\boldsymbol{x}_{1:k}\}, \tag{14}$$

where $\inf\{\cdot\}$ denotes the infimum of a discrete; t_l represents the time length of the multi-step prediction; $x_i(t_k + t_l)$ is the degradation path at time $t_k + t_l$, $\boldsymbol{x}_{1:k}$ denotes the historical measurements from t_1 to t_k; and $\mathcal{B}_i$ refers to a boundary set, containing a boundary, barrier, or failure threshold.

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RUL prediction

- the mean of the RUL is obtained by

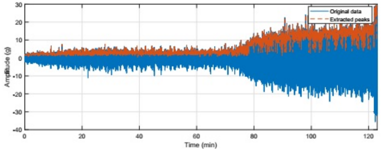
$$E_i(t_k) = \sum_{L_{ik}=0}^{+\infty} L_{ik} \cdot p_i(L_{ik}), \tag{15}$$

  where $p_i(L_{ik}) = \frac{\phi(g_{i,k+l})\Delta g_{i,k+l}}{1-\Phi(g_{i,k})}$, $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and cumulative distribution function of a standard normal random variable, respectively;
- the confidence interval for the prediction can also be obtained.

Table of Contents

Case study: Data

- A bearing dataset is used to demonstrate our proposed approach. Two accelerometers are placed on the bearings and positioned at 90° to each other, i.e., one is placed on the vertical axis and the other one is placed on the horizontal axis.
- Data from 78 minutes onwards are used for modelling.
- The operation is stopped when the amplitudes of the horizontal and vertical vibration signals are higher than 25g and 15g, respectively.



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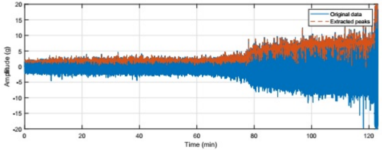

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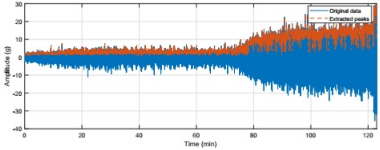

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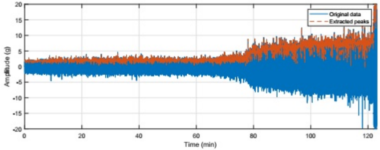

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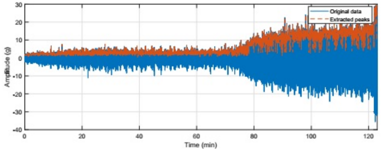

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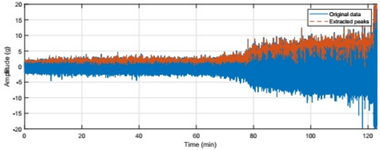

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Case study: Degradation Path Prediction

- The peaks during $80 - 100min$ are used as inputs and those during $80.6 - 100.6min$ are used as outputs
- For the sake of comparison, both MRVR and RVR are built on the data
- the predicted degradation path based on the RVR cannot follow the actual vertical amplitude so well as that based on the MRVR.
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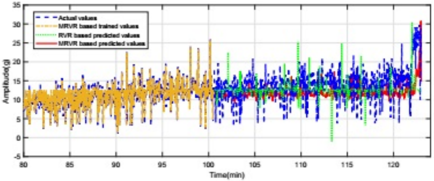

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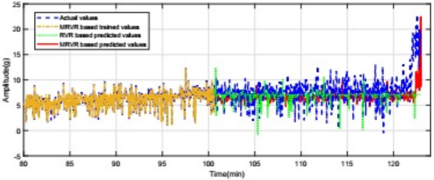

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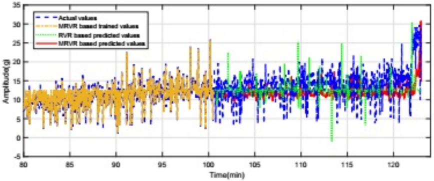

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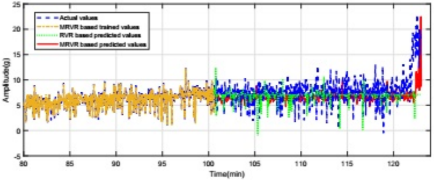

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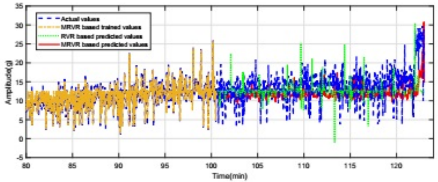

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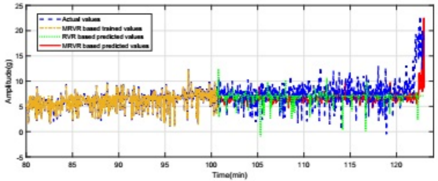

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RUL prediction and Performance metrics

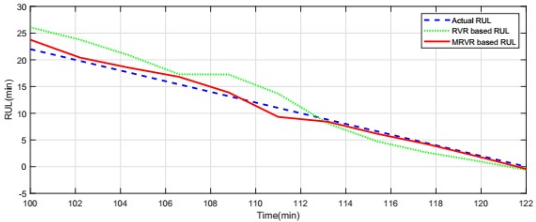

Figure: RUL Comparison

Performance metric	MRVR	RVR
MAE(min)	0.7960	2.3594
NRMSE(%)	8.7793	24.3193

Figure: Performance comparison

MAE=mean absolute error; NRMSE=Normalized Root Mean Relative Error

Conclusions

The MRVR outperforms!

Thank you all!

Questions?