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A Robust Approach to Hedging and Pricing in Imperfect Markets †

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Abstract: This paper proposes a model-free approach to hedging and pricing in the presence of market imperfections such as market incompleteness and frictions. The generality of this framework allows us to conduct an in-depth theoretical analysis of hedging strategies with a wide family of risk measures and pricing rules, and study the conditions under which the hedging problem admits a solution and pricing is possible. The practical implications of our proposed theoretical approach are illustrated with an application on hedging economic risk.

Keywords: imperfect markets; risk measures; hedging; pricing rule; quantile regression

JEL Classification: G11, G13, C22, E44

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1. Introduction

Hedging and pricing financial and economic variables in imperfect markets (incomplete markets and/or markets with frictions) proves to be a challenging problem. While pricing and hedging in complete and frictionless markets are typically carried out by a unique perfect replication of a contingent claim at a horizon time, the presence of market imperfections renders no unique solution to this problem. For example, when the no-arbitrage condition holds, the set of admissible stochastic discount factors for pricing financial variables is strictly positive, implying monotonicity of the pricing rules. By contrast, in the presence of market imperfections, the stochastic discount factors do not price the set of all possible payoffs (see Jouini and Kallal (1995a, 1995b, 1999)). In this case, the main problem lies in the existence of pricing rules that can be extended to the whole set of possible variables.

In this paper, we develop a unifying framework for hedging and pricing in imperfect markets. To this end, we account for market incompleteness and frictions by minimizing aggregate hedging costs that consist of costs associated with the risk of the non-hedged part and costs of purchasing the hedging strategy. This non-parametric or robust hedging approach is fairly general and can be used for various purposes such as hedging contingent claims and economic risk variables.

To position the contributions of this paper in the existing literature, we briefly review the main approaches to pricing and hedging in incomplete markets. First, there is a large literature on developing methods for economic hedging based on mean-variance utility (Balduzzi and Kallal (1997); Breeden et al. (1989); Balduzzi and Robotti (2001); Goorbergh et al. (2003); Vassalou (2003), among others). In these papers, the mimicking portfolios are constructed by means of an ordinary or weighted least squares projection of the risk variables on a set of security returns. Another line of research, that has been developed in mathematical finance, is devoted to replicating (or closely replicating) financial positions using contingent claims. For instance, hedging by super-replicating (also known as super hedging) is widely studied in the literature; see Karatzas and Shreve (1998). The idea is to find the time-zero value of a portfolio which pays off at least as much as the contingent claim. One possibility is to formulate the problem as mean-variance hedging; see Duffie and Richardson (1991); Schweizer (1992, 1995); Föllmer and Schweizer (1991); Gourieroux et al. (1998); Laurent and Pham (1999); Schäl (1994)). Alternatively, the hedging can be performed using more general decision making tools and risk measures. Föllmer and Leukert (2000) propose replicating the contingent claim by minimizing the probability of a shortfall. Similarly, Nakano (2004), Rudloff (2007, 2009) study the problem of minimizing the risk of a shortfall where the risk is measured by a general coherent or convex risk measure.

However, our paper is closely related to another strand of literature, mainly developed in the field of operations research, which is based directly on the concepts of hedging and minimization of risk rather than replication of contingent claims (see Assa and Balbás (2011); Balbás et al. (2009, 2010) or in a different discipline Smith and Nau (1995)). The main idea is that the financial practitioner tries to minimize the risk of his/her global position, given the budget constraint on a set of manipulatable positions (a set of accessible portfolios, for example). The analysis in this setup crucially depends on the sub-additivity property of coherent risk measures and capitalizes on its dual representation. Assa and Balbás (2011) characterize the existence of a solution to the hedging problem and show that a solution exists if and only if there is no costless risk-free position (arbitrage opportunity or *Good Deal*). A drawback of their analysis on coherent risk measures is that the hedging is no longer possible for a non-subadditive risk measure such as the popular Value-at-Risk. In this paper, we extend the analysis of Assa and Balbás (2011) to accommodate more general risk measures such as Value-at-Risk or risk measures related to Choquet expected utility Bassett et al. (2004). Our analysis demonstrates when pricing with Value-at-Risk is possible and attains the same *Good Deal* result as in the sub-additive case but only using the properties of cash-invariance and positive homogeneity. We also propose a simple method which is based on the fundamental idea of hedging by minimizing risk. While the focus in this paper is on the pricing part of the hedging problem and the extension of the pricing rule to the space of all financial and economic variables in imperfect markets, we also construct a set of market principles that are used to determine the existence of a solution to the hedging problem.

The rest of the paper is organized as follows. Section 2 introduces the notation, provides some preliminary definitions and states the main problem. Section 3 uses market principles to characterize the solutions to the hedging and pricing problems under generalized spectral risk measures. Section 4 discusses the practical implications of the main theoretical results for the purposes of hedging of economic risk. Section 5 concludes. The mathematical proofs are provided in Appendix A.

2. Preliminaries and Analytical Setup

We start by introducing the main terminology and notation for hedging and pricing financial or economic variables. We assume a finite probability space with a finite[1] event space $\Omega = \{\omega_1, \dots, \omega_n\}$. We denote the physical measure by $\mathbb{P}$, and the associated expectation by E. To simplify the discussion, we assume that $\mathbb{P}(\omega_i) = 1/n$ for all $i = 1, \dots, n$. Our theory is developed in a static setting and we only have time 0 and time T. Each random variable represents the random value on a variable at time T. We denote by $\mathbb{R}^n$ the set of all variables. The duality relation is expressed as $(x, y) \mapsto E(xy)$, $\forall x, y \in \mathbb{R}^n$. The risk measure and the pricing rule are expressed in terms of time-zero value and are real numbers.

[1] All of the results can be easily extended to a probability space with no atoms in an appropriate space—for instance, $L^2(\Omega)$.

Let $\mathcal{Y}$ be a subset of $\mathbb{R}^n$. In the subsequent discussion, we will assume that $\mathcal{Y}$ possesses one or several properties from the following list:

S1. Normality if $0 \in \mathcal{Y}$;
S2. Positive homogeneity if $\lambda\mathcal{Y} \subseteq \mathcal{Y}$, for all $\lambda > 0$;
S3. Translation-invariance if $\mathbb{R} + \mathcal{Y} \subseteq \mathcal{Y}$;
S4. Sub-additivity if $\mathcal{Y} + \mathcal{Y} \subseteq \mathcal{Y}$;
S5. Convexity if $\lambda\mathcal{Y} + (1-\lambda)\mathcal{Y} \subseteq \mathcal{Y}$.

2.1. Risk Measures

In what follows, we use risk measures to quantify the risk associated with the undiversifiable part of the market exposure.

A *risk measure* ϱ is a mapping from a set $\mathcal{D} \subseteq \mathbb{R}^n$ to the set of real numbers $\mathbb{R}$ which maps each random variable in $\mathcal{D}$ to a real number representing its risk. Each risk measure can have one or more of the following properties:

R1. $\varrho(0) = 0$;
R2. $\varrho(\lambda x) = \lambda\varrho(x)$, for all $\lambda > 0$ and $x \in \mathcal{D}$;
R3. $\varrho(x + c) = \varrho(x) - c$, for all $x \in \mathcal{D}$ and $c \in \mathbb{R}$;
R4. $\varrho(x) \leq \varrho(y)$, for all $x, y \in \mathcal{D}$ and $x \geq y$;
R5. $\varrho(x + y) \leq \varrho(x) + \varrho(y), \forall x, y \in \mathcal{D}$;
R6. $\varrho(\lambda x + (1-\lambda)y) \leq \lambda\varrho(x) + (1-\lambda)\varrho(y)$.

If ϱ satisfies properties R1, R2, R3, R5 or R6, $\mathcal{D}$ has to possess properties S1, S2, S3, S4, or S5, respectively. A risk measure is called an *expectation bounded risk* if it is defined on $\mathbb{R}^n$ and satisfies properties R1, R2, R3 and R5 above. The mean-variance risk measure defined as

$$MV_\delta(x) = \delta\sigma(x) - E(x),$$

where $\sigma(x)$ is the standard deviation of x and δ is a positive number representing the level of risk aversion, is an example of an expectation bounded risk.

An expectation bounded risk is called a *coherent risk measure* if it also satisfies property R4. Finally, a *convex risk measure* satisfies properties R1, R3, R4 and R6. Coherent and convex risk measures are introduced by Artzner et al. (1999) and Föllmer and Schied (2002), respectively.

One popular risk measure is the Value at Risk defined as

$$\mathrm{VaR}_\alpha(x) = -q_\alpha(x)\,, \forall x \in \mathbb{R}^n,$$

where $q_\alpha(x) = \inf\{a \in \mathbb{R} | \mathbb{P}[x \leq a] > \alpha\}$ denotes the α-th quantile of the distribution of x. Note that VaR_α is a decreasing risk measure which is neither a coherent risk measure nor an expectation bounded risk. In contrast, the Conditional Value at Risk (CVaR), expressed as the sum over all VaR below α percent

$$\mathrm{CVaR}_\alpha(x) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(x)d\beta, \tag{1}$$

is a coherent risk measure.

To further generalize the concept of a risk measure, consider the following family of risk measures.

Definition 1. *A risk measure is a generalized spectral risk measure if and only if there is a distribution* $\varphi : [0,1] \to \mathbb{R}_+$ *such that* $\int_0^1 \varphi(s)ds = 1$*, and*

$$\varrho_\varphi(x) = \int_0^1 \varphi(s)\mathrm{VaR}_s(x)ds. \tag{2}$$

One can readily see that ϱ_φ is law invariant, i.e., if x and x' are identically distributed, then we have $\varrho_\varphi(x) = \varrho_\varphi(x')$. Indeed, it can be shown that all law-invariant co-monotone additive coherent risk measures can be represented as (2); see Kusuoka (2001). Furthermore, Equation (2) describes a family of risk measures which are statistically robust. Cont et al. (2010) show that a risk measure $\varrho(x) = \int_0^1 \mathrm{VaR}_\beta(x)\varphi(\beta)d\beta$ is robust if and only if the support of φ is away from zero and one. For example, Value at Risk is a risk measure with this property.

An interesting fact about this type of risk measures is that it can be represented as infimum of a family of coherent risk measures.

Theorem 1. *If* $\varrho_\varphi(x) = \displaystyle\int_0^1 \mathrm{VaR}_\alpha(x)\varphi(\alpha)d\alpha$, *for a nonnegative distribution* φ *with* $\displaystyle\int_0^1 \varphi(s)ds = 1$, *then we have*

$$\varrho_\varphi(x) = \min\{\varrho(x) | \textit{ for all coherent risk measure } \varrho \textit{ such that } \varrho \geq \varrho_\varphi\}.$$

Proof. See Appendix A. □

This theorem provides a motivation for introducing another family of risk measures, called the *infimum risk measures*, which includes all coherent as well as spectral risk measures.

Definition 2. *Let* $\mathbb{D}$ *be a pointwise-closed set of risk measures on* $\mathcal{D}$*. Then, the infimum risk measure associated with* $\mathbb{D}$ *is defined as*

$$\varrho_{\mathbb{D}}(x) = \min_{\varrho \in \mathbb{D}} \varrho(x). \tag{3}$$

2.2. Pricing Rules

A *pricing rule* π is a mapping from $\mathcal{X} \subseteq \mathbb{R}^n$ to the set of real numbers $\mathbb{R}$ which maps each random variable in $\mathcal{X}$ to a real number representing its price. The pricing rule can possess one or more of the following properties:

P1. $\pi(0) = 0$;
P2. $\pi(\lambda x) = \lambda\pi(x)$, for all $\lambda > 0$ and $x \in \mathcal{X}$;
P3. $\pi(x + c) = \pi(x) + c$, for all $x \in \mathcal{X}$ and $c \in \mathbb{R}$ (cash-invariance);
P4. $\pi(x) \leq \pi(y)$, for all $x, y \in \mathcal{X}$ and $x \leq y$;
P5. $\pi(x + y) \leq \pi(x) + \pi(y)$, for all $x, y \in \mathcal{X}$;
P6. $\pi(\lambda x + (1 - \lambda y)) \leq \lambda\pi(x) + (1 - \lambda)\pi(y)$.

If π satisfies properties P1, P2, P3, P5 or P6, $\mathcal{X}$ has to satisfy properties S1, S2, S3, S4, or S5, respectively. A pricing rule is *super-additive* if $\pi(x + y) \geq \pi(x) + \pi(y)$, for all $x, y \in \mathcal{X}$.

A pricing rule that satisfies properties P1, P2, P3, P4 and P5 is called a sub-linear pricing rule. Any sub-linear pricing rule can be extended from $\mathcal{X}$ to $\mathbb{R}^n$ as follows

$$\tilde{\pi}(x) = \sup_{\{y \in \mathcal{X} | y \leq x\}} \pi(y). \tag{4}$$

Indeed, this supremum exists and is a finite number because (i) $\min(x) \in \{y \in \mathcal{X} | y \leq x\}$ and (ii) for any $x, y \in \mathcal{X}$ such that $y \leq x$, we have $\pi(y) \leq \max(x)$. It can be easily seen that $\tilde{\pi}$ is a sub-linear pricing rule on $\mathbb{R}^n$.

Moreover, any sub-linear pricing rule admits the following representation:

$$\tilde{\pi}(x) = \sup_{z \in \mathcal{R}} E(zx), \tag{5}$$

where $\mathcal{R}$ is given by

$$\mathcal{R} := \{z \in \mathbb{R}^n | E(zx) \leq \tilde{\pi}(x), \forall x \in \mathbb{R}^n\}. \tag{6}$$

Monotonicity implies that $z \geq 0, \forall z \in \mathcal{R}$ and translation-invariance implies $E(z) = 1, \forall z \in \mathcal{R}$. Therefore, $\mathcal{R}$ is a compact set.

In this paper, the set $\mathcal{R}$ represents the set of nonnegative stochastic discount factors induced by π and

$$\pi(x) = \tilde{\pi}(x) = \sup_{z \in \mathcal{R}} E(zx), \forall x \in \mathcal{X}. \tag{7}$$

Also, the condition $z > 0$ is equivalent to the no-arbitrage condition

$$\pi(x) \leq 0 \ \& \ x \geq 0 \Rightarrow x = 0. \tag{8}$$

Jouini and Kallal (1995a, 1995b, 1999) argue that for a wide range of market imperfections such as dynamic market incompleteness, short selling costs and constraints, borrowing costs and constraints, and proportional transaction costs, the pricing rule is sub-linear. Even though the set of sub-linear pricing rules is quite large, it does not cover some practically relevant pricing rules. For example, in a super-hedging context, ask and bid prices defined as

$$\pi^a(x) = \sup_{\mathbb{Q} \in \mathcal{R}} E^{\mathbb{Q}}[x], \tag{9}$$

and

$$\pi^b(x) = \inf_{\mathbb{Q} \in \mathcal{R}} E^{\mathbb{Q}}[x], \tag{10}$$

where $\mathcal{R}$ is the set of martingale measures of the normalized price processes of traded securities (see Jouini and Kallal (1995a) and Karatzas et al. (1991)), are of particular interest El Karoui and Quenez (1995). In this case, the bid price is a super-additive pricing rule which does not fulfill the sub-additivity conditions of the sub-linear pricing rule. Furthermore, in insurance applications, the pricing rules are not, in general, sub- or super-additive. As pointed out by Wang et al. (1997), the price of an insurance risk has representation as in Equation (2) with respect to a distorted probability. For this reason, we introduce the family of *infimum pricing rules* that subsumes both sub-linear and non-sub-linear pricing rules.

Definition 3. *Let $\mathbb{M}$ be a pointwise-closed set of pricing rules on $\mathcal{X}$. Then, the infimum risk measure associated with $\mathbb{M}$ is defined as*

$$\pi_{\mathbb{M}}(x) = \min_{\pi \in \mathbb{M}} \pi(x). \tag{11}$$

2.3. Projection

To put the subsequent discussion in the proper context, assume that we have a set of perfectly-hedged variables denoted by $\mathcal{X}$, where all members of $\mathcal{X}$ are priced according to the pricing rule $\pi : \mathcal{X} \to \mathbb{R}$. As an example, consider the case when $\mathcal{X}$ is equal to the set of all portfolios of given assets $(x_1, ..., x_N)$, i.e., $\mathcal{X} = \text{Span}(x_1, ..., x_N)$ or $\mathcal{X} = \text{Span}(x_1, ..., x_N)_+$ if the short-selling is forbidden. A variable y is perfectly-hedged if $y \in \mathcal{X}$. In this particular example, y is perfectly-hedged if there is a portfolio whose value is equal to y, i.e., there exist numbers $a_1, ..., a_N$ such that $y = a_1 x_1 + \cdots + a_N x_N$. If any variable y can be perfectly-hedged, we say that the market is complete. Otherwise, if there is at least one variable y whose risk cannot be diversified by the set of perfectly-hedged positions, the market is incomplete. This prompts the need to introduce the mapping (risk measure) ϱ from the set of all variables $\mathcal{D}$ to real numbers which measures the risk generated by the part that cannot be hedged.

We next introduce the idea of projection. Let us consider a financial position y in an incomplete market which has to be hedged or priced. To achieve this, we find a variable, among all perfectly-hedged variables in the set $\mathcal{X}$, that mimics y most closely. In other words, we want to project y on the set $\mathcal{X}$. Assume for a moment that we know this projection and denote it by $x \in \mathcal{X}$. Hence, y can be decomposed into two parts: a mimicking strategy (portfolio in our example) x which is *perfectly-hedged*, and an *unhedged* part $y - x$ which generates risk. The cost of the mimicking strategy (or perfectly-hedged) part is given by $\pi(x)$, and the risk generated by the unhedged part, which cannot be diversified by any member of $\mathcal{X}$, is measured by $\varrho(x - y)$. The idea of projection is to minimize the aggregate cost of the hedging given as $\pi(x) + \varrho(y - x)$. Therefore, one can state the problem as follows:

$$\inf_{x \in \mathcal{X}} \{\pi(x) + \varrho(x - y)\}. \tag{12}$$

In this case, the market imperfections are reflected by the (non-linear) pricing rule π and the risk measure ϱ which capture the market frictions and the market incompleteness, respectively.

Let us now look at this problem from a pricing point of view. Suppose that a financial practitioner wants to price the position (contingent claim, for example) y. While the pricing of y in complete markets can utilize directly the no-arbitrage approach, the pricing problem in incomplete markets is less straightforward as it needs to incorporate the cost of the unhedged part. As discussed above, the cost of forming the mimicking strategy x is given by $\pi(x)$ and the unhedged risk associated with the unhedged part of y is given by $\varrho(x - y)$. Then, the competitive price for position y can be defined as

$$\pi_\varrho(x) = \inf_{x \in \mathcal{X}} \{\pi\,(x) + \varrho\,(x - y)\}. \tag{13}$$

As we demonstrate below, if ϱ is a coherent risk measure and π a sub-linear pricing rule, π_ϱ satisfies all of the properties of a sub-linear pricing rule except for the normality condition. As a result, we need to ensure that the normality condition holds for π_ϱ to be a proper pricing rule.

Potential applications of this framework include hedging and pricing contingent claims, insurance underwriting, hedging of economic risk etc. It should be noted that a similar approach to pricing is adopted in Föllmer and Leukert (2000) and Rudloff (2007, 2009) but it is based on minimizing shortfall risk instead of minimizing aggregate cost as we do in this paper. In what follows, we refine the choice of pricing rules and risk measures and analyze their theoretical properties.

Remark 1. *In the literature mathematical finance there is usually an underlying asset, and all hedging strategies are designed to hedge a derivative on the underlying asset. This includes both complete and in-complete market approaches. In our work, we do not rely on any underlying asset. From that perspective, our approach can be regarded as an universal approach since we can compare prices of different assets without considering any specific underlying.*

Remark 2. *A major difference between a risk measure and a utility is that the value of a risk measure is not just simply a scalar, but it takes place in the domain of the risk measure. In other words, a risk measure ϱ is a mapping from $\mathcal{D}$ to $\mathbb{R} \subseteq \mathcal{D}$ where the set of real numbers are seen as almost surely constant random variables. For instance, if members of $\mathcal{D}$ represent the asset values or the returns, then the value of a risk measure is a number that is expressed in terms of capital or return, respectively. This fact plays an important role in our theory since in* (13) *in addition to finding an optimal hedging portfolio, we also find the associated price of a position of interest.*

3. Main Theoretical Results

In this section, we establish some market principles for general risk measures and pricing rules. The results are stated for two different categories: first, for risk measures and pricing rules which satisfy properties R1–R4 and P1–P4 (including non-sub-additive pricing rules and risk measures), and, second, for risk measures and pricing rules that satisfy properties R1, R2, R3, R5 and P1, P2, P3, P5, respectively (including non-monotone ones). Results for the second family make use of the dual representation of pricing rules and risk measures. We then study the conditions under which an arbitrage opportunity is generated.

3.1. Market Principles

We start with the following result for π_ϱ defined in (13).

Proposition 1. *Let*

$$\mathcal{X}_\varrho := \{x \in \mathbb{R}^n | \pi_\varrho(x) \in \mathbb{R}\}.$$

Then, the following statements hold:

1. *π_ϱ and $\mathcal{X}_\varrho$ are positive homogeneous if ϱ and π are.*
2. *π_ϱ and $\mathcal{X}_\varrho$ are translation-invariant if ϱ and π are.*
3. *π_ϱ and $\mathcal{X}_\varrho$ are sub-additive if ϱ and π are.*
4. *π_ϱ and $\mathcal{X}_\varrho$ are convex if ϱ and π are.*

   Furthermore,

5. *π_ϱ is monotone if ϱ and π are.*

Proof. See Appendix A. □

Note that Proposition 1 does not say anything about the first property of a pricing rule which warrants some further explanation. It turns out that for the first property of a pricing rule to hold, we need to guarantee that some conditions for $\mathcal{X}$, ϱ and π are satisfied. Below, we explicitly state these conditions as general pricing principles that are valid regardless of the type of pricing or pricing rule.

Normality (N). $\pi_\varrho(0) = 0$.

No Good Deal Assumption (NGD). There is no financial position x such that

$$\varrho(x) < 0\,, \; \pi(x) \leq 0.$$

Consistency Principle (CP). For any member $x \in \mathcal{X}$, π and π_ϱ are consistent, i.e.,

$$\pi(x) = \pi_\varrho(x).$$

Compatibility (C). For a risk measure ϱ and a pricing rule π, (12) has a finite infimum.

The first principle simply recognizes that the price of zero is always zero. The second principle states that any risk-free variable has a positive cost (see Cochrane and Saa-Requejo (2000)). The third principle is a consistency condition between a pricing rule π and π_ϱ over $\mathcal{X}$. The last principle points out that the hedging problem always yields a price.

3.2. Positive-Homogeneous and Monotone Risk and Pricing Rules

Next, we discuss the equivalence of the market principles for a risk measure ϱ and pricing rule π which satisfy properties R1–R4 and P1–P4.

Theorem 2. *Let us assume ϱ and π satisfy properties R1–R4 and P1–P4. Then,*

$$(CP) \Rightarrow (N) \Leftrightarrow (NGD) \Leftrightarrow (C).$$

Moreover, if $\mathcal{X}$ is a vector space and π is super-additive, we also have

$$(N) \Rightarrow (CP).$$

Proof. See Appendix A. □

The following corollary states the conditions under which π_ϱ is a pricing rule.

Corollary 1. *Given the notation above, $\pi_\varrho : \mathcal{X}_\varrho \to \mathbb{R}$ is a pricing rule if and only if (N) (or (NGD) or (C)) holds.*

3.3. Positive-Homogeneous and Sub-Additive Risk and Pricing Rules

In this section, we assume that the risk measure ϱ and the pricing rule π satisfy properties R1, R2, R3, R5 and P1, P2, P3, P5, respectively. In that case, we extend the range of the these mappings to $\mathbb{R} \cup \{+\infty\}$

$$\bar{\varrho}(x) = \begin{cases} \varrho(x) & x \in \mathcal{D} \\ +\infty & \text{otherwise} \end{cases}, \ \bar{\pi}(x) = \begin{cases} \pi(x) & x \in \mathcal{X} \\ +\infty & \text{otherwise.} \end{cases}$$

This extension allows us to use the dual representation of positive-homogeneous convex functions. Duality theory and sub-gradient analysis prove useful since the risk measures and pricing rules are usually not differentiable. First, we present conditions under which arbitrage opportunities do not exist in terms of the dual sets. Then, we characterize the solution to the hedging problem (12) and the pricing rule π_ϱ in (13).

We start by introducing some additional notation. From the theory of convex risk measures, any convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ has the following Fenchel-Moreau representation[2]

$$f(x) = \sup_{z \in \mathbb{R}^n} \{E(-zx) - f^*(z)\},$$

where $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the dual of f defined as

$$f^*(z) = \sup_{x \in \mathbb{R}^n} \{E(-zx) - f(x)\}.$$

It can be easily seen that for any positive-homogeneous function f, f^* is 0 on a convex closed set, denoted by Δ_f, and infinity otherwise. Therefore, the Fenchel-Moreau representation of a positive homogeneous function f has the form

$$f(x) = \sup_{z \in \Delta_f} E(-zx).$$

As an example, for any coherent risk measure ϱ, Δ_ϱ is a subset of the set of all probability measures, i.e., $\Delta_\varrho \subseteq \{z \in \mathbb{R}^n | z \geq 0, \sum z_i = 1\}$, and, therefore, it is compact (see Artzner et al. (1999)). In contrast, for any expectation bounded risk ϱ, $\Delta_\varrho \subseteq \{z \in \mathbb{R}^n | \sum z_i = 0\}$.

Now let us assume that, in general, ϱ and π are positive-homogeneous and sub-additive mappings. Since ϱ and π are positive-homogeneous and sub-additive, and because $\mathcal{X}$ and $\mathcal{D}$ are positive cones, their extensions are also positive-homogeneous and sub-additive. Then, we have the representations

$$\bar{\varrho}(x) = \sup_{z \in \Delta_{\bar{\varrho}}} E(-zx), \ \bar{\pi}(x) = \sup_{z \in \mathcal{R}_{\bar{\pi}}} E(zx) \ , \forall x \in \mathbb{R}^n, \tag{14}$$

for closed convex sets $\Delta_{\bar{\varrho}}$ and $\mathcal{R}_{\bar{\pi}}$.

In order to obtain the representations for $\bar{\varrho}$ and $\bar{\pi}$, we need to introduce the dual-polar of a scalar-cone of random payoffs. If A is a scalar-cone of a random payoff, the dual-polar of the set A is given by

$$A^{\perp} := \{z | E(zx) \leq 0 \, \forall x \in A\}.$$

[2] For technical reasons, we use $-z$ instead of z.

We then have the following proposition.

Proposition 2. *For any function* $f(x) := \sup_{z \in \Delta_f} E(zx)$*, for some set* Δ_f*, which is defined on a positive cone* A*, we have that*

$$\bar{f}(x) = \sup_{z \in \Delta_f + A^{\perp}} E(zx).$$

Proof. See Appendix A. □

Proposition 2 has the important implication that any risk measure $\varrho(x) = \sup_{z \in \Delta_\varrho} E(-zx)$ defined on $\mathcal{D}$, and pricing rule $\pi(x) := \sup_{z \in \mathcal{R}} E(zx)$ defined on $\mathcal{X}$, can be rewritten as

$$\varrho(x) = \sup_{z \in \Delta_{\bar{\varrho}}} E(-zx), \bar{\pi}(x) = \sup_{z \in \mathcal{R}_{\bar{\pi}}} E(zx),$$

where $\Delta_{\bar{\varrho}} = \Delta_\varrho - \mathcal{D}^{\perp}$ and $\mathcal{R}_{\bar{\pi}} = \mathcal{R} + \mathcal{X}^{\perp}$.

The following theorem states the main theoretical result of the paper.

Theorem 3. *Assume that the risk measure* $\varrho_{\mathbb{D}}$ *is defined as in* (3) *and the pricing rule* $\pi_{\mathbb{M}}$ *is defined as in* (11)*. Then, the following statements are equivalent:*

1. *The hedging problem* (12) *is finite.*
2. $\mathcal{R}_{\varrho,\pi} = (\Delta_\varrho - \mathcal{D}^{\perp}) \cap (\mathcal{R}_\pi + \mathcal{X}^{\perp}) \neq \Phi$, $\forall \varrho \in \mathbb{D}, \forall \pi \in \mathbb{M}$

   Furthermore, if condition 3 holds for π *and* ϱ*, these statements are equivalent to*
3. *There is no Good Deal in the market.*

In all cases, the price (13) *can be represented as*

$$(\pi_{\mathbb{D}})_{\varrho_{\mathbb{M}}}(x) = \inf_{\pi \in \mathbb{M}, \varrho \in \mathbb{D}} \pi_\varrho(x) = \inf_{\pi \in \mathbb{M}, \varrho \in \mathbb{D}} \sup_{z \in \mathcal{R}_{\pi,\varrho}} E(zx).$$

Proof. See Appendix A. □

In most cases, such as coherent risk measures and deviation measures of risk, the risk measure ϱ is defined on $\mathbb{R}^n$ meaning that $\mathcal{D}^{\perp} = \{0\}$. We then have the following corollary.

Corollary 2. *If* ϱ *is a coherent risk measure and* π *a sub-linear pricing rule, then there is no Good Deal if and only if* $\mathcal{R}_{\pi_\varrho} := \Delta_\varrho \cap (\mathcal{R}_\pi + \mathcal{X}^{\perp}) \neq \Phi$*, and the pricing rule is given by*

$$\pi_\varrho(y) = \sup_{z \in \mathcal{R}_{\pi_\varrho}} E(zx). \tag{15}$$

Remark 3. *Theorem 3 and Corollary 2 illustrate the generality of our approach compared to the existing literature. In the existing literature, the set of stochastic discount factors is constructed either parametrically (using, for example, a semi-martingale process), or empirically and a pricing rule* π *is then obtained by taking supremum of prices over a closed convex subset* $\mathcal{R}$*. In order to price all positions in the market, any stochastic discount factor* z' *is constructed as a positive and linear extension of* $z \in \mathcal{R}_\pi$*, i.e.,* $z'|_{\mathcal{X}} = z$*. Therefore, the set of stochastic discount factors is induced by the unique monotonic extension* $\tilde{\pi}$ *of* π *(for more details, see Theorem 1 in Jouini and Kallal (1995b)). By contrast, in our approach, the extension of the pricing rule is not constructed monotonically but it is obtained within the hedging problem and is affected, in general, by two additional factors:*

market incompleteness and frictions. In our approach, assuming that ϱ is defined on the whole space so that $\mathcal{D}^{\perp} = \{0\}$, the set of stochastic discount factors is equal to $\Delta_{\varrho} \cap (\mathcal{R}_{\pi} + \mathcal{X}^{\perp})$, which is expanded by adding $\mathcal{X}^{\perp}$ and contracted by intersecting with Δ_{ϱ}.

Remark 4. *Our method can reproduce the existing approach if we assume $\varrho(x) = \tilde{\pi}(-x)$. Indeed, our approach is able to reproduce the pricing rule $\tilde{\pi}$ if and only if the consistency principle holds. If the pricing rule is super-additive, this can be achieved if and only if $\pi(-x) \leq \varrho(x), \forall x \in \mathcal{X}$. This implies that $\mathcal{R} \subseteq \Delta_{\varrho}$. It can be easily verified that $x \mapsto \tilde{\pi}(-x)$ is the smallest risk measure for which the consistency principle holds. The mapping $x \mapsto \tilde{\pi}(-x)$ is the market measurement of risk and has been proposed by Assa and Balbás (2011). In this case, $\mathcal{D}^{\perp} = \{0\}$ and $\Delta_{\varrho} = \mathcal{R}$, which yields $\pi_{\varrho} = \pi$. Hence, the hedging problem becomes*

$$\begin{cases} \min\{\tilde{\pi}(y-x) + \pi(x)\} \\ x \in \mathcal{X}. \end{cases} \tag{16}$$

It is clear that since $\tilde{\pi}$ is sub-additive, $x = 0$ is a solution to this hedging problem. Therefore, the pricing rule $\pi_{\tilde{\pi}(-.)}$ equals $\tilde{\pi}$, which reproduces the existing approach in the literature.

Remark 5. *The convexity and regularity conditions (e.g., Inada conditions) can always guarantee the existence of a solution in a framework where agents based their decisions on a utility function. However, the are two main issues in this paper: first, in general, we do not have convexity, and second, regularity does not necessarily hold. As a result, the existence of the hedging strategy is not always guaranteed. However, as we argued above, the existence of a solution, which is in principle a technical issue, is equivalent to market principles, which have an economic interpretation. This is the main mathematical/economic challenge confronted in the paper.*

Remark 6. *One disadvantage of using non-convex framework is that the computational cost is usually higher, as we cannot easily rely on convex programming methods. In this paper, we show using a non-convex risk measure, VaR, will result in a hedging method that is based on quantile regression. For quantile regression in particular, there are lots of interesting approaches developed in practice and theory that can decrease the computational cost see for example Yu and Moyeed (2001), Koenker (2005) and Bernardi et al. (2015) .*

Remark 7. *While the extension of the hedging framework in this paper to a dynamic setting may, in principle, be possible, this proves to be highly non-trivial and challenging. The reason is that in a dynamic setup, one needs also to consider time-consistency of the risk measure and the pricing rule (as in Cheridito et al. (2006)). In particular for non-convex frameworks we have to introduce nested risk measures and pricing rules. This is beyond the scope of this paper and is left for future research.*

4. An Application to Hedging Economic Risk

4.1. Estimation Problem

In this section, we illustrate the practical relevance of our theoretical results in the context of hedging economic risk by highlighting the effect of different risk measures on hedging strategies and the role of $\mathcal{X}^{\perp}$. Our analysis of portfolios that track or hedge various economic risk variables follows largely Lamont (2001) and Goorbergh et al. (2003). While these papers employ the mean-variance (MV) framework for constructing the portfolio of assets, we consider the more general and robust CVaR and VaR risk measures. Let y_t denote an economic risk variable to be hedged at time t ($t = 1, 2, ..., T$), $x_t = (x_{t1}, ..., x_{tN})'$ be N securities (traded factors) at time t and $\mathcal{X} = \text{span}\langle x_1, \ldots, x_N \rangle$. The pricing rule is the expected value of the portfolio given by $\pi(x_t'\theta) = E(x_t'\theta)$, where $\theta = (\theta_1, \ldots, \theta_N)'$.

For the mean-variance risk measure, we have that $\varrho(x) = \delta\sigma(x) - E(x)$. To facilitate the comparison with the other risk measures, the risk aversion parameter δ is set equal to 1. By plugging $x = \sum \theta_i x_i - y$, the problem (13) reduces to the following OLS problem:

$$\min_{\theta} \frac{1}{T} \sum_{t=1}^{T} \left(\tilde{y}_t - \sum_{j=1}^{N} \theta_j \tilde{x}_{tj} \right)^2, \tag{17}$$

where $\tilde{y}_t = y_t - E(y_t)$ and $\tilde{x}_{tj} = x_{tj} - E(x_{tj})$.

For the CVaR risk measure, we rewrite the problem (13) with a risk measure $\varrho = \varrho_{\nu_\alpha}$ and a pricing rule $\pi = E$ as

$$\min_{\theta} \left\{ \mathrm{CVaR}_\alpha(x) \left(x_t'\theta - y_t \right) + E\left(x_t'\theta \right) \right\} \tag{18}$$

or, more conveniently, as

$$\min_{\theta} \left\{ \mathrm{CVaR}_{1-\alpha}(x) \left(y_t - x_t'\theta \right) + E\left(x_t'\theta \right) \right\}, \tag{19}$$

using that $\mathrm{CVaR}_\alpha(x)\left(x_t'\theta - y_t\right) = \mathrm{CVaR}_{1-\alpha}(x)\left(y_t - x_t'\theta\right)$. Then, using translation-invariance and Theorem 2 in Bassett et al. (2004), the problem (19) can be rewritten equivalently an $(1-\alpha)$-quantile regression problem:

$$\min_{\xi,\theta} \frac{1}{T} \sum_{t=1}^{T} \rho_{1-\alpha}\left(\tilde{y}_t - \xi - \tilde{x}_t'\theta \right), \tag{20}$$

where $\rho_{1-\alpha}(u) = u\left[(1-\alpha)\mathbb{I}\{u > 0\} - \alpha\mathbb{I}\{u \leq 0\}\right]$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function. Note that since 1 is trivially in the intersection of the sub-gradient set of these risk measures and $\mathcal{R}$, then it follows from Theorem 3 there is no Good Deal and the hedging problem has a solution.

For the VaR hedging problem, we simply minimize the aggregate hedging costs

$$\min_{\theta} \{ \mathrm{VaR}_{1-\alpha}(y_t - x_t'\theta) + E(x_t'\theta) \}.$$

One can easily show that the probability measure $\mathbb{P}$ belongs to the sub-gradient of any law-invariant risk measure which also has properties R2 and R5. Therefore, by using part 2 of Theorem 3, the risk measures MV and CVaR do not produce any Good Deal with the pricing rule E. For VaR, we use the No-Good-Deal assumption and the theoretical results developed in the previous section. Since $\mathcal{X}$ is a vector space and π is a linear function, then, according to Theorem 2, the No-Good-Deal assumption holds if and only if π_ϱ (here $E_{\mathrm{VaR_{ff}}}$) is consistent. Hence,

$$\min_{\theta} \{ \mathrm{VaR}_{1-\alpha}(y_t - x_t'\theta) + E(x_t'\theta) \} = E(y_t).$$

Remark 8. *Another advantage of our hedging framework is that the risk measures can be chosen in such a way that the hedging problem is statistically robust. For instance, as mentioned earlier, for the VaR or CVaR risk measures the hedging portfolio can be obtained within a robust quantile regression framework.*

4.2. Data Description

Our choice of economic risk variables and security returns is similar to Goorbergh et al. (2003). The data are at monthly frequency for the period February 1952–December 2012. The traded securities include the risk-free rate, four stock-market factors (Fama and French (1992), Carhart (1997)) and two bond-market factors proxied, respectively, by: (i) the one-month T-bill (from Kenneth French's website), denoted by *RF*, (ii) the excess return (in excess of the one-month T-bill rate) on the value-weighted stock market (NYSE-AMEX-NASDAQ) index (from Kenneth French's website), denoted by *MARKET*, (iii) the return difference between portfolios of stocks with small and large market capitalizations (from Kenneth French's website), denoted by *SMB*, (iv) the return difference between portfolios of stocks with high and low book-to-market ratios (from Kenneth French's website), denoted by *HML*, (v) the momentum factor defined as the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios (from Kenneth French's website), denoted by *MOM*, (vi) *TERM* defined as the difference between the yields of ten-year and one-year government bonds (from the Board of Governors of the Federal Reserve System), and (vii) *DEF* defined the difference between the yields of long-term corporate Baa bonds (from the Board of Governors of the Federal Reserve System) and long-term government bonds (from Ibbotson Associates).

The macroeconomic risk variables include (i) the inflation rate measured as monthly percentage changes in CPI for all urban consumers (all items, from the Bureau of Labor Statistics), denoted by *INF*, (ii) the real interest rate measured as the monthly real yield on the one-month T-bill (from CRSP, Fama Risk Free Rates), denoted by *RI*, (iii) the term spread measured as the difference between the 10-year Treasury (constant maturity) and 3-month (secondary market) T-bill rate (from the Board of Governors of the Federal Reserve System), denoted by *TS*, (iv) the default spread measured as the difference between corporate Baa and Aaa rated (by Moody's Investor Service) bonds (from the Board of Governors of the Federal Reserve System), denoted by *DS*, (v) the monthly dividend yield on value-weighted stock market portfolio (from the Center for Research in Security Prices, CRSP), denoted by *DIV*, and (vi) the monthly growth rate in real per capita total (seasonally-adjusted) consumption (from the Bureau of Economic Analysis), denoted by *CG*.

4.3. Results

In order to hedge against unexpected economic shocks, we follow Campbell (1996) and replace the variable y_t with the corresponding error term form a six-variable VAR(1) model of y_t ($y = [INF, RI, TS, DS, DIV, CG]$). For VaR and CVaR, we use $\alpha = 0.1$ and 0.05 (i.e., $1 - a = 0.9$ and 0.95). The standard errors for VaR and CVaR are computed by bootstrapping. Statistically significant coefficients at the 5% nominal level are reported in bold font. The results for hedging inflation, real interest rate, term spread, default spread, dividend yield and consumption growth using the three risk measures are presented in Tables 1 to 6, respectively. The last line in each table reports the computed price.

A number of interesting findings emerge from this hedging exercise. First, as it was noted in Section 4.1, if the pricing rule E is correctly specified, the price should equal $E(y)$ (in the VaR case we also need to know if y is fully hedged). Tables 1 to 6 reveal that in all cases, the prices are significantly different from $E(y)$, which is attributed to the unhedged part in pricing y. These results highlight the role of the set $\mathcal{X}^{\perp}$. Indeed, the true stochastic discount factor lies in the larger set $\mathcal{X}^{\perp} \cap \Delta_{\varrho}$ for MV and CVaR, while for VaR we have a family of Δ_{ϱ}'s as in part 2 of Theorem 3. Our theory suggests that the true SDF has to represented as $P + z$, where z belongs to $\mathcal{X}^{\perp}$.

Second, while there is agreement across the different risk measures in hedging term spread, dividend yield and, to some extent, consumption growth, the hedging of inflation, real interest rate and default spread exhibit substantial heterogeneity both across and within risk measures. For example, CVaR suggests that *RF*, *SMB* and *TERM* prove to be important factors for hedging inflation whereas the other risk measures indicate that these factors are largely insignificant. Furthermore, there are differences across the different quantile regressions for CVaR and in some cases, depending on the level of α, the investor needs to switch from 'long' to 'short' positions in order to hedge the underlying economic risk. This illustrates the potential of alternative risk measures for robustifying the performance of economic portfolios.

Table 1. Hedging of Inflation. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	**MV**	**CVaR$_{0.9}$**	**CVaR$_{0.95}$**	**VaR$_{0.9}$**	**VaR$_{0.95}$**
Intercept		**0.0026**	**0.0022**		
		(0.0001)	(0.0002)		
RF	0.0072	−**0.6737**	−**0.7844**	0.0058	0.0027
	(0.0558)	(0.0705)	(0.1106)	(0.0041)	(0.0019)
MARKET	−**0.0048**	−**0.0072**	−**0.0123**	−0.0038	−**0.0066**
	(0.0029)	(0.0030)	(0.0043)	(0.0013)	(0.0023)
SMB	−0.0008	**0.0131**	**0.0383**	−0.0008	−0.0003
	(0.0030)	(0.0048)	(0.0065)	(0.0005)	(0.0002)
HML	0.0022	0.0013	−0.0009	**0.0038**	0.0031
	(0.0042)	(0.0013)	(0.0006)	(0.0016)	(0.0023)
UMD	0.0015	0.0006	0.0002	0.0019	0.0023
	(0.0027)	(0.0006)	(0.0001)	(0.0012)	(0.0015)
TERM	0.0084	−**0.1427**	−**0.1440**	0.0025	0.0104
	(0.0109)	(0.0162)	(0.0263)	(0.0018)	(0.0070)
DEF	−**0.0265**	**0.1063**	**0.1370**	−**0.0246**	−**0.0227**
	(0.0242)	(0.0209)	(0.0369)	(0.0080)	(0.0117)
Price	**0.0023**	**0.0077**	**0.0093**	**0.0025**	**0.0037**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 2. Hedging of Real Interest Rate. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	**MV**	**CVaR$_{0.9}$**	**CVaR$_{0.95}$**	**VaR$_{0.9}$**	**VaR$_{0.95}$**
Intercept		**0.0021**	**0.0035**		
		(0.0001)	(0.0002)		
RF	−0.0304	−**0.1473**	−**0.6805**	−0.0147	−0.0296
	(0.0563)	(0.0621)	(0.1000)	(0.0063)	(0.0380)
MARKET	**0.0049**	−0.0020	0.0094	**0.0048**	0.0038
	(0.0028)	(0.0017)	(0.0045)	(0.0015)	(0.0032)
SMB	0.0013	−0.0009	0.0042	**0.0009**	0.0013
	(0.0029)	(0.0007)	(0.0033)	(0.0003)	(0.0020)
HML	−0.0029	−**0.0142**	**0.0188**	−**0.0031**	−0.0031
	(0.0042)	(0.0042)	(0.0072)	(0.0012)	(0.0037)
UMD	−0.0008	−**0.0346**	−0.0011	−0.0007	−0.0007
	(0.0027)	(0.0031)	(0.0008)	(0.0003)	(0.0013)
TERM	−0.0167	−**0.0664**	−**0.1187**	−**0.0284**	−0.0166
	(0.0109)	(0.0123)	(0.0239)	(0.0065)	(0.0133)
DEF	0.0205	**0.1095**	**0.2810**	**0.0222**	0.0226
	(0.0244)	(0.0194)	(0.0305)	(0.0073)	(0.0186)
Price	**0.0023**	**0.0075**	**0.0110**	**0.0027**	**0.0035**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 3. Hedging of Term Spread. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	**MV**	**CVaR$_{0.9}$**	**CVaR$_{0.95}$**	**VaR$_{0.9}$**	**VaR$_{0.95}$**
Intercept		−0.0000	−0.0000		
		(0.0012)	(0.0018)		
RF	**0.3958**	**0.3782**	**0.3696**	**0.3886**	**0.3883**
	(0.0922)	(0.0877)	(0.1094)	(0.0526)	(0.0898)
MARKET	0.0011	−0.0023	−0.0016	**0.0018**	0.0006
	(0.0043)	(0.0061)	(0.0095)	(0.0005)	(0.0006)
SMB	0.0034	0.0047	0.0033	**0.0026**	0.0028
	(0.0048)	(0.0092)	(0.0138)	(0.0008)	(0.0016)
HML	0.0071	0.0010	0.0013	**0.0091**	0.0078
	(0.0053)	(0.0227)	(0.0230)	(0.0021)	(0.0042)
UMD	−0.0038	−0.0017	−0.0015	**-0.0065**	-0.0051
	(0.0042)	(0.0069)	(0.0099)	(0.0018)	(0.0026)
TERM	**0.1346**	**0.1055**	**0.1024**	**0.1277**	**0.1532**
	(0.0163)	(0.0194)	(0.0242)	(0.0126)	(0.0251)
DEF	−**0.0691**	−**0.0789**	−0.0756	−**0.0926**	−**0.0571**
	(0.0296)	(0.0307)	(0.0341)	(0.0138)	(0.0268)
Price	**0.0033**	**0.0057**	**0.0082**	**0.0029**	**0.0044**
	(0.0000)	(0.0001)	(0.0002)	(0.0001)	(0.0002)

Table 4. Hedging of Default Spread. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	**MV**	**CVaR$_{0.9}$**	**CVaR$_{0.95}$**	**VaR$_{0.9}$**	**VaR$_{0.95}$**
Intercept		**0.0010**	**0.0011**		
		(0.0000)	(0.0001)		
RF	−**0.0616**	0.0358	−0.0017	−**0.0572**	−**0.0359**
	(0.0335)	(0.0214)	(0.0010)	(0.0119)	(0.0133)
MARKET	−0.0008	0.0002	**0.0115**	−**0.0011**	−0.0005
	(0.0018)	(0.0001)	(0.0026)	(0.0003)	(0.0002)
SMB	−0.0015	0.0030	−**0.0090**	−**0.0016**	−**0.0027**
	(0.0014)	(0.0015)	(0.0036)	(0.0005)	(0.0011)
HML	−0.0005	**0.0091**	**0.0177**	−0.0004	−**0.0001**
	(0.0023)	(0.0015)	(0.0040)	(0.0001)	(0.0001)
UMD	−0.0003	**0.0026**	**0.0097**	−**0.0002**	−**0.0000**
	(0.0012)	(0.0010)	(0.0026)	(0.0001)	(0.0000)
TERM	−**0.0147**	−**0.0153**	−0.0169	−**0.0135**	−0.0005
	(0.0049)	(0.0051)	(0.0082)	(0.0025)	(0.0003)
DEF	**0.0503**	**0.1078**	**0.0965**	**0.0498**	**0.0882**
	(0.0125)	(0.0063)	(0.0158)	(0.0052)	(0.0079)
Price	**0.0011**	**0.0036**	**0.0048**	**0.0009**	**0.0016**
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0001)

Table 5. Hedging of Dividend Yield. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	MV	CVaR$_{0.9}$	CVaR$_{0.95}$	VaR$_{0.9}$	VaR$_{0.95}$
Intercept		**0.0007**	**0.0011**		
		(0.0001)	(0.0001)		
RF	−**0.0735**	−0.0323	−**0.0794**	−**0.0774**	−**0.0736**
	(0.0148)	(0.0291)	(0.0352)	(0.0074)	(0.0104)
MARKET	−**0.0309**	−**0.0313**	−**0.0343**	−**0.0293**	−**0.0313**
	(0.0010)	(0.0015)	(0.0017)	(0.0012)	(0.0016)
SMB	−0.0000	0.0004	0.0000	−**0.0000**	−**0.0000**
	(0.0012)	(0.0007)	(0.0000)	(0.0000)	(0.0000)
HML	−**0.0028**	−0.0026	−0.0020	−**0.0029**	−**0.0032**
	(0.0014)	(0.0021)	(0.0019)	(0.0004)	(0.0005)
UMD	−**0.0012**	0.0016	−0.0028	−**0.0013**	−**0.0015**
	(0.0008)	(0.0012)	(0.0015)	(0.0002)	(0.0002)
TERM	−0.0036	**0.0146**	−**0.0163**	−**0.0035**	−**0.0040**
	(0.0029)	(0.0069)	(0.0067)	(0.0006)	(0.0008)
DEF	−0.0038	−0.0152	**0.0276**	−**0.0037**	**0.0010**
	(0.0044)	(0.0083)	(0.0105)	(0.0008)	(0.0002)
Price	**0.0007**	**0.0018**	**0.0028**	**0.0007**	**0.0010**
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)

Table 6. Hedging of Consumption Growth. The table reports the estimates and their corresponding bootstrap errors (based on 400 bootstrap replications) for different risk measures (mean-variance MV, conditional value-at-risk CVaR, and value-at-risk VaR). The bold font represents statistical significance (at the 5% nominal level) of individual coefficients except for the last row where the bold font signifies a statistically different price from $E(y)$.

Securities\ risk	MV	CVaR$_{0.9}$	CVaR$_{0.95}$	VaR$_{0.9}$	VaR$_{0.95}$
Intercept		**0.0059**	**0.0080**		
		(0.0003)	(0.0005)		
RF	−**0.2533**	−0.0510	0.0696	−**0.2882**	−0.1159
	(0.1112)	(0.0718)	(0.1184)	(0.0672)	(0.0788)
MARKET	0.0082	0.0079	−0.0067	**0.0074**	0.0048
	(0.0056)	(0.0062)	(0.0102)	(0.0023)	(0.0037)
SMB	**0.0256**	**0.0315**	**0.0467**	**0.0288**	**0.0237**
	(0.0080)	(0.0097)	(0.0169)	(0.0051)	(0.0094)
HML	0.0132	0.0079	−0.0096	**0.0034**	0.0108
	(0.0080)	(0.0089)	(0.0141)	(0.0020)	(0.0075)
UMD	−0.0034	**0.0334**	**0.0546**	−**0.0032**	−0.0032
	(0.0052)	(0.0073)	(0.0120)	(0.0015)	(0.0028)
TERM	−**0.0509**	−**0.0581**	−0.0070	−**0.0835**	−**0.0819**
	(0.0241)	(0.0223)	(0.0160)	(0.0165)	(0.0263)
DEF	−0.0568	0.0005	**0.1904**	−**0.0562**	−**0.0802**
	(0.0312)	(0.0007)	(0.0648)	(0.0178)	(0.0332)
Price	**0.0054**	**0.0159**	**0.0223**	**0.0064**	**0.0083**
	(0.0000)	(0.0001)	(0.0002)	(0.0001)	(0.0002)

5. Conclusions

In this paper, we develop a general framework for hedging and pricing that can for instance be used for hedging financial or economic variables in the presence of market incompleteness and frictions. The proposed approach to hedging and pricing is model-free and its generality allows us to accommodate a large family of risk measures and pricing rules. We augment this robust approach with a set of market principles to study the conditions under which the hedging problem admits a solution and pricing is possible. Our paper is the first to accommodate and analyze non-convex risk and pricing rules which are extensively used in risk management (such as Value at Risk and risk measures related to Choquet expected utility) and actuarial applications.

Furthermore, we illustrate the advantages of our proposed method for hedging economic risk using monthly U.S. data for the 1952–2012 period.

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Appendix A. Proofs of Propositions and Theorems

Appendix A.1. Proof of Theorem 1

From Delbaen (2002), the equality in Theorem 1 holds for $\varrho_\alpha = \mathrm{VaR}_\alpha$. Therefore, since the minimum is attained for VaR_α, for any α there exists $\varrho^\alpha \geq \mathrm{VaR}_\alpha$ such that $\varrho^\alpha(x_0) = \mathrm{VaR}_\alpha(x_0)$. Now introduce $\varrho(x) = \int_0^1 \varrho^\alpha(x)\varphi(\alpha)d\alpha$. It is easy to see that ϱ is a coherent risk measure such that $\varrho \geq \varrho_\varphi$ and $\varrho(x_0) = \varrho_\varphi(x_0)$, which proves the desired result.

Appendix A.2. Proof of Proposition 1

We only provide the proof of statement 1 since the proof of statement 2 follows very similar arguments. Let $g \in \mathcal{X}_\alpha$ and $t \in \mathbb{R}_+$. Then,

$$\pi_\varrho(tg) = \inf_{x\in\mathcal{X}}\{\varrho(x-tg)+\pi(x)\} = \inf_{tx\in\mathcal{X}}\{\varrho(tx-tg)+\pi(tx)\} = t\pi_\varrho(x) \in \mathbb{R}.$$

Using the same argument, one can show that for $g \in \mathcal{X}_\varrho$, $\pi_\varrho(x+c) = \pi_\varrho(x)+c$ for all $c \in \mathbb{R}$. Hence, we have that $g+c \in \mathcal{X}_\varrho$.

Now let $g \in \mathcal{X}_\varrho$ and $g \leq h$. Because ϱ is decreasing, we have that

$$\varrho(x-h)+\pi(x) \geq \varrho(x-g)+\pi(x).$$

By taking infimum on $\mathcal{X}$, we obtain that $\pi_\varrho(h) \in \mathbb{R}$.

Appendix A.3. Proof of Theorem 2

We begin by showing the equivalence between (N) and (NGD). To this end, we demonstrate that both of them are equivalent to the following inequality:

$$\varrho(x)+\pi(x) \geq 0\,,\ \forall x \in \mathcal{X}. \tag{A1}$$

First, we show that (N) is equivalent to (A1). Given (N), we have that $\pi_\varrho(0) = 0$ which, by construction, implies (A1). On the other hand, given (A1) it is easy to see that $\pi_\varrho(0) \geq 0$. In addition, by setting $x = 0$ in (A1), it follows that $\pi_\varrho(0) = 0$.

Second, we show the equivalence between (A1) and (NGD). Suppose that x is a Good Deal, i.e., $\varrho(x) < 0$ and $\pi(x) \leq 0$, which clearly implies $\varrho(x) + \pi(x) < 0$. On the other hand, if (A1) does not hold, we have that $\varrho(x) + \pi(x) < 0$ for some position x. By cash-invariance of π and ϱ, it is obvious that $x - \pi(x)$ is a Good Deal.

Next, we demonstrate the equivalence between (NGD) and (C). Assume that (NGD) does not hold. Then, there exists an x such that $\varrho(x) < 0$ and $\pi(x) \leq 0$. Let y be a variable and assume that $c \in \mathbb{R}$ is such that $y \leq c$. Since $tx - y \geq tx - c$ for all $t > 0$,

$$\begin{aligned} \varrho(tx - y) + \pi(tx) &\leq \varrho(tx - c) + \pi(tx) \\ &= \varrho(tx) + c + \pi(tx) \\ &= t(\varrho(x) + \pi(x)) + c \to -\infty, \end{aligned}$$

as t tends to $+\infty$. This shows that (12) does not have a finite infimum.

To establish (NGD) $\Rightarrow$ (C), assume that for a variable y, (12) does not have a finite infimum. Let $c \in \mathbb{R}$ be such that $c \leq y$. Since $x - c \geq x - y$ for all financial positions $x \in \mathcal{X}$, we have that

$$\begin{aligned} \varrho(x - c) \leq \varrho(x - y) &\Rightarrow \varrho(x) + c \leq \varrho(x - y) \\ &\Rightarrow \varrho(x) + \pi(x) + c \leq \varrho(x - y) + \pi(x). \end{aligned}$$

Since (12) is not bounded, then there exists an x such that $\varrho(x - y) + \pi(x) < c$. This yields $\varrho(x) + \pi(x) < 0$. Thus, it is clear that $\tilde{x} = x - \pi(x)$ is a Good Deal.

Finally, we show (N) $\Rightarrow$ (CP) when $\mathcal{X}$ is a vector space and π is super-additive. Let $y \in \mathcal{X}$ and suppose that (N) holds. Since $\mathcal{X}$ is a vector space, we have that, for a given x, $\mathcal{X} - x = \mathcal{X}$. Therefore, by construction,

$$\varrho(x - y) + \pi(x - y) \geq \pi_\varrho(0) = 0$$

and by super-additivity of π,

$$\varrho(x - y) + \pi(x) - \pi(y) \geq \varrho(x - y) + \pi(x - y) \geq 0$$

which implies that $\varrho(x - y) + \pi(x) \geq \pi(y)$. Therefore, $\pi_\varrho(y) = \pi(y)$.

Appendix A.4. Proof of Proposition 2

First, note that

$$\begin{aligned} \chi_A^*(z) &= \sup_x \{E(zx) - \chi_A(x)\} \\ &= \sup_{x \in A} E(zx) \qquad\qquad , \\ &= \chi_{A^\perp}(z). \end{aligned}$$

Hence, $\chi_A(x) = \sup\limits_{z \in A^\perp} E(zx)$. Then, we have

$$\begin{aligned} \bar{f}(x) &= f(x) + \chi_A(x) = \sup_{z \in \Delta_f} E(zx) + \sup_{z' \in A^\perp} E(z'x) \\ &= \sup_{(z,z') \in \Delta_f \times A^\perp} E((z + z')x) \\ &= \sup\nolimits_{z \in \Delta_f + A^\perp} E(zx). \end{aligned}$$

Appendix A.5. Proof of Theorem 3

First, we prove the result for sub-additive risk measures and pricing rules. The following proposition, which is a standard result in the literature on convex analysis, presents the necessary and sufficient conditions under which solution to the hedging problem exists.

Proposition A1. *Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two convex functions. Then, the following equality holds*

$$\inf_{x \in \mathbb{R}^n} \{f_1(y - x) + f_2(x)\} = (f_1^* + f_2^*)^*(x),$$

with the convention that $\sup(\Phi) = -\infty$.

In the particular case when $f_1 = \bar{\pi}$ and $f_2 = \bar{\varrho}$, we have

$$(f_1^* + f_2^*)(x) = (\chi_{\Delta_\varrho - \mathcal{D}} + \chi_{\mathcal{R} + \mathcal{X}^\perp})(x) = \chi_{(\Delta_\varrho - \mathcal{D}) \cap (\mathcal{R} + \mathcal{X}^\perp)}(x).$$

Therefore,

$$\inf_{x \in \mathcal{X}} \{\varrho(x - y) + \pi(x)\} = \sup_{z \in (\Delta_\varrho - \mathcal{D}) \cap (\mathcal{R} + \mathcal{X}^\perp)} E(zy).$$

This proves the existence of the infimum for the sub-additive case.

In the general case, we have

$$\begin{aligned}
\inf_{x \in \mathcal{X}} \{\varrho_{\mathbb{D}}(x - y) + \pi_{\mathbb{M}}(x)\} &= \inf_{x \in \mathcal{X}} \left\{ \inf_{\varrho \in \mathbb{D}} \varrho(x - y) + \inf_{\pi \in \mathbb{M}} \pi(x) \right\} \\
&= \inf_{x \in \mathcal{X}} \left\{ \inf_{\varrho \in \mathbb{D} \times \pi \in \mathbb{M}} \varrho(x - y) + \pi(x) \right\} \\
&= \inf_{\varrho \in \mathbb{D} \times \pi \in \mathbb{M}} \{ \inf_{x \in \mathcal{X}} \varrho(x - y) + \pi(x) \}.
\end{aligned}$$

This problem has a finite infimum if for every $\varrho \in \mathbb{D}$ and $\pi \in \mathbb{M}$, the inner problem $\inf_{x \in \mathcal{X}} \varrho(x - y) + \pi(x)$ is finite. Given the discussion above, this proves the statement of the theorem.

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