Multiple orthogonal polynomials with respect to Gauss' hypergeometric function

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Abstract

A new set of multiple orthogonal polynomials of both type I and type II with respect to two weight functions involving Gauss' hypergeometric function on the interval $(0,1)$ is studied. This type of polynomials have direct applications in the investigation of singular values of products of Ginibre matrices, in the analysis of rational solutions to Painlevé equations and are connected with branched continued fractions and total positivity problems in combinatorics. The pair of orthogonality measures is shown to be a Nikishin system and to satisfy a matrix Pearson-type differential equation. The focus is on the polynomials whose indexes lie on the step line, for which it is shown that differentiation on the variable gives a shift on the parameters, therefore satisfying Hahn's property. We obtain a Rodrigues-type formula for type I, while a more detailed characterisation is given for the type II polynomials (aka 2-orthogonal polynomials) which include: an explicit expression as a terminating hypergeometric series, a third-order differential equation, and a third-order recurrence relation. The asymptotic behaviour of their recurrence coefficients mimics those of Jacobi-Piñeiro polynomials, based on which, their zero asymptotic distribution and a Mehler-Heine asymptotic formula near the origin are given. Particular choices on the parameters degenerate in some known systems such as special cases of the Jacobi-Piñeiro polynomials, Jacobi-type 2-orthogonal polynomials, and components of the cubic decomposition of threefold symmetric Hahn-classical polynomials. Equally considered are confluence relations to other known polynomial sets, such as multiple orthogonal polynomials with respect to Tricomi functions.

Keywords: *Multiple orthogonal polynomials, Gauss hypergeometric function, Nikishin system, Rodrigues-type formula, generalised hypergeometric series, 2-orthogonal polynomials, Hahn classical*

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1 Introduction and motivation

The main aim of this paper is to investigate the multiple orthogonal polynomials with respect to two absolutely continuous measures supported on the interval $(0,1)$ and admitting an integral representation via weight functions $\mathscr{W}(x;a,b;c,d)$ and $\mathscr{W}(x;a,b+1;c+1,d)$, where

$$\mathscr{W}(x;a,b;c,d) = \frac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\delta)}\, x^{a-1}(1-x)^{\delta-1}\, {}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right), \tag{1.1}$$

with

$$a,b,c,d \in \mathbb{R}^+ \text{ such that } \min\{c,d\} > \max\{a,b\} \text{ and } \delta = c+d-a-b > 0. \tag{1.2}$$

The weight functions involve Gauss' hypergeometric function, which is defined, for parameters $\alpha,\beta \in \mathbb{C}$ and $\gamma \in \mathbb{C}\backslash\{-n : n \in \mathbb{N}\}$, by

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!}. \tag{1.3}$$

where $(z)_n$ denotes the Pochhammer symbol defined by

$$(z)_0 = 1 \quad \text{and} \quad (z)_n := z(z+1)\cdots(z+n-1), \quad n \in \mathbb{Z}^+.$$

The detailed knowledge of multiple orthogonal polynomials with respect to (generalised) hypergeometric functions has applications in random matrix theory, combinatorics, description of rational solutions to nonlinear differential difference equations, such as Painlevé equations, number theory, among other fields. For instance, the analysis of singular values of products of Ginibre matrices in [19, 18] uses multiple orthogonal polynomials associated with weight functions expressed in terms of Meijer G-functions, a class of weights to which the weight (1.1) belongs. Besides, these polynomials are linked with the branched continued fractions introduced in [29] as the generating functions of m-Dyck paths, for the purpose of solving total positivity problems involving combinatorially interesting sequences of polynomials. This connection, which leads to new results on both fields involved, will be further explored in forthcoming work.

The hypergeometric function defined by (1.3) converges absolutely for $|z| < 1$, and it is a solution of the hypergeometric differential equation

$$z(1-z)F''(z) + (\gamma - (\alpha+\beta+1)z)F'(z) - \alpha\beta F(z) = 0. \tag{1.4}$$

Recall the identity ${}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}$, which is valid for $(\gamma-\alpha-\beta) > 0$. When $(\gamma-\alpha-\beta) < 0$, we have $\lim\limits_{x\to 1^-}(1-x)^{-(\gamma-\alpha-\beta)}{}_2F_1(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}$. This yields $\lim\limits_{x\to 0^+}\mathscr{W}(x;a,b;c,d) = 0$.

Observe that $\mathscr{W}(x;a,b;c,d) = \mathscr{W}(x;a,b;d,c)$, which is a straightforward consequence of (1.1) and (1.3). In addition, the symmetry $\mathscr{W}(x;a,b;c,d) = \mathscr{W}(x;b,a;c,d)$ also holds, because using [8, Eq. 15.8.1] we have

$$ {}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right) = x^{b-a}{}_2F_1\left(\begin{matrix} d-a, c-a \\ \delta \end{matrix}; 1-x\right). $$

Under the assumptions (1.2), we have (see [4, Eq. 2.21.1.11] or [16, Eq. 7.512.4])

$$ \int_0^1 x^{a+n-1}(1-x)^{\delta-1}{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right)\mathrm{d}x = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(\delta)}{\Gamma(c+n)\Gamma(d+n)}. $$

Therefore, $\mathscr{W}(x;a,b;c,d)$ is a probability density function on the interval $(0,1)$ with moments

$$ \int_0^1 x^n \mathscr{W}(x;a,b;c,d)\mathrm{d}x = \frac{(a)_n(b)_n}{(c)_n(d)_n} \quad \text{for} \quad n \in \mathbb{N}. \tag{1.5} $$

Throughout the text, $\mathbb{N} = \mathbb{Z}_0^+ = \{0,1,2,\cdots\}$. When referring to $\{P_n(x)\}_{n\in\mathbb{N}}$ as a polynomial sequence it is assumed that P_n is a polynomial of a single variable with degree exactly n. We consistently deal with monic polynomials, unless stated otherwise.

Multiple orthogonal polynomials are a generalisation of (standard) orthogonal polynomials. We give a brief introduction to this topic here, further information can be found for instance in [17, Ch. 23] and [23].

The orthogonality conditions of multiple orthogonal polynomials are spread across a vector of $r \in \mathbb{Z}^+$ measures and they are polynomials on a single variable depending on a multi-index $\vec{n} = (n_0,\cdots,n_{r-1}) \in \mathbb{N}^r$ of length $|\vec{n}| = n_0+\cdots+n_{r-1}$. There are two types of multiple orthogonal polynomials with respect to a system of r measures $(\mu_0,\cdots,\mu_{r-1})$. When the number of measures is $r=1$, both types of multiple orthogonality reduce to standard orthogonality. A polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is orthogonal with respect to a measure μ if

$$ \int x^k P_n(x)\mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \le k \le n-1, \\ N_n \neq 0, & \text{if } n = k. \end{cases} $$

We focus on the case of $r=2$ measures but the definitions presented here are easily generalised for $r \geq 2$.

The *type I multiple orthogonal polynomials* for $\vec{n} = (n_0,n_1) \in \mathbb{N}^2$ are given by a vector of 2 polynomials $\left(A_{(n_0,n_1)}, B_{(n_0,n_1)}\right)$, with $\deg A_{(n_0,n_1)} \le n_0 - 1$ and $\deg B_{(n_0,n_1)} \le n_1 - 1$, satisfying the orthogonality and normalisation conditions

$$ \int x^k A_{(n_0,n_1)}(x)\mathrm{d}\mu_0(x) + \int x^k B_{(n_0,n_1)}(x)\mathrm{d}\mu_1(x) = \begin{cases} 0, & \text{if } 0 \le k \le n_0+n_1-2, \\ 1, & \text{if } k = n_0+n_1-1. \end{cases} \tag{1.6} $$

If the measures $\mu_0(x)$ and $\mu_1(x)$ are absolutely continuous with respect to a common positive measure μ, that is, if there exist weight functions $w_0(x)$ and $w_1(x)$ such that $\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x)$, for both $j \in \{0,1\}$, then the *type I function* is

$$ Q_{(n_0,n_1)}(x) = A_{(n_0,n_1)}(x)w_0(x) + B_{(n_0,n_1)}(x)w_1(x) \tag{1.7} $$

and the conditions in (1.6) become

$$\int x^k Q_{(n_0,n_1)}(x)\mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \le k \le n_0+n_1-2, \\ 1, & \text{if } k = n_0+n_1-1. \end{cases}$$

The *type II multiple orthogonal polynomial* for $\vec{n}=(n_0,n_1)\in\mathbb{N}^2$ is a monic polynomial $P_{(n_0,n_1)}$ of degree n_0+n_1 which satisfies, for both $j\in\{0,1\}$, the orthogonality conditions

$$\int x^k P_{(n_0,n_1)}(x)\mathrm{d}\mu_j(x) = 0, \quad 0\le k\le n_j-1. \tag{1.8}$$

The orthogonality conditions for both type I and type II multiple orthogonal polynomials give a non-homogeneous system of n_0+n_1 linear equations for the n_0+n_1 unknown coefficients of the vector of polynomials $\left(A_{(n_0,n_1)},B_{(n_0,n_1)}\right)$ in (1.6) or the polynomials $P_{(n_0,n_1)}(x)$ in (1.8). If the solution exists, it is unique and the corresponding matrices of the system for type I and type II are the transpose to each other. However it is possible that this system doesn't have a solution, unless further conditions are imposed (unlike standard orthogonality on the real line, the existence of such solutions is not a trivial matter). If there is a unique solution, then the multi-index $\vec{n}$ is called *normal* and if all multi-indices are normal, the system is a *perfect system*.

An example of systems known to be perfect are the Algebraic Tchebyshev systems, or simply *AT-systems* (see [28, Ch. 4]). A pair of measures (μ_0,μ_1) is an AT-system on an interval I for a multi-index $\vec{n}=(n_0,n_1)\in\mathbb{N}^2$ if the measures $\mu_0(x)$ and $\mu_1(x)$ are absolutely continuous with respect to a common positive measure μ on I, via weight functions $w_0(x)$ and $w_1(x)$, and the set of functions

$$\left\{w_0(x),xw_0(x),\cdots,x^{n_0-1}w_0(x),w_1(x),xw_1(x),\cdots,x^{n_1-1}w_1(x)\right\}$$

forms a Chebyshev system on I, meaning that for any polynomials p_0 and p_1 of degree not greater than n_0-1 and n_1-1, respectively, and not simultaneously equal to 0, the function $p_0(x)w_0(x)+p_1(x)w_1(x)$ has at most n_0+n_1-1 zeros on I. A vector of measures (μ_0,μ_1) is an AT-system on an interval I if it is an AT-system on I for every multi-index in $\mathbb{N}^2$.

Another special example of a perfect system is a *Nikishin system* (firstly introduced in [27]). A pair of measures (μ_0,μ_1) forms a Nikishin system (of order 2) if both measures are supported on an interval I_0 and there exists a positive measure σ on an interval I_1 with $I_0\cap I_1=\emptyset$ such that

$$\frac{\mathrm{d}\mu_1(x)}{\mathrm{d}\mu_0(x)} = \int_{I_1}\frac{\mathrm{d}\sigma(t)}{x-t}. \tag{1.9}$$

It was proved in [13] that every Nikishin system is perfect (see also [14] for the cases where the supports of the measures are unbounded or where consecutive intervals touch at one point). More precisely, it is proved in [13] and [14] that every Nikishin system is an AT-system, therefore it is perfect. Moreover, for any $(n_0,n_1)\in\mathbb{N}^2$ belonging to an AT-system on an interval I, the type I function for $Q_{(n_0,n_1)}$ defined by (1.7) has exactly n_0+n_1-1 sign changes on I and the type II multiple orthogonal polynomial $P_{(n_0,n_1)}$ has n_0+n_1 simple zeros on I which satisfy an interlacing property as there is always a zero of $P_{(n_0,n_1)}$ between two consecutive zeros of $P_{(n_0+1,n_1)}$ or $P_{(n_0,n_1+1)}$. As a Nikishin system is always an AT-system, the same properties hold for Nikishin systems.

The main contribution of this paper is on multi-indices on the step line. A multi-index $(n_0,n_1)\in\mathbb{N}^2$ is on the step line if either $n_0=n_1$ or $n_0=n_1+1$ (alternatively to the latter we could consider $n_1=n_0+1$, that change is equivalent to swapping the roles of the measures). For each $n\in\mathbb{N}$, there is a unique multi-index of length n on the step line of $\mathbb{N}^2$. More precisely, the multi-index of length n is $\vec{n}=(m,m)$, if $n=2m$, or $\vec{n}=(m+1,m)$, if $n=2m+1$. Therefore, when we only consider multi-indices on the step line, we can replace any multi-index by its length without any ambiguity.

For the type II multiple orthogonal polynomials on the step line, we obtain a polynomial sequence with exactly one polynomial of degree n for each $n\in\mathbb{N}$. These are often referred to as *d-orthogonal polynomials* (where d is the number of orthogonality measures), as introduced in [25]. In the case of $d=2$ measures, the type II multiple orthogonality conditions (1.8) on the step line correspond to say that if we set

$$P_{2m}(x)=P_{m,m}(x)\quad\text{and}\quad P_{2m+1}(x)=P_{m+1,m}(x),$$

then the polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is 2-*orthogonal* with respect to a pair of measures (μ_0,μ_1) if for each $j\in\{0,1\}$

$$\int x^k P_n(x)\mathrm{d}\mu_j(x)=\begin{cases}0, & \text{if } n\geq 2k+j+1,\\ N_n\neq 0, & \text{if } n=2k+j.\end{cases}$$

Straightforwardly from the definition (1.8) observe that $\{P_{(n,0)}\}_{n\geq 0}$ and $\{P_{(0,n)}\}_{n\geq 0}$ are (standard) orthogonal polynomial sequences with respect to the measures μ_0 and μ_1, respectively. As such, by the spectral theorem for orthogonal polynomials (*aka* Shohat-Favard theorem) : $\{P_{(n,0)}\}_{n\geq 0}$ and $\{P_{(0,n)}\}_{n\geq 0}$ are orthogonal if and only if there exist coefficient two pairs of coefficients $(\beta_n^{(0)},\gamma_n^{(0)})$ and $(\beta_n^{(1)},\gamma_n^{(1)})$ with $\gamma_n^{(j)}\neq 0$ for all $n\geq 1$ and each $j=1,2$ such that $\{P_{(n,0)}\}_{n\geq 0}$ and $\{P_{(0,n)}\}_{n\geq 0}$ respectively satisfy the second order recurrence relation

$$p_{n+1}(x)=(x-\beta_n^{(j)})p_n(x)-\gamma_n^{(1)}p_{n-1}(x),$$

with initial conditions $p_{-1}=0$ and $p_0=1$. Moreover, if the β-coefficients are all real and the γ-coefficients are all positive, then μ is a positive measure on the real line.

Multiple orthogonal polynomials also satisfy (nearest-neighbour) recurrence relations (see [34]). In particular, when the indexes lie on the step line, a polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is 2-orthogonal if and only if it satisfies a third order recurrence relation of the type

$$P_{n+1}(x)=(x-\beta_n)P_n(x)-\alpha_nP_{n-1}(x)-\gamma_{n-1}P_{n-2}(x), \tag{1.10}$$

with $\gamma_n\neq 0$, for all $n\geq 1$, and initial conditions $P_{-2}=P_{-1}=0$ and $P_0=1$.

The latter recurrence relation can be expressed, for each $n \in \mathbb{Z}^+$, as

$$\mathrm{H}_n \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} - P_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

involving the truncated lower-Hessenberg matrix

$$\mathrm{H}_n = \begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \beta_1 & 1 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 1 \\ 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}. \tag{1.11}$$

Therefore, the zeros of $P_n(x)$ correspond to the eigenvalues of the Hessenberg matrix H_n, which highlights the connection between multiple orthogonal polynomials and the spectral theory of non-selfadjoint operators explored in [1] and [33], among others.

For the type I multiple orthogonal polynomials on the step line for $r = 2$ measures, we have

$$\deg(A_n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{and} \quad \deg(B_n) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

that is, $\deg(A_n) = m-1$, if $n = 2m$ or $n = 2m-1$, and $\deg(B_n) = m-1$, if $n = 2m$ or $n = 2m+1$. Assuming that there exists a positive measure μ and a pair of weight functions (w_0, w_1) such that $\mathrm{d}\mu_0(x) = w_0(x)\mathrm{d}\mu(x)$ and $\mathrm{d}\mu_1(x) = w_1(x)\mathrm{d}\mu(x)$, the type I function on the step line is

$$Q_n(x) = A_n(x)w_0(x) + B_n(x)w_1(x) \tag{1.12}$$

and the orthogonality and normalisation conditions correspond to

$$\int x^k Q_n(x)\mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \leq k \leq n-2, \\ 1, & \text{if } k = n-1. \end{cases}$$

Further information about multiple orthogonal polynomials can be found for instance in [17, Ch. 23] and [23].

We start Section 2 by showing that the weight functions $\mathscr{W}(x;a,b;c,d)$ and $\mathscr{W}(x;a,b+1;c+1,d)$ in (1.1) form a Nikishin system (see Theorem 2.1). This readily implies that the multiple orthogonal polynomials of both type II and type I with respect to these weight functions exist and are unique for every multi-index $\vec{n} = (n_0, n_1) \in \mathbb{N}^2$ and their zeros satisfy the properties as those of an AT-system. Next we obtain a second order differential equation and a matrix differential equation satisfied by the weight functions (see Theorems 2.2 and 2.3, respectively), which we use to deduce differential properties for the multiple orthogonal polynomials of both type II and type I on the step line (see Theorem 2.4). More precisely, we show that the differentiation of both type II and type I polynomials on the step line gives a shift on the parameters as well as on the index. So this means that these multiple orthogonal polynomials satisfy the so called *Hahn's property*: the sequence of its derivatives is again multiple orthogonal. In particular, the type II polynomials stand as an example of a Hahn-classical 2-orthogonal family. Finally, we derive a Rodrigues-type formula for the type I functions on the step line (see Theorem 2.5) as well as a recursive relation generating the type I polynomials.

Section 3 is devoted to the characterisation of the 2-orthogonal polynomials with respect to the pair of weights $\left[\mathscr{W}(x;a,b;c,d),\mathscr{W}(x;a,b+1;c+1,d)\right]$. To begin with, in §3.1, we give an explicit expression for these polynomials as terminating generalised hypergeometric series, more precisely as ${}_3F_2$. Generalised hypergeometric series are formally defined by

$$ {}_pF_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\,, \tag{1.13}$$

where $p,q\in\mathbb{N}$, $z,\alpha_1,\cdots,\alpha_p\in\mathbb{C}$ and $\beta_1,\cdots,\beta_p\in\mathbb{C}\backslash\{-n:\ n\in\mathbb{N}\}$. If one of the parameters $\alpha_1,\cdots,\alpha_p$ is a non-positive integer, the series (1.13) terminates and defines a (hypergeometric type) polynomial. When the series does not terminate, it converges for all finite values of z if $p\leq q$ and on the open unit disk $|z|<1$ (with convergence on the unit circle depending on the parameters) if $p=q+1$ and it diverges for any $z\neq 0$ otherwise. When the series is convergent, the function defined by (1.13) is a solution to the generalised hypergeometric differential equation (see [8, Eq. 16.8.3])

$$\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\beta_1\right)\cdots\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\beta_q\right)\frac{\mathrm{d}}{\mathrm{d}z}\right]F(z)=\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\alpha_1\right)\cdots\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\alpha_p\right)\right]F(z). \tag{1.14}$$

Note that the latter reduces to (1.4) when $(p,q)=(2,1)$. Thus, based on the explicit expression for the 2-orthogonal polynomials we are able to describe them as a solution to a third order differential equation (of hypergeometric type) in §3.2 and in §3.3 as a solution to a third order recurrence relation. Particular choices on the parameters a,b,c,d of these polynomials result in known multiple orthogonal polynomials. So, in §3.5 we make the connection to the so-called Jacobi-type 2-orthogonal polynomials investigated in [21], where we pay particular attention to the case where all the coefficients are constant. The recurrence relation coefficients of the multiple orthogonal polynomials under analysis can be written as combinations of the coefficients of a branched continued fraction representation for a generalised hypergeometric function derived in [29]. Hence, these recurrence coefficients are real, positive and bounded, whose asymptotic behaviour coincides with the one of the recurrence relation coefficients of Jacobi-Piñeiro (type II) multiple orthogonal polynomials on the step line studied in [6]. As a consequence (see [5]), the two distinct polynomial sets also share the same ratio asymptotics and therefore the same asymptotic zero distribution as well as the same Mehler-Heine asymptotic near the endpoint at 0, as detailed in §3.4. Besides, other particular choices on the parameters a,b,c,d lead to the three components of certain 3-fold symmetric Hahn-classical 2-orthogonal polynomials on star-like sets that appeared in [24], as we explain in

§3.6. Finally, we establish confluence relations (or limiting relations on the parameters) to other Hahn-classical 2-orthogonal polynomials of hypergeometric type, such as the ones investigated in [22].

2 Differential properties and multiple orthogonality

This investigation starts with a pair of weight functions $\mathscr{W}(x;a,b;c,d)$ and $\mathscr{W}(x;a,b+1;c+1,d)$ defined in (1.1) subject to the constraints (1.2) on the parameters a,b,c and d. The goal is to describe multiple orthogonal polynomials of type I and type II with respect to these weights. Before doing so, we aim to prove that such sets of polynomials exist and are unique. A fact that is proved to be true, after it is shown in Theorem 2.1 in §2.1 that the vector of weights $\left[\mathscr{W}(x;a,b+1;c+1,d),\mathscr{W}(x;a,b;c,d)\right]$ forms a Nikishin system. The characterisation of the polynomials is guided by the algebraic and differential properties of the weights. As such, the technical results described in Theorem 2.3 (regarding the vector of weights) and Theorem 2.4 (for the differential properties of the polynomials of type II and the type I functions) form the basis of an explicit analysis carried on in §2.3 for the type I and in the next Section 3 for the type II polynomials.

2.1 Nikishin system

We show that the weight vector $\left[\mathscr{W}(x;a,b+1;c+1,d),\mathscr{W}(x;a,b;c,d)\right]$ forms a Nikishin system, which guarantees that both type I and II multiple orthogonal polynomials with respect to these weight functions exist and are unique for every multi-index $(n_0,n_1)\in\mathbb{N}^2$ as well as it implies that the type I multiple orthogonal polynomials $A_{(n_0,n_1)}$ and $B_{(n_0,n_1)}$ have degree exactly n_0-1 and n_1-1, respectively, and the type II multiple orthogonal polynomial $P_{(n_0,n_1)}$ has n_0+n_1 positive real simple zeros that satisfy the usual interlacing property: there is always a zero of $P_{(n_0,n_1)}$ between two consecutive zeros of $P_{(n_0+1,n_1)}$ or $P_{(n_0,n_1+1)}$.

To prove this result, we use the connection between continued fractions and Stieltjes transforms to guarantee the existence of an integral representation of the type in (1.9) for the ratio of the weight functions involved. For simplicity, we follow the notation for continued fractions used in [7]:

$$\underset{n=0}{\overset{\infty}{\mathbf{K}}}\left(\frac{a_n}{b_n}\right) := \cfrac{a_0}{b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cdots}}}. \tag{2.1}$$

Particularly relevant to this work are the so-called *Stieltjes continued fractions* or, simply, *S-fractions*, due to their connection with Stieltjes transforms which was firstly investigated in [32]. The continued fraction playing a role here is an example of a modified S-fraction which is obtained if, for some constants α_k, $k\in\mathbb{N}$, we set in (2.1), $a_0=\alpha_0$ and, for any $n\in\mathbb{N}$, $b_n=1$

and $a_{n+1} = \alpha_{n+1} z$, to obtain

$$F(z) = \cfrac{\alpha_0}{1+\cfrac{\alpha_1 z}{1+\cfrac{\alpha_2 z}{1+\cdots}}}. \tag{2.2}$$

The main result of this subsection is the following.

Theorem 2.1. *Let $\mathscr{W}(x;a,b;c,d)$ be given* (1.1) *under the assumptions* (1.2). *The ratio*

$$\frac{\mathscr{W}(x;a,b;c,d)}{\mathscr{W}(x;a,b+1;c+1,d)}$$

can be represented via the continued fraction (2.2) *with $z = x-1$ and $\alpha_n = (1-g_{n-1})g_n$, where $g_0 = 0$, $g_{2k+1} = \dfrac{c-b+k}{\delta+2k}$ and $g_{2k+2} = \dfrac{d-b+k}{\delta+2k+1}$ for $n \geq 1$ and $k \geq 0$. Moreover, there exist probability density functions σ in $(0,1)$ and θ in $(1,+\infty)$ such that*

$$\frac{\mathscr{W}(x;a,b;c,d)}{\mathscr{W}(x;a,b+1;c+1,d)} = \frac{c}{b}\int_0^1 \frac{\mathrm{d}\sigma(t)}{1+t(x-1)} = \frac{c}{b}\int_{-\infty}^{-1} \frac{\mathrm{d}\theta(-u)}{x-1-u} = \frac{c}{b}\int_{-\infty}^{0} \frac{\mathrm{d}\theta(1-s)}{x-s}. \tag{2.3}$$

Therefore, the vector of weight functions $\left[\mathscr{W}(x;a,b+1;c+1,d), \mathscr{W}(x;a,b;c,d)\right]$ forms a Nikishin system on the interval $(0,1)$.

Proof. Recalling (1.1),

$$\frac{\mathscr{W}(x;a,b;c,d)}{\mathscr{W}(x;a,b+1;c+1,d)} = \frac{c}{b}\,\frac{{}_2F_1(c-b,d-b;\delta;1-x)}{{}_2F_1(c-b,d-b-1;\delta;1-x)}, \quad \text{with } \delta = c+d-a-b. \tag{2.4}$$

Therefore, the ratio of weight functions above admits a representation similar to Gauss' continued fraction. More precisely, based on [29, Eq. (14.29)], the ratio of weights in (2.4) can be represented by a continued fraction of the type in (2.2), with $z = x-1$, and coefficients

$$\alpha_0 = \frac{c}{b};\ \alpha_1 = \frac{c-b}{\delta};\ \alpha_{2k+1} = \frac{(c-b+k)(c-a+k)}{(\delta+2k-1)(\delta+2k)}, k \geq 1;\ \alpha_{2k+2} = \frac{(d-b+k)(d-a+k)}{(\delta+2k)(\delta+2k+1)}, k \in \mathbb{N}.$$

Moreover, the coefficients α_n, $n \geq 1$, can be rewritten as $\alpha_n = (1-g_{n-1})g_n$, with $g_0 = 0$ and, for each $k \in \mathbb{N}$, $g_{2k+1} = \dfrac{c-b+k}{\delta+2k}$ and $g_{2k+2} = \dfrac{d-b+k}{\delta+2k+1}$ (see [20, Eqs. (2.7)-(2.8)]). Note that $0 < g_n < 1$, for all $n \geq 1$, and, as a result, the continued fraction described above is of the type in [37, Eq. 27.8]. Therefore, the first integral representation in (2.3) can be derived directly from [37, Eq. 67.5] and the second one can be deduced combining [37, Ths. 67.1 & 27.5], while the last equality in (2.3) is obtained via the change of variable $s = u+1$. $\square$

Under the additional assumption $b > a-1$ and using a recent result from Dyachenko and Karp in [11], the generating measure σ in (2.3) admits the following integral representation

$$\frac{\mathscr{W}(x;a,b;c,d)}{\mathscr{W}(x;a,b+1;c+1,d)} = \int_0^1 \frac{\lambda t^{c+d-2b-2}(1-t)^{b-a}\mathrm{d}t}{(1+t(x-1))\left|{}_2F_1(c-b,d-b-1;\delta;t^{-1})\right|^2} + K(c,d) \tag{2.5a}$$

with

$$\lambda = \frac{c\,(\Gamma(\delta))^2}{b\Gamma(c-b)\Gamma(d-b)\Gamma(d-a)\Gamma(c-a+1)} \quad\text{and}\quad K(c,d) = \begin{cases} 0, & \text{if } d \le c+1, \\ \frac{d-c-1}{d-1}, & \text{if } d \ge c+1. \end{cases}$$

The change of variable $t = \frac{1}{1-s}$ in (2.5a) gives

$$\frac{\mathscr{W}(x;a,b;c,d)}{\mathscr{W}(x;a,b+1;c+1,d)} = \int_{-\infty}^{0} \frac{\lambda\,(-s)^{b-a}(1-s)^{1-\delta}}{(x-s)\left|{}_2F_1\,(c-b,d-b-1;\delta;1-s)\right|^2}\mathrm{d}s + K(c,d). \tag{2.5b}$$

Hence, if $b > a-1$, the measures in the first and last integral representations in (2.3) can be explicitly represented by (2.5a) and (2.5b), respectively.

2.2 Differential properties

We start by describing the weight function $\mathscr{W}(x;a,b;c,d)$ in (1.1) as a solution to a second-order ordinary differential equation, to then describe the vector of weight functions

$$\overline{\mathscr{W}}(x;a,b;c,d) := \begin{bmatrix} \mathscr{W}(x;a,b;c,d) \\ \mathscr{W}(x;a,b+1;c+1,d) \end{bmatrix}, \tag{2.6}$$

as a solution to a system of first order differential equations in Theorem 2.3. A result that is crucial to obtain, in Theorem 2.4, differential properties on the system of multiple orthogonal polynomials of type II and the functions of type I, revealing their Hahn-classical property.

Proposition 2.2. *For* $a,b,c,d \in \mathbb{R}^+$ *such that* $\min\{c,d\} > \max\{a,b\}$, *let* $\mathscr{W}(x) := \mathscr{W}(x;a,b;c,d)$ *be the weight function defined by* (1.1). *Then*

$$(1-x)x^2\mathscr{W}''(x) + ((c+d-5)x-(a+b-3))\,x\,\mathscr{W}'(x) + ((a-1)(b-1)-(c-2)(d-2)x)\,\mathscr{W}(x) = 0. \tag{2.7}$$

Proof. We set $\delta = c+d-a-b$, $\lambda = \dfrac{\Gamma(c)\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(\delta)}$ and $F(z) = {}_2F_1\,(c-b,d-b;\delta;z)$ so that

$$\mathscr{W}(x) = \lambda x^{a-1}(1-x)^{\delta-1}F(1-x).$$

Differentiating this expression twice, we get

$$\mathscr{W}^{(j)}(x) = \lambda x^{a-1-j}(1-x)^{\delta-1-j}F_j(x), \quad j=0,1,2, \tag{2.8}$$

with $F_0(x) = F(1-x)$,

$$F_1(x) = \big((2+b-c-d)x+(a-1)\big)F(1-x) - x(1-x)F'(1-x)$$

and

$$\begin{aligned} F_2(x) = {}& x^2(1-x)^2F''(1-x) + 2x(1-x)\Big((c+d-b-2)x+(1-a)\Big)F'(1-x) \\ & + \Big((c+d-b-2)(c+d-b-3)x^2 - 2(a-1)(c+d-b-3)x + (a-1)(a-2)\Big)F(1-x). \end{aligned}$$

Recall (1.4) to derive that $F(1-x) = {}_2F_1(c-b, d-b; \delta; 1-x)$ satisfies

$$x(1-x)F''(1-x) = (c-b)(d-b)F(1-x) - ((c+d-2b+1)x + (b-a-1))F'(1-x),$$

which can be used to rewrite $F_2(x)$ as

$$F_2(x) = ((5-c-d)x + (a+b-3))\,F_1(x) + (1-x)\,((c-2)(d-2)x - (a-1)(b-1))\,F_0(x).$$

Combining the latter relation with (2.8), we derive (2.7). $\square$

Based on the second order differential equation (2.7) we deduce a system of first order differential equations for which the vector (2.6) is a solution.

Theorem 2.3. *Let* $\overline{\mathscr{W}}(x;a,b;c,d)$ *as defined in* (2.6) *subject to* (1.2). *Then, the following identities hold*

$$x\Phi(x)\overline{\mathscr{W}}(x;a,b;c,d) = \overline{\mathscr{W}}(x;a+1,b+1;d+1,c+2) \tag{2.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x\Phi(x)\overline{\mathscr{W}}(x;a,b;c,d)\Big) + \Psi(x)\overline{\mathscr{W}}(x;a,b;c,d) = 0, \tag{2.10}$$

where

$$\Phi(x) := \Phi(x;a,b;c,d) = \begin{bmatrix} \dfrac{c(c+1)d}{ab(c-b)} & -\dfrac{(c+1)d}{a(c-b)} \\ -\dfrac{c(c+1)d(d+1)}{ab(b+1)(d-a)}x & \dfrac{(c+1)d(d+1)}{a(b+1)(d-a)} \end{bmatrix} \tag{2.11}$$

and

$$\Psi(x) := \Psi(x;a,b;c,d) = \begin{bmatrix} -\dfrac{c(c+1)d}{a(c-b)} & \dfrac{c(c+1)d}{a(c-b)} \\ \dfrac{c(c+1)d^2(d+1)}{ab(b+1)(d-a)}x & -\dfrac{(c+1)d(d+1)}{(b+1)(d-a)} \end{bmatrix}.$$

Proof of Theorem 2.3. In order to prove (2.9), we need to check that

$$\begin{bmatrix} \mathscr{V}_0(x) \\ \mathscr{V}_1(x) \end{bmatrix} := x\Phi(x) \begin{bmatrix} \mathscr{W}(x;a,b;c,d) \\ \mathscr{W}(x;a,b+1;c+1,d) \end{bmatrix} = \begin{bmatrix} \mathscr{W}(x;a+1,b+1;d+1,c+2) \\ \mathscr{W}(x;a+1,b+2;d+2,c+2) \end{bmatrix}.$$

Firstly,

$$\begin{aligned} \mathscr{V}_0(x) &= x\frac{(c+1)d}{a(c-b)}\left(\frac{c}{b}\mathscr{W}(x;a,b;c,d) - \mathscr{W}(x;a,b+1;c+1,d)\right) \\ &= \frac{\Gamma(c+2)\Gamma(d+1)x^a(1-x)^{\delta-1}}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta)(c-b)}\left({}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right) - {}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta \end{matrix}; 1-x\right)\right). \end{aligned}$$

Based on [8, Eq. 15.5.15 & Eq. 15.5.16], we obtain, respectively,

$$(c-b)\,{}_2F_1\left(\begin{matrix}c-b+1,d-b\\ \delta+1\end{matrix};1-x\right)=\delta\,{}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta\end{matrix};1-x\right)-(d-a)\,{}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta+1\end{matrix};1-x\right)$$

and

$$\frac{d-a}{\delta}(1-x){}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta+1\end{matrix};1-x\right)={}_2F_1\left(\begin{matrix}c-b,d-b-1\\ \delta\end{matrix};1-x\right)-x\,{}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta\end{matrix};1-x\right).$$

Therefore, we can derive that

$$\frac{c-b}{\delta}(1-x)\,{}_2F_1\left(\begin{matrix}c-b+1,d-b\\ \delta+1\end{matrix};1-x\right)={}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta\end{matrix};1-x\right)-{}_2F_1\left(\begin{matrix}c-b,d-b-1\\ \delta\end{matrix};1-x\right)$$

and, as a result,

$$\mathscr{V}_0(x)=\frac{\Gamma(c+2)\Gamma(d+1)x^a(1-x)^\delta}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\,{}_2F_1\left(\begin{matrix}c-b+1,d-b\\ \delta+1\end{matrix};1-x\right)=\mathscr{W}(x;a+1,b+1;d+1,c+2). \tag{2.12}$$

Similarly, we have

$$\begin{aligned}\mathscr{V}_1(x)&=x\,\frac{(c+1)d(d+1)}{a(b+1)(d-a)}\Big(\mathscr{W}(x;a,b+1;c+1,d)-\frac{c}{b}x\mathscr{W}(x;a,b;c,d)\Big)\\&=\frac{\Gamma(c+2)\Gamma(d+2)x^a(1-x)^{\delta-1}}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta)(d-a)}\left({}_2F_1\left(\begin{matrix}c-b,d-b-1\\ \delta\end{matrix};1-x\right)-x\,{}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta\end{matrix};1-x\right)\right)\end{aligned}$$

and

$${}_2F_1\left(\begin{matrix}c-b,d-b-1\\ \delta\end{matrix};1-x\right)-x\,{}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta\end{matrix};1-x\right)=\frac{d-a}{\delta}(1-x){}_2F_1\left(\begin{matrix}c-b,d-b\\ \delta+1\end{matrix};1-x\right),$$

hence we get

$$\mathscr{V}_1(x)=\frac{\Gamma(c+2)\Gamma(d+2)x^a(1-x)^\delta}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta+1)}\,{}_2F_1\left(\begin{matrix}d-b,c-b\\ \delta+1\end{matrix};1-x\right)=\mathscr{W}(x;a+1,b+2;d+2,c+2).$$

In order to prove (2.10), we need to check that

$$\begin{bmatrix}\mathscr{V}_0'(x)\\ \mathscr{V}_1'(x)\end{bmatrix}=\begin{bmatrix}\dfrac{c(c+1)d}{a(c-b)}\Big(\mathscr{W}(x;a,b;c,d)-\mathscr{W}(x;a,b+1;c+1,d)\Big)\\ \dfrac{(c+1)d(d+1)}{(b+1)(d-a)}\Big(\mathscr{W}(x;a,b+1;c+1,d)-\dfrac{cd}{ab}x\mathscr{W}(x;a,b;c,d)\Big)\end{bmatrix}.$$

Recalling (2.12),

$$\begin{aligned}\mathscr{V}_0'(x)&=\mathscr{W}'(x;a+1,b+1;d+1,c+2)\\&=\frac{\Gamma(c+2)\Gamma(d+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a(1-x)^\delta\,{}_2F_1\left(\begin{matrix}c-b+1,d-b\\ \delta+1\end{matrix};1-x\right)\right),\end{aligned}$$

which is equivalent to

$$\mathscr{V}_0'(x) = \frac{\Gamma(c+2)\Gamma(d+1)x^{a-1}(1-x)^{\delta-1}}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\, G_0(x),$$

with

$$\begin{aligned} G_0(x) =& (a-(c+d-b)x)_2F_1\left(\begin{matrix} c-b+1, d-b \\ \delta+1 \end{matrix}; 1-x\right) \\ &- \frac{(c-b+1)(d-b)}{\delta+1} x(1-x)_2F_1\left(\begin{matrix} c-b+2, d-b+1 \\ \delta+2 \end{matrix}; 1-x\right). \end{aligned}$$

Using [8, Eq. 15.5.19],

$$\begin{aligned} &\frac{(c-b+1)(d-b)}{\delta+1} x(1-x)\,{}_2F_1\left(\begin{matrix} c-b+2, d-b+1 \\ \delta+2 \end{matrix}; 1-x\right) \\ &= \delta\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta \end{matrix}; 1-x\right) - \big((c+d-2b)x+(b-a)\big)\,{}_2F_1\left(\begin{matrix} c-b+1, d-b \\ \delta+1 \end{matrix}; 1-x\right). \end{aligned} \tag{2.13}$$

so that

$$\begin{aligned} G_0(x) &= -\delta\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta \end{matrix}; 1-x\right) + b(1-x)\,{}_2F_1\left(\begin{matrix} c-b+1, d-b \\ \delta+1 \end{matrix}; 1-x\right) \\ &= \frac{\delta}{c-b}\left(b\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right) - c\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta+1 \end{matrix}; 1-x\right)\right). \end{aligned}$$

Therefore,

$$\mathscr{V}_0'(x) = \frac{\Gamma(c+2)\Gamma(d+1)x^{a-1}(1-x)^{\delta-1}}{\Gamma(a+1)\Gamma(b)\Gamma(\delta)(c-b)}\left({}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right) - \frac{c}{b}\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta+1 \end{matrix}; 1-x\right)\right),$$

which implies that

$$\mathscr{V}_0'(x) = \frac{c(c+1)d}{a(c-b)}\Big(\mathscr{W}(x;a,b;c,d) - \mathscr{W}(x;a,b+1;c+1,d)\Big).$$

Similarly,

$$\mathscr{V}_1'(x) = \mathscr{W}'(x;a+1,b+2;d+2,c+2) = \frac{\Gamma(c+2)\Gamma(d+2)}{\Gamma(a+1)\Gamma(b+2)\Gamma(\delta+1)}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^a(1-x)^\delta\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta+1 \end{matrix}; 1-x\right)\right),$$

which is equivalent to

$$\mathscr{V}_1'(x) = \frac{\Gamma(c+2)\Gamma(d+1)x^{a-1}(1-x)^{\delta-1}}{\Gamma(a+1)\Gamma(b+1)\Gamma(\delta+1)}\, G_1(x),$$

with

$$G_1(x) = (a-(c+d-b)x)\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta+1 \end{matrix}; 1-x\right) - \frac{(c-b)(d-b)}{\delta+1} x(1-x)\,{}_2F_1\left(\begin{matrix} c-b+1, d-b+1 \\ \delta+2 \end{matrix}; 1-x\right).$$

Using (2.13) with a shift $c \to c-1$ and $a \to a-1$, we derive that

$$\begin{aligned} G_1(x) &= -\delta\,{}_2F_1\left(\begin{matrix} c-b-1, d-b-1 \\ \delta \end{matrix}; 1-x\right) + (b+1)(1-x)\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta+1 \end{matrix}; 1-x\right) \\ &= \frac{\delta}{d-a}\left(a\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta \end{matrix}; 1-x\right) - dx\,{}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right)\right). \end{aligned}$$

Therefore,

$$\mathscr{V}_1'(x) = \frac{\Gamma(c+2)\Gamma(d+2)x^{a-1}(1-x)^{\delta-1}}{\Gamma(a)\Gamma(b+2)\Gamma(\delta)(d-a)}\left({}_2F_1\left(\begin{matrix} c-b, d-b \\ \delta \end{matrix}; 1-x\right) - \frac{d}{a}x\,{}_2F_1\left(\begin{matrix} c-b, d-b-1 \\ \delta+1 \end{matrix}; 1-x\right)\right),$$

which implies that

$$\mathscr{V}_1'(x) = \frac{(c+1)d(d+1)}{(b+1)(d-a)}\Big(\mathscr{W}(x;a,b+1;c+1,d) - \frac{cd}{ab}x\mathscr{W}(x;a,b;c,d)\Big).$$

□

The latter result guarantees the Hahn-classical property of the multiple orthogonal polynomials investigated here. In fact, combining it with [22, Prop. 2.6 & 2.7] we show that the differentiation with respect to the variable of both type I and type II polynomials on the step line gives a shift on the parameters as well as on the index, as detailed in the next result.

Theorem 2.4. *For $a,b,c,d \in \mathbb{R}^+$ such that $\min\{c,d\} > \max\{a,b\}$, let $P_n(x;a,b;c,d)$ and $Q_n(x;a,b;c,d)$, with $n \in \mathbb{N}$, be, respectively, the type II multiple orthogonal polynomial and the type I function for the index of length n on the step line with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}(x;a,b;c,d)\right) = (n+1)P_n(x;a+1,b+1;d+1,c+2) \tag{2.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(Q_n(x;a+1,b+1;d+1,c+2)\big) = -n\,Q_{n+1}(x;a,b;c,d). \tag{2.15}$$

Proof. Let $\Phi(x)$ be defined by (2.11) and denote $\overline{\mathscr{W}}(x;a,b;c,d)$ by $\overline{\mathscr{W}}(x)$.

Since $\overline{\mathscr{W}}(x)$ satisfies the equation (2.10) and on account of the degrees of the polynomial entries in the matrices $\Phi(x)$ and $\Psi(x)$, then Proposition 2.6 in [22] ensures the 2-orthogonality of the polynomial sequence $\left\{(n+1)^{-1}P'_{n+1}(x;a,b;c,d)\right\}_{n\in\mathbb{N}}$ with respect to the vector of weights $x\Phi(x)\overline{\mathscr{W}}(x)$ whilst Proposition 2.7 in [22] implies that, if $R_n(x)$ is the type I function for the index of length n on the step line with respect to $x\Phi(x)\overline{\mathscr{W}}(x)$, then $-n^{-1}R'_n(x)$ is the type I function for the index of length $n+1$ on the step line with respect to the vector of weights $\overline{\mathscr{W}}(x)$.

Therefore, by virtue of (2.9), we conclude that both (2.14) and (2.15) hold. □

2.3 Type I multiple orthogonal polynomials

Due to the differential relation (2.15), the type I functions on the step line can be generated by concatenated differentiation of the weight function or, in other words, via a Rodrigues-type formula as it follows.

Theorem 2.5. *For $a,b,c,d\in\mathbb{R}^+$ such that $\min\{c,d\}>\max\{a,b\}$ and $n\in\mathbb{N}$, let $Q_{n+1}(x;a,b;c,d)$ be the type I function for the index of length $n+1$ on the step line with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. Then*

$$Q_{n+1}(x;a,b;c,d)=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathscr{W}\left(x;a+n,b+n;c+\left\lfloor\frac{n+1}{2}\right\rfloor+n,d+\left\lfloor\frac{n}{2}\right\rfloor+n\right)\right) \tag{2.16}$$

Proof. We proceed by induction on $n\in\mathbb{N}$.

For $n=0$, (2.16) reads as $Q_1(x;a,b;c,d)=\mathscr{W}(x;a,b;c,d)$, which trivially holds.

Using (2.15) and then evoking the assumption that (2.16) holds for a fixed $n\in\mathbb{N}$, we obtain

$$\begin{aligned}
Q_{n+2}(x;a,b;c,d)&=-\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_{n+1}(x;a+1,b+1;d+1,c+2)\right)\\
&=\frac{(-1)^{n+1}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathscr{W}\left(x;a+1+n,b+1+n;d+1+\left\lfloor\frac{n+1}{2}\right\rfloor+n,c+2+\left\lfloor\frac{n}{2}\right\rfloor+n\right)\right)\\
&=\frac{(-1)^{n+1}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathscr{W}\left(x;a+n+1,b+n+1;c+\left\lfloor\frac{n+2}{2}\right\rfloor+n+1,d+\left\lfloor\frac{n+1}{2}\right\rfloor+n+1\right)\right).
\end{aligned}$$

If we equate the first and latter members, we obtain (2.16) for $n+1$ and the result follows by induction. □

We continue with further properties reagrding type I polynomials (A_n,B_n) in (1.12) associated with the type I function $Q_n(x)$ in (2.16). In fact, Theorem 2.3 combined with the proof of [22, Prop. 2.7] leads to the following differential-difference relation between the pair of polynomials

$$(A_{n+1}(x),B_{n+1}(x)):=(A_{n+1}(x;a,b;c,d),B_{n+1}(x;a,b;c,d))$$

and

$$(C_n(x),D_n(x)):=(A_n(x;a+1,b+1;d+2,c+1),B_n(x;a+1,b+1;d+2,c+1)),$$

the polynomials on the step line satisfying multiple orthogonality relations of type I with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ and to $\overline{\mathscr{W}}(x;a+1,b+1;d+1,c+2)$:

$$A_{n+1}(x) = \frac{c(c+1)d}{nab}\left(-\frac{1}{c-b}\left(bC_n(x)+xC_n'(x)\right)+\frac{(d+1)x}{(b+1)(d-a)}\left(dD_n(x)+xD_n'(x)\right)\right) \tag{2.17a}$$

and

$$B_{n+1}(x) = \frac{(c+1)d}{na}\left(\frac{1}{c-b}\left(cC_n(x)+\frac{x}{b}C_n'(x)\right)-\frac{d+1}{(b+1)(d-a)}\left(aD_n(x)+cxD_n'(x)\right)\right) \tag{2.17b}$$

which hold for all $n \geq 1$.

Formulas (2.17a)-(2.17b) can be used to recursively generate type I polynomials with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. The latter can be written as follows

$$\frac{na}{c(c+1)d}\begin{bmatrix} b & 0 \\ 0 & c\end{bmatrix}\begin{bmatrix}A_{n+1}(x)\\B_{n+1}(x)\end{bmatrix} = \begin{bmatrix} b & -dx \\ -c & a\end{bmatrix} M \begin{bmatrix}C_n(x)\\D_n(x)\end{bmatrix} + x\begin{bmatrix} 1 & -x \\ -1/b & c\end{bmatrix} M \begin{bmatrix}C_n'(x)\\D_n'(x)\end{bmatrix}$$

where

$$M = \begin{bmatrix} -\frac{1}{c-b} & 0 \\ 0 & -\frac{(d+1)}{(b+1)(d-a)}\end{bmatrix}$$

or, equivalently,

$$\frac{nab}{(c+1)d}\begin{bmatrix}A_{n+1}(x)\\B_{n+1}(x)\end{bmatrix} = \begin{bmatrix} bc & -cdx \\ -bc & ba\end{bmatrix} M \begin{bmatrix}C_n(x)\\D_n(x)\end{bmatrix} + x\begin{bmatrix} c & -cx \\ -1 & bc\end{bmatrix} M \begin{bmatrix}C_n'(x)\\D_n'(x)\end{bmatrix}.$$

Thus, the type I polynomials can be generated by the rising operator

$$\mathscr{O}(a,b;c,d) = \frac{(c+1)d}{ab}\left(\begin{bmatrix} -\frac{bc}{c-b} & \frac{cdx}{c-b} \\ \frac{bc}{c-b} & -\frac{ab}{c-b}\end{bmatrix} + \begin{bmatrix} -\frac{c(d+1)}{(b+1)(d-a)}c & \frac{c(d+1)x}{(b+1)(d-a)} \\ \frac{(d+1)}{(b+1)(d-a)} & \frac{-bc(d+1)}{(b+1)(d-a)}\end{bmatrix} x\partial_x\right),$$

since we have

$$\begin{bmatrix}A_{n+1}(x;a,b;c,d)\\B_{n+1}(x;a,b;c,d)\end{bmatrix} = \frac{1}{n}\mathscr{O}\begin{bmatrix}A_n(x;a+1,b+1;d+2,c+1)\\B_n(x;a+1,b+1;d+2,c+1)\end{bmatrix}.$$

As a result, we obtain the matrix Rodrigues-type formula for type I polynomials

$$\begin{bmatrix}A_{n+1}(x;a,b;c,d)\\B_{n+1}(x;a,b;c,d)\end{bmatrix} = \frac{1}{n!}\left(\prod_{k=0}^{n-1}\mathscr{O}(a_k,b_k;c_k,d_k)\right)\begin{bmatrix}1\\0\end{bmatrix}, \quad n\in\mathbb{N},$$

where, as usual, the product of differential operators is understood as the composition, and the parameters involved are as follows $a_k = a+k$, $b_k = b+k$, $c_{2j} = c+3j$ and $c_{2j+1} = d+3j+2$, $d_{2j} = d+3j$ and $d_{2j+1} = c+3j+1$.

3 Characterisation of the type II polynomials

The type II multiple orthogonal polynomials on the step line are described in detail here. This characterisation includes: their explicit expression in Theorem 3.1, a third order linear differential equation with polynomial coefficients in Theorem 3.3, a third order recurrence in Theorem 3.4. The asymptotic properties of these polynomials are analysed in §3.4, which coincide with those observed for Jacobi-Piñeiro polynomials. We give the ratio asymptotics of two consecutive polynomials, the limiting zero distribution as well as a Mehler-Heine formula for the behaviour near the endpoint 0. At last, we analyse particular realisations of these polynomials. Namely, the connection to Jacobi-type 2-orthogonal polynomials, in §3.5, and the connection to the cubic components of Hahn-classical threefold symmetric polynomials in §3.6, where we also describe confluence relations to another (Hahn-classical) polynomials that are 2-orthogonal with respect to weights involving the confluent hypergeometric functions of the second kind.

3.1 Explicit expression

Based on the moments expression (1.5), we deduce an explicit representation for the type II multiple orthogonal polynomials on the step line with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ as generalised hypergeometric series.

Theorem 3.1. *For $a,b,c,d\in\mathbb{R}^+$ such that $\min\{c,d\}>\max\{a,b\}$, let $\{P_n(x):=P_n(x;a,b;c,d)\}_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. Then*

$$P_n(x)=\frac{(-1)^n\,(a)_n\,(b)_n}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}\,{}_3F_2\left(\begin{matrix}-n,c+\left\lfloor\frac{n}{2}\right\rfloor,d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix};x\right). \tag{3.1}$$

By definition of the generalised hypergeometric series, the latter formula is equivalent to

$$P_n(x)=\sum_{j=0}^{n}\tau_{n,j}x^{n-j},\quad\text{with}\quad \tau_{n,j}=\binom{n}{j}\frac{(-1)^j\,(a+n-j)_j\,(b+n-j)_j}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor+n-j\right)_j\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor+n-j\right)_j}. \tag{3.2}$$

To prove Theorem 3.1 we need to show that the sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ defined by (3.1) satisfies the 2-orthogonality conditions with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$, that is, we need to check that, for each $j\in\{0,1\}$,

$$\int_0^1 x^kP_n(x)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x=\begin{cases}0, & \text{if } n\geq 2k+j+1,\\ N_n(a,b;c,d)\neq 0, & \text{if } n=2k+j.\end{cases} \tag{3.3}$$

Actually, as we are dealing with a Nikishin system, the existence of a 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ is guaranteed. By virtue of the generalised hypergeometric differential equation (1.14), it is rather straightforward to show that the polynomials given by (3.1) satisfy the differential property (2.14) stated in Theorem 2.4. A property that a 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ must satisfy. Therefore, it would be sufficient to check the orthogonality conditions (3.3) when $k=0$ to then prove the result by induction on $n\in\mathbb{N}$ (the degree of the polynomials). However, we opt for checking that the polynomials $P_n(x)$ in (3.1) satisfy all the orthogonality conditions (3.3). On the one hand, this process enables us to show directly that the polynomials in (3.1) are indeed 2-orthogonal with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ without arguing with the Nikishin property. On the other hand, it provides a method to derive explicit expressions for the nonzero coefficients $N_n(a,b;c,d)$ in (3.3) which are used in Subsection 3.3 to obtain explicit expressions for the nonzero γ-coefficients in the third order recurrence relation (1.10) satisfied by these polynomials.

To compute the integrals in (3.3), we use the following auxiliary lemma, which can be found in [22, Lemma 3.2]. As mentioned therein, (3.5) was deduced in [26] and (3.4) can be obtained by taking the limit $\beta\to+\infty$ in (3.5).

Lemma 3.2. *Let n, p, and $m_1,\cdots,m_p$ be positive integers such that $m:=\sum_{i=1}^{p}m_i\leq n$ and $\beta,f_1,\cdots,f_p$ be complex numbers with positive real part. Then*

$$
{}_{p+1}F_p\left(\begin{matrix}-n,f_1+m_1,\cdots,f_p+m_p\\ f_1,\cdots,f_p\end{matrix};1\right)=\begin{cases}0 & \text{if } m<n,\\ \dfrac{(-1)^n n!}{(f_1)_{m_1}\cdots(f_p)_{m_p}} & \text{if } m=n.\end{cases}\tag{3.4}
$$

and

$$
{}_{p+2}F_{p+1}\left(\begin{matrix}-n,\beta,f_1+m_1,\cdots,f_p+m_p\\ \beta+1,f_1,\cdots,f_p\end{matrix};1\right)=\frac{n!\,(f_1-\beta)_{m_1}\cdots(f_p-\beta)_{m_p}}{(\beta+1)_n\,(f_1)_{m_1}\cdots(f_p)_{m_p}}.\tag{3.5}
$$

Proof of Theorem 3.1. Recalling the explicit expression for $P_n(x)$ given by (3.1), and the definition of the generalised hypergeometric series (1.13), we get, for both $j\in\{0,1\}$ and any $k,n\in\mathbb{N}$,

$$
\begin{aligned}
&\int_0^1 x^k P_n(x)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x\\
&=\frac{(-1)^n(a)_n(b)_n}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}\sum_{i=0}^{n}\frac{(-n)_i\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_i\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_i}{i!\,(a)_i\,(b)_i}\int_0^1 x^{k+i}\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x.
\end{aligned}
$$

Moreover, using the formula for the moments of the hypergeometric weight (1.5),

$$
\begin{aligned}
&\int_0^1 x^k P_n(x)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x\\
&=\frac{(-1)^n(a)_n(b)_n}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}\sum_{i=0}^{n}\frac{(-n)_i\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_i\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_i}{i!\,(a)_i\,(b)_i}\frac{(a)_{k+i}\,(b+j)_{k+i}}{(c+j)_{k+i}\,(d)_{k+i}}.
\end{aligned}
$$

Therefore, recalling again the definition of the generalised hypergeometric series (1.13), we have

$$
\begin{aligned}
&\int_0^1 x^k P_n(x)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x\\
&=\frac{(-1)^n(a)_n(b)_n(a)_k(b+j)_k}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n(c+j)_k(d)_k}\,{}_5F_4\left(\begin{matrix}-n,a+k,b+k+j,c+\left\lfloor\frac{n}{2}\right\rfloor,d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b,c+k+j,d+k\end{matrix};1\right).
\end{aligned}\tag{3.6}
$$

For any $n \in \mathbb{N}$, $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = n-1$ and if $n \geq 2k+j+1$, $j \in \{0,1\}$, then $\left\lfloor \frac{n}{2} \right\rfloor \geq k+j$ and $\left\lfloor \frac{n-1}{2} \right\rfloor \geq k$. Therefore, using (3.4) in Lemma 3.2, we deduce that, for both $j \in \{0,1\}$,

$$ {}_5F_4\left(\begin{matrix} -n, a+k, b+k+j, c+\left\lfloor \frac{n}{2} \right\rfloor, d+\left\lfloor \frac{n-1}{2} \right\rfloor \\ a, b, c+k+j, d+k \end{matrix} ; 1\right) = 0, \quad \text{for any} \quad n \geq 2k+j+1, $$

and, as a result,

$$ \int_0^1 x^k P_n(x) \mathscr{W}(x; a, b+j; c+j, d) \mathrm{d}x = 0, \quad \text{for any} \quad n \geq 2k+j+1. $$

Taking $j=0$ and $n=2k$ in (3.6),

$$ \int_0^1 x^k P_{2k}(x) \mathscr{W}(x; a, b; c, d) \mathrm{d}x = \frac{(a)_{2k}(b)_{2k}(a)_k(b)_k}{(c)_{3k}(d)_{3k-1}(d+k-1)} \, {}_4F_3\left(\begin{matrix} -2k, a+k, b+k, d+k-1 \\ a, b, d+k \end{matrix} ; 1\right). $$

and, using (3.5), we get

$$ {}_4F_3\left(\begin{matrix} -2k, a+k, b+k, d+k-1 \\ a, b, d+k \end{matrix} ; 1\right) = \frac{(2k)!\,(a-d+1-k)_k(b-d+1-k)_k}{(d+k)_{2k}(a)_k(b)_k} = \frac{(2k)!\,(d-a)_k(d-b)_k}{(a)_k(b)_k(d+k)_{2k}}. $$

Therefore,

$$ \int_0^1 x^k P_{2k}(x) \mathscr{W}(x; a, b; c, d) \mathrm{d}x = \frac{(2k)!\,(a)_{2k}(b)_{2k}(d-a)_k(d-b)_k}{(c)_{3k}(d)_{3k}(d+k-1)_{2k}} > 0, \tag{3.7} $$

and (3.3) holds for any $k, n \in \mathbb{N}$ when $j=0$.

Similarly, taking $j=1$ and $n=2k+1$ in (3.6),

$$ \int_0^1 x^k P_{2k+1}(x) \mathscr{W}(x; a, b+1; c+1, d) \mathrm{d}x = -\frac{(a)_{2k+1}(b)_{2k+1}(a)_k(b+1)_k}{(c+1)_{3k}(c+k)(d)_{3k+1}} \, {}_4F_3\left(\begin{matrix} -2k-1, a+k, b+k+1, c+k \\ a, b, c+k+1 \end{matrix} ; 1\right). $$

and, using again (3.5), we get

$$ {}_4F_3\left(\begin{matrix} -2k-1, a+k, b+k+1, c+k \\ a, b, c+k+1 \end{matrix} ; 1\right) = -\frac{(2k+1)!\,(c-a+1)_k(c-b)_{k+1}}{(a)_k(b)_{k+1}(c+k+1)_{2k+1}}, $$

so that

$$ \int_0^1 x^k P_{2k+1}(x) \mathscr{W}(x; a, b+1; c+1, d) \mathrm{d}x = \frac{(2k+1)!\,(a)_{2k+1}(b+1)_{2k}(c-a+1)_k(c-b)_{k+1}}{(c+1)_{3k+1}(c+k)_{2k+1}(d)_{3k+1}} > 0. \tag{3.8} $$

ensuring that (3.3) also holds for any $k, n \in \mathbb{N}$ when $j=1$. □

3.2 Differential equation

The type II multiple orthogonal polynomials of hypergeometric type described in (3.1) are solutions to the following third order differential equation.

Theorem 3.3. *For $a,b,c,d \in \mathbb{R}^+$ such that $\min\{c,d\} > \max\{a,b\}$, let $\{P_n(x) := P_n(x;a,b;c,d)\}_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. Then*

$$x^2(1-x)P_n'''(x) - x\varphi(x)P_n''(x) + \psi_n(x)P_n'(x) + n\lambda_n P_n(x) = 0, \tag{3.9}$$

with

$$\begin{aligned}
\varphi(x) &= (c+d+2)x - (a+b+1),\\
\psi_n(x) &= \big((n-1)(c+d+n) - \lambda_n\big)x + ab,\\
\lambda_n &= \left(c + \left\lfloor \frac{n}{2} \right\rfloor\right)\left(d + \left\lfloor \frac{n-1}{2} \right\rfloor\right).
\end{aligned}$$

Proof. Combining the explicit formula for the 2-orthogonal polynomials as terminating hypergeometric series (3.1) and the generalised hypergeometric differential equation (1.14), we obtain

$$\begin{aligned}
&\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x} + a\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + b\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]P_n(x)\\
&= \left[\left(x\frac{\mathrm{d}}{\mathrm{d}x} - n\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + c + \left\lfloor \frac{n}{2} \right\rfloor\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + d + \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right]P_n(x).
\end{aligned} \tag{3.10}$$

Expanding the left-hand side of (3.10), we get

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x} + a\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + b\right)\frac{\mathrm{d}}{\mathrm{d}x}\right]P_n(x) = x^2 P_n'''(x) + (a+b+1)xP_n''(x) + abP_n'(x).$$

Similarly, recalling that $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-1}{2} \right\rfloor = n-1$, for any $n \in \mathbb{N}$, we derive

$$\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x} + c + \left\lfloor \frac{n}{2} \right\rfloor\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + d + \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right]P_n(x) = x^2P_n''(x) + (c+d+n)xP_n'(x) + \varepsilon_n P_n(x).$$

Therefore, the right-hand side of (3.10) is

$$\begin{aligned}
&\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x} - n\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + c + \left\lfloor \frac{n}{2} \right\rfloor\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x} + d + \left\lfloor \frac{n-1}{2} \right\rfloor\right)\right]P_n(x)\\
&= x^3 P_n'''(x) + (c+d+2)x^2 P_n''(x) + (\lambda_n - (n-1)(c+d+n))\,xP_n'(x) - n\lambda_n P_n(x).
\end{aligned}$$

Finally, combining the expressions for both sides of (3.10), we derive the differential equation (3.9). $\square$

3.3 Recurrence relation

As a 2-orthogonal sequence, the hypergeometric type polynomials expressed by (3.1) satisfy a third order recurrence relation of the form

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x), \tag{3.11}$$

Our purpose here is to obtain explicit expressions for the recurrence coefficients involved. The linear independence of $\{x^n\}_{n\in\mathbb{N}}$ implies that we can equate their coefficients on both sides of the recurrence relation (3.11). After equating the coefficients of x^n and x^{n-1} we obtain, respectively,

$$\beta_n = \tau_{n,1} - \tau_{n+1,1} \quad \text{and} \quad \alpha_n = \tau_{n,2} - \tau_{n+1,2} - (\tau_{n,1})^2 + \tau_{n,1}\tau_{n+1,1},$$

where, based on (3.2), we have

$$\tau_{n,1} = -\frac{n(a+n-1)(b+n-1)}{\left(c+\left\lfloor\frac{n}{2}\right\rfloor+n-1\right)\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor+n-1\right)}$$

and

$$\tau_{n,2} = \frac{n(a+n-1)(b+n-1)(n-1)(a+n-2)(b+n-2)}{2\left(c+\left\lfloor\frac{n}{2}\right\rfloor+n-1\right)\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor+n-1\right)\left(c+\left\lfloor\frac{n}{2}\right\rfloor+n-2\right)\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor+n-2\right)}.$$

Hence we derive that, for each $k\in\mathbb{N}$,

$$\beta_{2k}(a,b;c,d) = \frac{(2k+1)(a+2k)(b+2k)}{(c+3k)(d+3k)} - \frac{2k(a+2k-1)(b+2k-1)}{(c+3k-1)(d+3k-2)}$$

and

$$\beta_{2k+1}(a,b;c,d) = \frac{(2k+2)(a+2k+1)(b+2k+1)}{(c+3k+2)(d+3k+1)} - \frac{(2k+1)(a+2k)(b+2k)}{(c+3k)(d+3k)}$$

as well as

$$\begin{aligned}\alpha_{2k+1}(a,b;c,d) = \frac{(2k+1)(a+2k)(b+2k)}{(c+3k)(d+3k)}\Bigg(&\frac{k(a+2k-1)(b+2k-1)}{(c+3k-1)(d+3k-1)}\\ &-\frac{(2k+1)(a+2k)(b+2k)}{(c+3k)(d+3k)} + \frac{(k+1)(a+2k+1)(b+2k+1)}{(c+3k+1)(d+3k+1)}\Bigg),\end{aligned}$$

and

$$\begin{aligned}\alpha_{2k+2}(a,b;c,d) = \frac{2(k+1)(a+2k+1)(b+2k+1)}{(c+3k+2)(d+3k+1)}\Bigg(&\frac{(2k+1)(a+2k)(b+2k)}{2(c+3k+1)(d+3k)}\\ &-\frac{2(k+1)(a+2k+1)(b+2k+1)}{(c+3k+2)(d+3k+1)} + \frac{(2k+3)(a+2k+2)(b+2k+2)}{2(c+3k+3)(d+3k+2)}\Bigg).\end{aligned}$$

The expressions for the coefficients γ_n in (3.11) could also be obtained in an analogous way after comparing the coefficients of x^{n-2}. However, it is easier to derive such expressions directly from the 2-orthogonality conditions, which, applied to the recurrence relation (3.11), imply that, for each $k \in \mathbb{N}$ and $j \in \{0,1\}$,

$$\gamma_{2k+1+j}(a,b;c,d) = \frac{\displaystyle\int_0^1 x^{k+1} P_{2k+2+j}(x;a,b;c,d)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x}{\displaystyle\int_0^1 x^{k} P_{2k+j}(x;a,b;c,d)\mathscr{W}(x;a,b+j;c+j,d)\mathrm{d}x}.$$

Based on the latter alongside with (3.7) and (3.8), we deduce that, for all $k \in \mathbb{N}$,

$$\gamma_{2k+1}(a,b;c,d) = \frac{(2k+1)_2\,(a+2k)_2\,(b+2k)_2\,(d-1+k)(d-a+k)(d-b+k)}{(c+3k)_3\,(d+3k-1)_3\,(d+3k)_3}.$$

and

$$\gamma_{2k+2}(a,b;c,d) = \frac{(2k+2)_2\,(a+2k+1)_2\,(b+2k+1)_2\,(c+k)(c-a+k+1)(c-b+k+1)}{(c+3k+1)_3\,(c+3k+2)_3\,(d+3k+1)_3}$$

As a consequence, we have just proved the following result.

Theorem 3.4. *For $a,b,c,d \in \mathbb{R}^+$ satisfying (1.2), let $\{P_n(x) := P_n(x;a,b;c,d)\}_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$. Then $\{P_n(x)\}_{n\in\mathbb{N}}$ satisfies the recurrence relation*

$$P_{n+1}(x) = (x-\beta_n)\,P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1} P_{n-2}(x), \tag{3.12}$$

where, for each $n \in \mathbb{N}$,

$$\beta_n = \frac{(n+1)(a+n)(b+n)}{\left(c'_{n-1}+n\right)\left(c'_n+n\right)} - \frac{n(a+n-1)(b+n-1)}{\left(c'_{n-1}+n-1\right)\left(c'_n+n-2\right)}, \tag{3.13a}$$

$$\alpha_{n+1} = \frac{(n+1)(a+n)(b+n)}{(c'_{n-1}+n)(c'_n+n)}\left(\frac{n(a+n-1)(b+n-1)}{2(c'_{n-1}+n-1)(c'_n+n-1)}\right.$$
$$\left. - \frac{(n+1)(a+n)(b+n)}{(c'_{n-1}+n)(c'_n+n)} + \frac{(n+2)(a+n+1)(b+n+1)}{2(c'_{n-1}+n+1)(c'_n+n+1)}\right).$$

and

$$\gamma_{n+1} = \frac{(n+1)_2\,(a+n)_2\,(b+n)_2\,(c'_n-1)(c'_n-a)(c'_n-b)}{\left(c'_{n-1}+n\right)_3\,(c'_n+n)_3\,(c'_n+n-1)_3}, \tag{3.13b}$$

with

$$c'_n = \begin{cases} c+k & \text{if } n = 2k-1, \\ d+k & \text{if } n = 2k. \end{cases} \tag{3.14}$$

With the purpose of rewriting the recurrence relation coefficients using more convenient expressions, we introduce a set of positive coefficients $(\lambda_k = \lambda_k(a,b;c,d))_{k\in\mathbb{N}}$, involving the c'_n introduced in the latter theorem and defined by

$$\begin{cases} \lambda_{3n} = \dfrac{n(b+n-1)(c'_n-a-1)}{(c'_n+n-2)(c'_n+n-1)(c'_{n-1}+n-1)}, \\ \lambda_{3n+1} = \dfrac{n(a+n)(c'_{n-1}-b)}{(c'_n+n-1)(c'_{n-1}+n-1)(c'_{n-1}+n)}, \\ \lambda_{3n+2} = \dfrac{(a+n)(b+n)(c'_n-1)}{(c'_n+n-1)(c'_n+n)(c'_{n-1}+n)}. \end{cases} \tag{3.15}$$

The coefficients above were obtained from [29, Th. 14.5] as the coefficients of a branched continued fraction representation for ${}_3F_2(a,b,1;c,d;t)$, the ordinary generating function of the moment sequence given by (1.5).

Observe that $\lambda_0 = \lambda_1 = 0$ and $\lambda_k > 0$, for all $k \geq 2$. In addition, $\lambda_k \to \dfrac{4}{27}$, as $k \to \infty$, and we have, for all $n \in \mathbb{N}$,

- $\beta_n = \lambda_{3n} + \lambda_{3n+1} + \lambda_{3n+2}$; (3.16)
- $\alpha_{n+1} = \lambda_{3n+1}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+3} + \lambda_{3n+2}\lambda_{3n+4}$; (3.17)
- $\gamma_{n+1} = \lambda_{3n+2}\lambda_{3n+4}\lambda_{3n+6}$. (3.18)

Therefore, we can rewrite Theorem 3.4 as the following result.

Theorem 3.5. *For $a,b,c,d \in \mathbb{R}^+$ satisfying* (1.2)*, let $\{P_n(x) := P_n(x;a,b;c,d)\}_{n\in\mathbb{N}}$ be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}(x;a,b;c,d)$ and let the coefficients λ_k, $k \in \mathbb{N}$, be defined by* (3.15) *Then $\{P_n(x)\}_{n\in\mathbb{N}}$ satisfies the recurrence relation* (3.12)*, with coefficients given by* (3.16)-(3.18)*. Therefore, the recurrence coefficients are real, positive and bounded with asymptotic behaviour*

$$\beta_n \to 3\left(\frac{4}{27}\right) = \frac{4}{9}, \quad \alpha_n \to 3\left(\frac{4}{27}\right)^2 = \frac{16}{243} \quad \textit{and} \quad \gamma_n \to \left(\frac{4}{27}\right)^3 = \frac{64}{19683}, \quad \textit{as } n \to \infty. \tag{3.19}$$

The expressions (3.16)-(3.18) for the recurrence coefficients lead to a decomposition of the lower-Hessenberg matrix (1.11) as a product of three bidiagonal matrices with positive entries in the nonzero diagonals. Thus, (1.11) is a special type of totally positive matrix, an oscillatory matrix (see [15]). As a result, we can conclude that the zeros of $P_n(x)$, which correspond to the eigenvalues of H_n, are real and positive as well as that the zeros of consecutive polynomials interlace, similarly to the main result of [31, §9.2]. Furthermore, applying [22, Th. 3.5] to this case, with $\beta = \frac{4}{9}$, $\alpha = \frac{16}{243}$ and $\gamma = \frac{64}{19683}$, we guarantee that the zeros have absolute value less than 1. Therefore, we have an alternative proof, independent of the system being Nikishin, that the zeros of $\{P_n(x;a,b;c,d)\}_{n\in\mathbb{N}}$ are all located in the interval $(0,1)$ and that the zeros of consecutive polynomials interlace.

3.4 Asymptotic behaviour. Connection with Jacobi-Piñeiro polynomials

Jacobi-Piñeiro polynomials are multiple orthogonal polynomials with respect to several classical Jacobi weights on the same interval. They are usually defined as the multiple orthogonal polynomials with respect to measures $(\mu_0,\cdots,\mu_{r-1})$ supported on the interval $(0,1)$, with $\mathrm{d}\mu_i(x)=x^{\alpha_i}(1-x)^{\beta}\mathrm{d}x$ for some $\beta,\alpha_1,\cdots,\alpha_r>-1$ such that $\alpha_i-\alpha_j\notin\mathbb{Z}$ for any $i\neq j$. These polynomials were introduced by Piñeiro in [30] with $\beta=0$. See [6] for a Rodrigues formula generating the type II Jacobi-Piñeiro polynomials as well as explicit expressions for the polynomials and for their recurrence relation coefficients.

The asymptotic behaviour of the recurrence relation coefficients in Theorem 3.4 coincides with the asymptotic behaviour obtained in [6] for the coefficients of the recurrence relation satisfied by the Jacobi-Piñeiro polynomials. Based on this relation, we show in this subsection that the polynomials investigated here share the ratio asymptotics, the asymptotic zero distribution and a Mehler-Heine asymptotic formula near the endpoint 0 with the Jacobi-Piñeiro polynomials. In fact, the Jacobi-Piñeiro polynomials originally studied by Piñeiro in [30] are a limiting case of the polynomials investigated here. Precisely, the choice of $c=a$ and $d=b+1$ gives $\mathscr{W}(x;a,b;c,d)=bx^{b-1}$ and $\mathscr{W}(x;a,b+1;c+1,d)=ax^{a-1}$, and, for this reason, the explicit formulas for the polynomials obtained in §3.1 and in [30] coincide.

Due to the asymptotic behaviour of the recurrence coefficients obtained in Theorem 3.4 and to the zeros of $P_n(x)$ being real, simple and interlacing with the zeros of $P_{n+1}(x)$, for each $n\in\mathbb{N}$, as previously shown, we can use [2, Lemma 3.2] and [5, Th. 3.1] to derive that

$$\lim_{n\to\infty}\frac{P_n(x)}{P_{n+1}(x)}=\rho(x):=\frac{27}{4}\left(\frac{3}{2}x^{\frac{1}{3}}\left(\mathrm{e}^{\frac{4\pi i}{3}}\left(-1+\sqrt{1-x}\right)^{\frac{1}{3}}+\mathrm{e}^{\frac{2\pi i}{3}}\left(-1-\sqrt{1-x}\right)^{\frac{1}{3}}\right)-1\right),$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,1]$. As explained in [5], the knowledge of this ratio asymptotic leads to prove that

$$\lim_{n\to\infty}\frac{P_n'(x)}{P_n(x)}=-\frac{\rho'(x)}{\rho(x)},\ \text{for}\ \ x\in\mathbb{C}\backslash[0,1],$$

which results in showing that there exists a limit for the normalised zero counting measure of $P_n(x)$

$$\nu_n:=\frac{1}{n}\sum_{P_n(x)=0}\delta_x,$$

as $n\to\infty$, in the sense of the weak limit of measures, ie $\lim_{n\to\infty}\int f\mathrm{d}\nu_n=\int f\mathrm{d}\nu$ for every bounded and continuous function f on $[0,1]$. Here δ_x is the Dirac point mass at x. As such, it was proved that

$$\lim_{n\to\infty}\int\frac{1}{x-t}d\nu_n(t)=-\frac{\rho'(x)}{\rho(x)},\ \text{for}\ \ x\in\mathbb{C}\backslash[0,1],$$

and, as shown in [5, Th. 2.1] the limiting measure ν has density

$$\frac{\mathrm{d}\nu}{\mathrm{d}x}=\begin{cases}\dfrac{\sqrt{3}}{4\pi}\,\dfrac{\left(1+\sqrt{1-x}\right)^{\frac{1}{3}}+\left(1-\sqrt{1-x}\right)^{\frac{1}{3}}}{x^{\frac{2}{3}}\sqrt{1-x}} & \text{if } x\in(0,1),\\ 0 & \text{elsewhere,}\end{cases}\tag{3.20}$$

which is the asymptotic zero distribution of $\{P_n(x)\}_{n\in\mathbb{N}}$.

Jacobi-Piñeiro polynomials (orthogonal with respect to two measures) on the step line and the polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ under analysis share the same ratio asymptotics and the asymptotic zero distribution because their recurrence coefficients have the same asymptotic behaviour and their zeros are simple, real, satisfy the interlacing property and are located on the interval $[0,1]$.

We also derive a Mehler-Heine asymptotic formula satisfied by the 2-orthogonal polynomials $P_n(x;a,b;c,d)$ near the origin, which give us more information about the zeros near the endpoint 0 of the orthogonality interval. For that purpose, we recall that the generalised hypergeometric series ${}_{p+1}F_q$ satisfies the confluent relation (see [8, Eq. 16.8.10])

$$\lim_{|\alpha|\to\infty}{}_{p+1}F_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p,\alpha\\ \beta_1,\cdots,\beta_q\end{matrix};\frac{z}{\alpha}\right)={}_pF_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};z\right),\tag{3.21}$$

whenever both sides of this relation are convergent. Moreover, we recall Theorem 3.1 to write

$$\frac{(-1)^n\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}P_n\left(\frac{z}{n^3};a,b;c,d\right)={}_3F_2\left(\begin{matrix}-n,c+\left\lfloor\frac{n}{2}\right\rfloor,d+\left\lfloor\frac{n-1}{2}\right\rfloor\\ a,b\end{matrix};\frac{z}{n^3}\right).$$

Clearly, $c+\left\lfloor\dfrac{n}{2}\right\rfloor,d+\left\lfloor\dfrac{n-1}{2}\right\rfloor\sim\dfrac{n}{2}$, as $n\to\infty$. So we apply (3.21) three consecutive times to the generalised hypergeometric series on the right-hand side of the latter equation to deduce a Mehler-Heine type formula near the endpoint 0

$$\lim_{n\to\infty}\frac{(-1)^n\left(c+\left\lfloor\frac{n}{2}\right\rfloor\right)_n\left(d+\left\lfloor\frac{n-1}{2}\right\rfloor\right)_n}{(a)_n\,(b)_n}P_n\left(\frac{z}{n^3};a,b;c,d\right)={}_0F_2\left(\begin{matrix}-\\ a,b\end{matrix};-\frac{z}{4}\right),$$

which converges uniformly on compact subsets of $\mathbb{C}$.

Note that the limit in this Mehler-Heine formula coincides with the limit in the Mehler-Heine formula for the Jacobi-Piñeiro polynomials obtained in [35, Th. 2], with $r=2$ and $q_1=q_2=1/2$.

Furthermore, we can derive a result about the asymptotic behaviour of the k-th smallest zero of $P_n(x;a,b;c,d)$, which also coincides with the one obtained in [35, §4] for the zeros of the 2-orthogonal Jacobi-Piñeiro polynomials. In fact, if we denote the zeros of $P_n(x;a,b;c,d)$ by $\left(x_k^{(n)}\right)_{1\leq k\leq n}$ and the zeros of the generalised hypergeometric series ${}_0F_2\,(-;a,b;-z)$, which are all real and positive, by $(f_k)_{k\in\mathbb{Z}^+}$, with the zeros written in increasing order for both cases, then we have

$$\lim_{n\to\infty}n^3x_k^{(n)}=4f_k.$$

3.5 Particular cases: Jacobi-type 2-orthogonal polynomials and a sequence with constant recurrence relation coefficients

Using the coefficients c'_n introduced in (3.14), the explicit expression for the type II polynomials given by (3.1) can be rewritten as

$$P_n(x;a,b;c,d) = \frac{(-1)^n\,(a)_n\,(b)_n}{\left(c'_{n-2}\right)_n\left(c'_{n-1}\right)_n}\,{}_3F_2\left(\begin{matrix}-n, c'_{n-2}, c'_{n-1}\\ a,b\end{matrix};x\right)$$

Furthermore, if $d = c+\dfrac{1}{2}$, then $c'_n = c+\dfrac{n+1}{2}$, for any $n\in\mathbb{N}$, and the expression above becomes

$$P_n\left(x;a,b;c,c+\frac{1}{2}\right) = \frac{(-4)^n\,(a)_n\,(b)_n}{(2c-1+n)_{2n}}\,{}_3F_2\left(\begin{matrix}-n, c+\frac{n-1}{2}, c+\frac{n}{2}\\ a,b\end{matrix};x\right). \tag{3.22}$$

The latter polynomials coincide, up to a linear transformation of the variable, with the Jacobi-type 2-orthogonal polynomials investigated in [21], with $c=\dfrac{\nu+1}{2}$.

A particular case of (3.22) which is worth of interest arises if we set $(a,b;c,d)=\left(\dfrac{4}{3},\dfrac{5}{3};2,\dfrac{5}{2}\right)$. This choice of parameters gives

$$P_n\left(x;\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right) = \frac{(n+1)(n+2)}{2}\left(\frac{-4}{27}\right)^n{}_3F_2\left(\begin{matrix}-n, \frac{n+3}{2}, \frac{n}{2}+2\\ \frac{4}{3},\frac{5}{3}\end{matrix};x\right), \tag{3.23}$$

where we have used $\left(\dfrac{4}{3}\right)_n\left(\dfrac{5}{3}\right)_n = \dfrac{(n+1)_{2n+2}}{2\cdot 27^n}$. So, we have $c'_n=\dfrac{n+5}{2}$, for any $n\in\mathbb{N}$, and the recurrence relation coefficients given by (3.13a)-(3.13b) are all constant and equal to the limits in (3.19), precisely we have:

$$\beta_n\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right)=\frac{4}{9},\quad \alpha_{n+1}\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right)=\frac{16}{243}\quad\text{and}\quad \gamma_{n+1}\left(\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right)=\frac{64}{19683},$$

for all $n\in\mathbb{N}$. Therefore, based on Theorem 3.4, the sequence $\left\{P_n\left(x;\dfrac{4}{3},\dfrac{5}{3};2,\dfrac{5}{2}\right)\right\}_{n\in\mathbb{N}}$ satisfies the third-order recurrence relation with constant coefficients

$$P_{n+1}(x) = \left(x-\frac{4}{9}\right)P_n(x) - \frac{16}{243}P_{n-1}(x) - \frac{64}{19683}P_{n-2}(x).$$

Finally, recall (2.6) and (1.1) and use [8, Eq. 15.4.9] to conclude that the polynomials in (3.23)-(3.24) are 2-orthogonal with respect to the vector of weights

$$\overline{\mathscr{W}}\left(x;\frac{4}{3},\frac{5}{3};2,\frac{5}{2}\right) = \begin{bmatrix}\dfrac{81\sqrt{3}}{16\pi}x^{\frac{1}{3}}\left(\left(1+\sqrt{1-x}\right)^{\frac{1}{3}}-\left(1-\sqrt{1-x}\right)^{\frac{1}{3}}\right)\\ \dfrac{243\sqrt{3}}{160\pi}x^{\frac{1}{3}}\left(\left(1+\sqrt{1-x}\right)^{\frac{4}{3}}-\left(1-\sqrt{1-x}\right)^{\frac{4}{3}}\right)\end{bmatrix}. \tag{3.24}$$

Observe the similarities between the orthogonality weights above and the asymptotic zero distribution (3.20).

3.6 Connection with other Hahn-classical 2-orthogonal polynomials

Particular choices on the parameters a,b,c and d of the 2-orthogonal polynomial sequence (3.1) appeared in [24] as the components of a certain family of *threefold symmetric Hahn-classical* 2-orthogonal polynomials on star-like sets. A polynomial sequence $\{S_n(x)\}_{n\in\mathbb{N}}$ is said to be threefold symmetric if

$$S_n\left(\mathrm{e}^{\frac{2\pi}{3}i}\,x\right) = \mathrm{e}^{\frac{2n\pi}{3}i} S_n(x) \text{ and } S_n\left(\mathrm{e}^{\frac{4\pi}{3}i}\,x\right) = \mathrm{e}^{\frac{4n\pi}{3}i} S_n(x). \text{ for all } n\in\mathbb{N}.$$

This means there exist three polynomial sequences $\left\{S_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$, with $k\in\{0,1,2\}$, which are called the *cubic components* of $\{S_n(x)\}_{n\in\mathbb{N}}$, such that

$$S_{3n+k}(x) = x^k S_n^{[k]}(x^3) \text{ for all } n\in\mathbb{N}.$$

As reported in [9] and studied in detail in [24], there are four distinct families of Hahn-classical threefold symmetric 2-orthogonal polynomials, up to a linear transformation of the variable. The four arising cases were therein denominated as A, B1, B2 and C. The polynomials in case A have no parameter dependence and their cubic components are particular cases of the 2-orthogonal polynomials with respect to Macdonald functions investigated in [3] and [36], while the polynomials in cases B1 and B2 depend on a parameter and their cubic components are particular cases of the 2-orthogonal polynomials with respect to weights involving confluent hypergeometric functions of the second kind. These components were investigated in [22]. At last, the cubic components of the polynomials in case C, depend on two parameters, are particular cases of the 2-orthogonal polynomials under analysis here.

Precisely, denoting the Hahn-classical threefold symmetric 2-orthogonal polynomials analysed in [24, §3.4] by $S_n(x;\mu,\rho)$ and their cubic components by $S_n^{[k]}(x;\mu,\rho)$, $k\in\{0,1,2\}$, and comparing the explicit expressions exhibited in [24, §3.4.1] with (3.1), we derive that, for each $\mu,\rho\in\mathbb{R}^+$ and $k\in\{0,1,2\}$,

$$S_n^{[k]}(x;\mu,\rho) = P_n\left(x;a_k,b_k;c_k,d_k\right),$$

with $(a_k,b_k;c_k,d_k)$ equal to $\left(\frac{1}{3},\frac{2}{3};\frac{\mu+2}{3},\frac{\rho}{3}+1\right)$, $\left(\frac{4}{3},\frac{2}{3};\frac{\rho}{3}+1,\frac{\mu+5}{3}\right)$ and $\left(\frac{4}{3},\frac{5}{3};\frac{\mu+5}{3},\frac{\rho}{3}+2\right)$, for $k=0,1,2$, respectively.

Furthermore, there are confluent relations between the 2-orthogonal polynomials analysed here and the ones investigated in [22]. These relations generalise the ones between case C and cases B1 and B2 in [24], similarly to how the confluent relations shown in [22, Section 3.5] generalise the ones between cases B1 and B2 and case A.

The 2-orthogonal polynomials investigated in [22] satisfy orthogonality conditions with respect to weight functions $\mathscr{V}(x;a,b;c)$ and $\mathscr{V}(x;a,b;c+1)$, supported in $\mathbb{R}^+$, with $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$ and

$$\mathscr{V}(x;a,b;c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-1}\,\mathbf{U}\left(c-b,a-b+1;x\right), \tag{3.25}$$

where $\mathbf{U}(\alpha,\beta;x)$ is the confluent hypergeometric function of the second kind, also known as the Tricomi function (see [8, §13] for the definition and some properties of the confluent hypergeometric functions). These 2-orthogonal polynomials have also appeared in [10].

As shown in [22, Th. 3.1], if we denote by $R_n^{[\varepsilon]}(x;a,b;c)$, with $\varepsilon \in \{0,1\}$, the 2-orthogonal polynomials with respect to $[\mathscr{V}(x;a,b;c+\varepsilon), \mathscr{V}(x;a,b;c+1-\varepsilon)]$, then

$$R_n^{[\varepsilon]}(x) = \frac{(-1)^n (a)_n (b)_n}{\left(c+\left\lfloor \frac{n+\varepsilon}{2} \right\rfloor\right)_n} {}_2F_2\left(\begin{matrix} -n, c+\left\lfloor \frac{n+\varepsilon}{2} \right\rfloor \\ a,b \end{matrix}; x\right).$$

The confluent relations are a straightforward consequence of the explicit expressions for the 2-orthogonal polynomials via the confluent relation for the generalised hypergeometric series (3.21). Naturally, limiting relations connecting the corresponding weight functions are obtained in a similar manner.

So, applying the confluent relation for the generalised hypergeometric series (3.21) to the polynomials defined by (3.1), we derive the confluent relations

$$\lim_{d\to\infty} P_n\left(\frac{x}{d};a,b;c,d\right) = R_n^{[0]}(x;a,b;c) \quad \text{and} \quad \lim_{c\to\infty} P_n\left(\frac{x}{c};a,b;c,d\right) = R_n^{[1]}(x;a,b;d-1).$$

We can also obtain similar confluent relations connecting the weight functions $\mathscr{W}(x;a,b;c,d)$, defined by (1.1), and $\mathscr{V}(x;a,b;c)$, defined as in (3.25). More precisely, we derive

$$\lim_{d\to\infty} \frac{1}{d}\mathscr{W}\left(\frac{x}{d};a,b;c,d\right) = \mathscr{V}(x;a,b;c) \quad \text{and} \quad \lim_{c\to\infty} \frac{1}{c}\mathscr{W}\left(\frac{x}{c};a,b;c,d\right) = \mathscr{V}(x;a,b;d),$$

as a consequence of combining the linear transformation of variable (see [8, Eq. 15.8.1])

$${}_2F_1\left(\begin{matrix} \alpha, \gamma-\beta \\ \gamma \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; \frac{z}{z-1}\right)$$

and the limiting relation between the hypergeometric and Tricomi functions (see [12, Eq. 6.8.1])

$$\lim_{\gamma\to\infty} {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; 1-\frac{\gamma}{x}\right) = x^{\alpha}\mathbf{U}(\alpha, \alpha-\beta+1;x).$$

Concluding remarks.

The main contribution of this paper is the analysis of the multiple orthogonal polynomials on the step line with respect to the Nikishin system obtained in Subsection 2.1. The study of the multiple orthogonal polynomials with respect to the same system for indices out of the step line and, in particular, the study of the (standard) orthogonal polynomials with respect to the weight function $\mathscr{W}(x;a,b;c,d)$ remains an open (and challenging) problem. The same holds in general when the weight function is a solution to a second (or higher) order differential equation. In spite of this, the knowledge of the multiple orthogonal polynomials whose indexes lie on the step line is a largely sufficient tool for its applicability to a number of related fields in mathematics. An example of this applicability is the newly found connection between multiple orthogonal polynomials and branched continued fractions which will be object of further research.

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