Canterbury Business School

Working Paper Series

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Working Paper No. 52
January 2004

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30 January 2004

Abstract

In this paper, we re-examine Data Envelopment Analysis models from perspectives of preference order, production set and performance measure. we investigate the relationship between Data Envelopment Analysis (DEA) and Multiple Criteria Decision Making Theory.

There are three key building blocks in a DEA model: preference order, production possibility set and performance measure. It is shown in this work that many known DEA models and new ones, can be derived via this approach.

1 Introduction

Data Envelopment Analysis (DEA) has become a standard non-parametric approach to productivity analysis, especially to relative efficiency analysis of Decision Making Units (DMUs). Since the introduction of the first DEA model CCR in 1978, it has been widely used in efficiency analysis of many business and industry evaluation procedures. Excellent literature surveys can be found in, for instance, [*Charnes et al.*1994] and [*Seiford* 1996]. The most well-known DEA models are the CCR model [*Charnes et al.* 1978], the BCC model [*Banker et al.* 1984], the Additive model [*Charnes et al.* 1985], and the Cone Ratio model [*Charnes et al.* 1989]. Most of these DEA models are designed to handle some "standard situation". Many non-standard applications have been transformed via changing either data structures or problem structures in order to apply these DEA models.

It is clear that there is no universal DEA model which is applicable to all or most of applications. Moreover most widely used efficiency notions are sensitive to data manipulations. Take the single-input and single-output case as an example, one always has

$$e \equiv \frac{Output}{Input} < \frac{Output + M}{Input + M}, \forall M > 0, e < 1$$

Thus even a simple data transformation changes efficiency unless the unit is 100% efficient. Also it is important to reflect the preferences of evaluators for some of the inputs or outputs, like profits in productivity analysis. One of the possible plausible approaches is to derive DEA models specially to suit a particular application.

Thus whenever possible a practitioner should select or derive a DEA model which is most suitable for his or her application. This calls for a more through understanding of the structures of DEA models.

It is clear that formulation of DEA models for a particular application may not be an easy task. Most of the fundamental DEA models are derived from economic efficiency theory including Debreu-Farrell efficiency, Pareto-Koopmans efficiency, and more general technical efficiency axiomatic approaches [*see Fare and Lovell* 1978 *and Russell* 1988].

In this paper, we examine the structures of DEA models and try to identify the building blocks that comprise them. Some ideas used in [*Liu and Sharp* 1999] in multiple criteria decision making theory have been adopted here. We actually identify three main building blocks on which most DEA model are based. Here we have no intention of exhausting every possible way of constructing these blocks, rather to show the essential ideas by giving some illustrative examples. We also examine and summarize the main functions of these blocks so that in many cases it should become relatively easy for practitioners to tune existing DEA models to new applications. However at present there is still no reliable way to construct the blocks to build new DEA models according to the requirements of a particular application. Nevertheless this study should pave the way for further investigations of this topic.

The first element of any DEA models is Preference. It is clear that decision making units are built or operated for some specific purposes. In order to be able to evaluate DMUs, we first have to know our "preference" in the input-output space (X, Y). To set up certain order relationship among the input-output possibilities, preference can be viewed as an order relation, that aims to clarify precise meaning of our fuzzy desires like "higher", "lower"; "better", or "worse". Any evaluation of performance efficiency has to be carried out against a preference measure.

However most of the preference in business applications are too weak to pick up "best" among finite set of DMUs since there are not enough peers for comparison. Very often one cannot find the peers for any DMU (X_i, Y_i) so that every unit would be classified as efficient. This is why we need to identify more peers. The production possibility set $P(\{(X_i, Y_i)\})$ is the set of all possible DMUs associated with the preference and the existing DMUs $\{(X_i, Y_i)\}$. Sometimes, they are referred to as "virtual" DMUs. If a real DMU (X_i, Y_i) is found to be "best" in $P(\{(X_i, Y_i)\})$, then it is considered to be efficient.

In order to find whether or not a particular DMU (X_i, Y_i) is "best" in $P(\{(X_i, Y_i)\})$, we need to use some performance measurement $m(\cdot, \cdot)$ which is compatible with the selected preferences, in the sense that $m((X, Y), (X_0, Y_0)) > 0$ if the DMU (X, Y) is better than (X_0, Y_0). Whether or not DMU (X_0, Y_0) is efficient can be found by solving a mathematical programming problem like:

$$max_{(X,Y) \in P(\{(X_i,Y_i)\})} m((X, Y), (X_0, Y_0))$$

This is essentially a DEA model. If the maximum is positive then the DMU (X_0, Y_0) is inefficient, and otherwise it is efficient.

The plan of the paper is as follows: In Section 2, we examine the notion of preference, which is important for goal setting. In Section 3 we examine the production possibility set. In Section 4 we discuss goals associated with DMUs. In Section 5 we identify and examine a performance measuring, merit function. In Section 6 we formulate a general DEA model.

2 Building Block 1 - Preferences

It is important to reflect decision makers' preference in production analysis. There are may existing approaches, see [*Allen et al* 1997, *Golany* 1988, *Halme et al* 1999, *Thanassoulis and Dyson* 1992, *Zhu* 1996]. In our view, preference could be introduced into DEA more directly. In fact, it forms a building block of DEA models. DMUs are expected to achieve certain goals. The evaluator has to have some criteria, in order to be able to say which unit may be doing better than another. Thus we need a preference structure.

To be more precise we need to introduce some basic ideas in multiple criteria decision making theory. Preference can be viewed as an order relation, and aims to clarify the precise meaning of our fuzzy concept like "higher", "lower"; "better", or "worse". With a preference selected, we can unambiguously state that one outcome is greater or lower than another.

Definition 2.1 *Let Y be a set and let $y_1, y_2 \in Y$. A preference or an order $\geq$ on Y is a subset of $Y \times Y$ denoted by $\{\geq\}$ such that $y_1 \geq y_2$ if and only if $(y_1, y_2) \in \{\geq\}$. Define $y_1 > y_2$ if and only if $y_1 \geq y_2$ and $y_1 \neq y_2$*

We require that preference is transitive, etc; see *Liu* (1985) and *Nemhauser* (1989) for the details. The most frequently used order in DEA is Pareto preference. Let $X = (x_1, , , x_n), Y = (y_1, , , y_n) \in R^n$. Then in Pareto preference, $X \geq Y$ if and only if $x_i \geq y_i$ (i=1,2,,n). $X \leq Y$ if and only if $-X \geq -Y$. The Pareto preference is most widely used in business and economical research. However we have to emphasis that other orders such as K-cone order and lexicographic order are also very useful in DEA model building. The former can, for instance, lead to the well-known Cone Ratio model [*Charnes et al*. 1989] and the latter may let us build DEA models which are able to express the preferences of the lexicographic order. Again readers are referred to *Liu* (1985) and *Nemhauser* (1989) for the more details on preference.

Example 1: K-cone order
Let K be a closed convex cone in R^n_+ such that 0$\in K$. Then an useful order, to be referred to as K-cone order or K-preference, is defined as:

$$X \geq Y \quad \text{if and only if} \quad X - Y \in K; \quad X \leq Y \quad \text{if and only if} \quad -(X - Y) \in K.$$

Thus if we take $K = \{X = (x_1, ..., x_n) \in R^n : x_i \geq 0\}$ then the K-cone order is just the Pareto order.

Example 2: Lexicographic Preference

A lexicographic ordering is sometimes very useful when the k-th component is overwhelmingly more important than the $k+1$-th component for $k = 1, 2, ..., n-1$. A lexicographic ordering preference is defined as follows: the outcome $Y = (y_1, , , y_n)$ is preferred to $X = (x_1, , , x_n)$ if and only if $y_1 > x_1$, or there is some $k \in (2, , , n)$ so that $y_k > x_k$ and $y_i = x_i$ for $i = 1, ..., k-1$.

2.1 Preference for input-output production systems

There are different ways to define preferences on input-output space (X, Y). In most applications, they are based on those of X and Y.

Assume that we have preferences $\geq$ on the input set $X \in R^m$ and output set $Y \in R^s$. Then naturally we have the following preference on the input-output production system.

Positive Input Positive Output (PIPO):
$(X, Y) \geq (W, Z)$ iff $X \leq W, Y \geq Z$. This is the most widely used preference.

Negative Input Positive Output (NIPO):
$(X, Y) \geq (W, Z)$ iff $X \geq W, Y \geq Z$.

Positive Input Negative Output (PINO):
$(X, Y) \geq (W, Z)$ iff $X \leq W, Y \leq Z$.

Negative Input Negative Output (NINO):
$(X, Y) \geq (W, Z)$ iff $X \geq W, Y \leq Z$.

Please note that negative input or output does not necessarily mean that input or output is negative. It simply means that they are undesirable for the DMUs. Foe instances they could be undesirable outputs like pollution produced by DMUs

Example 3: (A, B) matrix Preference
Let A be a $m \times m$ matrix. A preference on X can be defined via the matrix A such that X is preferred to W if and only if $AX \geq AW$ in the Pareto order. Similarly one can define a preference on Y via a $s \times s$ matrix B. Therefore under PIPO, $DMU(X, Y)$ is preferred to $DMU(W, Z)$ if and only if

$$AX \leq AW \text{ and } BY \geq BZ \text{ in the Pareto order.}$$

When A, B are the unit matrices. Then it is just the standard Pareto Preference. The following is a simple example.

In the discussion to follow, it is assumed that there are n DMUs to be evaluated. Each DMU consumes only two inputs to produce two outputs. Let the 2-dimensional vector

$X_j = (x_{1j}, x_{2j})^t$ denote the inputs of DMUj, and 2-dimensional vector $Y_j = (y_{1j}, y_{2j})^t$ denote the outputs of DMUj. Let

$$A = B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So DMU_i is preferred to DMU_j if and only if

$$\begin{aligned} x_{1i} + x_{2i} &\le x_{1j} + x_{2j} \\ x_{1i} - x_{2i} &\le x_{1j} - x_{2j} \end{aligned}$$

and

$$\begin{aligned} y_{1i} + y_{2i} &\ge y_{1j} + y_{2j} \\ y_{1i} - y_{2i} &\ge y_{1j} - y_{2j} \end{aligned}$$

Many preferences can be obtained this way. The Cone Ratio model is closely related to this preference.

Example 4: Economic order

Economic profit is defined to be the difference between the revenue that a DMU receives and the costs that it incurs. A basic assumption of most economic analyses of firm behaviour is that a DMU acts so as to maximise its profits. Here we assume that the DMU faces fixed prices for its inputs and outputs. Economists call a market where the individual producers take the prices as outside their control a competitive market.

Let the prices of the inputs be $W = (w_1, w_2, ..., w_m)$ and the prices of the outputs be $P = (p_1, p_2, ..., p_s)$. Normally the prices of both inputs and outputs are positive, however negative prices can be set for undesirable outputs. For example, pollution or emissions taxes (more generally, externality taxes) have been proposed by economists on numerous occasions as ways of reducing pollution or externalities in a flexible and efficient manner.

The profits the DMU_j receives, π_j, can be expressed as

$$\pi_j = PY_j - W\ X_j$$

The first term is the revenue, and the second term is the cost. A preference is defined such that DMU_i is preferred to DMU_j if and only if $\pi_i \ge \pi_j$.

Again suppose that the firm's (a special DMU) objective is to maximise its profit. An important implication of the DMU choosing a profit-maximisation production plan is that there is no way to produce the same amounts of outputs at a lower total input cost. Thus, cost minimisation is a necessary condition for profit maximisation. When a DMU is not a price taker in its output market, the profit function can no longer be used for analysis. Nevertheless, as long as the DMU is a price taker in its input market, the results from the cost minimization problem continue to be valid for the profit maximisation problem.

Furthermore often parts of the inputs and outputs are positive (desirable), but the other parts are negative (undesirable). This can be similarly treated. Mixed Input Mixed Output (MIMO):

$X = \begin{pmatrix} X^+ \\ X^- \end{pmatrix}, Y = \begin{pmatrix} Y^+ \\ Y^- \end{pmatrix}, W = \begin{pmatrix} W^+ \\ W^- \end{pmatrix}, Z = \begin{pmatrix} Z^+ \\ Z^- \end{pmatrix}$

Here we assume the first part is positive, and the second part is negative. Then we define the following preference:

$(X, Y) \geq (W, Z)$ iff $X^+ \leq W^+, X^- \geq W^-, Y^+ \geq Z^+ , Y^- \leq Z^-$.

3 Building Block 2 - Production possibility sets(PPS)

The production plan of a DMU is constrained to a given production possibility set representing essentially its limited technological knowledge. For a given DMU, a production plan is a specification of the quantities of all its inputs and outputs. A given production maybe technically possible or impossible for a DMU. The set of all the production possible for a DMU is called its Production Possibility Set. Formally $(X, Y) \in P$ if input X can produce output Y. We restate some commonly assumed properties of production possibility sets. The appropriateness of each these assumptions depends on the particular circumstances [$Mas - colell\ et\ al$ 1995]. We wish to emphasis that the preference ($\geq$) used in the assumptions may depend on what is being used in building block one. Thus they may have very different practical meanings with different preference specified.

CLOSED

The set P includes its boundary. Thus, the limit of a sequence of technologically feasible input-output vectors is also feasible. That is, $(X^n, Y^n) \to (X, Y)$ and $(X^n, Y^n) \in P$ imply $(X, Y) \in P$.

NO FREE LUNCH

It is impossible to produce something from nothing. That is, if $(X, Y) \in P$ and $X = 0$ then $Y = 0$.

FREE DISPOSAL

The property of free disposal holds if the absorption of any additional amounts of inputs without any reduction in output is always possible. Formally under PIPO,

$$\text{if } (X, Y) \in P \;\; and \;\; Z \geq X, W \leq Y \text{ then } (Z, W) \in P$$

The interpretation is that the extra amount of inputs or output be disposed of or eliminated at no cost.

In the discussion to follow, it is assumed that there are n DMUs to be evaluated. Each DMU consumes varying amounts of m-different inputs to produce s-different outputs. Let the m-dimensional vector $X_j = (x_{1j}, x_{2j}, ..., x_{mj})^t$ denote the inputs of DMUj, and s-dimensional vector $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})^t$ denote the outputs of DMUj. Mixture of input-output systems is given by

$$X(\lambda) = \sum_{1}^{n} \lambda_i X_i, \;\; Y(\lambda) = \sum_{1}^{n} \lambda_i Y_i, \lambda \in S,$$

where $\lambda = (\lambda_1, ..., \lambda_n)^t$ is sometimes referred to as the mixture vector in a technology set S to be specified in applications. The technology set S should at least contain the unit vectors $e_1 = (1, 0, ...0)^t, ..., e_n = (0, ..1)^t$

CONVEXITY

For a given preference with free disposal and convex technology set assumptions, the production possibility set for PIPO system is

$$P = \left\{(X, Y) \mid X \geq X(\lambda) \;\; and \;\; Y \leq Y(\lambda), \;\; \sum \lambda_j = 1, \lambda_j \geq 0 \; for \; all \; j\right\}$$

The observed activities of the DMUs (X_j, Y_j) $(j = 1, 2, ..., n)$ belong to P by letting $\lambda_j = 1, \lambda_i = 0$ $(i \neq j)$. In some applications, one may use different technology sets. For instance, assuming constant returns to scale, one can drop the constraint $\sum \lambda_j = 1$, or require $\sum \lambda_j \geq 1$ for decrease returns to scale.

For a given preference, let us examine Mixed Input Mixed Output (MIMO) system. Let:

$$X = \begin{pmatrix} X^+ \\ X^- \end{pmatrix}, X(\lambda) = \begin{pmatrix} X(\lambda)^+ \\ X(\lambda)^- \end{pmatrix}, Y = \begin{pmatrix} Y^+ \\ Y^- \end{pmatrix}, Y(\lambda) = \begin{pmatrix} Y(\lambda)^+ \\ Y(\lambda)^- \end{pmatrix}$$

Here we assume the first part of each vectors, i.e. X^+, Y^+, is positive (desirable) part , and the second part X^-, Y^- is negative (undesirable) part.

The production possibility set is

$$P = \Big\{(X,Y) \mid X^+ \geq X^+(\lambda),\ X^- \leq X^-(\lambda)\ and\ \ Y^+ \leq Y^+(\lambda), Y^- \geq Y^-(\lambda)\ \lambda \in S.\Big\}$$

4 Goal Associated with DMUs

Many multiple criteria decision making processes involve multiple objective optimisation, where instead of optimisation only one objective function as in the standard optimisation theory, multiple objective functions are considered. The most important example as far as this paper is concerned is the procedure to find efficient or inefficient DMUs as described in the introduction. In a sense, a (PIPO) DMU is efficient if it minimizes its inputs for a given output level, or it maximizes its outputs for a given input level. Multiple criteria optimisation is a difficult area and the theory and algorithms, by comparison with single objective optimisation, are far from complete. In particular, it is very rare that one is able to identity the true solutions. In multiple criteria decision theory, a different approach is adopted to seek "good enough" solutions. In this approach, instead of optimism multiple objective functions, we set up a group of goals to be achieved. It may be impossible to achieve all these goals simultaneously. It is thus important to investigate whether these goals can be simultaneously achieved, and furthermore whether these goals have been set properly or can be further improved, and if not, how to find some compromise solutions. These questions constitute a part of modern multiple criteria decision making studies, in particular the essential part of Goal Programming. The details of multiple criteria decision making theory and conventional goal programming theory may be found in, e.g., [*Liu* 1985]. Each DMU can be viewed as an input-output system, designed and operated to achieve a pre-set goal. In a sense, DEA aims to evaluate how well the goal is met by the system, and to identify possible avenues to improve the system performance whenever possible, see [*Liu and Sharp* 1999] for details.

In the following we apply the basic ideas introduced above to examine DMU systems, which can be viewed as input-output systems, which are fundamental to DEA theory. It is assumed that there are n decision-making units to be evaluated. Each DMU consumes varying amounts of m-different inputs to produce s-different outputs. Let the m-dimensional vector $X_j = (x_{1j}, x_{2j}, ..., x_{mj})^t$ denote the inputs of DMUj, and s-dimensional vector $Y_j = (y_{1j}, y_{2j}, ..., y_{sj})^t$ denote the outputs of DMUj with $j = 1, 2, ...n$. Therefore each of the DMUs is viewed as an input-output system with a goal pre-set by evaluators. In a sense, DEA evaluates the efficiency of the DMU by finding whether or not this goal level (or its performance) can be further improved. In the following we try to examine the DMUs according to their desired input-output response.

4.1 Goal setting

Setting up goals to suit a particular application is by no mean trivial, and has been extensively studied in the literature. Here we only examine some cases relevant to our later discussions. Let $\lambda = (\lambda_1, ..., \lambda_n)^t \in S$ as before.

Assume that we wish to maximise the input mixture (with respect to λ)

$$X(\lambda) = \sum_1^n \lambda_i X_i, \lambda \in S.$$

Instead of maximizing this quantity directly, we may first select an initial target $G^0 = (g_1^0, ..., g_m^0)^t \in R^m$, and then set a goal to find $\lambda = (\lambda_1, .., \lambda_n)$ such that

$$\sum_1^n \lambda_i X_i \geq G^0, \lambda \in S,$$

where $G^0 = (g_1^0, ..., g_m^0)^t$ is referred to as the goal level, and $\geq$ is in the sense of a selected preference. This type of goal will be referred to as positive response goal (PR), as a higher level of the goal is desirable in the selected preference. If a goal level is set too high, then the goal may not be achievable. Similarly we can set up the following goals:

$$\sum_1^n \lambda_i X_i \leq G^0, \lambda \in S.$$

and

$$\sum_1^n \lambda_i X_i = G^0, \lambda \in S.$$

These two types of goal will be referred to as negative response goal (NR) and targeted response goal (TR) respectively.

Assume that we use the Pareto preference. Then the above three basic types are:

$$(\sum_1^n \lambda_i X_i^1 \geq g_1^0, ..., \sum_1^n \lambda_i X_i^m \geq g_m^0), \lambda \in S, \qquad (PR)$$

or

$$(\sum_1^n \lambda_i X_i^1 \leq g_1^0, ..., \sum_1^n \lambda_i X_i^m \leq g_m^0), \lambda \in S. \quad (NR)$$

or

$$(\sum_1^n \lambda_i X_i^1 = g_1^0, ..., \sum_1^n \lambda_i X_i^m = g_m^0), \lambda \in S. \quad (TR)$$

Assume now we select a K-preference associated with a cone $K \subset R^m$. Then we can similarly set two typical multiple goals.

$$(\sum_1^n \lambda_i X_i^1 - g_1^0, ..., \sum_1^n \lambda_i X_i^m - g_m^0) \in K, \lambda \in S. \quad (KPR)$$

or

$$-(\sum_1^n \lambda_i X_i^1 - g_1^0, ..., \sum_1^n \lambda_i X_i^m - g_m^0) \in K, \lambda \in S. \quad (KNR)$$

We can also set mixed type of goal (PRNR) as

$$G^0 = (G_+^0, G_-^0)^t, X(\lambda) = (X(\lambda)^+, X(\lambda)^-)^t,$$

$$(X(\lambda)^+ \geq G_+^0, X(\lambda)^- \leq G_-^0), \lambda \in S.$$

4.2 Goal setting for input-output systems

Positive Input Positive Output (PIPO)

In many applications, a DMU is expected to yield a higher level of outputs when its input level is increased for selected preferences in the input and output spaces, if this unit is being operated efficiently. The efficiency of this DMU is judged by how much more output can be produced by the unit. The higher the extra output the unit yields, the more efficient it is considered to be. Such a DMU can be described by a PIPO system as defined in Section 2.1.

To be more precise we have to specify a preference and a goal to be achieved. Let $\geq$ be a selected preferences in the input and output spaces. We then set goal for a PIPO system as

$$(X(\lambda) \leq G_X^0, Y(\lambda) \geq G_Y^0), \lambda \in S,$$

where G_X^0, G_Y^0 are selected goal levels for $X(\lambda), Y(\lambda)$ respectively. Unless explicitly stated otherwise, in this paper we shall always use this goal type with a PIPO. Depending on the selected preference, the goal can have various practical interpretations. For instance, assuming K_{in}-order and K_{out}-order are selected for the input and output spaces, associated with $K_{in} \subset R^m$ and $K_{out} \subset R^n$, the precise meaning of the goal associated with a PIPO is then

$$(X(\lambda) - G_X^0 \in -K_{in}, Y(\lambda) - G_Y^0 \in K_{out}), \lambda \in S.$$

Of course one may select other preferences according to needs of an individual application, and set up different types of goals for a PIPO.

Positive Input Negative Output (PINO)

In some situations, an efficient DMU may be expected to yield a lower level of its outputs when its input level is increased. An example of such outputs is the pollution level of a factory, taking investment as the input. Such a DMU is described by a PINO system as defined in Section 2.1.

We define the goal type associated with a PINO as

$$(X(\lambda) \leq G_X^0, Y(\lambda) \leq G_Y^0), \lambda \in S.$$

Unless explicitly stated otherwise, we in this paper shall always use this goal type with a PINO. Again one may select other preference according to needs of an individual application.

Assume K_{in}-order and K_{out}-order are selected for the input and output spaces, associated with $K_{in} \subset R^m$ and $K_{out} \subset R^n$. The precise meaning of the goal associated with (NIPO) is then

$$(X(\lambda) - G_X^0 \in -K_{in}, Y(\lambda) - G_Y^0 \in -K_{out}), \lambda \in S.$$

The type of an input-output system does not depend on the sign of the input or output data, i.e., whether they are positive or negative.

Positive Input Targeted Output (PITO)

In some applications the desired outputs have a targeting type in the sense that the performance of the DMUs will be judged by the closeness of the outputs to a pre-set level (or interval). An example of such outputs is the level of a drug in a patient's blood. Such a DMU is referred to as Targeted Response System. As an example of possible extensions, we set the following goal type for a targeted response system with positive input:

$$(X(\lambda) \leq G_X^0, Y(\lambda) = G_Y^0), \lambda \in S.$$

This type goal has to be treated differently.

Mixed Input Mixed Output (MIMO)

In practical problems it is likely that the desired input-output relations of the DMUs exhibit more than one of the above response types. For instance, it may be that the outputs of the DMUs can be divided into two groups: desirable output1 and undesirable output2. Such a DMU is thus a mixed input and mixed output System (MIMO), as described in Section 2.1. For a power station, it is more practical to consider total amount of electricity produced, total profits, and overall pollution level as the outputs, instead considering only the pollution level. If one takes its staff numbers and investment as the inputs, then this system is a mixed response system.

It is sometimes possible to transfer a Non-PIPO to a PIPO, by redefining the outputs (e.g., by setting new-output = − old-output or new-output= $\frac{1}{old-output}$, etc). However such a transformation may completely change the nature of the original input-output system, so that the classic DEA models may not be suitable for the transferred problems, as no existing DEA models are fully transformation invariant. Such a situation was illustrated in [*Liu and Sharp* 1999]. Not only the efficiency scores (see [*Ali and Seiford* 1990]) but also the classification as efficient or inefficient (see [*Allen* 1998]), if a non-convex technology is used, may be changed by such translations. Hence one should use the original data whenever possible (see [*Pastor* 1996]). Furthermore, it may be very difficult or impossible to transfer between other types of goals and preferences.

Using the preference defined in Section 2.1 for MIMO systems and the PRNP goal type in Section 4.1, we can similarly set goal for a MIMO input-output system. We however omit further details here. An example can be found in [*Liu and Sharp* 1999].

5 Building Block 3 - Performance measure

According to the nature of an application, a suitable measure should be selected to quantify the extra performance achieved by a particular DMU with respect to the initial goal that has been set at the beginning of the evaluation period. Then one could measure the distance of a DMU to the frontier, and use it to quantify the efficiency differences among the DMUs.

It is natural and important to ask whether it is possible to express our preference over the outcomes in terms of numbers so that the larger the number the better the performance, and more importantly to quantify any extra achievement or performance that is beyond the initial goals. There is a school of thought that each outcome has "utility" which should be maximised according to the preference.

To some extents, such extra performances may be measured or reflected by a merit function associated with a particular goal such that the greater the extra performance, the larger (or smaller) the value of the function. Suppose that we select a preference $>$ on a set Z, A merit function $m(\cdot)$ is a function from Z to R^+ such that $m(z_1) > m(z_2)$ if $z_1 > z_2$; that is, it is a (strictly) monotone function on Z. We here only examine some merit functions closely related to DEA models. It therefore seems plausible to require them to satisfy some of the economic efficiency axiomatic conditions. We shall however not discuss this complex issue here (see, e.g., [*Ferrier et al.* 1994]). We shall only examine the case where the goal can be achieved, and wish to know how much extra performance beyond the initial goal is achievable and how to quantify this extra performance by using, for example, a suitable merit function.

5.1 Merit Functions

Let $S \subset R^n$ be a technology set and $\lambda = (\lambda_1, ..., \lambda_n)^t \in S$. Let G^0 be a goal level. Let $X(\lambda) = \sum_1^n \lambda_i X_i$ be a mix of inputs specified by an $\lambda \in S$. Assume that a preference is already selected, and unless explicitly stated otherwise, assume that if $s^+ = s_1^+ e_1 +, ..., + s_m^+ e_m \geq 0$, then $s_i^+ \geq 0$ for $i = 1, 2, ...m$. Note that even for commonly used preferences this may not hold. For the lexicographic preference, if $s^+ \geq 0$, one can only say $s_1^+ \geq 0$. We first examine merit measurements for the PR goal:

$$X(\lambda) = \sum_1^n \lambda_i X_i \geq G^0, \lambda \in S,$$

MF1). Additive Merit Function

Let $w_i > 0$ for $i = 1, 2, ..., m$ be such that $\sum_1^m w_i = 1$. We can define the following additive merit function for a mixture $X(\lambda)$

$$m(X(\lambda), G^0) = \max_{s^+ : X(\lambda) - s^+ \geq G^0} \sum_1^m w_i s_i^+$$

where $s^+ = s_1^+ e_1 +, ..., +s_m^+ e_m$. Often we shall write $m(X, G^0)$ as $m(X)$ as G^0 is always fixed. It is clear that in any K-order (thus including the Pareto-order) this function is strictly increasing for any $X \geq G^0$ in the sense that $m(X) > m(G^0) = 0$ if $X > G^0$. Therefore it is compatible with the commonly used Koopmans's efficiency. It is also clear that $m(X+Y) \leq m(X) + m(Y)$. For the Pareto Preference, we further have

$$m(X(\lambda), G^0) = \max_{s^+ : X(\lambda) - s^+ = G^0} \sum_1^m w_i s_i^+ = \sum_1^m w_i s_i^+$$

Componentwise,

$$m(X(\lambda), G^0) = \sum_{\sum_1^n x_i^1 \lambda_i - s_1^+ = g_1^0, ..., \sum_1^n x_i^m \lambda_i - s_m^+ = g_m^0, s_j^+ \geq 0, j=1,..,m} w_1 s_1^+ + ... w_m s_m^+,$$

Therefore we have the additive property: $m(X+Y) = m(X) + m(Y)$. The additive merit functions have been used in well-known Additive DEA Model.

MF2). Homogeneous Merit Function

Let $\Theta = (\theta_{i,j})_{n \times n}$ be a diagonal matrix such that $\theta_{i,i} \geq 1$. Assume that $g_i^0 > 0$ for $i = 1, 2, ..., m$. Let $w_i > 0$ for $i = 1, 2, ..., m$ be such that $\sum_1^m w_i = 1$. Define

$$m(X(\lambda), G^0) = \max_{\Theta : X(\lambda) \geq \Theta G^0} \sum_1^m w_i \theta_{i,i}.$$

where $G^0 = (g_1^0, ..., g_m^0)^t \in R^m$ is the goal level, and $\geq$ is in the sense of a selected preference. This function is strictly increasing for any $X \geq G^0$, in any K-order. It is clear that this merit function is positive homogeneous. For the Pareto Preference we further have

$$m(X(\lambda)) = \max_{\Theta : X(\lambda) = \Theta G^0} \sum_1^m w_i \theta_{i,i} = \sum_1^m w_i \theta_{i,i}.$$

MF3). Radial Merit Function

Let $\theta \geq 1$. Assume that $g_i^0 > 0$ for $i = 1, 2, ..., m$. Define

$$m(X(\lambda)) = \max_{\theta : X(\lambda) \geq \theta G^0} \theta.$$

This function is NOT strictly monotone under the Pareto Preference, though it is positive homogeneous. Thus although $X > G^0$ in the Pareto Preference (thus X should be better than G^0 in the Koopmans' efficiency notion), we may still have $m(X) = m(G^0) = 0$.

However one can define a new preference to make it strictly monotone as to be seen below.

MF4). Nearly Radial Merit Function

The above radial merit function is unfortunately not strictly monotone, and therefore may not be accurate enough to measure possible extra performance. One can define a nearly radial merit function which is strictly increasing as follows. Let $\theta \geq 1$. Let $s^+ = s_1^+ e_1 +, ..., + s_m^+ e_m$. Assume that $g_i^0 > 0$ for $i = 1, 2, ..., m$. Define

$$m(X(\lambda)) = \max_{\theta, s^+ : X(\lambda) - s^+ = \theta G^0} \theta + \epsilon \sum_1^m s_i^+,$$

where ϵ is a very small positive number. Here ϵ is understood as Non-Archimedean, so that this merit function actually has two components: $(\theta, \sum_1^m s_i^+)$. The second one is needed only when the first one is not able to measure extra performance. Thus strictly speaking, this merit function should be understood as a function from $R^m \to R^2$ with the lexicographic preference being used in R^2. Under Pareto Preference, we have

$$m(X(\lambda)) = \max_{\sum_1^n x_i^1 \lambda_i - s_1^+ = \theta g_1^0, ..., \sum_1^n x_i^m \lambda_i - s_m^+ = \theta g_m^0, s_j^+ \geq 0, \theta \geq 0} \theta + \epsilon (s_1^+ + ... + s_m^+),$$

This merit function has been used in the classic DEA models like CCR and BCC models.

It is important to note that the properties of a merit function much depend on the preference structure. For instance when the lexicographic order is used, the merit function may be very different from what we have seen for the Pareto Preference.

MF5). Additive Merit Function in Lexicographic Order

We have

$$m(X(\lambda)) = x_1(\lambda) - g_1^0, \;\; if \; x_1(\lambda) > g_1^0$$

otherwise

$$m(X(\lambda)) = x_k(\lambda) - g_k^0,$$

if there is some $k \in (2, , , .n)$ so that $y_k > x_k$ and $y_i = x_i$ for $i = 1, .., k-1$. For this merit function we have $X > G^0$ if and only if $m(X) > 0$ in the lexicographic order.

For a NR goal:

$$X(\lambda) \leq G^0, \; \lambda \in S,$$

we can similarly introduce merit functions as for PR gaols. For instance under Pareto Preference, we define the additive merit function:

$$m(X(\lambda)) = \sum_{\sum_1^n x_i^1\lambda_i + s_1^- = g_1^0, \ldots, \sum_1^n x_i^m\lambda_i + s_m^- = g_m^0, s_j^- \geq 0, j=1,..,m} w_1 s_1^- + \ldots w_m s_m^-,$$

and the almost radial merit function

$$m(X(\lambda)) = \max_{\sum_1^n x_i^1\lambda_i + s_1^- = \theta g_1^0, \ldots, \sum_1^n x_i^m\lambda_i + s_m^- = \theta g_m^0, s_j^- \geq 0, \theta \geq 0} 1 - \theta + \epsilon(s_1^- + \ldots. + s_m^-),$$

or

$$m(X(\lambda)) = \min_{\sum_1^n x_i^1\lambda_i + s_1^- = \theta g_1^0, \ldots, \sum_1^n x_i^m\lambda_i + s_m^- = \theta g_m^0, s_j^- \geq 0, \theta \geq 0} \theta - \epsilon(s_1^- + \ldots. + s_m^-),$$

where ϵ is a very small positive number. There are many more possible merit functions that can be introduced to suit a particular application. For instance, one can introduce more nonradial types of merit functions using the Fare-Lovell (or Russell) measure and the Zieschang measure (see [*Fare and Lovell* 1978] and [*Zieschang* 1984]). We now examine the mixture of the above two types.

Mixed Goal Types

In general let

$$G^0 = \begin{pmatrix} G_{PR}^0 \\ G_{NR}^0 \end{pmatrix},$$

and

$$X(\lambda) = \begin{pmatrix} X^{PR}(\lambda) \\ X^{NR}(\lambda) \end{pmatrix}.$$

Then a possible mixture goal setting reads:

$$X^{PR}(\lambda) \geq G_{PR}^0, \ and \ X^{NR}(\lambda) \leq G_{NR}^0.$$

Similarly we can introduce the additive merit function.

$$m(X(\lambda), G^0) = \sum_{X^{PR}(\lambda) - S_{PR} = G_{PR}^0, X^{NR}(\lambda) + S_{NR} = G_{NR}^0} (S_{PR} \cdot W^{PR} + S_{NG} \cdot W^{NR}),$$

where S_{PR} and S_{NR} are nonnegative vectors, and (W^{PR}, W^{NR}) are the positive weight vectors without any zero components. Let us note that up to now, λ is fixed in all the discussions above. We then can go further to find out the maximum extra performance of the whole mixture $X(\lambda), \lambda \in S$ for the given goal level. For instance using the additive merit function, this can naturally be found by solving the following linear programming:

$$\max_{\lambda \in S} m(X(\lambda), G^0) = \max_{X^{PR}(\lambda)-S_{PR}=G^0_{PR}, X^{NR}(\lambda)+S_{NR}=G^0_{NR}, \lambda \in S,} (S_{PR} \cdot W^{PR} + S_{NG} \cdot W^{NR})$$

Clearly this is one of the essential functions for any DEA models. One can also use the almost radial merit function if $G^0_{PR}, G^0_{NR} > 0$ have not any zero components, and find the possible extra performance for the whole mixture $X(\lambda), \lambda \in S$ by solving

$$\max \theta_{PR} + (1 - \theta_{NG}) + \epsilon(|S_{PR}|_1 + |S_{NR}|_1)$$

subject to

$$X^{PR}(\lambda) - S_{PR} = \theta_{PR} G^0_{PR},\ X^{NR}(\lambda) + S_{NR} = \theta_{NR} G^0_{NR}, \lambda \in S, \theta_{PR}, \theta_{NR} \geq 0,$$

where s_{PR} and s_{NR} are nonnegative vectors, and $| \cdot |_1$ is the l^1 norm.

The above merit functions can be readily extended to input-output systems. If we have merit functions $m_X(), m_Y()$ for input X and output Y respectively, then we can simply define

$$m((X,Y),(G^0_X, G^0_Y)) = m_X(X, G^0_X) + m_Y(Y, G^0_Y),$$

where G^0_X, G^0_Y are the goal levels set for X, Y respectively. If one needs to consider input or output orientation, one can define

$$m((X,Y),(G^0_X, G^0_Y)) = m_X(X, G^0_X) + \epsilon m_Y(Y, G^0_Y),$$

or

$$m((X,Y),(G^0_X, G^0_Y)) = \epsilon m_X(X, G^0_X) + m_Y(Y, G^0_Y),$$

where ϵ is a very small positive number, understood as Non-Archimedean.

6 General DEA models

Now suppose we wish to evaluate efficiency for the DMU (X_0, Y_0) among the DMUs (X_i, Y_i). We then first have to select a preference $\leq$, and a performance measurement $m()$ compatible with the preference. Finally we have to decide what production possibility set $P(\{(X_i, Y_i)\})$ to be used in this evaluation. Then to decide if the DMU (X_0, Y_0) is efficient or not, we only need to find out the maximum extra performance of the whole production possibility set $P(\{(X_i, Y_i)\})$ for the given goal level (X_0, Y_0) by solving the following programming (GDEA):

$$max_{(X,Y)\in P(\{(X_i,Y_i)\})} m((X,Y),(X_0,Y_0))$$

This is essentially a DEA model. If the maximum is positive then the DMU (X_0, Y_0) is inefficient, and otherwise it is efficient. Efficiency of the DMU may then be quantified via exploring the quantity $max_{(X,Y)\in P(\{(X_i,Y_i)\})} m((X,Y),(X_0,Y_0))$ according to respective economic theory. An immediate conclusion from our discusses is that for given preference and production set if a DMU is efficient for a particular strictly monotone merit function, then it remains efficient for any strictly monotone merit functions. Only the efficiency scores may be changed.

Clearly there does not exist enough space here to allow us to fully explore this framework, though we shall try to give some illustrations. Assume the Pareto preference is applied on the input-output space for PIPO DMUs (X_i, Y_i) to be evaluated. Assume the convex technology such that

$$P(\{(X_i,Y_i)\}) = \{(X,Y) : X \geq X(\lambda), Y \leq Y(\lambda), \lambda \geq 0, \sum \lambda_i = 1\}.$$

Select the additive merit function with the equal weights. Then the above (GDEA) reads

$$\max_{\sum_0^n X_i\lambda_i + s^- = X_0, \sum_0^n Y_i\lambda_i - s^+ = Y_0, s^+\geq 0, s^-\geq 0, \lambda\geq 0, \sum\lambda_i = 1} \sum_0^m s_i^+ + \sum_0^s s_i^-$$

This is, of course, the Additive DEA model.

If we use the almost radial merit function for the outputs instead and the additive merit function with the equal weights for the inputs, and adopt output orientation, then the above (GDEA) reads:

$$\max_{\theta, s^+, s^-, \lambda} \theta + \epsilon(\sum_0^m s_i^+ + \sum_0^s s_i^-)$$

subject to

$$\sum_0^n X_i\lambda_i + s^- = X_0,\ \sum_0^n Y_i\lambda_i - s^+ = \theta Y_0, s^+ \geq 0, s^- \geq 0, \lambda \geq 0, \sum \lambda_i = 1,$$

This is just the output orientation BCC model. Taking a different production possibility set, one then has the CCR model. If one selects a K-order or the matrix preference defined in Section 2.1, one can then derive the Cone Ratio model in [*Charnes et al*. 1989].

In general, the three building blocks have to be selected according to particular needs of applications. In [*Liu and Sharp* 1999], we derive a DEA model using the almost radial merit function for the inputs and the additive merit function with the equal weights for the outputs to evaluate some medical therapies, where two of the three outputs are undesirable, and some classic transformations were shown to be inappropriate.

Finally we mention how to derive DEA models for more non-standard applications, for example, to evaluate efficiency of the efforts that have been made by various countries to compete Olympic Metals. Here one should not directly apply the classic DEA models since a decision maker' preferences on the three outputs, which are the numbers of gold, silver and brown metals, are different. This problem has been considered in the literature, see [*Lins et al.* 2003] via weight restrictions. Here we think that it is not unreasonable for a decision maker to adopt the lexicographic preference on the outputs. Using the radial merit function in the lexicographic order for the outputs and the additive merit function for the inputs, one can derive new DEA models for such situations to directly reflect the decision maker's preference on the outputs in the new models. However here we omit further details which will be included in a forthcoming paper.

7 Conclusions

Preference, production possibility set and performance measure are three of the essential elements of any DEA models. These three Ps more or less decide a DEA model. In this paper, we have re-examined Data Envelopment Analysis from these three Ps' perspective. We have presented some examples of different preferences and performance measures, and discussed a possible framework to assemble them into DEA models.

Some DEA models with non-classic preferences and goals motivated from real applications will be discussed in forthcoming papers.

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