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Multiple orthogonal polynomials associated with confluent hypergeometric functions

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ABSTRACT. We introduce and analyse a new family of multiple orthogonal polynomials of hypergeometric type with respect to two measures supported on the positive real line which can be described in terms of confluent hypergeometric functions of the second kind. These two measures form a Nikishin system. Our focus is on the multiple orthogonal polynomials for indices on the step line. The sequences of the derivatives of both type I and type II polynomials with respect to these indices are again multiple orthogonal and they correspond to the original sequences with shifted parameters. For the type I polynomials, we provide a Rodrigues-type formula. We characterise the type II polynomials on the step line, also known as d-orthogonal polynomials (where d is the number of measures involved so that here $d = 2$), via their explicit expression as a terminating generalised hypergeometric series, as solutions to a third-order differential equation and via their recurrence relation. The latter involves recurrence coefficients which are unbounded and asymptotically periodic. Based on this information we deduce the asymptotic behaviour of the largest zeros of the type II polynomials. We also discuss limiting relations between these polynomials and the multiple orthogonal polynomials with respect to the modified Bessel weights. Particular choices on the parameters for the 2-orthogonal polynomials under discussion correspond to the cubic components of the already known threefold symmetric Hahn-classical multiple orthogonal polynomials on star-like sets.

Keywords: *Multiple orthogonal polynomials, confluent hypergeometric function, Nikishin system, Rodrigues formula, generalised hypergeometric series, 2-orthogonal polynomials, Hahn-classical, 3-fold symmetric.*

Mathematics Subject Classification 2000: Primary: 33C45, 42C05, Secondary: 33C10, 33C15, 33C20

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1 Introduction and motivation

The main aim of this paper is to investigate the *multiple orthogonal polynomials* with respect to two absolutely continuous measures supported on the positive real line and admitting an integral representation via weight functions $\mathscr{W}(x;a,b;c)$ and $\mathscr{W}(x;a,b;c+1)$ where

$$\mathscr{W}(x;a,b;c)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-1}\,\mathbf{U}(c-b,a-b+1;x)\,. \tag{1.1}$$

with $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$. The weight functions involve the *confluent hypergeometric function of the second kind* $\mathbf{U}(\alpha,\beta;x)$, also known as the *Tricomi function*, which is a solution of the second-order differential equation (see [6, Eq. 13.2.1])

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\beta-x)\frac{\mathrm{d}y}{\mathrm{d}x}-\alpha y=0, \tag{1.2}$$

and, provided that $\mathrm{Re}(\alpha)>0$ and $|\arg(x)|<\dfrac{\pi}{2}$, it admits the integral representation (see [6, Eq. 13.4.4])

$$\mathbf{U}(\alpha,\beta;x)=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(t+1)^{\beta-\alpha-1}e^{-tx}\mathrm{d}t, \tag{1.3}$$

provided that $\mathrm{Re}(\alpha)>0$ and $|\arg(x)|<\dfrac{\pi}{2}$, whilst (see [6, Eqs. 13.2.7 & 13.2.40])

$$\mathbf{U}(0,\beta;x)=1 \quad \text{and} \quad \mathbf{U}(\alpha,\beta;x)=x^{1-\beta}\,\mathbf{U}(\alpha-\beta+1,2-\beta;x)\,.$$

Note that the latter identity implies that $\mathscr{W}(x;a,b;c)=\mathscr{W}(x;b,a;c)$. Furthermore, the conditions $a,b,c\in\mathbb{R}^+$ and $c>\max\{a,b\}$ guarantee the convergence of the integral of the modified Tricomi weight over the positive real line. More precisely (see [12, Eq. (7.621.6)]),

$$\int_0^\infty \mathrm{e}^{-x}x^{a}\,\mathbf{U}(c-b,a-b+1;x)\,\mathrm{d}x=\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}. \tag{1.4}$$

Therefore the weight function $\mathscr{W}(x;a,b;c)$ is a probability density function whose moments are given by

$$\int_0^\infty x^n\,\mathscr{W}(x;a,b;c)\,\mathrm{d}x=\frac{(a)_n\,(b)_n}{(c)_n},\quad n\in\mathbb{N}, \tag{1.5}$$

where, as usual, $(z)_n$ denotes the Pochhammer symbol defined by

$$(z)_0=1 \quad \text{and} \quad (z)_n:=z(z+1)\cdots(z+n-1),\quad \text{if } n\in\mathbb{N}\backslash\{0\}. \tag{1.6}$$

Here and throughout the text, $\mathbb{N}=\mathbb{Z}_0^+=\{0,1,2,\cdots\}$. When referring to $\{P_n(x)\}_{n\in\mathbb{N}}$ as a polynomial sequence it is assumed that P_n is a polynomial of a single variable with degree exactly n and we consistently deal with monic polynomials, unless stated otherwise.

Research on multiple orthogonal polynomials has received a focus of attention in the past decennia, partly motivated by their applicability to different areas of mathematics and mathematical physics. In particular, they have been utilised in the description of rational solutions to Painlevé equations [4] as well as in random matrix theory. For instance, the investigation of singular values of products of Ginibre matrices uses multiple orthogonal polynomials associated with weight functions expressed in terms of Meijer G-functions [18]. If only two measures are involved, then those Meijer G-functions are hypergeometric or confluent hypergeometric functions. This research offers a thorough investigation of a collection of multiple orthogonal polynomials that fits within this category.

Multiple orthogonal polynomials are a generalisation of (standard) orthogonal polynomials. Their orthogonality measures are spread across a vector of $r \in \mathbb{Z}^+$ measures and they are polynomials on a single variable depending on a multi-index $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ of length $|\vec{n}| = n_0 + \cdots + n_{r-1}$. There are two types of multiple orthogonal polynomials with respect to a system of r measures $(\mu_0, \cdots, \mu_{r-1})$.

The *type I multiple orthogonal polynomials* for $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ are given by a vector $\left(A_{\vec{n}}^{(0)}, \cdots, A_{\vec{n}}^{(r-1)}\right)$ of r polynomials, with $\deg A_{\vec{n}}^{(j)} \leq n_j - 1$, for each $0 \leq j \leq r-1$, satisfying the orthogonality and normalisation conditions

$$\sum_{j=0}^{r-1} \int x^k A_{\vec{n}}^{(j)}(x) \mathrm{d}\mu_j(x) = \begin{cases} 0, & \text{if } 0 \leq k \leq |\vec{n}| - 2, \\ 1, & \text{if } k = |\vec{n}| - 1. \end{cases} \tag{1.7}$$

If the measures $\mu_j(x)$ are absolutely continuous with respect to a common positive measure μ, that is, if we can write $\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x)$, for each $0 \leq j \leq r-1$ and for some weight functions $w_j(x)$, then the type I function is

$$Q_{\vec{n}}(x) = \sum_{j=0}^{r-1} A_{\vec{n}}^{(j)}(x)\, w_j(x) \tag{1.8}$$

and the conditions in (1.7) become

$$\int x^k Q_{\vec{n}}(x) \mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \leq k \leq |\vec{n}| - 2, \\ 1, & \text{if } k = |\vec{n}| - 1. \end{cases} \tag{1.9}$$

In the case of $r = 2$ measures, we use the notation $A_{\vec{n}}$ for $A_{\vec{n}}^{(0)}$ and $B_{\vec{n}}$ for $A_{\vec{n}}^{(1)}$.

The *type II multiple orthogonal polynomial* for $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ consists of monic polynomials $p_{\vec{n}}$ of degree $|\vec{n}|$ which satisfies, for each $0 \leq j \leq r-1$, the orthogonality conditions

$$\int x^k p_{\vec{n}}(x) \mathrm{d}\mu_j(x) = 0, \quad 0 \leq k \leq n_j - 1. \tag{1.10}$$

For both types of multiple orthogonality, the case where the number of measures is $r = 1$ corresponds to standard orthogonality. A polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is orthogonal with respect to a measure μ if

$$\int x^k P_n(x) \mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \leq k \leq n-1, \\ N_n \neq 0, & \text{if } n = k. \end{cases} \tag{1.11}$$

The orthogonality conditions for type I and type II multiple orthogonal polynomials give a non-homogeneous system of $|\vec{n}|$ linear equations for the $|\vec{n}|$ unknown coefficients of the vector of polynomials $\left(A_{\vec{n}}^{(0)}, \cdots, A_{\vec{n}}^{(r-1)}\right)$ in (1.7) or the polynomials $P_{\vec{n}}(x)$ in (1.10). If the solution exists, it is unique and the corresponding matrices of the system for type I and type II are the transpose to each other. However it is possible that this system doesn't have a solution, unless further conditions are imposed (unlike standard orthogonality on the real line, the existence of such solutions is not a trivial matter). If there is a unique solution, then the multi-index $\vec{n}$ is called *normal* and if all multi-indices are normal, the system is a *perfect system*.

An example of systems known to be perfect are the Algebraic Tchebyshev systems, or simply *AT-systems*. A vector of measures $(\mu_0, \cdots, \mu_{r-1})$ is an AT-system on an interval Δ for a multi-index $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ if the measures $\mu_j(x)$ are absolutely continuous with respect to a common positive measure μ on Δ, that is, $\mathrm{d}\mu_j(x) = w_j(x)\mathrm{d}\mu(x)$, for each $j = 0, \cdots, r-1$ and for some weight functions $w_j(x)$, and the set of functions

$$\bigcup_{j=0}^{r-1} \left\{w_j(x), xw_j(x), \cdots, x^{n_j-1}w_j(x)\right\}$$

forms a Chebyshev system on Δ, that is, if for any polynomials f_j, $j = 0, \cdots, r-1$, of degree not greater than $n_j - 1$, and not all equal to 0, the function $\sum_{j=0}^{r-1} f_j(x)w_j(x)$ has at most $|\vec{n}| - 1$ zeros on Δ. A vector of measures $(\mu_0, \cdots, \mu_{r-1})$ is an AT-system on an interval Δ if it is an AT-system on Δ for every multi-index in $\mathbb{N}^r$.

Another special example of a perfect system is a *Nikishin system* (first introduced in [25]). We say that two measures (μ_0, μ_1) form a Nikishin system of order 2, if they are both supported on an interval Δ_0 and if there exists a positive measure σ on an interval Δ_1 with $\Delta_0 \cap \Delta_1 = \emptyset$ such that

$$\frac{\mathrm{d}\mu_1(x)}{\mathrm{d}\mu_0(x)} = \int_{\Delta_1} \frac{\mathrm{d}\sigma(t)}{x-t}. \tag{1.12}$$

The definition of a Nikishin system can be generalised to define a Nikishin system of $r > 2$ measures. It was proved in [10] that every Nikishin system is perfect (see also [11] for the cases where the supports of the measures are unbounded or where consecutive intervals touch at one point). More precisely, it is proved in [10] and [11] that every Nikishin system is an AT-system, therefore it is perfect. Moreover, for every AT-system and for any $\vec{n} \in \mathbb{N}^r$, the type I function for $Q_{\vec{n}}$ defined by (1.8) has exactly $|\vec{n}| - 1$ sign changes on Δ and the type II multiple orthogonal polynomial $P_{\vec{n}}$ has $|\vec{n}|$ simple zeros on Δ which satisfy an interlacing property as there is always a zero of $P_{\vec{n}}$ between two consecutive zeros of $P_{\vec{n}+\vec{e}_k}$, for each $0 \leq k \leq r-1$, where $\vec{e}_k \in \mathbb{N}^r$ is the multi-index that has all entries equal to 0 except the entry of index k which is equal to 1. As a Nikishin system is always an AT-system, the same properties hold for Nikishin systems.

The main contribution of this paper is on multi-indices on the step line. A multi-index $(n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ is on the step-line if $n_0 \geq n_1 \geq \cdots \geq n_{r-1} \geq n_0 - 1$ or, equivalently, if there exists $m \in \mathbb{N}$ and $0 \leq j \leq r-1$

such that

$$n_k = \begin{cases} m+1, & \text{if } 0 \le k < j, \\ m, & \text{if } j \le k \le r-1. \end{cases} \tag{1.13}$$

For any $r \in \mathbb{Z}^+$ and for each $n \in \mathbb{N}$, there is a unique multi-index of length n on the step line of $\mathbb{N}^r$. More precisely, if $n = rm + j$, with $m, j \in \mathbb{N}$ and $0 \le j \le r-1$, the multi-index of length n is $\vec{n} = (n_0, \cdots, n_{r-1}) \in \mathbb{N}^r$ with entries as described in (1.13). Hence, when the number of measures is fixed and we only consider multi-indices on the step line, we can replace the multi-index of the multiple orthogonal polynomials of both type I and type II by its length without any ambiguity. When $r = 2$, the indexes on the step line are illustrated in Figure 1.

For the type II multiple orthogonal polynomials on the step line, we obtain a polynomial sequence with exactly one polynomial of degree n for each $n \in \mathbb{N}$. These are often referred to as *d-orthogonal polynomials* (where d is the number of measures, so here $d = r$), as introduced in [21]. A great deal of research in multiple orthogonality is often specialised on d-orthogonality. For this, the corresponding polynomials satisfy higher order recurrence relations and they have been referred in the literature as vector orthogonal polynomials (see [16, 29]), as well as subcases of multidimensional orthogonal polynomials, Hermite-Padé polynomials or simultaneous orthogonal polynomials, among others.

In the case of $r = 2$ measures, the type II multiple orthogonality conditions (1.10) on the step line correspond to say that if we set

$$P_{2n}(x) = P_{n,n}(x) \quad \text{and} \quad P_{2n+1}(x) = P_{n+1,n}(x), \tag{1.14}$$

then the polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is 2-*orthogonal* with respect to a pair of measures (μ_0, μ_1):

$$\int x^k P_n(x) \mathrm{d}\mu_0(x) = \begin{cases} 0, & \text{if } n \ge 2k+1, \\ N_n \ne 0, & \text{if } n = 2k, \end{cases} \quad \text{and} \quad \int x^k P_n(x) \mathrm{d}\mu_1(x) = \begin{cases} 0, & \text{if } n \ge 2k+2, \\ N_n \ne 0, & \text{if } n = 2k+1. \end{cases} \tag{1.15}$$

In (1.14) and throughout we have considered the step line to be the lower step line as illustrated in Figure 1. If we were to consider the polynomials on the upper step line, then these happened to be 2-orthogonal with respect to the vector of measures (μ_1, μ_0).

There is a well-known connection between orthogonal polynomials and recurrence relations. The spectral theorem for orthogonal polynomials (also known as Shohat-Favard theorem) states that a polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ is orthogonal with respect to some measure μ if and only if it satisfies a second order recurrence relation of the form

$$p_{n+1}(x) = (x - \beta_n) p_n(x) - \gamma_n p_{n-1}(x), \tag{1.16}$$

with $\gamma_n \ne 0$, for all $n \ge 1$, and initial conditions $p_{-1} = 0$ and $p_0 = 1$. Moreover, if $\beta_n \in \mathbb{R}$ and $\gamma_{n+1} > 0$, for all $n \in \mathbb{N}$, then μ is a positive measure on the real line.

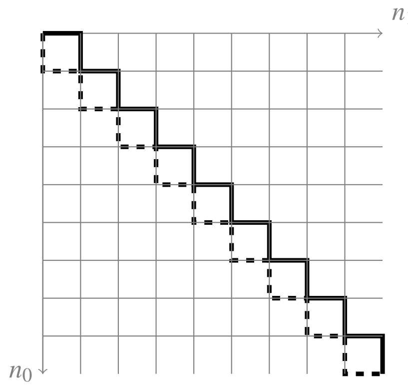

Figure 1: Upper and lower step line for the multi-index (n_0, n_1) in solid and dashed black line, respectively, when $r = 2$.

Multiple orthogonal polynomials also satisfy recurrence relations, known as nearest neighbour recurrence relations (see, for instance, [27]). In particular, when the multi-indexes lie on the step line, a polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is r-orthogonal if and only if it satisfies a higher order recurrence relation (precisely of order $r+1$) of the form

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \sum_{j=1}^{r} \gamma_{n+1-j}^{[j]} P_{n-j}(x), \tag{1.17}$$

with $\gamma_n^{[r]} \neq 0$, for all $n \geq 1$, and initial conditions $P_{-(r-1)} = \cdots = P_{-1} = 0$ and $P_0 = 1$, see [21, Th. 2.1]. Naturally, when $r = 2$, the relation (1.17) reduces to the third order recurrence relation

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \alpha_n P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x), \tag{1.18}$$

with $\gamma_n \neq 0$, for all $n \geq 1$, and initial conditions $P_{-2} = P_{-1} = 0$ and $P_0 = 1$. As such, a polynomial sequence $\{P_n(x)\}_{n\geq 0}$ satisfying the latter recurrence relation is 2-orthogonal, and therefore a pair of measures (μ_0, μ_1) exists so that (1.15) holds.

For the type I multiple orthogonal polynomials on the step line for $r = 2$ measures, we have, for each $n \in \mathbb{N}$,

$$\deg(A_n) \leq \left\lfloor \frac{n-1}{2} \right\rfloor \quad \text{and} \quad \deg(B_n) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1,$$

that is, $\deg(A_n) = m-1$, if $n = 2m$ or $n = 2m-1$, and $\deg(B_n) = m-1$, if $n = 2m$ or $n = 2m+1$. Moreover, assuming that there exists a positive measure μ and a pair of weight functions (w_0, w_1) such that we can write $\mathrm{d}\mu_0(x) = w_0(x)\mathrm{d}\mu(x)$ and $\mathrm{d}\mu_1(x) = w_1(x)\mathrm{d}\mu(x)$, the type I function is

$$Q_n(x) = A_n(x)w_0(x) + B_n(x)w_1(x) \tag{1.19}$$

and the orthogonality and normalisation conditions correspond to

$$\int x^k Q_n(x)\mathrm{d}\mu(x) = \begin{cases} 0, & \text{if } 0 \leq k \leq n-2, \\ 1, & \text{if } k = n-1. \end{cases} \tag{1.20}$$

For further information about multiple orthogonal polynomials and Nikishin systems, we refer to [14, Ch. 23] and [19].

In Section 2, we prove that the weight functions $\mathscr{W}(x;a,b;c)$ and $\mathscr{W}(x;a,b;c+1)$ defined in (1.1) form a Nikishin system. This readily imply that the multiple orthogonal polynomials of both type I and type II with respect to these weight functions exist and are unique for every multi-index $\vec{n}=(n_0,n_1)\in\mathbb{N}^2$ and their zeros satisfy the properties mentioned before for Nikishin systems (and AT-systems in general). Then we obtain differential equations satisfied by these weight functions, which we use to deduce differential properties for the multiple orthogonal polynomials of both type I and type II on the step line (see Theorem 2.9) and a Rodrigues-type formula for the type I polynomials (see Theorem 2.10).

Section 3 is devoted to the characterisation of the type II multiple orthogonal polynomials on the step-line (ie, the 2-*orthogonal* polynomial sequences). A remarkable property of these polynomials (a straightforward consequence of Theorem 2.9) is that they satisfy the so called *Hahn's property*, meaning that the sequence of its derivatives is again 2-orthogonal. As such, they stand as an example of a Hahn-classical 2-orthogonal family. Our detailed characterisation of these polynomials includes: an explicit expression for these polynomials as a terminating generalised hypergeometric series, more precisely a ${}_2F_2$ (see Theorem 3.1); explicit third order differential equation (in Theorem 3.3) as well as a third order recurrence relation (in Theorem 3.4) to which these type II polynomials on the step-line are a solution; an asymptotic upper bound for their largest zeros; limiting relations between these polynomials and multiple orthogonal polynomials with respect to weight functions involving the modified Bessel function of second kind $\mathrm{K}_\nu(x)$ (see (3.11)) studied in [3] and [28]. It turns out that each of the sequences of recurrence coefficients is unbounded and asymptotically periodic of period 2. As such, we believe this is the first explicit example of a Nikishin system associated with such periodic unbounded recurrence coefficients.

Earlier we mentioned generalised hypergeometric series, which are formally defined by

$$ {}_pF_q\left(\begin{matrix}\alpha_1,\cdots,\alpha_p\\ \beta_1,\cdots,\beta_q\end{matrix};z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}\,, \tag{1.21}$$

where $p,q\in\mathbb{N}$, $z,\alpha_1,\cdots,\alpha_p\in\mathbb{C}$ and $\beta_1,\cdots,\beta_p\in\mathbb{C}\backslash\{-n:\ n\in\mathbb{N}\}$. If one of the parameters $\alpha_1,\cdots,\alpha_p$ is a negative integer the series (1.21) terminates and defines a polynomial.

In Section 4 we explain that particular cases of the type II polynomials on the step line, characterised here, have appeared in [20] as the components of 3-fold symmetric Hahn-classical 2-orthogonal polynomials on star-like sets. A polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ is said to be 3-*fold symmetric* if, for any $n\in\mathbb{N}$,

$$P_n\left(\mathrm{e}^{\frac{2\pi i}{3}}x\right)=\mathrm{e}^{\frac{2n\pi i}{3}}P_n(x)\text{ and }P_n\left(\mathrm{e}^{\frac{4\pi i}{3}}x\right)=\mathrm{e}^{\frac{4n\pi i}{3}}P_n(x).$$

This property is commonly referred to as 2-symmetry, as introduced in [7, Definition 5.1]. We opt to follow the terminology in [20], as it gives a better picture of the intrinsic symmetry. The definition is equivalent to say that there exist three polynomial sequences $\left\{P_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$, each supra indexed with $k\in\{0,1,2\}$, which are called the cubic components of $\{P_n(x)\}_{n\in\mathbb{N}}$, such that

$$P_{3n+k}(x)=x^kP_n^{[k]}(x^3),\quad\text{for all}\quad n\in\mathbb{N}. \tag{1.22}$$

Whenever a 3-*fold symmetric* is 2-orthogonal, then each of its cubic components are 2-orthogonal polynomials [7, Théorème 5.2]. In this section we give a result of independent interest, Theorem 4.1, where we show that all the three cubic components of 3-fold symmetric Hahn-classical 2-orthogonal polynomials are themselves Hahn-classical.

The main contribution of this paper are the results in Sections 2 and 3 characterising multiple orthogonal polynomials with respect to the Nikishin system. The centre of the analysis is for the indices on the upper and lower step line (see Fig 1). The study of the multiple orthogonal polynomials with respect to the same system for indices out of the step line and, in particular, the study of the (standard) orthogonal polynomials with respect to the weight function $\mathscr{W}(x;a,b;c)$, defined by (1.1), remains an open (and challenging) problem. Partly this is due to the fact that when the weight function is a solution to a second order differential equation, then known techniques to obtain closed or explicit formulas for recurrence coefficients of standard orthogonal polynomials is, up to now, an onerous task. An example of such weights are those studied here and given in (1.1) or those expressed in terms of Bessel functions in (3.11). Notwithstanding, a deep grasp of the multiple orthogonal polynomials on the step line is at the core of applications. The present investigation focus essentially on the latter.

2 Multiple orthogonality

The starting point of this investigation is on the weight function $\mathscr{W}(x;a,b;c)$ in (1.1). The goal is to describe a system of multiple orthogonal polynomials with respect to $\mathscr{W}(x;a,b;c)$ and $\mathscr{W}(x;a,b;c+1)$. The first question to address is on whether such a system exists and, if so, whether it is unique. We are able to answer afirmatively to both issues because we are dealing with a Nikishin system of measures, as explained in Section 2.1. From this we want to move on to the characterisation of such system of polynomials. We succeed in doing so for the case where the indices lie on the step-line. Using key differential properties for the vector of weights, derived in Section 2.2, we obtain differential properties for the corresponding polynomials of both types in Section 2.3. We continue the analysis by providing a Rodrigues-type formula for the type I functions in Section 2.4. Concerning the type II, we defer their investigation to Section 3.

2.1 Nikishin system

The vector of weight functions $\left[\mathscr{W}(x;a,b;c)\,,\mathscr{W}(x;a,b;c+1)\right]$ forms a Nikishin system, as stated in Theorem 2.1. An important consequence of this result is that both type I and II multiple orthogonal polynomials with respect to the weight functions appearing on Theorem 2.1 exist and are unique for any multi-index $(n_0,n_1)\in\mathbb{N}^2$. Moreover, the type I multiple orthogonal polynomials $A_{(n_0,n_1)}$ and $B_{(n_0,n_1)}$ have degree exactly n_0-1 and n_1-1, respectively, and the type II multiple orthogonal polynomial $P_{(n_0,n_1)}$ has n_0+n_1 positive real simple zeros that satisfy an interlacing property: there is always a zero of $P_{(n_0,n_1)}$ between two consecutive zeros of $P_{(n_0+1,n_1)}$ or $P_{(n_0,n_1+1)}$.

On the one hand, the Nikishin property can be deduced through the connection between continued fractions and Stieltjes transforms, by guaranteeing the existence of a generating measure, as described in (1.12) . On the other hand, this property can be proved by providing an integral representation for the generating measure σ in (1.12) for this Nikishin system as given in Proposition 2.2.

To start with, we recall some properties from continued fractions and we follow the notation in [5] to describe a continued fraction:

$$\mathop{\mathbf{K}}_{n=0}^{\infty}\left(\frac{\alpha_n}{\beta_n}\right) := \cfrac{\alpha_0}{\beta_0+\cfrac{\alpha_1}{\beta_1+\cfrac{\alpha_2}{\beta_2+\cdots}}}. \tag{2.1}$$

Particularly relevant are the so-called *S-fractions* or *Stieltjes continued fractions*, which are obtained if in (2.1) we set, for any $n,k\in\mathbb{N}$, $\alpha_n=1$ and $\beta_{2k}=a_{2k}\,z$ and $\beta_{2k+1}=a_{2k+1}$, with $a_{2k},a_{2k+1}\in\mathbb{R}^+$, to obtain

$$\cfrac{1}{a_0z+\cfrac{1}{a_1+\cfrac{1}{a_2z+\cfrac{1}{a_3+\cdots}}}}\,. \tag{2.2}$$

Stieltjes showed in [26] that S-fractions can be represented as a Stieltjes transform of a measure with support in $(-\infty,0]$, that is, an integral of the form

$$\int_{-\infty}^{0}\frac{\mathrm{d}\sigma(-t)}{x-t}=\int_{0}^{\infty}\frac{\mathrm{d}\sigma(u)}{x+u}, \tag{2.3}$$

where σ is a non decreasing bounded function such that $\sigma(0)=0$ and $\lim\limits_{u\to\infty}\sigma(u)=\dfrac{1}{a_0}$.

Another special type of continued fractions, known as *J-fraction* and introduced by Jacobi, is obtained if, for some $c_n,b_n\in\mathbb{C}$, we set, in (2.1), $\alpha_0=c_0$, $\alpha_n=-c_n$, for any $n\geq 1$, and $\beta_n=z+b_n$, for all $n\in\mathbb{N}$, to obtain

$$\cfrac{c_0}{z+b_0-\cfrac{c_1}{z+b_1-\cfrac{c_2}{z+b_2-\cdots}}}. \tag{2.4}$$

If every $c_n,b_n\in\mathbb{R}^+$ then the J-fraction generated by them can be obtained by contraction from a S-fraction (see [26]) and, as a result, it can also be represented as a Stieltjes transform with support in $(-\infty,0]$ as in (2.3). Moreover, if (2.4) was obtained from (2.2) by contraction then $c_0=\dfrac{1}{a_0}$ hence $\lim\limits_{u\to\infty}\sigma(u)=c_0$ when we represent the continued fraction (2.4) as in (2.3). These results about continued fractions and Stieltjes transforms can also be found in [30, Ch. 13] and they are used here to prove the following result.

Theorem 2.1. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, let $\mathscr{W}(x;a,b;c)$ be defined by* (1.1)*, we have*

$$\frac{\mathscr{W}(x;a,b;c+1)}{\mathscr{W}(x;a,b;c)} = \frac{-c}{(c-a)(c-b)} \mathop{\mathbf{K}}_{n=0}^{\infty} \left(\frac{-(n+c-a)(n+c-b)}{x+2n+2c-a-b+1} \right), \tag{2.5}$$

and there exists a probability density function σ in $\mathbb{R}^+$ such that

$$\frac{\mathscr{W}(x;a,b;c+1)}{\mathscr{W}(x;a,b;c)} = c \int_{-\infty}^{0} \frac{\mathrm{d}\sigma(-t)}{x-t}. \tag{2.6}$$

As a result, the vector of weight functions $\left[\mathscr{W}(x;a,b;c), \mathscr{W}(x;a,b;c+1)\right]$ forms a Nikishin system in $\mathbb{R}^+$.

Proof. Following the definition of $\mathscr{W}(x;a,b;c)$,

$$\frac{\mathscr{W}(x;a,b;c+1)}{\mathscr{W}(x;a,b;c)} = \frac{c\,\mathbf{U}(c-b+1,a-b+1;x)}{\mathbf{U}(c-b,a-b+1;x)}. \tag{2.7}$$

According to [6, Eq. 13.3.7], we have

$$(c-a)(c-b)\mathbf{U}(c-b+1,a-b+1;x) = (x+2c-a-b-1)\mathbf{U}(c-b,a-b+1;x) - \mathbf{U}(c-b-1,a-b+1;x), \tag{2.8}$$

hence

$$\frac{\mathscr{W}(x;a,b;c+1)}{\mathscr{W}(x;a,b;c)} = \frac{c}{(c-a)(c-b)} \left(x+2c-a-b-1 - \frac{\mathbf{U}(c-b-1,a-b+1;x)}{\mathbf{U}(c-b,a-b+1;x)} \right). \tag{2.9}$$

Furthermore, based on [5, Eq. 16.1.20], we also have

$$\frac{\mathbf{U}(c-b-1,a-b+1;x)}{\mathbf{U}(c-b,a-b+1;x)} = x+2c-a-b-1 + \mathop{\mathbf{K}}_{n=0}^{\infty} \left(\frac{-(n+c-a)(n+c-b)}{x+2n+2c-a-b+1} \right). \tag{2.10}$$

Combining the two latter equations, we derive (2.5). Moreover, $n+c-a, n+c-b, 2n+2c-a-b+1 > 0$, for any $n \in \mathbb{N}$, because $c > \max\{a,b\}$. Therefore, (2.5) implies (2.6), which in turn shows that the vector of measures $\left[\mathscr{W}(x;a,b;c), \mathscr{W}(x;a,b;c+1)\right]$ forms a Nikishin system. $\square$

The generating measure σ in (2.6) can be found via the Stieltjes-Perron inversion formula. As such, we have

$$\frac{\mathrm{d}\sigma(t)}{\mathrm{d}t} = g(t) = \lim_{\varepsilon \to 0^+} \frac{c\,G(-t-i\varepsilon) - c\,G(-t+i\varepsilon)}{2\pi i} \tag{2.11}$$

where

$$G(x) = \frac{\mathscr{W}(x;a,b;c+1)}{c\mathscr{W}(x;a,b;c)} = \frac{\mathbf{U}(c-b+1,a-b+1;x)}{\mathbf{U}(c-b,a-b+1;x)},$$

and it is well defined if $c > \max\{a,b\}$ where $\mathbf{U}(c-b,a-b+1;x)$ has no zeros in the sector $|\arg x| < \pi$ (see [6, §13.9(ii)]). We use the identity [6, Eq. 13.3.10])

$$x\mathbf{U}(c-b+1,a-b+2;x) = \mathbf{U}(c-b,a-b+1;x) + (a-c)\mathbf{U}(c-b+1,a-b+1;x) \tag{2.12}$$

followed by the expression for the derivative of the function $\mathbf{U}(\alpha,\beta+1;x)$ [6, Eq. 13.3.22], to write

$$G(x)=\frac{1}{\alpha(\alpha-\beta)}\left(x\,\frac{\mathbf{U}'(\alpha,\beta+1;x)}{\mathbf{U}(\alpha,\beta+1;x)}+\alpha\right),$$

where $\alpha=c-b$ and $\beta=a-b$. Hence, (2.11) reads as

$$\begin{aligned}\frac{1}{c}g(t)&=\lim_{\varepsilon\to0^+}\frac{G(\mathrm{e}^{-i\pi}(t+i\varepsilon))-G(\mathrm{e}^{i\pi}(t-i\varepsilon))}{2\pi i}\\&=\lim_{\varepsilon\to0^+}\left(\frac{\mathrm{e}^{-\pi i}(t+i\varepsilon)\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\varepsilon))\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon)\right)}{2\pi i\alpha(\alpha-\beta)\left|\mathbf{U}(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon))\right|^2}\right.\\&\qquad\left.-\frac{\mathrm{e}^{\pi i}(t-i\varepsilon)\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\varepsilon)\right)\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon))}{2\pi i\alpha(\alpha-\beta)\left|\mathbf{U}(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon))\right|^2}\right).\end{aligned}$$

In [15, (3.4)-(3.5)], it was shown that, for noninteger values of β, we have

$$\lim_{\varepsilon\to0^+}\left(\frac{\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\varepsilon))\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon)\right)-\mathbf{U}\left(\alpha,\beta+1;\mathrm{e}^{-\pi i}(t+i\varepsilon)\right)\mathbf{U}'(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon))}{2\pi i\left|\mathbf{U}(\alpha,\beta+1;\mathrm{e}^{\pi i}(t-i\varepsilon))\right|^2}\right)$$

$$=\frac{-t^{-(\beta+1)}\,\mathrm{e}^{-t}}{\Gamma(\alpha)\Gamma(\alpha-\beta)\left|\mathbf{U}(\alpha,\beta+1;\mathrm{e}^{\pi i}t)\right|^2}.$$

The latter was obtained by expressing the function $\mathbf{U}$ as a linear combination of two independent solutions to the confluent differential equation as in [6, Eq. 13.2.42] to then use the expression for the wronskian of those two functions given in [6, Eq. 13.2.34]. As a result, we conclude

$$\frac{1}{c}g(t)=\frac{t^{\beta}\mathrm{e}^{-t}}{\Gamma(\alpha+1)\Gamma(\alpha-\beta+1)\left|\mathbf{U}(\alpha,\beta+1;\mathrm{e}^{\pi i}t)\right|^2},$$

from which we deduce the following result.

Proposition 2.2. *For $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$ and $a-b\notin\mathbb{Z}$, the relation* (2.6) *can be written as*

$$\frac{\mathscr{W}(x;a,b;c+1)}{\mathscr{W}(x;a,b;c)}=\frac{c\,\boldsymbol{U}(c-b+1,a-b+1;x)}{\boldsymbol{U}(c-b,a-b+1;x)}=\int_{-\infty}^{0}\frac{c(-t)^{b-a}e^{t}\left|\boldsymbol{U}(c-b,a-b+1;t)\right|^{-2}\mathrm{d}t}{(x-t)\Gamma(c-b+1)\Gamma(c-a+1)},\tag{2.13}$$

where $\mathscr{W}(x;a,b;c)$ is given by (1.1)*.*

2.2 Differential properties of the weight functions

From this point forth, we will index the vector of weights $\left[\mathscr{W}(x;a,b;c),\mathscr{W}(x;a,b;c+1)\right]$ with an extra parameter $d\in\{0,1\}$, by considering

$$\overline{\mathscr{W}}^{[d]}(x;a,b;c):=\begin{bmatrix}\mathscr{W}(x;a,b;c+d)\\\mathscr{W}(x;a,b;c+1-d)\end{bmatrix}.\tag{2.14}$$

The parameter $d \in \{0,1\}$ embodies the flip between the lower and the upper step line indexes of the corresponding multiple orthogonal polynomials of both types. As a consequence, if $\left\{P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ is the monic 2-orthogonal polynomial sequence and $Q_n^{[d]}(x;a,b;c)$ the type I function for the index of length n on the step-line for $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, then

$$P_{2n}^{[1]}(x;a,b;c) = P_{2n}^{[0]}(x;a,b;c) \quad \text{and} \quad Q_{2n}^{[1]}(x;a,b;c) = Q_{2n}^{[0]}(x;a,b;c).$$

There are further motivations for the introduction of this parameter d. Under the action of the derivative operator, the multiple orthogonal system for $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ bounces from the lower to the upper step line (and reciprocally) with shifted parameters, as perceivable in Theorem 2.9. A result that comes as a consequence of Theorem 2.5 where the vector of weights (2.14) is described as a solution to a matrix first order differential equation. Its structure fits into the category of Hahn-classical type vector of weights, in the sense expounded in [7]. Beforehand, in Proposition 2.3 we describe the weight function $\mathscr{W}(x;a,b;c)$ in (1.1) as a solution to a second-order differential equation.

Proposition 2.3. *Let $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$. Then $\mathscr{W}(x;a,b;c)$ defined in* (1.1) *satisfies the differential equation*

$$x^2\mathscr{W}''(x;a,b;c) + (x-(a+b-3))x\mathscr{W}'(x;a,b;c) + ((a-1)(b-1)-(c-2)x)\mathscr{W}(x;a,b;c) = 0. \tag{2.15}$$

Proof. To simplify the notation, let $\mathbf{U}(x) := \mathbf{U}(c-b,a+1-b;x)$ and $\mathscr{W}(x) := \mathscr{W}(x;a,b;c)$.

We differentiate (1.1) with respect to x to obtain

$$\mathscr{W}'(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-2}\left(x\mathbf{U}'(x) + (a-1-x)\mathbf{U}(x)\right). \tag{2.16}$$

Another differentiation brings

$$\mathscr{W}''(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-3}\Big(x^2\mathbf{U}''(x) + 2(a-1-x)x\mathbf{U}'(x) + \left(x^2-2(a-1)x+(a-1)(a-2)\right)\mathbf{U}(x)\Big).$$

Based on (1.2), we have

$$x\mathbf{U}''(x) = (x-a-1+b)\mathbf{U}'(x) + (c-b)\mathbf{U}(x)$$

so that

$$\mathscr{W}''(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-3}\Big((a+b-3-x)x\mathbf{U}'(x) + \left(x^2+(c-b-2a-2)x+(a-1)(a-2)\right)\mathbf{U}(x)\Big).$$

Finally, combining the latter expression, (2.16) and the definition of $\mathscr{W}(x)$, we deduce (2.15). $\square$

To prove Theorem 2.5 we need the following result, which allows us to write $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathscr{W}(x;a,b;c+d)\right)$, with $d \in \{0,1\}$, as a linear combination of $\mathscr{W}(x;a,b;c)$ and $\mathscr{W}(x;a,b;c+1)$.

Lemma 2.4. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, let $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, $d \in \{0,1\}$, be defined as in* (2.14). *Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathscr{W}(x;a,b;c)\right) = -(x+c-a-b)\mathscr{W}(x;a,b;c) + \frac{(c-a)(c-b)}{c}\mathscr{W}(x;a,b;c+1) \tag{2.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathscr{W}(x;a,b;c+1)\right) = c\mathscr{W}(x;a,b;c+1) - c\mathscr{W}(x;a,b;c). \tag{2.18}$$

Proof. From [6, Eq. 13.3.22],

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\mathbf{U}(c-b,a-b+1;x)\Big) = (b-c)\,\mathbf{U}(c-b+1,a-b+2;x)\,.$$

Therefore, recalling (2.16), we obtain

$$\mathscr{W}'(x;a,b;c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-2}\Big((b-c)x\mathbf{U}(c-b+1,a-b+2;x) + (a-1-x)\,\mathbf{U}(c-b,a-b+1;x)\Big).$$

and (2.12) implies

$$\mathscr{W}'(x;a,b;c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\,\mathrm{e}^{-x}x^{a-2}\Big((c-a)(c-b)\mathbf{U}(c-b+1,a-b+1;x) - (x+c-a-b+1)\,\mathbf{U}(c-b,a-b+1;x)\Big),$$

which leads to (2.17).

The shift $c \to c+1$ in (2.17) brings

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\mathscr{W}(x;a,b;c+1)\right) = -(x+c-a-b+1)\mathscr{W}(x;a,b;c+1) + \frac{(c-a+1)(c-b+1)}{c+1}\mathscr{W}(x;a,b;c+2).$$

Using (2.8) with $c \to c+1$, the last term in the latter expression becomes

$$\frac{(c-a+1)(c-b+1)}{c+1}\mathscr{W}(x;a,b;c+2) = (x+2c-a-b+1)\mathscr{W}(x;a,b;c+1) - c\mathscr{W}(x;a,b;c).$$

and, combining the two latter equations, we derive (2.18). □

Based on the previous, we can write the vector of weights $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ as a solution to a matrix first order equation of Pearson type. More precisely, we have:

Theorem 2.5. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, let $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, $d \in \{0,1\}$, be defined as in* (2.14). *Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\Phi^{[d]}(x;a,b;c)\overline{\mathscr{W}}^{[d]}(x;a,b;c)\right) + \Psi^{[d]}(x;a,b;c)\overline{\mathscr{W}}^{[d]}(x;a,b;c) = 0, \tag{2.19}$$

where

$$\Phi^{[0]}=\begin{bmatrix}0 & \dfrac{c+1}{ab}\\ \dfrac{c}{ab} & 0\end{bmatrix} \quad \textit{and} \quad \Psi^{[0]}(x)=\begin{bmatrix}\dfrac{c(c+1)}{ab} & -\dfrac{c(c+1)}{ab}\\ \dfrac{c}{ab}(x+c-a-b) & -\dfrac{(c-a)(c-b)}{ab}\end{bmatrix};$$

$$\Phi^{[1]}(x)=\begin{bmatrix}\dfrac{c+1}{ab} & 0\\ \dfrac{(c+1)(c+2)(x+2c-a-b+1)}{ab(c-a+1)(c-b+1)} & -\dfrac{c(c+1)(c+2)}{ab(c-a+1)(c-b+1)}\end{bmatrix} \quad \textit{and}$$

$$\Psi^{[1]}(x)=\begin{bmatrix}-\dfrac{c(c+1)}{ab} & \dfrac{c(c+1)}{ab}\\ -\dfrac{(c+1)^2(c+2)}{ab(c-a+1)(c-b+1)}\left(x+c-\dfrac{ab}{c+1}\right) & \dfrac{c(c+1)^2(c+2)}{ab(c-a+1)(c-b+1)}\end{bmatrix}.$$

Moreover,

$$x\Phi^{[d]}(x;a,b;c)\overline{\mathscr{W}}^{[d]}(x;a,b;c)=\overline{\mathscr{W}}^{[1-d]}(x;a+1,b+1;c+1+d). \tag{2.20}$$

Proof. To simplify the notation, let $\overline{\mathscr{W}}^{[d]}(x)=\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, $d\in\{0,1\}$. Then equations (2.17) and (2.18) in Lemma 2.4 can be rewritten, for $d\in\{0,1\}$, as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\overline{\mathscr{W}}^{[d]}(x;a,b;c)\right)=-\Omega^{[d]}(x)\overline{\mathscr{W}}^{[d]}(x;a,b;c),$$

with

$$\Omega^{[0]}(x)=\begin{bmatrix}-(x+c-a-b) & \dfrac{(c-a)(c-b)}{c}\\ -c & c\end{bmatrix} \quad \text{and} \quad \Omega^{[1]}(x)=\begin{bmatrix}c & -c\\ \dfrac{(c-a)(c-b)}{c} & -(x+c-a-b)\end{bmatrix}.$$

Multiplying the expressions for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\overline{\mathscr{W}}^{[d]}(x;a,b;c)\right)$ by $\Phi^{[d]}(x)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\Phi^{[d]}(x)\,\overline{\mathscr{W}}^{[d]}(x;a,b;c)\right)=\left(x\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi^{[d]}(x)\right)-\Phi^{[d]}(x)\Omega^{[d]}(x)\right)\overline{\mathscr{W}}^{[d]}(x),$$

which corresponds to (2.19), after observing that, for both $d\in\{0,1\}$,

$$\Psi^{[d]}(x)=\Phi^{[d]}(x)\Omega^{[d]}(x)-x\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi^{[d]}(x)\right).$$

Now, let

$$\begin{bmatrix}\mathscr{V}_0^{[d]}(x)\\ \mathscr{V}_1^{[d]}(x)\end{bmatrix}=x\Phi^{[d]}(x)\begin{bmatrix}\mathscr{W}(x;a,b;c+d)\\ \mathscr{W}(x;a,b;c+1-d)\end{bmatrix}, \quad d\in\{0,1\}.$$

In order to prove (2.20), we need to check that

$$\begin{bmatrix} \mathscr{V}_0^{[d]}(x) \\ \mathscr{V}_1^{[d]}(x) \end{bmatrix} = \begin{bmatrix} \mathscr{W}(x;a+1,b+1;c+2) \\ \mathscr{W}(x;a+1,b+1;c+1+2d) \end{bmatrix}, \quad d \in \{0,1\}.$$

Indeed, we have

$$\mathscr{V}_0^{[1]}(x) = \mathscr{V}_0^{[0]}(x) = \frac{c+1}{ab}\, x\mathscr{W}(x;a,b;c+1) = \frac{\Gamma(c+2)}{\Gamma(a+1)\Gamma(b+1)} \mathrm{e}^{-x}x^a\, \mathbf{U}(c-b+1,a-b+1;x),$$

hence

$$\mathscr{V}_0^{[1]}(x) = \mathscr{V}_0^{[0]}(x) = \mathscr{W}(x;a+1,b+1;c+2),$$

as well as

$$\mathscr{V}_1^{[0]}(x) = \frac{c}{ab}\, x\mathscr{W}(x;a,b;c) = \frac{\Gamma(c+1)}{\Gamma(a+1)\Gamma(b+1)} \mathrm{e}^{-x}x^a\, \mathbf{U}(c-b,a-b+1;x) = \mathscr{W}(x;a+1,b+1;c+1).$$

and

$$\begin{aligned} \mathscr{V}_1^{[1]}(x) &= \frac{(c+1)(c+2)x}{ab(c-a+1)(c-b+1)} \Big((x+2c-a-b+1)\mathscr{W}(x;a,b;c+1) - c\mathscr{W}(x;a,b;c) \Big) \\ &= \frac{\Gamma(c+3)}{\Gamma(a+1)\Gamma(b+1)} \mathrm{e}^{-x}x^a\, \frac{(x+2c-a-b+1)\mathbf{U}(c-b+1,a-b+1;x) - \mathbf{U}(c-b,a-b+1;x)}{(c-a+1)(c-b+1)}, \end{aligned}$$

which, recalling (2.8) (with the shift $c \to c+1$), can be rewritten as

$$\mathscr{V}_1^{[1]}(x) = \frac{\Gamma(c+3)}{\Gamma(a+1)\Gamma(b+1)} \mathrm{e}^{-x}x^a\, \mathbf{U}(c-b+2,a-b+1;x) = \mathscr{W}(x;a+1,b+1;c+3).$$

□

2.3 Differential properties of the multiple orthogonal polynomials

The main result of this section is Theorem 2.9, where we present a differential relation for multiple orthogonal polynomials on the step line of type II in (2.25) and of type I in (2.26). More precisely, we show that the differentiation with respect to the variable gives a shift on the parameters as well as on the index. Therefore, we can see these polynomials as part of the Hahn-classical family, since both type II and type I multiple orthogonal polynomials on the step line satisfy the Hahn-classical property.

To derive this theorem, we first prove Propositions 2.6 and 2.7 that give us differential properties for type II and type I multiple orthogonal polynomials on the step line in more general contexts. Proposition 2.6 is a consequence of the alternative characterisation of the Hahn-classical property for 2-orthogonal polynomials (ie multiple orthogonal polynomials on the step line) derived by Douak and Maroni in [7] (see also [8, Théorème 3.1] or [23, Proposition 6.2]). Here, we present an alternative proof, restricting ourselves to the use of weight functions instead of linear functionals. Incidentally, evoking similar arguments, Proposition 2.7 is an analogous result for type I polynomials, which we believe to be new.

Proposition 2.6. *Let $\overline{w}(x) = \begin{bmatrix} w_0(x) \\ w_1(x) \end{bmatrix}$ be a vector of weight functions satisfying a differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\Phi(x)\overline{w}(x)\right) + \Psi(x)\overline{w}(x) = 0, \tag{2.21}$$

with $\Phi(x) = \begin{bmatrix} \phi_{00} & \phi_{01} \\ \varphi(x) & \phi_{11} \end{bmatrix}$ and $\Psi(x) = \begin{bmatrix} \eta_0 & \eta_1 \\ \psi(x) & \xi \end{bmatrix}$, for constants ϕ_{00}, ϕ_{01}, ϕ_{11}, η_0, η_1 and ξ and polynomials φ and ψ such that $\deg\varphi \leq 1$ and $\deg\psi = 1$. Suppose that all multi-indices on the step-line are normal with respect to both $\overline{w}(x)$ and $x\Phi(x)\overline{w}(x)$ and let $\{P_n(x)\}_{n\in\mathbb{N}}$ be the 2-orthogonal polynomial sequence with respect to $\overline{w}(x)$. Then $\left\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}(x)\right)\right\}_{n\in\mathbb{N}}$ is 2-orthogonal with respect to $x\Phi(x)\overline{w}(x)$.

Proof. Let $\overline{v}(x) = x\Phi(x)\overline{w}(x) = \begin{bmatrix} v_0(x) \\ v_1(x) \end{bmatrix}$. Based on the assumption (2.21), we have $\frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{v}(x)\right) = -\Psi(x)\overline{w}(x)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k\overline{v}(x)\right) = x^k\left(k\Phi(x) - \Psi(x)\right)\overline{w}(x)$$

holds for any $k \in \mathbb{N}$, and it amounts to the same as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k v_0(x)\right) = x^k\Big(\left(k\phi_{00} - \eta_0\right)w_0(x) + \left(k\phi_{01} - \eta_1\right)w_1(x)\Big)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^k v_1(x)\right) = x^k\Big(\left(k\varphi(x) - \psi(x)\right)w_0(x) + \left(k\phi_{11} - \xi\right)w_1(x)\Big).$$

Performing integration by parts and then using the latter identities, we respectively have

$$\int_0^\infty x^k P'_{n+1}(x)v_0(x)\mathrm{d}x = \int_0^\infty \left(\eta_0 - k\phi_{00}\right)x^k P_{n+1}(x)w_0(x)\mathrm{d}x + \int_0^\infty \left(\eta_1 - k\phi_{01}\right)x^k P_{n+1}(x)w_1(x)\mathrm{d}x$$

and

$$\int_0^\infty x^k P'_{n+1}(x)v_1(x)\mathrm{d}x = \int_0^\infty \left(\psi(x) - k\varphi(x)\right)x^k P_{n+1}(x)w_0(x)\mathrm{d}x + \int_0^\infty \left(\xi - k\phi_{11}\right)x^k P_{n+1}(x)w_1(x)\mathrm{d}x,$$

which are valid for any $k \in \mathbb{N}$.

Arguing now with the 2-orthogonality of $\{P_n(x)\}_{n\in\mathbb{N}}$ with respect to $\overline{w}(x)$, combined with the degrees of φ and ψ not being greater than 1, we conclude that the polynomial sequence $\left\{\frac{1}{n+1}P'_{n+1}(x)\right\}_{n\in\mathbb{N}}$ is necessarily 2-orthogonal with respect to $\overline{v}(x)$. $\square$

A similar result can be deduced regarding multiple orthogonality of type I.

Proposition 2.7. *Let $\overline{w}(x)$ be a vector of weight functions satisfying* (2.21) *with* $\Phi(x)$ *and* $\Psi(x)$ *being two polynomial matrices as described in Proposition 2.6. Suppose that all multi-indices on the step-line are normal with respect to both* $\overline{w}(x)$ *and* $x\Phi(x)\overline{w}(x)$ *and let* $Q_n(x)$ *be the type I function for the index of length* n *on the step-line with respect to* $x\Phi(x)\overline{w}(x)$. *Then* $-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_n(x)\right)$ *is the type I function for the index of length* $n+1$ *on the step line with respect to* $\overline{w}(x)$.

Proof. By a simple integration and then by definition of $Q_n(x)$ we have

$$\int_0^\infty Q_n'(x)\mathrm{d}x = Q_n(x)\Big|_0^\infty = 0,$$

whilst, after performing integration by parts to then argue with the definition of $Q_n(x)$, we obtain

$$\int_0^\infty x^{k+1}Q_n'(x)\mathrm{d}x = -(k+1)\int_0^\infty x^k Q_n(x)\mathrm{d}x = \begin{cases} 0, & \text{if } 0\le k\le n-2,\\ -n, & \text{if } k=n-1.\end{cases}$$

Hence, it follows

$$\int_0^\infty -x^j\,\frac{Q_n'(x)}{n}\mathrm{d}x = \begin{cases} 0, & \text{if } 0\le j\le n-1,\\ 1, & \text{if } j=n.\end{cases}$$

Therefore it is sufficient to show that there are polynomials $A_{n+1}(x)$ and $B_{n+1}(x)$ such that

$$-\frac{1}{n}Q_n'(x) = A_{n+1}(x)w_0(x)+B_{n+1}(x)w_1(x) \quad \text{for all} \quad n\in\mathbb{N}, \tag{2.22}$$

and

$$\deg\left(A_{2m}(x)\right),\deg\left(B_{2m}(x)\right),\deg\left(B_{2m+1}(x)\right)\le m-1 \text{ and } \deg\left(A_{2m+1}(x)\right)\le m, \quad \text{for any} \quad m\in\mathbb{N}, \tag{2.23}$$

because this implies that $(A_{n+1}(x),B_{n+1}(x))$ is the vector of type I multiple orthogonal polynomials for the index of length $n+1$ on the step-line with respect to $\overline{w}(x)$. Consequently, this means that $-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_n(x)\right)$ is the type I function for the index of length $n+1$ on the step line with respect to $\overline{w}(x)$.

Consider the vector of weights $\overline{v}(x)=x\Phi(x)\overline{w}(x)$ and let $\overline{v}(x)=\begin{bmatrix} v_0(x)\\ v_1(x)\end{bmatrix}$. Then,

$$v_0(x) = x\left(\phi_{00}\,w_0(x)+\phi_{01}\,w_1(x)\right) \text{ and } v_1(x) = x\left(\varphi(x)w_0(x)+\phi_{11}\,w_1(x)\right). \tag{2.24a}$$

By virtue of equation (2.21), we have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\overline{v}(x)\right) = -\Psi(x)\overline{w}(x)$, which means

$$v_0'(x) = -\eta_0 w_0(x)-\eta_1 w_1(x) \text{ and } v_1'(x) = -\psi(x)w_0(x)-\xi\,w_1(x). \tag{2.24b}$$

For any $n\in\mathbb{N}$, let $(C_n(x),D_n(x))$ be the vector of type I multiple orthogonal polynomials for the index of length n on the step-line with respect to $\overline{v}(x)$. Then, by definition of the type I function,

$$Q_n(x) = C_n(x)v_0(x)+D_n(x)v_1(x),$$

with

$$\deg\left(C_{2m}(x)\right), \deg\left(D_{2m}(x)\right), \deg\left(D_{2m+1}(x)\right) \leq m-1 \text{ and } \deg\left(C_{2m+1}(x)\right) \leq m, \text{ for any } m \in \mathbb{N}.$$

Differentiating the expression for $Q_n(x)$, we obtain

$$Q_n'(x) = C_n'(x)v_0(x) + A_n(x)v_0'(x) + D_n'(x)v_1(x) + D_n(x)v_1'(x)$$

hence, using (2.24a) and (2.24b), we derive

$$\begin{aligned} Q_n'(x) = &\Big(\phi_{00}xC_n'(x) - \eta_0 C_n(x) + x\varphi(x)D_n'(x) - \psi(x)D_n(x)\Big)w_0(x) \\ &+ \Big(\phi_{01}xC_n'(x) - \eta_1 C_n(x) + \phi_{11}xD_n'(x) - \xi D_n(x)\Big)w_1(x). \end{aligned}$$

The uniqueness of type I multiple orthogonal polynomials leads to (2.22) where

$$A_{n+1}(x) = -\frac{1}{n}\Big(\phi_{00}xC_n'(x) - \eta_0 C_n(x) + x\varphi(x)D_n'(x) - \psi(x)D_n(x)\Big)$$

and

$$B_{n+1}(x) = -\frac{1}{n}\Big(\phi_{01}xC_n'(x) - \eta_1 C_n(x) + \phi_{11}xD_n'(x) - \xi D_n(x)\Big).$$

Finally, the conditions on the degrees of $C_n(x)$ and $D_n(x)$, combined with the degrees of Φ and Ψ not being greater than 1, imply that (2.23) holds. □

Remark 2.8. *As a straightforward consequence of Proposition 2.7, the type I multiple orthogonal polynomials $(A_n(x), B_n(x))$ and $(C_n(x), D_n(x))$ for $\overline{w}(x)$ and for $x\Phi(x)\overline{w}(x)$, respectively, are related by*

$$\begin{pmatrix} A_{n+1}(x) \\ B_{n+1}(x) \end{pmatrix} = x\Phi(x)^t \begin{pmatrix} {C_n}'(x) \\ D_n'(x) \end{pmatrix} - \Psi(x)^t \begin{pmatrix} C_n(x) \\ D_n(x) \end{pmatrix}, \quad \textit{for all} \quad n \geq 0,$$

where Φ^t and Ψ^t are the transpose of the matrices given in Proposition 2.6.

Combining Propositions 2.6 and 2.7 with Theorem 2.5, we deduce differential and difference properties for type I and type II multiple polynomials, which are described in the following result.

Theorem 2.9. *For $a, b, c \in \mathbb{R}^+$ such that $c > \max\{a, b\}$, let $\left\{P_n^{[d]}(x; a, b; c)\right\}_{n\in\mathbb{N}}$ and $\left\{Q_n^{[d]}(x; a, b; c)\right\}_{n\in\mathbb{N}}$ be the sequences of type II multiple orthogonal polynomials and type I functions, respectively, on the step-line with respect to $\overline{\mathscr{W}}^{[d]}(x; a, b; c)$, defined as in (2.14). Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[d]}(x; a, b; c)\right) = (n+1)P_n^{[1-d]}(x; a+1, b+1; c+1+d), \tag{2.25}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_n^{[1-d]}(x; a+1, b+1; c+1+d)\right) = -n\,Q_{n+1}^{[d]}(x; a, b; c). \tag{2.26}$$

Proof. Let $\Phi^{[d]}(x;a,b;c)$ be defined as in Theorem 2.5. Then Proposition 2.6 ensures that $\left\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[d]}(x;a,b;c)\right)\right\}_{n\in\mathbb{N}}$ is 2-orthogonal with respect to $x\Phi^{[d]}(x;a,b;c)\overline{\mathscr{W}}^{[d]}(x;a,b;c)$. Besides, from Proposition 2.7, we know that, if $\tilde{Q}_n^{[d]}(x;a,b;c)$ is the type I function for the index of length n on the step-line with respect to $x\Phi^{[d]}(x;a,b;c)\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, then $-\frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left(\tilde{Q}_n^{[d]}(x;a,b;c)\right)$ is the type I function for the index of length $n+1$ on the step line with respect to the vector of weights $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$. Therefore, by virtue of (2.20), we conclude that both (2.25) and (2.26) hold. □

The differentiable properties described in Theorem 2.9 are the main pillars for further characterisation of the multiple orthogonal polynomials under analysis. These intrinsic properties resemble those found within the context of the very classical standard orthogonal polynomials.

2.4 Type I multiple orthogonal polynomials

Let us revisit Proposition 2.7 and Remark 2.8 for the case where the vector of weights $\overline{w}(x)$ is replaced by $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ defined by (2.14). We recall (2.20) in Theorem 2.5 and (2.26) in Theorem 2.9 to conclude that if $\left(A_n^{[d]}(x;a,b;c),B_n^{[d]}(x;a,b;c)\right)$ is the vector of type I multiple orthogonal polynomials on the step line for $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, then

$$\begin{pmatrix} A_{n+1}^{[d]}(x;a,b;c) \\ B_{n+1}^{[d]}(x;a,b;c)\end{pmatrix} = \left(x\Phi(x)^t\frac{\mathrm{d}}{\mathrm{d}x}-\Psi(x)^t\right)\begin{pmatrix} A_n^{[1-d]}(x;a+1,b+1;c+d) \\ B_n^{[1-d]}(x;a+1,b+1;c+d)\end{pmatrix}, \quad \text{for all} \quad n\geq 0.$$

The type I multiple orthogonal functions on the step line can be generated by concatenated differentiation of the weight function or, in other words, via a Rodrigues-type formula.

Theorem 2.10. *For $a,b,c\in\mathbb{R}^+$ such that $c>\max\{a,b\}$, $\delta\in\{0,1\}$ and $n\in\mathbb{N}$, let $\left\{Q_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ be the sequence of type I functions on the step-line with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$. Then, for any $n\in\mathbb{N}$,*

$$Q_{n+1}^{[d]}(x;a,b;c)=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathscr{W}\left(x;a+n,b+n;c+n+\left\lfloor\frac{n+1+d}{2}\right\rfloor\right)\right). \tag{2.27}$$

Proof. We proceed by induction on $n\in\mathbb{N}$. For $n=0$, the relation (2.27) reads as

$$Q_1^{[d]}(x;a,b;c)=\mathscr{W}(x;a,b;c+d), \tag{2.28}$$

which holds trivially.

Using (2.26) and then evoking the assumption that (2.27) holds for a fixed $n \in \mathbb{N}$, we obtain

$$\begin{aligned} Q_{n+2}^{[d]}(x;a,b;c) &= -\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(Q_{n+1}^{[1-d]}(x;a+1,b+1;c+1+d)\right) \\ &= \frac{(-1)^{n+1}}{(n+1)!}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(\mathscr{W}\left(x;a+n+1,b+n+1;c+n+1+\left\lfloor\frac{n+2+d}{2}\right\rfloor\right)\right). \end{aligned}$$

If we equate the first and latter members, we readily see that (2.27) holds for $n+1$ and, as a result, we can state that it holds for all $n \in \mathbb{N}$ by induction. $\square$

From this point forth the focus will be on the type II multiple orthogonal polynomials.

3 Characterisation of the type II multiple orthogonal polynomials

One of the defining properties of the four families of the very classical orthogonal polynomials (of Hermite, Laguerre, Bessel and Jacobi) is that the sequence of their derivatives is also orthogonal (ie, they satisfy Hahn's property). Together, these four families share a number of other properties. We highlight two of them. They are polynomial solutions to a second order linear differential equation with polynomial coefficients (the so called Bochner's differential equation). Their orthogonality weight functions are solutions to a first order homogeneous linear differential equation with polynomial coefficients (commonly referred to as the Pearson equation).

The type II multiple polynomials on the step line $\left\{P_n^{[d]}(x)\right\}_{n\in\mathbb{N}}$ orthogonal for $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ in (2.14) also satisfy a number of properties that resemble those found among the classical polynomials. The relation between $\left\{P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ and its sequence of derivatives given in (2.25) clearly shows that $\left\{P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ satisfy the Hahn's property. Additionally, we show in Section 3.2 that they are solution to a third order linear differential equation with polynomial coefficients and we have described the vector of weight functions to be a solution to a first order homogeneous linear matrix differential equation resembling a matrix version of the Pearson equation. Therefore, it makes all sense to perceive these polynomials as (Hahn)-classical polynomials in the context of multiple orthogonality. (However, within this context, it is worth to note that in the literature there are other notions of "classical".) As a matter of fact, such properties are also shared by other Hahn-classical 2-orthogonal polynomials. In [20] the equivalence between these three properties was proved for the threefold symmetric case.

3.1 Explicit expression

Based on (1.5), we deduce an explicit expression for the type II multiple orthogonal polynomials on the step line $\left\{P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ as a generalised hypergeometric function.

Theorem 3.1. *Let $a,b,c>0$ such that $c>\max\{a,b\}>0$ and $d\in\{0,1\}$. If the polynomial sequence $\left\{P_n^{[d]}(x):=P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$ is monic and 2-orthogonal with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, then*

$$P_n^{[d]}(x)=\frac{(-1)^n\,(a)_n\,(b)_n}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)_n}\,{}_2F_2\left(\begin{matrix}-n,c+\left\lfloor\frac{n+d}{2}\right\rfloor\\ a,b\end{matrix};x\right) \tag{3.1a}$$

or, equivalently,

$$P_n^{[d]}(x)=\sum_{j=0}^{n}\tau_{n,j}^{[d]}x^{n-j},\quad \text{with}\quad \tau_{n,j}^{[d]}=(-1)^j\binom{n}{j}\frac{(a+n-j)_j\,(b+n-j)_j}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor+n-j\right)_j}. \tag{3.1b}$$

To prove this theorem we need to check that $\left\{P_n^{[d]}(x)\right\}_{n\in\mathbb{N}}$ in (3.1b) satisfies the 2-orthogonality conditions with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$, which are:

$$\int_0^\infty x^kP_n^{[d]}(x)\mathscr{W}(x;a,b;c+d)\mathrm{d}x=\begin{cases}0, & \text{if } n\geq 2k+1,\\ N_n\neq 0, & \text{if } n=2k,\end{cases} \tag{3.2a}$$

and

$$\int_0^\infty x^kP_n^{[d]}(x)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x=\begin{cases}0, & \text{if } n\geq 2k+2,\\ N_n\neq 0, & \text{if } n=2k+1,\end{cases} \tag{3.2b}$$

where it is understood that $N_n:=N_n^{[d]}(a,b;c)\neq 0$ for all $n\in\mathbb{N}$.

Actually, as we are dealing with a Nikishin system, the existence of a 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ is guaranteed. By virtue of the known differential formula for a generalised hypergeometric series (see [6, Eq. 16.3.1]), it is rather straightforward to show that the polynomials given by (3.1a) satisfy the differential property (2.25) stated in Proposition 2.6. A property that a 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ must satisfy. Therefore, it would be sufficient to check the orthogonality conditions (3.2a)-(3.2b) when $k=0$ to then prove the result by induction on $n\in\mathbb{N}$ (the degree of the polynomials).

However, we opt for checking that the polynomials $P_n^{[d]}(x)$ in (3.1a) satisfy all the orthogonality conditions (3.2a)-(3.2b). On the one hand, this process enables us to show directly that the polynomials in (3.1a) are indeed 2-orthogonal with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$ without arguing with the Nikishin property. On the other hand, it provides a method to derive the following explicit expressions for the nonzero coefficients N_n in (3.2a)-(3.2b):

$$N_{2k}^{[d]}(a,b;c)=\begin{cases}\dfrac{(2k)!\,(a)_{2k}\,(b)_{2k}}{(c)_{3k}} & \text{if}\quad d=0,\\[2ex] \dfrac{(2k)!\,(a)_{2k}\,(b)_{2k}\,(c-a+1)_k\,(c-b+1)_k}{(c+k)_{2k}\,(c+1)_{3k}} & \text{if}\quad d=1.\end{cases} \tag{3.2c}$$

and

$$N_{2k+1}^{[d]}(a,b;c) = \begin{cases} -\dfrac{(2k+1)!\,(a)_{2k+1}\,(b)_{2k+1}\,(c)_k\,(c-a+1)_k\,(c-b+1)_k}{(c)_{3k+1}\,(c+1)_{3k+1}}, & \text{if } d=0, \\ \dfrac{(2k+1)!\,(a)_{2k+1}\,(b)_{2k+1}}{(c)_{3k+2}}, & \text{if } d=1. \end{cases} \tag{3.2d}$$

Note that $N_{2k+j}^{[d]}(a,b;c)$ is negative when $j=1$ and $d=0$ and positive for all other cases. The explicit expression for $N_n^{[d]}(a,b;c)$ readily gives an explicit expression for the nonzero γ-coefficients in the third order recurrence relation (1.18) satisfied by these polynomials. Such relation is discussed in Section 3.3.

In order to prove Theorem 3.1, we need the following result.

Lemma 3.2. *Let n, p, and $m_1,\cdots,m_p$ be positive integers such that $m := \sum_{i=1}^{p} m_i \le n$ and $\beta, f_1,\cdots,f_p$ be complex numbers with positive real part. Then*

$${}_{p+1}F_p\left(\begin{matrix} -n, f_1+m_1,\cdots,f_p+m_p \\ f_1,\cdots,f_p \end{matrix}; 1\right) = \begin{cases} 0 & \text{if } m<n, \\ \dfrac{(-1)^n n!}{(f_1)_{m_1}\cdots(f_p)_{m_p}} & \text{if } m=n. \end{cases} \tag{3.3}$$

and

$${}_{p+2}F_{p+1}\left(\begin{matrix} -n, \beta, f_1+m_1,\cdots,f_p+m_p \\ \beta+1, f_1,\cdots,f_p \end{matrix}; 1\right) = \frac{n!\,(f_1-\beta)_{m_1}\cdots(f_p-\beta)_{m_p}}{(\beta+1)_n\,(f_1)_{m_1}\cdots(f_p)_{m_p}}. \tag{3.4}$$

Proof. Formula (3.4) was deduced by Minton in [24] (see also [17]) and (3.3) can be obtained by taking the limit $\beta \to +\infty$ in (3.4). □

Proof of Theorem 3.1. We evaluate the left hand side of (3.2a) for any $k,n \in \mathbb{N}$, by using the expression for the moments (1.5) and the polynomial expansion (3.1b). This successively gives

$$\begin{aligned} \int_0^\infty x^k P_n^{[d]}(x;a,b;c)\mathscr{W}(x;a,b;c+d)\mathrm{d}x &= \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j} \frac{(a+j)_{n-j}\,(b+j)_{n-j}}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor+j\right)_{n-j}} \frac{(a)_{k+j}\,(b)_{k+j}}{(c+d)_{k+j}} \\ &= \frac{(-1)^n\,(a)_n\,(b)_n\,(a)_k\,(b)_k}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)_n (c+d)_k} \sum_{j=0}^{n} \frac{(-n)_j\,(a+k)_j\,(b+k)_j\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)_j}{j!\,(a)_j\,(b)_j\,(c+k+d)_j} \\ &= \frac{(-1)^n\,(a)_n\,(b)_n\,(a)_k\,(b)_k}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)_n (c+d)_k}\, {}_4F_3\left(\begin{matrix} -n, a+k, b+k, c+\left\lfloor\frac{n+d}{2}\right\rfloor \\ a, b, c+k+d \end{matrix}; 1\right). \end{aligned}$$

Based on (3.3) in Lemma 3.2, we obtain

$${}_4F_3\left(\begin{matrix} -n, a+k, b+k, c+\left\lfloor\frac{n+d}{2}\right\rfloor \\ a, b, c+k+d \end{matrix}; 1\right) = 0, \quad \text{for any} \quad n \ge 2k+1,$$

so that

$$\int_0^\infty x^k P_n^{[d]}(x)\mathscr{W}(x;a,b;c+d)\mathrm{d}x = 0, \quad \text{for any} \quad n \geq 2k+1. \tag{3.5a}$$

Besides,

$$\int_0^\infty x^k P_{2k}^{[d]}(x;a,b;c)\mathscr{W}(x;a,b;c+d)\mathrm{d}x = \frac{(a)_{2k}(b)_{2k}(a)_k(b)_k}{(c+k)_{2k}(c+d)_k}\, {}_4F_3\left(\begin{matrix}-2k, a+k, b+k, c+k \\ a+1, b+1, c+k+d\end{matrix}; 1\right).$$

For the $d=0$ the latter hypergeometric series simplifies to a ${}_3F_2$ which, on account of the identity (3.3), can be evaluated to

$${}_3F_2\left(\begin{matrix}-2k, a+k, b+k \\ a, b\end{matrix}; 1\right) = \frac{(2k)!}{(a)_k(b)_k},$$

whilst for the case where $d=1$ we have to use (3.4) to get

$${}_4F_3\left(\begin{matrix}-2k, a+k, b+k, c+k \\ a, b, c+k+1\end{matrix}; 1\right) = \frac{(2k)!\,(c-a+1)_k(c-b+1)_k}{(a)_k(b)_k(c+k+1)_{2k}}.$$

As a result, we have

$$\int_0^\infty x^k P_{2k}^{[d]}(x;a,b;c)\mathscr{W}(x;a,b;c+d)\mathrm{d}x = N_{2k}^{[d]}(a,b;c), \tag{3.5b}$$

with the values of $N_{2k}^{[d]}(a,b;c)$ being given by (3.2c), ensuring that (3.2a) holds for any k and n.

Using similar arguments as before, we recall (1.5) to write the left hand side of (3.2b) as

$$\int_0^\infty x^k P_n^{[d]}(x;a,b;c)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x = \frac{(-1)^n(a)_n(b)_n(a)_k(b)_k}{\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)_n(c+1-d)_k}\, {}_4F_3\left(\begin{matrix}-n, a+k, b+k, c+\left\lfloor\frac{n+d}{2}\right\rfloor \\ a, b, c+k+1-d\end{matrix}; 1\right).$$

For any $n \geq 2k+2$, identity (3.3) in Lemma 3.2 implies that

$$\int_0^\infty x^k P_n^{[d]}(x)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x = 0. \tag{3.6a}$$

When $n = 2k+1$, in order to evaluate the terminating hypergeometric series in the latter expression, we use (3.4) for the case where $d=0$ and we use (3.3) when $d=1$ to derive

$$\int_0^\infty x^k P_{2k+1}^{[d]}(x;a,b;c)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x = N_{2k+1}^{[d]}(a,b;c), \tag{3.6b}$$

with the values of $N_{2k+1}^{[d]}(a,b;c)$ being given by (3.2d), ensuring that (3.2b) holds for any k and n. □

3.2 Differential equation

The type II multiple orthogonal polynomials of hypergeometric type described in (3.1a) are solutions to a third order differential equation. The structure of this differential equation resembles the structure of differential equations satisfied by other 2-orthogonal polynomials satisfying the Hahn property, that is, Hahn-classical polynomials. Examples of such polynomials can be found for instance in [3, 7, 28] among other works.

In a way this differential equation can be seen as a Bochner type differential equation satisfied by all the classical (standardly) orthogonal polynomials.

Theorem 3.3. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, let $\left\{P_n^{[d]}(x) := P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$, $d \in \{0,1\}$, be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$. Then*

$$x^2 \frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(P_n^{[d]}(x)\right) - x\varphi(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[d]}(x)\right) + \psi_n^{[d]}(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[d]}(x)\right) + n\left(c + \left\lfloor \frac{n+d}{2} \right\rfloor\right) P_n^{[d]}(x) = 0, \tag{3.7}$$

with $\varphi(x) = x-(a+b+1)$ and $\psi_n^{[d]}(x) = \left(\left\lfloor \frac{n+1-d}{2}\right\rfloor - (c+1)\right)x + ab$.

Proof. Combining the explicit formula for the polynomials given by (3.1a) and the generalised hypergeometric differential equation [6, Eq. 16.8.3], we obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a-1\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+b-1\right)\right]P_n^{[d]}(x) = \left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}+c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\right]P_n^{[d]}(x). \tag{3.8}$$

Moreover, we have

$$\begin{aligned}&\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a-1\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+b-1\right)\right]\left(P_n^{[d]}(x)\right)\\ &= x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[d]}(x)\right) + (a+b-1)x\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[d]}(x)\right) + (a-1)(b-1)P_n^{[d]}(x)\end{aligned}$$

and, observing that $c + \left\lfloor\frac{n+d}{2}\right\rfloor - n + 1 = c + 1 - \left\lfloor\frac{n+1-d}{2}\right\rfloor$, the right-hand side of (3.8) is

$$\begin{aligned}&\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}+c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}-n\right)\right]\left(P_n^{[d]}(x)\right)\\ &= x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[d]}(x)\right) + \left(c+1-\left\lfloor\frac{n+1-d}{2}\right\rfloor\right)x\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[d]}(x)\right) - n\left(c+\left\lfloor\frac{n+d}{2}\right\rfloor\right)P_n(x).\end{aligned}$$

Furthermore, differentiating our expression for $\left[\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a-1\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+b-1\right)\right]\left(P_n^{[d]}(x)\right)$, we obtain

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}}{\mathrm{d}x}+a-1\right)\left(x\frac{\mathrm{d}}{\mathrm{d}x}+b-1\right)\right]P_n(x) = x^2\frac{\mathrm{d}^3}{\mathrm{d}x^3}\left(P_n^{[d]}(x)\right) + (a+b+1)x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(P_n^{[d]}(x)\right) + ab\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n^{[d]}(x)\right).$$

Finally, combining the former and the latter expressions, we derive the differential equation (3.7). ☐

The alternative representation (3.8) for the differential equation (3.7) highlights some symmetrical properties of these polynomials that may be worth to explore.

3.3 Recurrence relation

One of the main features of the 2-orthogonal polynomials is the third order recurrence relation. The hypergeometric type polynomials defined by (3.1a) necessarily satisfy a recurrence relation of the type

$$P_{n+1}^{[d]}(x) = \left(x - \beta_n^{[d]}\right) P_n^{[d]}(x) - \alpha_n^{[d]} P_{n-1}^{[d]}(x) - \gamma_{n-1}^{[d]} P_{n-2}^{[d]}(x). \tag{3.9}$$

The key point here is to obtain explicit expressions for the recurrence coefficients triplet $(\beta_n^{[d]}, \alpha_n^{[d]}, \gamma_n^{[d]})$ in (3.9). For simplification, we have written $\beta_n^{[d]} := \beta_n^{[d]}(a,b;c)$, $\alpha_n^{[d]} := \alpha_n^{[d]}(a,b;c)$ and $\gamma_n^{[d]} := \gamma_n^{[d]}(a,b;c)$.

Their expressions can be derived through the explicit expression given in (3.1a) or in (3.1b). As the polynomials are terminating generalised hypergeometric series, one can use some well known symbolic computation packages available in Maple (for instance) to deduce a difference equation on the parameter of the hypergeometric function corresponding to the index of the polynomial. Ultimately, this gives a recurrence relation and expressions for the recurrence coefficients. However, for a matter of completion, we deduce expressions for the recurrence coefficients via a standard algebraic method. In the recurrence relation (3.9) we replace the polynomials $P_{n+1-j}^{[d]}(x)$ (with $j = 0,1,2,3$) by their corresponding expansion expression (3.1b). The linear independence of $\{x^n\}_{n\in\mathbb{N}}$ implies that we can equate the expressions of both sides of the recurrence relation. After equating the coefficients of x^n in (3.9), we obtain

$$\beta_n^{[d]} = \tau_{n,1}^{[d]} - \tau_{n+1,1}^{[d]}.$$

Similarly, a comparison of the coefficients of x^{n-1}, combined with the latter formula, brings the identity

$$\alpha_n^{[d]} = \tau_{n,2}^{[d]} - \tau_{n+1,2}^{[d]} - \left(\tau_{n,1}^{[d]}\right)^2 + \tau_{n,1}^{[d]} \tau_{n+1,1}^{[d]}.$$

Based on the expression for these τ-coefficients in (3.1b), we have

$$\tau_{n,1}^{[d]}(a,b;c) = -\frac{n(a+n-1)(b+n-1)}{c + \left\lfloor \frac{n+d}{2} \right\rfloor + n - 1}$$

and

$$\tau_{n,2}^{[d]}(a,b;c) = \frac{1}{2} \frac{n(a+n-1)(b+n-1)(n-1)(a+n-2)(b+n-2)}{\left(c + \left\lfloor \frac{n+d}{2} \right\rfloor + n - 1\right)\left(c + \left\lfloor \frac{n+d}{2} \right\rfloor + n - 2\right)},$$

which leads to

$$\beta_{2m+d}^{[d]}(a,b;c) = \frac{(2m+d+1)(a+2m+d)(b+2m+d)}{c+3m+2d} - \frac{(2m+d)(a+2m+d-1)(b+2m+d-1)}{c+3m+2d-1} \tag{3.10a}$$

and

$$\beta^{[1-d]}_{2m+d}(a,b;c) = \frac{(2m+d+1)(a+2m+d)(b+2m+d)}{c+3m+d+1} - \frac{(2m+d)(a+2m+d-1)(b+2m+d-1)}{c+3m+d-1}, \tag{3.10b}$$

as well as

$$\alpha^{[1-d]}_{2m+d}(a,b;c+d) = \alpha^{[d]}_{2m+d}(a,b;c) \tag{3.10c}$$

$$= \frac{(2m+d)(a+2m+d-1)(b+2m+d-1)}{c+3m+2d-1}\left(\frac{(2m+d-1)(a+2m+d-2)(b+2m+d-2)}{2(c+3m+2d-2)}\right. \tag{3.10d}$$
$$\left. - \frac{(2m+d)(a+2m+d-1)(b+2m+d-1)}{c+b+3m+2d-1} + \frac{(2m+d+1)(a+2m+d)(b+2m+d)}{2(c+3m+2d)}\right).$$

The expressions for the coefficients $\gamma^{[d]}_n$ could also be obtained in an analogous way after comparing the coefficients of x^{n-2} in (3.9). However, it is rather easier from the computational point of view, to derive such expressions directly from the 2-orthogonality conditions. Indeed, the 2-orthogonality conditions applied to the recurrence relation (3.9), straightforwardly imply that

$$\gamma^{[d]}_{2n+1}(a,b;c) = \frac{\int_0^\infty x^{n+1} P^{[d]}_{2n+2}(x;a,b;c)\mathscr{W}(x;a,b;c+d)\mathrm{d}x}{\int_0^\infty x^n P^{[d]}_{2n}(x;a,b;c)\mathscr{W}(x;a,b;c+d)\mathrm{d}x}$$

and

$$\gamma^{[d]}_{2n+2}(a,b;c) = \frac{\int_0^\infty x^{n+1} P^{[d]}_{2n+3}(x;a,b;c)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x}{\int_0^\infty x^n P^{[d]}_{2n+1}(x;a,b;c)\mathscr{W}(x;a,b;c+1-d)\mathrm{d}x}.$$

Based on the latter alongside with (3.2a)-(3.2d), we deduce

$$\gamma^{[d]}_{2m+d}(a,b;c) = \frac{(2m+d)_2\,(a+2m+d-1)_2\,(b+2m+d-1)_2\,(c+m+d-1)(c-a+m+d)(c-b+m+d)}{(c+3m+2d-2)_3\,(c+b+3m+2d-1)_3}, \tag{3.10e}$$

and

$$\gamma^{[1-d]}_{2m+d}(a,b;c) = \frac{(2m+d)_2\,(a+2m+d-1)_2\,(b+2m+d-1)_2}{(c+3m+d-1)_3}. \tag{3.10f}$$

As a consequence, we have just proved the following result.

Theorem 3.4. *For $a,b,c \in \mathbb{R}^+$ such that $c > \max\{a,b\}$, let $\left\{P^{[d]}_n(x) := P^{[d]}_n(x;a,b;c)\right\}_{n\in\mathbb{N}}$, $d \in \{0,1\}$, be the monic 2-orthogonal polynomial sequence with respect to $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$. Then the recurrence relation*

(3.9) *holds and the recurrence coefficients are given by* (3.10a)-(3.10f) *with* $\gamma_{n+1}^{[d]}(a,b;c) > 0$ *for all* $n \in \mathbb{N}$. *Furthermore, the coefficients have the following asymptotic behaviour of period* 2*:*

$$\beta_{2m+d}^{[d]}(a,b;c) \sim \frac{28}{9}m, \quad \beta_{2m+d}^{[1-d]}(a,b;c) \sim \frac{20}{9}m,$$

$$\alpha_{2m+d}^{[1-d]}(a,b;c+d) = \alpha_{2m+d}^{[d]}(a,b;c) \sim \frac{208}{81}m^2,$$

$$\gamma_{2m+d}^{[d]}(a,b;c) \sim \frac{2^6}{3^6}m^3 \quad \textit{and} \quad \gamma_{2m+d}^{[1-d]}(a,b;c) \sim \frac{2^6}{3^3}m^3, \quad \textit{as} \quad m \to \infty.$$

3.4 Asymptotic behaviour of the largest zero

We have already stated that, because $\mathscr{W}(x;a,b;c)$ and $\mathscr{W}(x;a,b;c+1)$ form a Nikishin system, then, if $\{P_n(x)\}_{n\in\mathbb{N}}$ is the 2-orthogonal polynomial sequence with respect to these weight functions, P_n has n real positive simple zeros and the zeros of consecutive polynomials interlace as there is always a zero of P_n between two consecutive zeros of P_{n+1}. An asymptotic upper bound for the largest zero of 2-orthogonal polynomial sequences is intimately related to the asymptotic behaviour of their recurrence relation coefficients, as explained in the following theorem.

Theorem 3.5. *Let* $\{P_n(x)\}_{n\in\mathbb{N}}$ *be a* 2*-orthogonal polynomial sequence satisfying* (1.18) *with* $\gamma_n > 0$*, for all* $n \in \mathbb{N}$*. Suppose there exists a non-decreasing positive sequence* $g : \mathbb{N} \to \mathbb{R}^+$ *and real constants* $\gamma > 0$ *and* $\alpha, \beta \geq 0$ *so that*

$$|\beta_n| \leq \beta\left(g(n) + o\left(g(n)\right)\right), \quad |\alpha_n| \leq \alpha\left((g(n))^2 + o\left((g(n))^2\right)\right), \textit{ and } |\gamma_n| \leq \gamma\left((g(n))^3 + o\left((g(n))^3\right)\right),$$

and such that $\Delta := \gamma^2 - \dfrac{\alpha^3}{27} > 0$*. If* $x_n^{(n)}$ *denotes the largest zero in absolute value of* $P_n(x)$*, then, with* $\tau = \sqrt[3]{\gamma + \sqrt{\Delta}} + \sqrt[3]{\gamma - \sqrt{\Delta}}$*, we have*

$$\left|x_n^{(n)}\right| \leq \left(\frac{3}{2}\tau + \beta + \frac{\alpha}{2\tau}\right)g(n) + o\left(g(n)\right), \quad \textit{as } n \to +\infty.$$

Recalling the asymptotic behaviour for the recurrence coefficients obtained in Theorem 3.4, finding an asymptotic upper bound for the largest zero of $P_n^{[d]}(x;a,b;c)$ is an immediate consequence of Theorem 3.5.

Corollary 3.6. *For* $a,b,c \in \mathbb{R}^+$ *such that* $c > \max\{a,b\}$*, let* $\left\{P_n^{[d]}(x;a,b;c)\right\}_{n\in\mathbb{N}}$*,* $d \in \{0,1\}$*, be the monic* 2*-orthogonal polynomial sequence with respect to* $\overline{\mathscr{W}}^{[d]}(x;a,b;c)$*. Then* $P_n^{[d]}$ *has n simple real positive zeros and, if we denote by* $x_n^{(n)}$ *the largest zero of* $P_n^{[d]}(x;a,b;c)$*, then*

$$x_n^{(n)} < M \cdot n + o(n), \quad \textit{as} \quad n \to +\infty,$$

where $M = \dfrac{3}{2}\tau + \beta + \dfrac{\alpha}{2\tau} \approx 3.484$, *with* $\alpha = \dfrac{52}{81}$, $\beta = \dfrac{14}{9}$, $\gamma = \dfrac{8}{27}$, $\Delta = \gamma^2 - \dfrac{\alpha^3}{27} = \dfrac{1119104}{14348907} > 0$ *and* $\tau = \sqrt[3]{\gamma + \sqrt{\Delta}} + \sqrt[3]{\gamma - \sqrt{\Delta}}$.

We illustrate the latter result in Figure 2, produced in *Maple*. The curve $y = 3.484n$ gives clearly an upper bound for the largest zero of $P_n^{[d]}(x;a,b;c)$ for each $d \in \{0,1\}$. As already explained, the even order polynomials do not depend on d. Therefore, the zeros of $P_{2n}^{[0]}$ and $P_{2n}^{[1]}$ coincide, but a similar remark does not apply for the odd order polynomials because $P_{2n+1}^{[0]} \neq P_{2n+1}^{[1]}$. A sharper upper bound could be obtained if we consider further terms in the estimation and adapting the proof accordingly. For the purpose of this investigation this is not so relevant.

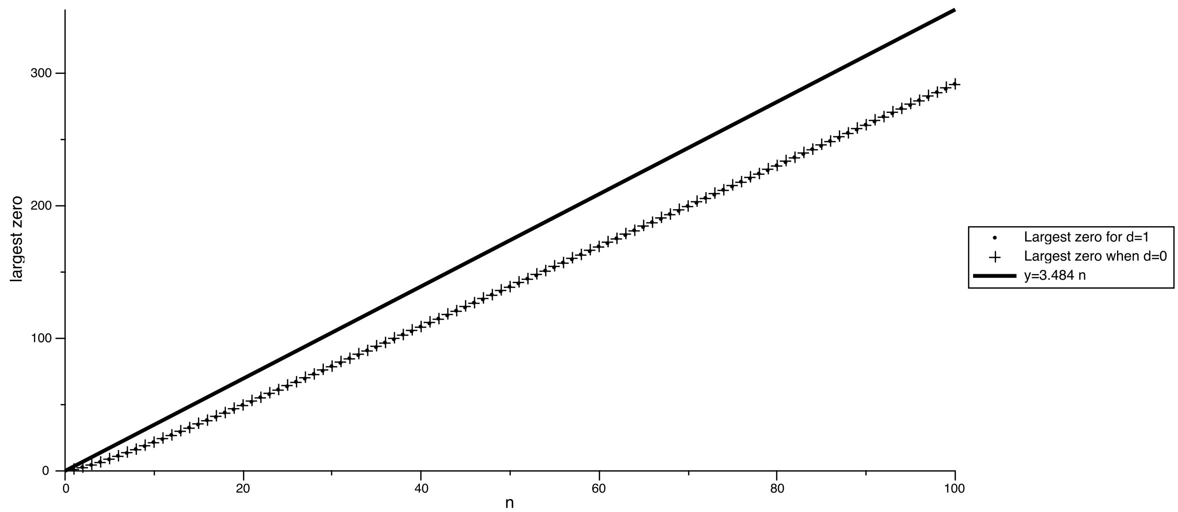

Figure 2: Joint plots of the largest zeros of $P_n^{[d]}(x;3,2.5,7.5)$ for $d=0$ (crosses) and $d=1$ (dots) for each $n = 1,\ldots,100$ with the upper bound curve $y = 3.484n$ in solid line.

Observe that Theorem 3.5 is a generalisation of Theorem 2.2 in [20], since the latter can be obtained from the former by setting $\alpha = \beta = 0$ and $g(n) = n^{\lambda}$ with $\lambda \geq 0$. Its proof is inspired on the one for [20, Th. 2.2].

Proof of Theorem 3.5. Consider the Hessenberg matrix

$$\mathrm{H}_n = \begin{bmatrix} \beta_0 & 1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \beta_1 & 1 & 0 & \cdots & 0 \\ \gamma_1 & \alpha_2 & \beta_2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \gamma_{n-3} & \alpha_{n-2} & \beta_{n-2} & 1 \\ 0 & \cdots & 0 & \gamma_{n-2} & \alpha_{n-1} & \beta_{n-1} \end{bmatrix}$$

so that the recurrence relation can be expressed as

$$\mathrm{H}_n \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} = x \begin{bmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_{n-2}(x) \\ P_{n-1}(x) \end{bmatrix} - P_n(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and each zero of P_n is an eigenvalue of the matrix H_n. Then, if $\rho(\mathrm{H}_n) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathrm{H}_n\}$ is the spectral radius of the matrix H_n, $\left|x_n^{(n)}\right| < \rho(\mathrm{H}_n)$.

Moreover, $\rho(\mathrm{H}_n)$ is bounded from above by the matrix norm (see [13, Section 5.6])

$$\|\mathrm{H}_n\|_S = \|S^{-1}\mathrm{H}_n S\|_\infty = \max_{1\le i\le n} \left\{ \sum_{j=1}^{n} |(S^{-1}\mathrm{H}_n S)_{i,j}| \right\},$$

for a non-singular $n\times n$ matrix S. In particular, if we set $S = \mathrm{diag}(s_1,\cdots,s_n)$, with $\det(S) = s_1\cdots s_n \neq 0$, then

$$\begin{aligned}\|\mathrm{H}_n\|_S = \max_{1\le i\le n} \Bigg\{ &\left|\frac{s_2+\beta_0 s_1}{s_1}\right|, \left|\frac{s_3+\beta_1 s_2+\alpha_1 s_1}{s_2}\right| \left|\frac{s_4+\beta_2 s_3+\alpha_2 s_2+\gamma_1 s_1}{s_3}\right|, \cdots, \\ &\left|\frac{s_n+\beta_{n-2}s_{n-1}+\alpha_{n-2}s_{n-2}+\gamma_{n-3}s_{n-3}}{s_{n-1}}\right|, \left|\frac{\beta_{n-1}s_n+\alpha_{n-1}s_{n-1}+\gamma_{n-2}s_{n-2}}{s_n}\right| \Bigg\}.\end{aligned}$$

If we take $s_k = \prod_{j=1}^{k} (s\,g(j)) = s^k \prod_{j=1}^{k} g(j) > 0$, for some $s>0$, then, using the asymptotic behaviour of the recurrence coefficients in (3.9), as $k\to+\infty$ we have

$$\begin{aligned}\left|\frac{s_k+\beta_{k-2}s_{k-1}+\alpha_{k-2}s_{k-2}+\gamma_{k-3}s_{k-3}}{s_{k-1}}\right| &\le \frac{s_k}{s_{k-1}} + |\beta_{k-2}| + |\alpha_{k-2}|\,\frac{s_{k-2}}{s_{k-1}} + |\gamma_{k-3}|\,\frac{s_{k-3}}{s_{k-1}} \\ &\le s g(k) + \beta g(k-2) + \alpha\,\frac{(g(k-2))^2}{s g(k-1)} + \gamma\,\frac{(g(k-3))^3}{s^2 g(k-2) g(k-1)} + o\left(g(k)\right) \\ &\le \left(s+\beta+\frac{\alpha}{s}+\frac{\gamma}{s^2}\right) g(k) + o\left(g(k)\right).\end{aligned}$$

As a result,

$$\|\mathrm{H}_n\|_S \le \left(s+\beta+\frac{\alpha}{s}+\frac{\gamma}{s^2}\right) g(n) + o\left(g(n)\right),\ n\to+\infty.$$

To find a sharper upper bound for $\|\mathrm{H}_n\|_S$ $\left(\text{and, as a consequence, for } \left|x_n^{(n)}\right|\right)$ given by this formula we need to find the minimum value of $f(s) = s+\beta+\dfrac{\alpha}{s}+\dfrac{\gamma}{s^2}$ on $\mathbb{R}^+$. With that purpose, we look for the roots of $f'(s) = 1-\dfrac{\alpha}{s^2}-\dfrac{2\gamma}{s^3} = \dfrac{1}{s^3}\left(s^3-\alpha s-2\gamma\right)$. Due to the condition $\Delta>0$, we know that f' has one real root and two complex roots. Moreover, the real root is $\tau = \sqrt[3]{\gamma+\sqrt{\Delta}} + \sqrt[3]{\gamma-\sqrt{\Delta}} > 0$ (where we are taking real and positive square and cubic roots). Furthermore, $f''(s) = \dfrac{2\alpha}{s^3} + \dfrac{6\gamma}{s^4}$ hence $f''(\tau) > 0$ and, consequently, the choice $s = \tau$ gives a minimum value to $f(s) = s + \beta + \dfrac{\alpha}{s} + \dfrac{\gamma}{s^2}$. Finally, $f'(\tau) = 0$ implies $\dfrac{2\gamma}{\tau^3} = 1 - \dfrac{\alpha}{\tau}$ therefore $f(\tau) = \dfrac{3}{2}\tau + \beta + \dfrac{\alpha}{2\tau}$, which implies the result. □

3.5 Confluence relation with the modified Bessel weights

There is a clear relation by confluence of these multiple orthogonal polynomials to those studied independently in [3] and [28]. In the latter, the study addressed multiple orthogonal polynomials of both types, while the former concentrated on the type II. In both works, the focus was on weights involving the modified Bessel functions and defined on the positive real line, for parameters $a, b \in \mathbb{R}^+$, as follows

$$\mathscr{V}(x;a,b) = \frac{2}{\Gamma(a)\Gamma(b)}\, x^{\frac{a+b}{2}-1} \mathrm{K}_{a-b}\left(2\sqrt{x}\right), \tag{3.11}$$

where, as mentioned in the introduction, $\mathrm{K}_\nu(x)$ is the modified Bessel function of second kind (also known as Macdonald function).

Let $\{R_n(x;a,b)\}_{n\in\mathbb{N}}$ be the 2-orthogonal polynomial sequence with respect to the vector of weight functions $[\mathscr{V}(x;a,b), \mathscr{V}(x;a,b+1)]$ supported on the positive real line. Similar to the 2-orthogonal polynomials with respect to the modified Tricomi weights, this sequence can also be explicitly represented as a sequence of generalised hypergeometric polynomials by

$$R_n(x;a,b) = (-1)^n (a)_n (b)_n \, {}_1F_2\left(\begin{matrix} -n \\ a, b \end{matrix}; x\right). \tag{3.12}$$

As a direct consequence of this representation, combined with the differential formula for the generalised hypergeometric series (see [6, Eq. 16.3.1]), the sequence of derivatives of $\{R_n(x;a,b)\}_{n\in\mathbb{N}}$ is also 2-orthogonal and it corresponds to the same sequence with shifted parameters. More precisely, for any $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(R_{n+1}(x;a,b)\right) = (n+1)R_n(x;a+1,b+1). \tag{3.13}$$

So, it fits within the family of Hahn-classical 2-orthogonal polynomials.

To obtain a limiting relation between the 2-orthogonal polynomials with respect to the modified Bessel and Tricomi weights, recall that, if $p \le q$, then the generalised hypergeometric series ${}_{p+1}F_q$ satisfies the confluent relation (see [6, Eq. 16.8.10])

$$\lim_{|\alpha|\to\infty} {}_{p+1}F_q\left(\begin{matrix} \alpha_1, \cdots, \alpha_p, \alpha \\ \beta_1, \cdots, \beta_q \end{matrix}; \frac{x}{\alpha}\right) = {}_pF_q\left(\begin{matrix} \alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q \end{matrix}; x\right). \tag{3.14}$$

Using this formula to compare (3.1a) and (3.12), we obtain the confluent relation

$$\lim_{c\to\infty} P_n^{[d]}\left(\frac{x}{c};a,b,c\right) = R_n\left(x;a,b\right). \tag{3.15}$$

As expected, we can also obtain an equivalent confluent relation for the weight functions $\mathscr{V}(x;a,b)$ and $\mathscr{W}(x;a,b;c)$ as

$$\lim_{c\to\infty} \frac{1}{c}\mathscr{W}\left(\frac{x}{c};a,b,c\right) = \mathscr{V}\left(x;a,b\right),$$

which is obtained after taking $\nu = b-a$ in the following limiting relation between the modified Bessel function and the Tricomi function (see [9, p.266])

$$\lim_{c\to\infty} \Gamma(c+\nu)\mathbf{U}\left(a,1-\nu;\frac{x}{c}\right) = 2x^{\frac{\nu}{2}}\mathrm{K}_{\nu}\left(x\right).$$

4 Connection with Hahn-classical 3-fold symmetric polynomials

There are four distinct families of Hahn-classical threefold symmetric 2-orthogonal polynomials, up to a linear transformation of the variable. A fact that was highlighted in [7] and all these families were studied in detail in [20]. The four arising cases were therein denominated as A, B1, B2 and C. The simplest is case A, which consists of 2-orthogonal Appell polynomials, with no parameter dependence, and whose cubic components are particular cases of the 2-orthogonal polynomials mentioned in Section 3.5 involving modified Bessel weights and previously studied in [28] and [3]. The cases B1 and B2 have a richer structure, depend on a parameter and are related to each other via differentiation articulated with parameter shift. Their three cubic components are particular cases of the 2-orthogonal polynomials studied in Section 3. In other words, particular choices on the parameters (a,b,c,d) in (3.1a) allows to describe the three cubic components $P_n^{[k]}$ as in (1.22) for these two cases. More precisely, for each of the cubic components indexed with $k\in\{0,1,2\}$ we have

$$P_n^{[k]}(x;\mu) = P_n^{[d_k]}\left(x;a_k,b_k,\frac{\mu}{3}+b_k\right) \quad \text{in case B1}$$

and

$$P_n^{[k]}(x;\rho) = P_n^{[1-d_k]}\left(x;a_k,b_k,\frac{\rho-1}{3}+a_k\right) \quad \text{in case B2,}$$

where

$$(a_0,b_0)=\left(\frac{1}{3},\frac{2}{3}\right),\ (a_1,b_1)=\left(\frac{4}{3},\frac{2}{3}\right),\ (a_2,b_2)=\left(\frac{4}{3},\frac{5}{3}\right) \quad \text{and} \quad d_k=\frac{1-(-1)^k}{2}. \tag{4.1}$$

As reported above, the cubic components $P_n^{[k]}(x)$ for case A are obtained from particular choices on the parameters of the 2-orthogonal polynomials $\{R_n(x;a,b)\}_{n\in\mathbb{N}}$ in (3.12). Precisely, for each $k\in\{0,1,2\}$ we have $P_n^{[k]}(x) = R_n\left(\frac{x}{9};a_k,b_k\right)$ with (a_k,b_k) as in (4.1). As expected, the confluent relation (3.15) generalises a limiting relation observed in [7] for Hahn-classical 3-fold symmetric 2-orthogonal polynomials: by taking $\mu,\rho\to\infty$ in cases B1 and B2, respectively, leads to case A.

Another observation lies on the fact that the cubic decomposition of threefold symmetric 2-orthogonal polynomials preserves the Hahn-classical property. It is certainly true for cases A and B, if we take into account the identities (3.13) and (2.25), respectively. This property is rather intrinsic to all threefold symmetric 2-orthogonal polynomials Hahn-classical polynomials, as we show below in Theorem 4.1. As a consequence, the cubic components in case C are also part of the Hahn-classical family. A further benefit from this result is on the techniques involved. Among other things, they can be adapted to prove analogous results regarding Hahn-classical polynomials with respect to other annihilating operators such as the q-derivative.

Theorem 4.1. *Let $\{P_n(x)\}_{n\in\mathbb{N}}$ be a 3-fold symmetric Hahn-classical 2-orthogonal polynomial sequence satisfying (1.18) with $\beta_n=\alpha_n=0$ and $\gamma_{n+1}>0$, for all $n\in\mathbb{N}$. Then each of the three cubic components $\left\{P_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$ given by (1.22), with $k\in\{0,1,2\}$, is Hahn-classical.*

The proof of Theorem 4.1 consists of showing that, under the assumptions, $\left\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{[k]}(x)\right\}_{n\in\mathbb{N}}$ is also 2-orthogonal. The result has already been proved in [7, Corollaire 5.1] for the case $k=0$ and here we extend for the remaining cubic components (when $k=1$ or 2). To do so, we first need to derive the orthogonality weights for the cubic components of a 3-fold symmetric Hahn-classical 2-orthogonal polynomial sequence, which are explained in Proposition 4.4. For that purpose, we recall two auxiliary results obtained in [20] and [21], respectively. The first providing the structure of the 2-orthogonality measures for threefold symmetric Hahn-classical polynomials. The second to give the structure for the measures, written in terms of linear functionals, associated with the corresponding cubic components. We recall the following result, which is written here with a minor correction: it should be taken the maximum of the limit of the even and odd order subsequences of the coefficients γ_n.

Proposition 4.2. *(cf. [20, Theorem 3.3]) Let $\{P_n(x)\}_{n\in\mathbb{N}}$ be a 3-fold symmetric Hahn-classical 2-orthogonal polynomial sequence satisfying (1.18) with $\beta_n=\alpha_n=0$ and $\gamma_{n+1}>0$, for all $n\in\mathbb{N}$. Then $\{P_n(x)\}_{n\in\mathbb{N}}$ is 2-orthogonal with respect to a pair of measures (μ_0,μ_1) supported on $S_3=\bigcup_{k=0}^{2}[0,\gamma\omega^k]$ with $\omega=\mathrm{e}^{\frac{2\pi i}{3}}$ and admitting the integral representations*

$$\int_{S_3} f(x)\mathrm{d}\mu_j(x)=\frac{1}{3}\left(\int_0^{\gamma} f(x)\,\mathscr{U}_j(x)\mathrm{d}x+\omega^{2-j}\int_0^{\gamma\omega} f(x)\,\mathscr{U}_j(\omega^2x)\mathrm{d}x+\omega^{j+1}\int_0^{\gamma\omega^2} f(x)\,\mathscr{U}_j(\omega x)\mathrm{d}x\right), \quad (4.2)$$

for each $j\in\{0,1\}$, where $\gamma=\dfrac{27}{4}\lim\limits_{n\to\infty}\gamma_n$ and $\mathscr{U}_j:[0,\gamma]\to\mathbb{R}$ are twice differentiable functions satisfying the matrix differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi(x)\begin{bmatrix}\mathscr{U}_0(x)\\ \mathscr{U}_1(x)\end{bmatrix}\right)+\Psi(x)\begin{bmatrix}\mathscr{U}_0(x)\\ \mathscr{U}_1(x)\end{bmatrix}=0, \quad (4.3)$$

where

$$\Phi(x)=\begin{bmatrix}\Theta_1 & (1-\Theta_1)\,x\\ \dfrac{2\,(1-\Theta_2)}{\gamma_1}\,x^2 & 2\Theta_2-1\end{bmatrix}\quad \text{and}\quad \Psi(x)=\begin{bmatrix}0 & 1\\ \dfrac{2}{\gamma_1}\,x & 0\end{bmatrix},\tag{4.4}$$

for some constants Θ_1 *and* Θ_2 *such that* $\Theta_1,\Theta_2\neq\dfrac{n-1}{n}$, *for any* $n\geq 1$.

In [7, Théorème 5.2] is shown that the cubic components of a threefold symmetric 2-orthogonal sequence are also 2-orthogonal. The structure of the vector of two linear functionals for which the cubic components are 2-orthogonal polynomial is also explained. We recall this result in Lemma 4.3. Beforehand, and for a matter of completeness, we note that for any measure μ such that all moments exist and are finite, we can naturally define in $\mathscr{P}'$ (the dual space of the vector space of polynomials $\mathscr{P}$) a linear functional u such that $\langle u,p\rangle=\int p(x)\mathrm{d}\mu(x)$, for all $p\in\mathscr{P}$. Given a polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}}$ in $\mathscr{P}$, we can build its corresponding dual sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{P}'$ through $\langle u_n,P_m\rangle=\delta_{nm}$.

Lemma 4.3. *[7, Proposition 6.2] (cf. [20, Lemma 2.3.]) Suppose* $\{P_n(x)\}_{n\in\mathbb{N}}$ *is a threefold symmetric 2-orthogonal polynomial sequence with respect to a pair of linear functionals* (u_0,u_1) *and let* $\{u_n\}_{n\in\mathbb{N}}$ *be the corresponding dual sequence. Then, there exist three cubic components* $\left\{P_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$, *with* $k\in\{0,1,2\}$, *satisfying* (1.22) *and they are 2-orthogonal with respect to the vector of linear functionals* $\left(u_0^{[k]},u_1^{[k]}\right)$ *given by* $u_0^{[k]}=\sigma_3(x^k u_k)$ *and* $u_1^{[k]}=\sigma_3(x^k u_{k+3})$, *where* $\sigma_3:\mathscr{P}'\to\mathscr{P}'$ *represents the linear operator defined in* $\mathscr{P}'$ *by* $\langle\sigma_3(v),p(x)\rangle:=\langle v,f(x^3)\rangle$ *for any* $v\in\mathscr{P}'$ *and* $p\in\mathscr{P}$.

As explained in [22, 23], all elements of the dual sequence of a 2-orthogonal polynomial sequence can be written as a combination of the first two elements. Namely, there exists polynomials $E_n(x)$, $a_n(x)$, $F_n(x)$ and $b_n(x)$ such that $\deg E_n=\deg F_n=n$, $\deg \mathrm{a}_n\leq n$ and $\deg \mathrm{b}_n\leq n$ such that

$$u_{2n}=E_n(x)u_0+\mathrm{a}_{n-1}(x)u_1 \text{ and } u_{2n+1}=\mathrm{b}_n(x)u_0+F_n(x)u_1,\quad \text{for all}\quad n\in\mathbb{N}.$$

We have just used the product of a polynomial f by a linear functional u, which is defined by duality: $\langle fu,p\rangle:=\langle u,fp\rangle$, for any $p\in\mathscr{P}$. The polynomials E_n, F_n, a_n and b_n satisfy recursive relations, which can be found in [22]. For our purpose, we need the expressions of the following ones:

$$\begin{cases} u_2=E_1u_0+\mathrm{a}_0u_1=\frac{x}{\gamma_1}u_0,\\ u_3=\mathrm{b}_1u_0+F_1u_1=-\frac{1}{\gamma_2}u_0+\frac{x}{\gamma_2}u_1=\frac{1}{\gamma_2}\left(xu_1-u_0\right),\\ u_4=E_2u_0+\mathrm{a}_1u_1=\frac{x^2}{\gamma_1\gamma_3}u_0-\frac{1}{\gamma_3}u_1=\frac{1}{\gamma_1\gamma_3}\left(x^2u_0-\gamma_1u_1\right),\\ u_5=\mathrm{b}_2u_0+F_2u_1=-\left(\frac{1}{\gamma_1\gamma_4}+\frac{1}{\gamma_2\gamma_4}\right)xu_0+\frac{x^2}{\gamma_2\gamma_4}u_1=\frac{1}{\gamma_2\gamma_4}\left(x^2u_1-\left(1+\frac{\gamma_2}{\gamma_1}\right)xu_0\right).\end{cases}\tag{4.5}$$

The latter allows us to describe for each $k\in\{0,1,2\}$ the vector of functionals $\left(u_0^{[k]},u_1^{[k]}\right)$ in Lemma 4.3, which are used to derive the following result.

Proposition 4.4. *Suppose* $\{P_n(x)\}_{n\in\mathbb{N}}$ *is a* 3*-fold symmetric polynomial sequence satisfying* (1.18)*, with* $\beta_n = \alpha_n = 0$ *and* $\gamma_{n+1} > 0$*, whose* 2*-orthogonality measures* μ_0 *and* μ_1 *admit the integral representations given by* (4.2)*. Then the cubic components* $\left\{P_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$*,* $k \in \{0,1,2\}$*, are* 2*-orthogonal with respect to the pairs of measures* $\left(\mu_0^{[k]},\mu_1^{[k]}\right)$ *admitting the integral representation*

$$\int f(x)\mathrm{d}\mu_j^{[k]}(x) = \int_0^{\gamma^3} f(x)\,\mathscr{U}_j^{[k]}(x)\mathrm{d}x, \quad j=0,1, \tag{4.6}$$

where the weight functions $\mathscr{U}_j^{[k]}(x)$ *are*

$$\begin{cases} \mathscr{U}_0^{[0]}(x) = \dfrac{1}{3}x^{-\frac{2}{3}}\mathscr{U}_0\left(x^{\frac{1}{3}}\right) \text{ and } \mathscr{U}_1^{[0]}(x) = \dfrac{1}{3\gamma_2}\left(x^{-\frac{1}{3}}\mathscr{U}_1\left(x^{\frac{1}{3}}\right) - x^{-\frac{2}{3}}\mathscr{U}_0\left(x^{\frac{1}{3}}\right)\right), & \text{(4.7a)} \\ \mathscr{U}_0^{[1]}(x) = \dfrac{1}{3}x^{-\frac{1}{3}}\mathscr{U}_1\left(x^{\frac{1}{3}}\right) \text{ and } \mathscr{U}_1^{[1]}(x) = \dfrac{1}{3\gamma_1\gamma_3}\left(x^{\frac{1}{3}}\mathscr{U}_0\left(x^{\frac{1}{3}}\right) - \gamma_1 x^{-\frac{1}{3}}\mathscr{U}_1\left(x^{\frac{1}{3}}\right)\right), & \text{(4.7b)} \\ \mathscr{U}_0^{[2]}(x) = \dfrac{1}{3\gamma_1}x^{\frac{1}{3}}\mathscr{U}_0\left(x^{\frac{1}{3}}\right) \text{ and } \mathscr{U}_1^{[2]}(x) = \dfrac{1}{3\gamma_2\gamma_4}\left(x^{\frac{2}{3}}\mathscr{U}_1\left(x^{\frac{1}{3}}\right) - \left(1+\dfrac{\gamma_2}{\gamma_1}\right)x^{\frac{1}{3}}\mathscr{U}_0\left(x^{\frac{1}{3}}\right)\right). & \text{(4.7c)} \end{cases}$$

Proof. Note that it is sufficient to prove (4.6) for $f(x) = x^n$, for all $n \in \mathbb{N}$. Observe that the integral representations given by (4.2) imply that, for $j \in \{0,1\}$ and $n \in \mathbb{N}$, $\langle u_j, x^{3n+k}\rangle = 0$, if $k \in \{0,1,2\}\backslash\{j\}$, and

$$\langle u_j, x^{3n+j}\rangle = \int_0^{\gamma} x^{3n+j}\,\mathscr{U}_j(x)\mathrm{d}x = \frac{1}{3}\int_0^{\gamma^3} t^{n+\frac{j-2}{3}}\,\mathscr{U}_j\left(t^{\frac{1}{3}}\right)\mathrm{d}t. \tag{4.8}$$

Then, using Lemma 4.3, we have

$$\int x^n \mathrm{d}\mu_0^{[0]}(x) = \langle \sigma_3(u_0), x^n\rangle = \langle u_0, x^{3n}\rangle = \frac{1}{3}\int_0^{\gamma} x^{n-\frac{2}{3}}\mathscr{W}_0\left(x^{\frac{1}{3}}\right)\mathrm{d}x \tag{4.9}$$

and

$$\int x^n \mathrm{d}\mu_0^{[1]}(x) = \langle \sigma_3(xu_1), x^n\rangle = \langle xu_1, x^{3n}\rangle = \langle u_1, x^{3n+1}\rangle = \frac{1}{3}\int_0^{\gamma} x^{n-\frac{1}{3}}\mathscr{U}_1\left(x^{\frac{1}{3}}\right)\mathrm{d}x, \tag{4.10}$$

which give the expressions for $\mathscr{U}_0^{[0]}(x)$ and $\mathscr{U}_0^{[1]}(x)$ in (4.7a) and (4.7b), respectively.

Additionally, using the expressions for the elements of the dual sequence u_2, u_3, u_4 and u_5 given in (4.5), we successively get:

$$\begin{aligned} &\int x^n \mathrm{d}\mu_0^{[2]}(x) = \langle \sigma_3(x^2u_2), x^n\rangle = \langle x^2u_2, x^{3n}\rangle = \frac{1}{\gamma_1}\langle x^3u_0, x^{3n}\rangle = \frac{1}{\gamma_1}\langle u_0, x^{3n+3}\rangle, \\ &\int x^n \mathrm{d}\mu_1^{[0]}(x) = \langle \sigma_3(u_3), x^n\rangle = \langle u_3, x^{3n}\rangle = \frac{1}{\gamma_2}\left(\langle u_1, x^{3n+1}\rangle - \langle u_0, x^{3n}\rangle\right), \\ &\int x^n \mathrm{d}\mu_1^{[1]}(x) = \langle \sigma_3(xu_4), x^n\rangle = \langle xu_4, x^{3n}\rangle = \frac{1}{\gamma_3}\left(\frac{1}{\gamma_1}\langle u_0, x^{3n+3}\rangle - \langle u_1, x^{3n+1}\rangle\right), \\ &\int x^n \mathrm{d}\mu_1^{[2]}(x) = \langle \sigma_3(x^2u_5), x^n\rangle = \langle x^2u_5, x^{3n}\rangle = \frac{1}{\gamma_2\gamma_4}\left(\langle u_1, x^{3n+4}\rangle - \left(1+\frac{\gamma_2}{\gamma_1}\right)\langle u_0, x^{3n+3}\rangle\right), \end{aligned}$$

which, because of the last identities in (4.9)-(4.10), lead to

$$\mathscr{U}_0^{[2]}(x) = \frac{x}{\gamma_1}\,\mathscr{U}_0^{[0]}(x), \quad \mathscr{U}_1^{[0]}(x) = \frac{1}{\gamma_2}\left(\mathscr{U}_0^{[1]}(x) - \mathscr{U}_0^{[0]}(x)\right), \quad \mathscr{U}_1^{[1]}(x) = \frac{1}{\gamma_3}\left(\mathscr{U}_0^{[2]}(x) - \mathscr{U}_0^{[1]}(x)\right),$$
$$\mathscr{U}_1^{[2]}(x) = \frac{x}{\gamma_2\gamma_4}\left(\mathscr{U}_0^{[1]}(x) - \left(1+\frac{\gamma_2}{\gamma_1}\right)\mathscr{U}_0^{[0]}(x)\right).$$

Finally, (4.7a)-(4.7c) follow directly from the latter identities, after we take into account the already obtained expressions for $\mathscr{U}_0^{[0]}(x)$ and $\mathscr{U}_0^{[1]}(x)$. □

We can now prove the main result in this section.

Proof of Theorem 4.1. If $\{P_n(x)\}_{n\in\mathbb{N}}$ is a 3-fold symmetric Hahn-classical 2-orthogonal polynomial sequence, then the sequence of derivatives $\left\{Q_n(x) := \frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}\left(P_n(x)\right)\right\}_{n\in\mathbb{N}}$ is also 3-fold symmetric and 2-orthogonal and, recalling Lemma 4.3, the same holds for the cubic components $\left\{Q_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$, $k\in\{0,1,2\}$. As a result, it is straightforward to check that Theorem 4.1 is valid for $k=0$, that is, $\left\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{[0]}(x)\right\}_{n\in\mathbb{N}}$ is a 2-orthogonal polynomial sequence because

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(P_{n+1}^{[0]}(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(P_{3n+3}\left(x^{\frac{1}{3}}\right)\right) = \frac{x^{-\frac{2}{3}}}{3}\,P'_{3n+3}\left(x^{\frac{1}{3}}\right) = (n+1)x^{-\frac{2}{3}}Q_{3n+2}\left(x^{\frac{1}{3}}\right) = (n+1)Q_n^{[2]}(x). \quad (4.11)$$

This observation was already made by Douak and Maroni in [7, Corollaire 5.1]. However, an analogous procedure does not give an obvious way to conclude about the 2-orthogonality of $\left\{\frac{1}{n+1}\frac{\mathrm{d}}{\mathrm{d}x}P_{n+1}^{[k]}(x)\right\}_{n\in\mathbb{N}}$, for $k\in\{1,2\}$. So we take a different approach to prove this.

According to Proposition 2.6, to prove the Hahn-classical character of $\left\{P_n^{[k]}(x)\right\}_{n\in\mathbb{N}}$, it is sufficient to find matrices $\Phi^{[k]}(x) = \begin{bmatrix}\phi_{00} & \phi_{01}\\ \varphi(x) & \phi_{11}\end{bmatrix}$ and $\Psi^{[k]}(x) = \begin{bmatrix}\eta_0 & \eta_1\\ \psi(x) & \xi\end{bmatrix}$, for some constants ϕ_{00}, ϕ_{01}, ϕ_{11}, η_0, η_1 and ξ and polynomials φ and ψ with $\deg\varphi\le 1$ and $\deg\psi = 1$, such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\Phi^{[k]}(x)\overline{\mathscr{U}}^{[k]}(x)\right) + \Psi^{[k]}(x)\overline{\mathscr{U}}^{[k]}(x) = 0, \quad (4.12)$$

where we use the notation $\overline{\mathscr{U}}^{[k]}(x) := \begin{bmatrix}\mathscr{U}_0^{[k]}(x)\\ \mathscr{U}_1^{[k]}(x)\end{bmatrix}$. Similarly, we also consider $\overline{\mathscr{U}}(x) := \begin{bmatrix}\mathscr{U}_0(x)\\ \mathscr{U}_1(x)\end{bmatrix}$.

To find the matrices $\Phi^{[k]}(x)$ and $\Psi^{[k]}(x)$, we start by rewriting formulas (4.7b) and (4.7c) as

$$\overline{\mathscr{U}}^{[k]}(s) = \frac{1}{3}\,T_k(s)\overline{\mathscr{U}}\left(s^{\frac{1}{3}}\right),$$

where

$$T_1(s) = \begin{bmatrix} 0 & s^{-\frac{1}{3}} \\ \frac{1}{\gamma_1\gamma_3} s^{\frac{1}{3}} & -\frac{1}{\gamma_3} s^{-\frac{1}{3}} \end{bmatrix} \text{ and } T_2(s) = \begin{bmatrix} \frac{1}{\gamma_1} s^{\frac{1}{3}} & 0 \\ -\frac{1}{\gamma_4}\left(\frac{1}{\gamma_2}+\frac{1}{\gamma_1}\right) s^{\frac{1}{3}} & \frac{1}{\gamma_2\gamma_4} s^{\frac{2}{3}} \end{bmatrix}.$$

These equations are naturally equivalent to

$$\overline{\mathscr{U}}\left(s^{\frac{1}{3}}\right) = 3\, T_k^{-1}(s)\overline{\mathscr{U}}^{[k]}(s),$$

with

$$T_1^{-1}(s) = \begin{bmatrix} \gamma_1 s^{-\frac{1}{3}} & \gamma_3\gamma_1 s^{-\frac{1}{3}} \\ s^{\frac{1}{3}} & 0 \end{bmatrix} \text{ and } T_2^{-1}(s) = \begin{bmatrix} \gamma_1 s^{-\frac{1}{3}} & 0 \\ (\gamma_1+\gamma_2)\, s^{-\frac{2}{3}} & \gamma_2\gamma_4 s^{-\frac{2}{3}} \end{bmatrix}.$$

If we consider the change of variable $s = x^{\frac{1}{3}}$ in matrix differential equation (4.3) and then use the previous formula, we obtain, for both $k \in \{1,2\}$,

$$9x^{\frac{2}{3}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi\left(x^{\frac{1}{3}}\right) T_k^{-1}(x)\overline{\mathscr{U}}^{[k]}(x)\right) + 3\Psi\left(x^{\frac{1}{3}}\right) T_k^{-1}(x)\overline{\mathscr{U}}^{[k]}(x) = 0, \tag{4.13}$$

or, equivalently,

$$3x^{\frac{2}{3}}\Phi\left(x^{\frac{1}{3}}\right) T_k^{-1}(x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathscr{U}}^{[k]}(x)\right) + \left(\Psi\left(x^{\frac{1}{3}}\right) T_k^{-1}(x) + 3x^{\frac{2}{3}} \frac{\mathrm{d}}{\mathrm{d}x}\left(\Phi\left(x^{\frac{1}{3}}\right) T_k^{-1}(x)\right)\right)\overline{\mathscr{U}}^{[k]}(x) = 0. \tag{4.14}$$

For $k = 1$, relation (4.14) reads as

$$\begin{aligned} 3&\begin{bmatrix} (1-\Theta_1)x^{\frac{4}{3}} + \Theta_1\gamma_1 x^{\frac{1}{3}} & \Theta_1\gamma_1\gamma_3 x^{\frac{1}{3}} \\ x & 6(1-\Theta_2)\gamma_3 x \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathscr{U}}^{[1]}(x)\right) \\ &+ \begin{bmatrix} (3-2\Theta_1)x^{\frac{1}{3}} - \Theta_1\gamma_1 x^{-\frac{2}{3}} & -\Theta_1\gamma_1\gamma_3 x^{-\frac{2}{3}} \\ 3 & -2(2\Theta_2-1)\gamma_3 \end{bmatrix}\overline{\mathscr{U}}^{[1]}(x) = 0, \end{aligned}$$

which, after a multiplication by $\begin{bmatrix} 0 & 1 \\ x^{\frac{2}{3}} & 0 \end{bmatrix}$, leads to (4.12) for $k = 1$, with

$$\Phi^{[1]}(x) = 3\begin{bmatrix} \frac{1}{2(2\Theta_2-1)\gamma_3} & \frac{1-\Theta_2}{2\Theta_2-1} \\ (1-\Theta_1)x + \Theta_1\gamma_1 & \Theta_1\gamma_1\gamma_3 \end{bmatrix} \text{ and } \Psi^{[1]}(x) = \begin{bmatrix} 0 & 1 \\ (4\Theta_1-3)x - 4\Theta_1\gamma_1 & -4\Theta_1\gamma_1\gamma_3 \end{bmatrix}.$$

As a result, $\left\{P_n^{[1]}(x)\right\}_{n\in\mathbb{N}}$ is a Hahn-classical 2-orthogonal polynomial sequence.

To prove the result for $k=2$, we start by using the relation $\gamma_1 = \dfrac{1}{3}(4\Theta_1 - 3)\gamma_2$ (see [20, Theorem 3.2] with $n=0$) to rewrite

$$T_2^{-1}(s) = \begin{bmatrix} \dfrac{1}{3}(4\Theta_1 - 3)\gamma_2 s^{-\frac{1}{3}} & 0 \\ \dfrac{4}{3}\Theta_1\gamma_2 s^{-\frac{2}{3}} & \gamma_2\gamma_4 s^{-\frac{2}{3}} \end{bmatrix}.$$

Therefore (4.14) becomes

$$\begin{aligned} 3 &\begin{bmatrix} \dfrac{1}{3}\Theta_1\gamma_2 x^{\frac{1}{3}} & \gamma_2\gamma_4(1-\Theta_1)x^{\frac{1}{3}} \\ 2(1-\Theta_2)x + \dfrac{4}{3}\gamma_2(2\Theta_2 - 1) & \gamma_2\gamma_4(2\Theta_2 - 1) \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}x}\left(\overline{\mathscr{U}}^{[2]}(x)\right) \\ &+ \begin{bmatrix} \Theta_1\gamma_2 x^{-\frac{2}{3}} & \Theta_1\gamma_2\gamma_4 x^{-\frac{2}{3}} \\ 2(2-\Theta_2) + \dfrac{8}{3}\gamma_2(1-2\Theta_2)x^{-1} & -2\gamma_2\gamma_4(2\Theta_2 - 1)x^{-1} \end{bmatrix} \overline{\mathscr{U}}^{[2]}(x) = 0, \end{aligned}$$

which, after a multiplication by $\begin{bmatrix} x^{\frac{2}{3}} & 0 \\ 0 & x \end{bmatrix}$, corresponds to (4.12) for $k=2$, with

$$\Phi^{[2]}(x) = \begin{bmatrix} \dfrac{\Theta_1\gamma_2}{3\gamma_1\gamma_4} & (1-\Theta_1)\dfrac{\gamma_2}{\gamma_1} \\ \varphi^{[2]}(x) & 3(2\Theta_2 - 1)\gamma_2\gamma_4 \end{bmatrix} \quad \text{and} \quad \Psi^{[2]}(x) = \begin{bmatrix} 0 & 1 \\ \psi^{[2]}(x) & -5(2\Theta_2 - 1)\gamma_2\gamma_4 \end{bmatrix},$$

where $\varphi^{[2]}(x) = 6(1-\Theta_2)x + 4(2\Theta_2 - 1)\gamma_2$ and $\psi^{[2]}(x) = 2(5\Theta_2 - 4)x - \dfrac{20}{3}\Theta_1(2\Theta_2 - 1)\gamma_2$.
Hence, $\left\{P_n^{[2]}(x)\right\}_{n\in\mathbb{N}}$ is Hahn-classical. □

We have shown here that the type II multiple orthogonal polynomials characterised in Section 3 generalise the cubic components of cases B1 and B2 of the Hahn-classical 3-fold symmetric 2-orthogonal polynomials in a similar way to how the type II multiple orthogonal polynomials on the step line with respect to the modified Bessel weights, defined by (3.11), generalise the cubic components of case A. As proved in Theorem 4.1, the cubic components of case C are again Hahn-classical. It remains an open question if there is an analogous generalisation for these components and, in case there is one, if that generalisation can be such the differentiation gives a shift on parameters. In this scenario, we also expect the confluence relations between case C and cases B1 and B2 to be preserved. We defer this investigation to a forthcoming work.

The study we have carried on Section 4 can be adjusted to more than two measures. Taking into account results given in [1, 2, 8], the structure of the proof of Theorem 4.1 can be (routinely) adapted to study each of the $(d+1)$ components of Hahn-classical d-orthogonal polynomials satisfying a $(d+1)$-fold symmetry (therefore satisfying a three term recurrence relation of order $d+1$).

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