Linearisability and Integrability of Discrete Dynamical Systems from Cluster and LP Algebras

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Well, they surrounded the house, they smoked him out, took him off in chains
The sky turned black and bruised and we had months of heavy rains
Now, the ravens nest in the rotted roof of Chenoweth's old place
And no one's asking Cal about that scar upon his face
'Cause there's nothin' strange about an axe with bloodstains in the barn
There's always some killin' you got to do around the farm
A murder in the red barn, a murder in the red barn...

Waits

Abstract

From the bipartite belt of a cluster algebra one may obtain generalisations of frieze patterns. It has been proven that linear relations exist within these frieze patterns if the associated quiver is, up to mutation equivalence, Dynkin or affine. The second chapter of this work is devoted to reproving this fact, for $\tilde{D}$ and $\tilde{E}$ types, using alternative methods to the known proof, allowing much more detail. We prove the existence of periodic quantities for affine ADE friezes with periods that mirror the widths of the tubes of their Auslander-Reiten quivers. Furthermore we interpret these friezes as discrete dynamical systems, given by a generalised cluster map. We prove the integrability of a reduction of this cluster map for each $\tilde{E}$ type and for $\tilde{D}_N$ where N is odd.

In our third chapter we consider recurrences that lie beyond cluster algebras, in LP algebras, named because mutation in these algebras has the Laurent property, like cluster algebras. We examine two particular examples of these recurrences and show that they can be linearised. We also show that they can be obtained by reductions of lattice equations. Finally we consider a 2-dimensional version of the Laurent property and give large sets of initial values such that these lattice equations possess this generalised Laurent property.

Acknowledgements

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Contents

Chapter 1

Introduction and background

Cluster algebras were defined in the early 2000s by Fomin and Zelevinsky [14] in, to use those author's own words, "an attempt to create an algebraic framework for dual canonical bases and total positivity in semisimple groups". This, and a series of following papers [3, 16, 17], the second of which written with Berenstein, laid the groundwork for the nascent field.

Cluster algebras were soon found to be related to diverse areas of mathematics including Teichmüller theory [13, 23], representation theory [5, 6, 31], frieze patterns [2, 8, 9, 38] and integrable systems [19, 20].

While at first cluster algebras seem unwieldy and their construction perverse, one can argue that their beautiful properties justify these peculiarities. In [16], for example, it is shown that the "finite type" cluster algebras are classified in the same way as the Cartan-Killing classification of semisimple complex Lie algebras. Another remarkable and surprising result, from [14], is the Laurent property, that every cluster variable can be written as a Laurent polynomial in a set of initial cluster variables. In [15] the authors generalised this statement to situations beyond cluster algebras. Lam and Pylyavskyy took these ideas further still in [34] by defining LP algebras, where the Laurent property holds by construction. Many examples of recurrences appearing in LP algebras are given in [1], two of which motivate the work of the latter part of this thesis.

Cluster algebras can also give rise to recurrence relations and discrete dynamical systems. Due to the Laurent property, the iterates of these recurrences are Laurent polynomials in the initial cluster variables. Moreover, in [19] it was shown that the iterates of the (nonlinear) recurrences associated with affine A quivers satisfy linear relations, and the associated dynamical systems are integrable. We concern ourselves with generalising these results for affine D and E types.

The first chapter of this work is to motivate the theory of cluster algebras and give necessary background information for the results obtained in latter chapters. Section 1.1 is devoted to defining (coefficient free) cluster algebras, stating some fundamental theorems and elucidating the relationship between cluster algebras and root systems, quiver representations and friezes. In Section 1.2 we discuss how to obtain dynamical systems from cluster algebras. We give examples and review the results of [19]. In Section 1.3 we define Liouville integrability and review the integrability of the cluster map from [19]. Finally Section 1.4 is used to define LP algebras and show how one may obtain recurrence relations from them.

The second chapter gives the results of [40], where we generalised the cluster map of [19] to obtain a generalised frieze sequence for affine D and E type quivers. We then obtain periodic quantities for this cluster map and linear relations between the frieze entries, extending the results of [33] which were obtained using other methods. Finally we prove the integrability of this cluster map for each affine E type and affine D_N where N is odd.

In the third chapter we review the author's share of the results from [26]. We examine and linearise the "Little Pi" recurrence of [1]. This recurrence was shown in [30] to be a standing wave reduction of a 2-d lattice equation. We apply similar methods to another lattice equation obtained indirectly from the "Extreme polynomial" of [1]. Using the algorithm of [41] we produce sets of initial values for these two equations that give well defined solutions on the entire $\mathbb{Z}^2$ lattice. We then prove that the Laurent property holds for these sets of values and give an example where it doesn't hold.

To aid the examiners of this thesis, and with no intended arrogance, the author would like to stress that the results of Sections 2 and 3 are entirely his own.

1.1 Cluster algebras

We first define quiver mutation and cluster mutation allowing us to define cluster algebras. We then give some standard results and demonstrate the links between cluster algebras and root systems, quiver representations and friezes.

1.1.1 Quiver mutation

Definition 1.1 (Quivers and quiver mutation)**.** A quiver Q is a directed graph where multiple edges are allowed. We refer to the sets of vertices and edges as Q_0 and Q_1, respectively. Here we disallow loops or two-cycles, shown in Figure 1.1. Quiver mutation at a vertex k is defined in three steps:

1. For each path $i \to k \to j$ add a new arrow $i \to j$.
2. Reverse the direction of all arrows entering or exiting k.
3. Delete all two-cycles that have appeared.

The mutation is denoted μ_k and the resulting quiver $Q' := \mu_k(Q)$.

$$1 \rightleftarrows 2 \qquad\qquad 3 \circlearrowright$$

Figure 1.1: Disallowed subquivers

Definition 1.2. The adjacency matrix for a quiver Q is $B = (b_{ij})$ where

$$b_{ij} := \text{ number of arrows } i \to j \text{ in } Q.$$

We consider an arrow $j \to i$ to be a negative arrow $i \to j$. Hence

$$-b_{ij} := \text{ number of arrows } j \to i \text{ in } Q.$$

Since we do not allow loops, $b_{ii} = 0$ and B is skew-symmetric.

We remark than there is a bijection between quivers and adjacency matrices, so we may use these interchangeably. Defining B' to be the adjacency matrix for $Q' = \mu_k(Q)$ for some k and we write $\mu_k(B) = B' = (b'_{ij})$, i.e.

$$b'_{ij} := \text{ number of arrows } i \to j \text{ in } Q',$$

then each b'_{ij} can be expressed in terms of the entries of B via

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise.} \end{cases} \tag{1.1}$$

This follows from the definition of quiver mutation.

Example 1.3. *Let Q be the top-left quiver of Figure 1.2. We'll demonstrate μ_3. Step 1 adds an arrow $1 \to 2$ as there is a two path $1 \to 3 \to 2$. Step 2 then reverses the arrows at 3. Finally step 3 removes the two-cycle $1 \to 2 \to 1$. The before and after B matrices are*

$$B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \qquad \mu_3(B) = B' := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

1.1.2 Cluster mutation

In addition to quiver mutation, we also have the notion of cluster mutation. For a quiver with N vertices we take indeterminates $x_1, x_2, \ldots, x_N$, called cluster variables. We can consider each x_i to be attached to the vertex i. We allow the

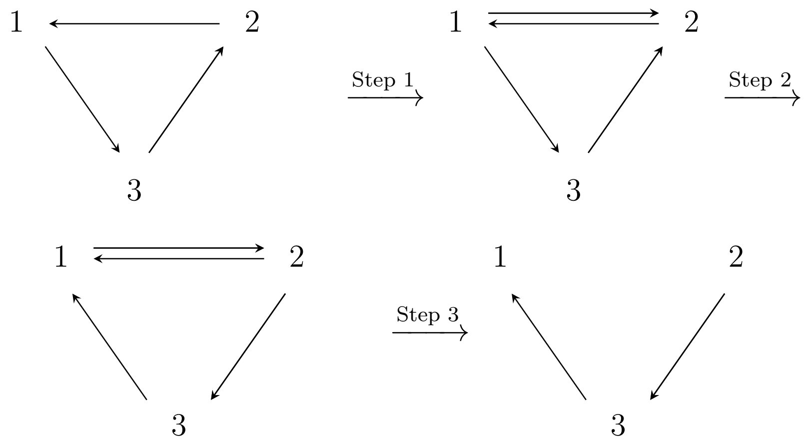


FIGURE 1.2: Example of quiver mutation. We apply μ_3 to the top-left quiver.

mutation μ_k to also act on the cluster variables as follows:

$$x_i' = \begin{cases} \frac{1}{x_i}(\prod_{j\to i} x_j + \prod_{j\leftarrow i} x_j) & i = k, \\ x_i & i \neq k \end{cases} \tag{1.2}$$

where $\prod_{j\to i}$ means the product is taken over all arrows from j to i in Q.

Remark 1.4. Often cluster algebras are considered with a further set of coefficient variables, denoted y_i, that are also mutated, with a distinct formula to (1.2), designed to mimic the Y- systems appearing in [44]. Moreover, these coefficient variables affect the x variable mutation. In this work we shall consider each y_i to be 1, giving us the simpler x mutation (1.2).

Example 1.5. *The mutation μ_3 in Example 1.3 fixes x_1 and x_2 but*

$$\mu_3(x_3) = \frac{1}{x_3}(x_1 + x_2).$$

We define the set $\mathbf{x} := \{x_1, \dots, x_N\}$ to be a cluster and the pair $(\mathbf{x}, B)$ to be a seed. Since μ_k fixes all but one of the cluster variables it gives a new cluster:

$$\mathbf{x}' := \mu_k(\mathbf{x}) = (x_1, \dots, x_k', \dots, x_N). \tag{1.3}$$

Collecting these discussions we define mutation of seeds as follows.

Definition 1.6. The mutation μ_k acts on seeds as

$$\mu_k : (\mathbf{x}, B) \mapsto (\mathbf{x}', B')$$

where B' is given indirectly in Definition 1.1 or directly in (1.1) and $\mathbf{x}'$ is given by (1.2) and (1.3).

1.1.3 The cluster algebra and basic results

We first list two immediate properties of mutation. Firstly that mutation is an involution on seeds, in that

$$\mu_k^2(Q) = Q, \qquad \mu_k^2(\mathbf{x}) = \mathbf{x}$$

where μ_k^2 means we apply μ_k twice. Secondly we note that if there is no edge between vertex i and vertex j in Q then μ_i and μ_j commute. That is to say if $b_{ij} = 0$ then

$$\mu_j \mu_i(\mathbf{x}, B) = \mu_i \mu_j(\mathbf{x}, B).$$

We say that two quivers Q and Q' are mutation equivalent if one can obtain Q' from a finite sequence of mutations applied to Q.

Definition 1.7. The set of quivers that are mutation equivalent to Q is called the mutation class of Q.

One question to ask is when is a mutation class finite, with the following answer proved in [12].

Theorem 1.8. *A quiver has finite mutation class if and only if the quiver can be obtained from a triangulation of a bordered two-dimensional surface or the quiver is mutation equivalent to one of* 11 *exceptional quivers.*

We note that the $A, D, \tilde{A}$ and $\tilde{D}$ type quivers can be obtained from surfaces and that the E and $\tilde{E}$ are included in the 11 exceptional quivers. The corresponding diagrams for these (where arrows are replaced with directionless edges) are shown in Figures 1.3 and 1.4.

Definition 1.9 (The cluster algebra)**.** Given an initial seed $(\mathbf{x}_0, B)$, the cluster algebra $\mathcal{A}(\mathbf{x}_0, B)$ is the algebra, over $\mathbb{C}$, generated by all of the cluster variables in each of the seeds mutation equivalent to $(\mathbf{x}_0, B)$. In fact, this does not depend on the choice of initial seed, so we may write instead $\mathcal{A}(B)$.

Definition 1.10 (Cluster algebras of finite or infinite type)**.** The cluster algebra is said to be of finite type if there are only finitely many seeds, and of infinite type otherwise.

Example 1.11. *The smallest non-trivial example is mutation of the A_2 quiver:*

$$1 \longrightarrow 2$$

In this case we can calculate all of the cluster variables. Recall that mutation is an involution, so an arbitrary composition of mutations will alternate in μ_1 and μ_2. Each of these mutations simply reverses the arrow in the quiver. Denoting the initial cluster variables x_1 and x_2 we can, without loss of generality, perform μ_1 first (performing μ_2 first is tantamount to relabelling $x_1 \leftrightarrow x_2$). Performing μ_1 and μ_2 in turn we get

$$x_1' := \mu_1(x_1) = \frac{1}{x_1}(1 + x_2),$$

$$x_2' := \mu_2(x_2) = \frac{1}{x_2}(1 + x_1') = \frac{1 + x_1 + x_2}{x_1 x_2},$$

$$x_1'' := \mu_1(x_1') = \frac{1}{x_1'}(1 + x_2') = \frac{x_1}{1 + x_2}\left(1 + \frac{1 + x_1 + x_2}{x_1 x_2}\right) = \frac{1 + x_1}{x_2},$$

$$\mu_2(x_2') = \frac{1}{x_2'}(1 + x_1'') = \frac{x_1 x_2}{1 + x_1 + x_2}\left(1 + \frac{1 + x_1}{x_2}\right) = x_1.$$

We see that $\mu_2(x_2') = x_1$. Another mutation will show that $x_1''' = x_2$. This means that there are only 5 cluster variables in this case, so this cluster algebra is of finite type.

Definition 1.12. If we replace the arrows in a quiver with directionless edges we call the result a diagram. If we take a diagram and choose directions for its edges we call the resulting quiver an orientation of the diagram.

The classification of cluster algebras of finite type was given in [16].

Theorem 1.13. *A cluster algebra is of finite type if and only if the associated quiver is mutation equivalent to an orientation of a simply laced Dynkin diagram, i.e. of type A, D, or E.*

The ADE type Dynkin diagrams are given in Figure 1.3. The subscript denotes the number of vertices. No two diagrams are mutation equivalent.

Lemma 1.14. *Every pair of orientations of a tree (in particular, Dynkin diagrams) are mutation equivalent. In fact, it is enough to mutate only at sinks or sources.*

Proof. This is proved by induction on the number of vertices. Let our two orientations be Q_1 and Q_2. Since Q_1 is a tree it has a leaf v connected to a single other vertex v'. By the inductive step $Q_1 \backslash \{v\}$ is mutation equivalent to $Q_2 \backslash \{v\}$ by a sequence of mutations μ', mutating only at sinks or sources. When considering μ' as a mutation of Q_1 it is possible that some mutations at v' will occur when v' is not a sink or source, due to the direction of the arrows between v and v'. However, by mutating at v wherever necessary (inserting some copies of μ_v inside μ' in the right places), we can ensure that we will always mutate at v' when it is a sink or source. This augmented mutation is a mutation equivalence of Q_1 and Q_2, up to possible direction of the arrows between v and v', which is easily fixed by a μ_v. □

Remark 1.15. There are, in fact, also Dynkin diagrams of types B, C, F and G, each of which has a double or triple edge. The ADE types drawn here are called simply laced because there is at most one edge between any two vertices. Here we will only deal with simply laced types so will use the terms “Dynkin diagram” and “simply laced Dynkin diagram” interchangeably.

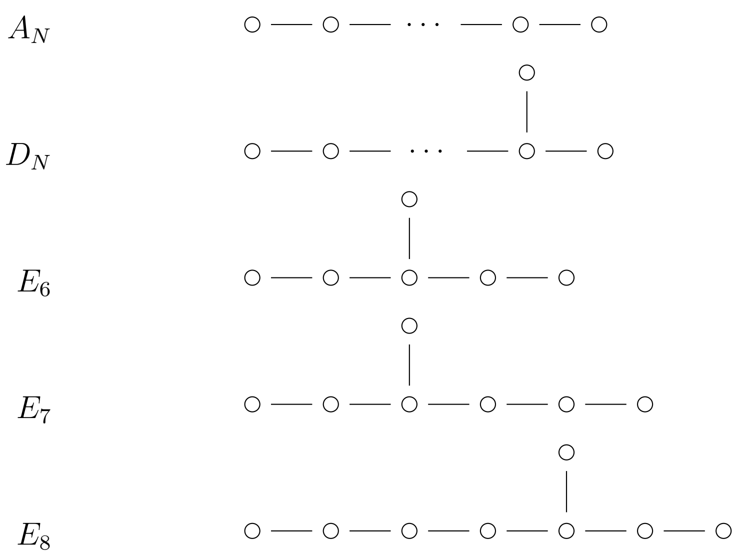


FIGURE 1.3: The Dynkin Diagrams

Example 1.11 also demonstrates a fundamental property of cluster algebras. Miraculously the non-monomial denominators in x_1'' and $\mu_2(x_2')$ cancel to leave a monomial.

Theorem 1.16 ([14], Laurent phenomenon)**.** *Every cluster variable can be expressed as a Laurent polynomial in the initial variables, with coefficients in $\mathbb{Z}$, i.e.*

$$x \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

for any cluster variable x, where $x_1, \ldots, x_n$ are the initial cluster variables.

In the finite type case we have more information about the monomials that appear in the denominators, involving the root system associated to the Dynkin diagram, which we discuss in the following subsection. A corresponding interpretation in terms of the cluster category may be found in [7].

1.1.4 Root systems

The definitions and results in this section are from [10] unless otherwise stated. Our goal is to define root systems and, in the ADE cases, link these to cluster algebras of finite type.

Definition 1.17. Let Q be a graph with vertices $1, 2, \ldots, n$ and $\alpha \in \mathbb{Z}^n$. We define a symmetric bilinear form on $\mathbb{Z}^n$ by

$$(e_i, e_j) = \begin{cases} -n_{ij} & i \neq j, \\ 2 - 2n_{ii} & i = j \end{cases}$$

where e_i is the ith basis vector and $n_{ij} = n_{ji}$ is the number of edges between vertices i and j. The Tits form is given by

$$q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{i \leq j} n_{ij}\alpha_i\alpha_j$$

and we observe that $q(\alpha) = \frac{1}{2}(\alpha, \alpha)$. The radical of q is defined as

$$\mathrm{rad}(q) = \{\alpha \in \mathbb{Z}^n \mid (\alpha, -) = 0\}.$$

Definition 1.18. We say that q is positive definite if

$$q(\alpha) > 0 \qquad \forall \alpha \in \mathbb{Z}^n \backslash \{0\}$$

and q is positive semi-definite if

$$q(\alpha) \geq 0 \qquad \forall \alpha \in \mathbb{Z}^n.$$

The quadratic form q gives the following equivalent definition of Dynkin and affine quivers.

Theorem 1.19. *For a connected quiver Q, the quadratic form q is positive definite if and only if Q is an orientation of a simply laced Dynkin diagram and q is positive semi-definite if and only if Q is an orientation of a simply laced affine diagram. In the latter case* $\mathrm{rad}\, q = \mathbb{Z}\delta$. *The simply laced affine diagrams with the components δ_i of the vector δ at each vertex i are shown in Figure 1.4*

Throughout this paper we have used a tilde to denote the affine version of the diagram or quiver. We note that, for affine types, the subscript is one less than the number of vertices.

Definition 1.20. An extending vertex of a affine diagram is a vertex that can be removed, along with any adjacent edges, to give the corresponding Dynkin diagram. These are the vertices i with $\delta_i = 1$.

For the rest of this section we shall only be discussing Dynkin or affine diagrams.

Definition 1.21. The set of roots of a diagram is

$$\{0 \neq \alpha \in \mathbb{Z}^n \mid q(\alpha) \leq 1\}.$$

We say that a root is real if $q(\alpha) = 1$ and imaginary if $q(\alpha) = 0$.

Lemma 1.22. *We have that*

$$q(\alpha) = 0 \Leftrightarrow \alpha \in \operatorname{rad} q.$$

Due to Theorem 1.19 and this lemma we have that a Dynkin digram has no imaginary roots and the imaginary roots of a affine diagram are precisely $\mathbb{Z}\delta$.

Proposition 1.23. *Each of the non-zero components of a root have the same sign.*

Because of this result we can call each root positive or negative, depending on the sign of one of the non-zero components.

Definition 1.24 (Simple roots, almost positive roots)**.** Let α_i be the ith standard basis vector for $\mathbb{Z}^n$. Since the diagrams we are dealing with have no edges from a vertex to itself, $q(\alpha_i) = 1$, so each α_i is a root and we call these the simple roots. The set of almost positive roots is $\Phi_{>0} \cup \{-\alpha_i\}_{i=1,\dots,n}$, where $\Phi_{>0}$ is the set of positive roots.

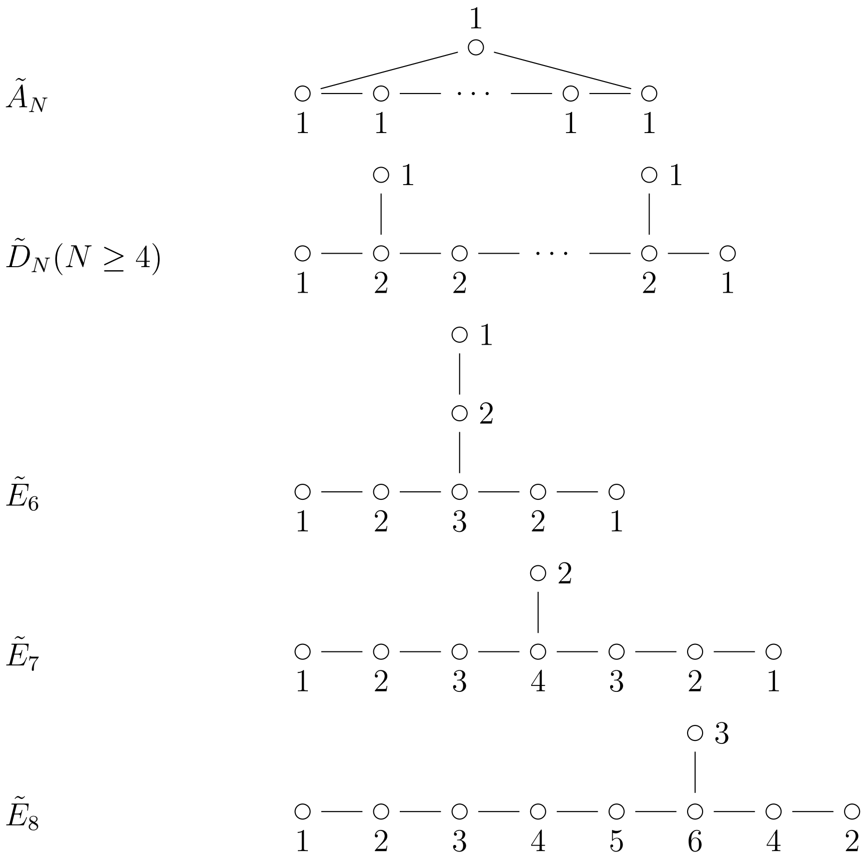


FIGURE 1.4: The simply laced affine diagrams with labels δ_i

Of course, the α_i form a basis for $\mathbb{Z}^n$ so every root α may be written as

$$\alpha = \sum_i a_i \alpha_i$$

with $a_i \in \mathbb{Z}$. Moreover, the positive roots have each $a_i \geq 0$.

Theorem 1.25. *[16]. For cluster algebras of finite type, there is a bijection between the cluster variables and the almost positive roots of the root system associated to the Dynkin diagram. Let the simple roots be $\{\alpha_i\}$, then the bijection is given by*

$$\alpha = \sum_i a_i \alpha_i \mapsto \frac{P_\alpha}{\prod_i x_i^{a_i}}$$

where P_α is a polynomial in the initial variables x_i. Note that

$$-\alpha_i \to \frac{1}{x_i^{-1}} = x_i$$

i.e. the almost positive roots are sent to the initial variables.

Example 1.26. *In Example 1.11 we found* 5 *cluster variables. Let* α_1 *and* α_2 *be the simple roots for* A_2*, then the almost positive roots are* $\pm\alpha_1$*,* $\pm\alpha_2$ *and* $\alpha_1+\alpha_2$*. The bijection sends*

$$-\alpha_1 \mapsto x_1, \qquad -\alpha_2 \mapsto x_2, \qquad \alpha_1 \mapsto x_1' = \frac{1+x_2}{x_1},$$

$$\alpha_2 \mapsto x_1'' = \frac{1+x_1}{x_2}, \qquad \alpha_1+\alpha_2 \mapsto x_2' = \frac{1+x_1+x_2}{x_1x_2}.$$

We have seen that Dynkin diagrams parametrise cluster finite cluster algebras. In fact they also parametrise representation finite quivers, which we now explain.

1.1.5 Quiver representations

The results and definitions from this section are from [31] and [10]. We define quiver representations before stating the link between Dynkin diagrams and representation finite quivers.

Definition 1.27. A representation V of a quiver Q is a vector space V_i for each vertex $i = 1, \ldots, n$ and a linear map $\theta_a : V_i \to V_j$ for each arrow $a : i \to j$. The dimension vector of V is given by

$$\underline{\dim}(V) = (\dim(V_1), \dim(V_2), \ldots, \dim(V_n)).$$

Definition 1.28. Let V be a representation as above and W a representation given by vector spaces W_i and morphisms θ'_a. A morphism of representations $f : V \to W$ is a set of linear maps $f_i : V_i \to W_i$ such that

$$\begin{tikzcd} V_i \arrow[r, "\theta_a"] \arrow[d, "f_i"] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "\theta'_a"] & W_j \end{tikzcd}$$

commutes for all arrows $a \in Q_1$. A morphism f is called an isomorphism if each component f_i is invertible. Two morphisms are composed in the obvious way, giving the category of quiver representations rep(Q).

Definition 1.29. Let V be a representation of Q. V' is called a subrepresentation if $V_i' \subset V_i$ for each i and $\theta_a(V_i') \subset V_j'$ for each arrow $a : i \to j$. One may consider the maps in the subrepresentation as being given by restricting θ_a to V_i'. A representation V is called simple if it has precisely two subrepresentations, 0 and V.

Definition 1.30. The direct sum of two representations V and W is given by $(V \bigoplus W)_i = V_i \bigoplus W_i$ and $(\theta \bigoplus \theta')_a = \theta_a \bigoplus \theta_a'$. Explicitly for vectors $v_i \in V_i$ and $w_i \in W_i$ and an arrow $a : i \to j$ we have

$$(\theta \bigoplus \theta')_a(v_i, w_i) = (\theta_a \bigoplus \theta_a')(v_i, w_i) = (\theta_a(v_i), \theta_a'(w_i)).$$

A representation $V \neq 0$ is called indecomposable if $V \cong V' \bigoplus V''$ implies that $V' = 0$ or $V'' = 0$.

Definition 1.31. A quiver Q is called representation finite if it has only finitely many indecomposable representations up to isomorphism.

Theorem 1.32 (From [21], stated in [31], Theorem 5.3)**.** *Let Q be a connected quiver and k algebraically closed. The following are equivalent*

1. *Q is representation finite,*
2. *The quadratic form q_Q is positive definite,*
3. *Q is an orientation of a simply laced Dynkin diagram*

and in this case the map $V \mapsto \underline{dim}(V)$ gives a bijection between the isomorphism classes of indecomposable representations and the set of positive roots.

Note that the equivalence of statements 1 and 2 is Theorem 1.19. Combining the bijections of Theorems 1.32 and 1.25 we obtain the following:

Theorem 1.33. *There is a bijection between the isomorphism classes of indecomposable representations and the non-initial cluster variables.*

Remark 1.34. Though it is beyond the scope of the thesis to give a detailed construction, we briefly describe Auslander-Reiten (AR) quivers. The vertices of these quivers are given by the isomorphism classes of indecomposable representations. The set of arrows is given by "irreducible maps" between these. The regular representations are defined to be those that are neither projective nor injective. For representation of affine quivers, the components of the AR quiver containing the regular modules are known as "tubes". The relevance of this discussion will become apparent in Section 2.1 where we compare these tubes to periodic quantities appearing in certain cluster algebras. We refer to [29] for an accessible introduction to AR theory.

1.1.6 Friezes and cluster algebras

Coxeter-Conway friezes, as arrays of integers in the plane, were defined by Coxeter in [9]. Many details are given in the review article [37]. We instead start by defining frieze sequences from quivers and show how to obtain these sequences via cluster mutation. We then discuss the relationship with Coxeter-Conway friezes.

Definition 1.35. The frieze sequence for an acyclic quiver Q with adjacency matrix $B = (b_{ij})$ is a sequence X_n^k for pairs $(k, n) \in V \times \mathbb{Z}$ given by

$$X_n^k X_{n+1}^k = 1 + \left(\prod_{i \to k} (X_n^i)^{|b_{ik}|}\right)\left(\prod_{k \to i} (X_{n+1}^i)^{|b_{ik}|}\right) \tag{1.4}$$

where the products is taken over all arrows in Q (there is no quiver mutation involved for this definition). We take initial variables X_0^k for each $k \in Q_0$.

The link between friezes and cluster algebras, which we now demonstrate, is not immediately obvious.

To make the construction cleaner we relabel the vertices of Q so that each vertex i is a sink in the full subquiver with vertices greater than or equal to i. That is, if we remove all vertices labelled less than i then i becomes a sink. This is possible for any acyclic quiver.

Example 1.36. *We relabel the A_4 quiver $1 \leftarrow 2 \rightarrow 3 \rightarrow 4$ so that it becomes $1 \leftarrow 4 \rightarrow 3 \rightarrow 2$.*

In this case the frieze sequence (1.4) becomes

$$X_n^k X_{n+1}^k = 1 + \left(\prod_{k<i} (X_n^k)^{|b_{ik}|}\right)\left(\prod_{i<k} (X_{n+1}^k)^{|b_{ik}|}\right) \tag{1.5}$$

since $i \rightarrow k$ in Q if and only if $k < i$ with $|b_{ik}| > 0$.

To obtain (1.5) from cluster mutations we take Q, labelled as above, and apply the sequence of mutations $\mu := \mu_N \mu_{N-1} \dots \mu_1$. Due to the relabelling each mutation will be applied at a sink so we simply reverse each arrow twice, returning the original quiver. Because we only use sink mutation, each new cluster variable x'_k will be given by a relation of the form

$$x'_k x_k = 1 + \prod$$

for a product Π. This product will be taken over neighbouring cluster variables. The x_i with $i < k$ will have already been mutated. This gives

$$x'_k x_k = 1 + \prod_{i<k} {x'_i}^{|b_{ik}|} \prod_{k<i} x_i^{|b_{ik}|}$$

where the $|b_{ik}|$ may be taken in the initial quiver, as these numbers do not change at any point under the sequence of mutations. This is of the right form, (1.5). We define the initial cluster variables $X_0^k := x_k$ and the once-mutated variables $X_k^1 := x'_k$. Since μ returns the original quiver we can keep applying it. Defining recursively $X_n^k := \mu^n(X_0^k)$ for all $n \in \mathbb{Z}$ we arrive at (1.5).

Example 1.37. *Returning to the relabelled A_4 quiver $1 \leftarrow 4 \rightarrow 3 \rightarrow 2$ we define initial cluster variables X_0^k for $k = 1, 2, 3, 4$. The first mutation μ_1 gives $X_1^1 := \mu_1(X_0^1)$, where*

$$X_1^1 X_0^1 = 1 + X_0^4,$$

and the new quiver $1 \to 4 \to 3 \to 2$*. Next* μ_2 *gives* $X_1^2 := \mu_2(X_0^2)$ *with*

$$X_1^2 X_0^2 = 1 + X_0^3,$$

and the quiver $1 \to 4 \to 3 \leftarrow 2$*. Note that now vertex* 3 *is a sink, so we have*

$$X_1^3 X_0^3 = 1 + X_1^2 X_0^4$$

and $1 \to 4 \leftarrow 3 \to 2$*. The* X_1^2 *term appears because we have already mutated at vertex* 2*. Finally, vertex* 4 *is a now a sink and* μ_4 *gives*

$$X_1^4 X_0^4 = 1 + X_1^1 X_1^3$$

and returns the original quiver. Further applications of $\mu := \mu_4\mu_3\mu_2\mu_1$ *will give cluster variables* X_n^k*, for all* $n \in \mathbb{Z}$*, defined by*

$$X_{n+1}^1 X_n^1 = 1 + X_n^4, \qquad X_{n+1}^2 X_n^2 = 1 + X_n^3,$$

$$X_{n+1}^3 X_n^3 = 1 + X_{n+1}^2 X_n^4, \qquad X_{n+1}^4 X_n^4 = 1 + X_{n+1}^1 X_{n+1}^3, \tag{1.6}$$

which are precisely the frieze relations (1.5) for this quiver.

We now describe friezes and their relation with frieze sequences.

Definition 1.38. A (Coxeter-Conway) frieze is a planar array such that:

- The 0th and $(m+3)$rd rows consist only of zeroes.
- The 1st and $(m+2)$nd rows consist only of ones.
- We display each row so that its entries sit "in between" those of the row above (see Figure 1.5) such that each diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

$$
\begin{array}{llllllllll}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\quad\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\qquad\dots & 4 & 1 & 2 & 2 & 2 & 1 & 4 & 1 & \dots \\
\quad\qquad\dots & 3 & 1 & 3 & 3 & 1 & 3 & 3 & 1 & \\
\qquad\qquad\dots & 2 & 1 & 4 & 1 & 2 & 2 & 2 & 1 & \\
\quad\qquad\qquad\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\
\qquad\qquad\qquad\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & &
\end{array}
$$

FIGURE 1.5: A frieze of width 3

$$
\begin{array}{lllll}
\dots & 1 & 1 & 1 & \dots \\
\quad\dots & X_n^1 & X_{n+1}^1 & X_{n+2}^1 & \dots \\
\qquad\dots & X_n^2 & X_{n+1}^2 & X_{n+2}^2 & \dots \\
\quad\qquad\dots & X_n^3 & X_{n+1}^3 & X_{n+2}^3 & \dots \\
\qquad\qquad\dots & X_n^4 & X_{n+1}^4 & X_{n+2}^4 & \dots \\
\quad\qquad\qquad\dots & \ddots & \ddots & \ddots & \\
\qquad\qquad\qquad\dots & X_n^N & X_{n+1}^N & X_{n+2}^N & \\
\quad\qquad\qquad\qquad\dots & 1 & 1 & &
\end{array}
$$

FIGURE 1.6: The frieze corresponding to the A_N quiver.

satisfies the "unimodular rule": $ad - bc = 1$.

We call m the width of the frieze, the number of rows in between the rows of ones.

We give an example of a frieze in Figure 1.5. We now demonstrate how to obtain these from cluster algebras. We take the A_N quiver:

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \dots \leftarrow N \tag{1.7}$$

The sequence of mutations $\mu_N \dots \mu_2\mu_1$ generates the frieze sequence given by

$$X_{n+1}^1 X_n^1 = 1 + X_n^2, \qquad X_{n+1}^N X_n^N = 1 + X_{n+1}^{N-1},$$

$$X_{n+1}^k X_n^k = 1 + X_{n+1}^{k-1} X_n^{k+1} \qquad \text{for} \quad k = 2, 3 \dots, N-1. \tag{1.8}$$

One can then construct the generic frieze of width N, Figure 1.6, which satisfies the unimodular rule due to (1.8).

Example 1.39. *The frieze of Figure 1.5 can be constructed from the A_3 quiver*

$$1 \to 2 \to 3$$

taking initial cluster variables $X_0^1 = 4, X_0^2 = 3$ and $X_0^3 = 2$.

Since every entry of the frieze is a cluster variable in the A_N cluster algebra, we know that, due to Theorem 1.13, there are finitely many (different) frieze entries. Hence the frieze will be periodic. This was known to Coxeter, whose proof we give here.

We label entries by $(r, s) \in \mathbb{Z} \times \mathbb{Z}$ with $r \leq s \leq r + m + 3$ as follows:

$$\begin{array}{ccccccccccccc}
\dots & & (-1,-1) & & & & (0,0) & & (1,1) & & (2,2) & & \dots \\
& \dots & & & (-1,0) & & & (0,1) & & (1,2) & & (2,3) & \\
& & \dots & & & & (-1,1) & & (0,2) & & (1,3) & & (2,4) \\
& & & & & & & \ddots & & \ddots & & \ddots &
\end{array}$$

The boundary rows are given by

$$(r, r) = (r, r + m + 3) = 0, \qquad (r, r + 1) = (r, r + m + 2) = 1 \tag{1.9}$$

for each $r \in \mathbb{Z}$. The unimodular rule becomes

$$(r, s)(r + 1, s + 1) - (r + 1, s)(r, s + 1) = 1.$$

It is clear that the entire frieze is determined by one diagonal, for example the one given by $(-1, s)$. One then calculates the entries of the diagonal $(0, s)$ starting with $(0, 2)$ and working south-east. Successive rows are calculated the same way. To fill the row to the left we start with $(-2, m - 1)$ in the row $(-1, s)$ and move north-west. A result of [9] allows us to express any frieze entry in terms of those on two adjacent diagonals:

$$(r, s) = (-1, r)(0, s) - (-1, s)(0, r), \tag{1.10}$$

for $0 \leq r, s \leq m+2$. This formula is proved by induction. We then arrive at the following result:

Theorem 1.40. *The frieze is periodic with period $m+3$, in that*

$$(r, s) = (r+m+3, s+m+3)$$

for each $(r, s) \in \mathbb{Z} \times \mathbb{Z}$.

Proof. By (1.10) and (1.9) we have

$$(r, m+2) = (-1, r)(0, m+2) - (-1, m+2)(0, r) = (-1, r)$$

for $0 \leq r \leq m+2$. This formula is invariant under horizontal shifts, giving

$$(r+a, m+2+a) = (-1+a, r+a)$$

for each $a \in \mathbb{Z}$. We relabel $r \mapsto s-r-1$ and set $a = 1+r$ to arrive at

$$(r, s) = (s, r+m+3) \tag{1.11}$$

hence

$$(r+m+3, s+m+3) = (s, r+m+3) = (r, s).$$

□

Example 1.41. *The frieze of Figure 1.5 has period $m+3=6$, we have coloured two equal diagonals in red to highlight this. This frieze also exhibits the glide reflection symmetry of Equation 1.11, which is seen by comparing the red diagonals with the blue.*

1.2 Discrete dynamical systems from cluster algebras

In this section we will review definitions and results from [19, 20] about periodic quivers and how we can obtain discrete dynamical systems from them.

Informally, a quiver is periodic if, after a sequence of mutations, the quiver looks the same, up to relabelling of the vertices.

Definition 1.42 (Periodic quiver, [20])**.** Putting the N vertices of the quiver Q on a circle we label such that as we travel clockwise around the circle the labels increase, except for 1 which follows N (as the numbers 1 to 12 are set into a clock face). We define the permutation

$$\rho : (1, 2, \ldots, N) \to (2, 3, \ldots, N, 1).$$

Q is said to have period m if a composition of mutations $\mu := \mu_m \mu_{m-1} \ldots \mu_1$ acts as

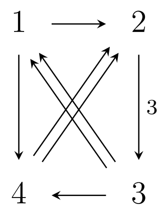
$$\mu(Q) = \rho^m(Q) \tag{1.12}$$

That is, the composition rotates the labels anticlockwise m times.

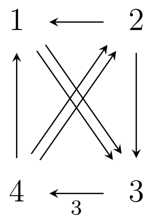
One may, of course, generalise this to allow any permutation, not just cyclic ones, as in [39]. Period 1 quivers, where this sequence involves just one mutation, were classified in [20].

Remark 1.43. If we consider the N vertices of Q to be equally spaced on a circle, with this clockwise ascending labelling, one can consider ρ as a rotation of the vertices by $2\pi/N$ anticlockwise, keeping the arrows fixed. Equivalently ρ is a rotation of the arrows by $2\pi/N$ clockwise with fixed vertices.

Example 1.44. *We demonstrate this with the Somos-4 quiver, studied in [19], where the "3" between vertices 2 and 3 denotes a triple arrow.*

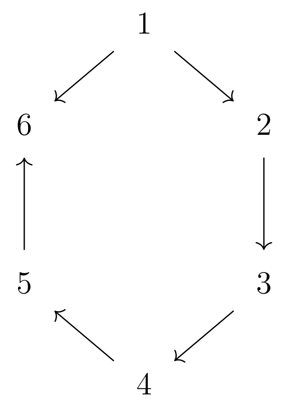
$Q =$ 

Performing μ_1 results in the quiver

$\mu_1(Q) =$ 

and we see that relabelling $\rho : (1, 2, 3, 4) \to (2, 3, 4, 1)$ in Q gives the same quiver $\mu_1(Q)$. Hence Q is a period 1 quiver.

Example 1.45. *The $\tilde{A}_{5,1}$ quiver is:*



which, again, is simply rotated by μ_1 so is a period 1 quiver.

In these examples we have $\mu_1(Q) = \rho(Q)$, hence these quivers are period 1. Observe that $\mu_{\rho(1)} = \mu_2$ will rotate the quiver once more. Moreover the cluster mutation formula for vertex 2 in $\rho(Q)$ is the same as for vertex 1 in Q. Explicitly if

$$x_1'x_1 = F(x_2, x_3, \ldots, x_N)$$

then

$$x_2' x_2 = F(x_3, x_4, \dots, x_1')$$

so when we apply the sequence of mutations $\mu_1, \mu_2, \mu_3 \dots$ the same polynomial F will always appear. From this we can define a map

$$\varphi : \begin{pmatrix} x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+N-1} \end{pmatrix} \mapsto \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+N-1} \\ x_{n+N} \end{pmatrix} \qquad x_{n+N} := \frac{F(x_{n+1}, x_{n+2}, \dots, x_{n+N-1})}{x_n} \tag{1.13}$$

for all $n \in \mathbb{Z}$, with F independent of n. This is called the cluster map, as defined in [19]. We can consider this as a recurrence relation for x_n, with initial values $x_0, x_1, \dots, x_{N-1}$.

Example 1.46. *For the Somos-4 quiver mutation at vertex* 1 *gives*

$$x_1' x_1 = F(x_2, x_3, x_4) = x_2 x_4 + x_3^2$$

so the form of (1.13) is

$$x_{n+4} = \frac{x_{n+1} x_{n+3} + x_{n+2}^2}{x_n}$$

which generates the iterates of the Somos-4 recurrence, justifying the name of the quiver.

Example 1.47. *The mutation μ_1 in the $\tilde{A}_{5,1}$ quiver gives the relation*

$$x_1' x_1 = 1 + x_6 x_2$$

from which we have the recurrence

$$x_{n+7} = \frac{1 + x_{n+1} x_{n+6}}{x_n}. \tag{1.14}$$

Since we can generate every iterate of (1.13) from cluster mutation, we have the following result that follows from Theorem 1.16.

Theorem 1.48. *Each iterate of the recurrence (1.13) is a Laurent polynomial in the N initial values.*

1.2.1 Period 1 quivers from $\tilde{A}_N$ diagrams

We now describe how to obtain period 1 quivers and cluster maps from certain orientations of the $\tilde{A}_N$ quivers, generalising the quiver of Example 1.45 and the corresponding recurrence (1.14). We remark that here our quivers will have labels $0, 1, \ldots, N-1$.

Firstly we take the diagram in Figure 1.7, with $N := p+q$ vertices, each labelled kp for $k = 0, 1, \ldots, (p+q-1)$, with p and q coprime. We then reduce the labels mod N such that each label k satisfies $0 \leq k < N$ and orient so that the arrows point from the lower labelled vertex to the higher. We give an example in Figure 1.8.

$$\begin{tikzcd}
& 0 \arrow[dl, dash] \arrow[dr, dash] & \\
(p+q-1)p \arrow[d, dash] & & p \arrow[d, dash] \\
(p+q-2)p \arrow[d, dash] & & 2p \arrow[d, dash] \\
(p+q-3)p \arrow[d, dash] & & 3p \arrow[d, dash] \\
\vdots \arrow[d, dash] & & \vdots \arrow[d, dash] \\
(q+1)p \arrow[dr, dash] & & (q-1)p \arrow[dl, dash] \\
& qp &
\end{tikzcd}$$

FIGURE 1.7: The first step to our orientation of the $\tilde{A}_N$ quiver.

Lemma 1.49. *The $\tilde{A}_N$ quiver with the above orientation is period* 1.

$$\begin{tikzcd}
 & 0 \arrow[dl] \arrow[dr] & \\
8 & & 7 \arrow[d] \\
1 \arrow[u] \arrow[d] & & 14 \\
9 & & 6 \arrow[u] \arrow[d] \\
2 \arrow[u] \arrow[d] & & 13 \\
10 & & 5 \arrow[u] \arrow[d] \\
3 \arrow[u] \arrow[ddr] & & 12 \\
 & & 4 \arrow[u] \arrow[dl] \\
 & 11 &
\end{tikzcd}$$

FIGURE 1.8: Our orientation of the $\tilde{A}_{15}$ quiver, with $p = 7$ and $q = 8$.

Proof. We take l such that $1 = lp \mod N$. One can see that moving each vertex of the diagram in Figure 1.7 by l steps anticlockwise has the effect of relabelling each vertex kp by $kp \mapsto (k + l)p$. After reducing mod N we have that $kp \mapsto (k + l)p = kp + 1 \mod N$. Hence this rotation is tantamount to adding 1 to each vertex label, except for $N - 1$ which becomes 0, and is now a sink.

On the other hand the mutation μ_0 will only reverse the arrows at 0, causing it to becomes a sink. One can see this that this mutation has the same affect as the above rotation. $\square$

The cluster map (1.13) now gives the recurrence

$$x_{n+N}x_n = x_{n+p}x_{n+q} + 1, \tag{1.15}$$

where $p + q = N$.

Remark 1.50. These quivers are constructed in [20], by summing the orbits of B matrices of single arrows under the conjugation action of the following matrix

$$\tau := \tau_N := \begin{pmatrix} 0 & \dots & & & & -1 \\ 1 & 0 & \dots & & & 0 \\ 0 & 1 & 0 & \dots & & 0 \\ & 0 & 1 & 0 & \dots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & \dots & 0 & 1 & 0 \end{pmatrix} \tag{1.16}$$

which is $N \times N$.

1.2.2 Periodic quantities and linear relations for the $\tilde{A}_N$ cluster map

Here we review results about (1.15) from [19, 20], where the authors obtained periodic quantities for this recurrence and linear relations for the iterates with constant coefficients. We briefly formalise our definition of "periodic quantity".

The map (1.13) induces an automorphism φ^*, a "shift", on the field of rational functions

$$\mathbb{C}(\{x_k\}_{k=0,\dots,N-1}) \tag{1.17}$$

by $\varphi^*(x_n) = x_{n+1}$ for each $n \in \mathbb{Z}$.

Definition 1.51. An element of the field (1.17) is said to be periodic, with period p, if it is fixed by $(\varphi^*)^p$. Non-constant period one elements are called invariants or first integrals.

We begin with the following result known as the Desanot-Jacobi identity, which is part of the Dodgson condensation algorithm [11]. A proof may be found in [4].

Lemma 1.52. *For an $N \times N$ matrix M we have*

$$|M||M_{1,N}^{1,N}| = |M_1^1||M_N^N| - |M_N^1||M_1^N| \tag{1.18}$$

where M_j^i denotes the matrix M with row i and column j deleted and $M_{1,N}^{1,N}$ is the matrix M with rows and columns 1 and N deleted.

Example 1.53. *For $N = 3$ and $M = (x_{ij})$ we have*

$$\begin{vmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{vmatrix} x_{1,1} = \begin{vmatrix} x_{0,0} & x_{0,1} \\ x_{1,0} & x_{1,1} \end{vmatrix} \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{vmatrix} - \begin{vmatrix} x_{1,0} & x_{1,1} \\ x_{2,0} & x_{2,1} \end{vmatrix} \begin{vmatrix} x_{0,1} & x_{0,2} \\ x_{1,1} & x_{1,2} \end{vmatrix}. \tag{1.19}$$

Theorem 1.54. *The recurrence (1.15) has period q and p quantities given by*

$$J_n := \frac{x_{n+2p} + x_n}{x_{n+p}}, \qquad \tilde{J}_n := \frac{x_{n+2q} + x_n}{x_{n+q}} \tag{1.20}$$

respectively.

Proof. We apply Lemma 1.52 to the matrix

$$\tilde{\Psi}_n := \begin{pmatrix} x_n & x_{n+p} & x_{n+2p} \\ x_{n+q} & x_{n+N} & x_{n+N+p} \\ x_{n+2q} & x_{n+N+q} & x_{n+2N} \end{pmatrix} \tag{1.21}$$

giving

$$|\tilde{\Psi}_n| x_{n+N} = |\Psi_n||\Psi_{n+N}| - |\Psi_{n+p}||\Psi_{n+q}| \tag{1.22}$$

where we have defined

$$\Psi_n := \begin{pmatrix} x_n & x_{n+p} \\ x_{n+q} & x_{n+N} \end{pmatrix}$$

Now the recurrence (1.15) is equivalent to $|\Psi_n| = 1$. Hence $|\tilde{\Psi}_n| = 0$. Next we take a scaled right kernel vector $(1, -J_n, C_n)^T$ for Ψ_n. In particular

$$\begin{pmatrix} x_n & x_{n+p} \\ x_{n+q} & x_{n+N} \end{pmatrix} \begin{pmatrix} 1 \\ -J_n \end{pmatrix} = C_n \begin{pmatrix} x_{n+2p} \\ x_{n+N+p} \end{pmatrix}.$$

From this we can quickly determine, using (1.15), that $C_n = 1$. The kernel equation $\Psi_n(1, -J_n, 1)^T = 0$ gives us

$$J_n = \frac{x_{n+2p} + x_n}{x_{n+p}} = \frac{x_{n+2p+q} + x_{n+q}}{x_{n+N}}$$

hence J_n is period q. By the same argument we have a left kernel vector $(1, -\tilde{J}_n, 1)$ with

$$\tilde{J}_n = \frac{x_{n+2q} + x_n}{x_{n+q}}$$

period p. □

This result immediately give the periodic coefficient linear relations

$$x_{n+2p} - J_n x_{n+p} + x_n = 0, \qquad x_{n+2q} - \tilde{J}_n x_{n+q} + x_n = 0. \tag{1.23}$$

The following result uses these to give constant coefficient linear relations.

Theorem 1.55. *The iterates of the recurrence (1.15) satisfy the constant coefficient linear relation*

$$x_{n+2pq} - \mathcal{K} x_{n+pq} + x_n = 0 \tag{1.24}$$

for a constant $\mathcal{K}$.

Proof. The linear relations (1.23) give that $\Psi_n L_n = \Psi_{n+p}$ where

$$L_n := \begin{pmatrix} 0 & -1 \\ 1 & J_n \end{pmatrix}$$

We define

$$M_n := L_n L_{n+p} L_{n+2p} \dots L_{n+(q-1)p} \tag{1.25}$$

hence $\Psi_n M_n = \Psi_{n+qp}$. On the other hand we have

$$\tilde{L}_n \Psi_n = \Psi_{n+q}, \qquad \tilde{M}_n \Psi_n = \Psi_{n+pq}$$

where

$$\tilde{L}_n := \begin{pmatrix} 0 & 1 \\ -1 & \tilde{J}_n \end{pmatrix}, \qquad \tilde{M}_n := \tilde{L}_{n+(p-1)q} \dots \tilde{L}_{n+p} \tilde{L}_n.$$

The relation $\Psi_n M_n = \tilde{M}_n \Psi_n$ gives $\Psi_n M_n \Psi_n^{-1} = \tilde{M}_n$, hence $\mathrm{tr}(M_n) = \mathrm{tr}(\tilde{M}_n)$. We also see from (1.25) that

$$M_{n+p} = L_{n+p} L_{n+2p} \dots L_{n+(q-1)p} L_{n+qp} = L_{n+p} L_{n+2p} \dots L_{n+(q-1)p} L_n$$

since L_n is period q. Taking the trace and using its cyclic property gives

$$\mathrm{tr}(M_{n+p}) = \mathrm{tr}(L_{n+p} L_{n+2p} \dots L_{n+(q-1)p} L_n)$$

$$= \mathrm{tr}(L_n L_{n+p} L_{n+2p} \dots L_{n+(q-1)p}) = \mathrm{tr}(M_n)$$

hence $\mathrm{tr}(M_n)$ is period p. By an identical argument, applied to $\tilde{M}_n$, we see that $\mathrm{tr}(\tilde{M}_n) = \mathrm{tr}(M_n)$ is period q. Since p and q are coprime we have that $\mathcal{K} := \mathrm{tr}(M_n)$ is constant. We apply the Cayley-Hamilton theorem to M_n:

$$M_n^2 - \mathcal{K} M_n - I = 0,$$

since $|M_n| = 1$, and multiply by Ψ_n on the left, giving

$$\Psi_{n+2qp} - \mathcal{K} \Psi_{n+qp} + \Psi_n = 0.$$

The top left entry of this matrix equation gives the theorem. □

1.2.3 Frieze sequences, periodic quivers and a generalised cluster map

Above we saw that for a quiver Q the frieze sequence (1.5) (after a relabelling) can be obtained from the sequence of mutations $\mu_N \dots \mu_2 \mu_1$. In particular

$$\mu_N \dots \mu_2 \mu_1(Q) = Q,$$

hence we can see Q as a period N quiver, as in Definition 1.42, with the permutation $\rho = 1$. We can also generalise the cluster map (1.13) to

$$\varphi : \begin{pmatrix} X_n^1 \\ X_n^2 \\ X_n^3 \\ \vdots \\ X_n^N \end{pmatrix} \mapsto \begin{pmatrix} X_{n+1}^1 \\ X_{n+1}^2 \\ X_{n+1}^3 \\ \vdots \\ X_{n+1}^N \end{pmatrix} \tag{1.26}$$

where each X_{n+1}^k is given in terms of $X_n^1, X_n^2, \ldots, X_n^N$ by the frieze formula (1.5). The form of cluster mutation means that each new variable can be written as a birational function of the previous ones, and since (1.5) is obtained as a composition of mutations the map (1.26) is birational. A large part of Chapter 2 will be devoted to the study of this map where Q is of affine D or E type.

Remark 1.56. In [39], Nakanishi obtains T- and Y-systems from periodic quivers. These are discrete dynamical systems for the cluster and coefficient variables, respectively. It is known [17] that a solution of the T- system will give a solution of the corresponding Y-system. The converse is not true, however, which is remedied by the introduction of T_z-systems in [25]. The cluster map (1.26) may be seen as a special case of the T-systems in Nakanishi's construction, with the permutation $\rho = 1$ and trivial coefficient variables.

Remark 1.57. Any cluster map or, more generally, T- or Y-system, obtained from a quiver in the mutation class of a Dynkin quiver is necessarily periodic by Theorem 1.13. Furthermore, it has been proved [32] that Y-systems from pairs of Dynkin diagrams are also periodic.

1.2.4 Linear relations within frieze sequences

The existence of linear relations (with periodic and constant coefficients) for the recurrence (1.15) of the cluster map (1.13) leads one to ask if linear relations exist for the frieze sequences (1.5) that come from generalised cluster maps (1.26). If the quiver is of Dynkin type, then due to Theorem 1.13 the frieze sequence is periodic, so linear relations exist and are trivial.

Definition 1.58. We say that the frieze sequences X_n^k satisfy (constant coefficient) linear relations if, for each $k \in Q_0$,

$$\sum_{n=0}^{m} \alpha_n X_n^k = 0$$

for some $m \in \mathbb{N}$ and constants α_n. Note that this sum involves only the cluster variables that appear at a single vertex.

It was proved in [2] that, setting Dynkin types aside, if there are linear relations between the variables in the frieze sequence then the corresponding diagram is affine simply laced. These are shown in Figure 1.4. The converse for $\tilde{A}$ and $\tilde{D}$ types was also proved in [2], that the associated frieze sequences satisfy constant coefficient linear relations. In [33], using a representation theoretic approach, the authors proved this for all $\tilde{A}\tilde{D}\tilde{E}$ quivers, with some of these linear relations given explicitly.

Theorem 1.59. *The frieze sequences satisfy constant coefficient linear relations if and only if the quiver is (up to mutation) an orientation of an ADE (Dynkin, Figure 1.3) or an $\tilde{A}\tilde{D}\tilde{E}$ (affine, Figure 1.4) diagram.*

We adopt a different method of proof in Chapter 2. We adapt the methods of [19] by constructing more complicated analogues of (1.21), for affine D and E types, and applying the Desanot-Jacobi identity. This allows us a uniform expression for these results. Using these methods we also find periodic quantities for the frieze sequences, analogous to those of Theorem 1.54. In our work we also prove the existence of linear relations with periodic quantities as coefficients. Similarly to the $\tilde{A}$ case, the map (1.26) induces an automorphism φ^* on the field of rational functions

$$\mathbb{C}(\{X_0^k\}_{k=1,\dots,N+1}) \tag{1.27}$$

by $\varphi^*(X_n^k) = X_{n+1}^k$ for each $n \in \mathbb{Z}$. Again, we consider an element of the field (1.27) to be periodic, with period p, if it is fixed by $(\varphi^*)^p$.

1.3 Liouville integrability

In broad terms a dynamical system is called integrable if it is "solvable" in some sense. There are many competing definitions for integrability, one of which is known as Liouville integrability. This is our chosen definition. Roughly, a symplectic map φ is Liouville integrable if there are "enough" first integrals for φ that commute with respect to a non-degenerate Poisson bracket. A symplectic map is a map that preserves a certain 2-form (a symplectic form).

The goal of this section is to describe the missing ingredients for this definition as well as give an account of the proof of the integrability of the cluster map (1.13) in [19] for even N.

Firstly we define Poisson brackets. It has been shown [22] that cluster algebras possess a space of Poisson brackets that are compatible with mutation. In our work we demand more, that the bracket is invariant under mutation. We show, with an example, that this may lead to cases where the only brackets is trivial. We then work towards the definition of two-forms. It was shown in [23] that there is a natural two-form ω for cluster algebras that is compatible with mutation. Unfortunately for all $\tilde{D}$ and $\tilde{E}$ type quivers, and for $\tilde{A}_N$ when N is odd, this 2-form is degenerate, so cannot serve as our symplectic form. We then describe a process [19] for finding a reduced set of variables, a reduced cluster map $\hat{\varphi}$ and a reduced symplectic form $\hat{\omega}$ such that $\hat{\varphi}$ is symplectic with respect to $\hat{\omega}$. From this $\hat{\omega}$ we can obtain a non-degenerate Poisson bracket.

We then give a definition of Liouville integrability for systems of recurrence relations and give some $\tilde{A}$ type examples of integrability, one of which involves the reduced cluster map described above. Finally we describe the general proof in $\tilde{A}$ type, including a brief discussion on bi-Hamiltonian systems.

1.3.1 Poisson brackets

Definition 1.60. Let A be an algebra over a field k. A Poisson bracket is a binary map $\{\cdot,\cdot\} : A \times A \to A$ such that:

- $\{f, g\} = -\{g, f\}$ (anticommutativity)
- $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$ (linearity)
- $\{fg, h\} = f\{g, h\} + \{f, h\}g$ (Leibniz rule)
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity)

for all $f, g, h \in A$ and $\alpha, \beta \in k$. We call an algebra with a Poisson bracket a Poisson algebra.

Taking local coordinates $\{x_i\}$ we have the following expression for the Poisson bracket of two rational functions

$$\{f, g\} = \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\{x_i, x_j\} = (\nabla f)P(\nabla g)^T$$

where $P = (p_{i,j})$ is the Poisson matrix given by $p_{i,j} = \{x_i, x_j\}$.

Definition 1.61. A function C such that $\{C, \cdot\} = 0$ is called a Casimir for the bracket $\{\cdot,\cdot\}$. Equivalently $(\nabla C)P = 0$.

Definition 1.62. A Poisson bracket is called log-canonical if, with local coordinates $\{x_i\}$, it is of the form

$$\{x_i, x_j\} = c_{i,j}x_ix_j \tag{1.28}$$

with each $c_{i,j}$ in the field k. Taking the $\{x_i\}$ to be cluster variables, this bracket is said to be compatible with mutation if, after an arbitrary mutation, the bracket is still log-canonical:

$$\{x_i', x_j'\} = c_{i,j}'x_i'x_j',$$

where the prime denotes the new cluster variables, x_i', and new coefficients, $c_{i,j}'$.

It was shown in [22] that cluster algebras have a space of log-canonical Poisson brackets, with $c_{i,j} \in \mathbb{Z}$.

In the context of the period 1 cluster map (1.13) we have initial cluster variables $x_0, \dots, x_N$ and a bracket (1.28). In our work we will demand that the coefficients $c_{i,j}$ are invariant with respect to this map, equivalently

$$c_{i,j} = c_{i+n,j+n} \tag{1.29}$$

for all $n \in \mathbb{Z}$. As we demonstrate with the following example, it may be that case that this is too restrictive, in that the only Poisson bracket is trivial.

Example 1.63. *The recurrence from the $\tilde{A}_3$ quiver is*

$$x_{n+3}x_n = 1 + x_{n+1}x_{n+2} \tag{1.30}$$

where, by construction, $x_3 = x_0'$. By (1.29) the Poisson matrix may be taken to be

$$C = \begin{bmatrix} 0 & c_{0,1} & c_{0,2} \\ -c_{0,1} & 0 & c_{0,1} \\ -c_{0,2} & -c_{0,1} & 0 \end{bmatrix}$$

Applying $\{\cdot, x_i\}$ to (1.30), with n set to zero for brevity, we have

$$\{x_0, x_i\}x_3 + \{x_3, x_i\}x_0 = \{x_2, x_i\}x_1 + \{x_1, x_i\}x_2$$

or, equivalently,

$$x_0x_3x_i(c_{0,i} + c_{3,i}) = x_1x_2x_i(c_{2,i} + c_{1,i}).$$

We cancel x_i and replace x_0x_3 using (1.30):

$$x_1x_2(c_{2,i} + c_{1,i} - c_{0,i} - c_{3,i}) = c_{0,i} + c_{3,i},$$

hence

$$0 = c_{0,i} + c_{3,i} = c_{2,i} + c_{1,i}.$$

One can set $i = 1, 2$ here to see $c_{0,1} = c_{0,2} = 0$ so $C = 0$.

To avoid these problems we instead work with a 2-form in the initial variables, whose construction we now explain.

1.3.2 Symplectic forms

To define symplectic forms we first define the tensor algebra, of which the exterior algebra is a quotient. Symplectic forms live in the exterior algebra of certain vector spaces.

Definition 1.64. The nth tensor power of a vector space V over a field k is the vector space defined as

$$T^n V := V^{\otimes n} := \{(v_1, v_2, \ldots, v_n) \in V \times V \times \ldots \times V\}/\sim$$

with addition defined component wise:

$$(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

and multiplication by a scalar $\lambda \in k$ is defined subject to the relations

$$\lambda(v_1, v_2, \ldots, v_n) \sim (\lambda v_1, v_2 \ldots, v_n) \sim (v_1, \lambda v_2 \ldots, v_n) \sim \ldots \sim (v_1, v_2 \ldots, \lambda v_n).$$

The equivalence class of $(v_1, v_2, \ldots, v_n)$ under this identification is defined to be

$$v_1 \otimes v_2 \otimes \ldots \otimes v_n.$$

The tensor algebra $T(V)$ is defined as

$$T(V) := \bigoplus_{n=0}^{\infty} T^n V.$$

Definition 1.65. The exterior algebra $\bigwedge(V)$ is the tensor algebra modulo the relations

$$v_1 \otimes v_2 \otimes \ldots \otimes v_n \sim \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(n)}$$

for any permutation σ. The equivalence classes under this identification are denoted

$$v_1 \wedge v_2 \wedge \ldots \wedge v_n.$$

Definition 1.66. For each point p on a manifold M, we let T_pM be the tangent space to M at p, with dual $T_p^*M := (T_pM)^*$. A (differential) k-form β smoothly assigns, for each $p \in M$, an element

$$\beta_p \in \bigwedge^k T_p^* M.$$

In other words, β_p is an alternating multilinear map acting on k-tuples of tangent vectors in T_pM. The set of k-forms forms a vectors space, denoted $\Omega^k(M)$.

Definition 1.67. The exterior derivative d is the unique linear map

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for each k, such that $d(d\beta) = 0$ and

$$d(\beta_1 \wedge \beta_2) = d\beta_1 \wedge \beta_2 + (-1)^p(\beta_1 \wedge d\beta_2)$$

for all forms $\beta, \beta_1\ \beta_2$ where β_1 is a p-form. A form β is said to be closed if $d\beta = 0$.

Definition 1.68. A 2-form ω is said to be degenerate at a point $p \in M$ if there exists an X such that

$$\omega(X, Y) = 0 \quad \forall\, Y \in T_pM.$$

Equivalently the linear map $\omega(X, -) : T_pM \to \mathbb{R}$ given by

$$\omega(X, -) : Y \mapsto \omega(X, Y)$$

is the zero map. A 2-form is non-degenerate if it is not degenerate at any $p \in M$. A symplectic form ω is a 2-form that is closed and non-degenerate. If ω is closed but degenerate then we call it pre-symplectic.

Definition 1.69. A manifold M with a symplectic form ω is called a symplectic manifold. A map between symplectic manifolds

$$f : (M, \omega) \to (M', \omega')$$

is called a symplectic map if $f^*(\omega') = \omega$. Here the pullback $f^*(\omega')$ acts on pairs of tangent vectors as

$$f^*(\omega')\,(X, Y) = \omega'\,(Df(X), Df(Y))$$

for tangent vectors X and Y on M, and the differential Df acts on tangents vectors such that

$$(Df(X))g = X(g \circ f)$$

for all $g : M' \to \mathbb{R}$.

Lemma 1.70. *A generic 2-form $\tau := \sum_{i<j} f_{ij} x_i \wedge x_j$ acts on pairs of tangent vectors*

$$X := \sum_i \alpha_i \frac{\partial}{\partial x_i}, \qquad Y := \sum_i \beta_i \frac{\partial}{\partial x_i} \tag{1.31}$$

as

$$\tau(X, Y) = \underline{\alpha} F \underline{\beta}^T$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ and F is the $n \times n$ matrix with entries f_{ij}. Moreover, τ is degenerate at p if the matrix f_{ij} has determinant zero at p.

Proof. Firstly we note that, for $k < l$ and $i < j$,

$$x_k \wedge x_l \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \begin{vmatrix} \frac{\partial x_k}{\partial x_i} & \frac{\partial x_k}{\partial x_j} \\ \frac{\partial x_l}{\partial x_i} & \frac{\partial x_l}{\partial x_j} \end{vmatrix} = \delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il} = \delta_{ik}\delta_{jl}$$

so

$$\tau\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\sum_{k,l} f_{kl}x_k\wedge x_l\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\sum_{k,l} f_{kl}(\delta_{ik}\delta_{jl})=f_{ij}.$$

Since τ is bilinear we have

$$\tau(X,Y)=\sum_{i,j}\alpha_i\beta_j\tau\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\sum_{i,j}\alpha_i\beta_j f_{ij}.$$

the right hand side of which is equal to $\underline{\alpha}F\underline{\beta}^T$. We now see that the linear map $\tau(X,-)$ is given by $\underline{\alpha}F$ which maps $\underline{\beta}$ to $\underline{\alpha}F\underline{\beta}^T$. Hence the 2-form is degenerate if and only if there exists an $\underline{\alpha}$ in the left kernel of F, if and only if F has determinant 0. □

Finally, before applying these ideas to cluster algebras, we explain how a symplectic form gives rise to a non-degenerate Poisson bracket.

Definition 1.71. For a fixed 2-form τ we define a linear map $I : TM \to T^*M$ by

$$I(X)=\tau(-,X)$$

where $X\in TM$. If τ is non-degenerate then we have an inverse $I^{-1} : T^*M \to TM$. In this case we define a Poisson bracket

$$\{f,g\}=\tau(I^{-1}df, I^{-1}dg) \tag{1.32}$$

for functions f, g.

In local coordinates the above bracket (1.32) is given by

$$\{f,g\}=(\nabla f)F^{-1}(\nabla g)^T \tag{1.33}$$

where F is the matrix of Lemma 1.70.

For a quiver Q with exchange matrix $B=(b_{ij})$ the natural pre-symplectic form is

$$\omega := \sum_{1\le i<j\le n}\frac{b_{ij}}{x_ix_j}dx_i\wedge dx_j \tag{1.34}$$

which was shown in [23] to be compatible with cluster mutation, in the sense that it transforms as

$$\mu_k^*(\omega) = \sum_{1\le i<j\le n} \frac{b'_{ij}}{x'_i x'_j} dx'_i \wedge dx'_j,$$

where the prime denotes the images of the cluster variables and exchange matrix entries under the mutation μ_k.

Lemma 1.72. *The 2-form ω is always closed and is non-degenerate if and only if B is.*

Proof. We note that

$$d(\log(x_i)) = \frac{dx_i}{x_i}$$

so one may write ω as

$$\omega = \sum_{i<j} b_{ij} d\log(x_i) \wedge d\log(x_j)$$

which is clearly closed. With this expression in mind, we see that due to Lemma 1.70, with $f_{ij} \mapsto b_{ij}$ and $dx_i \mapsto d(\log(x_i))$, ω is degenerate if and only if B has determinant 0. □

Theorem 1.73. *If the matrix B is non-degenerate then the cluster map (1.13) is symplectic.*

Proof. The cluster map acts on the 2-form (1.34) as

$$\varphi^*\omega := \sum_{1\le i<j\le n} \frac{b_{ij}}{\varphi^*(x_i)\varphi^*(x_j)} \varphi^*(dx_i) \wedge \varphi^*(dx_j)$$

and $\varphi^*(x_i) = x_i \circ \varphi = x_{i+1}$ and $\varphi^* dx_i = d(\varphi^* x_i) = dx_{i+1}$. Hence

$$\varphi^*\omega = \sum_{1\le i<j\le n} \frac{b_{ij}}{x_{i+1}x_{j+1}} dx_{i+1} \wedge dx_{j+1}.$$

To show that this is equal to ω one needs to replace the x_{n+1} and dx_{n+1} terms using the cluster map. We do not replicate these calculations here, which are given in [19]. □

1.3.3 The cluster map as a symplectic map

If ω is symplectic then by Theorem 1.73 the cluster map (1.13) is symplectic and we have a Poisson bracket of the form (2.7) where the c_{ij} are the entries of the matrix B^{-1}. To remedy the problem cases, where ω is degenerate, we first project to a lower dimensional space where φ reduces to a symplectic map. This gives a new, symplectic, 2-form

$$\hat{\omega} = \sum_{i<j} \frac{\hat{b}_{ij}}{y_i y_j} dy_i \wedge dy_j \tag{1.35}$$

with reduced variables $\{y_i\}$. The proof of the existence and a construction of this map are given in [19].

Theorem 1.74. *For* $\mathrm{rank}(B) := 2m < N$ *there exists a rational map* π*, given by (1.37) and (1.38) and a symplectic birational map* $\hat{\varphi}$ *such that*

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\varphi} & \mathbb{C}^N \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{C}^{2m} & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2m} \end{array} \tag{1.36}$$

commutes and $\pi^* \hat{\omega} = \omega$*.*

To construct this map we take the vectors $v_1, \dots, v_{2m}$ to be a basis for $\mathrm{im}(B)$ and project:

$$\begin{aligned} \pi : \mathbb{C}^N &\to \mathbb{C}^{2m} \\ \mathbf{x} &\mapsto \mathbf{y} \end{aligned} \tag{1.37}$$

where

$$y_j := \prod_i x_i^{(v_j)_i} \tag{1.38}$$

and $(v_j)_i$ is the ith component of v_j.

Lemma 1.75. *The map* $\hat{\varphi}$ *is symplectic with respect to the reduced 2-form* $\hat{\omega}$*.*

Proof. We have

$$(\hat{\varphi}^*\hat{\omega}) \circ \pi = \hat{\omega} \circ \hat{\varphi} \circ \pi = \hat{\omega} \circ \pi \circ \varphi = (\pi^*\hat{\omega}) \circ \varphi = \omega \circ \varphi = \varphi^*(\omega) = \omega = \pi^*\hat{\omega} = \hat{\omega} \circ \pi$$

where the second equality uses the commutativity of (1.36), the forth uses $\pi^*\hat{\omega} = \omega$ and the sixth uses the fact that ω is preserved by φ^*. □

The proof of the preceding theorem considers invariants of the scaling transformations given by

$$(x_1, \ldots, x_n) \mapsto (\lambda^{u_1} x_1, \ldots, \lambda^{u_N} x_n) \tag{1.39}$$

for any $\lambda \in \mathbb{C}^\star$ and $u = (u_1, \ldots, u_N) \in \ker(B)$.

Lemma 1.76. *The y_i in (1.38) are a complete set of scaling invariants for (1.39). Moreover, (1.35) is symplectic and the coefficients $\hat{b}_{ij}$ are given by the entries of $\hat{B}$, which satisfies*

$$B^\natural = A^{-T} B A^{-1} = \begin{bmatrix} \hat{B} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\hat{B}$ is a non-degenerate, skew symmetric $2m \times 2m$ matrix. Here we have taken the sets of vectors $\{v_1, v_2, \ldots, v_{2m}\}$ and $\{v_{2m+1}, \ldots, v_N\}$ to be bases for the image and kernel of B respectively and the matrix A has the vectors $v_1, \ldots, v_N$ as rows.

Once we have found the symplectic form $\hat{\omega}$ we can take the matrix inverse $\hat{C} := \hat{B}^{-1}$ to give coefficients of a non-degenerate, log-canonical, Poisson bracket on the reduced variables

$$\{y_i, y_j\} = \hat{c}_{i,j} y_i y_j.$$

as in Definition 1.71 and equation (1.33).

1.3.4 Liouville integrability for dynamical systems from systems of recurrence relations

A system of recurrence relations

$$X^i_{n+1} = \tilde{F}_i(X^1_n \dots X^N_n), \qquad i = 1, \dots, N$$

with initial conditions $X^1_0, X^2_0, \dots, X^N_0$ yields a discrete dynamical system:

$$\varphi : \begin{pmatrix} X^1_n \\ X^2_n \\ \vdots \\ X^N_n \end{pmatrix} \mapsto \begin{pmatrix} X^1_{n+1} \\ X^2_{n+1} \\ \vdots \\ X^N_{n+1} \end{pmatrix} \tag{1.40}$$

with the same initial data. In our work each $\tilde{F}_i$ will be a rational function, hence $X^i_n \in \mathbb{C}(X^1_0, \dots, X^N_0)$ and we extend φ to a $\mathbb{C}(X^1_0, \dots, X^N_0)$ homomorphism. Examples of this include the cluster maps (1.13) and (1.26). We take the following definition of integrability from [35, 42].

Definition 1.77. A $N = 2m$ dimensional map (1.40) is said to be Liouville integrable if it is symplectic and if there exists m independent Poisson commuting first integrals $\mathcal{I}_1, \dots \mathcal{I}_m$. That is,

$$\{\mathcal{I}_i, \mathcal{I}_j\} = 0$$

for all i, j. Here by symplectic we mean that $\varphi^* \omega = \omega$ where ω is a symplectic form.

1.3.5 Some examples for $\tilde{A}$ type quivers

Here we give two examples of cluster maps arising from affine A type quivers. Will we prove that the first is integrable and that the second has a reduction which is also integrable.

Example 1.78. *The $\tilde{A}_{3,1}$ quiver has B matrix*

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

which, up to scaling, is its own inverse, hence we can take B as the coefficients for a log-canonical Poisson bracket on the variables $\{x_0, x_1, x_2, x_3\}$. The cluster map for this quiver is given by the recurrence

$$x_{n+4}x_n = x_{n+1}x_{n+3} + 1$$

for which we have the period 3 quantities

$$J_n := \frac{x_{n+2} + x_n}{x_{n+1}}.$$

Writing J_0, J_1 and J_2 as functions of the initial variables we can calculate, with the aid of a computer, the Poisson brackets between these J_i. In this example we arrive at

$$\{J_0, J_1\} = 2J_0J_1 - 2, \qquad \{J_0, J_2\} = -2J_0J_2 + 2$$

which is enough to calculate $\{J_i, J_j\}$ for each $i, j = 1, 2, 3$. We see that the J_i generate a Poisson subalgebra of dimension 3. Since the cluster map is of dimension 4 we need two commuting first integrals to prove integrability. In this low dimensional case it is easy enough to check that

$$\{\Sigma, \Pi\} = 0, \qquad \Sigma := J_0 + J_1 + J_2, \qquad \Pi := J_0J_1J_2$$

and these are first integrals due to the periodicity of J_i.

Example 1.79. *The $\tilde{A}_{2,1}$ quiver gives the recurrence*

$$x_{n+3}x_n = x_{n+2}x_{n+1} + 1 \tag{1.41}$$

and has B matrix

$$B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

which is singular. Hence we cannot define a Poisson bracket directly as described above, see Example 1.63. Following the construction in Lemma 1.76 we take image vectors, v_1 and v_2, and a kernel vector, v_3

$$v_1 = (1, 1, 0) \qquad v_2 = (0, 1, 1) \qquad v_3 = (1, -1, 1)$$

then the matrices A and $B^\natural = A^{-T} B A^{-1}$ are

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad B^\natural = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced coordinates are $y_1 = x_1 x_2$ and $y_2 = x_2 x_3$. One can see from (1.41) that

$$x_{n+3} x_{n+2} x_{n+1} x_n = (x_{n+2} x_{n+1})^2 + x_{n+2} x_{n+1}.$$

With $y_n := x_{n+1} x_n$ for all n, this gives the reduced cluster map

$$\hat{\varphi} : \begin{pmatrix} y_n \\ y_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}$$

where

$$y_{n+2} = \frac{y_{n+1}^2 + y_{n+1}}{y_n}.$$

The matrix $\hat{B}$, which is the upper left 2×2 matrix in $B^\natural$, gives the 2-form

$$\omega = d \log y_1 \wedge d \log y_2$$

We have $\hat{B}^{-1} = -B$ which gives the Poisson structure (after scaling by -1)

$$\{y_1, y_2\} = y_1 y_2. \tag{1.42}$$

To prove integrability we again look at

$$J_n := \frac{x_{n+2} + x_n}{x_{n+1}}$$

which, in this case, is period 2*. Note that, since the reduced map has dimension* 2*, we only need one conserved quantity to prove integrability. The kernel vector gives the scaling*

$$\lambda : (x_1, x_2, x_3) \mapsto (\lambda x_1, \lambda^{-1} x_2, \lambda x_3).$$

We would like to write J_0 and J_1 in terms of y_1 and y_2 to use the reduced bracket (1.42) but we see that J_n is not preserved under λ, so this is not possible, in fact,

$$J_1 \mapsto \lambda^2 J_1 \qquad J_2 \mapsto \lambda^{-2} J_2$$

hence the product $J_1 J_2$ is fixed under the scaling. Clearly $J_1 J_2$ is constant, which proves integrability.

1.3.6 Integrability of the $\tilde{A}_N$ cluster map

Here we exhibit the proof from [19] of the integrability of the cluster map, for even N,

$$\varphi : \begin{pmatrix} x_n \\ x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+N-1} \end{pmatrix} \mapsto \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+N-1} \\ x_{n+N} \end{pmatrix} \qquad x_{n+N} := \frac{x_{n+1}x_{n+N-1} + 1}{x_n} \tag{1.43}$$

which is (1.15) with $p = 1$, arising from the $\tilde{A}$ type affine diagrams as described in Section 1.2.1. Here the B matrix is non-singular and has inverse given by

$$C = \tau_n^T + (\tau_n^T)^3 + \ldots + (\tau_n^T)^{N-1}$$

where τ is given by (1.16). The Poisson structure (1.28) is then

$$\{x_i, x_j\} = \mathrm{sgn}(j-i)x_i x_j$$

where sgn denotes the sign function. To prove integrability we look at the $N-1$ periodic functions

$$J_n = \frac{x_{n+2} + x_n}{x_{n+1}}$$

from (1.54).

Lemma 1.80. *For N even and $p = 1$ the periodic functions J_n form a Poisson subalgebra with brackets given by*

$$\{J_i, J_j\} = 2\mathrm{sgn}(j-i)(-1)^{i+j+1} J_i J_j + 2(\delta_{i,j+1} - \delta_{i+1,j} + \delta_{i+q-1,j} - \delta_{i,j+q-1}) \quad (1.44)$$

The construction of commuting first integrals for this bracket is the subject of the following subsection.

1.3.6.1 Bi-Hamiltonian structures

The arguments here are directly from [18]. We first prove that (1.44) is the sum of two Poisson brackets. We then use this to prove that the homogenous components of the constant $\mathcal{K}$ from Theorem 1.24 give enough commuting first integrals for (1.44).

Definition 1.81. Two Poisson brackets are said to be compatible if their sum is also a Poisson bracket.

We observe that (1.44) is the sum of a constant term and a quadratic term. We define $\{-,-\}_0$ as

$$\{J_i, J_j\}_0 := 2(\delta_{i,j+1} - \delta_{i+1,j} + \delta_{i+q-1,j} - \delta_{i,j+q-1})$$

which is easily verified as a Poisson bracket. Now the leftmost expression in (1.44) remains a Poisson bracket under the scaling $J_i \to \lambda J_i$ for each i and arbitrary λ, so

$$\{J_i, J_j\} = 2\mathrm{sgn}(j-i)(-1)^{i+j+1}J_iJ_j + 2\lambda^{-2}(\delta_{i,j+1} - \delta_{i+1,j} + \delta_{i+q-1,j} - \delta_{i,j+q-1})$$

is also a Poisson bracket. Sending $\lambda \to \infty$ proves that

$$\{J_i, J_j\}_2 := 2\mathrm{sgn}(j-i)(-1)^{i+j+1}J_iJ_j$$

is also Poisson bracket. Hence we have

$$\{J_i, J_j\} = \{J_i, J_j\}_2 + \{J_i, J_j\}_0$$

giving the compatibility of the brackets on the right hand side. The following result of [19] gives a bi-Hamiltonian ladder of functions [36] which will later provide our commuting first integrals for (1.43).

Theorem 1.82. *The function $\mathcal{K}$ of Theorem 1.24 is a Casimir for the bracket (1.44). Moreover $\mathcal{K}$ is of the form*

$$\mathcal{K} = \mathcal{I}_{N/2} - \mathcal{I}_{(N/2)-1} \ldots \pm \mathcal{I}_1$$

where each $\mathcal{I}_i$ is of degree $2i-1$ and $\mathcal{I}_{N/2}$ and $\mathcal{I}_1$ are Casimirs for $\{-,-\}_2$ and $\{-,-\}_0$ respectively.

Due to the homogeneity of $\mathcal{I}_i$ and the two brackets we have that these functions satisfy

$$\{\mathcal{I}_{i+1}, -\}_0 = \{\mathcal{I}_i, -\}_2 \tag{1.45}$$

We firstly show that each of these $\mathcal{I}_i$ is constant, provided $\mathcal{I}_0$ is.

Lemma 1.83. *If $\mathcal{I}_0$ is constant then so is $\mathcal{I}_i$ for each i.*

Proof. We prove this inductively, with the base case given by assumption. We have

$$\{\mathcal{I}_{i+1}, J_j\}_0 = \{\mathcal{I}_i, J_j\}_2$$

for all j. Applying the Poisson algebra homomorphism φ^* to this we have

$$\{\varphi^*(\mathcal{I}_{i+1}), J_{j+1}\}_0 = \{\varphi^*(\mathcal{I}_i), J_{j+1}\}_2 = \{\mathcal{I}_i, J_{j+1}\}_2 = \{\mathcal{I}_{i+1}, J_{j+1}\}_0$$

where for the second equality we have used that $\mathcal{I}_i$ is constant, the induction assumption and the third equality is given by (1.45). Since this relation holds for all j we must have that $\varphi^*(\mathcal{I}_{i+1}) = \mathcal{I}_{i+1}$. □

The following lemma and theorem prove that the $\mathcal{I}_i$ commute with respect to each Poisson bracket, hence providing $N/2$ first integrals for the dynamical system. From this we have Liouville integrability.

Lemma 1.84. *We have*

$$\{\mathcal{I}_i, \mathcal{I}_j\}_0 = \{\mathcal{I}_i, \mathcal{I}_{j-1}\}_2 = \{\mathcal{I}_{i+1}, \mathcal{I}_{j-1}\}_0$$

Proof. The second equality is precisely (1.45). For the first equality we use

$$\{\mathcal{I}_i, \mathcal{I}_j\}_0 = -\{\mathcal{I}_j, \mathcal{I}_i\}_0 = -\{\mathcal{I}_{j-1}, \mathcal{I}_i\}_2 = \{\mathcal{I}_i, \mathcal{I}_{j-1}\}_2$$

□

Theorem 1.85. *The functions $\mathcal{I}_i$ commute with respect to each bracket:*

$$\{\mathcal{I}_i, \mathcal{I}_j\}_0 = \{\mathcal{I}_i, \mathcal{I}_j\}_2 = 0$$

Hence $\{\mathcal{I}_i, \mathcal{I}_j\} = 0$ and these $N/2$ commuting constants prove Liouville integrability for the system (1.43).

Proof. Without loss of generality we can take $i < j$. If $j - i$ is even then we have an l such that $i + l = j - l$ and

$$\{\mathcal{I}_i, \mathcal{I}_j\}_0 = \{\mathcal{I}_i, \mathcal{I}_{j-1}\}_2 = \{\mathcal{I}_{i+1}, \mathcal{I}_{j-1}\}_0 = \ldots = \{\mathcal{I}_{i+l}, \mathcal{I}_{j-l}\}_0 = 0.$$

If $j-i$ is odd then we have an l such that $i+l=j-l-1$ and

$$\{\mathcal{I}_i,\mathcal{I}_j\}_0=\{\mathcal{I}_i,\mathcal{I}_{j-1}\}_2=\ldots=\{\mathcal{I}_{i+l},\mathcal{I}_{j-l-1}\}_2=0$$

a similar proof holds for the other bracket. □

1.4 Laurent phenomenon algebras and period 1 seeds

Here we give the background and motivation for the first part of Chapter 3, where we examine recurrences outside of cluster algebras that have the Laurent property. This section is a review of relevant information from [1, 34].

Due to the definition of cluster mutation, each recurrence from a period 1 quiver is of the form

$$x_{n+m}x_n=P(x_{n+1},\ldots x_{m+n-1}) \tag{1.46}$$

where P is a binomial. The Laurent property of cluster algebras, Theorem 1.16, ensures that each iterate of these recurrences can be written as a Laurent polynomial in the initial values $x_0,\ldots,x_{m-1}$. Examples are known, however, of recurrences of the form (1.46) satisfying the Laurent property but with P no longer binomial [1, 27]. Some of these can be explained by Laurent Phenomenon (LP) algebras [34], which generalise cluster mutation, allowing for non-binomial exchange relations while preserving the Laurent property. LP algebras, like cluster algebras, have clusters of m cluster variables. We also have seeds, each of which is a cluster with m "exchange polynomials" and mutation allows us to obtain new seeds, using these exchange polynomials. The construction is designed to preserve the Laurent property of cluster algebras (hence the name). We also have a concept of periodic seeds, similar to the periodic quivers from cluster algebras.

1.4.1 Construction of the LP algebra

Definition 1.86. An (LP) seed $(\mathbf{x}, \mathbf{P})$ is a collection of m algebraically independent elements, the cluster variables, $\mathbf{x} := \{x_1, \ldots x_m\}$ and m exchange polynomials $\mathbf{P} := \{P_1, \ldots P_m\}$. Each P_i must

- be irreducible in $\mathbb{Z}[x_1, \ldots, x_m]$
- not contain the variable x_i.

Each $\mathbf{x}$ is called a cluster.

Remark 1.87. In order to follow the notation used in [15] we use two different hats. The smaller one over a polynomial denotes a new polynomial. A larger one over a variable denotes the omission of that variable.

Definition 1.88. For each seed $(\mathbf{x}, \mathbf{P})$ we may perform a different mutation for each $k \in \{1, \ldots, m\}$, giving us a new seeds $\mu_k((\mathbf{x}, \mathbf{P})) = (\mathbf{x}', \mathbf{P}')$ This process is defined in the following steps:

1. We first define the "exchange Laurent polynomials"

$$\{\hat{P}_1, \ldots, \hat{P}_m\} \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

   which are the unique polynomials such that

   - $\hat{P}_j = x_1^{a_1} \ldots \widehat{x_j^{a_j}} \ldots x_n^{a_m} P_j$ for each j and $a_i \in \mathbb{Z}_{\leq 0}$ for each i.
   - for $i \neq j$ we have

$$\hat{P}_i|_{x_j \leftarrow \frac{P_j}{x}} \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x^{\pm 1}, x_{j+1}^{\pm 1}, \ldots, x_m^{\pm 1}]$$

     and this polynomial is not divisible by P_j in this ring.

2. Our new cluster is $\mathbf{x}' = \{x_1, \ldots, x_k', \ldots, x_m\}$ where $x_k' := \frac{\hat{P}_k}{x_k}$.

3. Now, for each j, define polynomials

$$G_j := P_j\left(x_k \leftarrow \frac{\hat{P}_k|_{x_j\leftarrow 0}}{x'_k}\right)$$

denoting the replacement of x_k in P_j

4. For each j, remove all common factors with $\hat{P}_k|_{x_j\leftarrow 0}$ from G_j in the UFD $\mathbb{Z}[x_1,\ldots,\widehat{x_k},\ldots,\widehat{x_j},\ldots,x_m]$. Denote these new polynomials by H_j.

5. The new exchange polynomials are $P'_j = H_jM_j$ where M_j is the unique Laurent monomial in $\mathbb{Z}[x_0,\ldots,x_{k-1},x'_k,x_{k+1},\ldots,x_m]$ such that P'_j is not divisible by any Laurent monomial in this new ring.

6. The new seed is $(\mathbf{x}',\mathbf{P}') = (\{x_1,\ldots,x'_k,\ldots,x_m\},\{P'_1,\ldots,P'_m\})$

Definition 1.89. Two seeds are said to be mutation equivalent if one can be obtained from the other via a finite sequence of mutations. It is useful to pick one seed $(\mathbf{x},\mathbf{P})$ to be the "initial seed". The LP algebra $\mathcal{A}$ is the subalgebra of $\mathbb{Q}(x_1,\ldots,x_m)$ generated by the cluster variables in seeds that are mutation equivalent to an initial seed. Evidently this does not depend on the choice of initial seed.

The convoluted construction of LP mutation is to ensure that the proof of the Laurent property of cluster algebras, via the caterpillar lemma in [14], is still valid in the more general LP case.

Theorem 1.90 ([15], Theorem 5.1)**.** *Each of the cluster variables in the LP algebra is a Laurent polynomial in the cluster variables of an initial seed, i.e. for any seed* $(\{x_0,\ldots x_m\},\mathbf{P})$ *we have* $\mathcal{A}\subset\mathbb{Z}[x_0^{\pm1},\ldots x_m^{\pm1}]$.

1.4.2 Recurrence relations from period 1 seeds

We now show how to obtain recurrences of the form (1.46) where P has more than two terms, from periodic seeds in LP algebras, in the special case where the period is 1. These may be used to show that the iterates of certain recurrence relations may be obtained via LP mutation, hence satisfying the conditions of Theorem 1.90. This proves the Laurent property for these recurrences.

Definition 1.91 (Period 1 seed)**.** Let

$$\{(x_1,\ldots,x_m),(P_1,\ldots,P_m)\}, \qquad \{(x'_2,\ldots,x'_m,x'_1),(P'_2,\ldots,P'_m,P'_1)\}$$

be a seed and that seed after mutation at 1 respectively. The seed is called period 1 if $P'_i = \mathcal{S}P_{i-1}$ for i from 2 to m and $P'_1 = \mathcal{S}P_m$, where the shift operator $\mathcal{S}$ increases the subscripts on each of the x_i appearing by one. Note the reordering of the seed after mutation.

Proposition 1.92. *If* $\{(x_1,\ldots,x_m),(P_1,\ldots,P_m)\}$ *is a period* 1 *seed with* $P_1 = P(x_2,\ldots,x_m)$ *then the iterates of recurrence*

$$x_n x_{n+m} = P(x_{n+1},\ldots,x_{n+m-1}) \tag{1.47}$$

are Laurent polynomials in the first m *variables.*

Proof. Since $P'_2 = \mathcal{S}P_1$ we have $P'_2 = P(x_3,\ldots x_{m+1})$. Mutating a period 1 seed

$$\{(x_1,\ldots,x_m),(P_1,\ldots,P_m)\}$$

at 1 gives $x'_1 = \frac{P(x_2,\ldots,x_m)}{x_1}$ which we'll define as x_{m+1} and agrees with the proposed recurrence (1.47). Now mutating at 2 gives

$$x'_2 = \frac{P'_2(x_3,\ldots,x_{m+1})}{x_2} = \frac{P(x_3,\ldots,x_{m+1})}{x_2}.$$

This we define as x_{m+2}. One can see that composing consecutive mutations will give precisely the iterates of (1.47). Since these iterates are given by compositions of mutations they belong to the LP algebra generated by the seed, hence are Laurent polynomials in the initial variables by Theorem 1.90. □

1.4.3 The Little Pi recurrence as a period 1 seed

In Chapter 3 one of the recurrences we shall be concerned with is the "Little Pi" recurrence:

$$x_n x_{n+2k+l} = x_{n+2k}x_{n+l} + ax_{n+k} + ax_{n+k+l} \tag{1.48}$$

which was shown in [1] to possess the Laurent property, by obtaining the iterates from LP mutation of a period 1 seed. Here we give some details of this construction.

Remark 1.93. There is an inconsistency in the notation used in the papers referenced here. In [15], a cluster of size m is labelled from 1 to m. This is the notation we have used above. Unfortunately in [1] the labels run from 0 to $m-1$. This convention does make thing neater, however, so we will employ it for this section.

In order to apply Proposition 1.92 the authors of [1] construct the "intermediate polynomials", the other polynomials that appear in the period 1 seed, such that the shifting conditions hold. This construction is split into 4 cases. Only the polynomials P_j for $j \in J := \{0, k, 2k, l, k+l\}$ will be given. To find the intermediate polynomial P_i, take the largest $j \in J$ with $j \leq i$ and shift P_j up by $i-j$. Note that in all cases $P_0 = P$ and n has been set to 0 for readability.

- If $l > 2k$

$$P_k = ax_0x_{2k} + ax_{2k}x_l + x_0x_{3k}x_{k+l} + a^2_{k+l},$$

$$P_{2k} = ax_0x_{3k} + ax_{l-k}x_{k+l} + x_0x_{l-2k}x_{k+l} + a^2x_0,$$

$$P_l = ax_kx_{l-k} + ax_{l-k}x_{k+l} + x_0x_{l-2k}x_{k+l} + a^2x_0,$$

$$P_{k+l} = ax_0 + ax_l + x_kx_{l-k}.$$

- If $l = 2k$

$$P_k = ax_0x_{2k} + ax^2_{2k} + x_0x^2_{3k} + a^2x_{3k},$$

$$P_{2k} = ax^2_k + ax_kx_{3k} + x^2_0x_{3k} + a^2x_0,$$

$$P_{3k} = ax_0 + ax_{2k} + x^2_k.$$

- If $2k > l > k$

$$P_k = ax_0x_{2k} + ax_{2k}x_l + x_0x_{3k}x_{k+l} + a^2x_{k+l},$$

$$P_{k+l} = x_0x_{l-k}x_{k+l} + x_0x_{2l-k}x_{k+l} + x_{l-k}x_kx_{2l} + x_{l-k}x_{k+l}x_{2l} + ax_0x_{2l},$$

$$P_{2k} = ax_kx_{3k-l} + ax_kx_{3k} + x_0x_kx_{3k} + a^2x_{2k-l},$$

$$P_{k+l} = ax_0 + ax_l + x_kx_{l-k}.$$

- If $k > l$

$$P_l = x_{2l}x_k + x_{2l}x_{k+l} + x_0x_{2k} + x_0x_{k+2l},$$

$$P_k = x_0x_{k+l}x_{2k-l} + x_0x_{k+l}x_{2k} + x_0x_{k-l}x_{2k} + x_lx_{k-l}x_{2k} + ax_{k-l}x_{k+l},$$

$$P_{k+l} = x_lx_k + x_{k+2l}x_l + x_0x_{k+2l} + x_kx_{2l},$$

$$P_{2k} = ax_{k-l} + ax_k + x_0x_{2k-l}.$$

1.5 Thesis outline

With the preliminaries dealt with, we can begin to present the new results of this thesis.

In Chapter 2 we obtain cluster maps (1.26) for affine D and E type quivers and find periodic quantities and linear relations for these, analogously to affine A type, as given in Section 1.2. We also prove the integrability of a reduction of these dynamical systems for affine D_N with odd N and for each affine E type, using the results and ideas of Section 1.3.

In Chapter 3 we study two non-linear recurrences that appear in LP algebras, including the Little Pi from Section 1.4. We find linear relations for these, also using the Desanot-Jacobi identity. We also prove that they can each be obtained as reductions of 2-dimensional recurrences. Finally we construct sets of initial values such that these 2-d recurrences still possess the Laurent property.

Chapter 2

Linearisability and integrability of cluster maps from $\tilde{D}$ and $\tilde{E}$ type quivers

2.1 Summary of results

The results of this chapter come from analysing the frieze sequence (1.4) (or generalised cluster map (1.26)) of an orientation of $\tilde{D}$ and $\tilde{E}$ type quivers. Our goal is to find analogues to the periodic quantities, (1.20), the linear relations (1.23, 1.24) the Poisson structure (1.44) and integrability of the $\tilde{A}$ type quivers. We summarise our results here.

Theorem 2.1. *For each extending vertex k, the variables that live there satisfy the constant coefficient linear relation*

$$X^k_{n+2b} - \mathcal{K}X^k_{n+b} + X^k_n = 0. \tag{2.1}$$

Here b depends on the quiver, the values of which are given in Figure 2.1, and $\mathcal{K}$ is invariant under the shift φ^*. The extending vertices for an affine quiver are those labelled 1 *in Figure 1.4.*

Quiver		b
	$\tilde{A}_{p,q}$	$\mathrm{lcm}(p,q)$
$\tilde{D}_N$	N even	$N-2$
$\tilde{D}_N$	N odd	$2N-4$
	$\tilde{E}_6$	6
	$\tilde{E}_7$	12
	$\tilde{E}_8$	30

FIGURE 2.1: Values of b for the $\tilde{A}\tilde{D}\tilde{E}$ quivers.

Quiver	Period	Quantity
$\tilde{A}_{p,q}$	p	J_n
	q	$\tilde{J}_n$
$\tilde{D}_N$	$N-2$	J_n
	2	X_n^1/X_n^2
	2	X_n^N/X_n^{N+1}
$\tilde{E}_6$	3	J_n
	3	$\tilde{J}_n$
	2	K_n
$\tilde{E}_7$	4	J_n
	3	K_n
	2	$\tilde{K}_n$
$\tilde{E}_8$	5	J_n
	3	K_n
	2?	$\tilde{K}_n$

FIGURE 2.2: Periodic quantities found for the $\tilde{A}\tilde{D}\tilde{E}$ quivers.

We remark that in [33], for $\tilde{D}$ and $\tilde{E}$ types, $\mathcal{K}$ is given by X_δ, or a function thereof, defined as the image of any module with dimension δ under the Caldero–Chapoton map [6], where δ gives the radical of the symmetrized Euler form [10]. The values δ_i of δ at each vertex i are given in Figure 1.4.

In order to prove Theorem 2.1 for the $\tilde{D}$ and $\tilde{E}$ quivers we first find periodic quantities for their frieze sequences which give linear relations with periodic coefficients for the extending vertices.

Theorem 2.2. *The frieze sequences for the $\tilde{A}\tilde{D}\tilde{E}$ quivers have the periodic quantities given in Figure 2.2.*

Here we find only the period 5 and 3 quantities, J_n and K_n respectively, for the $\tilde{E}_8$ quiver. The question mark is Conjecture 2.39 below. As mentioned in Remark 1.34 the Auslander-Reiten quivers for affine type have regular representations appearing in tubes, whose widths can be found in [10]. We remark that these widths match precisely with the periods in our Figure 2.2. This is why we conjecture the missing period for $\tilde{E}_8$.

For each quiver at least one of the quantities in Figure 2.2 gives a linear relation of the form (1.23) between the variables X_n^k at only one vertex k. The rest of the quantities give linear relations that involve variables that live at different extending vertices. Focussing on the former, we derive relations that are remarkably similar to the $\tilde{A}$ type mutation relation (1.15), two for $\tilde{E}_8$ (plus a third conjectured relation), and one for each of the remaining quivers.

Theorem 2.3. *For the $\tilde{D}_N$, with N even, and $\tilde{E}$ quivers we have the following $\tilde{A}$ type recurrence:*

$$X^k_{n+a+p}X^k_n = X^k_{n+a}X^k_{n+p} + \gamma^k_n. \tag{2.2}$$

Here k is an extending vertex and γ^k_n is period a. For $\tilde{D}_N$ with N odd we have instead

$$X^1_{n+a+p}X^1_n = (\lambda_{n+1})^2 X^1_{n+a}X^1_{n+p} + \gamma^1_n, \tag{2.3}$$

and corresponding results for the other extending vertices, where λ_{n+1} is period 2 and γ^1_n is period a. The values of a and p are given in Figure 2.3.

As for the presymplectic form (1.34), in our cases, this will be degenerate. We instead project to a lower dimensional space with a reduced cluster map and a symplectic form. We have the following result about the integrability of this reduction.

Theorem 2.4. *The reduced cluster map for the $\tilde{D}_N$ quivers, where N is odd, and for the $\tilde{E}$ type quivers is Liouville integrable.*

We provide an example of integrability for $\tilde{D}_6$ and leave the proof for general even N open.

Quiver	a	p
$\tilde{D}_N \quad N$ even	1	$N-2$
$\tilde{D}_N \quad N$ odd	1	$2N-4$
$\tilde{E}_6$	3	2
$\tilde{E}_7$	4	3
$\tilde{E}_8$	6	5
	10	3
	15?	2?

FIGURE 2.3: Values of a and p for the $\tilde{D}\tilde{E}$ quivers.

2.2 Source-sink orientations

Any tree, in particular the $\tilde{D}$ and $\tilde{E}$ diagrams, can be oriented such that each vertex is a sink or a source. These quivers are bipartite, and we define

$$\mu_{\text{sink}} = \prod_{k \text{ a sink}} \mu_k, \qquad \mu_{\text{source}} = \prod_{k \text{ a source}} \mu_k$$

the composition of mutations at each sink and at each source respectively. The order in the products is inconsequential since there are no arrows between two sinks or two sources. For the sequence of mutations giving the cluster map (1.26) we take $\mu = \mu_{\text{source}} \circ \mu_{\text{sink}}$ and φ may be written as the composition $\varphi = \varphi_{\text{source}}\varphi_{\text{sink}}$ where

$$\varphi_{\text{sink}}(X_n^i) = \begin{cases} X_{n+1}^i & i \text{ a sink} \\ X_n^i & i \text{ a source} \end{cases}$$

and

$$\varphi_{\text{source}}(X_n^i) = \begin{cases} X_n^i & i \text{ a sink} \\ X_{n+1}^i & i \text{ a source} \end{cases}$$

In this case the frieze relations (1.4) become

$$\begin{aligned} X^i_{n+1}X^i_n &= 1+\prod_{j\to i} X^j_n, \qquad & i \text{ a sink} \\ X^i_{n+1}X^i_n &= 1+\prod_{j\to i} X^j_{n+1}, \qquad & i \text{ a source} \end{aligned} \tag{2.4}$$

where the products are taken over arrows in the initial quiver. The frieze relations will give the generalised cluster map (1.26).

Remark 2.5. The cluster variables constructed this way, from a bipartite quiver, form the bipartite belt of the cluster algebra, defined in [17].

2.3 Integrability of the generalised cluster map

In our source-sink oriented quivers we use a slightly different 2-form,

$$\omega := \sum_{\substack{i \text{ a sink} \\ j \text{ a source}}} \frac{b_{i,j}}{X^i_n X^j_n} dX^i_n \wedge dX^j_n,$$

equivalent to (1.34) up to relabelling. The first step to proving integrability is to check that the generalised cluster map (1.26) is presymplectic, i.e. that it preserves this 2-form, or equivalently that ω is independent of n.

Lemma 2.6. *The cluster map (1.26), φ, associated with a frieze sequence preserves the symplectic form, i.e. $\varphi^*\omega = \omega$.*

Proof. Since it shifts the frieze variables and fixes the quiver, the cluster map acts as

$$\varphi^*\omega = \sum_{\substack{i \text{ a sink} \\ j \text{ a source}}} \frac{b_{i,j}}{X^i_{n+1} X^j_{n+1}} dX^i_{n+1} \wedge dX^j_{n+1}.$$

This, however may be decomposed, as above, into $\varphi = \varphi_{\text{source}}\varphi_{\text{sink}}$. Since, for all sinks i, the map φ_{sink} sends $X^i_n \mapsto X^i_{n+1}$ and $b_{i,j} \mapsto b_{j,i} = -b_{i,j}$ we have

$$\varphi^*_{\text{sink}}(\omega) = -\sum_{\substack{i \text{ a sink} \\ j \text{ a source}}} \frac{b_{i,j}}{X^i_{n+1} X^j_n} dX^i_{n+1} \wedge dX^j_n$$

where, and throughout this proof, we consider the sum over sinks and sources in the original quiver Q. For a general sink i the mutation relation is

$$X^i_{n+1}X^i_n = 1 + \prod_{l\to i} X^l_n.$$

Since

$$dX^i_{n+1} = \sum_k \frac{\partial X^i_{n+1}}{\partial X^k_n} dX^k_n = \frac{\partial X^i_{n+1}}{\partial X^i_n} dX^i_n + \sum_{k\to i} \frac{\partial X^i_{n+1}}{\partial X^k_n} dX^k_n$$

we calculate

$$\frac{\partial X^i_{n+1}}{\partial X^i_n} = -\frac{1+\prod_{l\to i} X^l_n}{(X^i_n)^2} = -\frac{X^i_{n+1}X^i_n}{(X^i_n)^2} = -\frac{X^i_{n+1}}{X^i_n}$$

and, for $k \to i$,

$$\frac{\partial X^i_{n+1}}{\partial X^k_n} = \frac{1}{X^i_n}\frac{\partial}{\partial X^k_n}\prod_{l\to i} X^l_n = \frac{b_{k,i}}{X^i_n}\frac{\prod_{l\to i} X^l_n}{X^k_n} = \frac{b_{k,i}(X^i_{n+1}X^i_n - 1)}{X^i_n X^k_n}.$$

Collecting these gives

$$\frac{dX^i_{n+1}}{X^i_{n+1}} = -\frac{1}{X^i_n} dX^i_n + \frac{(X^i_{n+1}X^i_n - 1)}{X^i_n X^i_{n+1}} \sum_{k \text{ a source}} b_{k,i}\frac{dX^k_n}{X^k_n}.$$

Finally

$$\varphi^*_{\text{sink}}(\omega) = -\sum_{\substack{i \text{ a sink}\\ j \text{ a source}}} \frac{b_{i,j}}{X^i_{n+1}X^j_n} dX^i_{n+1}\wedge dX^j_n =$$

$$\sum_{\substack{i \text{ a sink}\\ j \text{ a source}}} \frac{b_{i,j}}{X^i_n X^j_n} dX^i_n \wedge dX^j_n - \sum_{\substack{i \text{ a sink}\\ j \text{ a source}}} b_{i,j}\frac{X^i_{n+1}X^i_n - 1}{X^i_n X^i_{n+1}} \sum_{k \text{ a source}} b_{k,i}\frac{dX^k_n \wedge dX^j_n}{X^k_n X^j_n},$$

the second term of which may be written

$$\sum_{i \text{ a sink}} \frac{X^i_{n+1}X^i_n - 1}{X^i_n X^i_{n+1}} \left(\sum_{\substack{j \text{ a source}\\ k \text{ a source}}} b_{i,j}b_{k,i}\frac{dX^k_n \wedge dX^j_n}{X^k_n X^j_n} \right).$$

We see that in the bracketed sum every summand is counted twice with opposite sign, giving zero. Hence

$$\sum_{i\to j} b_{i,j} \frac{dX^i_{n+1}\wedge dX^j_n}{X^i_{n+1}X^j_n} = \sum_{j\to i} b_{i,j} \frac{dX^i_n \wedge dX^j_n}{X^i_n X^j_n}$$

so $\varphi^*_{\text{sink}}(\omega) = \omega$. A similar proof shows that $\varphi^*_{\text{source}}(\omega) = \omega$ so the decomposition $\varphi = \varphi_{\text{source}} \circ \varphi_{\text{sink}}$ gives the result. □

The cluster map in our cases is only pre-symplectic, as the B matrices for the $\tilde{D}$ and $\tilde{E}$ quivers are degenerate. Following the reduction procedure described in Subsection 1.3.3, for each n we take reduced coordinates $\{y^j_n\}_{j=1,\dots,2m}$, where $\text{rank}(B) = 2m$, and

$$y^j_n := \prod_i (X^i_n)^{v_{j,i}}.$$

giving, for each n, the two form

$$\hat{\omega} = \sum_{i<j} \frac{\hat{b}_{ij}}{y^i_n y^j_n} dy^i_n \wedge dy^j_n, \tag{2.5}$$

which is symplectic with $\pi^* \circ \hat{\omega} = \omega$. Here the coefficients $\hat{b}_{ij}$ are given by the entries of $\hat{B}$ which satisfies

$$A^{-T}BA^{-1} = \begin{pmatrix} \hat{B} & 0 \\ 0 & 0 \end{pmatrix}$$

where the matrix A has the vectors $v_1, \dots, v_{N+1}$ as rows, and the sets of vectors $\{v_1, \dots, v_{2m}\}$ and $\{v_{2m+1}, \dots, v_{N+1}\}$ are bases for $\text{im}\, B$ and $\ker B$ respectively. The form of $\hat{B}$ and the reduced coordinates depend on the choice of basis for the image and kernel of B. The following theorem is the analogue of Theorem 2.6 in [19].

Theorem 2.7. *For the $\tilde{D}\tilde{E}$ type quivers the cluster map φ reduces to a map $\hat{\varphi}$:*

$$\hat{\varphi}:\begin{pmatrix} y_n^1 \\ y_n^2 \\ \vdots \\ y_n^{2m} \end{pmatrix} \mapsto \begin{pmatrix} y_{n+1}^1 \\ y_{n+1}^2 \\ \vdots \\ y_{n+1}^{2m} \end{pmatrix} \tag{2.6}$$

such that $\pi \circ \varphi = \hat{\varphi} \circ \pi$. The map $\hat{\varphi}$ is birational and symplectic with respect to the two form (1.35).

Proof. The property $\pi \circ \varphi = \hat{\varphi} \circ \pi$ and birationality of the maps $\hat{\varphi}$ is shown as they are constructed, case by case, in Section 2.6. That $\hat{\varphi}$ is symplectic is the same as in [19] Lemma 2.9. □

Since $\hat{\omega}$ is symplectic, it defines a non-degenerate Poisson bracket:

$$\{y_n^i, y_n^j\} = c_{ij} y_n^i y_n^j, \qquad \text{where} \qquad C = (c_{ij}) := \hat{B}^{-1}. \tag{2.7}$$

Our aim is to prove Liouville integrability of the map (2.6), by finding enough involutive first integrals with respect to this bracket.

2.4 Method of constructing periodic quantities for the $\tilde{D}$ and $\tilde{E}$ quivers

Here we'll discuss our general process for finding periodic quantities for the frieze sequences for $\tilde{D}$ and $\tilde{E}$ quivers and we show how to find constant coefficient linear relations from these. The key to our results, for $\tilde{D}$ type, is to write each of the mutation relations (2.4) as a 2×2 matrix with determinant -1:

$$-1 = \begin{vmatrix} \star & X_{n+1}^k \\ X_n^k & \star \end{vmatrix}, \tag{2.8}$$

where the product of the stars will give the product on the right hand side of (2.4). For the $\tilde{E}$ quivers we have taken the determinant in (2.8) to be $+1$, but the procedure is equivalent, up to swapping matrix columns. We then construct a 3×3 matrix such that each connected 2×2 determinant inside is of the form (2.8). By Dodgson condensation (for 3×3 matrices), Lemma 1.52, the 3×3 matrix will have determinant zero.

In our cases the 3×3 matrix will have a kernel vector of the form $(1,-\alpha,1)^T$. We then add rows to the bottom (or top) of this matrix that preserve this kernel vector. Our main method for doing this will be such that, using (2.8), the new row will create two new connected 2×2 matrices with equal determinant. By the following lemma this newly formed 4×3 matrix will have the same kernel vector $(1,-\alpha,1)^T$.

Lemma 2.8. *Let M be a 3×3 matrix*

$$M := \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{pmatrix}$$

such that $(1,-\alpha,1)^T$ is in the kernel of the first two rows. We define the determinants $\delta_{i,j}$ by

$$\delta_{i,j} := \begin{vmatrix} x_{i,j} & x_{i,j+1} \\ x_{i+1,j} & x_{i+1,j+1} \end{vmatrix}.$$

If $x_{1,1}\neq 0$ and $\delta_{1,0}=\delta_{1,1}\neq 0$ then $(1,-\alpha,1)^T$ is in the kernel for all three rows of M.

Proof. By (1.18), and since $\delta_{1,0}=\delta_{1,1}\neq 0$, we have

$$|M| = 0 \Leftrightarrow \delta_{0,0}\delta_{1,1} - \delta_{1,0}\delta_{0,1} = 0 \Leftrightarrow \delta_{0,0} - \delta_{0,1} = 0.$$

Now the matrix

$$\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ -1 & 0 & 1 \end{pmatrix}$$

has a kernel vector $(1, -\alpha, 1)^T$ and determinant zero. We expand this determinant along the bottom row to see that $\delta_{0,0} = \delta_{0,1}$. Hence $|M| = 0$. Taking a kernel vector $(A, B, 1)^T$ for M we have $A = \frac{\delta_{1,1}}{\delta_{1,0}} = 1$ by Cramer's rule and $B = -\alpha$. $\square$

After adding one row our 4×3 matrix will be of the form

$$\begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ x_{3,0} & x_{3,1} & x_{3,2} \end{pmatrix}$$

so that the bottom 3×3 matrix satisfies the conditions of Lemma 2.8, hence the 4×3 matrix has $(1, -\alpha, 1)^T$ as a kernel vector. As in (2.8) for the $x_{i,j}$ we will take cluster variables X_n^k, or functions thereof. We continue adding rows in this way (except for a few exceptional rows in the $\tilde{D}$ cases) until we find that either α is periodic, in the $\tilde{D}$ cases, or one of the matrix entries is periodic, in the $\tilde{E}$ cases, each with respect to the shift φ^*.

2.4.1 Linear relations with fixed coefficients

The periodic quantities we find immediately give linear relations with periodic coefficients, which in turn give linear relations with constant coefficients, as for the $\tilde{A}$ type quivers. In each case we'll have $\Psi_n L_n = \Psi_{n+q}$ for some integer q, where Ψ_n is a 2×2 matrix with cluster variable entries and L_n is a 2×2 matrix with periodic elements, i.e. $L_{n+p} = L_n$ for some p. Taking $m := \text{lcm}(p, q)$ we have

$$\Psi_n M_n = \Psi_{n+m}, \qquad M_n := L_n L_{n+q} \dots L_{n+m-q}. \tag{2.9}$$

Theorem 2.9. *The matrix* Ψ_n *satisfies*

$$\Psi_{n+2m} - \mathrm{tr}\,(M_n)\Psi_{n+m} + \det(M_n)\Psi_n = 0, \tag{2.10}$$

where $\mathrm{tr}\,(M_n)$ *is period* q*.*

Proof. Applying the Cayley-Hamilton theorem to M_n gives

$$M_n^2 - \mathrm{tr}(M_n)M_n + \det(M_n)I = 0.$$

Multiplying by Ψ_n from the left gives the result. □

In each of the cases we deal with, however, we find that $\mathcal{K} := \mathrm{tr}(M_n)$ is invariant (under shifts in n) and that $\det(M_n) = 1$. Hence (2.10) reduces to $\Psi_{n+2m} - \mathcal{K}\Psi_{n+m} + \Psi_n = 0$ and any entry of this matrix equation will give us constant coefficient linear relations between cluster variables.

2.5 Linear relations in the $\tilde{D}$ and $\tilde{E}$ quivers

Here we apply the ideas of the previous section to the $\tilde{D}$ and $\tilde{E}$ quivers. For $\tilde{D}_N$ we deal with different parities separately. As well as those stated above we have sporadic appearances of other linear relations.

2.5.1 Quivers of $\tilde{D}$ type

We'll orient the $\tilde{D}_N$ diagram as in Figure 2.4. The two sided arrows are used on the right-hand side to indicate that the direction depends on the parity of N, i.e. for a given N one of the arrowheads needs to be deleted on each double arrow, ensuring that each vertex is a sink or source. We deal with the case $N = 4$ separately, in Section 2.5.1.2.

$$\begin{tikzcd}
X^1 \arrow[dr] & & & & & & & & X^N \arrow[dl, leftrightarrow] \\
& X^3 & X^4 \arrow[l] \arrow[r] & X^5 & X^6 \arrow[l] \arrow[r] & \cdots \arrow[r, leftrightarrow] & X^{N-2} \arrow[r, leftrightarrow] & X^{N-1} \arrow[dr, leftrightarrow] & \\
X^2 \arrow[ur] & & & & & & & & X^{N+1}
\end{tikzcd}$$

FIGURE 2.4: The $\tilde{D}_N$ Quiver.

We use the sequence of mutations described in Section 2.2. For example, mutating at vertex 3 gives

$$X_{n+1}^3 = \frac{1}{X_n^3}(1 + X_n^1 X_n^2 X_n^4) \tag{2.11}$$

and then at vertex 1

$$X_{n+1}^1 = \frac{1}{X_n^1}\left(1 + X_{n+1}^3\right) = \frac{1}{X_n^1}\left(1 + \frac{1}{X_n^3}(1 + X_n^1 X_n^2 X_n^4)\right).$$

In general the relations are

$$\begin{aligned}
X_{n+1}^1 &= \frac{1}{X_n^1}(1 + X_{n+1}^3), & X_{n+1}^2 &= \frac{1}{X_n^2}(1 + X_{n+1}^3), & X_{n+1}^3 &= \frac{1}{X_n^3}(1 + X_n^1 X_n^2 X_n^4), \\
X_{n+1}^i &= \frac{1}{X_n^i}(1 + X_{n+1}^{i-1} X_{n+1}^{i+1}), & & 3 < i < N-1 & & \text{for } i \text{ even}, \\
X_{n+1}^i &= \frac{1}{X_n^i}(1 + X_n^{i-1} X_n^{i+1}), & & 3 < i < N-1 & & \text{for } i \text{ odd}.
\end{aligned} \tag{2.12}$$

Towards the right end of the diagram we need to distinguish two cases, depending on the parity of N. When N is even we have

$$\begin{aligned}
X_{n+1}^{N-1} &= \frac{1}{X_n^{N-1}}(1 + X_n^{N-2} X_n^N X_n^{N+1}), \\
X_{n+1}^N &= \frac{1}{X_n^N}(1 + X_{n+1}^{N-1}), & X_{n+1}^{N+1} &= \frac{1}{X_n^{N+1}}(1 + X_{n+1}^{N-1}).
\end{aligned} \tag{2.13}$$

and when N is odd we have

$$X_{n+1}^{N-1} = \frac{1}{X_n^{N-1}}(1 + X_{n+1}^{N-2} X_{n+1}^{N} X_{n+1}^{N+1}),$$
$$X_{n+1}^{N} = \frac{1}{X_n^N}(1 + X_n^{N-1}), \qquad X_{n+1}^{N+1} = \frac{1}{X_n^{N+1}}(1 + X_n^{N-1}). \quad (2.14)$$

Lemma 2.10. *We note that from (2.12) at the extending vertices* 1 *and* 2 *we have*

$$\frac{X_{n+1}^1}{X_{n+1}^2} = \frac{X_n^2}{X_n^1} = \ldots = \left(\frac{X_0^1}{X_0^2}\right)^{(-1)^n},$$

as well as a corresponding result for vertices N and $N+1$ from Equations 2.13 and 2.14. In particular X_n^1/X_n^2 and X_n^{N+1}/X_n^N are period 2.

The 3×3 matrix and kernel vector from Lemma 2.8 are given in the following lemma.

Lemma 2.11. *The matrix*

$$M := \begin{pmatrix} X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ X_{n+2}^3 & X_{n+1}^4 & X_{n+1}^5 \\ X_{n+2}^4 & X_{n+2}^5 & X_{n+1}^6 \end{pmatrix}$$

has kernel vector $(1, -J_n, 1)^T$, where

$$J_n := \frac{X_{n+1}^1 X_{n+1}^2 + X_n^4}{X_{n+1}^3}.$$

Proof. Each of the cluster relations, for example (2.11), can be written as

$$-1 = \begin{vmatrix} \star & \star \\ \star & \star \end{vmatrix}$$

for appropriate 2×2 matrices, so each connected 2×2 determinant inside M is equal to -1. We see from (2.12) that

$$\left(1, -\frac{X_{n+1}^1 X_{n+1}^2 + X_n^4}{X_{n+1}^3}, 1\right)^T$$

is in the kernel of the first two rows of M. Hence by Lemma 2.8 this vector is in the kernel of M. $\square$

We can extend M to involve the variables at each vertex, ensuring that each connected 2×2 matrix has determinant -1, as described in Section 2.4. By repeated applications of Lemma 2.8 the vector $(1, -J_n, 1)$ will be in the kernel of each newly added row. For example, after the first application we have

$$M := \begin{pmatrix} X_{n+1}^1 X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\ X_{n+2}^3 & X_{n+1}^4 & X_{n+1}^5 \\ X_{n+2}^4 & X_{n+2}^5 & X_{n+1}^6 \\ X_{n+3}^5 & X_{n+2}^6 & X_{n+2}^7 \end{pmatrix},$$

with the newly formed connected 2×2 matrices

$$\begin{pmatrix} X_{n+2}^4 & X_{n+2}^5 \\ X_{n+3}^5 & X_{n+2}^6 \end{pmatrix}, \qquad \begin{pmatrix} X_{n+2}^5 & X_{n+1}^6 \\ X_{n+2}^6 & X_{n+2}^7 \end{pmatrix}$$

each having determinant -1, which follows from the mutation relations (2.12). The kernel vector $(1, -J_n, 1)$ for this 4×3 matrix gives us a new equation

$$\begin{pmatrix} X_{n+3}^5 & X_{n+2}^6 & X_{n+2}^7 \end{pmatrix} \begin{pmatrix} 1 \\ -J_n \\ 1 \end{pmatrix} = 0.$$

This particular relation is not so important, we simply state it to stress that we arrive at a new equation for each row we add. We can use the Equations 2.12 to add $N-6$ rows to M in the same way:

$$M := \begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\ X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\ \vdots & \vdots & \vdots \\ X^{N-3}_{n+l} & X^{N-2}_{n+l-h} & X^{N-1}_{n+l-1} \\ X^{N-2}_{n+l+1-h} & X^{N-1}_{n+l} & X^N_{n+l-h}X^{N+1}_{n+l-h} \end{pmatrix}.$$

Here the subindex in the leftmost column increases by 1 only at each sink, so l is 1 greater than the number of sinks from X^3 to X^{N-3} inclusive. The value of h is 0 if N is odd and 1 if N is even. To proceed we can use the identities:

$$J_n = \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n+1}X^2_{n+1} + X^4_n}{X^3_{n+1}} = \frac{X^1_{n-1}X^2_{n-1} + X^4_n}{X^3_n}, \tag{2.15}$$

which follow from some simple manipulation of (2.12), to add two rows (shown in black) to the top of the following matrix while preserving our kernel vector. Note that these new connected 2×2 minors don't have determinant -1. This gives us

$$M := \begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ \vdots & \vdots & \vdots \\ X^{N-2}_{n+l+1-h} & X^{N-1}_{n+l} & X^N_{n+l-h}X^{N+1}_{n+l-h} \end{pmatrix}.$$

In order to extend the bottom in this way we'll need to use similar identities to (2.15) for the vertices at the right end of the diagram, but these depend on the parity of N, so now we split the problem in to two cases.

Before we do this we note that for small N the above process may be not defined. We deal with the case $N = 4$ in Section 2.5.1.2. For $N = 5$ one should start with the matrix

$$\begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1}X^6_{n+1} \end{pmatrix}$$

and use (2.15) to add rows above. Rows below should then be added as described in Section 2.5.1.3. For $N = 6$ add rows above the matrix

$$\begin{pmatrix} X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ X^4_{n+2} & X^5_{n+2} & X^6_{n+1}X^7_{n+1} \end{pmatrix}$$

with (2.15) and add rows below as described in Section 2.5.1.1.

2.5.1.1 The even N case with $N > 4$.

Here the vertices N and $N+1$ are sources, $l = \frac{N}{2} - 1$ and $h = 1$. We now establish why J_n is of such importance.

Lemma 2.12. *The quantity of (2.15), $J_n = \frac{X^1_{n+1}+X^1_{n-1}}{X^2_n}$, is periodic with period* $2l = N - 2$

Proof. The quiver is symmetric about the centre vertex so the relations

$$\frac{X^{N+1}_{n+1} + X^{N+1}_{n-1}}{X^N_n} = \frac{X^{N+1}_{n+1}X^N_{n+1} + X^{N-2}_n}{X^{N-1}_{n+1}} = \frac{X^{N+1}_{n-1}X^N_{n-1} + X^{N-2}_n}{X^{N-1}_n} \tag{2.16}$$

mirror (2.15). We use these to extend M:

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ \vdots & \vdots & \vdots \\ X^{N-2}_{n+l} & X^{N-1}_{n+l} & X^N_{n+l-1}X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1} & X^N_{n+l} & X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1}X^N_{n+l+1} & X^{N-1}_{n+l+1} & X^{N-2}_{n+l} \end{pmatrix}.$$

Now the bottom row here is the counterpart to the top row of (2.5.1) from the opposite end of the quiver. Hence we can add extra rows below in exactly the same way in which we started:

$$\begin{pmatrix} X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\ X^1_{n+1} & X^2_n & X^1_{n-1} \\ X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\ X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\ \vdots & \vdots & \vdots \\ X^{N+1}_{n+l+1} & X^N_{n+l} & X^{N+1}_{n+l-1} \\ X^{N+1}_{n+l+1}X^N_{n+l+1} & X^{N-1}_{n+l+1} & X^{N-2}_{n+l} \\ X^{N-1}_{n+l+2} & X^{N-2}_{n+l+1} & X^{N-3}_{n+l+1} \\ X^{N-2}_{n+l+2} & X^{N-3}_{n+l+2} & X^{N-4}_{n+l+1} \\ \vdots & \vdots & \vdots \\ X^5_{n+2l} & X^4_{n+2l-1} & X^3_{n+2l-1} \\ X^4_{n+2l} & X^3_{n+2l} & X^2_{n+2l-1}X^1_{n+2l-1} \\ X^1_{n+2l+1} & X^2_{n+2l} & X^1_{n+2l-1} \end{pmatrix},$$

where the final row follows from (2.15). Comparing the second and last rows we can see that

$$J_n = \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n+2l+1} + X^1_{n+2l-1}}{X^2_{n+2l}} = J_{n+2l}.$$

□

Theorem 2.13. *For even $N > 4$ the constant coefficient linear relation for the variables at the extending vertices is*

$$X^k_{n+2N-4} - \mathcal{K}X^k_{n+N-2} + X^k_n = 0, \tag{2.17}$$

by which we mean that $k \in \{1, 2, N, N+1\}$ and $\mathcal{K}$ is invariant.

Proof. Defining

$$\Psi_n := \begin{pmatrix} X^1_n & X^2_{n+1} \\ X^1_{n+N-2} & X^2_{n+N-1} \end{pmatrix}, \quad \tilde{L}_n := \begin{pmatrix} 0 & -1 \\ 1 & J_n \end{pmatrix},$$

then

$$\Psi_n \tilde{L}_n = \begin{pmatrix} X^2_{n+1} & X^1_{n+2} \\ X^2_{n+N-1} & X^1_{n+N} \end{pmatrix},$$

so $\Psi_n \tilde{L}_n \tilde{L}_{n+1} = \Psi_{n+2}$. Calling $L_n := \tilde{L}_n \tilde{L}_{n+1}$ allows us to apply Theorem 2.9 with $q = 2$ and $p = m = N - 2$. Explicitly $M_n = L_n L_{n+2} \dots L_{n+N-4}$. Since $\tilde{L}_{n+N-2} = \tilde{L}_n$ and the trace is fixed under cyclic permutations we have that $\mathrm{tr}(M_n) = \mathcal{K}$ is invariant. We also have $|M_n| = 1$. The theorem gives $\Psi_{n+2N-4} - \mathcal{K}\Psi_{n+N-2} + \Psi_n = 0$ and the top left entry of this matrix equation gives the linear relation (2.17) for $k = 1$. By symmetry this holds for the other extending vertices. □

We now give the following result, one of the cases in Theorem 2.3.

Corollary 2.14. *For even $N > 4$ the X^1_n variables satisfy the $\tilde{A}$ type recurrence*

$$X^1_{n+N-1} X^1_n = X^1_{n+N-2} X^1_{n+1} + \gamma^1,$$

where γ^1 is invariant.

Proof. Taking determinants in $\Psi_n \tilde{L}_n = \Psi_{n+1}$ we have that

$$X^1_n X^2_{n+N-1} - X^2_{n+1} X^1_{n+N-2} = X^2_{n+1} X^1_{n+N} - X^1_{n+2} X^2_{n+N-1},$$

and by Lemma 2.10 we can write X^2_n as $X^1_n \lambda_n$ where $\lambda_n := \left(\frac{X^2_0}{X^1_0}\right)^{(-1)^n}$ which is period 2. When replacing each X^2_n, each λ will appear with the same subscript and we cancel them to give the result. □

2.5.1.2 The case $N = 4$.

In this case the mutation relations are given by

$$
\begin{aligned}
X^i_{n+1} &= \frac{1}{X^i_n}(1 + X^3_{n+1}), && \text{for } i = 1, 2, 4, 5, \\
X^3_{n+1} &= \frac{1}{X^3_n}(1 + X^1_n X^2_n X^4_n X^5_n).
\end{aligned}
$$

The analogue of (2.15) is

$$
J_n := \frac{X^1_{n+1} + X^1_{n-1}}{X^2_n} = \frac{X^1_{n+1}X^2_{n+1} + X^4_n X^5_n}{X^3_{n+1}} = \frac{X^1_{n-1}X^2_{n-1} + X^4_n X^5_n}{X^3_n}
$$

and the analogous result

$$
\sigma(J_n) = \frac{X^4_{n+1} + X^4_{n-1}}{X^5_n} = \frac{X^4_{n+1}X^5_{n+1} + X^1_n X^2_n}{X^3_{n+1}} = \frac{X^4_{n-1}X^5_{n-1} + X^1_n X^2_n}{X^3_n}
$$

is given by the permutation $\sigma := (14)(25)$. A simple calculation shows that $J_n = \sigma(J_{n-1})$ hence J_n is period $N - 2 = 2$. Since the expression $J_n = \frac{X^1_{n+1}+X^1_{n-1}}{X^2_n}$ is the same as the other cases, Theorem 2.13 and Corollary 2.14 hold for $N = 4$.

2.5.1.3 The odd N case.

Here vertices N and $N + 1$ are sinks, $l = \frac{N-3}{2}$, and $h = 0$. We can start with the matrix

$$
\begin{pmatrix}
X^4_n & X^3_n & X^1_{n-1}X^2_{n-1} \\
X^1_{n+1} & X^2_n & X^1_{n-1} \\
X^1_{n+1}X^2_{n+1} & X^3_{n+1} & X^4_n \\
X^3_{n+2} & X^4_{n+1} & X^5_{n+1} \\
X^4_{n+2} & X^5_{n+2} & X^6_{n+1} \\
X^5_{n+3} & X^6_{n+2} & X^7_{n+2} \\
\vdots & \vdots & \vdots \\
X^{N-3}_{n+l} & X^{N-2}_{n+l} & X^{N-1}_{n+l-1} \\
X^{N-2}_{n+l+1} & X^{N-1}_{n+l} & X^N_{n+l}X^{N+1}_{n+l}
\end{pmatrix}.
$$

The analogue for (2.16) is

$$\frac{X_{n+1}^{N+1}+X_{n-1}^{N+1}}{X_n^N}=\frac{X_{n+1}^{N+1}X_{n+1}^{N}+X_n^{N-2}}{X_n^{N-1}}=\frac{X_{n-1}^{N+1}X_{n-1}^{N}+X_n^{N-2}}{X_{n-1}^{N-1}},$$

which we use to add the first two black rows to (2.18). The following rows are again constructed in the same way we started:

$$\begin{pmatrix}
X_n^4 & X_n^3 & X_{n-1}^1X_{n-1}^2 \\
X_{n+1}^1 & X_n^2 & X_{n-1}^1 \\
X_{n+1}^1X_{n+1}^2 & X_{n+1}^3 & X_n^4 \\
X_{n+2}^3 & X_{n+1}^4 & X_{n+1}^5 \\
X_{n+2}^4 & X_{n+2}^5 & X_{n+1}^6 \\
X_{n+3}^5 & X_{n+2}^6 & X_{n+2}^7 \\
\vdots & \vdots & \vdots \\
X_{n+l}^{N-3} & X_{n+l}^{N-2} & X_{n+l-1}^{N-1} \\
X_{n+l+1}^{N-2} & X_{n+l}^{N-1} & X_{n+l}^{N}X_{n+l}^{N+1} \\
X_{n+l+2}^{N+1} & X_{n+l+1}^{N} & X_{n+l}^{N+1} \\
X_{n+l+2}^{N+1}X_{n+l+2}^{N} & X_{n+l+1}^{N-1} & X_{n+l+1}^{N-2} \\
X_{n+l+2}^{N-1} & X_{n+l+2}^{N-2} & X_{n+l+1}^{N-3} \\
\vdots & \vdots & \vdots \\
X_{n+2l+1}^5 & X_{n+2l}^4 & X_{n+2l}^3 \\
X_{n+2l+1}^4 & X_{n+2l+1}^3 & X_{n+2l}^2X_{n+2l}^1 \\
X_{n+2l+2}^1 & X_{n+2l+1}^2 & X_{n+2l}^1
\end{pmatrix}. \tag{2.18}$$

From the kernel for this matrix we see that

$$J_n := \frac{X_{n+1}^1+X_{n-1}^1}{X_n^2}$$

has period $2l+1=N-2$, the same as in the even N case.

Theorem 2.15. *For odd N the linear relation for the extending vertex variables is*

$$X_{n+4N-8}^k-\mathcal{K}X_{n+2N-4}^k+X_n^k=0. \tag{2.19}$$

Proof. As above we have

$$\Psi_n := \begin{pmatrix} X_n^1 & X_{n+1}^2 \\ X_{n+N-2}^1 & X_{n+N-1}^2 \end{pmatrix}, \quad \tilde{L}_n := \begin{pmatrix} 0 & -1 \\ 1 & J_{n+1} \end{pmatrix}$$

and

$$\Psi_n \tilde{L}_n = \begin{pmatrix} X_{n+1}^2 & X_{n+2}^1 \\ X_{n+N-1}^2 & X_{n+N}^1 \end{pmatrix}.$$

Once again we define $L_n := \tilde{L}_n \tilde{L}_{n+1}$ so $\Psi_n L_n = \Psi_{n+2}$. Applying Theorem 2.9 with $q = 2$, $p = N - 2$ and $m = 2N - 4$ yields $\Psi_{n+4N-8} - \mathcal{K}\Psi_{n+2N-4} + \Psi_n = 0$, from which we can extract the linear relation. The arguments that $\mathrm{tr}(M_n)$ is invariant and $|M_n| = 1$ are identical to the ones from the N even case. □

The following result is Equation 2.3 in Theorem 2.3.

Corollary 2.16. *The X_n^1 variables satisfy the A type recurrence*

$$X_{n+N-1}^1 X_n^1 = X_{n+N-2}^1 X_{n+1}^1 \lambda_{n+1}^2 + \gamma_n^1,$$

where both γ_n^1 and λ_n are period 2.

Proof. By taking determinants in $\Psi_n L_n = \Psi_{n+2}$ we see that

$$X_n^1 X_{n+N-1}^2 - X_{n+1}^2 X_{n+N-2}^1$$

is period 2. By Lemma 2.10 we can replace X_{n+N-1}^2 and X_{n+1}^2 to give that

$$X_n^1 X_{n+N-1}^1 \lambda_n - X_{n+1}^1 X_{n+N-2}^1 \lambda_{n+1} \tag{2.20}$$

is period 2, where we have again defined $\lambda_n := \left(\frac{X_0^2}{X_0^1}\right)^{(-1)^n}$, which is also period 2. We define expression (2.20) as γ_n^1/λ_{n+1} and rearrange to give the corollary. □

$$\begin{tikzcd} a \arrow[r] & b & c \arrow[l] \arrow[r] \arrow[d] & f & g \arrow[l] \\ & & d & & \\ & & e \arrow[u] & & \end{tikzcd}$$

FIGURE 2.5: The $\tilde{E}_6$ quiver.

2.5.2 The $\tilde{E}_6$ quiver

The $\tilde{E}_6$ diagram is given in Figure 2.5 with a source-sink orientation. Since we have a fixed number of vertices we can label each using different letters, making calculations clearer. In our previous notation we would have had, for example, $a_n = X_n^1$. The sequence of mutations, μ, gives the following recurrence relations.

$$\begin{aligned} &a_{n+1}a_n = 1 + b_n, && b_{n+1}b_n = 1 + a_{n+1}c_{n+1}, && c_{n+1}c_n = 1 + b_n d_n f_n, \\ &d_{n+1}d_n = 1 + c_{n+1}e_{n+1}, && e_{n+1}e_n = 1 + d_n, && \\ &f_{n+1}f_n = 1 + c_{n+1}g_{n+1}, && g_{n+1}g_n = 1 + f_n. && \end{aligned} \tag{2.21}$$

The diagram has S_3 symmetry generated by (for example) the reflections $(bf)(ag)$ and $(bd)(ae)$. It will be useful to name some of the group elements.

$$\sigma_1 := (bf)(ag), \qquad \sigma_2 := (bd)(ae), \qquad \sigma_3 := (bdf)(aeg)$$

2.5.2.1 Periodic quantities

Using the mutation equations, (2.21), we form the following matrix, which has determinant 0 by (1.18). We add labels to the right of the matrix so we may refer to each row:

$$M := \begin{array}{c} \begin{pmatrix} 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \end{pmatrix} \end{array} \begin{array}{c} -1 \\ 0 \\ 1 \end{array}$$

After scaling we take a kernel vector $(1, -a_n, 1)^T$ since, as before, we can see that the third entry of this vector is equal to the first by (2.21). We can add rows to M satisfying the conditions of Theorem 2.8:

$$M := \begin{pmatrix} \star & d_{n-3}/f_{n-2} & e_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2}f_{n-2} \\ a_{n-2} & b_{n-2} & c_{n-1} \\ 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_{n+1} & a_{n+2} \\ d_{n+1}f_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2}/f_{n+1} & \star \end{pmatrix} \begin{array}{c} -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

until we reach rows ± 4 where, in order to use the d mutation relation, we have to divide in our middle entry. The starred entries of M can be filled, preserving the kernel, by insisting that the right 2×2 minor of rows 3 and 4 have determinant 1, ditto for the left 2×2 minor of rows -4 and -3.

Lemma 2.17. *Setting the rightmost entry of row 4 to $\frac{g_{n+4}}{f_{n+1}}$ and the leftmost entry of row -4 to $\frac{g_{n-4}}{f_{n-2}}$ will preserve the kernel.*

Proof. We just prove the statement for row 4. In order for the 2×2 minor to have determinant 1 we need to set the blank entry to

$$\frac{1 + b_{n+2}d_{n+2}/f_{n+1}}{c_{n+2}} = \frac{f_{n+1} + b_{n+2}d_{n+2}}{c_{n+2}f_{n+1}},$$

but we have

$$\frac{f_{n+1}+b_{n+2}d_{n+2}}{c_{n+2}}=\frac{f_{n+1}+\frac{c_{n+3}c_{n+2}-1}{f_{n+2}}}{c_{n+2}}=\frac{f_{n+1}f_{n+2}-1+c_{n+3}c_{n+2}}{c_{n+2}f_{n+2}}$$

$$=\frac{c_{n+2}g_{n+2}+c_{n+3}c_{n+2}}{c_{n+2}f_{n+2}}=\frac{g_{n+2}+c_{n+3}}{f_{n+2}}.$$

Now from the kernel of row -2 we have $a_{n-2}-a_nb_{n-2}+c_{n-1}=0$, to which we apply the permutation σ_1 and shift to see $\frac{g_{n+2}+c_{n+3}}{f_{n+2}}=g_{n+4}$. □

Using this Lemma we can fill in rows ±4. We then use vertex e mutation to partially add rows ±5:

$$M:=\begin{array}{c}\begin{pmatrix} \star & e_{n-3} & f_{n-2} \\ g_{n-4}/f_{n-2} & d_{n-3}/f_{n-2} & e_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2}f_{n-2} \\ \vdots & \vdots & \vdots \\ d_{n+1}f_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2}/f_{n+1} & g_{n+4}/f_{n+1} \\ f_{n+1} & e_{n+3} & \star \end{pmatrix}\end{array}\begin{array}{c} -5 \\ -4 \\ -3 \\ \\ 3 \\ 4 \\ 5 \end{array}$$

Now again we require that both

$$\begin{vmatrix} \star & e_{n-3} \\ g_{n-4}/f_{n-2} & d_{n-3}/f_{n-2} \end{vmatrix}=1, \qquad \begin{vmatrix} d_{n+2}/f_{n+1} & g_{n+4}/f_{n+1} \\ e_{n+3} & \star \end{vmatrix}=1.$$

We simply define quantities that satisfy these relations

$$J_n:=\frac{f_{n+1}+e_{n+3}g_{n+4}}{d_{n+2}}, \qquad \tilde{J}_n:=\frac{f_{n-2}+e_{n-3}g_{n-4}}{d_{n-3}}$$

and fill the blank entries with them. The reason for naming these J_n and $\tilde{J}_n$ will be made apparent in the following lemma. We will need the equations from the kernel for rows ±5:

$$f_{n+1}-a_ne_{n+3}+J_n=0, \qquad \tilde{J}_n-a_ne_{n-3}+f_{n-2}=0. \tag{2.22}$$

Lemma 2.18. *We have the following expressions for J_n and $\tilde{J}_n$ in terms of the extending variables:*

$$J_n = \frac{a_n + g_{n+4}}{e_{n+2}}, \qquad \tilde{J}_n = \frac{a_n + g_{n-4}}{e_{n-2}} \tag{2.23}$$

and each of these has period 3.

Proof. From the kernel for row 4 and vertex e mutation we have

$$f_{n+1} = \frac{a_n d_{n+2} - g_{n+4}}{e_{n+2}} = \frac{a_n(e_{n+3}e_{n+2} - 1) - g_{n+4}}{e_{n+2}} = a_n e_{n+3} - \frac{a_n + g_{n+4}}{e_{n+2}}.$$

Comparing this with the first equation of (2.22) gives $J_n = \frac{a_n+g_{n+4}}{e_{n+2}}$. The corresponding $\tilde{J}_n$ result is proved similarly. Comparing the expressions in (2.23) one sees that $\sigma_1(J_n) = \tilde{J}_{n+4}$ and applying σ_2 to (2.22) we have $\sigma_2(J_n) = \tilde{J}_{n+3}$. Composing these gives $\sigma_2\sigma_1(J_n) = J_{n+1}$. The observation that $\sigma_2\sigma_1$ has order 3 gives the periodicity of J_n and $\tilde{J}_n$. $\square$

2.5.2.2 Constant coefficient linear relations

Theorem 2.19. *The recurrences for the variables attached at the extending vertices satisfy the constant coefficient linear relations*

$$x_{n+12} - \mathcal{K}x_{n+6} + x_n = 0, \tag{2.24}$$

i.e. $x \in \{a, e, g\}$ and $\mathcal{K}$ is invariant.

Proof. Defining

$$\Psi_n := \begin{pmatrix} e_{n+5} & a_{n+3} \\ e_{n+2} & a_n \end{pmatrix}, \qquad \tilde{L}_n := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix}$$

we have $\Psi_n\tilde{L}_n = \sigma_3(\Psi_{n+2})$ which gives $\Psi_{n+6} = \Psi_n\tilde{L}_n\tilde{L}_{n+2}\tilde{L}_{n+4}$. We define $L_n := \tilde{L}_n\tilde{L}_{n+2}\tilde{L}_{n+4}$ and apply Theorem 2.9 with $p = 3$ and $q = 6$. This gives, since

$|L_n| = 1$,

$$\Psi_{n+12} - tr(L_n)\Psi_{n+6} - \Psi_n = 0.$$

Note that here $L_n = M_n$. Any entry of this matrix equation will give (2.24), since $\mathcal{K} := \mathrm{tr}(L_n)$ is invariant as $\mathrm{tr}(L_{n+1}) = \mathrm{tr}(\tilde{L}_{n+1}\tilde{L}_n\tilde{L}_{n+2}) = \mathrm{tr}(\tilde{L}_n\tilde{L}_{n+1}\tilde{L}_{n+2}) = \mathrm{tr}(L_n)$. □

In this case we have slightly more information. We have a period 2 quantity K_n which we use to find a linear relation for the b, d and f vertices. We also prove a relation between K_n and $\mathcal{K}$.

Lemma 2.20. *The following expressions, each in terms of variables at only one vertex, are fixed by all permutations:*

$$\frac{a_{n-3} + a_{n+3}}{a_n} = \frac{e_{n-3} + e_{n+3}}{e_n} = \frac{g_{n-3} + g_{n+3}}{g_n}. \tag{2.25}$$

Proof. From (2.22) we have $J_n - \tilde{J}_{n+3} = a_n e_{n+3} - a_{n+3}e_n$, but the left hand side is period 3 so $a_n e_{n+3} - a_{n+3}e_n = a_{n+3}e_{n+6} - a_{n+6}e_{n+3}$ which we rearrange for the result. □

The following lemma is equivalent to the $\tilde{E}_6$ part of Theorem 2.3.

Lemma 2.21. *The quantity*

$$K_n := \frac{a_{n-3} + a_{n+3}}{a_n} \tag{2.26}$$

has period 2*.*

Proof. Since $J_n = \frac{a_n + g_{n+4}}{e_{n+2}}$ is period 3 we have

$$a_n e_{n+5} + e_{n+5}g_{n+4} = a_{n+3}e_{n+2} + e_{n+2}g_{n+7}.$$

Applying $(\varphi^*)^3 + 1$ to this equation we get

$$e_{n+5}(a_n + a_{n+6} + g_{n+4} + g_{n+10}) = a_{n+3}(e_{n+2} + e_{n+8}) + g_{n+7}(e_{n+2} + e_{n+8}),$$

which we divide by $a_{n+3}e_{n+5}g_{n+7}$, noting that by (2.25) K_n is fixed by permutations,

$$\frac{K_{n+3}}{g_{n+7}} + \frac{K_{n+7}}{a_{n+3}} = \frac{K_{n+5}}{g_{n+7}} + \frac{K_{n+5}}{a_{n+3}}.$$

Assuming we may divide by $K_{n+5} - K_{n+3}$ this is equivalent to

$$\frac{g_{n+7}(K_{n+7} - K_{n+5})}{K_{n+5} - K_{n+3}} = a_{n+3},$$

but here the left hand side is fixed under σ_2, so we have $a_{n+3} = e_{n+3}$. This is a contradiction since a_{n+3} and e_{n+3} are independent variables, since setting $n = -3$ means that a_{n+3} and e_{n+3} would appear in a set of arbitrary initial variables. Hence the division by $K_{n+5} - K_{n+3}$ was invalid, giving $K_{n+5} - K_{n+3} = 0$. □

Lemma 2.22. *The period 2 quantity K_n and the invariant $\mathcal{K}$ are related via $\mathcal{K} = K_0K_1 - 2$.*

Proof. From (2.26) we have

$$\begin{pmatrix} a_{n+5} & a_{n+2} \\ a_{n+3} & a_n \end{pmatrix} \begin{pmatrix} K_1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{n+8} & a_{n+5} \\ a_{n+6} & a_{n+4} \end{pmatrix}.$$

Considering the product $\begin{pmatrix} K_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} K_0 & 1 \\ -1 & 0 \end{pmatrix}$ we apply an argument similar to Theorem 2.9 and compare with (2.24) to arrive at the result. □

We can use K_n to find a constant coefficient linear relation for the vertices adjacent to the extending ones.

Theorem 2.23. *We have the linear relation $x_n - (\mathcal{K}+1)x_{n+3} + (\mathcal{K}+1)x_{n+6} - x_{n+9} = 0$, for $x \in \{b, d, f\}$.*

Proof. From (2.21) We have

$$\frac{b_n - b_{n+9}}{b_{n+3} - b_{n+6}} = \frac{a_n a_{n-1} - a_{n+9}a_{n+8}}{a_{n+3}a_{n+2} - a_{n+6}a_{n+5}}.$$

Using (2.26) to replace everything in the numerator on the right hand side, i.e.

$$a_{n-1} = K_{n+1}a_{n+2} - a_{n+5}, \qquad a_n = K_n a_{n+3} - a_{n+6},$$

$$a_{n+8} = K_n a_{n+5} - a_{n+2}, \qquad a_{n+9} = K_{n+1}a_{n+6} - a_{n+3},$$

we can rearrange to give

$$\frac{b_n - b_{n+9}}{b_{n+3} - b_{n+6}} = K_n K_{n+1} - 1 = \mathcal{K} + 1.$$

This proves the theorem for $x = b$. The other two cases follow by symmetry. □

2.5.3 The $\tilde{E}_7$ quiver

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \arrow[r] & d & f \arrow[l] \arrow[r] & g & h \arrow[l] \\
 & & & e \arrow[u] & & &
\end{tikzcd}$$

FIGURE 2.6: The $\tilde{E}_7$ quiver.

The mutation relations for the $\tilde{E}_7$ quiver, Figure 2.6, are

$$\begin{aligned}
a_{n+1}a_n &= 1 + b_n, & b_{n+1}b_n &= 1 + a_{n+1}c_{n+1}, \\
c_{n+1}c_n &= 1 + b_n d_n, & d_{n+1}d_n &= 1 + c_{n+1}e_{n+1}f_{n+1}, \\
e_{n+1}e_n &= 1 + d_n, & f_{n+1}f_n &= 1 + d_n g_n, \\
g_{n+1}g_n &= 1 + f_{n+1}h_{n+1}, & h_{n+1}h_n &= 1 + g_n.
\end{aligned} \tag{2.27}$$

This diagram has S_2 symmetry, the generator of which we denote $\sigma := (ah)(bg)(cf)$.

2.5.3.1 Periodic quantities

We begin with the matrix centred at vertex a with kernel vector $(1, -a_n, 1)^T$:

$$\begin{array}{cc} \begin{pmatrix} \star & f_{n-3}/e_{n-2} & g_{n-3} \\ c_{n-3} & d_{n-3} & e_{n-2}f_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2} \\ a_{n-2} & b_{n-2} & c_{n-1} \\ 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_{n+1} & a_{n+2} \\ d_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2}f_{n+2} & d_{n+2} & c_{n+3} \\ g_{n+2} & f_{n+3}/e_{n+2} & \star \end{pmatrix} & \begin{array}{c} -5 \\ -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \end{array}$$

The construction of which is routine until we reach the stars in rows ± 5.

Lemma 2.24. *Setting the lower right and upper left entries to be*

$$\frac{e_{n+2} + c_{n+3}f_{n+3}}{d_{n+2}e_{n+2}} = \frac{e_{n+4}}{e_{n+2}}, \qquad \frac{e_{n-2} + f_{n-3}c_{n-3}}{d_{n-3}e_{n-2}} = \frac{e_{n-4}}{e_{n-2}}.$$

respectively preserves the kernel.

Proof. The proof of these equalities is the same argument as that used for Lemma 2.17. □

We can now replace the stars, adding to the rows marked ± 5 below. The rows labelled ± 6 are to be filled using the mutation relation for g, giving us another two starred entries:

$$\begin{array}{cl}\begin{pmatrix} \star & g_{n-4} & e_{n-2}h_{n-3} \\ e_{n-4}/e_{n-2} & f_{n-3}/e_{n-2} & g_{n-3} \\ \vdots & \vdots & \vdots \\ g_{n+2} & f_{n+3}/e_{n+2} & e_{n+4}/e_{n+2} \\ e_{n+2}h_{n+3} & g_{n+3} & \star \end{pmatrix} & \begin{array}{c} -6 \\ -5 \\ \\ 5 \\ 6 \end{array}\end{array}$$

which we can fill with the following result.

Lemma 2.25. *Due to the relations*

$$\frac{e_{n+2}+e_{n+4}g_{n+3}}{f_{n+3}e_{n+2}} = \frac{a_{n+6}}{e_{n+2}}, \qquad \frac{e_{n-2}+e_{n-4}g_{n-4}}{e_{n-2}f_{n-3}} = \frac{a_{n-6}}{e_{n-2}}$$

we can replace the stars with a_{n-6} and a_{n+6} respectively.

Proof. From the kernel for row -5 we have $e_{n-4} - f_{n-3}a_n + g_{n-3}e_{n-2} = 0$, which we shift upwards by 6 and substitute into our expression to get the first equality. The second is proved similarly. □

We now use the mutation relation for h to add most of rows ± 7:

$$\begin{array}{cl}\begin{pmatrix} & h_{n-4}/e_{n-2} & 1 \\ a_{n-6} & g_{n-4} & e_{n-2}h_{n-3} \\ \vdots & \vdots & \vdots \\ e_{n+2}h_{n+3} & g_{n+3} & a_{n+6} \\ 1 & h_{n+4}/e_{n+2} & \end{pmatrix} & \begin{array}{c} -7 \\ -6 \\ \\ 6 \\ 7 \end{array}\end{array}$$

This is the last row we need to add. We define the quantities

$$J_n := \frac{e_{n+2}+a_{n+6}h_{n+4}}{g_{n+3}}, \qquad \tilde{J}_n := \frac{e_{n-2}+a_{n-6}h_{n-4}}{g_{n-4}}$$

that sit in the blank corners, such that the kernel is preserved. As we shall see, these J_n and $\tilde{J}_n$ are also periodic.

Theorem 2.26. *J_n and $\tilde{J}_n$ satisfy*

$$J_n = \tilde{J}_{n+6} \tag{2.28}$$

Proof. Multiplying the numerator and denominator of J_n by g_{n+2} we get

$$J_n = \frac{g_{n+2}e_{n+2} + g_{n+2}a_{n+6}h_{n+4}}{g_{n+2}g_{n+3}}. \tag{2.29}$$

From the kernel for row -6 we have

$$a_n g_{n-4} = a_{n-6} + e_{n-2}h_{n-3},$$

which we shift upwards by 6 then multiply by h_{n+4} to give

$$a_{n+6}g_{n+2}h_{n+4} = a_n h_{n+4} + e_{n+4}h_{n+3}h_{n+4} = a_n h_{n+4} + e_{n+4}(1 + g_{n+3}), \tag{2.30}$$

where the second equality uses h mutation. We also have, from the kernel for row 5,

$$e_{n+2}g_{n+2} = a_n f_{n+3} - e_{n+4}. \tag{2.31}$$

Substituting (2.30) and (2.31) into (2.29) gives

$$J_n = \frac{a_n(f_{n+3} + h_{n+4}) + e_{n+4}g_{n+3}}{g_{n+2}g_{n+3}}. \tag{2.32}$$

From a shift of row 2 we have $c_{n+3} - a_{n+2}b_{n+3} + a_{n+4} = 0$ to which we can apply σ to get $f_{n+3} - h_{n+2}g_{n+3} + h_{n+4} = 0$. This is used to replace the bracketed term in (2.32), giving

$$J_n = \frac{a_n h_{n+2} + e_{n+4}}{g_{n+2}}$$

which is $\tilde{J}_{n+6}$. □

Lemma 2.27. *The permutation σ acts on J_n and $\tilde{J}_n$ by $\sigma(J_n) = \tilde{J}_{n+4} = J_{n+2}$, therefore J_n and $\tilde{J}_n$ are period 4.*

Proof. From rows ±7 we have

$$J_n = a_n h_{n+4} - e_{n+2}, \qquad \tilde{J}_n = a_n h_{n-4} - e_{n-2},$$

which gives the first equality. Using this we can apply σ to (2.28) to get $J_{n+2} = \tilde{J}_{n+4}$. □

Lemma 2.28. *We have the following expression, fixed under σ, in terms of the variables at only one extending vertex:*

$$K_n := \frac{a_{n+8} + a_n}{a_{n+4}} = \frac{h_{n+8} + h_n}{h_{n+4}}. \tag{2.33}$$

Proof. Since $J_n = a_n h_{n+4} - e_{n+2}$ and $J_{n+2} = \sigma(J_n) = a_{n+4} h_n - e_{n+2}$ we have

$$J_n - J_{n+2} = a_n h_{n+4} - a_{n+4} h_n.$$

The left hand side of this is period 4 so $a_n h_{n+4} - a_{n+4} h_n$ is also period 4, which is equivalent to (2.33). □

We have the following more useful expression for J_n which we will use to calculate linear relations.

Theorem 2.29. *The quantity J_n may be written*

$$J_n = \frac{a_{n+6} + a_n}{h_{n+3}} = \frac{h_{n+8} + h_{n+2}}{a_{n+5}}. \tag{2.34}$$

Proof. From row 6 we see

$$\begin{aligned} 0 = e_{n+2} h_{n+3} - a_n g_{n+3} + a_{n+6} &= e_{n+2} h_{n+3} - a_n h_{n+4} h_{n+3} + a_n + a_{n+6} \\ &= -h_{n+3} J_n + a_n + a_{n+6}. \end{aligned}$$

We apply σ to this equation, noting that by Lemma 2.27 $\sigma(J_n) = J_{n+2}$, to get the result. □

Proposition 2.30. *The quantity K_n is period 3, hence the a variables satisfy the linear relation*

$$a_{n+8} - K_n a_{n+4} + a_n = 0$$

and the $\tilde{E}_7$ part of Theorem 2.3. We also have that the determinant

$$\tilde{K}_n := \begin{vmatrix} a_n & h_{n+3} \\ a_{n+4} & h_{n+7} \end{vmatrix}$$

is period 2 with $\sigma(\tilde{K}_n) = \tilde{K}_{n+1}$.

Proof. The matrix

$$\begin{pmatrix} a_n & h_{n+3} & a_{n+6} \\ a_{n+4} & h_{n+7} & a_{n+10} \\ a_{n+8} & h_{n+11} & a_{n+14} \end{pmatrix}$$

has a right kernel vector $(1, -J_n, 1)^T$ by (2.34), hence has determinant zero. Taking a left kernel vector $(A_n, -\tilde{K}_n, 1)$, we have that

$$\sigma(A_n) = A_{n+3}, \qquad \sigma(\tilde{K}_n) = \tilde{K}_{n+3}, \qquad A_n = \frac{\delta_{n+4}}{\delta_n}, \qquad \sigma(\delta_n) = \delta_{n+3}.$$

where we have defined

$$\delta_n := \begin{vmatrix} a_n & h_{n+3} \\ a_{n+4} & h_{n+7} \end{vmatrix}$$

Since this vector is in the kernel we have

$$\frac{a_n + a_{n+8}}{a_{n+4}} = \frac{(1 - A_n)a_n}{a_{n+4}} + \tilde{K}_n.$$

Due to (2.33) the left hand side is fixed by σ so we can equate

$$\frac{(1 - A_n)a_n}{a_{n+4}} + \tilde{K}_n = \frac{(1 - A_{n+3})h_n}{h_{n+4}} + \tilde{K}_{n+3}. \tag{2.35}$$

Shifting this by 3 gives

$$\frac{(1 - A_{n+3})a_{n+3}}{a_{n+7}} + \tilde{K}_{n+3} = \frac{(1 - A_n)h_{n+3}}{h_{n+7}} + \tilde{K}_n \tag{2.36}$$

and subtracting (2.35) from (2.36) we get

$$(1 - A_{n+3})\left(\frac{a_{n+3}}{a_{n+7}} - \frac{h_n}{h_{n+4}}\right) = (1 - A_n)\left(\frac{h_{n+3}}{h_{n+7}} - \frac{a_n}{a_{n+4}}\right)$$

so

$$\frac{(1 - A_{n+3})\delta_{n+3}}{a_{n+7}h_{n+4}} = \frac{(1 - A_n)\delta_n}{a_{n+4}h_{n+7}}. \tag{2.37}$$

We now rearrange this, assuming $A_n \neq 0$

$$\frac{(1 - A_{n+3})\delta_{n+3}}{(1 - A_n)\delta_n} = \frac{a_{n+7}h_{n+4}}{a_{n+4}h_{n+7}}.$$

Since σ inverts the left hand side, it also inverts the right, so we have

$$\frac{a_{n+7}h_{n+4}}{a_{n+4}h_{n+7}} = \frac{a_{n+7}h_{n+10}}{a_{n+10}h_{n+7}} \Rightarrow \frac{h_{n+4}}{a_{n+4}} = \frac{h_{n+10}}{a_{n+10}}. \tag{2.38}$$

Row 7 of our matrix gives

$$J_n = \frac{a_n h_{n+4}}{e_{n+2}} - 1.$$

Since $\sigma(J_n) = J_{n+2}$ we calculate

$$0 = J_{n+2} - \sigma(J_n) = \frac{a_{n+2}h_{n+6}}{e_{n+4}} - \frac{a_{n+4}h_n}{e_{n+2}}$$

or, equivalently

$$\frac{h_{n+6}}{h_n} = \frac{a_{n+4}e_{n+4}}{a_{n+2}e_{n+2}}.$$

We can now use (2.38) to show that the left hand side is σ invariant, hence the right is too, giving

$$\frac{a_{n+4}}{a_{n+2}} = \frac{h_{n+4}}{h_{n+2}}$$

which is demonstrably false. Therefore the division by $(1 - A_n)$ in (2.37) was invalid, and $A_n = 1$. Hence

$$\tilde{K}_n = \frac{a_n + a_{n+8}}{a_{n+4}} = \frac{h_n + h_{n+8}}{h_{n+4}} = \tilde{K}_{n+3}$$

where the second equality follows from (2.33) and the third from $\sigma(\tilde{K}_n) = \tilde{K}_{n+3}$. This gives

$$\frac{a_n + a_{n+8}}{a_{n+4}} = \frac{a_{n+3} + a_{n+11}}{a_{n+7}}$$

hence $a_n a_{n+7} - a_{n+3} a_{n+4}$ is period 4 which is the result of Theorem 2.3. Finally we see that δ_n is period 4 since $A_n = 1$. It is also period 6 since $\sigma(\delta_n) = \delta_{n+3}$, hence is period 2. □

Finally we conjecture an alternate expression for $\tilde{K}_n$.

Conjecture 2.31. The period 2 quantity $\tilde{K}_n$ can be expressed as

$$\tilde{K}_n = \frac{a_{n+12} + a_n}{h_{n+6}}.$$

2.5.3.2 Constant coefficient linear relations

Theorem 2.32. *We have the constant coefficient linear relation*

$$x_{n+24} - \mathcal{K} x_{n+12} + x_n = 0$$

where $x \in \{a, h\}$.

Proof. We define the matrices

$$\Psi_n := \begin{pmatrix} a_{n+5} & h_{n+2} \\ h_{n+3} & a_n \end{pmatrix}, \qquad \tilde{L}_n := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix},$$

then, due to (2.34) we have,

$$\Psi_n \tilde{L}_{n+2} = \sigma(\Psi_{n+3}), \qquad \Psi_n \tilde{L}_{n+2} \tilde{L}_{n+3} = \Psi_{n+6}.$$

Now we can set $L_n := \tilde{L}_{n+2} \tilde{L}_{n+3}$ and use Theorem 2.9 with $M_n = L_n L_{n+6}$. Again we call $\mathcal{K} := tr(M_n)$, which, by the same argument used for $\tilde{E}_6$, is invariant. □

2.5.4 The $\tilde{E}_8$ quiver

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \arrow[r] & d & e \arrow[l] \arrow[r] & f & h \arrow[l] \arrow[r] & i \\
& & & & & g \arrow[u] & &
\end{tikzcd}$$

FIGURE 2.7: The $\tilde{E}_8$ quiver.

The $\tilde{E}_8$ is given in Figure 2.7. The recurrence relations are

$$\begin{aligned}
a_{n+1}a_n &= 1 + b_n, & b_{n+1}b_n &= 1 + a_{n+1}c_{n+1}, & c_{n+1}c_n &= 1 + b_n d_n, \\
d_{n+1}d_n &= 1 + c_{n+1}e_{n+1}, & e_{n+1}e_n &= 1 + d_n f_n, & f_{n+1}f_n &= 1 + e_{n+1}g_{n+1}h_{n+1}, \\
g_{n+1}g_n &= 1 + f_n, & h_{n+1}h_n &= 1 + f_n i_n, & i_{n+1}i_n &= 1 + h_{n+1}.
\end{aligned} \tag{2.39}$$

2.5.4.1 Periodic quantities

We begin with the matrix centred at vertex a, denoted M_a, with kernel vector $(1, -a_n, 1)^T$, whose construction is identical to the $\tilde{E}_6$ and $\tilde{E}_7$ cases.

$$M_a := \begin{array}{c} \begin{pmatrix} \star & h_{n-4}/g_{n-3} & i_{n-4} \\ e_{n-4} & f_{n-4} & g_{n-3}h_{n-3} \\ d_{n-4} & e_{n-3} & f_{n-3} \\ c_{n-3} & d_{n-3} & e_{n-2} \\ b_{n-3} & c_{n-2} & d_{n-2} \\ a_{n-2} & b_{n-2} & c_{n-1} \\ 1 & a_{n-1} & b_{n-1} \\ a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_{n+1} & a_{n+2} \\ d_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2} & c_{n+3} \\ f_{n+2} & e_{n+3} & d_{n+3} \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ i_{n+3} & h_{n+4}/g_{n+3} & \star \end{pmatrix} \end{array} \begin{array}{c} -7 \\ -6 \\ -5 \\ -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}$$

Lemma 2.33. *The stars in rows ± 7 are found via the relations*

$$\frac{g_{n+3} + e_{n+4}h_{n+4}}{f_{n+3}g_{n+3}} = \frac{g_{n+5}}{g_{n+3}}, \qquad \frac{g_{n-3} + e_{n-4}h_{n-4}}{g_{n-3}f_{n-4}} = \frac{g_{n-5}}{g_{n-3}}.$$

The proof of this is analogous to Lemma 2.17.

We can now fill in rows ± 7 and add some of rows ± 8 via i mutation:

$$M_a = \begin{array}{c} \begin{pmatrix} \star & i_{n-5} & g_{n-3} \\ g_{n-5}/g_{n-3} & h_{n-4}/g_{n-3} & i_{n-4} \\ e_{n-4} & f_{n-4} & g_{n-3}h_{n-3} \\ \vdots & \vdots & \vdots \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ i_{n+3} & h_{n+4}/g_{n+3} & g_{n+5}/g_{n+3} \\ g_{n+3} & i_{n+4} & \star \end{pmatrix} \end{array} \begin{array}{c} -8 \\ -7 \\ -6 \\ \\ 6 \\ 7 \\ 8 \end{array}$$

Lemma 2.34. *The starred entries can be replaced by $a_{n\pm 8}$ respectively, preserving the kernel, because we have*

$$\frac{g_{n+3} + g_{n+5}i_{n+4}}{h_{n+4}} = a_{n+8}, \qquad \frac{g_{n-3} + g_{n-5}i_{n-5}}{h_{n-4}} = a_{n-8}.$$

Proof. From row -7 we have $g_{n-5} - a_n h_{n-4} + g_{n-3}i_{n-4} = 0$, which we shift up 8 and substitute into our expression to get the first result. $\square$

We've extended M_a as much as necessary. In this case we'll also need to construct a matrix centred at vertex i:

$$M_i := \left(\begin{array}{ccc}
\tilde{J}_n & b_{n-4} & a_{n-3}g_{n-1} \\
a_{n-8}/g_{n-1} & c_{n-3}/g_{n-1} & b_{n-3} \\
i_{n-5} & d_{n-3} & c_{n-2}g_{n-1} \\
g_{n-3}/g_{n-1} & e_{n-2}/g_{n-1} & d_{n-2} \\
h_{n-2} & f_{n-2} & e_{n-1}g_{n-1} \\
i_{n-2} & h_{n-1} & f_{n-1} \\
1 & i_{n-1} & h_n \\
i_n & 1 & 0 \\
h_{n+1} & i_{n+1} & 1 \\
f_{n+1} & h_{n+2} & i_{n+2} \\
e_{n+2}g_{n+2} & f_{n+2} & h_{n+3} \\
d_{n+2} & e_{n+3}/g_{n+2} & g_{n+4}/g_{n+2} \\
c_{n+3}g_{n+2} & d_{n+3} & i_{n+5} \\
b_{n+3} & c_{n+4}/g_{n+2} & a_{n+9}/g_{n+2} \\
a_{n+4}g_{n+2} & b_{n+4} & J_n
\end{array}\right)
\begin{array}{c}
-7 \\ -6 \\ -5 \\ -4 \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}$$

The entries of rows ± 4 and ± 5 were simplified by results similar to Lemmas 2.33 and 2.34 respectively. For rows ± 6 we use the following equalities:

$$\frac{g_{n+2} + c_{n+4}i_{n+5}}{d_{n+3}g_{n+2}} = \frac{a_{n+9}}{g_{n+2}}, \qquad \frac{g_{n-1} + c_{n-3}i_{n-5}}{d_{n-3}g_{n-1}} = \frac{a_{n-8}}{g_{n-1}}.$$

We've also defined

$$J_n := \frac{g_{n+2} + a_{n+9}b_{n+4}}{c_{n+4}}, \qquad \tilde{J}_n := \frac{g_{n-1} + a_{n-8}b_{n-4}}{c_{n-3}}.$$

The following lemma proves one of the $\tilde{E}_8$ relations given in Theorem 2.3.

Lemma 2.35. *The expressions J_n and $\tilde{J}_n$ satisfy*

$$J_n = \tilde{J}_n = a_{n-3}a_{n+4} - i_n, \qquad J_n = J_{n+5}.$$

We also have the following expression which we will use to apply Theorem 2.9:

$$J_{n+3} = \frac{a_{n+12} + a_n}{a_{n+6}}. \tag{2.40}$$

Proof. From row 7 of M_i:

$$J_n = i_n b_{n+4} - a_{n+4} g_{n+2} = i_n(a_{n+5}a_{n+4} - 1) - a_{n+4}g_{n+2} =$$

$$a_{n+4}(i_n a_{n+5} - g_{n+2}) - i_n = a_{n+4}a_{n-3} - i_n$$

where the last equality uses row -8 of M_a. A similar calculation gives

$$\tilde{J}_n = a_{n-3}a_{n+4} - i_n$$

which is the first result. For the second we note that

$$a_{n+2}a_{n+9} = a_{n+2}(i_n c_{n+4} - g_{n+2}b_{n+3}) = i_n(d_{n+3} + b_{n+4}) - g_{n+2}(a_{n+4} + c_{n+3}) \tag{2.41}$$

where the first equality uses row 6 of M_i and the second rows 2 and 3 of M_a. From row 5 of M_i we have

$$i_{n+5} = i_n d_{n+3} - c_{n+3}g_{n+2} \tag{2.42}$$

so

$$J_{n+5} = a_{n+2}a_{n+9} - i_{n+5} = i_n b_{n+4} - a_{n+4}g_{n+2} = J_n$$

where the first equality is the first part of this lemma and the second equality uses (2.41) and (2.42). The last equality again uses row 7 of M_i. As for the last result, we have

$$a_{n+12} = c_{n+7}i_{n+3} - b_{n+3}g_{n+2} = a_n b_{n+6} - a_{n+6}i_{n+3} =$$

$$a_n(-1 + a_{n+6}a_{n+7}) - a_{n+6}i_{n+3} = -a_n + a_{n+6}(a_n a_{n+7} - i_{n+3}) = -a_n + a_{n+6}J_{n+3}.$$

The first equality comes from row 6 of M_i and the second from rows -2 and -8 of M_a. The third uses a mutation and the fifth the first result of this lemma. $\square$

To prove the existence of the period 3 quantity we will use the following matrix, also beginning at vertex a, but with alternate rows 7 and 8.

$$\tilde{M}_a := \begin{array}{c} \begin{pmatrix} a_n & 1 & 0 \\ b_n & a_{n+1} & 1 \\ c_{n+1} & b_1 & a_{n+2} \\ d_{n+1} & c_{n+2} & b_{n+2} \\ e_{n+2} & d_{n+2} & c_{n+3} \\ f_{n+2} & e_{n+3} & d_{n+3} \\ g_{n+3}h_{n+3} & f_{n+3} & e_{n+4} \\ 1 & g_{n+4}/h_{n+3} & i_{n+5}/h_{n+3} \\ h_{n+3} & g_{n+4} & i_{n+5} \end{pmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Lemma 2.36. *We have*

$$a_n a_{n+13} - i_{n+6} = J_{n+3}J_{n+4} - 1.$$

Proof. From (2.40) we have the first equality of the following:

$$a_n a_{n+13} = J_{n+4}a_n a_{n+7} - a_n a_{n+1} = J_{n+4}(J_{n+3} + i_{n+3}) - 1 - b_n. \tag{2.43}$$

The second equality uses $J_{n+3} = a_n a_{n+7} - i_{n+3}$ and a mutation. Now from row -7 of M_i we have $i_{n+3}J_{n+4} - i_{n+3}i_{n+4}b_n + i_{n+3}a_{n+1}g_{n+3} = 0$. We replace the $i_{n+3}i_{n+4}$ term to get

$$i_{n+3}J_{n+4} - b_n = b_n h_{n+4} - i_{n+3}a_{n+1}g_{n+3}.$$

The right-hand side of (2.43) now becomes

$$J_{n+3}J_{n+4} - 1 + b_n h_{n+4} - i_{n+3}a_{n+1}g_{n+3}. \tag{2.44}$$

Row 7 of M_a yields

$$i_{n+3}a_{n+1}g_{n+3} = a_n a_{n+1}h_{n+4} - a_{n+1}g_{n+5} = h_{n+4} + b_n h_{n+4} - a_{n+1}g_{n+5},$$

so (2.44) is equal to

$$J_{n+3}J_{n+4} - 1 + a_{n+1}g_{n+5} - h_{n+4}. \tag{2.45}$$

Finally we have $h_{n+4} - a_{n+1}g_{n+5} + i_{n+6}$ from row 8 of $\tilde{M}_a$ so (2.45) becomes $J_{n+3}J_{n+4} - 1 - i_{n+6}$ which completes the proof. □

The following theorem proves the other $\tilde{E}_8$ relation given in Theorem 2.3.

Theorem 2.37. *The quantity*

$$K_n := \frac{a_{n+20} + a_n}{a_{n+10}}$$

is period 3.

Proof. By Lemma 2.36 the expression $a_n a_{n+13} - i_{n+6}$ is period 5. In particular

$$a_n a_{n+13} - i_{n+6} = a_{n+10}a_{n+23} - i_{n+16}.$$

We also have $J_{n+1} = a_{n+3}a_{n+10} - i_{n+6} = a_{n+13}a_{n+20} - i_{n+16}$ so

$$a_n a_{n+13} - a_{n+3}a_{n+10} = a_{n+10}a_{n+23} - a_{n+13}a_{n+20},$$

which can be factored to give the theorem. □

2.5.4.2 Constant coefficient linear relations

Theorem 2.38. *The constant coefficient relation for the a variables is*

$$a_{n+60} - \mathcal{K}a_{n+30} + a_n = 0,$$

where $\mathcal{K}$ is invariant.

Proof. Defining

$$\Psi_n := \begin{pmatrix} a_{n+11} & a_{n+5} \\ a_{n+6} & a_n \end{pmatrix}, \qquad L_{n-3} := \begin{pmatrix} J_n & 1 \\ -1 & 0 \end{pmatrix}$$

gives $\Psi_n L_n = \Psi_{n+6}$ so again we apply Theorem 2.9 with $p = 5$, $q = 6$ and $M_n := L_n L_{n+6} \dots L_{n+24}$ to arrive at the above result. □

Finally, we state that we were unable to prove the following conjecture to complete the table in Figure 2.2, but we give an ansatz.

Conjecture 2.39. The following expression is period 2:

$$\tilde{K}_n := \frac{a_{n+30} + a_n}{a_{n+15}}.$$

2.6 Integrability for $\tilde{D}$ and $\tilde{E}$ quivers

In this section we construct reduced cluster maps, (2.6), and find log-canonical Poisson structures for the reduced variables, (2.7). By examining the Poisson subalgebras generated by the J_n, and in some cases K_n, we find enough commuting first integrals, defined in terms of these periodic quantities, to prove the integrability of the reduced systems. We do this for $\tilde{D}_N$ with N odd and for each $\tilde{E}$ type system. The proof for $\tilde{D}_N$ with N even, however, was too complicated so the problem remains open.

2.6.1 Integrability for $\tilde{D}_N$

The B matrix for the $\tilde{D}$ type diagrams, oriented as above, is

$$B = \begin{pmatrix} \begin{matrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & \pm1 & 0 & 0 & 0 \\ \mp1 & 0 & \mp1 & 0 & 0 \\ 0 & \pm1 & 0 & \pm1 & \pm1 \\ 0 & 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 & 0 \end{matrix} \end{pmatrix}$$

Where the signs depend on the parity of N. The first two rows show that B is singular, so we first project to a lower dimensional space. Since the new coordinates are given in terms of the image of B, which is different for different parities of N, we'll deal with the two cases separately.

Since the Poisson bracket on the reduced variables commutes with the shift operator φ^*, calculating the brackets between cluster variables with the same subscripts is straightforward. When calculating the brackets between J variables, however, different subscripts will appear.

2.6.2 The N odd case

Let $\{e_i\}$ be standard basis vectors. Here the kernel of B is spanned by $e_1 - e_2$ and $e_N - e_{N+1}$ and the image is spanned by

$$e_1 + e_2, \quad e_3, \quad e_4, \quad \ldots \quad , e_{N-1}, \quad e_N + e_{N+1}.$$

We take reduced variables $p_n := X_n^1 X_n^2$, $q_n := X_n^N X_n^{N+1}$ and leave $X_n^3, \dots X_n^{N-1}$ fixed. This gives a reduced cluster map

$$\varphi : \begin{pmatrix} p_n \\ X_n^3 \\ \vdots \\ X_n^{N-1} \\ q_n \end{pmatrix} \mapsto \begin{pmatrix} p_{n+1} \\ X_{n+1}^3 \\ \vdots \\ X_{n+1}^{N-1} \\ q_{n+1} \end{pmatrix}, \tag{2.46}$$

where the X_{n+1}^i are defined as in (2.12) and (2.14) for $i = 4, \dots, N-2$. The remaining relations are

$$p_{n+1} = \frac{1}{p_n}\left(1 + X_{n+1}^3\right)^2, \qquad X_{n+1}^3 = \frac{1}{X_n^3}\left(1 + p_n X_n^4\right),$$

$$q_{n+1} = \frac{1}{q_n}\left(1 + X_n^{N-1}\right)^2, \qquad X_{n+1}^{N-1} = \frac{1}{X_n^{N-1}}\left(1 + q_{n+1} X_{n+1}^{N-2}\right).$$

Our goal is to prove the integrability of the map (2.46). The kernel and image of B are used to define A:

$$A := \begin{pmatrix} e_1 + e_2 \\ e_3 \\ \vdots \\ e_{N-1} \\ e_N + e_{N+1} \\ e_1 - e_2 \\ e_N - e_{N+1} \end{pmatrix},$$

which gives $\hat{B}$ and then $C = (c_{ij})$, the Poisson matrix for the reduced variables:

$$c_{ij} = \begin{cases} (-1)^{(j-i+1)/2} & j \text{ even, } i \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

This gives the Poisson bracket, as in (2.7), between y variables, where

$$y_n^1 := p_n, \quad y_n^2 := X_n^3, \quad y_n^3 := X_n^4, \quad \dots \quad y_n^{N-2} := X_n^{N-1}, \quad y_n^{N-1} := q_n.$$

For our calculations, however, we'll stick to using X, p and q. Since each J_n is preserved by scaling (1.39) for each kernel vector of B we can express them in terms of the reduced variables, hence we can calculate brackets between the J_n.

Theorem 2.40. *The Poisson structure for the J variables, defined in Lemma 2.12, is given by*

$$\{J_0, J_{-k}\} = (-1)^k J_0 J_{-k} + \delta_{1,k} - \delta_{N-3,k}, \tag{2.47}$$

for $k = 1, ..., N-3$, where $\delta_{i,j}$ is the Kronecker delta. We can apply the shift φ^*, a Poisson algebra homomorphism, to this relation to obtain the brackets between the remaining J variables.

Proof. We use (2.18) to express J_n in plenty of different ways, i.e. for each row

$$J_n = \frac{1^{\text{st}} \text{ entry} + 3^{\text{rd}} \text{ entry}}{2^{\text{nd}} \text{ entry}}.$$

Shifting these expressions by appropriate amounts allows us to write each J_n in terms of reduced variables with subscript 0 or 1, simplifying calculations. We have

$$J_0 = \frac{p_1 + X_0^4}{X_1^3}, \qquad J_{\frac{-N+3}{2}} = \frac{X_1^{N-2} + q_0}{X_0^{N-1}}$$

and for $k = 1, \dots, \frac{N-5}{2}$

$$J_{-k} = \frac{X_1^{2k+2} + X_0^{2k+4}}{X_1^{2k+3}}.$$

We first calculate

$$\{J_0, J_{-k}\} = \left\{ \frac{p_1 + X_0^4}{X_1^3}, \frac{X_1^{2k+2} + X_0^{2k+4}}{X_1^{2k+3}} \right\} = \frac{1}{X_1^3 X_1^{2k+3}} \{p_1 + X_0^4, X_1^{2k+2} + X_0^{2k+4}\}$$

$$-\frac{J_{-k}}{X_1^3 X_1^{2k+3}} \{p_1 + X_0^4, X_1^{2k+3}\} - \frac{J_0}{X_1^{2k+3} X_1^3} \{X_1^3, X_1^{2k+2} + X_0^{2k+4}\}.$$

The issue now is calculating the brackets between elements with unequal subscripts. For example

$$\{p_1, X_0^{2k+4}\} = \left\{ p_1, \frac{1 + X_1^{2k+3} X_1^{2k+5}}{X_1^{2k+4}} \right\} = \frac{1}{X_1^{2k+4}} \{p_1, X_1^{2k+3} X_1^{2k+5}\} =$$

$$\frac{X_1^{2k+3}}{X_1^{2k+4}}\{p_1, X_1^{2k+5}\} + \frac{X_1^{2k+5}}{X_1^{2k+4}}\{p_1, X_1^{2k+3}\} = 0.$$

After some work we have

$$\{J_0, J_{-k}\} = \delta_{k,1} + (-1)^k J_0 J_{-k},$$

while the bracket

$$\{J_0, J_{\frac{N-3}{2}}\} = (-1)^{(N-3)/2} J_0 J_{\frac{N-3}{2}}$$

is found similarly. □

Theorem 2.41. *The reduced cluster map (2.46) is Liouville integrable.*

Proof. The bracket (2.47) can be written as the sum of the two Poisson brackets

$$\{J_0, J_{-k}\}_0 := \delta_{1,k} - \delta_{N-3,k}, \qquad \{J_0, J_{-k}\}_2 := (-1)^k J_0 J_{-k}.$$

This bi-Hamiltonian structure allows for the construction of $(N-1)/2$ commuting first integrals, as in the $\tilde{A}$ case, as the Poisson subalgebra (2.47) is the same, up to scaling. □

2.6.3 The case $N = 6$

For general even N the calculations were too much to bear. Instead we will work with the example $\tilde{D}_6$. The B matrix and our choice of A are

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

This gives the Poisson matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Our reduced variables are

$$y_n^1 := p_n := X_n^1 X_n^2 X_n^4, \quad y_n^2 := X_n^3, \quad y_n^3 := X_n^5, \quad y_n^4 := q_n := X_n^4 X_n^6 X_n^7,$$

giving a reduced cluster map

$$\varphi : \begin{pmatrix} p_n \\ X_n^3 \\ X_n^5 \\ q_n \end{pmatrix} \mapsto \begin{pmatrix} p_{n+1} \\ X_{n+1}^3 \\ X_{n+1}^5 \\ q_{n+1} \end{pmatrix}, \tag{2.48}$$

where

$$p_{n+1} = \frac{1}{p_n}\left(1 + X_{n+1}^3\right)^2 \left(1 + X_{n+1}^3 X_{n+1}^5\right), \qquad X_{n+1}^3 = \frac{1}{X_n^3}\left(1 + p_n\right)$$

$$X_{n+1}^5 = \frac{1}{X_n^5} = \left(1 + q_n\right), \qquad q_{n+1} = \frac{1}{q_n}\left(1 + X_{n+1}^3 X_{n+1}^5\right)\left(1 + X_{n+1}^5\right)^2.$$

Now we can express the J variables as, for example

$$J_n = \frac{X_{n+1}^1 X_{n+1}^2 + X_n^4}{X_{n+1}^3}, \qquad J_{n-1} = \frac{X_{n+1}^4 + X_n^6 X_n^7}{X_{n+1}^5}. \tag{2.49}$$

The scaling (1.39), for the kernel vector $(1, 0, 0, -1, 0, 0, 1)^T$, acts as

$$X_{n+1}^1 \mapsto \lambda X_{n+1}^1, \quad X_{n+1}^4 \mapsto \lambda^{-1} X_{n+1}^4, \quad X_{n+1}^7 \mapsto \lambda X_{n+1}^7$$

and fixes the remaining X^i_{n+1}. From the expression $X^4_{n+1}X^4_n = 1 + X^3_{n+1}X^5_{n+1}$ and that of J_n in (2.49) we see that

$$X^4_n \mapsto \lambda X^4_n, \qquad J_n \mapsto \lambda J_n.$$

Since J_n is not preserved by this scaling it cannot be written in terms of the reduced variables. However, one sees from the second expression in (2.49), that J_{n-1} transforms to $\lambda^{-1}J_{n-1}$. We instead define $J'_n := J_nJ_{n-1}$ and look at the Poisson subalgebra generated by $\{J'_{n-i}\}^3_{i=0}$. One can check that these generators are preserved by the scaling associated with the other two kernel vectors of B, so can be written in terms of the reduced variables. Indeed, for example, we have

$$J'_n = \frac{1}{X^3_{n+1}X^5_{n+1}}\left(p_{n+1} + q_n + 1 + X^3_{n+1}X^5_{n+1} + \frac{p_{n+1}}{q_{n+1}}(1 + X^5_{n+1})^2\right).$$

We find similar expressions for the other J' and rewrite so that the same subscript appears throughout. With the aid of a computer we arrive at

$$\{J'_0, J'_{-1}\} = -J'_0J'_{-1} + J'_0 + J'_{-1}, \qquad \{J'_0, J'_{-2}\} = J'_{-3} - J'_{-1}.$$

We remark that this Poisson subalgebra structure also appears in the 6 dimensional reduced system from the $\tilde{E}_7$ quiver. Here however, unlike the $\tilde{E}_7$ case, the J_n generated enough independent, commuting first integrals to prove integrability:

$$J'_0 + J'_1 + J'_2 + J'_3, \qquad J'_0J'_1J'_2J'_3.$$

2.6.4 Integrability for $\tilde{E}_6$

For the $\tilde{E}_6$ quiver, we have the B matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For which we can take image vectors

$$e_1 + e_3, e_2, e_3 + e_5, e_4, e_3 + e_6, e_7,$$

giving reduced coordinates

$$y_n^1 := a_n c_n, \qquad y_n^2 := b_n, \qquad y_n^3 := c_n e_n,$$

$$y_n^4 := d_n, \qquad y_n^5 := c_n g_n, \qquad y_n^6 := f_n.$$

The reduced cluster map

$$\varphi : \begin{pmatrix} y_n^1 \\ y_n^2 \\ \vdots \\ y_n^6 \end{pmatrix} \mapsto \begin{pmatrix} y_{n+1}^1 \\ y_{n+1}^2 \\ \vdots \\ y_{n+1}^6 \end{pmatrix} \tag{2.50}$$

is given by

$$y_{n+1}^1 = \frac{1}{y_n^1}\left(1 + y_n^2\right)\left(1 + y_n^2 y_n^4 y_n^6\right), \qquad y_{n+1}^2 = \frac{1}{y_n^2}\left(1 + y_{n+1}^1\right),$$

$$y_{n+1}^3 = \frac{1}{y_n^3}\left(1 + y_n^2 y_n^4 y_n^6\right)\left(1 + y_n^4\right), \qquad y_{n+1}^4 = \frac{1}{y_n^4}\left(1 + y_{n+1}^3\right),$$

$$y_{n+1}^5 = \frac{1}{y_n^5}\left(1 + y_n^2 y_n^4 y_n^6\right)\left(1 + y_n^6\right), \qquad y_{n+1}^6 = \frac{1}{y_n^6}\left(1 + y_{n+1}^5\right).$$

The image vectors and a kernel vector $(e_1 - e_3 + e_5 + e_7)$ are used as a basis for the matrix A. We may then calculate the Poisson matrix, after scaling,

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We can explicitly write J_0 as

$$J_0 = \frac{y_0^1 y_0^5 + y_0^1 y_0^6 + y_0^1 + y_0^5 + (1 + y_0^6)(y_0^2 y_0^4 y_0^6 + 1)}{y_0^2 y_0^5 y_0^6}$$

and find similar expressions for each of $J_1, J_2, \tilde{J}_0, \tilde{J}_1, \tilde{J}_2$ from the S_3 action. The first row of the Poisson matrix is

$$\{J_0, J_1\} = J_0 J_1 - 1, \qquad \{J_0, J_2\} = -J_0 J_2 + 1, \qquad \{J_0, \tilde{J}_i\} = 0$$

for $i = 1, 2, 3$. From this we can find each bracket, noting that the tilde is an automorphism. The algebra is the direct sum of the subalgebras generated by the J_i and the $\tilde{J}_i$ and these summands are isomorphic. We take independent commuting first integrals

$$J_0 + J_1 + J_2, \qquad J_0 J_1 J_2, \qquad \tilde{J}_0 + \tilde{J}_1 + \tilde{J}_2,$$

proving the integrability of (2.50).

2.6.5 Integrability for $\tilde{E}_7$

In this case we have the B matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Our choice of image vectors are

$$e_1 + e_3, e_2, e_3 + e_5 + e_6, e_4, e_6 + e_8, e_7$$

and reduced coordinates

$$y_n^1 := a_n c_n, \qquad y_n^2 := b_n, \qquad y_n^3 := c_n e_n f_n,$$

$$y_n^4 := d_n, \qquad y_n^5 := f_n h_n, \qquad y_n^6 := g_n.$$

Here the reduced cluster map is given by

$$y_{n+1}^1 = \frac{1}{y_n^1}\left(1 + y_n^2\right)\left(1 + y_n^2 y_n^4\right), \qquad y_{n+1}^2 = \frac{1}{y_n^2}\left(1 + y_{n+1}^1\right)$$

$$y_{n+1}^3 = \frac{1}{y_n^3}\left(1 + y_n^2 y_n^4\right)\left(1 + y_n^4\right)\left(1 + y_n^4 y_n^6\right), \qquad y_{n+1}^4 = \frac{1}{y_n^4}\left(1 + y_{n+1}^3\right)$$

$$y_{n+1}^5 = \frac{1}{y_n^5}\left(1 + y_n^4 y_n^6\right)\left(1 + y_n^6\right), \qquad y_{n+1}^6 = \frac{1}{y_n^6}\left(1 + y_{n+1}^5\right).$$

The matrix B has kernel vectors $e_1 - e_3 + e_6 - e_8$ and $e_1 - e_3 + e_5$ so we construct A as usual and find

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The kernel vectors give the scaling symmetries

$$\lambda_1 : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ e_n \\ \lambda^{(-1^n)} f_n \\ g_n \\ \lambda^{(-1^{n+1})} h_n \end{pmatrix}, \qquad \lambda_2 : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ \lambda^{(-1^n)} e_n \\ f_n \\ g_n \\ h_n \end{pmatrix}.$$

Our periodic quantities are fixed by λ_1 but not by λ_2. We have

$$\lambda_2 : \begin{pmatrix} J_0 \\ J_1 \\ J_2 \\ J_3 \end{pmatrix} \mapsto \begin{pmatrix} \lambda J_0 \\ \lambda^{-1} J_1 \\ \lambda J_2 \\ \lambda^{-1} J_3 \end{pmatrix},$$

so we can instead take $J_i' := J_i J_{i+1}$ for $i = 0, 1, 2$ with each J_i' fixed under λ_1 and λ_2. We have, for example,

$$J_0' = J_0 J_1 = \frac{1}{y_0^2 y_0^3 (y_0^4)^2 y_0^5 y_0^6} (y_0^1 y_0^3 + y_0^1 y_0^4 y_0^6 + y_0^3 + y_0^1 + (y_0^4 y_0^6 + 1)(y_0^2 y_0^4 + 1)) \times$$

$$(y_0^4 (y_0^4 ((y_0^6)^2 + y_0^6) + y_0^5 + (y_0^6)^2 + 2y_0^6 + 1) + (y_0^5 + y_0^6 + 1)(y_0^3 + 1).$$

Now we can calculate

$$\{J_0', J_1'\} = J_0'J_1' - J_0' - J_1', \qquad \{J_0', J_2'\} = J_1' - J_3',$$

from which the other brackets follow. We have commuting first integrals

$$J_0' + J_1' + J_2' + J_3', \qquad J_0'J_2' + J_1'J_3'.$$

Similar calculations give that the first integral $K_0 + K_1 + K_2$ is independent of, and commutes with, the previous two. This proves integrability of the reduced cluster map for $\tilde{E}_7$.

2.6.6 Integrability for $\tilde{E}_8$

Here the B matrix is

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

with image vectors

$$e_1 + e_3, e_2, e_3 + e_5, e_4, e_5 + e_7 + e_8, e_6, e_8, e_9$$

and kernel vector $e_1 - e_3 + e_5 - e_7$. These give the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The reduced coordinates are

$$y_n^1 := a_n c_n, \quad y_n^2 := b_n, \quad y_n^3 := c_n e_n, \quad y_n^4 := d_n,$$

$$y_n^5 := e_n g_n h_n, \quad y_n^6 := f_n, \quad y_n^7 := h_n, \quad y_n^8 := i_n,$$

with reduced cluster map $\hat{\varphi}$, given by

$$y_{n+1}^1 = \frac{1}{y_n^1}\left(1 + y_n^2\right)\left(1 + y_n^2 y_n^4\right), \qquad y_{n+1}^2 = \frac{1}{y_n^2}\left(1 + y_{n+1}^1\right),$$

$$y_{n+1}^3 = \frac{1}{y_n^3}\left(1 + y_n^2 y_n^4\right)\left(1 + y_n^4 y_n^6\right), \qquad y_{n+1}^4 = \frac{1}{y_n^4}\left(1 + y_{n+1}^3\right),$$

$$y_{n+1}^5 = \frac{1}{y_n^5}\left(1 + y_n^4 y_n^6\right)\left(1 + y_n^6\right)\left(1 + y_n^6 y_n^8\right), \qquad y_{n+1}^6 = \frac{1}{y_n^6}\left(1 + y_{n+1}^5\right),$$

$$y_{n+1}^7 = \frac{1}{y_n^7}\left(1 + y_n^6 y_n^8\right), \quad y_{n+1}^8 = \frac{1}{y_n^8}\left(1 + y_{n+1}^7\right).$$

The single kernel vector generates the scaling

$$\lambda : \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \\ e_n \\ f_n \\ g_n \\ h_n \\ i_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{(-1^n)} a_n \\ b_n \\ \lambda^{(-1^{n+1})} c_n \\ d_n \\ \lambda^{(-1^n)} e_n \\ f_n \\ \lambda^{(-1^n)} g_n \\ h_n \\ i_n \end{pmatrix},$$

for which the period 5 quantity

$$J_n = \frac{a_{n+12} + a_n}{a_{n+6}}$$

is fixed. Again we compute the brackets

$$\{J_0, J_1\} = J_0 J_1 - 1, \qquad \{J_0, J_2\} = -J_0 J_2,$$

and find the three commuting first integrals

$$J_0+J_1+J_2+J_3+J_4, \qquad J_0J_1J_2J_3J_4, \qquad J_0J_1J_2+J_1J_2J_3+J_2J_3J_4+J_3J_4J_0+J_4J_0J_1.$$

From the period 3 quantities we construct a final first integral $K_0+K_1+K_2$ which commutes with the other three, proving integrability.

2.7 Conclusion and outlook

In this chapter we have found periodic quantities for the frieze sequences from affine D and E type quivers and found linear relations for the cluster variables living at extending vertices. We also proved the integrability of the reduced cluster map for E type and for D_N with N odd.

The first question to ask is if integrability still holds for D_N with N even. As demonstrated in Section 2.6.3, answering this question requires either someone with more patience than the author or a novel approach. We can also ask about linear relations for non-extending vertices, which appear to be more complicated than the extending case, for example in Theorem 2.23.

Finally it would be satisfying to explain the appearance of the periodic quantities, shown in Figure 2.2, in the tubes of the associated Auslander-Reiten quivers.

Chapter 3

Linearisability and the Laurent property for two lattice equations and their reductions

We'll begin this chapter by examining the Little Pi recurrence

$$x_n x_{n+2k+l} = x_{n+2k} x_{n+l} + a x_{n+k} + a x_{n+k+l} \tag{3.1}$$

for a fixed parameter a and positive integers k and l. In [1], it was shown that this can be obtained from a period one seed in LP algebras, we gave some details of this construction in Subsection 1.4.2. In particular this recurrence satisfies the Laurent property. Here we'll prove that this recurrence is linearisable, with the constant coefficient linearisation

$$x_{n+6kl} - \mathcal{K} x_{n+4kl} + \mathcal{K} x_{n+2kl} - x_n = 0 \tag{3.2}$$

if $2k$ and l are coprime, and a counterpart relation

$$x_{n+6kl} - \mathcal{A} x_{n+5kl} + \mathcal{B} x_{n+4kl} - \mathcal{C} x_{n+3kl} + \mathcal{B} x_{n+2kl} - \mathcal{A} x_{n+kl} + x_n = 0$$

if $\gcd(2k, l) = 2$. All other cases can be reduced to one of these. On the way to these results we find various periodic quantities and two linear relations with these periodic objects appearing as coefficients. We note that the special case of (3.1), where $l = 1$, is the Heideman-Hogan recurrence defined in [24] and studied in [28] and [43]. Our results resolve some conjectures of [43]. The Little Pi recurrence was recently found as a reduction of the two dimensional lattice equation

$$u_{s+2,t+1}u_{s,t} = u_{s,t+1}u_{s+2,t} + u_{s+1,t} + u_{s+1,t+1} \tag{3.3}$$

in [30], along with the results of Proposition 3.5 and Theorem 3.7. In Section 3.3 we apply this approach to a generalised "Extreme polynomial"

$$(x_{n+k+2l} + x_{n+k} + a)x_{n+l} = (x_{n+2l} + x_n + a)x_{n+k+l} \tag{3.4}$$

the special case of which was studied in [43] and [1] and shown to satisfy the Laurent property. We show that the 6-point 2-dimensional recurrence

$$(u_{s+1,t+2} + u_{s+1,t} + a)u_{s,t+1} = (u_{s,t+2} + u_{s,t} + a)u_{s+1,t+1} \tag{3.5}$$

satisfies two linear relations, each with coefficients fixed in one of the lattice directions. After reduction these become linear relations with periodic coefficients for (3.4) which gives the constant coefficient linear relation

$$x_{n+3kl} - \mathcal{K}x_{n+2kl} + \mathcal{K}x_{n+kl} - x_n = 0$$

which is of the same form as (3.2). In Subsection 3.4 we use the methods of [41] to construct sets of initial values for (3.3) and (3.5) that give well defined solutions on the entire lattice. We then consider a 2-dimensional version of the Laurent property and prove that this holds for large sets of initial values.

3.1 Linear relations with periodic coefficients

The form of the Little Pi recurrence we shall work with is

$$x_n x_{n+2k+l} = x_{n+2k} x_{n+l} + x_{n+k} + x_{n+k+l} \tag{3.6}$$

Note that scaling each $x_n \to a x_n$ in (3.1) gives the recurrence (3.6), so these two forms are equivalent. We can find linear relations with periodic coefficients from the left and rights kernels of the following matrix:

$$\hat{\Psi}_n := \begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} & x_{n+6k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} & x_{n+6k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} & x_{n+6k+2l} \\ x_{n+3l} & x_{n+2k+3l} & x_{n+4k+3l} & x_{n+6k+3l} \end{bmatrix}$$

First we need to prove that $\hat{\Psi}_n$ has determinant 0, via proving that its 3×3 analogue

$$\Psi_n := \begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{bmatrix}$$

has non-zero periodic determinant. The non-zero determinant ensures that $\hat{\Psi}_n$ is the smallest matrix of its shape with determinant zero. To tidy the following calculations we'll define

$$z_n := x_n + x_{n+l}.$$

It will be useful to note two identities which follow from (3.6)

$$z_n x_{n+2k+l} = x_{n+l} z_{n+2k} + z_{n+k}, \tag{3.7}$$

$$z_n x_{n+2k} = x_n z_{n+2k} - z_{n+k}. \tag{3.8}$$

Lemma 3.1. *The* 3×3 *determinant*

$$\delta_n := |\Psi_n| = \begin{vmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{vmatrix}$$

has period k.

Proof. The first thing to notice from (3.6) is that

$$\begin{vmatrix} x_n & x_{n+2k} \\ x_{n+l} & x_{n+2k+l} \end{vmatrix} = z_{n+k}$$

so now we may use Dodgson condensation to write

$$\delta_n = \frac{1}{x_{n+2k+l}} \begin{vmatrix} z_{n+k} & z_{n+3k} \\ z_{n+k+l} & z_{n+3k+l} \end{vmatrix}. \tag{3.9}$$

We'll write the 2×2 determinant in (3.9) as δ'_{n+k}. Scaling the first column by x_{n+3k+l} gives

$$x_{n+3k+l}\delta'_{n+k} = \begin{vmatrix} z_{n+k}x_{n+3k+l} & z_{n+3k} \\ z_{n+k+l}x_{n+3k+l} & z_{n+3k+l} \end{vmatrix}.$$

Here we can use (3.7) and (3.8) on the left column

$$x_{n+3k+l}\delta'_{n+k} = \begin{vmatrix} x_{n+k+l}z_{n+3k} + z_{n+2k} & z_{n+3k} \\ x_{n+k+l}z_{n+3k+l} - z_{n+2k+l} & z_{n+3k+l} \end{vmatrix} = \begin{vmatrix} z_{n+2k} & z_{n+3k} \\ -z_{n+2k+l} & z_{n+3k+l} \end{vmatrix} \tag{3.10}$$

By the same token, but manipulating the right column in (3.9), we have

$$x_{n+k+l}\delta'_{n+k} = \begin{vmatrix} z_{n+k} & -z_{n+2k} \\ z_{n+k+l} & z_{n+2k+l} \end{vmatrix}$$

Shifting this up by k and comparing with (3.10) we arrive at

$$\frac{\delta'_{n+k}}{x_{n+2k+l}} = \frac{\delta'_{n+2k}}{x_{n+3k+l}}$$

so these ratios are k periodic. $\square$

Lemma 3.2. *The determinant δ_n is non-zero.*

Proof.

$$\delta_{n-k} = 0 \iff \delta'_n = 0 \iff \begin{vmatrix} z_n & -z_{n+k} \\ z_{n+l} & z_{n+k+l} \end{vmatrix} = 0$$
$$\iff z_n z_{n+k+l} + z_{n+k} z_{n+l} = 0. \tag{3.11}$$

If (3.11) is true then it should be true for any initial values. It is clear from the recurrence that if we set $x_i > 0$ for $i = 0, \ldots, 2k+l-1$ then $z_n > 0$ for all n. Hence each of the summands in (3.11) is zero, but again this is impossible due to having $z_n > 0$. $\square$

Theorem 3.3. *The 4×4 determinant*

$$|\hat{\Psi}_n| := \begin{vmatrix} x_n & x_{n+2k} & x_{n+4k} & x_{n+6k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} & x_{n+6k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} & x_{n+6k+2l} \\ x_{n+3l} & x_{n+2k+3l} & x_{n+4k+3l} & x_{n+6k+3l} \end{vmatrix}$$

is zero.

Proof. We can expand this determinant, again using Dodgson condensation:

$$|\hat{\Psi}_n| = \frac{\delta_{n+k}\delta_{n+2k+l} - \delta_{n+k+l}\delta_{n+2k}}{z_{n+3k+l}}$$

but δ is k periodic so $|\hat{\Psi}_n| = 0$. $\square$

Now that we have $|\hat{\Psi}_n| = 0$ we can examine the kernels.

Remark 3.4. The kernel of $\hat{\Psi}_n$ is one dimensional. If it were two (or more) dimensional then we would get a kernel vector for Ψ_n, contradicting Lemma 3.2.

Proposition 3.5. *The iterates of (3.6) satisfy*

$$x_{n+6k} - K_n^{(3)} x_{n+4k} + K_n^{(2)} x_{n+2k} - x_n = 0$$

and

$$x_{n+3l} + \gamma_n x_{n+2l} + \beta_n x_{n+l} + \alpha_n x_n = 0$$

where $K_n^{(2)}$ *and* $K_n^{(3)}$ *are period* l*,* α_n *is period* k*, and* β_n *and* γ_n *are period* $2k$*.*

Proof. Let $(K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, 1)^T$ be in the kernel of $\hat{\Psi}_n$. We are justified in scaling the last entry to 1 due to Lemma 3.2. From the first 3 rows of

$$\hat{\Psi}_n(K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, 1)^T = 0 \tag{3.12}$$

we get the matrix equation

$$\begin{bmatrix} x_n & x_{n+2k} & x_{n+4k} \\ x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \end{bmatrix} \begin{bmatrix} K_n^{(1)} \\ K_n^{(2)} \\ K_n^{(3)} \end{bmatrix} = - \begin{bmatrix} x_{n+6k} \\ x_{n+6k+l} \\ x_{n+6k+2l} \end{bmatrix} \tag{3.13}$$

and by Cramer's rule we have

$$K_n^{(1)} = \frac{-\delta_{n+2k}}{\delta_n} = -1.$$

The last 3 rows of (3.12) give

$$\begin{bmatrix} x_{n+l} & x_{n+2k+l} & x_{n+4k+l} \\ x_{n+2l} & x_{n+2k+2l} & x_{n+4k+2l} \\ x_{n+3l} & x_{n+2k+3l} & x_{n+4k+3l} \end{bmatrix} \begin{bmatrix} K_n^{(1)} \\ K_n^{(2)} \\ K_n^{(3)} \end{bmatrix} = - \begin{bmatrix} x_{n+6k+l} \\ x_{n+6k+2l} \\ x_{n+6k+3l} \end{bmatrix} \tag{3.14}$$

The equations (3.13) and (3.14) tell us that $K_n^{(2)}$ and $K_n^{(3)}$ are l periodic. Now let

$$\hat{\Psi}_n^T(\alpha_n, \beta_n, \gamma_n, 1)^T = 0. \tag{3.15}$$

The same arguments as above give

$$\alpha_n = -\frac{\delta_{n+l}}{\delta_n}$$

and that α_n is k periodic and β_n and γ_n are $2k$ periodic. □

We can derive some relations between these coefficients from (3.13) and (3.14) and the corresponding equations for the left kernel of $\hat{\Psi}_n$.

Proposition 3.6. *The periodic coefficients satisfy* $K^{(2)}_{n+k} = -K^{(3)}_n$

Proof. From the first 2 rows of (3.12)

$$\begin{bmatrix} x_{n+2k} & x_{n+4k} \\ x_{n+2k+l} & x_{n+4k+l} \end{bmatrix} \begin{bmatrix} K^{(2)}_n \\ K^{(3)}_n \end{bmatrix} = \begin{bmatrix} x_n & x_{n+6k} \\ x_{n+l} & x_{n+6k+l} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving for $K^{(2)}_n$ and $K^{(3)}_n$:

$$K^{(2)}_n = \frac{x_n x_{n+4k+l} - x_{n+l} x_{n+4k} + z_{n+5k}}{z_{n+3k}},$$

$$K^{(3)}_n = \frac{x_{n+2k+l} x_{n+6k} - x_{n+2k} x_{n+6k+l} - z_{n+k}}{z_{n+3k}}$$

from which we get

$$z_{n+5k} K^{(2)}_{n+2k} + z_{n+3k} K^{(3)}_n = z_{n+7k} - z_{n+k}$$

The z variables satisfy the same matrix equation (3.13), due to the l periodicity of $K^{(2)}$ and $K^{(3)}$, so

$$z_{n+5k} K^{(2)}_{n+2k} + z_{n+3k} K^{(3)}_n = -K^{(2)}_{n+k} z_{n+3k} - -K^{(3)}_{n+k} z_{n+5k}.$$

If we assume that $K^{(2)}_{n+k} + K^{(3)}_n \neq 0$ (it is clear that $z_{n+5k} \neq 0$) then

$$\frac{K^{(2)}_{n+2k} + K^{(3)}_{n+k}}{K^{(2)}_{n+k} + K^{(3)}_n} = -\frac{z_{n+3k}}{z_{n+5k}}$$

the left hand side is period l so the right hand side should be too, i.e.

$$\frac{z_{n+3k}}{z_{n+5k}} = \frac{z_{n+3k+l}}{z_{n+5k+l}} \qquad \Longleftrightarrow \qquad \begin{vmatrix} z_{n+3k} & z_{n+3k+l} \\ z_{n+5k} & z_{n+5k+l} \end{vmatrix} = 0$$

but this determinant is δ'_{n+3k} from (3.9). Now Lemma 3.2 gives $\delta'_{n+3k} \neq 0$, a contradiction. Hence $K^{(2)}_{n+k} + K^{(3)}_n = 0$. □

Theorem 3.7. *These two propositions combine to give the period l linear relation*

$$-x_n + K_n x_{n+2k} - K_{n+k} x_{n+4k} + x_{n+6k} = 0$$

where we have set $K_n := K^{(2)}_n$.

Proposition 3.8. *The entries of the left kernel of $\hat{\Psi}_n$ satisfy both of the following:*

$$\alpha_n = \beta_n + \gamma_{n+k} - 1, \qquad \alpha_{n+l} = \frac{\beta_{n+l} + \beta_{n+k+l}}{\gamma_n + \gamma_{n+k}} \tag{3.16}$$

Proof. From the left kernel analogue of (3.14)

$$\begin{bmatrix} x_{n+l} & x_{n+2l} \\ x_{n+2k+l} & x_{n+2k+2l} \end{bmatrix} \begin{bmatrix} \beta_n \\ \gamma_n \end{bmatrix} = - \begin{bmatrix} x_n & x_{n+3l} \\ x_{n+2k} & x_{n+2k+3l} \end{bmatrix} \begin{bmatrix} \alpha_n \\ 1 \end{bmatrix}$$

so we can express β and γ as

$$\beta_n = \frac{\alpha_n (x_{n+2k} x_{n+2l} - x_n x_{n+2k+2l}) + z_{n+k+2l}}{z_{n+k+l}},$$

$$\gamma_n = \frac{(x_{n+2k+l} x_{n+3l} - x_{n+l} x_{n+2k+3l}) + \alpha_n z_{n+k}}{z_{n+k+l}}.$$

Shifting β_n up by l we can equate the bracketed terms

$$\alpha_{n+l}\gamma_n z_{n+k+l} - \alpha_n \alpha_{n+l} z_{n+k} = \beta_{n+l} z_{n+k+2l} - z_{n+k+3l}. \tag{3.17}$$

Now if we write the $z's$ in terms of $x's$ and replace the x_{n+k+4l} that appears as

$$x_{n+k+4l} = -\alpha_{n+l} x_{n+k+l} - \beta_{n+k+l} x_{n+k+2l} - \gamma_{n+k+l} x_{n+k+3l}$$

then (3.17) becomes

$$
\begin{aligned}
&-\alpha_n\alpha_{n+l}x_{n+k}+(\alpha_{n+l}\gamma_n-\alpha_n\alpha_{n+l}-\alpha_{n+l})x_{n+k+l}\\
&\quad+(\alpha_{n+l}\gamma_n-\beta_{n+l}-\beta_{n+k+l})x_{n+k+2l}+(1-\beta_{n+l}-\gamma_{n+k+l})x_{n+k+3l}=0 \quad (3.18)
\end{aligned}
$$

Since the kernel of $\hat{\Psi}_n$ is one dimensional we can scale and equate coefficients in (3.18) and an appropriate shift of (3.15) to get three equations

$$\alpha_n=\beta_n+\gamma_{n+k}-1,$$

$$\gamma_{n+k}\alpha_{n+l}=\beta_{n+l}+\beta_{n+k+l}-\alpha_{n+l}\gamma_n,$$

$$\alpha_{n+l}=\beta_{n+l}+\gamma_{n+k+l}-1.$$

The third of these is simply a shift of the first and the second can be rearranged to give the result. □

3.2 Linear relations with fixed coefficients

The two kernel equations of Proposition 3.5 can be used to shift Ψ_n by $2k$ and by l respectively, i.e. if we define two matrices:

$$L_n:=\begin{bmatrix}0&0&1\\1&0&-K_n\\0&1&K_{n+k}\end{bmatrix}\qquad \tilde{L}_n:=\begin{bmatrix}0&1&0\\0&0&1\\-\alpha_n&-\beta_n&-\gamma_n\end{bmatrix}$$

then

$$\Psi_nL_n=\Psi_{n+2k},\qquad \tilde{L}_n\Psi_n=\Psi_{n+l}.$$

The point of this is that if we define

$$M_n:=L_nL_{n+2k}L_{n+4k}\dots L_{n+2kl-2k},\qquad \tilde{M}_n:=\tilde{L}_{n+2kl-l}\dots\tilde{L}_{n+2l}\tilde{L}_{n+l}\tilde{L}_n \quad (3.19)$$

then multiplication by M_n will shift Ψ_n by $2k$ upwards l times:

$$\Psi_n M_n = \Psi_{n+2kl}$$

and multiplication by $\tilde{M}_n$ (on the other side) will shift Ψ_n by l upwards $2k$ times:

$$\tilde{M}_n \Psi_n = \Psi_{n+2kl}$$

we will use this to calculate properties of the trace of M_n. The Cayley-Hamilton theorem, as well as knowledge of the trace and determinant of M_n, will give the constant coefficient linear relations.

Remark 3.9. If $d := \gcd(k, l) > 1$ then the recurrence (3.6) splits into d copies of itself, so without loss of generality we can take $d = 1$. In this case if l is even then $\gcd(2k, l) = 2$ and $\operatorname{lcm}(2k, l) = kl$. If l is odd then $\gcd(2k, l) = 1$ and $\operatorname{lcm}(2k, l) = 2kl$.

With this remark in mind, we'll deal with the different cases separately.

3.2.1 The case $\gcd(2k, l) = 1$

Here $\operatorname{lcm}(2k, l) = 2kl$ and we define M_n and $\tilde{M}_n$ as above. Note that due to the l periodicity of L_n and the cyclicity of trace, $\operatorname{tr}(M_n)$ has period l. Similarly $\operatorname{tr}(\tilde{M}_n)$ has period $2k$. We have

$$\Psi_{n+2kl} = \Psi_n M_n = \tilde{M}_n \Psi_n \tag{3.20}$$

so $\operatorname{tr}(M_n) = \operatorname{tr}(\tilde{M}_n)$ has period $\gcd(2k, l) = 1$, hence is a constant which we'll call $\mathcal{K}$. Also we'll define $\tilde{\mathcal{K}} := \operatorname{tr}(M_n^{-1}) = \operatorname{tr}(\tilde{M}_n^{-1})$ which is constant by (3.20). In fact these constants are equal:

$$\mathcal{K} = \tilde{\mathcal{K}}$$

the proof of which appears in [26], which we omit here.

Theorem 3.10. *If* $\gcd(2k, l) = 1$ *then the iterates of (3.6) satisfy*

$$x_{n+6kl} - \mathcal{K}x_{n+4kl} + \mathcal{K}x_{n+2kl} - x_n = 0$$

where $\mathcal{K}$ *is a constant, defined above.*

Proof. By the Cayley-Hamilton theorem

$$M_n^3 - \mathrm{tr}(M_n)M_n^2 + cM_n - I = 0 \tag{3.21}$$

for some c. Applying M_n^{-3} to (3.21) gives

$$M_n^{-3} - cM_n^{-2} + \mathrm{tr}(M_n)M_n^{-1} - I = 0$$

and we see that $c = \mathrm{tr}(M_n^{-1}) = \mathrm{tr(M_n)}$. Multiplying (3.21) by Ψ_n from the left yields

$$\Psi_{n+6kl} - \mathrm{tr}(M_n)\Psi_{n+4kl} + \mathrm{tr}(M_n)\Psi_{n+2kl} - \Psi_n = 0$$

and the top leftmost entry of this matrix equation gives the linear relation. □

3.2.2 The case $\mathbf{gcd(2k, l) = 2}$

In this case $\mathrm{lcm}(2k, l) = kl$. While the definition for M_n (3.19) is valid, each matrix in the product would appear twice (M_n would be a perfect square) so we can instead take a product of half as many factors, ditto for $\tilde{M}_n$. The total shift for Ψ_n is now kl instead of $2kl$:

$$M_n := L_n L_{n+2k} ... L_{n+kl-4k} L_{n+kl-2k},$$

$$\tilde{M}_n := \tilde{L}_{n+kl-l}\tilde{L}_{n+kl-2l} ... \tilde{L}_{n+l}\tilde{L}_n,$$

$$\Psi_{n+kl} = \Psi_n M_n = \tilde{M}_n \Psi_n. \tag{3.22}$$

Again $\mathrm{tr}(M_n)$ has period $2k$ and $\mathrm{tr}(\tilde{M}_n)$ has period l, but now $\mathrm{tr}(M_n) = \mathrm{tr}(\tilde{M}_n)$ has period $\gcd(2k, l) = 2$. We let $\mathcal{K}_n := \mathrm{tr}(M_n)$ and $\mathcal{K}_n^{-1} := \mathrm{tr}(M_n^{-1})$. In this case we have

$$\mathcal{K}_n = \tilde{\mathcal{K}}_{n+1}$$

the proof of which we also omit.

Proposition 3.11. *The iterates of (3.6) satisfy*

$$x_{n+3kl} - \mathcal{K}_n x_{n+2kl} + \mathcal{K}_{n+1} x_{n+kl} - x_n = 0 \tag{3.23}$$

Proof. The proof here is the same as in Theorem 3.10, but with different traces appearing. □

Theorem 3.12. *We have a constant coefficient relation*

$$x_{n+6kl} - \mathcal{A}x_{n+5kl} + \mathcal{B}x_{n+4kl} - \mathcal{C}x_{n+3kl} + \mathcal{B}x_{n+2kl} - \mathcal{A}x_{n+kl} + x_n = 0$$

where

$$\mathcal{A} := \mathcal{K}_n + \mathcal{K}_{n+1}, \qquad \mathcal{B} := \mathcal{K}_n\mathcal{K}_{n+1} + \mathcal{K}_n + \mathcal{K}_{n+1}, \qquad \mathcal{C} := \mathcal{K}_n^2 + \mathcal{K}_{n+1}^2 + 2.$$

Proof. Let $\mathcal{S}$ be the shift operator sending $x_n \mapsto x_{n+1}$ for all n. Then applying

$$\mathcal{S}^{3kl} - \mathcal{K}_{n+1}\mathcal{S}^{2kl} + \mathcal{K}_n\mathcal{S}^{kl} - 1$$

to equation (3.23) gives the result. □

3.3 Reductions of a 6-point lattice equation

Recently in [30] the Little Pi family of recurrences was obtained as a reduction of a lattice equation in two dimensions, namely

$$u_{s+2,t+1}u_{s,t} = u_{s,t+1}u_{s+2,t} + u_{s+1,t} + u_{s+1,t+1}. \tag{3.24}$$

By imposing the constraint

$$u_{s,t} = u_{s+l,t-k}$$

for integers k and l one obtains the travelling wave reduction

$$u_{s,t} = x_n, \qquad n = sk + tl \tag{3.25}$$

where the new dependent variable satisfies the Little Pi recurrence (3.1). Various features of the 6-point equation (3.24), including the Laurent property and its linearisation, were studied in [30] and used to derive some of the properties of the reductions (3.1). Here we find the analogue of these results for another 6-point equation:

$$(u_{s+1,t+2} + u_{s+1,t} + a)u_{s,t+1} = (u_{s,t+2} + u_{s,t} + a)u_{s+1,t+1} \tag{3.26}$$

and its reduction

$$(x_{n+k+2l} + x_{n+k} + a)x_{n+l} = (x_{n+2l} + x_n + a)x_{n+k+l}. \tag{3.27}$$

Remark 3.13. The “Extreme Polynomial” in [43] and [1] is defined as

$$x_{n+k+1}x_n = x_{n+k}x_{n+1} + a\sum_{i=1}^{k} x_{n+i} + B$$

for constants a and B. Shifting this up 1 and taking a difference yields

$$(x_{n+k+2} + x_{n+k} + a)x_{n+1} = (x_{n+2} + x_n + a)x_{n+k+1}$$

which is a special case, with $l = 1$, of the one-dimensional recurrence (3.27) that is found as a reduction of (3.26).

Proposition 3.14. *The 6-point equation (3.26) satisfies the linear relations*

$$u_{s,t+3} - (J(t+1)+1)u_{s,t+2} + (J(t)+1)u_{s,t+1} - u_{s,t} = 0, \tag{3.28}$$

$$u_{s+3,t} + A(s)u_{s+2,t} + B(s)u_{s+1,t} + C(s)u_{s,t} = 0 \tag{3.29}$$

where $J(t)$ is independent of s and $A(s), B(s)$ and $C(s)$ are independent of t.

Proof. Dividing (3.26) we immediately find a quantity which is invariant under shifts in the s direction:

$$J(t) := \frac{u_{s,t+2} + u_{s,t} + a}{u_{s,t+1}},$$

giving a linear relation

$$u_{s,t+2} - J(t)u_{s,t+1} + u_{s,t} + a = 0. \tag{3.30}$$

Shifting this in t and taking a difference leads to

$$u_{s,t+3} - (J(t+1)+1)u_{s,t+2} + (J(t)+1)u_{s,t+1} - u_{s,t} = 0.$$

Recalling that these coefficients are fixed under s shift we have

$$\begin{bmatrix} u_{s,t} & u_{s,t+1} & u_{s,t+2} & u_{s,t+3} \\ u_{s+1,t} & u_{s+1,t+1} & u_{s+1,t+2} & u_{s+1,t+3} \\ u_{s+2,t} & u_{s+2,t+1} & u_{s+2,t+2} & u_{s+2,t+3} \\ u_{s+3,t} & u_{s+3,t+1} & u_{s+3,t+2} & u_{s+3,t+3} \end{bmatrix} \begin{bmatrix} -1 \\ J(t)+1 \\ -J(t)-1 \\ 1 \end{bmatrix} = 0.$$

We see that this 4×4 matrix has determinant zero so we may take a left kernel vector $(C(s), B(s), A(s), 1)$ with coefficients invariant under t shifts. This kernel gives the second linear relation. □

3.3.1 Linear relations for the reduction

Theorem 3.15. *The linear relations with periodic coefficients for the reduction (3.27) are*

$$x_{n+3l} - (J_{n+l}+1)x_{n+2l} + (J_n+1)x_{n+l} - x_n = 0, \tag{3.31}$$

$$x_{n+3k} + A_n x_{n+2k} + B_n x_{n+k} + C_n x_n = 0. \tag{3.32}$$

Proof. The travelling wave reduction (3.25) applied to (3.26) gives the one dimensional recurrence (3.27). After this reduction $J(t)$ becomes

$$J_n := \frac{x_{n+2l} + x_n + a}{x_{n+l}} \tag{3.33}$$

which is period k. $A(s), B(s)$ and $C(s)$ become period l quantities (denoted A_n, B_n, C_n) and our linear relations (3.28) and (3.29) become (3.31) and (3.32). $\square$

Theorem 3.16. *The iterates of (3.27) satisfy the linear relation*

$$x_{n+3kl} - \mathcal{K}x_{n+2kl} + \mathcal{K}x_{n+kl} - x_n = 0$$

with constant $\mathcal{K}$.

Proof. Defining, as before,

$$\Psi_n := \begin{bmatrix} x_n & x_{n+l} & x_{n+2l} \\ x_{n+k} & x_{n+k+l} & x_{n+k+2l} \\ x_{n+2k} & x_{n+2k+l} & x_{n+2k+2l} \end{bmatrix} \qquad L_n := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -J_n - 1 \\ 0 & 1 & J_{n+l} + 1 \end{bmatrix}$$

$$\tilde{L}_n := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -A_n & -B_n & -C_n \end{bmatrix}$$

Then we have $\Psi_{n+l} = \Psi_n L_n$ and $\Psi_{n+k} = \tilde{L}_n \Psi_n$. We are again justified in taking $\gcd(k, l) = 1$ so

$$\Psi_{n+kl} = \Psi_n M_n = \tilde{M}_n \Psi_n$$

where

$$M_n := L_n L_{n+l} \dots L_{n+(k-1)l}, \qquad \tilde{M}_n := \tilde{L}_{n+k(l-1)} \tilde{L}_{n+k(l-2)} \dots \tilde{L}_n.$$

We can now follow the same procedure as in the Little Pi case to get the result with

$$\mathcal{K} := \mathrm{tr}(M_n) = \mathrm{tr}(L_n L_{n+l} \dots L_{n+(k-1)l}).$$

□

3.4 Laurentness for linearisable lattice equations

Here we'll discuss the Laurent property for lattice equations. We construct a family of bands of initial values, and some special sets as well, that give well defined solutions on the whole lattice. Given some conditions on the initial values linear relations with coefficients fixed in one lattice direction can be used to prove the Laurent property. We show that the band sets of initial values satisfy these criteria.

Definition 3.17. We call the set of initial values for a lattice equation I, and the Laurent ring generated by these $\mathcal{L}$. That is,

$$\mathcal{L} := \mathbb{Z}[I, I^{-1}]$$

where $I^{-1} = \{1/u : u \in I\}$. We say that a 2-dimensional lattice equation satisfies the Laurent property, or is Laurent, with this I if

$$u_{s,t} \in \mathcal{L}[A]$$

for all s and t and a set of parameters A.

3.4.1 Construction of band sets of initial values

In [41] an algorithm is given to find a unique (in almost all cases) solution to a lattice equation of an arbitrary stencil if I is a band of initial values. We apply this to the 6-point "domino" stencil that (3.26) in defined on.

First we define the lines L_1 and L_2, each with positive rational gradient, such that $L_2 = L_1 + (1, -2)$. An example with gradient $1/3$ is shown in Figure 3.1. We will consider the points on L_1 to each be the top left corner of our 6 point

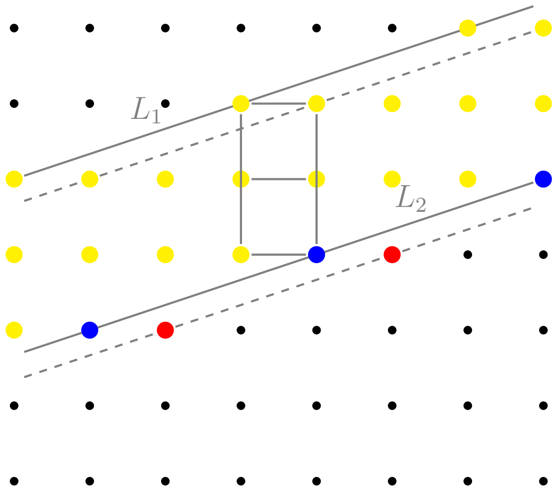


FIGURE 3.1: Initial values on the band with gradient 1/3

domino, hence L_2 will pass through the points diagonally opposite. Taking initial values between these lines and on L_1 (but not on L_2) allows us to find a unique solution of (3.26) for each choice of gradient. The first step is to calculate the values on L_2, drawn in blue, using the yellow initial values. We then shift the lines perpendicularly to their direction until they pass through another point of the domino. These are the dashed lines. The new L_2 will pass through the next points to be calculated, drawn in red. This process is continued until we fill the whole lattice below L_1. We can shift the lines in the opposite direction to fill the lattice above L_1.

3.4.2 The Laurent property for equation (3.26)

To prove the Laurent property we will use the linear relation (3.30), but first we must prove that the coefficients $J(t)$ are in the Laurent ring.

Lemma 3.18. *If we have an $\tilde{s}$ such that $u_{\tilde{s},t+1} \in I$ and*

$$\{u_{\tilde{s},t}, u_{\tilde{s},t+2}\} \subset \mathcal{L}[a]$$

then $J(t) \in \mathcal{L}[a]$.

Proof. Since $J(t)$ is independent of s we may shift it until $\tilde{s}$ appears, so

$$J(t) = \frac{u_{\tilde{s},t+2} + u_{\tilde{s},t} + a}{u_{\tilde{s},t+1}}$$

which is in $\mathcal{L}[a]$ by assumption. □

Theorem 3.19. *If, given some I, Lemma 3.18 holds for each t and for each s we have $\tilde{t}$ such that*

$$\{u_{s,\tilde{t}}, u_{s,\tilde{t}+1}\} \subset \mathcal{L}[a]$$

then (3.26) has the Laurent property for this I.

Proof. For each s we use induction on t and the relation

$$u_{s,t+2} = J(t)u_{s,t+1} - u_{s,t} - a$$

noting that, by the lemma, $J(t)$ is in the Laurent ring. The base case for the induction is given by

$$u_{s,\tilde{t}+2} = J(\tilde{t})u_{s,\tilde{t}+1} - u_{s,\tilde{t}} - a$$

this proves Laurentness for $t > \tilde{t} + 1$. The proof for $t < \tilde{t}$ is similar. □

Theorem 3.20. *The Laurent property for (3.26) holds for the band sets of initial values described in Section 3.4.1.*

Proof. To calculate $u_{s,t}$ we only have to divide by $u_{s+1,t+1}$ and vice-versa, so we know all the values we calculate are Laurent until we have to divide at one of the blue points (see Figure 3.1) and the corresponding points above L_1. These we mark in green in Figure 3.2. We draw L_2' parallel to and below L_2 through the first non-green point and L_1' parallel to and above L_1 through the last green point. Equivalently

$$L_2' = L_2 + (-1,-1), \qquad L_1' = L_1 + (-1,1)$$

hence $L_2' = L_1' + (1,-4)$. Since the minimal distance between L_1' and L_2' is $\sqrt{17} > 4$ any horizontal or vertical line that intersects I will intersect at least 4 elements

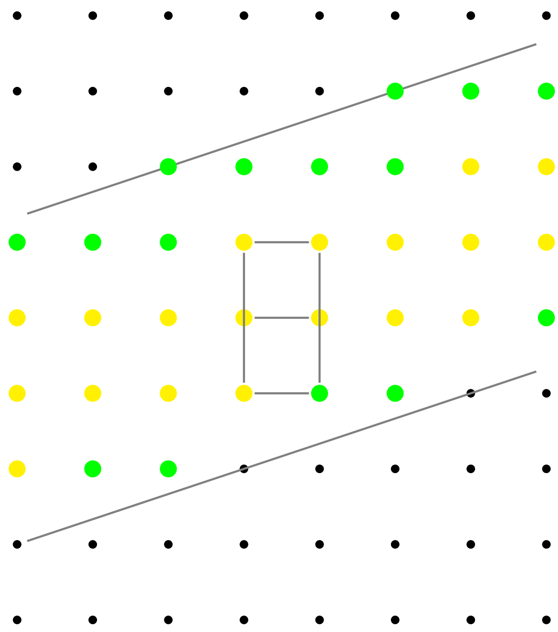


Figure 3.2: The points marked green are in the Laurent ring

of $\mathcal{L}$. For Lemma 3.18 we take horizontal lines with height t and see that they intersect at least 4 green or yellow points, at least one of which will be yellow. Hence $J(t) \in \mathcal{L}$ for all t. For Theorem 3.19 we take vertical lines for each s and see that these intersect at least two green or yellow points. Hence the conditions of the theorem hold and we have the Laurent property for these initial values. $\square$

In the special case where the gradient is 0 it is prescribed in [41] that we take an extra line of initial values perpendicular to L_1 and L_2, shown in Figure 3.3, for which the equation also has the Laurent property. We note that Laurentness does not hold for all well posed initial value problems, for example the yellow set shown in Figure 3.4. In this case one can see from the form of (3.26) that to calculate the value of $u_{s,t}$ at the blue node we must divide by a polynomial (not a monomial) in the surrounding initial values.

Remark 3.21. The Laurent property in the reduced case is easily seen from (3.33). In fact, since J_n is period k, the only initial variables that can appear in the denominator are $x_l, x_{l+1}, \ldots, x_{k+l-1}$. In particular, setting each of these to be 1 will give a polynomial sequence in the remaining initial values.

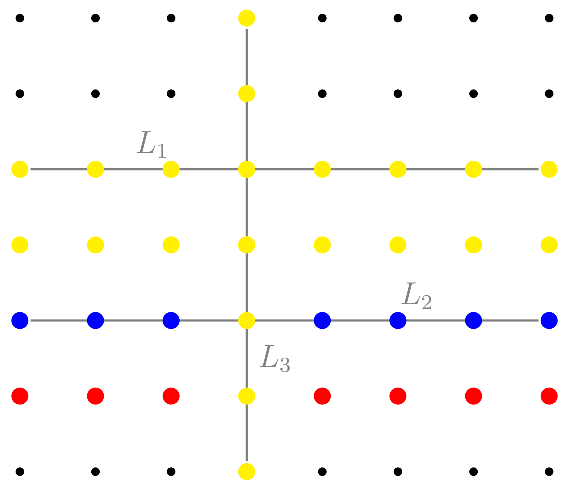


FIGURE 3.3: Initial values on the band with gradient 0 with Laurentness

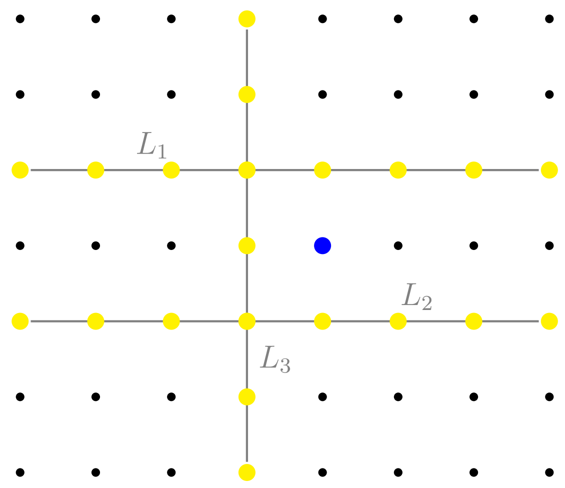


FIGURE 3.4: An example of initial values without the Laurent property

3.4.3 The Laurent property for the lattice Little Pi

The Laurent property for the lattice equation (3.24) was proved in [30] for the initial values

$$I = \{u_{s,0}, u_{0,t}, u_{1,t} : s, t \in \mathbb{N}\}.$$

Here we extend I to $\mathbb{Z} \times \mathbb{Z}$ and also switch the s and t coordinates to conform with the format of Equation (3.26). Hence we work with the lattice equation

$$u_{s+1,t+2}u_{s,t} = u_{s+1,t}u_{s,t+2} + u_{s,t+1} + u_{s+1,t+1} \tag{3.34}$$

we have drawn the I from [30] in yellow in Figure 3.5. Again we define the Laurent ring $\mathcal{L}$. Here we offer a different proof of Laurentness, similar to the proof for (3.26). From Theorem 2.1 and Proposition 2.6 respectively in [30] we have

$$u_{s,t+6} - \beta_{t+1}u_{s,t+4} + \beta_t u_{s,t+2} - u_{s,t} = 0 \tag{3.35}$$

and that β_t is given by

$$\beta_t = \frac{1 + u_{0,t}u_{0,t+3} + u_{0,t+1}u_{0,t+4} + u_{0,t+2}u_{0,t+5}}{u_{0,t+2}u_{0,t+3}} \tag{3.36}$$

Note that in this expression s has been set to zero since β is fixed under s shifts. Similarly to the proof of Theorem 3.20 we colour the values that only require division by elements of I in green in Figure 3.5. Due to the shape of (3.34) we end up with more green vertices than we had for (3.26).

Proposition 3.22. *The lattice equation (3.34) has the Laurent property for the initial values*

$$I = \{u_{s,0}, u_{s,1}, u_{0,t} : s, t \in \mathbb{Z}\}.$$

Proof. Again we fix s and use induction on t. The induction start is given the vertical line of 6 values in $\mathcal{L}$, shown in yellow and green in Figure 3.5. We can see from (3.36) that $\beta_t \in \mathcal{L}$ for all t for this I. □

For the band sets of initial values we have to work harder.

Lemma 3.23. *If, for a set of initial values I, we have an $\tilde{s}$ such that*

$$\{u_{\tilde{s},t}, u_{\tilde{s},t+1}, u_{\tilde{s},t+4}, u_{\tilde{s},t+5}\} \subset \mathcal{L}$$

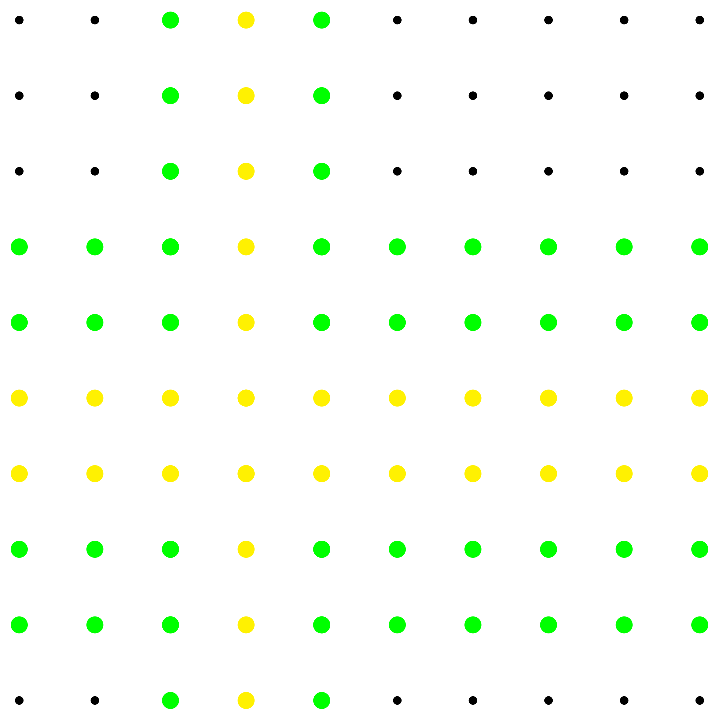


FIGURE 3.5: The yellow dots are initial values and the green are Laurent in the initial values

and

$$\{u_{\tilde{s},t+2}, u_{\tilde{s},t+3}\} \subset I$$

then $\beta_t \in \mathcal{L}$.

Proof. We shift the expression for β_t from (3.36) in the s direction until $u_{\tilde{s},t+2}$ and $u_{\tilde{s},t+3}$ appear in the denominator. □

Theorem 3.24. *If the conditions of Lemma 3.23 hold for all t and for all s we have a $\tilde{t}$ such that*

$$\{u_{s,\tilde{t}}, u_{s,\tilde{t}+1}, u_{s,\tilde{t}+2}, u_{s,\tilde{t}+3}, u_{s,\tilde{t}+4}, u_{s,\tilde{t}+5}\} \subset \mathcal{L}$$

then equation (3.34) has the Laurent property.

Proof. The proof is the same as for Proposition (3.22). □

Theorem 3.25. *The equation (3.34) has the Laurent property if I is a band of initial values.*

Proof. Similarly to the proof of Theorem 3.20 we have

$$L_1' = L_1 + (-1, -2), \qquad L_2' = L_2 + (1, -2)$$

so $L_2' = L_1' + (3, -6)$ and the minimal distance between them is $\sqrt{45} > 6$ so for any vertical or horizontal line intersecting the lattice we have at least 6 consecutive values in $\mathcal{L}$, and at least 2 of these will be neighbours and in I. $\square$

3.5 Conclusion and outlook

In this section we have linearised two recurrences from LP algebras and proved that they can be obtained as reductions of 2-dimensional recurrences. We also constructed sets of initial values such that the 2-dimensional recurrences satisfy a 2-dimensional Laurent property.

One open problem would be to construct more, or all, recurrences in LP algebras that can be linearised. One could also ask if there are recurrences with the Laurent property that live outside of LP algebras.

Additionally we could ask which other Laurent property recurrences can be obtained from reductions of higher dimensional recurrences, and for which initial values these too have the Laurent property.

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