Fault-Tolerant Control for Systems with Unmatched Actuator Faults and Disturbances

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***Abstract*— A fault-tolerant control (FTC) scheme for a class of nonlinear systems with unmatched actuator redundancy and unmatched disturbances is proposed in this note. A methodology to construct unified smooth sliding mode control laws and update laws is proposed such that the equivalent injections of the first-order time derivatives of the unmatched actuator faults and unmatched disturbances can appear in the unmatched channels. The unmatched actuator faults and unmatched disturbances are completely canceled by these equivalent injections. Based on this methodology and using the backstepping design procedure, a set of smooth FTC sliding surfaces, FTC laws and update laws are then designed. With the help of the FTC law selecting mechanism, the output tracking errors of the closed-loop FTC system converge to zero asymptotically, and time-varying faults and disturbances are reconstructed. A simulation example is presented to illustrate the effectiveness of the proposed FTC method.**

***Index Terms*— Fault-tolerant control, unmatched actuator faults, unmatched disturbances.**

I. Introduction

Fault-tolerant control (FTC) deals with the design of feedback control algorithms for systems with potential malfunctions in actuators, sensors and other components, thereby providing an effective way to improve the reliability and safety for critical systems such as aircrafts, high-speed trains and nuclear power stations [1]. Actuator faults are more destructive because they may cause control loss and even breakdown of the whole systems. Tolerating actuator faults has attracted many efforts of the control community in the past decades, and many effective actuator FTC approaches have been proposed such as adaptive control approaches [2], [3], multi-model control approaches [4], [5], sliding mode control approaches [6], [7], [8], robust control approaches [9], [10], control allocation approached [11], performance-based approach [12] and so on.

Generally speaking, actuator faults can be divided into two categories [13]: loss of effectiveness faults and stuck faults. However, due to complete actuation losses and undesirable float inputs, actuator stuck faults are more serious. Using the strategy that reconfiguring the healthy actuators to compensate for the vacancies left by the actuation losses and to reject the effects caused by the undesirable float inputs, the adaptive fault compensation approach is proposed in [14] and most recently developed in [3] and [15]. Also, based on this compensation strategy, sliding mode compensation schemes are proposed in [6] and developed more recently in [7], [16] and [17]. However, in these results, the considered systems are required to have matched actuator redundancy. Actuator stuck faults in systems with unmatched actuator redundancy, which are referred to as *unmatched actuator faults*, have not been fully studied. Tolerating unmatched actuator faults remains an open issue because unmatched actuator faults will introduce unmatched unknown inputs and may also convert the matched disturbances to the unmatched ones. Handling unmatched unknown inputs and disturbances is still challenging for the control community.

In this paper, we consider some key issues, unmatched actuator faults and unmatched disturbances, in actuator FTC topics, which have not been fully concerned so far in the most recent literature [3], [10] and [16]. Based on the differential geometry theories, a set of fault mode sets is proposed to group all tolerable actuator faults into distinct groups. In the presence of any fault mode belonging to one fault mode set, the system's relative degree from the outputs to the healthy actuators is fixed. However, the unmatched actuator faults and disturbances may arise in the faulty systems. By developing particular structures of the update laws and the sliding mode control laws, the equivalent injections of the first-order derivatives of the faults and disturbances arise in the unmatched channels, which facilitates complete rejection of the faults and disturbances. The advantage of this methodology is that it can not only asymptotically stabilize the closed-loop system but also exactly reconstruct the faults and disturbances. Based on this methodology, for each fault mode set, a smooth FTC sliding surface, a sliding mode control law, and a bank of update laws are designed based on the backstepping design procedure. Each update law can reconstruct a fault or a disturbance with the reconstruction error converging to zero asymptotically. Each smooth FTC sliding surface can guarantee that all states of the associated sliding motion are uniformly bounded, and the output tracks the reference signals asymptotically. Moreover, each sliding mode control law can guarantee that the associated sliding motion occurs in a finite time, and the closed-loop faulty system maintains on the associated sliding surface thereafter.

Different from the most recent literature [16] and [17], the unmatched actuator faults and disturbances are considered in this paper. In comparison to [3], this paper deals with a class of nonlinear systems and does not require restrictions on the fault modes of maintaining some properties. Furthermore, the methodology developed in this paper can completely remove the unmatched actuator faults and disturbances in the unmatched channels. Moreover, the FTC scheme in this paper can accurately reconstruct the time-varying faults and disturbances.

Notation: For a square matrix $A \in \mathbb{R}^{n\times n}$, $\underline{\lambda}(A)$ represents the minimum eigenvalue of A, and $(A)_{ij}$ represents the ith row and jth column element of A. For any vector $x = \text{col}(x_1, \cdots, x_n) \in \mathbb{R}^n$,

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$|x| = \mathrm{col}(|x_1|, \cdots, |x_n|)$. In addition, the relative degree used in this note is defined in [18].

II. Problem Formulation

Consider a class of nonlinear over-actuated systems

$$\begin{aligned}\dot{\zeta} =& f(\zeta) + \sum\nolimits_{i=1}^{m} g_i(\zeta)u_i(t) + \sum\nolimits_{i=1}^{n} D_i(\zeta)d_i(t),\\ y =& h(\zeta),\end{aligned} \tag{1}$$

where $\zeta \in \Omega \subset \mathbb{R}^n$ (the region Ω is a neighborhood of the origin), $y \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$, $i = 1, \cdots, m$ are the state vector, output and control inputs, respectively. The nonlinearities $f(\zeta)$ and $g_i(\zeta)$, $i = 1, \cdots, m$ are known smooth vector fields, and $h(\zeta)$ is a known scalar smooth function. Each $d_i(t) \in \mathbb{R}$ is an unknown time-varying parameter, while $D_i(\cdot) \in \mathbb{R}^n$ is the known distribution vector. The terms $D_i(\zeta)d_i(t)$, $i = 1, \cdots, n$, may represent not only external and internal disturbances, but also system uncertainties. Without loss of generality, it is assumed that all $d_i(t)$ have been well partitioned such that the relative degree of the triple $(f(\zeta), D_i(\zeta), h(\zeta))$ is i. Note that all $D_i(\zeta)d_i(t)$, $i = 1, \cdots, n$, are not required to be matched because each vector $D_i(\zeta)$ is not required to be parallel to any $g_j(\zeta)$ for $j = 1, \cdots, m$.

Remark 1. Many uncertainties such as polytopic type uncertainties (see chapter 3 of [19]) can be expressed in the form of $D_i(\zeta)d_i(t)$. The vector field $f(\zeta)$ represents the dynamics of the physical process, and $g_i(\zeta)$ and $D_i(\zeta)$ are distribution vectors that represent the directions that $u_i(t)$ and $d_i(t)$ enter the system respectively. Typically, they are completely determined by the physical structure of the system and are independent of external components. Thus, $f(\zeta)$, $g_i(\zeta)$ and $D_i(\zeta)$ describe the system's inherent properties and are not affected by actuator faults since actuators are typically external components with respect to the system. ∇

Unmatched Actuator Redundancy. This paper considers the case that for all $g_i(\cdot)$, $i = 1, \cdots, m$, $\mathrm{rank}([g_1(\zeta), \cdots, g_m(\zeta)]) = m$, which means that actuators $u_1, \cdots, u_m$ do not have matched redundancy (see [20]). Suppose that each triple $(f(\zeta), g_i(\zeta), h(\zeta))$ has a known relative degree r_i. Then, for any two actuators u_i and u_j, $i \neq j$ in system (1), the fact that if there exists a scalar function $\alpha_{ij}(\zeta) \neq 0$ such that $g_i(\zeta) = \alpha_{ij}(\zeta)g_j(\zeta)$, then $r_i = r_j$ holds, and its converse-negative that if $r_i \neq r_j$, then $g_i(\zeta) \neq \alpha_{ij}(\zeta)g_j(\zeta)$ for any $\alpha_{ij}(\zeta) \neq 0$ also holds. Thus, $r_1 \neq r_2 \neq \cdots \neq r_m$. Without loss of generality, it is assumed that $r_1 < r_2 < \cdots < r_m \leq n$ throughout the paper. We refer to this as unmatched actuator redundancy.

Actuator Stuck Faults. As modeled in [21], the output of the ith actuator with a potential stuck fault is expressed as

$$u_i(t) = \mu_{ai}(t) + \rho_i(\bar{u}_i(t) - \mu_{ai}(t)) = (1 - \rho_i)\mu_{ai}(t) + \rho_i\bar{u}_i(t), \tag{2}$$

where $\mu_{ai}(t)$ is the applied input of the ith actuator, $\bar{u}_i(t)$ is the time-varying stuck value and ρ_i is the ith actuator fault indicator where

$$\rho_i = \begin{cases} 1, & \text{if the } i\text{th actuator is locked at } \bar{u}_i(t),\\ 0, & \text{otherwise.}\end{cases}$$

This paper considers the scenarios of multiple faults but at least one healthy actuator. A set including all considered fault modes can be defined as follows:

$$\Sigma = \left\{ \begin{array}{l} \rho = \mathrm{diag}\{\rho_1, \cdots, \rho_m\} : \rho_1 = \cdots = \rho_{i-1} = 1 \text{ or } 0,\\ \rho_i = 0,\ \rho_j = 1 \text{ or } 0,\ i \in \{1, \cdots, m\},\ j = i + 1, \cdots, m \end{array} \right\} \tag{3}$$

It is worthy pointing out that this set Σ characterizes fault scenarios considered in many existing literature such as [21] and [22].

Objective. The objective of this paper is to design FTC laws $u_{ai}(t)$, $i = 1, \cdots, m$ for system (1) such that in the presence of actuator stuck faults (2) with ρ belonging to Σ, the remaining healthy actuators can still ensure that

- all states in the closed-loop system are uniformly bounded,
- the output $y(t)$ tracks the reference signal $y_r(t)$ with $\lim_{t\to\infty} y(t) - y_r(t) = 0$,

and design update laws to reconstruct $\bar{u}_i(t)$, $i = 1, \cdots, m$ and as many $d_j(t)$, $j = 1, \cdots, n$ as possible such that their reconstruction errors converge to zero asymptotically.

III. Preliminaries

In this section, a fault mode grouping scheme will be proposed, and then based on this scheme, feedback linearization schemes and FTC strategy are due to formulated.

A. Fault Mode Grouping

There are $2^m - 1$ fault modes are considered and included in Σ, which indicates that $2^m - 1$ FTC laws would need to be designed to handle each possible case if one FTC law is designed distinctly for each fault mode. This motivates the development of fault mode grouping scheme to reduce the required number of FTC laws. To this end, the actuation scheme for the system (1) is presented. To manage all the actuators of the over-actuated system (1), a proportional actuation scheme is used and given as

$$\mu_{ai}(t) = \pi_i\mu(t),\ \ i = 1, \cdots, m, \tag{4}$$

where π_i is a positive constant and $\mu(t)$ is the control law to be designed. To simplify the notations, u_i, μ_{ai} and μ are used in the remaining sections to replace $u_i(t)$, $\mu_{ai}(t)$ and $\mu(t)$, respectively.

The fault mode grouping sets are ready to be proposed as follows: for any $i \in \{1, \cdots, m\}$,

$$\Sigma_{(i)} = \left\{ \begin{array}{l} \rho = \mathrm{diag}\{\rho_1, \cdots, \rho_m\} : \rho_1 = \cdots = \rho_{i-1} = 1,\\ \rho_i = 0,\ \rho_j = 1 \text{ or } 0,\ \ j = i + 1, \cdots, m \end{array} \right\}. \tag{5}$$

Then, one has

$$\Sigma = \bigcup_{i=1}^{m} \Sigma_{(i)},\ \ \Sigma_{(i)} \cap \Sigma_{(j)} = \emptyset,\ \ \forall\ i \neq j.$$

Moreover, by submitting (4) to (2), the description for the system (1) under the actuator stuck faults (2) with $\rho \in \Sigma_{(i)}$ can be obtained by

$$\begin{aligned}\dot{\zeta} =& f(\zeta) + \sum_{l=i}^{m} g_l(\zeta)(1 - \rho_l)\pi_l\mu + \sum_{l=1}^{i-1} g_l(\zeta)\bar{u}_l(t) + \sum_{j=1}^{n} D_j(\zeta)d_j(t),\\ y =& h(\zeta).\end{aligned} \tag{6}$$

Then, it is ready to show the following lemma.

Lemma 1. *For any $i \in \{1, \cdots, m\}$ and all $\rho \in \Sigma_{(i)}$, the triples $(f(\zeta), \sum_{l=i}^{m} g_l(\zeta)(1 - \rho_l)\pi_l, h(\zeta))$ have a common relative degree γ_i.* ∇

Proof. This lemma can be easily verified by

$$L_{\sum_{l=i}^{m} g_l(\zeta)(1-\rho_l)\pi_l} L_f^{r_i-1} h(\zeta) = \pi_i L_{g_i} L_f^{r_i-1} h(\zeta) \neq 0,$$

where the notation L_f^i denotes ith-order Lie derivative [18]. $\square$

Based on Lemma 1 and [18], for all $\rho \in \Sigma_{(i)}$, the triples $(f(\zeta), \sum_{l=i}^{m} g_l(\zeta)(1 - \rho_l)\pi_l, h(\zeta))$ have a common linearization law, which will be specified later. This shows that m, rather than $2^m - 1$, FTC laws are needed, which is also the main advantage of the fault mode grouping scheme characterized by (5).

B. Feedback Linearization and Preliminary Assumptions

Supposed that μ_i is the common feedback linearization law for all $\rho \in \Sigma_{(i)}$. Then, based on [18], μ_i can be designed as

$$\mu_i = b_i^{-1}(z_i,\eta_i)\left(\nu_i - a_i(z_i,\eta_i)\right), \tag{7}$$

where ν_i is the virtual FTC law to be designed, $\mathrm{col}(z_i,\eta_i) = \Phi_i(\zeta)$ is a diffeomorphism, and

$$\begin{aligned} a_i(\cdot) &= L_f^{r_i-1}h(\zeta)|_{\zeta=\Phi_i^{-1}(z_i,\eta_i)}, \\ b_i(\cdot) &= \pi_i L_{g_i}L_f^{r_i-1}h(\zeta)|_{\zeta=\Phi_i^{-1}(z_i,\eta_i)}. \end{aligned}$$

Thus, in the coordinates z_i and η_i, system (6) has the following normal form

$$\begin{aligned} \dot z_{ij} &= z_{ij+1} + d_{ij}(z_i,\eta_i,d_j(t),\bar u(t)), \quad j = 1,\cdots,r_i-1, \\ \dot z_{ir_i} &= \nu_i + d_{ir_i}(z_i,\eta_i,d_{r_i}(t),\bar u(t)), \quad (8) \\ \dot\eta_i &= \Psi_i(z_i,\eta_i,d_{r_i+1}(t),\cdots,d_n(t),\bar u(t)), \quad (9) \\ y &= z_{i1}, \end{aligned}$$

where $z_i = \mathrm{col}(z_{i1},\cdots,z_{ir_i}) \in \mathbb{R}^{r_i}$, $\eta_i \in \mathbb{R}^{n-r_i}$ and $\bar u(t) = \mathrm{col}(\bar u_1(t),\cdots,\bar u_m(t))$. Moreover, $d_{ij}(\cdot)$, $j = 1,\cdots,r_i-1$ are unmatched with respect to ν_i, while $d_{ir_i}(\cdot)$ is matched. It is worth pointing out that since there is no control input in the unmatched channel $\dot z_{ij}$, the unmatched $d_{ij}(\cdot)$ is not easy to reject and also increases the difficulties related to the design of the control law ν_i (see, e.g., [23]). Since

$$\begin{aligned} L_{g_l}L_f^{r_l-1}h(\zeta) &\neq 0, \quad l = 1,\cdots,i-1, \\ L_{D_j}L_f^{j}h(\zeta) &\neq 0, \quad j = 1,\cdots,r_i, \end{aligned}$$

$d_{ij}(\cdot)$ can be written as

$$d_{ij}(\cdot) = \Theta_{ij}(z_i,\eta_i,t)\theta_{ij}(t), \quad j = 1,\cdots,r_i, \tag{10}$$

where

$$\Theta_{ij}(\cdot) = \begin{cases} \left[L_{g_l}L_f^{j-1}h(\zeta),\ \ L_{D_j}L_f^{j-1}h(\zeta)\right] \Big|_{\zeta=\Phi_i^{-1}(z_i,\eta_i)}, & j = r_l, \\ \left[0,\ \ L_{D_j}L_f^{j-1}h(\zeta)\right] \Big|_{\zeta=\Phi_i^{-1}(z_i,\eta_i)}, & \text{otherwise}, \end{cases}$$

$$\theta_{ij}(t) = \begin{cases} \mathrm{col}(\bar u_l(t),\ d_j(t)), & j = r_l, \\ \mathrm{col}(0,\ d_j(t)), & \text{otherwise}. \end{cases} \tag{11}$$

The unknown time-varying variables $\theta_{ij}(t)$ are assumed to satisfy the following assumptions.

Assumption 1. For $i = 1,\cdots,m$, there exist known constants $\bar\theta_{ij}$ such that $\|\theta_{ij}(t)\| \le \bar\theta_{ij}$ for $j = 1,\cdots,r_i$. ∇

Assumption 2. The unknown time-varying variables $\theta_{ij}(t)$, $j = 1,\cdots,r_i$ are parameterized by

$$\dot\omega_{ij}(t) = M_{ij}(t)\omega_{ij}(t), \quad \theta_{ij}(t) = C_{ij}\omega_{ij}(t), \tag{12}$$

where $\omega_{ij}(0)$ are unknown, and $M_{ij}(t)$ and C_{ij} are user-specified to generate the basic functions[1]. ∇

Remark 2. It is worth pointing out that $\theta_{ij}(t)$ generated by (12) is unknown even though both $M_{ij}(t)$ and C_{ij} are known because the initial condition $\omega_{ij}(0)$ is not known. The dynamical equation (12) is widely used to describe unknown inputs in the existing literature such as [24]. In fact, (12) is an alternative parameterization for faults and disturbances, and is an extension of the linear parameterizations used in [3]. For some special matrices $M_{ij}(t)$, the solution of the time-varying system (12) can be obtained by $\theta_{ij}(t) = C_{ij}\exp\left(\int_0^t M_{ij}(\tau)d\tau\right)\omega(0)$ which can approximate a large number of practical actuator stuck faults. For example, if $M_{ij}(t) = 0$, then $\theta_{ij}(t) = C_{ij}\omega(0)$ represents the constant stuck faults such as a faulty scenario that an aircraft control surface locks at an unknown fixed position, and if $M_{ij}(t) = \sin(t)\left[\begin{smallmatrix} a & -a \\ a & -a \end{smallmatrix}\right]$, the parameters $\theta_{ij}(t) = C_{ij}(I_2 + \left[\begin{smallmatrix} a & -a \\ a & -a \end{smallmatrix}\right]\sin(t))\omega(0)$ can express periodic sinusoidal stuck faults. ∇

Remark 3. The dynamical system given in (12) can accurately approximate $\theta_{ij}(t)$ by choosing $M_{ij}(t)$ with sufficiently rich frequencies. An example is to take a periodic $\theta_{ij}(t)$. To determine $M_{ij}(t)$ for the periodic $\theta_{ij}(t)$, a prior condition is to know the fundamental frequencies of $\theta_{ij}(t)$ (see, e.g., [25]), which typically relies on the engineers' experiences. If this information is not available, the periodic $\theta_{ij}(t)$ can be accurately approximated by choosing $M_{ij}(t)$ with as rich frequencies as possible. However, in practice, there exists a trade-off between computational complexity and approximation accuracy. In addition, a time-varying matrix $M_{ij}(t)$ can generate basic functions with richer frequencies than a constant matrix, which facilitates a more accurate approximation of $\theta_{ij}(t)$, and is also the main reason to use a time-varying matrix $M_{ij}(t)$ rather than a constant one. ∇

In addition, the zero dynamics described by (9) are required to satisfy the following assumption.

Assumption 3. For $i = 1,\cdots,m$, the zero dynamics $\dot\eta_i = \Psi_i(0,\eta_i,d_{r_i+1}(t),\cdots,d_n(t),\bar u(t))$ are input-to-state stable (ISS) with respect to input $d_{r_i+1}(t),\cdots,d_n(t)$ and $\bar u(t)$. ∇

Remark 4. Assumption 3 is similar to the conditions employed in [14] and [7]. It is actually a minimum phase assumption, which is also needed in the nominal case (no faults). ∇

Denote $\hat\theta_{ij}(t)$ as the estimates of $\theta_{ij}(t)$ for $j = 1,\cdots,r_i-1$, and $\tilde\theta_{ij}(t)$ as the estimation errors, where $\tilde\theta_{ij}(t) = \theta_{ij}(t) - \hat\theta_{ij}(t)$. To simplify the notations, $\hat\theta_{ij}$ and $\tilde\theta_{ij}$ are used throughout this paper instead of $\hat\theta_{ij}(t)$ and $\tilde\theta_{ij}(t)$ respectively. Let

$$x_{ij} = z_{ij} - x_{ij}^*(x_{i1},\cdots,x_{i(j-1)},\hat\theta_{i1},\cdots,\hat\theta_{i(j-1)}) - y_r^{(j-1)}(t) \tag{13}$$

for $j = 1,\cdots,r_i-1$, where $x_{i1}^*(\cdot) = 0$ and $x_{i2}^*(\cdot),\cdots,x_{i(r_i-1)}^*(\cdot)$ are smooth virtual control laws to be designed, and let

$$s_i = z_{ir_i} - \sigma_i(x_{i1},\cdots,x_{i(r_i-1)},\tilde\theta_{i1},\cdots,\tilde\theta_{i(r_i-1)}) - y_r^{(r_i-1)}(t), \tag{14}$$

where $\sigma_i(\cdot)$ will be constructed recursively in the next section. In the coordinates $x_{i1},\cdots,x_{i(r_i-1)},s_i$, system (8)-(9) becomes

$$\begin{aligned} \dot x_{ij} &= x_{i(j+1)} + x_{i(j+1)}^*(\cdot) - \dot x_{ij}^*(\cdot) + \Theta_{ij}(\cdot)\theta_{ij}(t), \\ &\quad j = 1,\cdots,r_i-2, \\ \dot x_{i(r_i-1)} &= s_i + \sigma_i(\cdot) - \dot x_{i(r_i-1)}^*(\cdot) + \Theta_{i(r_i-1)}(\cdot)\theta_{i(r_i-1)}(t), \\ \dot s_i &= \nu_i - \dot\sigma_i(\cdot) + \Theta_{ir_i}(\cdot)\theta_{ir_i}(t) - y_r^{(r_i)}(t), \quad (15) \\ \dot\eta_i &= \Psi_i(x_{i1},\cdots,x_{i(r_i-1)},s_i,\eta_i,d_{r_i+1}(t),\cdots,d_n(t),\bar u(t)), \quad (16) \\ y &= x_{i1} + y_r(t). \end{aligned}$$

Remark 5. Take the first row in (15) as an example. There is rare result to converge x_{ij} and $\tilde\theta_{ij}$ to zero simultaneously. Suppose that $x_{i(j+1)}^*(\cdot)$ and $\dot{\hat\theta}_{ij}$ are designed as the general form $q(x_{ij},\hat\theta_{ij}) + \dot x_{ij}^*(\cdot) - x_{i(j+1)}$ and $-p(x_{ij},\hat\theta_{ij})$ respectively where $p(x_{ij},\hat\theta_{ij})$ is continuous and $p(0,\theta_{ij}(t)) \neq 0$. Note that the scopes of the variables $p(\cdot)$ and $q(\cdot)$ are limited to this remark. Then, $\dot x_{ij} = q(x_{ij},\theta_{ij}(t) - \tilde\theta_{ij}) + \Theta_{ij}(\cdot)\theta_{ij}(t)$, $\dot{\tilde\theta}_{ij} = \dot\theta_{ij}(t) + p(x_{ij},\theta_{ij}(t) - \tilde\theta_{ij})$ where $q(0,\theta_{ij}(t)) + \Theta_{ij}(\cdot)\theta_{ij}(t)$ is matched with respect to the controller $q(\cdot)$, while $\dot\theta_{ij}(t) + p(0,\theta_{ij}(t))$ is unmatched. To the best of the authors' knowledge, due to the unvanishing unmatched $\dot\theta_{ij}(t) + p(0,\theta_{ij}(t))$, there is still no available smooth $q(x_{ij},\hat\theta_{ij})$ and $p(x_{ij},\hat\theta_{ij})$ to simultaneously converge x_{ij} and $\tilde\theta_{ij}$ to zero asymptotically. ∇

[1]The description for basic functions here can be found in [21] and [3]. In linear parameterizations, the richer frequencies the basic functions have, the more accurately we can approximate the original signals.

C. FTC Strategy

Suppose that a set of ν_i in (7) for $i = 1, \cdots, m$ has been determined where each ν_i is first designed to stabilize the system (15) and distinctly well accommodate the unmatched stuck faults $\rho \in \Sigma_{(i)}$. A set of u_i, $i = 1, \cdots, m$ are then constructed based on ν_i where each u_i distinctly tolerate the fault modes in $\Sigma_{(i)}$. Thus, all the u_i can be fully determined. Furthermore, all the u_i with $i = 1, \cdots, m$ perform through the selection of the switching mechanism of the active FTC architecture used in [26] and shown in Fig. 1.

The switching mechanism in Fig. 1 is generated based on the fault detection and isolation (FDI) schemes (see [27]). Similar to [26], it is assumed that the used FDI schemes can provide accurate and quick fault diagnosis such that the switching signals can select and activate μ_i correctly and sufficiently fast after faults occur. The issue, which is not considered in this work is the stability of the closed-loop system after the occurrence of the fault but before it is detected and isolated. Also, we do not consider the actuator saturation problem for healthy actuators.

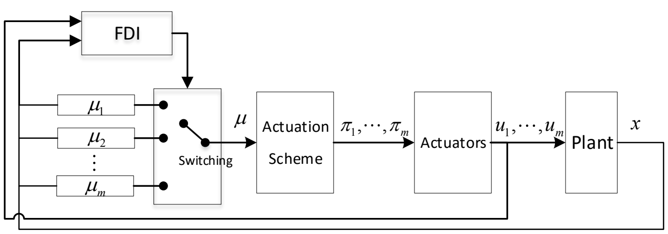

Fig. 1. Fault-tolerant control architecture.

IV. Main Results

Based on the FTC strategy presented in the previous section, the objective of this paper now becomes to design ν_i for system (15)-(16) such that $\lim_{t\to\infty} y(t) - y_r(t) = 0$, and update laws $\hat{\theta}_{ij}$ for $j = 1, \cdots, r_i - 1$ such that $\lim_{t\to\infty} \tilde{\theta}_{ij}(t) = 0$. The major challenge is to handle the unmatched $\Theta_{ij}(\cdot)\theta_{ij}(t)$, $j = 1, \cdots, r_i - 1$ in the unmatched channels $\dot{x}_{ij}$ with respect to ν_i. In this section, a methodology to completely cancel $\Theta_{ij}(\cdot)\theta_{ij}(t)$ in the unmatched channels will be developed using sliding mode control. Intuitively, using the equivalent injections of $\dot{\theta}_{ij}(t)$ during the sliding motion associated with a particularly designed sliding surface, $\dot{\theta}_{ij}(t)$ can be assigned to the unmatched channels $\dot{x}_{ij}$, and then by designing proper gains, $\Theta_{ij}(\cdot)\theta_{ij}(t)$ can be canceled by the equivalent injections under Assumption 2.

A. Sliding Surface Design

A recursive process to construct the sliding function $\sigma(\cdot)$ in (14) will be developed in this section based on the backstepping design procedure. To this end, some preliminary results are presented. The virtual control laws $x^*_{i(j+1)}$ and update laws $\hat{\theta}_{ij}$, $j = 1, \cdots, r_i - 1$ for the system (15) are proposed, respectively, by

$$\begin{aligned} x^*_{i(j+1)}(\cdot) =& k_{ij1}x_{ij} + Y_{ij1}\hat{\theta}_{ij} + \dot{x}^*_{ij}(\cdot) - k_{ij3}\tanh\left(x_{ij}/\delta_{ij}(t)\right), && (17)\\ \dot{\hat{\theta}}_{ij} =& -K_{ij2}x_{ij} - Y_{ij2}\hat{\theta}_{ij} - K_{ij4}\tanh\left(x_{ij}/\delta_{ij}(t)\right), && (18)\end{aligned}$$

where $\delta_{ij}(t)$ is determined by

$$\dot{\delta}_{ij}(t) = -k_{ij3}\delta_{ij}(t), \quad k_{ij3} > 0, \quad \delta_{ij}(0) > 0, \tag{19}$$

the parameters $k_{ij1} \in \mathbb{R}$, $K_{ij2} \in \mathbb{R}^2$, $Y_{ij1} \in \mathbb{R}^{1\times 2}$, $Y_{ij2} \in \mathbb{R}^{2\times 2}$, $k_{ij3} \in \mathbb{R}$, $K_{ij4} \in \mathbb{R}^2$ are selected such that

- there exist $\alpha_{ij} \in \mathbb{R}^2$ and the Hurwitz and Metzler matrix $\bar{A}_{ij}$ such that for $l,\ \kappa = 1, 2, 3,\ l \neq \kappa$,

$$(A_{ij})_{ll} \leq (\bar{A}_{ij})_{ll}, \ |(A_{ij})_{l\kappa}| \leq (\bar{A}_{ij})_{l\kappa}, \tag{20}$$

  where

$$A_{ij} = \begin{bmatrix} -(\alpha_{ij}Y_{ij1} + Y_{ij2}) & \Xi_{ij} \\ -Y_{ij1} & k_{ij1} + Y_{ij1}\alpha_{ij} \end{bmatrix} \tag{21}$$

  with $\Xi_{ij} = \alpha_{ij}k_{ij1} + K_{ij2} + (\alpha_{ij}Y_{ij1} + Y_{ij2})\alpha_{ij}$.
- the $\underline{\lambda}(\bar{Q}_{ij})$ satisfies

$$\underline{\lambda}(\bar{Q}_{ij}) \geq 2(r_i - j + \beta_{ij1} + \beta_{ij2}), \tag{22}$$

  where

$$-\bar{Q}_{ij} = \bar{A}^T_{ij}P_{ij} + P_{ij}\bar{A}_{ij} \tag{23}$$

  with P_{ij} being the Lyapunov matrix, and β_{ij1} and β_{ij2} are to be specified later.
- the following relations hold:

$$\begin{aligned} &(\alpha_{ij}Y_{ij1} + Y_{ij2} + \alpha_{ij}\Theta_{ij}(\cdot))C_{ij} + C_{ij}M_{ij}(t) = 0, && (24)\\ &k_{ij3} \geq \|Y_{ij1} + \Theta_{ij}(\cdot)\| \cdot \bar{\theta}_{ij}, \quad K_{ij4} = \alpha_{ij}k_{ij3}. && (25)\end{aligned}$$

Based on the result in the positive system theories [28], P_{ij} in (23) should be a diagonal positive definite matrix. Without loss of generality, it is assumed throughout this paper that $P_{ij} = \mathrm{diag}\{P_{ij1}, 1\}$ with $P_{ij1} \in \mathbb{R}^{2\times 2}$. Let $-Q_{ij} = A^T_{ij}P_{ij} + P_{ij}A_{ij}$. Then it follows from (20) that for any vector $x \in \mathbb{R}^3$,

$$-x^T Q_{ij} x \leq -|x|^T \bar{Q}_{ij}|x|. \tag{26}$$

As first shown in [29], for any $\delta(t) > 0$ and any $x \in \mathbb{R}$, hyperbolic function $\tanh(x/\delta(t))$ satisfies

$$0 \leq |x| - x\tanh\left(x/\delta(t)\right) \leq \epsilon\delta(t), \tag{27}$$

where $\epsilon = 0.2785$ is given in [29]. In addition, for any $\alpha \in \mathbb{R}^2$, an invertible matrix $T(\alpha)$ is defined as

$$T(\alpha) = \begin{bmatrix} 1 & 0 \\ \alpha & I_2 \end{bmatrix}, \tag{28}$$

which will be used for coordinate transformation.

Now, we are ready to show the recursive design procedure of sliding surface, that is design $\sigma_i(\cdot)$ in (14).

Step 1: The smooth virtual control law $x^*_{i2}(\cdot)$ and update law $\dot{\hat{\theta}}_{i1}$ are constructed respectively based on (17) and (18) with $j = 1$. Using variable α_{i1}, a new coordinate $\mathrm{col}(\xi_{i1}, x_{i1}) = T(\alpha_{i1})\mathrm{col}(x_{i1}, \tilde{\theta}_{i1})$ is introduced where $T(\cdot)$ is defined in (28), and in this new coordinate,

$$\begin{aligned} \dot{\xi}_{i1} =& \alpha_{i1}x_{i2} - (\alpha_{i1}Y_{i11} + Y_{i12})\xi_{i1} + \Xi_{i1}x_{i1}\\ &+ (\alpha_{i1}Y_{i11} + Y_{i12} + \alpha_{i1}\Theta_{i1}(\cdot))\theta_{i1}(t) + \dot{\theta}_{i1}(t)\\ &- (\alpha_{i1}k_{i13} - K_{i14})\tanh\left(x_{i1}/\delta_{i1}(t)\right), && (29)\\ \dot{x}_{i1} =& x_{i2} - Y_{i11}\xi_{i1} + (k_{i11} + Y_{i11}\alpha_{i1})x_{i1} + (Y_{i11} + \Theta_{i1}(\cdot))\theta_{i1}(t)\\ &- k_{i13}\tanh\left(x_{i1}/\delta_{i1}(t)\right), && (30)\end{aligned}$$

where, according to (24) and (25), the terms $(\alpha_{i1}Y_{i11} + Y_{i12} + \alpha_{i1}\Theta_{i1}(\cdot))\theta_{i1}(t) + \dot{\theta}_{i1}(t)$ and $(\alpha_{i1}k_{i13} - K_{i14})\tanh\left(x_{i1}/\delta_{i1}(t)\right)$ in (29) are identically zero. By choosing $V_{i1} = \frac{1}{2}\xi^T_{i1}P_{i11}\xi_{i1} + \frac{1}{2}x^2_{i1} + \epsilon\delta_{i1}(t)$ as a candidate Lyapunov function, the time derivative of V_{i1} along (29) and (30) is obtained by

$$\begin{aligned} \dot{V}_{i1} =& \xi^T_{i1}P_{i11}\dot{\xi}_{i1} + x_{i1}\dot{x}_{i1} + \epsilon\dot{\delta}_{i1}(t)\\ =& -\frac{1}{2}\mathrm{col}(\xi_{i1}, x_{i1})^T Q_{i1}\mathrm{col}(\xi_{i1}, x_{i1}) + \xi^T_{i1}P_{i11}\alpha_{i1}x_{i2} + x_{i1}x_{i2}\\ &+ x_{i1}(Y_{i11} + \Theta_{i1}(\cdot))\theta_{i1}(t) - x_{i1}k_{i13}\tanh\left(x_{i1}/\delta_{i1}(t)\right) + \epsilon\dot{\delta}_{i1}(t).\end{aligned}$$

Based on (26) and (22) with $\beta_{i11} = \beta_{i12} = 0$, it has

$$-\mathrm{col}(\xi_{i1}, x_{i1})^T Q_{i1}\mathrm{col}(\xi_{i1}, x_{i1}) \le -|\mathrm{col}(\xi_{i1}, x_{i1})|^T \bar{Q}_{i1}|\mathrm{col}(\xi_{i1}, x_{i1})| \le -2(r_i - 1)(\|\xi_{i1}\|^2 + x_{i1}^2).$$

Also, based on (25) and (27), it has

$$x_{i1}(Y_{i11} + \Theta_{i1}(\cdot))\theta_{i1}(t) - x_{i1}k_{i13}\tanh(x_{i1}/\delta_{i1}(t)) \le k_{i13}\epsilon\delta_{i1}(t),$$

and further, it follows from (19) that

$$x_{i1}(Y_{i11} + \Theta_{i1}(\cdot))\theta_{i1}(t) - x_{i1}k_{i13}\tanh(x_{i1}/\delta_{i1}(t)) + \epsilon\dot{\delta}_{i1}(t) \le 0.$$

Therefore, it is concluded that

$$\dot{V}_{i1} \le -(r_i - 1)(\|\xi_{i1}\|^2 + x_{i1}^2) + \xi_{i1}^T P_{i11}\alpha_{i1}x_{i2} + x_{i1}x_{i2}. \tag{31}$$

Inductive step: Suppose that at step $j-1$, $j \in \{3, \cdots, r_i - 2\}$, there are a positive definite Lyapunov function $V_{i(j-1)}$ and a set of continuous virtual laws $x_{i1}^*, \cdots, x_{ij}^*(\cdot)$ and update laws $\dot{\hat{\theta}}_{i1}, \cdots, \dot{\hat{\theta}}_{i(j-1)}$ in the form of (17) and (18) respectively such that

$$\dot{V}_{i(j-1)} \le -\sum_{k=1}^{j-1}(r_i - j + 1)(\|\xi_{ik}\|^2 + x_{ik}^2) + \xi_{i(j-1)}^T P_{i(j-1)1}\alpha_{i(j-1)}x_{ij} + x_{i(j-1)}x_{ij}. \tag{32}$$

Obviously, inequality (32) reduces to (31) when $j = 2$.

At the jth step, consider a candidate Lyapunov function $V_{ij} = V_{i(j-1)} + \frac{1}{2}\xi_{ij}^T P_{ij1}\xi_{ij} + \frac{1}{2}x_{ij}^2 + \epsilon\delta_{ij}(t)$ where $\mathrm{col}(\xi_{ij}, x_{ij}) = T(\alpha_{ij})\mathrm{col}(x_{ij}, \tilde{\theta}_{ij})$ with $T(\cdot)$ being defined in (28). By choosing $x_{i(j+1)}^*(\cdot)$ and $\dot{\hat{\theta}}_{ij}$ based on (17) and (18) respectively, due to (20), (24) and (25),

$$\begin{aligned}\dot{V}_{ij} \le& \dot{V}_{i(j-1)} - \frac{1}{2}\mathrm{col}(\xi_{ij}, x_{ij})^T Q_{ij}\mathrm{col}(\xi_{ij}, x_{ij}) \\ &+ \xi_{ij}^T P_{ij1}\alpha_{ij}x_{i(j+1)} + x_{ij}x_{i(j+1)} + x_{ij}(Y_{ij1} + \Theta_{ij}(\cdot))\theta_{ij}(t) \\ &- x_{ij}k_{ij3}\tanh\left(x_{ij}/\delta_{ij}(t)\right) + \epsilon\dot{\delta}_{ij}(t).\end{aligned} \tag{33}$$

For the items $\xi_{i(j-1)}^T P_{i(j-1)1}\alpha_{i(j-1)}x_{ij}$ and $x_{i(j-1)}x_{ij}$ in (32), there exist positive constants β_{ij1} and β_{ij2} such that

$$|\xi_{i(j-1)}^T P_{i(j-1)1}\alpha_{i(j-1)}x_{ij}| \le \sum_{k=1}^{j-1}\|\xi_{ik}\|^2 + \beta_{ij1}x_{ij}^2, \tag{34}$$

$$|x_{i(j-1)}x_{ij}| \le \sum_{k=1}^{j-1}x_{ik}^2 + \beta_{ij2}x_{ij}^2. \tag{35}$$

By substituting (34) and (35) into (32), it obtains

$$\dot{V}_{i(j-1)} \le -\sum_{k=1}^{j-1}(r_i - j)(\|\xi_{ik}\|^2 + x_{ik}^2) + (\beta_{ij1} + \beta_{ij2})x_{ij}^2. \tag{36}$$

Then substituting (36) into (33), and using the analogous procedure used in obtaining (31), it yields

$$\begin{aligned}\dot{V}_{ij} \le& \dot{V}_{i(j-1)} - \frac{1}{2}\underline{\lambda}(\bar{Q}_{ij})(\|\xi_{ij}\|^2 + x_{ij}^2) + \xi_{ij}^T P_{ij1}\alpha_{ij}x_{i(j+1)} + x_{ij}x_{i(j+1)} \\ \le& -\sum_{k=1}^{j}(r_i - j)(\|\xi_{ik}\|^2 + x_{ik}^2) + \xi_{ij}^T P_{ij1}\alpha_{ij}x_{i(j+1)} + x_{ij}x_{i(j+1)}.\end{aligned} \tag{37}$$

It should be pointed out that in order to cancel the last term of (36) using $-\frac{1}{2}\underline{\lambda}(\bar{Q}_{ij})(\|\xi_{ij}\|^2 + x_{ij}^2)$ in (37), $\underline{\lambda}(\bar{Q}_{ij})$ needs to satisfy (22).

Now, we are ready to design $\sigma_i(\cdot)$ and the sliding surface. The function $\sigma_i(\cdot)$ in (14) is chosen as

$$\sigma_i(\cdot) = x_{ir_i}^*(\cdot), \tag{38}$$

where $x_{ir_i}^*(\cdot)$ and $\dot{\hat{\theta}}_{i(r_i-1)}$ are given in (17) and (18) respectively, and the corresponding sliding surface is then determined by

$$\mathcal{S}_i = \left\{\left(x_{i1}, \cdots, x_{i(r_i-1)}, s_i, \tilde{\theta}_{i1}, \cdots, \tilde{\theta}_{i(r_i-1)}\right) | s_i = 0\right\}. \tag{39}$$

Thus, the sliding motion associated with $\mathcal{S}_i$ is determined by the dynamics of $x_{i1}, \cdots, x_{i(r_i-2)}, \tilde{\theta}_{i1}, \cdots, \tilde{\theta}_{i(r_i-1)}$ given in aforementioned recursive procedure and $x_{i(r_i-1)}$. Since on the sliding surface, $\dot{s}_i = s_i = 0$, it follows from (15) and (38) that

$$\dot{x}_{i(r_i-1)} = x_{ir_i}^*(\cdot) - \dot{x}_{i(r_i-1)}^* + \Theta_{i(r_i-1)}(\cdot)\theta_{i(r_i-1)}(t). \tag{40}$$

The stability of the associated sliding motion is analyzed as follows. By introducing the new coordinate $\mathrm{col}(\xi_{i(r_i-1)}, x_{i(r_i-1)}) = T(\alpha_{i(r_i-1)})\mathrm{col}(x_{i(r_i-1)}, \tilde{\theta}_{i(r_i-1)})$ where $T(\cdot)$ defined in (28), a candidate Lyapunov function of the sliding motion is chosen as $V_{i(r_i-1)} = V_{i(r_i-2)} + \frac{1}{2}\xi_{i(r_i-1)}^T P_{i(r_i-1)1}\xi_{i(r_i-1)} + \frac{1}{2}x_{i(r_i-1)}^2 + \epsilon\delta_{i(r_j-1)}(t)$. Using the analogous procedure used in obtaining (37), we can obtain that the time derivative of $V_{i(r_i-1)}$ along the sliding motion satisfies

$$\dot{V}_{i(r_i-1)} \le -\sum_{k=1}^{r_i-1}(\|\xi_{ik}\|^2 + x_{ik}^2). \tag{41}$$

Thus, based on the LaSalle-Yoshizawa lemma (see Theorem 2.1 in [30]), it can be concluded that ξ_{ij} and x_{ij}, $j = 1, \cdots, r_i - 1$ are uniformly bounded and $\lim_{t\to\infty}\xi_{ij} = 0$, $\lim_{t\to\infty}x_{ij} = 0$. Since $T(\cdot)$ is invertible, $\tilde{\theta}_{ij}$, $j = 1, \cdots, r_i - 1$, are uniformly bounded and further, $\lim_{t\to\infty}\tilde{\theta}_{ij} = 0$ and it follows from (13) that $\lim_{t\to\infty}y(t) - y_r(t) = \lim_{t\to\infty}z_{i1} - y_r(t) = \lim_{t\to\infty}x_{i1} = 0$. Moreover, based on the dynamics of ξ_{ij} and x_{ij} for $j = 1, \cdots, r_i - 1$, $\dot{\xi}_{ij}$ and $\dot{x}_{ij}$ are uniformly bounded, which, based on (17) and (18), results in that $\dot{\hat{\theta}}_{ij}$ and $x_{i(j+1)}^*(\cdot)$, $j = 1, \cdots, r_i - 1$ are uniformly bounded. Thus, z_{ij}, $j = 1, \cdots, r_i - 1$ are uniformly bounded. In addition, since during the sliding motion, $s_i = 0$, it follows from (14) that $z_{ir_i} = x_{ir_i}^*(\cdot) + y_r^{(r_i-1)}(t)$, which implies that z_{ir_i} is also uniformly bounded.

Remark 6. It can be seen from the ***Step 1*** in the recursive procedure that inequalities (20) are used to stabilize the system (29)-(30), and (24) and (25) are used to cancel the unmatched disturbances in (29). It is worth pointing out that even if all the elements of A_{i1} are time-varying and state-related, the conditions in (20) can still guarantee the stability. This implies that α_{i1}, Y_{i11} and Y_{i12} can be designed as time-varying or state-related functions to satisfy (24) and (25), and further, $M_{i1}(t)$ in (24) can also be extended to the state-related case, i.e. $M_{i1}(\zeta, t)$. Therefore, the developed conditions (20)-(25) can be extended the case that $\dot{\omega}_{ij}(t) = M_{ij}(z_i, t)\omega_{ij}(t)$, $\theta_{ij}(\zeta, t) = C_{ij}\omega_{ij}(t)$, and further, the developed sliding surface is applicable to the system (1) with $d_i(t) = d_i(\zeta, t)$. ∇

B. Sliding Mode Control Law Design

To drive s_i in (15) to the sliding surface $\mathcal{S}_i$, the discontinuous control law v_i is designed as

$$v_i = k_{ir_i1}s_i + y_r^{(r_i)}(t) + \dot{\sigma}_i(\cdot) - (\|\Theta_{ir_i}(\cdot)\|\bar{\theta}_{ir_i} + k_{ir_i2})\mathrm{sign}(s_i) \tag{42}$$

where k_{ir_i1} and k_{ir_i2} are chosen to satisfy $k_{ir_i1} < 0$ and $k_{ir_i2} > 0$. Then, $s_i\dot{s}_i = -k_{ir_i1}s_i^2 + s_i\Theta_{ir_i}(\cdot)\theta_{ir_i}(t) - (\|\Theta_{ir_i}(\cdot)\|\bar{\theta}_{ir_i} + k_{ir_i2})s_i\cdot\mathrm{sign}(s_i) \le -k_{ir_i2}|s_i|$. Thus, the reachability condition is satisfied, which means s_i is driven to the sliding surface $\mathcal{S}_i$ given in (39) in finite time and remains on it thereafter.

Hence, a theorem is ready to be presented as follows:

Theorem 1. *Under Assumptions 3 and 1, the update laws $\dot{\hat{\theta}}_{ij}$ given in (18) and control law v_i in (42) for system (8)-(9) can guarantee that for all actuator stuck fault modes $\rho \in \Sigma_{(i)}$ defined in (5),*

- *$z_{i1}, \cdots, z_{ir_i}, \tilde{\theta}_{i1}, \cdots, \tilde{\theta}_{i(r_i-1)}$ are uniformly bounded,*
- *the estimate errors satisfy $\lim_{t\to\infty}\theta_{ij}(t) - \hat{\theta}_{ij} = 0$ for $j = 1, \cdots, r_i - 1$,*
- *the tracking error satisfies $\lim_{t\to\infty}y(t) - y_r(t) = 0$,*
- *the closed-loop system is driven to the sliding surface $\mathcal{S}_i$ in finite time and remains on it thereafter.*

∇

Remark 7. From Theorem 1, for $\rho \in \Sigma_{(i)}$, $\theta_{ij}(t)$, $j = 1, \cdots, r_i - 1$ are reconstructed by $\hat{\theta}_{ij}$. It follows from (11) that

$$\hat{u}_l(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\theta}_{ij}, \; j = r_l, \; l = 1, \cdots, i-1, \tag{43}$$

$$\hat{d}_j(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{\theta}_{ij}, \; j = 1, \cdots, r_i - 1. \tag{44}$$

This means that the stuck values $\bar{u}_1(t), \cdots, \bar{u}_{i-1}(t)$ and part of disturbances $d_1(t), \cdots, d_{r_i-1}(t)$ are reconstructed. ∇

V. Simulation

Consider a nonlinear system

$$\begin{aligned} \dot{\zeta} &= f(\zeta) + \sum\nolimits_{i=1}^{2} g_i u_i + \sum\nolimits_{i=1}^{4} D_i d_i(t), \\ y &= h(\zeta) = \zeta_3 \end{aligned}$$

where $\zeta = [\zeta_1, \zeta_2, \zeta_3, \zeta_4]^T$, $f(\zeta) = [\zeta_3^2 - 3\zeta_1 - \zeta_4, -\zeta_1, -\zeta_2, -5\zeta_4 - \zeta_1 \sin(\zeta_1)]^T$, $g_1 = [0, -6.3045, 0, 0]^T$, $g_2 = [1, 0, 0, 0]^T$, $D_1 = [0, 0, -3.5544, 0]^T$, $D_2 = [0, 6.4653, 0, 0]^T$, $D_3 = [1, 0, 0, 0]^T$ and $D_4 = [0, 0, 0, 1]^T$. This system has unmatched actuator redundancy, which is indicated by the different relative degrees between $(f(\zeta), g_1, h(\zeta))$ and $(f(\zeta), g_2, h(\zeta))$ where $r_1 = 2$ and $r_2 = 3$. The disturbances $d_i(t)$ for $i = 1, \cdots, 4$ have been well partitioned and the relative degree of the triple $(f(\zeta), D_i, h(\zeta))$ is i. Moreover, proportional factors π_1 and π_2 of the actuation scheme (4) are respectively given by $\pi_1 = 1$ and $\pi_2 = 1$. The reference signal $y_r(t)$ is given by $y_r(t) = 10\sin(t)$.

The sequence of the calculations in the sequel follows the presentation sequence of the theoretical part to make them consistent with each other. Based on the fault mode grouping in (5), all the tolerable fault modes are grouped by $\Sigma_{(1)} = \{\rho : \rho_1 = 0, \; \rho_2 = 0 \text{ or } 1\}$ with common relative degree being r_1, and $\Sigma_{(2)} = \{\rho : \rho_1 = 1, \; \rho_2 = 0\}$ with common relative degree being r_2. For all actuator fault modes $\rho \in \Sigma_{(1)}$, there exists a coordinate transformation $\mathrm{col}(z_1, \eta_1) = \Phi_1(\zeta) = [\zeta_3, -\zeta_2, \zeta_1, \zeta_3^2 - \zeta_4]^T$ and a feedback linearization law $\mu_1 = 0.1586\,(\nu_1 - \eta_{11})$ such that in the new coordinates $z_1 \in \mathbb{R}^2$ and $\eta_1 \in \mathbb{R}^2$,

$$\begin{aligned} \dot{z}_{11} &= z_{12} + d_{11}, \quad \dot{z}_{12} = \nu_1 + d_{12}, \\ \dot{\eta}_1 &= \Psi_1(\cdot) \\ &= \begin{bmatrix} \eta_{12} - 3\eta_{11} + d_3(t) \\ 2z_{11}z_{12} + 5z_{11}^2 - 5\eta_{12} + \eta_{11}\sin(\eta_{11}) - 7.1088 z_{11} d_1(t) - d_4(t) \end{bmatrix}, \end{aligned}$$

where $\eta_1 = \mathrm{col}(\eta_{11}, \eta_{12})$ is the state of the zero dynamics. It can be verified that the zero dynamics satisfy Assumption 3. Moreover, $d_{11} = -3.5544 d_1(t)$ and $d_{12} = -6.4653 d_2(t)$. Based on (10), d_{11} and d_{12} can be respectively parameterized by $d_{11}(\cdot) = \Theta_{11}\theta_{11}(t)$ and $d_{12}(\cdot) = \Theta_{12}\theta_{12}(t)$ where $\Theta_{11} = [0, -3.5544]$ and $\Theta_{12} = [0, 6.4653]$. Both θ_{11} and θ_{12} can be roughly bounded by 10 and Assumption 1 is satisfied if $|d_1(t)| \le 10$ and $|d_2(t)| \le 10$. Suppose that $d_1(t)$ is produced by $\dot{d}_1(t) = -0.1\sin(t) d_1(t)$ with unknown $d_1(0)$. Then, the matrix $M_{11}(t)$ of (12) in Assumption 2 can be determined by

$$M_{11}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -0.1\sin(t) \end{bmatrix}.$$

Based on the recursive backstepping design procedure presented in the theoretical part,

$$\begin{aligned} \sigma_1(\cdot) &= x_{12}^*(\cdot), \\ x_{12}^*(\cdot) &= -4.2124 x_{11} + [0, 8.8818 \times 10^{-16}]\hat{\theta}_{11} - 50\tanh\left(x_{11}/\delta_{11}(t)\right), \\ \dot{\hat{\theta}}_{11} &= [0, -0.6580]^T x_{11} - Y_{112}\hat{\theta}_{11} - 50[0, 1]^T \tanh\left(x_{11}/\delta_{11}(t)\right), \end{aligned}$$

where $\dot{\delta}_{11}(t) = -50\delta_{11}(t)$ with $\delta_{11}(0) = 2$ and

$$Y_{112} = \begin{bmatrix} 0 & 0 \\ 0 & 3.5544 + 0.1\sin(t) \end{bmatrix}.$$

Thus, s_1 and ν_1 are respectively designed as

$$\begin{aligned} s_1 &= z_{12} - \dot{x}_{12}^*(\cdot) - 10\cos(t), \\ \nu_1 &= -20 s_1 - 10\sin(t) + \dot{\sigma}_1(\cdot) - 100\mathrm{sign}(s_1). \end{aligned}$$

For all actuator fault modes $\rho \in \Sigma_{(2)}$, there exists a coordinate transformation $\mathrm{col}(z_2, \eta_2) = \Phi_2(\zeta) = [\zeta_3, -\zeta_2, \zeta_1, \zeta_4]^T$ and a feedback linearizaiton law $\mu_2 = \nu_2 - z_{21}^2 + 3z_{23} - \eta_2$ such that

$$\begin{aligned} \dot{z}_{21} &= z_{22} + d_{21}, \quad \dot{z}_{22} = z_{23} + d_{22}, \\ \dot{z}_{23} &= \nu_2 + d_{23}, \quad \dot{\eta}_2 = \Psi_2(\cdot) = -5\eta_2 - z_{23}\sin(z_{23}) + d_4(t), \end{aligned}$$

where η_2 is the state of the zero dynamics. It can be verified that the zero dynamics satisfy Assumption 3. Moreover, $d_{21} = -3.5544 d_1(t)$, $d_{22} = 6.3045\bar{u}_1(t) - 6.4653 d_2(t)$ and $d_{23} = d_3(t)$. Based on (10), d_{21}, d_{22} and d_{23} can be parameterized by $d_{21}(\cdot) = \Theta_{21}\theta_{21}(t)$, $d_{22}(\cdot) = \Theta_{22}\theta_{22}(t)$ and $d_{23}(\cdot) = \Theta_{23}\theta_{23}(t)$, respectively, where $\Theta_{21} = [0, -3.5544]$, $\Theta_{22} = [6.3045, -6.4653]$ and $\Theta_{23} = [0, 1]$. The variables are bounded by $\|\theta_{21}\| \le 10$, $\|\theta_{22}\| \le 10$ and $\|\theta_{23}\| \le 3$ and Assumption 1 is satisfied if $|d_1(t)| \le 10$, $\|[\bar{u}_1(t), d_2(t)]\| \le 10$ and $|d_3(t)| \le 3$. Suppose that $d_2(t)$ and $\bar{u}_1(t)$ are generated by

$$\begin{bmatrix} \dot{d}_2(t) \\ \dot{\bar{u}}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 0.8395 - 0.1\cos(3t) \\ -9.4179 & -0.1\sin(2t) \end{bmatrix} \begin{bmatrix} d_2(t) \\ \bar{u}_1(t) \end{bmatrix}$$

with unknown $d_2(0)$ and $\bar{u}_1(0)$. Then, the matrices $M_{21}(t)$, $M_{22}(t)$ of (12) in Assumption 2 can be determined by

$$M_{21}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -0.1\sin(t) \end{bmatrix}, \; M_{22}(t) = \begin{bmatrix} 0 & 0.8395 - 0.1\cos(3t) \\ -9.4179 & -0.1\sin(2t) \end{bmatrix}.$$

Thus, we can calculate that

$$\begin{aligned} x_{22}^*(\cdot) &= -4.2124 x_{21} + [0, 8.8818 \times 10^{-16}]\hat{\theta}_{21} - 50\tanh\left(x_{21}/\delta_{21}(t)\right), \\ \dot{\hat{\theta}}_{21} &= [0, -0.6580]^T x_{21} - Y_{212}\hat{\theta}_{21} - 50[0, 1]^T \tanh\left(x_{21}/\delta_{21}(t)\right), \end{aligned}$$

where $\dot{\delta}_{21}(t) = -50\delta_{21}(t)$ with $\delta_{21}(0) = 2$ and

$$Y_{212} = \begin{bmatrix} 0 & 0 \\ 0 & 3.5544 + 0.1\sin(t) \end{bmatrix},$$

$$\begin{aligned} x_{23}^*(\cdot) &= -45.0153 x_{22} + [-0.4627, 0.3890] \times 10^{-12}\hat{\theta}_{22} + \dot{x}_{22}^*(\cdot) \\ &\quad - 350\tanh\left(x_{22}/\delta_{22}(t)\right), \\ \dot{\hat{\theta}}_{22} &= [-37.8347, 54.5962]^T x_{22} - Y_{222}\hat{\theta}_{22} \\ &\quad - 350[3, -2]^T \tanh\left(x_{22}/\delta_{22}(t)\right), \end{aligned}$$

where $\dot{\delta}_{22}(t) = -350\delta_{21}(t)$ with $\delta_{22}(0) = 20$ and

$$Y_{222} = \begin{bmatrix} 18.9135 & -19.3959 + 0.1\cos(3t) \\ -12.6090 & 12.9306 + 0.1\sin(2t) \end{bmatrix}.$$

Moreover,

$$\begin{aligned} \sigma_2(\cdot) &= x_{23}^*(\cdot), \\ s_2 &= z_{23} - \dot{x}_{23}^*(\cdot) + 10\sin(t), \\ \nu_2 &= -40 s_2 - 10\cos(t) + \dot{\sigma}_2(\cdot) - 350\mathrm{sign}(s_2). \end{aligned}$$

As a conclusion, it can be summarized that for $\rho \in \Sigma_{(1)}$,

$$\mu_1 = 0.1586\left(-20 s_1 - 10\sin(t) + \dot{\sigma}_1(\cdot) - 100\mathrm{sign}(s_1) - \eta_{11}\right),$$

and for $\rho \in \Sigma_{(2)}$,

$$\begin{aligned} \mu_2 = &-40 s_2 - 10\cos(t) + \dot{\sigma}_2(\cdot) - 350\mathrm{sign}(s_2) \\ &- z_{21}^2 + 3z_{23} - \eta_2. \end{aligned}$$

For simulation purpose, some information of disturbances and stuck faults are given as follows: $d_1(0) = -1.109$, $d_2(0) = -3.233$, $d_3(0) = 2\sin(t)$, $\bar{u}_1(0) = 1.629$ and $d_4(t) = 1.5\sin(2t)$, and an actuator stuck fault occurs on u_1 for $t \ge 5$. Thus, for $t < 5$, $\rho_1 = 0$ and $\rho_2 = 0$, $u_1 = u_{a1}$ and $u_2 = u_{a2}$ while for $t \ge 5$, $\rho_1 = 1$ and $\rho_2 = 0$, $u_1 = \bar{u}_1(t)$ and $u_2 = u_{a2}$.

The simulated fault mode belongs to $\Sigma_{(2)}$. Based on the FTC strategy, for $t < 5$, $\mu = \mu_1$ is selected by the switching mechanism in Fig. 1, and then $u_{a1} = \pi_1\mu_1 = \mu_1$ and $u_{a2} = \pi_2\mu_1 = \mu_1$. Thus,

$$\begin{aligned} u_1 &= 0.1586(-20s_1 - 10\sin(t) + \dot{\sigma}_1(\cdot) - 100\text{sign}(s_1) - \eta_{11}), \\ u_2 &= 0.1586(-20s_1 - 10\sin(t) + \dot{\sigma}_1(\cdot) - 100\text{sign}(s_1) - \eta_{11}), \\ \hat{d}_1 &= [0, 1]\,\hat{\theta}_{11}. \end{aligned}$$

Moreover, for $t \geq 5$, $\mu = \mu_2$ is selected. Thus, $u_{a2} = \pi_2\mu_2 = \mu_2$, and

$$\begin{aligned} u_1 &= \bar{u}_1(t), \\ u_2 &= -40s_2 - 10\cos(t) + \dot{\sigma}_2(\cdot) - 350\text{sign}(s_2) - z_{21}^2 + 3z_{23} - \eta_2, \\ \hat{d}_1 &= [0, 1]\,\hat{\theta}_{21},\;\; \hat{\bar{u}}_2 = [1, 0]\,\hat{\theta}_{22},\;\; \hat{d}_2 = [0, 1]\,\hat{\theta}_{22}. \end{aligned}$$

The simulation results are shown in Figs. 2-5. It can be seen from Fig. 2 that the tracking error $y - y_r(t)$ converges to zero asymptotically no matter the occurrence of the unmatched actuator fault. Also, all ζ_1, ζ_2 and ζ_4 in Fig. 2 are uniformly bounded. Figs. 3 and 4 illustrate the reconstructions for stuck value $\bar{u}_1(t)$ and disturbances $d_1(t)$ and $d_2(t)$, respectively. Fig. 5 shows the time responses of the control inputs u_1 and u_2. For $t < 5$, it can be seen from Fig. 5 that both u_1 and u_2 work normally, and $d_1(t)$ is reconstructed in Fig. 4. When $t \geq 5$, u_1 and u_2 are adjusted to new operating points in Fig. 5 to tolerate the unmatched actuator fault, and $\bar{u}_1(t)$, $d_1(t)$ and $d_2(t)$ are also reconstructed in Figs. 3 and 4, with errors converging to zero asymptotically.

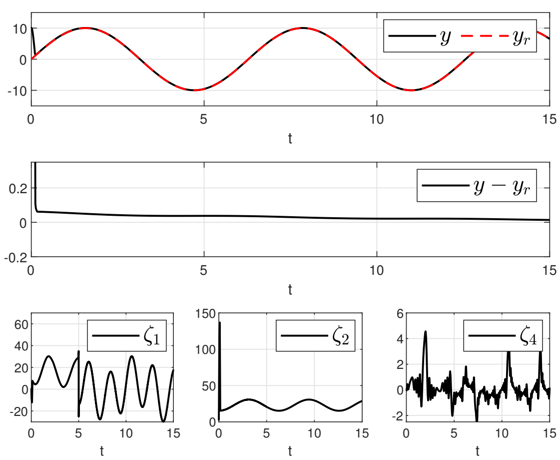

Fig. 2. Time responses of y, $y_r(t)$, $y - y_r(t)$, ζ_1, ζ_2 and ζ_4.

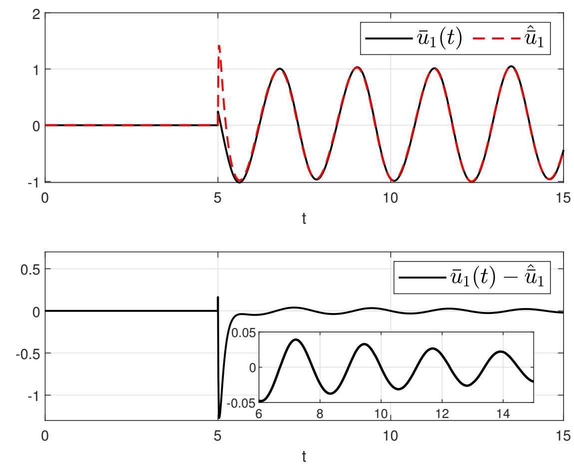

Fig. 3. Time responses of $\hat{\bar{u}}_1$, $\bar{u}_1(t)$ and reconstruction error $\bar{u}_1(t) - \hat{\bar{u}}_1$.

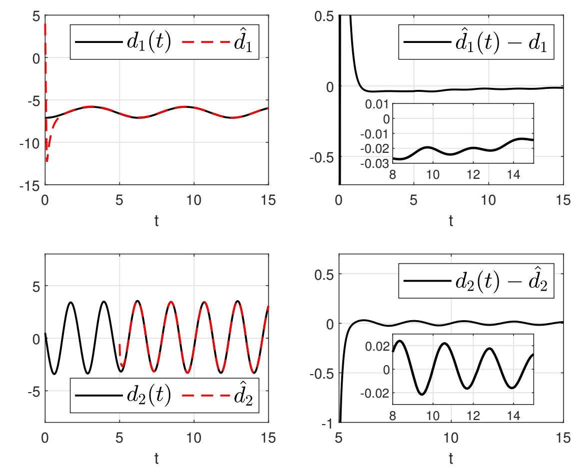

Fig. 4. Time responses of $d_1(t)$, $\hat{d}_1$, $d_2(t)$, $\hat{d}_2$, and reconstruction errors $d_1(t) - \hat{d}_1$ and $d_2(t) - \hat{d}_2$.

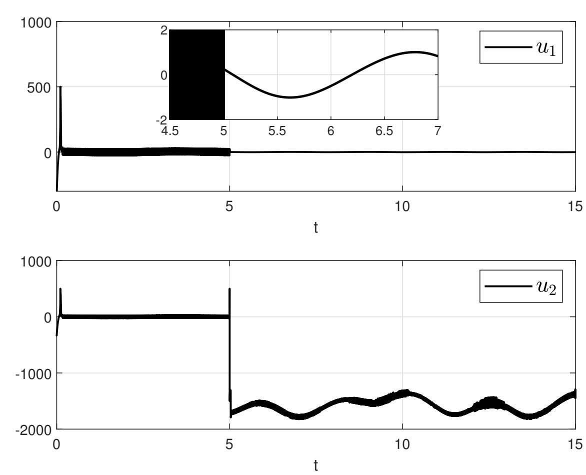

Fig. 5. Time responses of u_1 and u_2.

VI. Conclusion

This paper has proposed a sliding mode control based actuator FTC scheme for nonlinear systems in the presence of unmatched actuator stuck faults and disturbances. A new methodology to design controllers and update laws has been developed to deal with the unmatched unknown inputs. Based on this methodology, smooth FTC sliding surfaces and FTC sliding mode control laws were proposed using the backstepping design procedure, which accommodate unmatched actuator stuck faults and disturbances effectively. Finally, a simulation example was presented to illustrate the effectiveness. In our future work, the following issues will be considered :

- The non-minimum phase problem caused by actuator stuck faults will be addressed, which has been considered in [3]. This is because actuator stuck faults will cause relative degree changing which may lead to the non-minimum phase problem.

- The switching signal in Fig. 1 will be generated by easier ways rather than by the complex FDI schemes, such as Nussbaum gain function used in [4].
- Uncertainties included in $f(\zeta)$, $g_i(\zeta)$ and $D_i(\zeta)$ will be considered, which can actually be embedded into $d_i(t)$ in (1) and alternatively expressed as $d_i(\zeta, t)$ as in [16].

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