THE CO-PIERI RULE FOR STABLE KRONECKER COEFFICIENTS

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Abstract. We generalise the lattice word condition from Young tableaux to all Kronecker tableaux and hence calculate a large new family of stable Kronecker coefficients.

Perhaps the last major open problem in the complex representation theory of symmetric groups is to describe the decomposition of a tensor product of two simple representations. The coefficients describing the decomposition of these tensor products are known as the Kronecker coefficients and they have been described as 'perhaps the most challenging, deep and mysterious objects in algebraic combinatorics' [34]. More recently, these coefficients have provided the centrepiece of Geometric Complexity Theory (GCT), a "new hope" [16] for settling the P versus NP problem [30]. It was recently shown that GCT requires not only to understand the positivity, but also precise information on the explicit values of these coefficients [10]. The positivity of Kronecker coefficients is equivalent to the existence of certain quantum systems [22, 12, 11] and they have been used to understand entanglement entropy [13]. Much recent progress has focussed on the stability properties enjoyed by Kronecker coefficients [4, 7, 28, 39, 41].

Whilst a complete understanding of the Kronecker coefficients seems out of reach, the purpose of this paper is to attempt to understand the *stable* Kronecker coefficients in terms of oscillating tableaux. Oscillating tableaux hold a distinguished position in the study of tensor product decompositions [42, 38, 18] but surprisingly they have never before been used to calculate Kronecker coefficients of symmetric groups. In this work, we see that the oscillating tableaux defined as paths on the graph given in Figure 1 (which we hereafter refer to as standard Kronecker tableaux) provide bases of certain modules for the partition algebra, $P_s(n)$, which is closely related to the symmetric group. We hence add a new level of structure to the classical picture — this extra structure is the key to our main result: the co-Pieri rule for stable Kronecker coefficients.

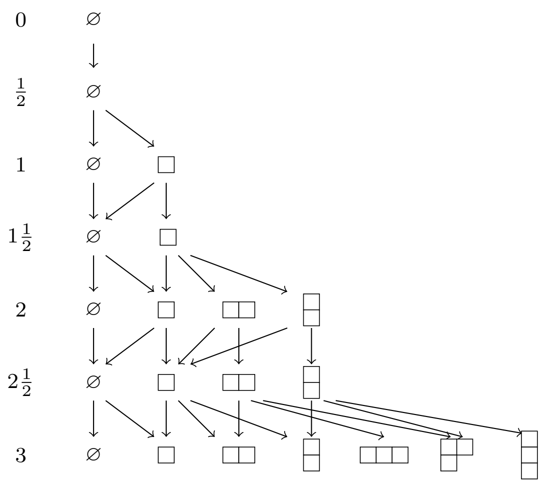

Figure 1. The first three layers of the branching graph $\mathcal{Y}$

A momentary glance at the graph given in Figure 1 reveals a very familiar subgraph: namely Young's graph (with each level doubled up). The stable Kronecker coefficients labelled by triples from this subgraph are well-understood — the values of these coefficients can be calculated via a tableaux counting algorithm known as the Littlewood–Richardson rule [24] (see Theorems 1.6 and 1.15). This rule has long served as the hallmark for our understanding (or lack thereof) of Kronecker coefficients. The Littlewood–Richardson rule was discovered as a rule of two halves (as we explain below). In this paper we succeed in generalising one half of this rule to all Kronecker tableaux, and thus solve one half of the *stable* Kronecker problem. Our main result unifies and vastly generalises the work of Littlewood–Richardson [25] and many other authors [36, 37, 6, 9, 27]. Most promisingly, our result counts explicit homomorphisms and thus works on a structural level, above any description of a family of Kronecker coefficients since those first considered by Littlewood–Richardson over eighty years ago [25].

In more detail, given a triple of partitions (λ, ν, μ) and with $|\mu| = s$, we have constructed a skew $P_s(n)$-module spanned by the Kronecker tableaux from λ to ν of length s, which we denote by $\Delta_s(\lambda \to \nu)$ (see [5]). For $\lambda = \varnothing$ and $n \geqslant 2s$ these modules provide a complete set of non-isomorphic $P_s(n)$-modules (and we drop the partition $\varnothing$ from the notation). The stable Kronecker coefficients are then interpreted as the dimensions,

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\lambda \to \nu))) \tag{†}$$

for $n \geqslant 2s$. Restricting to the Young subgraph, or equivalently to a triple (λ, ν, μ) of so-called maximal depth such that $|\lambda| + |\mu| = |\nu|$, these modules specialise to be the usual simple and skew modules for the symmetric group and hence the multiplicities $\overline{g}(\lambda, \nu, \mu)$ are the Littlewood–Richardson coefficients $c(\lambda, \nu, \mu)$. Thus we naturally recover, in this context, the well-known fact that the Littlewood–Richardson coefficients appear as the subfamily of stable Kronecker coefficients labelled by triples of maximal depth. The tableaux counted by the Littlewood–Richardson rule satisfy two conditions: the semistandard and the lattice word conditions [21, (16.4)]. This rule naturally decomposes into two halves: the Pieri rule (Theorem 1.9) encapsulates the semistandard condition (for triples in which the lattice word condition is vacuous) and the co-Pieri rule (Theorem 1.10) which encapsulates the lattice word condition (for triples in which the semistandard condition is vacuous). While the restrictions on partitions in both rules is technical, each rule represents one half of the full Littlewood–Richardson rule. This article generalises the classical co-Pieri rule to stable Kronecker coefficients.

Main Theorem. *Let (λ, ν, μ) be a co-Pieri triple or a triple of maximal depth. Then the stable Kronecker coefficient $\overline{g}(\lambda, \nu, \mu)$ is equal to the number of semistandard Kronecker tableaux of shape $\lambda \to \nu$ and weight μ whose reverse reading word is a lattice word.*

The observant reader will notice that the statement above describes the Littlewood–Richardson coefficients uniformly as part of a far broader family of stable Kronecker coefficients (and is the first result in the literature to do so). Whilst the classical Pieri rule is elementary, it served as a first step towards understanding the full Littlewood–Richardson rule; indeed Knutson–Tao–Woodward have shown that the Littlewood–Richardson rule follows from the Pieri rule by associativity [23]. We hope that our generalisation of the co-Pieri rule will prove equally useful in the study of stable Kronecker coefficients.

The definition of semistandard Kronecker tableaux naturally generalises the classical notion of semistandard Young tableaux as certain "orbits" of paths on the branching graph given in Figure 1 (see Section 1.2 and Definition 4.1). The lattice word condition is identical to the classical case once we generalise the dominance order to all steps in the branching graph $\mathcal{Y}$ to define the reverse reading word of a semistandard Kronecker tableau (see Definition 2.5 and Section 5).

Examples of co-Pieri triples. Given that the Kronecker coefficients are some of the most difficult objects in algebraic combinatorics, it is unsurprisingly that we have had to develop a vast new wealth of difficult and intricate combinatorics in this paper. For an introduction to our combinatorics, we refer the reader to the slides from the first author's recent mini-course of three lectures at CIRM, Luminy which can be found at [1] and to the extended abstract of this paper [3] (which reviews the combinatorics of some elementary examples of co-Pieri triples in great detail). We have included a further wealth of examples of both *stable* Kronecker and *non-stable* Kronecker coefficients in Section 6. For the ease of the reader, we now list a few elementary examples of (infinite families of) co-Pieri triples:

- *(i)* λ and ν are one-row partitions and μ is arbitrary. This family has been extensively studied over the past thirty years and there are many distinct combinatorial descriptions of some or all of these coefficients [2, 36, 37, 6, 9, 27], none of which generalises.
- *(ii)* the two skew partitions $\lambda \setminus (\lambda \cap \nu)$ and $\nu \setminus (\lambda \cap \nu)$ have no two boxes in the same column and $|\mu| = \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$. It is easy to see that if, in addition, (λ, ν, μ) is a triple of maximal depth, then this case specialises to the classical co-Pieri triples.
- *(iii)* $\lambda = \nu = (dl, d(l-1), \dots, 2d, d)$ for any $l, d \geqslant 1$ and $|\mu| \leqslant d$.

(*iv*) The full definition of a co-Pieri triple is given in Definition 3.10. It can be encapsulated by the idea that we only wish to consider triples (λ, ν, s) for which the action of the algebra is by permutation matrices.

As already observed, our description covers the family of stable Kronecker coefficients labelled by co-Pieri triples uniformly along with the Littlewood–Richardson coefficients. In order to demonstrate the uniformity of our approach, we now illustrate how to calculate $\overline{g}((2,1),(3,3,2),(2,2,1)) = 1$ and $\overline{g}((4),(5),(2,2,1)) = 1$. The former is an example of a triple of maximal depth (and so is calculated by the Littlewood–Richardson rule) and the latter is an example of a coefficient indexed by two one-row partitions. In both cases, there is a unique semistandard Kronecker tableau whose reverse reading word is a lattice word (under the dominance ordering on Kronecker tableaux). Each of these semistandard tableaux is an orbit consisting of four individual standard Kronecker tableaux. These tableaux are pictured in Figure 2: notice that λ and ν appear at the top and bottom of the diagram in Figure 2 and that the partition μ determines the orbit — which we depict as a dashed series of rectangular frames. This is explained in detail Sections 1, 2, 5 and 6 of the paper (but we hope this lightly sketched example helps the reader). We have included a third example in Figure 2 of a co-Pieri triple as in (*ii*), to help the reader get a more general picture (the corresponding stable Kronecker coefficient is calculated in Section 6).

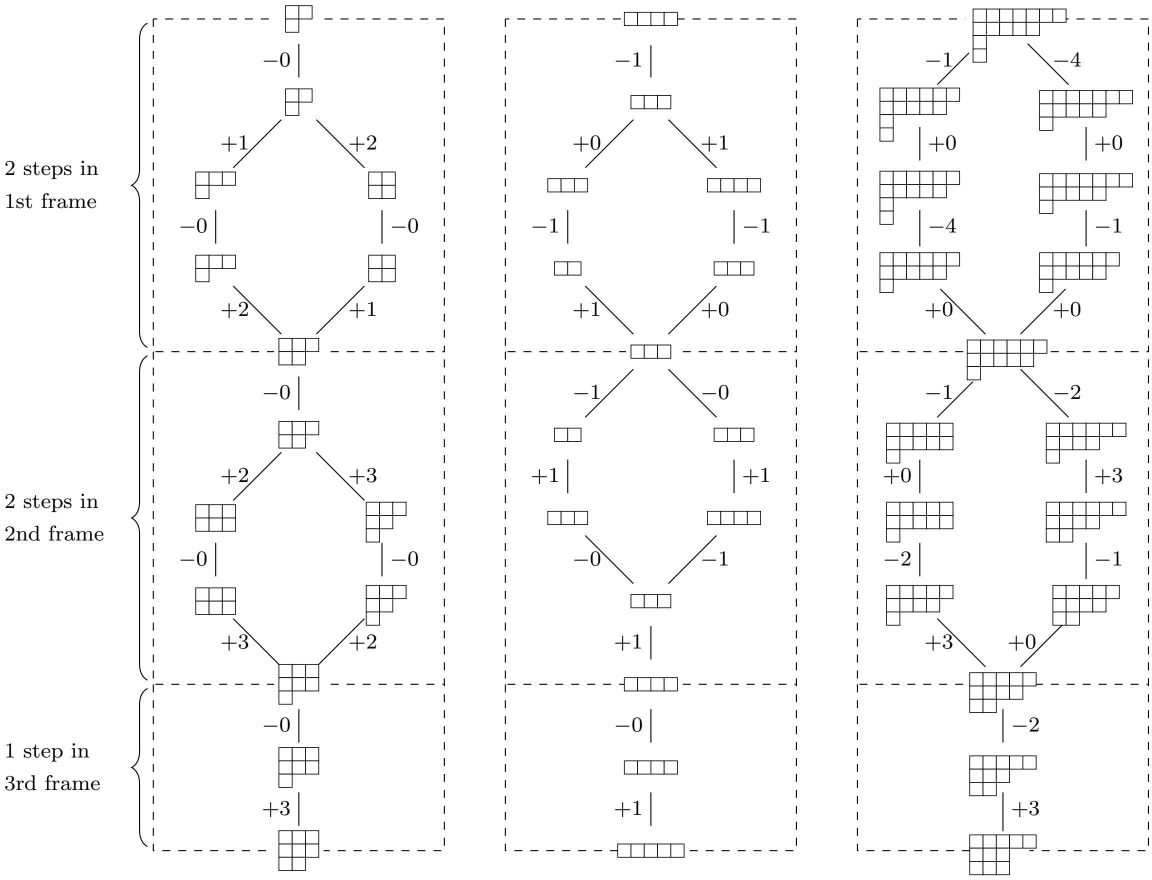

FIGURE 2. Three examples of semistandard Kronecker tableaux of weight $\mu = (2,2,1)$. The number of steps in the ith frame is μ_i. The first is a triple of maximal depth, the latter two are co-Pieri triples.

For $\lambda = (2,1)$ and $\nu = (3,2,2)$, the (integral) steps taken in the semistandard tableau on the left of Figure 2 are to add a box in the first row, add two boxes in the second row, and two in the third row

$$a(1) = (-0,+1) \quad a(2) = (-0,+2) \quad a(2) = (-0,+2) \quad a(3) = (-0,+3) \quad a(3) = (-0,+3).$$

We record the steps according to the dominance ordering for Kronecker tableaux ($a(1) < a(2) < a(3)$) and then we refine this by recording the frames in which these steps occur in weakly decreasing fashion, as follows

$$\left(\begin{array}{c|cc|cc} a(1) & a(2) & a(2) & a(3) & a(3) \\ 1 & 2 & 1 & 3 & 2 \end{array} \right).$$

This should be very familiar to experts, who will also recognise that the resulting word is a lattice word. For $\lambda = (4)$ and $\nu = (5)$, the steps taken in the semistandard Kronecker tableau in the middle of Figure 2 are to remove a box from the first row, do two "dummy" steps in the first row, and add two boxes in the first row

$$r(1) = (-1, +0) \quad d(1) = (-1, +1) \quad d(1) = (-1, +1) \quad a(1) = (-0, +1) \quad a(1) = (-0, +1).$$

We record the steps according to our 'generalised dominance ordering' for Kronecker tableaux $r(1) < d(1) < a(1)$ (see Definition 2.5) and we refine this by recording the frames in which these steps occur backwards,

$$\left(\begin{array}{c|cc|cc} r(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 2 & 1 & 3 & 2 \end{array} \right)$$

and notice that the second row is again a lattice word (and identical to the previous example!).

Structure of the paper. In Section 1 we recall the classical tableaux combinatorics of the Littlewood–Richardson rule; we re-cast the notion of a semistandard tableau in a manner which will be generalisable from the symmetric group to the partition algebra setting. We then recall some well-known facts concerning Kronecker coefficients which will be used in what follows. In Section 2, we define a standard Kronecker tableau of shape $\lambda \to \nu$ to be a path from λ to ν in the branching graph of the partition algebra. For triples of maximal depth, our definition specialises to the usual definition of (skew) Young tableaux.

In Section 3 we describe the action of the partition algebra on skew cell modules of shape $\lambda \to \nu$ in the case of co-Pieri triples. That we can understand the action of the partition algebra in this case is the crux of this paper. However, for reasons of readability (particularly for a non-diagrammatic audience) we have delayed some of the proofs to Appendices A and B. In Section 4, we define a semistandard Kronecker tableau of shape $\lambda \to \nu$ and weight μ to be an orbit of standard Kronecker tableaux under the action of the corresponding Young subgroups $\mathfrak{S}_\mu$. For a triple of partitions of maximal depth, our construction specialises to the usual definition of semistandard Young tableaux. In the case that (λ, ν, μ) is a co-Pieri triple we are able to provide an elegant combinatorial description of these semistandard Kronecker tableaux.

In Section 5, using an ordering on the steps in the branching graph of the partition algebra we define the reverse reading word of a semistandard Kronecker tableau. We hence extend the classical lattice word condition to semistandard Kronecker tableaux. When (λ, ν, μ) is a co-Pieri triple of partitions, we show that the corresponding stable Kronecker coefficient is equal to the number of semistandard Kronecker tableaux whose reverse reading word is a lattice word, generalising the Littlewood–Richardson rule to give the co-Pieri rule for stable Kronecker coefficients. Section 6 is dedicated to providing examples of Kronecker coefficients which can be calculated using our main theorem.

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1. The Littlewood–Richardson and Kronecker coefficients

The combinatorics underlying the representation theory of the partition algebras and symmetric groups is based on compositions and partitions. A composition λ of n, denoted $\lambda \vDash n$, is a sequence of non-negative integers which sum to n. If the sequence is weakly decreasing, we write $\lambda \vdash n$ and refer to λ as a partition of n. We let $\mathscr{P}_n$ denote the set of all partitions of n. We let $\varnothing$ denote the unique partition of 0. Given a partition, $\lambda = (\lambda_1, \lambda_2, \dots)$, the associated Young diagram is the set of nodes

$$[\lambda] = \left\{ (i, j) \in \mathbb{Z}^2_{>0} \mid j \leqslant \lambda_i \right\}.$$

We will often identify a partition with its Young diagram. We define the length, $\ell(\lambda)$, of a partition λ, to be the number of non-zero parts. Given $\lambda = (\lambda_1, \lambda_2, \dots)$, we let $|\lambda|_a = \sum_{i \geqslant 1}^{a} \lambda_i$ for $a \in \mathbb{Z}_{>0}$ and $|\lambda| = \sum_{i \geqslant 1} \lambda_i$. We formally set $|\lambda|_0 = \lambda_0 = 0$. Given two partitions λ, μ we say that λ dominate μ and write $\lambda \trianglerighteq \mu$ if $|\lambda| < |\mu|$ or if $|\lambda| = |\mu|$ and $|\lambda|_a \geqslant |\mu|_a$ for all $a \in \mathbb{Z}_{>0}$.

Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ a partition and n an integer, define

$$\lambda_{[n]} = (n - |\lambda|, \lambda_1, \lambda_2, \dots, \lambda_\ell).$$

Given $\lambda_{[n]}$ a partition of n, we say that the partition has depth equal to $|\lambda|$. Given two compositions λ and ν, we write $\lambda \subseteq \nu$ if $\lambda_i \leqslant \nu_i$ for all $i \geqslant 1$. For λ a partition and ν a composition such that $\lambda \subseteq \nu$, we define the skew diagram, denoted $\nu \setminus \lambda$, to be the set difference between the Young diagrams of λ and ν. If $|\nu| - |\lambda| = s$, we will also write $\nu \setminus \lambda \vdash s$ and call $\nu \setminus \lambda$ is a skew partition of s.

1.1. **Young tableaux combinatorics and the Littlewood–Richardson rule.** Given $\lambda \vdash r - s, \nu \vdash r$ such that $\lambda \subseteq \nu$ we define a standard Young tableau of shape $\nu \setminus \lambda$ to be a filling of the boxes of $\nu \setminus \lambda$, with the entries $1, \dots, s$ in such a way that the entries are increasing along the rows and columns of $\nu \setminus \lambda$.

Example 1.1. The six standard Young tableaux of shape $(5, 3, 1) \setminus (4, 2)$ are depicted in Figure 3.

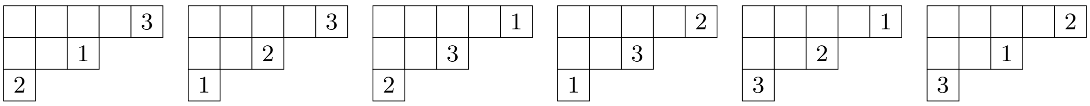


FIGURE 3. The standard Young tableaux $\mathsf{s}_1, \mathsf{s}_2, \mathsf{t}_1, \mathsf{t}_2, \mathsf{u}_1, \mathsf{u}_2$ of shape $(5, 3, 1) \setminus (4, 2)$.

Given $\lambda \vdash r - s, \nu \vdash r, \mu = (\mu_1, \mu_2, \dots, \mu_\ell) \vDash s$ such that $\lambda \subseteq \nu$ we define a Young tableau of shape $\nu \setminus \lambda$ and weight μ to be a filling of the boxes of $\nu \setminus \lambda$ with the entries

$$\underbrace{1, \dots, 1}_{\mu_1}, \underbrace{2, \dots, 2}_{\mu_2}, \dots, \underbrace{\ell, \dots, \ell}_{\mu_\ell}$$

in such a way that the entries are weakly increasing along the rows and columns. We say that the Young tableau is semistandard if, in addition, the entries are strictly increasing along the columns of $\nu \setminus \lambda$. In the case that $\lambda \vdash r - s, \nu \vdash r$ and $\mu = (1^s)$, we note that these are the standard Young tableaux of shape $\nu \setminus \lambda$.

One should think of a Young tableau of weight μ as an $\mathfrak{S}_\mu$-orbit of standard Young tableaux; we shall now make this idea more precise. Let s be a standard Young tableau of shape $\nu \setminus \lambda$ and let μ be a composition. Then define $\mu(\mathsf{s})$ to be the Young tableau of weight μ obtained from s by replacing each of the entries $|\mu|_{c-1} < i \leqslant |\mu|_c$ in s by the entry c for $c \geqslant 1$. We identify a Young tableau, S, of weight μ with the set of standard Young tableaux, $\mu^{-1}(\mathsf{S}) = \{\mathsf{s} \mid \mu(\mathsf{s}) = \mathsf{S}\}$. The set $\mu^{-1}(\mathsf{S})$ forms the basis of a cyclic $\mathfrak{S}_\mu$-module with generator given by any element $\mathsf{s} \in \mu^{-1}(\mathsf{S})$ (see [29, Chapter 4] for more details).

Example 1.2. The three semistandard Young tableaux of shape $(5, 3, 1) \setminus (4, 2)$ and weight $(2, 1)$ are depicted in Figure 4. We have that $\mu(\mathsf{s}_1) = \mu(\mathsf{s}_2) = \mathsf{S}$, $\mu(\mathsf{t}_1) = \mu(\mathsf{t}_2) = \mathsf{T}$, and $\mu(\mathsf{u}_1) = \mu(\mathsf{u}_2) = \mathsf{U}$. In each case, the non-trivial element $s_1 \in \mathfrak{S}_{(2,1)} \subseteq \mathfrak{S}_3$ acts by permuting these pairs of Young tableaux (and therefore acts trivially on the orbits sums in each case).

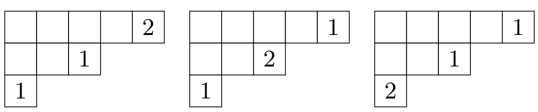


FIGURE 4. The semistandard Young tableaux $\mathsf{S}, \mathsf{T}, \mathsf{U}$ of shape $(5, 3, 1) \setminus (4, 2)$ and weight $(2, 1)$.

Example 1.3. An example of a semistandard Young tableaux, S, of shape $(9, 8, 6, 3) \setminus (6, 4, 3)$ and weight $(5, 5, 3)$ is given by the leftmost Young tableau depicted in Figure 5. Two standard Young tableaux, s and t, of shape $(9, 8, 6, 3) \setminus (6, 4, 3)$ are depicted in Figure 5. For $\mu = (5, 5, 3)$, we have that $\mu(\mathsf{s}) = \mu(\mathsf{t}) = \mathsf{S}$.

Definition 1.4. Given a semistandard Young tableau of shape $\nu \setminus \lambda$ and weight μ, we define the reverse reading word to be the sequence of integers obtained by reading the entries of the Young tableau from right-to-left along successive rows (beginning with the first row).

						1	1	2
				2	2	2	3	
			2	3	3			
1	1	1						

						1	2	6
				7	8	9	11	
			10	12	13			
3	4	5						

						1	4	8
				6	7	10	13	
			9	11	12			
2	3	5						

FIGURE 5. Three semistandard Young tableau of shape $(9,8,6,3) \setminus (6,4,3)$. The first, S, is of weight $(5,5,3)$ and the second, s, and third, t, are standard Young tableaux.

Example 1.5. The reverse reading words of the standard Young tableaux in Example 1.1 are

$$(3,1,2) \quad (3,2,1) \quad (1,3,2) \quad (2,3,1) \quad (1,2,3) \quad (2,1,3)$$

respectively. The reverse reading word of the semistandard Young tableau S in Example 1.3 is $(2,1,1,3,2,2,2,3,3,2,1,1,1)$.

The representation theory of the symmetric group $\mathfrak{S}_r$ over the rational field $\mathbb{Q}$ is semisimple. For each $\nu \vdash r$, we have a corresponding Specht module $\mathbf{S}(\nu)$ which has a basis indexed by all standard Young tableaux of shape ν. The set $\{\mathbf{S}(\nu) \mid \nu \in \mathscr{P}_r\}$ forms a complete set of non-isomorphic simple $\mathbb{Q}\mathfrak{S}_r$-modules. More generally, for $s \leqslant r$ and $\lambda \vdash r-s$ with $\lambda \subseteq \nu$, we have a corresponding skew Specht module $\mathbf{S}(\nu \setminus \lambda)$ for $\mathbb{Q}\mathfrak{S}_s$ which has a basis indexed by standard Young tableaux of shape $\nu \setminus \lambda$ [35].

Theorem 1.6. [20] *Let $\lambda \vdash r-s$, $\mu \vdash s$ and $\nu \vdash r$ and suppose that $\lambda \subseteq \nu$. We define the Littlewood–Richardson coefficients to be the multiplicities,*

$$c(\lambda,\nu,\mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_{r-s}\times\mathfrak{S}_s}(\mathbf{S}(\lambda) \boxtimes \mathbf{S}(\mu), \mathbf{S}(\nu)\downarrow^{\mathfrak{S}_r}_{\mathfrak{S}_{r-s}\times\mathfrak{S}_s}) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathbf{S}(\mu), \mathbf{S}(\nu \setminus \lambda)). \tag{1.1}$$

The Littlewood–Richardson coefficient, $c(\lambda,\nu,\mu)$, is equal to the number of Young tableaux of shape $\nu \setminus \lambda$ and weight μ satisfying the following two conditions,

(1) the Young tableau is semistandard*;*
(2) the reverse reading word of the Young tableau is a lattice word*, that is, for each positive integer j, starting from the first entry of the word to any other place in word, there are at least as many entries equal to j as there are equal to $(j+1)$.*

Example 1.7. The Young tableau of shape $(9,8,6,3) \setminus (6,4,3)$ and weight $(5,5,3)$ depicted in Figure 5 is semistandard but its $(5,5,3)$-reverse reading word is not a lattice word.

Example 1.8. The three semistandard Young tableaux of shape $(5,3,1) \setminus (4,2)$ and weight $(2,1)$ are depicted in Figure 4. Only the latter two of these Young tableaux satisfy condition (2) of Theorem 1.6. Therefore $c((5,3,1),(4,2),(2,1)) = 2$.

A famous precursor to the full Littlewood–Richardson rule was provided by Pieri's rule. In this case, we assume that the weight partition $\mu = (s)$. This is equivalent to all Young tableaux of weight μ (and any arbitrary fixed shape) satisfying condition (2) of Theorem 1.6. Therefore the following rule, while elementary, serves as a first step towards understanding condition (1) of Theorem 1.6.

Theorem 1.9 (The classical Pieri rule)**.** *Let $\lambda \vdash r-s$ and $\nu \vdash r$ be such that $\lambda \subseteq \nu$. We have that*

$$c(\lambda,\nu,(s)) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\mathfrak{S}_s}(\mathbf{S}((s)), \mathbf{S}(\nu \setminus \lambda))$$

is equal to the number of semistandard Young tableaux *of shape $\nu \setminus \lambda$ and weight (s). The number of such Young tableaux is equal to 1 (respectively 0) if ν is (respectively is not) obtained from λ by adding a total of s nodes, no two of which appear in the same column.*

We now consider a dual to the above case, which we refer to as the co-Pieri rule. Here we assume that the Young diagram of $\nu \setminus \lambda$ consists of no two nodes in the same column. This is equivalent to all Young tableaux of shape $\nu \setminus \lambda$ (and any arbitrary fixed weight) satisfying condition (1) of Theorem 1.6. Therefore the following rule serves as a first step towards understanding condition (2) of Theorem 1.6.

Theorem 1.10 (The classical Co-Pieri rule)**.** *Suppose that $\lambda \subseteq \nu$ and that $\nu \setminus \lambda$ is a skew partition of s with no two nodes in the same column. We have that*

$$c(\lambda,\nu,\mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\mathfrak{S}_s}(\mathbf{S}(\mu), \mathbf{S}(\nu \setminus \lambda))$$

is equal to the number of Young tableaux *of shape $\nu \setminus \lambda$ and weight μ whose reverse reading word is a* lattice word.

To reiterate, Theorem 1.9 describes precisely the set of Littlewood–Richardson coefficients which can be calculated without mention of the lattice word condition; whilst Theorem 1.10 describes precisely the set of Littlewood–Richardson coefficients which can be calculated without mention of the semistandardness condition.

1.2. **Young tableaux combinatorics revisited.** In the next section, we shall see that the Littlewood–Richardson coefficients appear as a subfamily of the wider class of (stable) Kronecker coefficients. The purpose of this paper is to generalise the combinatorics of standard and semistandard Young tableaux from this subclass to the study of all (stable) Kronecker coefficients. In order to illustrate how we shall proceed, we first recast the pictorial Young tableaux described earlier in the setting of the branching graph of the symmetric groups.

The branching graph of the symmetric groups encodes the induction and restriction of Specht modules for the tower of symmetric groups. For $k \in \mathbb{Z}_{\geqslant 0}$, the set of vertices on the kth level are given by the set of partitions of k. There is an edge $\lambda \to \mu$ if μ is obtained from λ by adding a box in the ith row for some $i \geqslant 1$ in which case we write $\mu = \lambda + \varepsilon_i$. The first few levels of this graph are given in Figure 6.

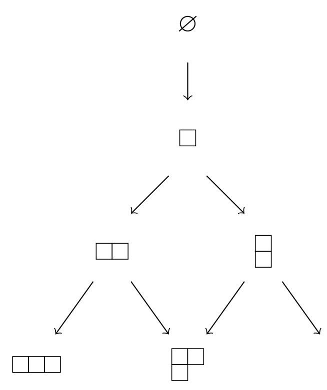
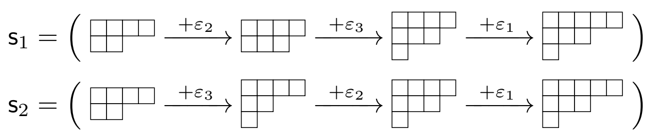


FIGURE 6. The first few levels of the branching graph of the symmetric groups.

One can then identify any skew standard Young tableau of shape $\nu \setminus \lambda$ with a path from λ to ν in the branching graph; this is done simply by adding nodes in the prescribed order. This is best illustrated via an example.

Example 1.11. Let $\lambda = (4, 2)$ and $\nu = (5, 3, 1)$. We have six standard Young tableaux of shape $\nu \setminus \lambda$. Two of these Young tableaux are as follows:



These paths correspond with the two leftmost Young tableaux (also labelled by s_1 and s_2) depicted in Figure 3.

We now wish to re-imagine the notion of a semistandard Young tableaux in this setting. Recall that a Young tableau of weight μ is merely a picture which encodes an $\mathfrak{S}_\mu$-orbit of standard Young tableaux. We shall picture a Young tableau, S, of weight μ simply as the corresponding set of paths $\mu^{-1}(\mathsf{S})$ in the branching graph. In order to highlight the weight of the Young tableau, we shall decorate the graph with a corresponding series of frames. An illustrative example is given in Figure 7. A Young tableau is semistandard (in the classical picture) if and only if the entries are strictly increasing along the columns; equivalently the successive differences between partitions on the edges of the frame have no two nodes in the same column. While we have refrained from being too precise here, a more general definition of such a tableau is made in Section 4.

We leave the reinterpretation of the reverse reading word in this setting to Section 5.

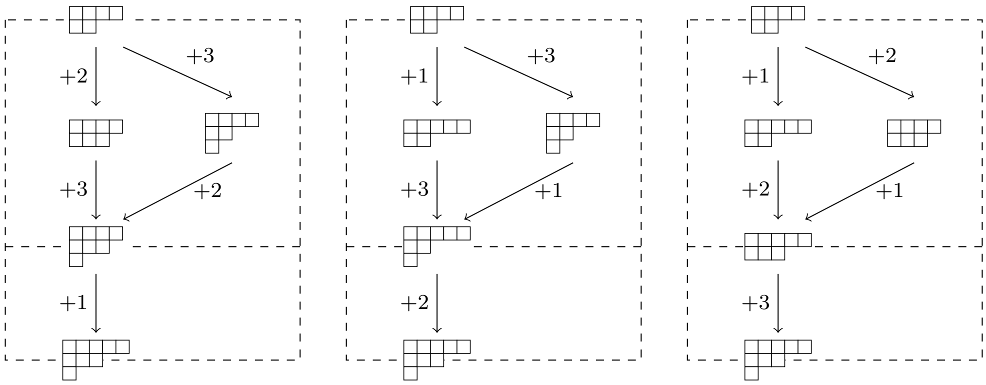


FIGURE 7. The 3 semistandard Young tableaux $\mathsf{S}, \mathsf{T}, \mathsf{U}$ of shape $(5,3,1) \setminus (4,2)$ and weight $(2,1)$.

1.3. **The Kronecker coefficients.** We now introduce the Kronecker coefficients and illustrate how they generalise the Littlewood–Richardson coefficients discussed above. Given $\lambda, \mu, \nu \vdash r$ we define the associated Kronecker coefficient to be the multiplicity

$$g(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_r}(\mathbf{S}(\nu), \mathbf{S}(\lambda) \otimes \mathbf{S}(\mu))).$$

For $\alpha \subset \lambda$,$\beta \subset \mu$ with $|\lambda \setminus \alpha| = |\mu \setminus \beta| = s$ and $\nu \vdash s$ we extend this notation to skew Specht modules in the obvious way,

$$g(\lambda \setminus \alpha, \nu, \mu \setminus \beta) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_s}(\mathbf{S}(\nu), \mathbf{S}(\lambda \setminus \alpha) \otimes \mathbf{S}(\mu \setminus \beta))).$$

Given $\lambda = (\lambda_1, \lambda_2, \ldots)$ a partition and n sufficiently large, we set $\lambda_{[n]} := (n - |\lambda|, \lambda_1, \lambda_2, \ldots)$. It was discovered by Murnaghan in [31] that the sequence of integers $\{g(\lambda_{[n]}, \mu_{[n]}, \nu_{[n]})\}_{n \in \mathbb{Z}_{\geqslant 0}}$ stabilises as $n \gg 0$ with stable limit that is denoted $\overline{g}(\lambda, \nu, \mu)$ and called the stable Kronecker coefficient. In other words,

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_n}(\mathbf{S}(\nu_{[n]}), \mathbf{S}(\lambda_{[n]}) \otimes \mathbf{S}(\mu_{[n]})))$$

for $n \gg 0$. Murnaghan also observed that

$$\overline{g}(\lambda, \nu, \mu) \neq 0 \quad \text{ implies } |\mu| \leqslant |\lambda| + |\nu|, \ |\nu| \leqslant |\lambda| + |\mu| \text{ and } |\lambda| \leqslant |\mu| + |\nu|. \tag{1.2}$$

The (stable) Kronecker coefficients have been studied extensively (see for example [31, 32, 8, 22, 43]). Recent work [6, 7, 4] has shown that the stable Kronecker coefficients can serve as an important stepping stone towards understanding the general case.

The search for a positive combinatorial formula of the Kronecker coefficients has been described by Richard Stanley as 'one of the main problems in the combinatorial representation theory of the symmetric group', [40]. While this is a very difficult problem, there are many useful descriptions of the Kronecker coefficients which *do* involve cancellations; chief among these is the following recursive description.

Theorem 1.12. [14, 2.3]. *Given $\lambda_{[n]}, \mu_{[n]}, \nu_{[n]} \vdash n$ such that $|\mu| = s$, we have that*

$$g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) = \sum_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} g(\lambda_{[n]} \setminus \alpha, \nu_{[n]} \setminus \alpha, \mu) - \sum_{\substack{\beta \in P(n,\mu) \\ \beta \neq \mu_{[n]}}} g(\lambda_{[n]}, \nu_{[n]}, \beta) \tag{1.3}$$

where $P(n, \mu)$ is the set of partitions of n obtained by adding a total of $n - s$ boxes to μ so that no two of which are in the same column. In particular, if $s < |\lambda_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]})|$ then $g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) = 0$ and if $s = |\lambda_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]})|$ then

$$g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) = g(\lambda_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]}), \nu_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]}), \mu). \tag{1.4}$$

Corollary 1.13. *Let λ, ν, μ be partitions with $\overline{g}(\lambda, \nu, \mu) \neq 0$. Then we have*

$$\max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\} \leqslant |\mu| \leqslant |\lambda| + |\nu|.$$

Proof. This follows directly from (1.2) and Theorem 1.12, noting that $\max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\} = |\lambda_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]})|$. $\square$

Finally, we conclude this section by realising the Littlewood–Richardson coefficients as a subset of the wider family of stable Kronecker coefficients.

Definition 1.14. Let λ, ν, μ be partitions. We say that (λ, ν, μ) is a triple of partitions of maximal depth if $|\nu| = |\lambda| + |\mu|$. We also call (λ, ν, s) a triple of of maximal depth if $|\nu| = |\lambda| + s$.

Theorem 1.15. [24, 32] *For (λ, ν, μ) a triple of partitions of maximal depth, $\overline{g}(\lambda, \nu, \mu) = c(\lambda, \nu, \mu)$.*

2. The partition algebra and Kronecker tableaux

We now define the partition algebra $P_r(n)$ for $r, n \in \mathbb{N}$. Although it can be defined over any field, in this paper we consider $P_r(n)$ over the rational field $\mathbb{Q}$. As a vector space, it has a basis given by all set-partitions of $\{1, 2, \dots, r, \overline{1}, \overline{2}, \dots, \overline{r}\}$. We call a part of a set-partition a block. For example,

$$d = \{\{\overline{1}, \overline{2}, \overline{4}, 2, 5\}, \{\overline{3}\}, \{\overline{5}, \overline{6}, \overline{7}, 3, 4, 6, 7\}, \{\overline{8}, 8\}, \{1\}\},$$

is a set-partition (for $r = 8$) with 5 blocks. To define the multiplication on $P_r(n)$, it is helpful to represent a set-partition by an partition diagram consisting of a frame with r distinguished points on the northern and southern boundaries, which we call vertices. We number the northern vertices from left to right by $\overline{1}, \overline{2}, \dots, \overline{r}$ and the southern vertices similarly by $1, 2, \dots, r$ and connect two vertices by an edge if they belong to the same block. Note that such a diagram is not uniquely defined, two diagrams representing the set-partition d above are given in Figure 8.

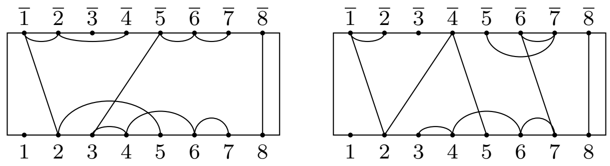


FIGURE 8. Two representatives of the set-partition d.

We define the product $x \cdot y$ of two diagrams x and y using the concatenation of x above y, where we identify the southern vertices of x with the northern vertices of y. If there are t connected components consisting only of middle vertices, then the product is set equal to n^t times the diagram with the middle components removed. Extending this by linearity defines the multiplication on $P_r(n)$. It is easy to see that $P_r(n)$ is generated (as an algebra) by the elements $s_{k,k+1}$, $p_{k+\frac{1}{2}}$ $(1 \leqslant k \leqslant r-1)$ and p_k $(1 \leqslant k \leqslant r)$ depicted in Figure 9. We define $*$ to be the anti-involution given by flipping a diagram through its horizontal axis.

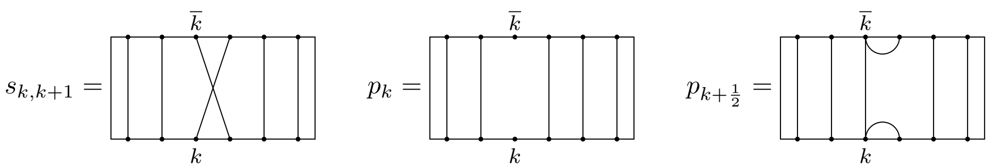


FIGURE 9. Generators of $P_r(n)$

The elements $s_{k,k+1}$ $(1 \leqslant k \leqslant r-1)$ generate the subalgebra $\mathbb{Q}\mathfrak{S}_r \subset P_r(n)$. In particular, each permutation $\sigma \in \mathfrak{S}_r$ corresponds to the set partition $\{\{\overline{1}, \sigma(1)\}, \{\overline{2}, \sigma(2)\}, \dots, \{\overline{r}, \sigma(r)\}\}$.

2.1. **Standard Kronecker tableaux.** The branching graph, $\mathcal{Y}$, of the partition algebras encodes the induction and restriction of cell modules for the tower of partition algebras. We will construct the cell modules explicitely later in this section.

For $k \in \mathbb{Z}_{\geqslant 0}$, we denote by $\mathscr{P}_{\leqslant k}$ the set of partitions of degree less or equal to k. Now the set of vertices on the kth and $(k + \frac{1}{2})$th levels of $\mathcal{Y}$ are given by

$$\mathcal{Y}_k = \{(\lambda, k - |\lambda|) \mid \lambda \in \mathscr{P}_{\leqslant k}\} \qquad \mathcal{Y}_{k+\frac{1}{2}} = \{(\lambda, k - |\lambda|) \mid \lambda \in \mathscr{P}_{\leqslant k}\}.$$

The edges of $\mathcal{Y}$ are as follows,

- for $(\lambda, l) \in \mathcal{Y}_k$ and $(\mu, m) \in \mathcal{Y}_{k+\frac{1}{2}}$ there is an edge $(\lambda, l) \to (\mu, m)$ if $\mu = \lambda$, or if μ is obtained from λ by removing a box in the ith row for some $i \geqslant 1$; we write $\mu = \lambda - \varepsilon_0$ or $\mu = \lambda - \varepsilon_i$, respectively.
- for $(\lambda, l) \in \mathcal{Y}_{k+\frac{1}{2}}$ and $(\mu, m) \in \mathcal{Y}_{k+1}$ there is an edge $(\lambda, l) \to (\mu, m)$ if $\mu = \lambda$, or if μ is obtained from λ by adding a box in the ith row for some $i \geqslant 1$; we write $\mu = \lambda + \varepsilon_0$ or $\mu = \lambda + \varepsilon_i$, respectively.

When it is convenient, we decorate each edge with the index of the node that is added or removed when reading down the diagram. The first few levels of $\mathcal{Y}$ are given in Figure 1. When no confusion is possible, we identify $(\lambda, l) \in \mathcal{Y}_k$ with the partition λ.

Definition 2.1. Given $\lambda \in \mathscr{P}_{r-s} \subseteq \mathcal{Y}_{r-s}$ and $\nu \in \mathscr{P}_{\leqslant r} \subseteq \mathcal{Y}_r$, we define a standard Kronecker tableau of shape $\lambda \to \nu$ and degree s to be a path t of the form

$$\lambda = \mathsf{t}(0) \to \mathsf{t}(\tfrac{1}{2}) \to \mathsf{t}(1) \to \cdots \to \mathsf{t}(s - \tfrac{1}{2}) \to \mathsf{t}(s) = \nu, \tag{2.1}$$

in other words t is a path in the branching graph which begins at λ and terminates at ν. We let $\mathrm{Std}_s(\lambda \to \nu)$ denote the set of all such paths. If $\lambda = \emptyset \in \mathcal{Y}_0$ then we write $\mathrm{Std}_r(\nu)$ instead of $\mathrm{Std}_r(\varnothing \to \nu)$.

For $\lambda \in \mathcal{Y}_{r-s}$, $\nu \in \mathcal{Y}_r$, $\mathsf{s} \in \mathrm{Std}_{r-s}(\lambda)$ and $\mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu)$, we denote the composition of these paths by $\mathsf{s} \circ \mathsf{t} \in \mathrm{Std}_r(\nu)$. Also, for $\mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu)$ as in (2.1) and $0 \leqslant m < m' \leqslant s$ we denote by $\mathsf{t}[m, m']$ the truncation $\mathsf{t}(m) \to \mathsf{t}(m + \frac{1}{2}) \to \cdots \to \mathsf{t}(m')$.

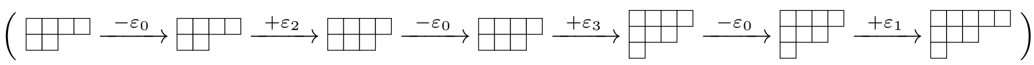
Note that we have used the notation $\lambda \to \nu$, instead of $\nu \setminus \lambda$, as we do not have $\lambda \subseteq \nu$ in general.

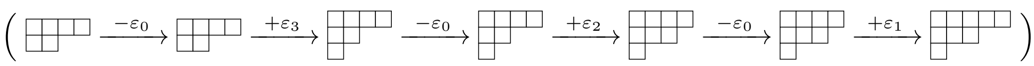
Remark 2.2. For (λ, ν, s) a triple of maximal depth, the set $\mathrm{Std}_s(\lambda \to \nu)$ can be identified with the set of standard skew Young tableau of shape $\nu \setminus \lambda$ for the symmetric group (see Example 2.4 below).

We now extend the dominance order on partitions to the set of standard Kronecker tableaux.

Definition 2.3. For $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu)$, we write $\mathsf{s} \trianglerighteq \mathsf{t}$ if $\mathsf{s}(k) \trianglerighteq \mathsf{t}(k)$ for $k = 1, \ldots, s$.

Example 2.4. Let $\lambda = (4, 2)$ and $\nu = (5, 3, 1)$. We have six standard Kronecker tableaux of shape $\lambda \to \nu$ and degree 3. Two of these tableaux are as follows:

$\mathsf{s}_1 =$ 

$\mathsf{s}_2 =$ 

We remark that $\mathsf{s}_1 \rhd \mathsf{s}_2$. These paths correspond with the two leftmost Young tableaux (also labelled by s_1 and s_2) depicted in Figure 3 and Example 1.11.

One can think of a path $\mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu)$ as a sequence of partitions; or equivalently, as the sequence of boxes added and removed. We shall refer to a pair of steps, $(-\varepsilon_a, +\varepsilon_b)$, between consecutive integral levels of the branching graph as an integral step in the branching graph. We order integral steps as follows.

Definition 2.5. We define three types of integral step in the branching graph of $P_r(n)$

(1) move-up $m{\uparrow}(p, q) := (-\varepsilon_p, +\varepsilon_q)$ for $p > q$,
(2) dummy $d(t) := (-\varepsilon_t, +\varepsilon_t)$, and
(3) move-down $m{\downarrow}(u, v) := (-\varepsilon_u, +\varepsilon_v)$ for $u < v$.

We order them as follows

$$m{\uparrow}(p, q) < d(t) < m{\downarrow}(u, v)$$

and we refine this to a total order by setting

- $m{\uparrow}(p, q) < m{\uparrow}(p'q')$ if $q < q'$ or $q = q'$ and $p > p'$;
- $d(t) < d(t')$ if $t > t'$;
- $m{\downarrow}(u, v) < m{\downarrow}(u'v')$ if $u > u'$ or $u = u'$ and $v < v'$.

We sometimes let $a(i) := m{\downarrow}(0, i)$ (respectively $r(i) := m{\uparrow}(i, 0)$) and think of this as adding (respectively removing) a box.

2.2. **The Murphy basis.** We shall now recall from [15] the construction of an integral basis of the partition algebra indexed by (pairs of) paths in the branching graph. This basis captures much of the representation theoretic structure of $P_r(n)$ and naturally generalises Murphy's basis of $\mathbb{Z}\mathfrak{S}_r$ [33]. The construction inductively associates two elements of the partition algebra, called branching coefficients, to each edge in the branching graph. These branching coefficients are defined using the following elements.

Definition 2.6. We define elements of $P_r(n)$ as follows

$$e^{(l)}_{k+\frac{1}{2}} = \underbrace{p_{k-l+\frac{3}{2}} \cdots p_{k-\frac{1}{2}}\ p_{k+\frac{1}{2}}}_{l \text{ factors}} \qquad e^{(l)}_{k} = \underbrace{p_{k-l+1} \cdots p_{k-1} p_k}_{l \text{ factors}} \qquad s_{l,k} = \underbrace{s_{l,l+1} \cdots s_{k-1,k}}_{k-l \text{ factors}}$$

for $1 \leqslant l \leqslant k < r$ in the first case and Let $1 \leqslant l \leqslant k \leqslant r$ in the second and third cases. If $k = 0$ or $l = 0$, we let $e^{(l)}_k = e^{(l)}_{k-\frac{1}{2}} = 1$. For $1 \leqslant k < l \leqslant r$ we let $s_{l,k} = s^{-1}_{k,l}$. If $l = 0$ or $k = 0$, we let $s_{l,k} = 1$ and if $l < 0$ or $k < 0$ we let $s_{l,k} = 0$. These elements are depicted in Figure 10.

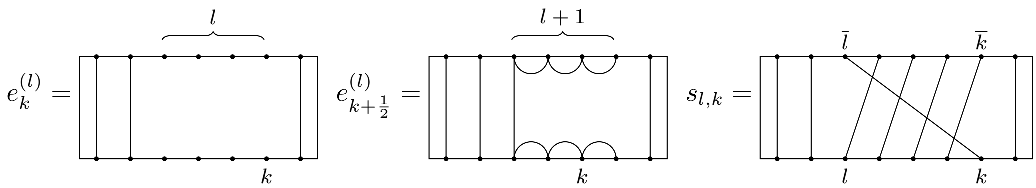


FIGURE 10. The elements $e^{(l)}_k$ and $e^{(l)}_{k+\frac{1}{2}}$ and $s_{l,k}$. (If $k = 0$ or $l = 0$, we let $e^{(l)}_k = e^{(l)}_{k+\frac{1}{2}} = 1$.)

The following definition is lifted from [15, Section 6.5] (but we have applied the involution $*$ to their conventions, in other words we have flipped the diagrams through their horizontal axes).

Definition 2.7. Let $0 \leqslant k \leqslant r-1$ and t be a standard Kronecker tableau of degree r such that

$$\mathsf{t}(k) \xrightarrow{-a} \mathsf{t}(k+\tfrac{1}{2}) \xrightarrow{+b} \mathsf{t}(k+1)$$

for $a, b \geqslant 0$. We set $\mathsf{t}(k) = \lambda$, $\mathsf{t}(k+\frac{1}{2}) = \mu$, $\mathsf{t}(k+1) = \nu$ and we define the up branching coefficients,

$$u_{\mathsf{t}(k)\to\mathsf{t}(k+\frac{1}{2})} = e^{(k-|\mu|)}_{k+\frac{1}{2}} s_{|\lambda|,|\lambda|_a} \quad \text{and} \quad u_{\mathsf{t}(k+\frac{1}{2})\to\mathsf{t}(k+1)} = e^{(k+1-|\nu|)}_{k+1} \left(\sum_{i=0}^{\nu_b - 1} s_{|\nu|_b - i, |\nu|_b} \right) s_{|\nu|_b, |\nu|}$$

and the down branching coefficients,

$$d_{\mathsf{t}(k)\to\mathsf{t}(k+\frac{1}{2})} = e^{(k-|\lambda|)}_{k} \left(\sum_{i=0}^{\lambda_a - 1} s_{|\lambda|_a - i, |\lambda|_a} \right) s_{|\lambda|_a, |\lambda|} \quad \text{and} \quad d_{\mathsf{t}(k+\frac{1}{2})\to\mathsf{t}(k+1)} = e^{(k-|\mu|)}_{k+\frac{1}{2}} s_{|\nu|,|\nu|_b}.$$

Note that in the above definition we take the convention that $s_{a,b} = 1$ if either $a = 0$ or $b = 0$ (recall that for any partition λ we set $\lambda_0 = |\lambda|_0 = 0$). Moreover, we set any empty summation equal to 1 (in particular any sum of the form $\sum_{i=0}^{-1}$ is equal to 1). Given $\nu \in \mathcal{Y}_r$ and $\mathsf{t} \in \mathrm{Std}_r(\nu)$ we let

$$d_{\mathsf{t}} = d_{\mathsf{t}(0)\to\mathsf{t}(\frac{1}{2})} d_{\mathsf{t}(\frac{1}{2})\to\mathsf{t}(1)} \cdots d_{\mathsf{t}(r-\frac{1}{2})\to\mathsf{t}(r)} \quad \text{and} \quad u_{\mathsf{t}} = u_{\mathsf{t}(r-\frac{1}{2})\to\mathsf{t}(r)} \cdots u_{\mathsf{t}(\frac{1}{2})\to\mathsf{t}(1)} u_{\mathsf{t}(0)\to\mathsf{t}(\frac{1}{2})}.$$

We now provide two examples of branching coefficients to illustrate the manner in which adding/removing a box in the branching graph is reflected in this construction.

Example 2.8. Intuitively, one should think of the "adding a box" up-coefficients as first pulling the strand past lower rows and then averaging over the length of the row in which the box is added. For example if $\mathsf{t}(4+\frac{1}{2}) = ((2,2), 4+\frac{1}{2})) \in \mathcal{Y}_{4+\frac{1}{2}}$ and $\mathsf{t}(5) = ((3,2),5)) \in \mathcal{Y}_5$ then the new box (corresponding to the 5th strand) is moved past the second row (corresponding to the 3 and 4 strands) and added to the first row (by averaging over the first two strands). This is depicted in Figure 11 below.

Example 2.9. Intuitively, one should think of the "removing a box" up-coefficients as removing a box from the end of a row. For example if $\mathsf{t}(5) = ((3,2),5) \in \mathcal{Y}_{4+\frac{1}{2}}$ and $\mathsf{t}(5+\frac{1}{2}) = ((2,2), 5+\frac{1}{2})) \in \mathcal{Y}_5$ then we remove the box at the end of row 1 (corresponding to the 3 strand) and then add enough arcs along the top row. This is depicted in Figure 12 below.

$u_{((2^2),4+\frac{1}{2})\to((3,2),5)} =$

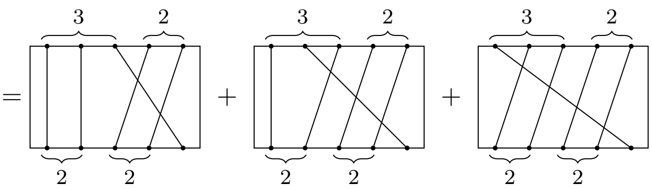


FIGURE 11. The up branching coefficient for $\mathsf{t}(4+\frac{1}{2}) = ((2,2), 4+\frac{1}{2})) \in \mathcal{Y}_{4+\frac{1}{2}}$ and $\mathsf{t}(5) = ((3,2),5)) \in \mathcal{Y}_5$. The bottom of the diagram corresponds to the partition (2^2) on 4 strands and the top corresponds to $(3,2)$ on 5 strands. In particular in this example the term $e_{k+1}^{(k+1-|\nu|)} = 1$.

$u_{((3,2),5)\to((2^2),5+\frac{1}{2})} = e^{(1)}_{5+\frac{1}{2}} s_{5,3} =$

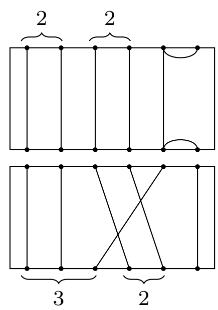


FIGURE 12. The up branching coefficient for $\mathsf{t}(5) = ((2,2),5)) \in \mathcal{Y}_5$ and $\mathsf{t}(5+\frac{1}{2}) = ((2^2), 5+\frac{1}{2})) \in \mathcal{Y}_{5+\frac{1}{2}}$. The bottom of the diagram corresponds to the partition $(3,2)$ on 4 strands and the top corresponds to (2^2) on 5 strands (note we have one strand too many and we fill this up with an arc using the quasi-idempotent).

The motivation for constructing these branching coefficients comes from the fact that they give an integral cellular basis for the partition algebra.

Theorem 2.10 ([15, Section 6.5])**.** *The algebra $P_r(n)$ has an integral basis*

$$\{d_{\mathsf{s}} u_{\mathsf{t}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\nu), \nu \in \mathscr{P}_{\leqslant r}\}.$$

Moreover, if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\nu)$ for some $\nu \in \mathscr{P}_{\leqslant r}$, and $a \in P_r(n)$ then there exist scalars $r_{\mathsf{tu}}(a)$, which do not depend on s, such that

$$d_{\mathsf{s}} u_{\mathsf{t}} a = \sum_{\mathsf{u} \in \mathrm{Std}_r(\nu)} r_{\mathsf{tu}}(a) d_{\mathsf{s}} u_{\mathsf{u}} \pmod{P_r^{\rhd \nu}(n)}, \tag{2.2}$$

where $P_r^{\rhd\nu}(n)$ is the $\mathbb{Q}$-submodule of $P_r(n)$ spanned by

$$\{d_{\mathsf{q}} u_{\mathsf{r}} \mid \mu \rhd \nu \text{ and } \mathsf{q}, \mathsf{r} \in \mathrm{Std}_r(\mu)\}.$$

Finally, we have that $(d_{\mathsf{s}} u_{\mathsf{t}})^* = d_{\mathsf{t}} u_{\mathsf{s}}$, for all $\nu \in \mathscr{P}_{\leqslant r}$ and all $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\nu)$. Therefore the algebra is cellular, in the sense of [19].

Remark 2.11. The subalgebra spanned by $\{d_{\mathsf{s}} u_{\mathsf{t}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\alpha), \alpha \in \mathscr{P}_{\leqslant r-1} \subset \mathscr{P}_{\leqslant r}\}$ is equal to the 2-sided ideal generated by the element $p_r \in P_r(n)$ depicted in Figure 9. The resulting integral cellular structure on the quotient $\mathbb{Q}\mathfrak{S}_r \cong P_r(n)/P_r(n)p_rP_r(n)$ is the basis of [33].

Lemma 2.12. *For any $\nu = (\nu_1, \dots, \nu_\ell) \in \mathscr{P}_{\leqslant r}$, if we take s to be the Kronecker tableau of the form*

$$\underbrace{a(1) \circ \cdots \circ a(1)}_{\nu_1} \circ \underbrace{a(2) \circ \cdots \circ a(2)}_{\nu_2} \circ \cdots \circ \underbrace{a(\ell) \circ \cdots \circ a(\ell)}_{\nu_\ell} \circ \circ \underbrace{d(0) \circ d(0) \circ \cdots \circ d(0)}_{r-|\nu|}$$

then for any $\mathsf{t} \in \mathrm{Std}_r(\nu)$, we have that $d_{\mathsf{s}} u_{\mathsf{t}} = x_{(\nu,r)} d^*_{\mathsf{t}} = u_{\mathsf{t}}$ where $x_{(\nu,r)} = e_r^{(r-|\nu|)} \sum_{g \in \mathfrak{S}_\nu} g$.

Proof. We have that $d_{\mathsf{s}} = e^{(r-1-|\nu|)}_{r-1} e^{(r-1-|\nu|)}_{r-\frac{1}{2}}$. Now, for any $\mathsf{t} \in \mathrm{Std}_r(\nu)$, we have

$$u_{\mathsf{t}} = e_r^{r-|\nu|} \sum_{g \in \mathfrak{S}_\nu} g d^*_{\mathsf{t}}$$

by [15, Lemma A.1] and [15, Section 6]. So we have

$$d_{\mathsf{s}} u_{\mathsf{t}} = e_{r-1}^{(r-1-|\nu|)} e_{r-\frac{1}{2}}^{(r-1-|\nu|)} e_r^{r-|\nu|} \sum_{g \in \mathfrak{S}_\nu} g d_{\mathsf{s}}^* = e_r^{r-|\nu|} \sum_{g \in \mathfrak{S}_\nu} g d_{\mathsf{s}}^* = u_{\mathsf{t}}$$

as required. □

Thus, using Theorem 2.10 and Lemma 2.12 we can make the following definition.

Definition 2.13. Given any $\nu \in \mathscr{P}_{\leqslant r}$, the cell module $\Delta_r(\nu)$ is the right $P_r(n)$-module with basis $\{m_{\mathsf{t}} = u_{\mathsf{t}} + P_r^{\rhd \nu}(n) \mid \mathsf{t} \in \mathrm{Std}_r(\nu)\}$. The action of $P_r(n)$ on $\Delta_r(\nu)$ is given by

$$m_{\mathsf{t}} a = \sum_{\mathsf{u} \in \mathrm{Std}_r(\nu)} r_{\mathsf{tu}}(a) m_{\mathsf{u}},$$

where the scalars $r_{\mathsf{tu}}(a)$ are the scalars appearing in equation (2.2).

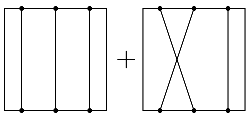
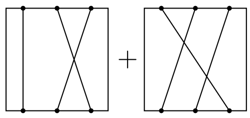
Remark 2.14. For $\nu \in \mathscr{P}_r \subseteq \mathcal{Y}_r$ the module $\Delta_r(\nu)$ is isomorphic to the Specht module $\mathbf{S}(\nu)$ of $\mathfrak{S}_r$ lifted to $P_r(n)$ via the isomorphism $\mathbb{Q}\mathfrak{S}_r \cong P_r(n)/P_r(n)p_rP_r(n)$.

Example 2.15. Let us start with the cell module $\Delta_3((2,1)) = \mathbf{S}((2,1))$ lifted to $P_3(n)$. There are two standard Kronecker tableaux in $\mathrm{Std}_3((2,1))$, namely

$$\mathsf{s} = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \xrightarrow{-\varepsilon_0} \square \xrightarrow{+\varepsilon_1} \ydiagram{2} \xrightarrow{-\varepsilon_0} \ydiagram{2} \xrightarrow{+\varepsilon_2} \ydiagram{2,1} \right)$$

$$\mathsf{t} = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \xrightarrow{-\varepsilon_0} \square \xrightarrow{+\varepsilon_2} \ydiagram{1,1} \xrightarrow{-\varepsilon_0} \ydiagram{1,1} \xrightarrow{+\varepsilon_1} \ydiagram{2,1} \right).$$

All the branching coefficients coreesponding to edges in these tableaux are equal to 1 except for $u_{\mathsf{s}(1+\frac{1}{2}) \to \mathsf{s}(2)} = 1 + s_{1,2}$ and $u_{\mathsf{t}(2+\frac{1}{2}) \to \mathsf{t}(3)} = (1 + s_{1,2})s_{2,3}$. So we have

$u_{\mathsf{s}} =$  $\qquad u_{\mathsf{t}} =$ 

and $\{m_{\mathsf{s}} = u_{\mathsf{s}} + P_3^{\rhd(2,1)}(n), m_{\mathsf{t}} = u_{\mathsf{t}} + P_3^{\rhd(2,1)}(n)\}$ form a basis for $\Delta_3((2,1))$ (which is indeed the classical Murphy basis).

Example 2.16. We now use Definition 2.7 to construct a basis for the cell module $\Delta_2((1))$ for $P_2(n)$. We have three elements $\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3$ in $\mathrm{Std}_2((1))$ totally ordered by $\mathsf{t}_1 \rhd \mathsf{t}_2 \rhd \mathsf{t}_3$ and given by

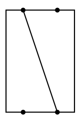
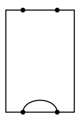
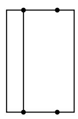
$$\mathsf{t}_1 = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_0} \varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \right) \quad \mathsf{t}_2 = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \xrightarrow{-\varepsilon_1} \varnothing \xrightarrow{+\varepsilon_1} \square \right)$$

$$\mathsf{t}_3 = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \xrightarrow{-\varepsilon_0} \square \xrightarrow{+\varepsilon_1} \square \right)$$

Then we have

$$u_{\mathsf{t}_1} = e_2^{(1)} e_{1+\frac{1}{2}}^{(1)} e_1^{(1)} e_{\frac{1}{2}}^{(0)} \qquad u_{\mathsf{t}_2} = e_2^{(1)} e_{1+\frac{1}{2}}^{(1)} e_1^{(0)} e_{\frac{1}{2}}^{(0)} \qquad u_{\mathsf{t}_3} = e_2^{(1)} e_{1+\frac{1}{2}}^{(0)} e_1^{(0)} e_{\frac{1}{2}}^{(0)}$$

Concatenating these diagrams we get

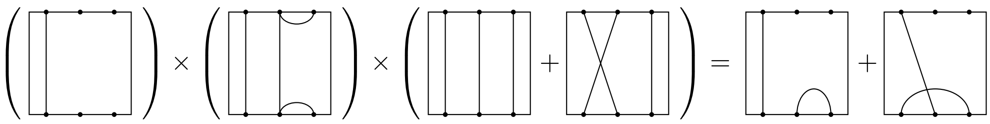
$u_{\mathsf{t}_1} =$  $\qquad u_{\mathsf{t}_2} =$  $\qquad u_{\mathsf{t}_3} =$ 

The corresponding $\{m_{\mathsf{t}_1}, m_{\mathsf{t}_2}, m_{\mathsf{t}_3}\}$ give a basis for $\Delta_2((1))$. Note that the action of $P_2(n)$ on these elements is given by concatenation from below on the u_{t_i}'s.

Example 2.17. We now construct the basis element for the cell module $\Delta_3((1))$ labelled by the standard Kronecker tableau

$$\mathsf{v} = \left(\varnothing \xrightarrow{-\varepsilon_0} \varnothing \xrightarrow{+\varepsilon_1} \square \xrightarrow{-\varepsilon_0} \square \xrightarrow{+\varepsilon_1} \ydiagram{2} \xrightarrow{-\varepsilon_1} \square \xrightarrow{+\varepsilon_0} \square \right).$$

Using the definition of the branching coefficients we get $u_{\mathsf{v}} = e_3^{(2)} e_{2+\frac{1}{2}}^{(1)}(1 + s_{1,2})$. By concatenating the diagrams we obtain



Example 2.18. We refer to Example 3.6 for a larger example for $P_6(n)$.

2.3. **Skew cell modules.** In what follows, we view $P_s(n)$ as a subalgebra of $P_r(n)$ via the embedding

$$P_s(n) \cong \mathbb{Q} \otimes P_s(n) \hookrightarrow P_{r-s}(n) \otimes P_s(n) \hookrightarrow P_r(n).$$

We now recall the definition of skew modules for $P_s(n)$. This family of modules were first introduced (in the more general context of diagram algebras) in [5]. Given $\nu \in \mathscr{P}_{\leqslant r}$, we let $\mathsf{t}^\nu \in \mathrm{Std}_r(\nu)$ denote the Kronecker tableau of the form

$$\underbrace{d(0) \circ d(0) \circ \dots \circ d(0)}_{r-|\nu|} \circ \underbrace{a(1) \circ \dots \circ a(1)}_{\nu_1} \circ \underbrace{a(2) \circ \dots \circ a(2)}_{\nu_2} \circ \cdots$$

which is maximal in the dominance ordering on $\mathrm{Std}_r(\nu)$.

Example 2.19. For $\nu = (2,1) \in \mathscr{P}_{\leqslant 5} \subseteq \mathcal{Y}_5$, the Kronecker tableau t^ν is equal to

$$\left(\varnothing, \varnothing, \varnothing, \varnothing, \Box\,,\ \Box\,,\ \Box\Box\,,\ \Box\Box\,,\ \begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}\,,\ \begin{smallmatrix}\Box&\Box\\ \Box&\end{smallmatrix}\right)$$

Definition 2.20. Given $\lambda \in \mathscr{P}_{r-s} \subseteq \mathcal{Y}_{r-s}$ and $\nu \in \mathscr{P}_{\leqslant r} \subseteq \mathcal{Y}_r$, define

$$\Delta_r(\nu; \rhd\lambda) = \mathrm{span}_{\mathbb{Q}}\{m_{\mathsf{t}} \mid \mathsf{t}(r-s) \rhd \lambda\} \qquad \Delta_r(\nu; \mathsf{t}^\lambda) = \mathrm{span}_{\mathbb{Q}}\{m_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}_r(\nu), \mathsf{t}[0, r-s] = \mathsf{t}^\lambda\}$$

then $\Delta_r(\nu; \rhd\lambda)$ and $\Delta_r(\nu; \mathsf{t}^\lambda) + \Delta_r(\nu; \rhd\lambda)$ are $P_s(n)$-submodules of $\Delta_r(\nu)\downarrow_{P_s(n)}$. We define the skew cell module

$$\Delta_s(\lambda \to \nu) = (\Delta_r(\nu; \mathsf{t}^\lambda) + \Delta_r(\nu; \rhd\lambda))/\Delta_r(\nu; \rhd\lambda).$$

Remark 2.21. It follows from Definition 2.13 that we can realise the skew cell module as a subquotient of the algebra $P_r(n)$ as follows. Define

$$P_{r,s}^{\rhd(\lambda\to\nu)} = P_r(\nu) + \mathrm{span}_{\mathbb{Q}}\{u_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}_r(\nu), \mathsf{t}(r-s) \rhd \lambda\},$$

then

$$\Delta_s(\lambda \to \nu) = \mathrm{span}_{\mathbb{Q}}\{u_{\mathsf{t}^\lambda \circ \mathsf{s}} + P_{r,s}^{\rhd(\lambda\to\nu)} \mid \mathsf{s} \in \mathrm{Std}_s(\lambda \to \nu)\}.$$

Remark 2.22. The basis of $\Delta_s(\lambda \to \nu)$ is indexed by the elements of $\mathrm{Std}_s(\lambda \to \nu)$ and if (λ, ν, s) is triple of maximal depth, this module is isomorphic to $\mathbf{S}(\nu \setminus \lambda)$, the skew Specht module for $\mathfrak{S}_s$, lifted to $P_s(n)$.

We can now reinterpret the stable Kronecker coefficients in the context of partition algebras as follows.

Theorem 2.23. [4, 5] *Let $\lambda \in \mathscr{P}_{r-s}$, $\mu \in \mathscr{P}_s$ and $\nu \in \mathscr{P}_{\leqslant r}$. Then we have*

$$\overline{g}(\lambda,\nu,\mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_{r-s}(n)\times P_s(n)}(\Delta_{r-s}(\lambda) \boxtimes \Delta_s(\mu), \Delta_r(\nu)\downarrow)) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\lambda \to \nu)))$$

for all $n \gg 0$.

Remark 2.24. Using Remark 2.22 and (1.6) we recover Theorem 1.15. So the Littlewood–Richardson coefficients appear naturally as a subclass of the stable Kronecker coefficients in the context of the partition algebra.

3. Skew modules and co-Pieri triples for partition algebras

To describe the action of the generators of the partition algebra on the Murphy basis is very difficult in general. We define co-Pieri triples as the triples for which we can fully understand the algebra action in terms of permutation matrices (precisely generalising the triples of Theorem 1.10). In order to make this paper suitable for a 'Kronecker audience', we have postponed the diagrammatic proofs to appendices A and B of this paper.

Definition 3.1. Fix $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $1 \leqslant k \leqslant r-1$ and suppose that

$$\mathsf{t}(k-1) \xrightarrow{-t} \mathsf{t}(k-\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}(k) \xrightarrow{-v} \mathsf{t}(k+\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}(k+1).$$

We define $\mathsf{t}_{k\leftrightarrow k+1} \in \mathrm{Std}_r(\nu)$ to be the tableau, if it exists, determined by $\mathsf{t}_{k\leftrightarrow k+1}(l) = \mathsf{t}(l)$ for $l \neq k, k \pm \frac{1}{2}$ and

$$\mathsf{t}_{k\leftrightarrow k+1}(k-1) \xrightarrow{-v} \mathsf{t}_{k\leftrightarrow k+1}(k-\tfrac{1}{2}) \xrightarrow{+w} \mathsf{t}_{k\leftrightarrow k+1}(k) \xrightarrow{-t} \mathsf{t}_{k\leftrightarrow k+1}(k+\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}_{k\leftrightarrow k+1}(k+1).$$

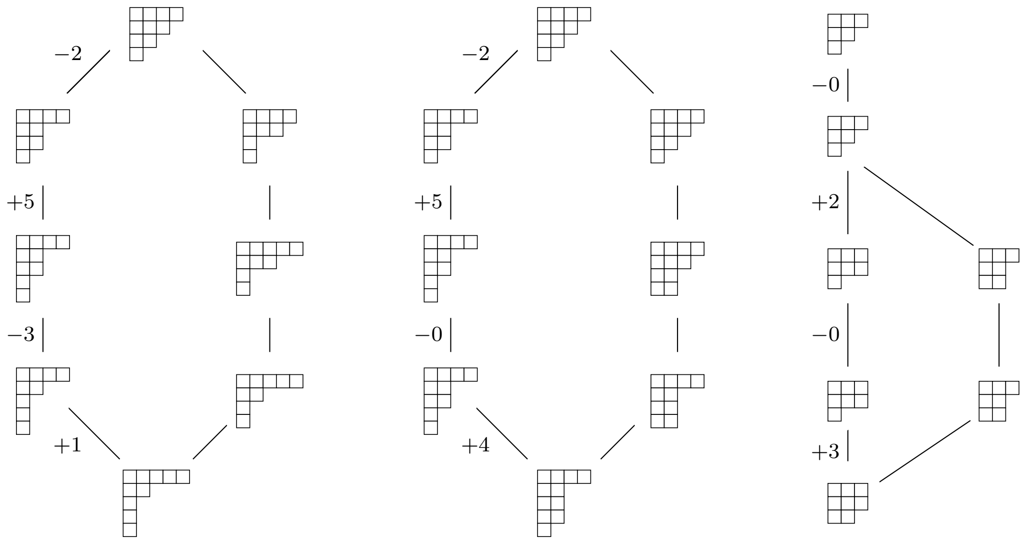


FIGURE 13. Examples of the pairs of paths t and $\mathsf{t}_{k\leftrightarrow k+1}$ in $\mathcal{Y}$.

In this section, we will discuss explicitly the action of $s_{k,k+1}$ on u_{t} for all paths $\mathsf{t} \in \mathrm{Std}(\nu)$ such that the path $\mathsf{t}_{k\leftrightarrow k+1}$ exists.

Before stating the main result, we need one more piece of notation.

Definition 3.2. For $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $1 \leqslant k \leqslant r$ with

$$\mathsf{t}(k-\tfrac{1}{2}) \xrightarrow{+u} \mathsf{t}(k) \xrightarrow{-u} \mathsf{t}(k+\tfrac{1}{2})$$

for $u > 0$, we define $\mathsf{s} = \mathsf{e}_k(\mathsf{t}) \in \mathrm{Std}_r(\nu)$ by $\mathsf{s}(l) = \mathsf{t}(l)$ for $l \neq k$ and

$$\mathsf{s}(k-\tfrac{1}{2}) \xrightarrow{+L} \mathsf{s}(k) \xrightarrow{-L} \mathsf{s}(k+\tfrac{1}{2}) \tag{3.1}$$

where $L = \ell(\mathsf{t}(k-\frac{1}{2})) + 1$. If $\mathsf{t}(k-\frac{1}{2}) \neq \mathsf{t}(k+\frac{1}{2})$, then $\mathsf{e}_k(\mathsf{t})$ is undefined.

Theorem 3.3. *Fix $1 \leqslant k \leqslant r$ and let $\mathsf{t} \in \mathrm{Std}_r(\nu)$. If $\mathsf{t}_{k\leftrightarrow k+1}$ exists, then*

$$(u_{\mathsf{t}})s_{k,k+1} = u_{\mathsf{t}_{k\leftrightarrow k+1}} + u_{\mathsf{e}_k(\mathsf{t})} - u_{\mathsf{e}_k(\mathsf{t}_{k\leftrightarrow k+1})},$$

where we take the convention that $u_{\mathsf{e}_k(\mathsf{v})} = 0$ whenever the path $\mathsf{e}_k(\mathsf{v})$ is undefined for $\mathsf{v} \in \mathrm{Std}_r(\nu)$.

Proof. The proof of this result is an in depth diagrammatic calculation which we have postponed to Appendix A. □

As before, we identify $P_s(n)$ as a subalgebra of $P_r(n)$ via the embedding $P_s(n) \cong \mathbb{Q} \otimes P_s(n) \subseteq P_{r-s}(n) \otimes P_s(n) \subseteq P_r(n)$, that is we view each partition diagram in $P_s(n)$ as a set-partition of $\{r-s+1, \dots, r, \overline{r-s+1}, \dots, \overline{r}\}$. We also assume throughout this section that $n \gg r$. We have seen in Section 2 that

$$g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) = \overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\lambda \to \nu)))$$

for any triple of partitions $(\lambda, \nu, \mu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r} \times \mathscr{P}_s$. Now, as $|\mu| = s$ we have that the ideal $P_s(n)p_rP_s(n) \subset P_s(n)$ annihilates $\Delta_s(\mu)$ and so

$$\dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\lambda \to \nu)P_s(n)p_rP_s(n))) = 0$$

and thus we wish to consider the quotient of $\Delta_s(\lambda \to \nu)$ by the submodule $\Delta_s(\lambda \to \nu)P_s(n)p_rP_s(n)$. This leads us to the following definition.

Definition 3.4. We define the $\mathsf{Dvir\ radical}$ of the skew module $\Delta_s(\lambda \to \nu)$ by

$$\mathsf{DR}_s(\lambda \to \nu) = \Delta_s(\lambda \to \nu)P_s(n)p_rP_s(n) \subseteq \Delta_s(\lambda \to \nu)$$

and set

$$\Delta_s^0(\lambda \to \nu) = \Delta_s(\lambda \to \nu)/\mathsf{DR}_s(\lambda \to \nu).$$

By definition, we have that

$$\overline{g}(\lambda,\nu,\mu) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s(\lambda \to \nu))) = \dim_{\mathbb{Q}}(\mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s^0(\lambda \to \nu))) \tag{3.2}$$

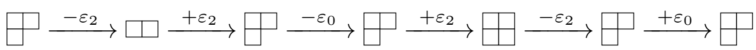
for any $\mu \in \mathscr{P}_s$. Thus, in order to understand the coefficients $\overline{g}(\lambda,\nu,\mu)$, we need to construct a basis for the modules $\Delta_s^0(\lambda \to \nu)$ and to describe the $P_s(n)$-action on this basis. Towards that end, we make the following definition.

Definition 3.5. Given $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ we set

$$\mathrm{Std}_s^0(\lambda \to \nu) = \left\{ m_{\mathsf{t}} \;\middle|\; \begin{array}{l} \mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu),\ \mathsf{t} \text{ has no steps of the form } (-\varepsilon_0, +\varepsilon_0), \\ \mathsf{t} \text{ has at most } \lambda_i \text{ steps of the form } (-\varepsilon_i, +\varepsilon_j) \text{ for } j \geqslant 0 \end{array} \right\} \tag{3.3}$$

and we let $\mathrm{DR{-}Std}_s(\lambda \to \nu) := \mathrm{Std}_s(\lambda \to \nu) \setminus \mathrm{Std}_s^0(\lambda \to \nu)$.

Example 3.6. Let $\nu = \lambda = (2,1)$ and $s = 3$. The path $\mathsf{t} \in \mathrm{Std}_3(\lambda \to \nu)$ given by



belongs to DR-$\mathrm{Std}_3(\lambda \to \nu)$. To see this note that

$$\sharp\{\text{steps of the form } (-\varepsilon_2, +\varepsilon_j) \text{ for } j \geqslant 0 \text{ in } \mathsf{t}\} = 2 > 1 = \lambda_2.$$

Now the element $u_{\mathsf{t}^\lambda \circ \mathsf{t}}$ is depicted in Figure 14, below. We see that every elementary diagram in this sum has at most 2 blocks with both an element from $\{4,5,6\}$ and an element from $\{\overline{1},\overline{2},\dots,\overline{6}\} \cup \{1,2,3\}$. This means that $u_{\mathsf{t}^\lambda \circ \mathsf{t}} \in P_6(n)p_6P_3(n)$ and therefore $u_{\mathsf{t}^\lambda \circ \mathsf{t}} \in \mathsf{DR}_3(\lambda \to \nu)$.

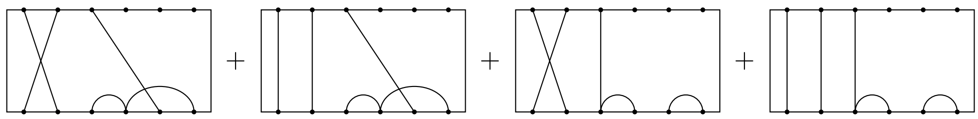


FIGURE 14. The element $u_{\mathsf{t}^\lambda \circ \mathsf{t}}$ from Example 3.6.

Example 3.6 generalises to give the following proposition.

Proposition 3.7. *If* $\mathsf{t} \in \mathrm{DR{-}Std}_s(\lambda \to \nu)$ *then* $u_{\mathsf{t}^\lambda \circ \mathsf{t}} + P_{r,s}^{\rhd(\lambda\to\nu)}(n) \in \mathsf{DR}_s(\lambda \to \nu)$.

Proof. This is another in depth diagrammatic calculation which is deferred to Appendix B. ☐

Remark 3.8. If (λ,ν,s) is a triple of maximal depth then $\mathrm{Std}_s^0(\lambda \to \nu) = \mathrm{Std}_s(\lambda \to \nu)$.

Remark 3.9. By Proposition 3.7, we know that $\mathrm{Std}_s^0(\lambda \to \nu)$ indexes a spanning set for the module $\Delta_s^0(\lambda \to \nu)$ for any $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$. In particular, if $s > |\lambda| + |\nu|$ or $s < \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$ then $\mathrm{Std}_s^0(\lambda \to \nu) = \emptyset$ and $\overline{g}(\lambda,\nu,\mu) = 0$ for all $\mu \vdash s$.

Definition 3.10. We say that $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ form a co-Pieri triple (λ,ν,s) if $\mathrm{Std}_s^0(\lambda \to \nu) \neq \emptyset$,

(C1) $\mathsf{s}_{k\leftrightarrow k+1}$ exists for all $\mathsf{s} \in \mathrm{Std}_s^0(\lambda \to \nu)$ and $1 \leqslant k \leqslant s-1$ and
(C2) $\Delta_s^0(\lambda \to \nu)$ has basis indexed by $\mathrm{Std}_s^0(\lambda \to \nu)$.

We will also refer to any triple of the form (λ,ν,μ) with $\mu \vdash s$ as a co-Pieri triple.

We will shortly provide a combinatorial classification of co-Pieri triples. Before embarking on this, we observe a simple corollary of the definition which explains our interest in these triples — namely, we can fully understand the $P_s(n)$-action for such triples. To simplify the notation for the basis elements of the skew module $\Delta_s(\lambda \to \nu)$ we set

$$m_{\mathsf{t}} := u_{\mathsf{t}^\lambda \circ \mathsf{t}} + P_{r,s}^{\rhd(\lambda\to\nu)}(n)$$

for all $\mathsf{t} \in \mathrm{Std}_s(\lambda \to \nu)$.

Corollary 3.11. *Let (λ,ν,s) be a co-Pieri triple. Then we have that*

$$\{m_{\mathsf{t}} + \mathsf{DR}_s(\lambda\to\nu) \mid \mathsf{t}\in \mathrm{Std}^0_s(\lambda\to\nu)\}$$

is a basis for $\Delta^0_s(\lambda\to\nu)$ and the $P_s(n)$-action on $\Delta^0_s(\lambda\to\nu)$ is as follows:

$$(m_{\mathsf{t}} + \mathsf{DR}_{\mathsf{s}}(\lambda\to\nu))s_{k,k+1} = m_{\mathsf{t}_{k\leftrightarrow k+1}} + \mathsf{DR}_{\mathsf{s}}(\lambda\to\nu) \tag{3.4}$$

for $1\leqslant k<s$,

$$(m_{\mathsf{t}} + \mathsf{DR}_{\mathsf{s}}(\lambda\to\nu))p_{k,k+1} = 0 \quad \textit{and} \quad (m_{\mathsf{t}} + \mathsf{DR}_{\mathsf{s}}(\lambda\to\nu))p_{k} = 0$$

for all $1\leqslant k<s$ and $1\leqslant k\leqslant s$, respectively.

Proof. This follows immediately from Definition 3.10 and Theorem 3.3 □

For triples of maximal depth, it is clear that (C2) is empty and so (λ,ν,s) is a co-Pieri triple if and only if all $\lambda\to\nu$-tableaux are standard, if and only if there are no two nodes in the same column of $\nu\setminus\lambda$. Indeed, condition $(C1)$ says that all tableaux (not belonging to the Dvir radical) are standard and condition $(C2)$ simply says that these tableaux give the required basis. While the definition of a co-Pieri triple is simple and intuitive (and generalises the classical definition of a co-Pieri triple in the most natural possible way), we shall now see that the classification of general co-Pieri triples is difficult and technical.

Theorem 3.12 (Classification of co-Pieri triples)**.** *Let $s\in\mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu)\in\mathscr{P}_{r-s}\times\mathscr{P}_{\leqslant r}$ be such that $\mathrm{Std}^0_s(\lambda\to\nu)\neq\emptyset$. We have that (λ,ν,s) is a* co-Pieri triple *if and only if*

$$\text{(coP)}\begin{cases} s=1, \textit{ or} \\ s>1 \textit{ and if } \max\{\ell(\lambda),\ell(\nu)\}\geqslant 2 \textit{ then } s\leqslant \max\{|\lambda\setminus(\lambda\cap\nu)|,|\nu\setminus(\lambda\cap\nu)|\}+\mathrm{minmax}(\lambda,\nu)\end{cases}$$

where

$$\mathrm{minmax}(\lambda,\nu)=\min\{\min\{\lambda_{i-1},\nu_{i-1}\}-\max\{\lambda_i,\nu_i\}\mid 2\leqslant i\leqslant \max\{\ell(\lambda),\ell(\nu)\}\}.$$

Remark 3.13. Note that if (λ,ν,s) satisfies $\mathrm{Std}^0_s(\lambda\to\nu)\neq\emptyset$ and (coP) then the skew partitions $\lambda\setminus(\lambda\cap\nu)$ and $\nu\setminus(\lambda\cap\nu)$ contain no two nodes in the same column. To see this, observe that $\mathrm{minmax}(\lambda,\nu)<0$ precisely when one of these skew partitions has two nodes in the same column. On the other hand, $\mathrm{Std}^0_s(\lambda\to\nu)\neq\emptyset$ implies that $s\geqslant\max\{|\lambda\setminus(\lambda\cap\nu)|,|\nu\setminus(\lambda\cap\nu)|\}$. Thus we must have $\mathrm{minmax}(\lambda,\nu)\geqslant 0$.

Example 3.14. Note that any triple (λ,ν,s) with $\ell(\lambda)=\ell(\nu)=1$ is a co-Pieri triple. We calculate the corresponding Kronecker coefficients labelled by two two-line partitions in Section 6.

Example 3.15. For $d,\ell,m\geqslant 0$, we define the partition

$$\rho = d(\ell,\ell-1,\dots,2,1)+(m^l)$$

As $\mathrm{minmax}(\rho,\rho)=d$ we have that (ρ,ρ,s) with any $s\leqslant d$ is a co-Pieri triple.

Example 3.16. Let λ and ν be any pair of partitions such that $\lambda\setminus(\lambda\cap\nu)$ and $\nu\setminus(\lambda\cap\nu)$ are skew partitions with no two nodes in the same column and let $s=\max\{|\lambda\setminus(\lambda\cap\nu)|,|\nu\setminus(\lambda\cap\nu)|\}$. Then (λ,ν,s) is a co-Pieri triple. This clearly includes the triples of Theorem 1.10 as a subcase. For another example, $((10,5,2),(8,3,3,2),4)$ is such a co-Pieri triple.

Example 3.17. Let $\lambda=(4,2)$ and $\nu=(4,3,1)$. We have that $\max\{|\lambda\setminus(\lambda\cap\nu)|,|\nu\setminus(\lambda\cap\nu)|\}=2$ and $\mathrm{minmax}(\lambda,\nu)=1$. Therefore (λ,ν,s) is a co-Pieri triple for $s=2$ or 3.

Lemma 3.18. *Let $s\in\mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu)\in\mathscr{P}_{r-s}\times\mathscr{P}_{\leqslant r}$ with $\mathrm{Std}^0_s(\lambda\to\nu)\neq\emptyset$. Assume that (λ,ν,s) satisfies* (coP)*. Let $n\gg r$ and $\alpha\subseteq\lambda_{[n]}\cap\nu_{[n]}$ be any composition of $n-s$, say*

$$\alpha=(\alpha_1,\alpha_2,\dots)=\lambda_{[n]}-\varepsilon_{i_1}-\varepsilon_{i_2}-\cdots-\varepsilon_{i_s}=\nu_{[n]}-\varepsilon_{j_1}-\varepsilon_{j_2}-\cdots-\varepsilon_{j_s}.$$

Define the composition

$$\beta=(\beta_1,\beta_2,\dots)=\lambda_{[n]}+\varepsilon_{j_1}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}=\nu_{[n]}+\varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_s}.$$

Then for all $c\geqslant 1$ we have

$$\alpha_c\geqslant\beta_{c+1}.$$

In particular, $\alpha\subseteq\lambda_{[n]}\cap\nu_{[n]}$ is a partition and $\lambda_{[n]}\setminus\alpha$ and $\nu_{[n]}\setminus\alpha$ have no two nodes in the same column.

Proof. First note that as $n \gg r$, $\alpha_1 \geqslant \beta_2$. If $\ell(\lambda) = \ell(\nu) = 1$ then $\alpha_2 \geqslant \beta_3 = 0$ and for $c \geqslant 3$ we have $\alpha_c = \beta_{c+1} = 0$ so we are done. Now assume $\max\{\ell(\lambda), \ell(\nu)\} \geqslant 2$. Define multi-sets

$$I = \{i_1, i_2, \ldots, i_s\} \quad \text{and} \quad J = \{j_1, j_2, \ldots, j_s\}.$$

For $c \geqslant 2$, define $|I|_c = \sharp\{i_k \in I \mid i_k = c\}$ and define $|J|_c$ and $|I \cap J|_c$ similarly. Now,

$$\alpha_c = \lambda_{c-1} - |I|_c = \lambda_{c-1} - |I \setminus (I \cap J)|_c - |I \cap J|_c$$

$$\beta_{c+1} = \lambda_c + |J|_{c+1} = \lambda_c + |J \setminus (I \cap J)|_c + |I \cap J|_{c+1}.$$

Note that

$$|I \setminus I \cap J|_c = \begin{cases} \lambda_{c-1} - \nu_{c-1} & \text{if } \lambda_{c-1} - \nu_{c-1} \geqslant 0 \\ 0 & \text{otherwise,} \end{cases} \qquad |J \setminus I \cap J|_{c+1} = \begin{cases} \nu_c - \lambda_c & \text{if } \nu_c - \lambda_c \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\lambda_c - |I \setminus I \cap J|_c = \min\{\lambda_{c-1}, \nu_{c-1}\}, \qquad \lambda_c + |J \setminus I \cap J|_{c+1} = \max\{\lambda_c, \nu_c\},$$

and we get

$$\begin{aligned} \alpha_c - \beta_{c+1} &= \min\{\lambda_{c-1}, \nu_{c-1}\} - \max\{\lambda_c, \nu_c\} - |I \cap J|_c - |I \cap J|_{c+1} \\ &\geqslant \min\{\lambda_{c-1}, \nu_{c-1}\} - \max\{\lambda_c, \nu_c\} - |I \cap J|. \end{aligned}$$

Now,

$$|I \cap J| = s - \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}.$$

So we get

$$\alpha_c - \beta_{c+1} \geqslant \min\{\lambda_{c-1}, \nu_{c-1}\} - \max\{\lambda_c, \nu_c\} + \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\} - s.$$

Using (coP), we get that $\alpha_c - \beta_{c+1} \geqslant 0$ for $2 \leqslant c \leqslant \max\{\ell(\lambda), \ell(\nu)\}$. Now, if $c > \max\{\ell(\lambda), \ell(\nu)\}$ then $\beta_{c+1} = 0$ and so $\alpha_c \geqslant \beta_{c+1} = 0$ as required. □

We define $\mathrm{Std}_s^+(\lambda \to \nu) = \mathrm{Std}_s(\lambda \to \nu) \setminus (\cup_{i \geqslant 1} \mathrm{DR}^i(\lambda \to \nu))$.

Lemma 3.19. *Let $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda, \nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ with $\mathrm{Std}_s^0(\lambda \to \nu) \neq \emptyset$. Assume that (λ, ν, s) satisfies* (coP)*. Then we have a bijective map*

$$\varphi_s : \bigsqcup_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} \mathrm{Std}_s(\alpha \to \nu_{[n]}) \times \mathrm{Std}_s(\lambda_{[n]} \to \alpha) \to \mathrm{Std}_s^+(\lambda \to \nu) \tag{3.5}$$

where a given pair on the lefthand-side is necessarily of the form

$$(\mathsf{s}, \mathsf{t}) = ((-\varepsilon_0, +\varepsilon_{j_1}, -\varepsilon_0, +\varepsilon_{j_2}, \ldots, -\varepsilon_0, +\varepsilon_{j_s}), (-\varepsilon_{i_1}, +\varepsilon_0, -\varepsilon_{i_2}, +\varepsilon_0, \ldots, -\varepsilon_{i_s}, +\varepsilon_0)),$$

with $i_l, j_l \neq 0$ for all $1 \leqslant l \leqslant s$, and such a pair of tableaux is sent to

$$\varphi_s(\mathsf{s}, \mathsf{t}) = (-\varepsilon_{i_1 - 1}, +\varepsilon_{j_1 - 1}, -\varepsilon_{i_2 - 1}, +\varepsilon_{j_2 - 1}, \ldots, -\varepsilon_{i_s - 1}, +\varepsilon_{j_s - 1}) \in \mathrm{Std}_s^+(\lambda \to \nu).$$

Moreover, given any $\varphi_s(\mathsf{s}, \mathsf{t}) = \mathsf{u} \in \mathrm{Std}_s^+(\lambda \to \nu)$ and any $1 \leqslant k \leqslant s-1$ we have that $\varphi(\mathsf{s}_{k \leftrightarrow k+1}, \mathsf{t}_{k \leftrightarrow k+1}) = \mathsf{u}_{k \leftrightarrow k+1} \in \mathrm{Std}_s^+(\lambda \to \nu)$ and hence (C1) *holds.*

Proof. We first show that for any $\alpha \vdash n-s$ with $\alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}$ and $(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}_s(\alpha \to \nu_{[n]}) \times \mathrm{Std}_s(\lambda_{[n]} \to \alpha)$ we have $\varphi_s(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}_s(\lambda \to \nu)$. Write

$$\mathsf{s} = (-\varepsilon_0, +\varepsilon_{j_1}, -\varepsilon_0, +\varepsilon_{j_2}, \ldots, -\varepsilon_0, +\varepsilon_{j_s}) \qquad \mathsf{t} = (-\varepsilon_{i_1}, +\varepsilon_0, -\varepsilon_{i_2}, +\varepsilon_0, \ldots, -\varepsilon_{i_s}, +\varepsilon_0).$$

So we have

$$\alpha = \lambda_{[n]} - \varepsilon_{i_1} - \varepsilon_{i_2} - \ldots - \varepsilon_{i_s} = \nu_{[n]} - \varepsilon_{j_1} - \varepsilon_{j_2} - \ldots - \varepsilon_{j_s}.$$

Setting

$$\beta = \lambda_{[n]} + \varepsilon_{j_1} + \varepsilon_{j_2} + \ldots + \varepsilon_{j_s}$$

and using Lemma 3.18 we get

$$\alpha_i \geqslant \beta_{i+1} \quad \forall i \geqslant 1.$$

In order to prove that $\mathsf{u} = \varphi_s(\mathsf{s}, \mathsf{t}) \in \mathrm{Std}_s(\lambda \to \nu)$ we need to show that for all $1 \leqslant l \leqslant s-1$ we have that $\gamma(l) := \lambda_{[n]} + \sum_{k=1}^{l}(-\varepsilon_{i_k} + \varepsilon_{j_k})$ and $\gamma'(l) = \lambda_{[n]} + \sum_{k=1}^{l-1}(-\varepsilon_{i_k} + \varepsilon_{j_k}) - \varepsilon_{i_l}$ are partitions. But for $\gamma = \gamma(l)$ or $\gamma'(l)$ we have

$$\gamma_i \geqslant \alpha_i \geqslant \beta_{i+1} \geqslant \gamma_{i+1} \quad \forall i \geqslant 1.$$

So we are done. Now $\varphi_s(\mathsf{s},\mathsf{t}) \in \mathrm{Std}_s^+(\lambda \to \nu)$ follows directly from the fact that α is a partition. Moreover, it is clear that the map φ_s is injective and that $\varphi_s(\mathsf{s}_{k\leftrightarrow k+1},\mathsf{t}_{\leftrightarrow k+1}) = \mathsf{u}_{k\leftrightarrow k+1}$ by definition.

It remains to show that φ_s is surjective. Given

$$\mathsf{u} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, -\varepsilon_{i_2}, +\varepsilon_{j_2}, \dots, -\varepsilon_{i_s}, +\varepsilon_{j_s}) \in \mathrm{Std}_s^+(\lambda \to \nu),$$

we set $\alpha = \min_n(\mathsf{u}) := \lambda_{[n]} - \varepsilon_{i_1+1} - \varepsilon_{i_2+1} - \dots - \varepsilon_{i_s+1} = \nu_{[n]} - \varepsilon_{j_1+1} - \varepsilon_{j_2+1} - \dots - \varepsilon_{j_s+1}$. As $\mathsf{u} \in \mathrm{Std}_s^+(\lambda \to \nu)$ we have that α must be a composition of $n-s$. Using Lemma 3.18, we know that $\alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}$ is in fact a partition and that $\lambda_{[n]} \setminus \alpha$ and $\nu_{[n]} \setminus \alpha$ contain no two boxes in the same column. It follows that

$$\mathsf{s} := (-\varepsilon_0, +\varepsilon_{j_1+1}, -\varepsilon_0, +\varepsilon_{j_2+1}, \dots, -\varepsilon_0, +\varepsilon_{j_s+1}) \in \mathrm{Std}_s(\alpha \to \nu_{[n]}) \quad \text{and}$$

$$\mathsf{t} := (-\varepsilon_{i_1+1}, +\varepsilon_0, -\varepsilon_{i_2+1}, +\varepsilon_0, \dots, -\varepsilon_{i_s+1}, +\varepsilon_0) \in \mathrm{Std}_s(\lambda_{[n]} \to \alpha)$$

satisfy $\varphi_s(\mathsf{s},\mathsf{t}) = \mathsf{u}$ as required. □

The next proposition gives a representation theoretic interpretation (for co-Pieri triples) of Dvir's recursive formula for calculating Kronecker coefficients (and hence justifies the name 'Dvir radical').

Proposition 3.20. *Let $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ with $\mathrm{Std}_s^0(\lambda \to \nu) \neq \emptyset$. Assume that (λ,ν,s) satisfies* (coP). *Then there is a surjective $P_s(n)$-homomorphism*

$$\overline{\varphi}_s : \bigoplus_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} \Delta_s(\alpha \to \nu_{[n]}) \otimes \Delta_s(\lambda_{[n]} \to \alpha) \to \Delta_s^0(\lambda \to \nu) \tag{3.6}$$

given by

$$\overline{\varphi}_s(m_{\mathsf{s}} \otimes m_{\mathsf{t}}) = m_{\varphi_s(\mathsf{s},\mathsf{t})} + \mathsf{DR}_s(\lambda \to \nu)$$

for all $\mathsf{s} \in \mathrm{Std}_s(\alpha \to \nu_{[n]})$ and $\mathsf{t} \in \mathrm{Std}_s(\lambda_{[n]} \to \alpha)$ (where $P_s(n)$ acts diagonally on the module on the lefthand-side). Furthermore, the kernel of this homomorphism is spanned by

$$\{m_{\mathsf{s}} \otimes m_{\mathsf{t}} \mid \varphi_s(\mathsf{s},\mathsf{t}) \in \mathrm{DR}^0\text{-}\mathrm{Std}_s(\lambda \to \nu)\}. \tag{3.7}$$

and hence the set

$$\{m_{\mathsf{u}} + \mathsf{DR}_s(\lambda \to \nu) \mid \mathsf{u} \in \mathrm{Std}_s^0(\lambda \to \nu)\}$$

form a basis for $\Delta_s^0(\lambda \to \nu)$, i.e. (C2) *holds.*

Proof. By Lemma 3.19 and Proposition 3.7, it is clear that $\overline{\varphi}_s$ is a surjective map. The generators p_k and $p_{k,k+1}$ act as zero on both modules. Using Theorem 3.3, the action of $\mathfrak{S}_s$ on skew cell modules and Lemma 3.19 we have that the action of $s_{k,k+1}$ also coincide under the map $\overline{\varphi}_s$. Thus $\overline{\varphi}_s$ is a surjective $P_s(n)$-homomorphism. It remains to show that its kernel has the required form. As p_k and $p_{k,k+1}$ act as zero, we can view $\overline{\varphi}_s$ as a homorphism of $\mathfrak{S}_s$-modules. As such we have

$$\Delta_s^+(\lambda \to \nu) := \bigoplus_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} \Delta_s(\alpha \to \nu_{[n]}) \otimes \Delta_s(\lambda_{[n]} \to \alpha) \cong \bigoplus_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]} \\ \mu \vdash s}} g(\lambda_{[n]} \setminus \alpha, \nu_{[n]} \setminus \alpha, \mu)\, \mathsf{S}(\mu). \tag{3.8}$$

On the other hand, recall that we have

$$\Delta_s^0(\lambda \to \nu) = \bigoplus_{\mu \vdash s} g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]})\, \mathsf{S}(\mu). \tag{3.9}$$

Now, note that $\Delta_s^+(\lambda \to \nu)$ decomposes as

$$\Delta_s^+(\lambda \to \nu) = \bigoplus_{0 \leqslant m \leqslant s} V_s^m \tag{3.10}$$

where V_s^m is spanned by all $m_{\mathsf{s}} \otimes m_{\mathsf{t}}$ such that $\varphi_s(\mathsf{s},\mathsf{t})$ has precisely m integral steps of the form $(-\varepsilon_0, +\varepsilon_0)$. In particular we have that V_s^0 is spanned by all $m_{\mathsf{s}} \otimes m_{\mathsf{t}}$ with $\varphi_s(\mathsf{s},\mathsf{t}) \in \mathrm{Std}_s^0(\lambda \to \nu)$. We claim that

$$\ker(\overline{\varphi}_s) = \bigoplus_{0 < m \leqslant s} V_s^m.$$

By Proposition 3.7 we know that

$$\bigoplus_{0 < m \leqslant s} V_s^m \subseteq \ker(\overline{\varphi}_s). \tag{3.11}$$

We will prove that in fact we have equality, in other words $V_s^0 \cong \Delta_s^0(\lambda \to \nu)$. We proceed by induction on s. If $s = \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$ then $\mathrm{Std}_s^+(\lambda \to \nu) = \mathrm{Std}_s^0(\lambda \to \nu)$ and so $\bigoplus_{1<m\leqslant s} V_s^m = 0$. Moreover, in this case equation (3.8) gives

$$\begin{aligned}\Delta_s^+(\lambda \to \nu) &\cong \sum_{\mu \vdash s} g(\lambda_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]}), \nu_{[n]} \setminus (\lambda_{[n]} \cap \nu_{[n]}), \mu_{[n]}) \mathsf{S}(\mu)\\ &= \sum_{\mu \vdash s} g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) \mathsf{S}(\mu)\\ &\cong \Delta_s^0(\lambda \to \nu), \end{aligned} \tag{3.12}$$

so we are done in this case. Now let $s > \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$ and assume that the result holds for all $s' < s$. Note that for $m > 0$ we have

$$V_s^m \cong (V_{s-m}^0 \boxtimes \mathsf{S}(m)) \uparrow_{\mathfrak{S}_{s-m} \times \mathfrak{S}_m}^{\mathfrak{S}_s}, \tag{3.13}$$

and by induction, we have

$$V_{s-m}^0 \cong \bigoplus_{\beta \vdash s-m} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]})\, \mathsf{S}(\beta) \tag{3.14}$$

for $m > 0$. Using the Littlewood–Richardson rule, we have

$$\begin{aligned} V_s^m &\cong \bigoplus_{\beta \vdash s-m} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]}) (\mathsf{S}(\beta) \boxtimes \mathsf{S}(m)) \uparrow_{\mathfrak{S}_{s-m} \times \mathfrak{S}_m}^{\mathfrak{S}_s}\\ &\cong \bigoplus_{\substack{\beta \vdash s-m\\ \mu \in P(s,\beta)}} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]})\, \mathsf{S}(\mu). \end{aligned} \tag{3.15}$$

for $m > 0$. Note that $\mu \in P(s, \beta)$ if and only if $\beta_{[n]} \in P(n, \mu)$. This follows from the fact that $\mu \in P(s, \beta)$ if and only if $\mu_i \geqslant \beta_i \geqslant \mu_{i+1}$ for all $i \geqslant 1$, the fact that $\beta_{[n]} \in P(n, \mu)$ if and only if $\mu_i \geqslant (\beta_{[n]})_{i+1} \geqslant \mu_{i+1}$ for all $i \geqslant 1$, and noting that $(\beta_{[n]})_{i+1} = \beta_i$. Thus we get

$$\begin{aligned} \bigoplus_{0<m\leqslant s} V_s^m &\cong \bigoplus_{0<m\leqslant s} \bigoplus_{\substack{\beta \vdash s-m\\ \mu \in P(s,\beta)}} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]})\, \mathsf{S}(\mu)\\ &= \bigoplus_{\substack{0<m\leqslant s\\ \mu \vdash s}} \bigoplus_{\substack{\beta \vdash s-m\\ \beta_{[n]} \in P(n,\mu)}} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]})\, \mathsf{S}(\mu)\\ &= \bigoplus_{\mu \vdash s} \bigoplus_{\substack{\beta_{[n]} \in P(n,\mu)\\ \beta_{[n]} \neq \mu_{[n]}}} g(\lambda_{[n]}, \nu_{[n]}, \beta_{[n]})\, \mathsf{S}(\mu). \end{aligned} \tag{3.16}$$

Combining this with equation (3.8) we get

$$\begin{aligned} V_s^0 &\cong \bigoplus_{\mu \vdash s} \left([\Delta_s^+(\lambda \to \nu) : \mathsf{S}(\mu)] - \sum_{0<m\leqslant s} [V_s^m : \mathsf{S}(\mu)] \right) \mathsf{S}(\mu)\\ &= \bigoplus_{\mu \vdash s} \left(\sum_{\substack{\alpha \vdash n-s\\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} g(\lambda_{[n]} \setminus \alpha, \nu_{[n]} \setminus \alpha, \mu) - \sum_{\substack{\beta_{[n]} \in P(n,\mu)\\ \beta_{[n]} \neq \mu_{[n]}}} g(\lambda_{[n]}, \mu_{[n]}, \beta_{[n]}) \right) \mathsf{S}(\mu)\\ &= \bigoplus_{\mu \vdash s} g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]})\, \mathsf{S}(\mu) \end{aligned}$$

where the last equality follows by using Dvir's recursive formula. Finally using equation (3.9) we deduce that $V_s^0 \cong \Delta_s^0(\lambda \to \nu)$ as required. □

Lemma 3.21. *Suppose that $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda, \nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ satisfies (C1). Then neither of the skew-partitions $\nu \setminus (\lambda \cap \nu)$ or $\lambda \setminus (\lambda \cap \nu)$ contains two nodes in the same column.*

Proof. For $s = 1$, the result is clear. We assume $s > 1$. We assume that one of the skew partitions $\nu \setminus (\lambda \cap \nu)$ or $\lambda \setminus (\lambda \cap \nu)$ does contain two nodes in the same column. (Recall that $\max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\} \leqslant s \leqslant |\lambda| + |\nu|$ by Remark 3.9 and our assumption that $\mathrm{Std}_s(\lambda \to \nu) \neq \emptyset$). We first assume that $s' = \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$. We let $\mathsf{u} \in \mathrm{Std}_{s'}^0(\lambda \to \nu)$ be any path of the form

$$\mathsf{u} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, -\varepsilon_{i_2}, +\varepsilon_{j_2}, \dots, -\varepsilon_{i_{s'}}, +\varepsilon_{j_{s'}}) \tag{3.17}$$

such that the nodes $-\varepsilon_{i_k}$ and $-\varepsilon_{i_{k+1}}$ (respectively $+\varepsilon_{j_k}$ and $+\varepsilon_{j_{k+1}}$) are removed (respectively added) in the same column for some $1 \leqslant k < s$. Such a pair of nodes exists by our assumption on λ and ν. Note that we can also assume that the tableau u given in equation (3.17) satisfies $i_l, j_l \neq 0$ for all $1 \leqslant l \leqslant \min\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$ (we will use this fact later in the proof). Now the sequence

$$(-\varepsilon_{i_1}, +\varepsilon_{j_1}, \dots, -\varepsilon_{i_{k+1}}, +\varepsilon_{j_{k+1}}, -\varepsilon_{i_k}, +\varepsilon_{j_k}, \dots, -\varepsilon_{i_{s'}}, +\varepsilon_{j_{s'}})$$

is not an element of $\mathrm{Std}_{s'}(\lambda \to \nu)$, and so $\mathsf{u}_{k\leftrightarrow k+1}$ does not exist. Therefore (λ, ν, s') is not a co-Pieri triple, as required.

We shall now consider larger values of $s \in \mathbb{N}$ by *inflating* the tableau in equation (3.17). For s satisfying

$$s' < s \leqslant |\lambda \setminus (\lambda \cap \nu)| + |\nu \setminus (\lambda \cap \nu)|,$$

we have $s - s' \leqslant \min\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$, so we can inflate the tableau u given in equation (3.17) to get $\bar{\mathsf{u}} \in \mathrm{Std}_s(\lambda \to \nu)$ by setting $\bar{\mathsf{u}}$ to be the tableau

$$(\underbrace{-\varepsilon_{i_1}, +\varepsilon_0, \dots, -\varepsilon_{i_{s-s'}}, +\varepsilon_0}_{2(s-s')}, \underbrace{-\varepsilon_0, +\varepsilon_{j_1}, \dots, -\varepsilon_0, +\varepsilon_{j_{s-s'}}}_{2(s-s')}, -\varepsilon_{i_{s-s'+1}}, +\varepsilon_{j_{s-s'+1}}, \dots, -\varepsilon_{i_{s'}}, +\varepsilon_{j_{s'}}) \tag{3.18}$$

if the nodes $-\varepsilon_{i_k}$ and $-\varepsilon_{i_{k+1}}$ are *removed* from the same column or $\bar{\mathsf{u}}$ to be the tableau

$$(\underbrace{-\varepsilon_0, +\varepsilon_{j_1}, \dots, -\varepsilon_0, +\varepsilon_{j_{s-s'}}}_{2(s-s')}, \underbrace{-\varepsilon_{i_1}, +\varepsilon_0, \dots, -\varepsilon_{i_{s-s'}}, +\varepsilon_0}_{2(s-s')}, -\varepsilon_{i_{s-s'+1}}, +\varepsilon_{j_{s-s'+1}}, \dots, -\varepsilon_{i_{s'}}, +\varepsilon_{j_{s'}}) \tag{3.19}$$

if the nodes $+\varepsilon_{j_k}$ and $+\varepsilon_{j_{k+1}}$ are *added* in the same column. In either case, we have that $\bar{\mathsf{u}}_{k\leftrightarrow k+1}$ does not exist, as before. Finally, assume

$$|\lambda \setminus (\lambda \cap \nu)| + |\nu \setminus (\lambda \cap \nu)| \leqslant s \leqslant |\lambda| + |\nu|.$$

We let $\lambda \cap \nu = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$. We let a denote the sequence of steps obtained from deleting the middle $t = (2|\alpha| + |\lambda \setminus (\lambda \cap \nu)| + |\nu \setminus (\lambda \cap \nu)| - s)$ integral steps from

$$\underbrace{a(1) \circ a(1) \circ \cdots a(1)}_{\alpha_1} \circ \cdots \circ \underbrace{a(\ell) \circ a(\ell) \circ \cdots a(\ell)}_{\alpha_\ell} \circ \underbrace{r(\ell) \circ r(\ell) \circ \cdots r(\ell)}_{\alpha_\ell} \circ \cdots \circ \underbrace{r(1) \circ r(1) \circ \cdots r(1)}_{\alpha_1} \tag{3.20}$$

or

$$\underbrace{a(1) \circ a(1) \circ \cdots a(1)}_{\alpha_1} \circ \cdots \circ \underbrace{a(\ell) \circ a(\ell) \circ \cdots a(\ell)}_{\alpha_\ell - 1} \circ d(\ell) \circ \underbrace{r(\ell) \circ r(\ell) \circ \cdots r(\ell)}_{\alpha_\ell - 1} \circ \cdots \circ \underbrace{r(1) \circ r(1) \circ \cdots r(1)}_{\alpha_1} \tag{3.21}$$

for t even or odd respectively. As $\alpha \subseteq \nu$ is a partition, we have that a is a standard tableau of degree $s - |\lambda \setminus (\lambda \cap \nu)| - |\nu \setminus (\lambda \cap \nu)|$ beginning and terminating at ν. Finally if $\bar{\mathsf{u}}$ is the tableau of degree $|\lambda \setminus (\lambda \cap \nu)| + |\nu \setminus (\lambda \cap \nu)|$ as in equation (3.18) or equation (3.19), then

$$\mathsf{v} = \bar{\mathsf{u}} \circ \mathsf{a} \in \mathrm{Std}^0_s(\lambda \to \nu) \text{ and } \mathsf{v}_{k\leftrightarrow k+1} \notin \mathrm{Std}^0_s(\lambda \to \nu)$$

for $1 \leqslant k \leqslant s'$ as before, as required. □

Proposition 3.22. *Let $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda, \nu) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r}$ with $\mathrm{Std}^0_s(\lambda \to \nu) \neq \emptyset$. If (λ, ν, s) satisfies (C1) and (C2), then (λ, ν, s) satisfies (coP).*

Proof. Using Lemma 3.21 we can assume that neither of the skew partitions $\nu \setminus (\lambda \cap \nu)$ or $\lambda \setminus (\lambda \cap \nu)$ contain two nodes in the same column, i.e. $\mathrm{minmax}(\lambda, \nu) \geqslant 0$.

Throughout the proof, we let $s' = \max\{|\lambda \setminus (\lambda \cap \nu)|, |\nu \setminus (\lambda \cap \nu)|\}$.

We will prove this result by contrapositive. Suppose that (λ, ν, s) does not satisfy (coP). Then $s > 1$, $\max\{\ell(\lambda), \ell(\nu)\} \geqslant 2$ and $s' + \mathrm{minmax}(\lambda, \nu) + 1 \leqslant s \leqslant |\lambda| + |\nu|$. We pick $c \geqslant 2$ minimal such that $\mathrm{minmax}(\lambda, \nu) = \min\{\lambda_{c-1}, \nu_{c-1}\} - \max\{\lambda_c, \nu_c\}$.

Case I. $\mathrm{minmax}(\lambda, \nu) = 0$. By the minimality of c we can find $\mathsf{u} \in \mathrm{Std}^0_{s'}(\lambda \to \nu)$ and $0 \leqslant k \leqslant s'$ such that the $(c-1)$th and cth rows of either $\mathsf{u}(k)$ or $\mathsf{u}(k + 1/2)$ have the same length. We choose k minimal with this property. Let $s = s' + 1$. By the minimality of c, we have that

$$\mathsf{v} = \mathsf{u}[0, k] \circ \underbrace{(-i_{k+1}, +(c-1), -(c-1), +j_{k+1})}_{\text{important}} \circ \mathsf{u}[k+1, s']$$

belongs to $\mathrm{Std}^0_s(\lambda \to \nu)$. If we swap the two important integral steps of v we obtain a sequence which does not belong to $\mathrm{Std}_s(\lambda \to \nu)$. This violates condition $(C1)$. One can inflate the tableau v as in equation (3.18) or equation (3.19) and/or by concatenating with a path of the form in equation (3.20) and (3.21) to obtain an element of $\mathrm{Std}^0_t(\lambda \to \nu)$ for any $s \leqslant t \leqslant |\lambda| + |\nu|$ which violates (C1).

Cases II and III. For the remainder of the proof we set $k = \max\{0, \lambda_{c-1} - \nu_{c-1}, \nu_c - \lambda_c\}$. We let $\mathsf{u} \in \mathrm{Std}^0_{s'}(\lambda \to \nu)$ denote any path in which all steps of the form $-\varepsilon_{c-1}$ or $+\varepsilon_c$ occur in the first k integral steps and all steps of the form $+\varepsilon_{c-1}$ or $-\varepsilon_c$ occur in the final $s' - k$ integral steps. That such a tableau exists follows from our assumption that s' is minimal such that $\mathrm{Std}_{s'}(\lambda \to \nu) \neq \emptyset$ (so no step can be added and removed in the same row).

Case II. $\mathrm{minmax}(\lambda, \nu) > 0$ and $c < \max\{\ell(\lambda), \ell(\nu)\}$. Let $s = s' + \mathrm{minmax}(\lambda, \nu) + 1$. For $\mathrm{minmax}(\lambda, \nu)$ even, we let v denote the following tableau

$$\mathsf{u}[0,k] \circ \underbrace{m{\downarrow}(c-1,c) \circ \cdots \circ m{\downarrow}(c-1,c)}_{\mathrm{minmax}(\lambda,\nu)/2-1} \circ \underbrace{d(c-1) \circ m{\downarrow}(c-1,c)}_{\text{important}} \circ \underbrace{m{\uparrow}(c,c-1) \circ \cdots \circ m{\uparrow}(c,c-1)}_{\mathrm{minmax}(\lambda,\nu)/2} \circ \mathsf{u}[k,s'].$$

We have that $\mathsf{v} \in \mathrm{Std}^0_s(\lambda \to \nu)$. For $\mathrm{minmax}(\lambda, \nu)$ odd, we let v denote the following tableau

$$\mathsf{u}[0,k] \circ \underbrace{m{\downarrow}(c-1,c) \circ \cdots \circ m{\downarrow}(c-1,c)}_{(\mathrm{minmax}(\lambda,\nu)-1)/2} \circ \underbrace{m{\uparrow}(c,c-1) \circ m{\downarrow}(c-1,c)}_{\text{important}} \circ \underbrace{m{\uparrow}(c,c-1) \circ \cdots \circ m{\uparrow}(c,c-1)}_{(\mathrm{minmax}(\lambda,\nu)-1)/2} \circ \mathsf{u}[k,s']$$

We have that $\mathsf{v} \in \mathrm{Std}^0_s(\lambda \to \nu)$. In both cases, if we swap the two important integral steps in the tableau v we obtain a sequence which does not belong to $\mathrm{Std}_s(\lambda \to \nu)$. This violates (C1). Again, we can inflate v as in Case I to get an element of $\mathrm{Std}^0_t(\lambda \to \nu)$ for any $s \leqslant t \leqslant |\lambda| + |\nu|$ which also violates (C1).

Case III. $\mathrm{minmax}(\lambda, \nu) > 0$ and $c = \max\{\ell(\lambda), \ell(\nu)\}$. For $s = s' + 2\,\mathrm{minmax}(\lambda, \nu) + 1$. We let v denote the following tableau

$$\mathsf{u}[0,k] \circ \underbrace{r(c-1) \circ \cdots \circ r(c-1)}_{\mathrm{minmax}(\lambda,\nu)-1} \circ \underbrace{d(c-1) \circ r(c-1)}_{\text{important}} \circ \underbrace{a(c-1) \circ \cdots \circ a(c-1)}_{\mathrm{minmax}(\lambda,\nu)} \circ \mathsf{u}[k,s'].$$

We have that $\mathsf{v} \in \mathrm{Std}^0_s(\lambda \to \nu)$. If we swap the two important integral steps in the tableau v, we obtain a sequence which does not belong to $\mathrm{Std}_s(\lambda \to \nu)$. This violates condition $(C1)$. Moreover we can inflate v as in Case I to show that (λ, ν, s) does not satisfy (C1) for any $s' + 2\,\mathrm{minmax}(\lambda, \nu) + 1 \leqslant s \leqslant |\lambda| + |\nu|$.

It remains to consider the case $s' + \mathrm{minmax}(\lambda, \nu) + 1 \leqslant s \leqslant s' + 2\,\mathrm{minmax}(\lambda, \nu)$. We will show that (λ, ν, s) does not satisfy (C2). We begin with the case $s = s' + \mathrm{minmax}(\lambda, \nu) + 1$. We shall see that the map of equation (3.5) is well-defined and injective, but no longer surjective.

Let $\alpha \subset \lambda_{[n]} \cap \nu_{[n]}$ with $\alpha \vdash n - s$. Let $\mathsf{s} \in \mathrm{Std}_s(\alpha \to \nu_{[n]})$ and $\mathsf{t} \in \mathrm{Std}_s(\lambda_{[n]} \to \alpha)$ and write $\mathsf{v} = \varphi_s(\mathsf{s}, \mathsf{t})$ defined as in equation (3.5). We need to show that $\mathsf{v} \in \mathrm{Std}_s(\lambda \to \nu)$. Using the same notation as in Lemma 4.19, following its proof, and using the fact that $s \leqslant s' + \min\{\lambda_{i-1}, \nu_{i-1}\} - \max\{\lambda_i, \nu_i\}$ for all $i \neq c$ (by minimality of c) we obtain that $\alpha_i \geqslant \beta_{i+1}$ for all $i \neq c$ and $\alpha_c \geqslant \beta_{c+1}$. Now following the proof of Lemma 4.20 this implies that $\mathsf{v}(l)_{i-1} \geqslant \mathsf{v}(l)_i$ for all $i \neq c$ and $\mathsf{v}(l)_{c-1} \geqslant \mathsf{v}(l)_c - 1$ for all $1 \leqslant l \leqslant s$. Now suppose, for a contradiction that $\mathsf{v}(k)_{c-1} = \mathsf{v}(k)_c - 1$. Then we must have $\mathsf{v}(k)_{c-1} = \alpha_c = \mathsf{s}(k)_c$ and $\mathsf{v}(l)_c = \beta_{c+1} = \mathsf{s}(l)_{c+1}$, contradicting the fact that s is a standard tableau. Thus φ_s is well-defined. Injectivity is obvious by definition.We now show that there is some $\bar{\mathsf{u}} \in \mathrm{Std}^0_s(\lambda \to \nu)$ which is not in the image of φ_s. Recall, we picked $\mathsf{u} \in \mathrm{Std}^0_{s'}(\lambda \to \nu)$ such that all steps of the form $-\varepsilon_{c-1}$ or $+\varepsilon_c$ occur in the first k integral steps and all steps of the form $+\varepsilon_{c-1}$ or $-\varepsilon_c$ occur in the final $s' - k$ integral steps; this ensures that $\mathsf{u}(k)_{c-1} - \mathsf{u}(k)_c = \mathrm{minmax}(\lambda, \nu)$. Now consider the tableau

$$\bar{\mathsf{u}} = \mathsf{u}[0,k] \circ \underbrace{d(c-1) \circ \cdots \circ d(c-1)}_{\mathrm{minmax}(\lambda,\nu)+1} \circ \mathsf{u}[k,s']$$

which belongs to $\mathrm{Std}^0_s(\lambda \to \nu)$. Suppose for a contradiction that $\bar{\mathsf{u}} = \varphi_s(\mathsf{s}, \mathsf{t})$ for some standard tableaux s and t. Now if $\lambda_c = \nu_c$ then $\alpha = \min_n(\mathsf{u})$ is not a partition so u cannot be in the image of φ_s. If $\lambda_c > \nu_c$ then $\mathsf{t}(k + \mathrm{minmax}(\lambda, \nu) + 1)$ is not a partition and if $\nu_c > \lambda_c$ then $\mathsf{s}(k)$ is not a partition. In all cases we see that $\bar{\mathsf{u}} \in \mathrm{Std}^0_s(\lambda \to \nu)$ is not in the image of φ_s. Now we can decompose

$$\bigoplus_{\substack{\alpha \vdash n-s \\ \alpha \subseteq \lambda_{[n]} \cap \nu_{[n]}}} \Delta_s(\alpha \to \nu_{[n]}) \otimes \Delta_s(\lambda_{[n]} \to \alpha) = \bigoplus_{0 \leqslant m \leqslant s} V_s^m$$

as in (4.11). The fact that the map φ_s is not surjective implies that $|\mathrm{Std}_s^0(\lambda \to \nu)| > \dim V_s^0$. Now if we follow (4.16) – (4.22), noting that $(\lambda, \nu, s-m)$ satisfies (coP) for $m > 0$, we obtain

$$|\mathrm{Std}_s^0(\lambda \to \nu)| > \dim V_s^0 = \dim \Delta_s^0(\lambda \to \nu).$$

This implies that (C2) is not satisfied, as required.

More precisely, we know that there must be some element $\sum_{\mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu)} r_{\mathsf{t}} u_{\mathsf{t}} \in \mathsf{DR}_s(\lambda \to \nu)$ for $r_{\mathsf{t}} \in \mathbb{Q}$.

We now consider (λ, ν, s) for $s' + \mathrm{minmax}(\lambda, \nu) + 1 + k = s \leqslant s' + 2\,\mathrm{minmax}(\lambda, \nu)$. Let $\nu' = \nu - k\varepsilon_{c-1}$ and $\lambda' = \lambda - k\varepsilon_{c-1}$. Notice that $s - k = \max\{|\lambda' \setminus (\lambda' \cap \nu')|, |\nu' \setminus (\lambda' \cap \nu')|\} + \mathrm{minmax}(\lambda', \nu') + 1$. By the above, there exists $a \in P_{s-k}(n) p_{r-k} P_{s-k}(n)$ and $\mathsf{s} \in \mathrm{Std}_{s-k}^0(\lambda' \to \nu')$ such that

$$u_{\mathsf{s}} a = \sum_{\mathsf{t} \in \mathrm{Std}_{s-k}^0(\lambda' \to \nu')} r_{\mathsf{t}} u_{\mathsf{t}} \in \mathsf{DR}_{s-k}(\lambda' \to \nu')$$

with some $r_{\mathsf{t}} \neq 0$. Now, for any tableau $\mathsf{v} \in \mathrm{Std}_{s-k}(\lambda' \to \nu')$ we can inflate the tableau v to obtain

$$\bar{\mathsf{v}} = \underbrace{r(c-1) \circ \cdots \circ r(c-1)}_{k} \circ \mathsf{v} \circ \underbrace{a(c-1) \circ \cdots \circ a(c-1)}_{k} \in \mathrm{Std}_{s+k}^0(\lambda \to \nu).$$

Similarly, given $a \in P_{s-k}(n)$ we let $\bar{a} \in P_{s+k}(n)$ denote the image of a under the embedding $P_{s-k}(n) \to P_k(n) \times P_{s-k}(n) \times P_k(n)$. By [5, Corollary 3.12] we have that

$$u_{\bar{\mathsf{s}}} \bar{a} = \sum_{\mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu')} r_{\mathsf{t}} u_{\bar{\mathsf{t}}} + \sum_{\substack{\mathsf{w} \in \mathrm{Std}_{s+k}(\lambda \to \nu) \\ \mathsf{w}(s) \rhd \nu'}} q_{\mathsf{w}} u_{\mathsf{w}} \in \mathsf{DR}_{s+k}(\lambda \to \nu)$$

which again violates $(C2)$. This completes the proof. □

4. Semistandard Kronecker tableaux

Recall from equation (3.2) that for any $(\lambda, \nu, s) \in \mathscr{P}_{r-s} \times \mathscr{P}_{\leqslant r} \times \mathbb{Z}_{>0}$ and any $\mu \vdash s$ we have

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{P_s(n)}(\Delta_s(\mu), \Delta_s^0(\lambda \to \nu)) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{Q}\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\lambda \to \nu)),$$

where $\mathbb{Q}\mathfrak{S}_s$ is viewed as the quotient of $P_s(n)$ by the ideal generated by p_r. Now for each $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash s$ we have an associated Young permutation module,

$$\mathsf{M}(\mu) = \mathbb{Q} \otimes_{\mathfrak{S}_\mu} \mathbb{Q}\mathfrak{S}_s$$

where $\mathfrak{S}_\mu = \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_l} \subseteq \mathfrak{S}_s$. It is well known that there is a surjective homomorphism $\mathsf{M}(\mu) \to \mathsf{S}(\mu)$. Moreover, for any $\tau \vdash s$, the multiplicity of $\mathsf{S}(\tau)$ as a composition factor of $\mathsf{M}(\mu)$ is given by the number of semistandard Young tableaux of shape τ and weight μ. So, as a first step towards understanding the stable Kronecker coefficients, it is natural to consider

$$\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\lambda \to \nu)).$$

In the case of triples of maximal depth, this dimension is given by the number of semistandard Young tableaux of shape $\lambda \to \nu$ and weight μ. We now extend this result by defining semistandard Kronecker tableaux and show that, in the case of co-Pieri triples, the number of such tableaux give the required dimension. In fact, we explicitly construct these homomorphisms directly from the associated tableaux.

4.1. **Semistandard Kronecker tableaux for co-Pieri triples.** We start with a definition of semistandard Kronecker tableaux, generalising the classical definition of semistandard Young tableaux.

Definition 4.1. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vDash s$, $\lambda \in \mathscr{P}_{r-s}$, $\nu \in \mathscr{P}_{\leqslant r}$ and let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu)$.

(1) For $1 \leqslant k < s$ we write $\mathsf{s} \overset{k}{\sim} \mathsf{t}$ if $\mathsf{s} = \mathsf{t}_{k \leftrightarrow k+1}$.
(2) We write $\mathsf{s} \overset{\mu}{\sim} \mathsf{t}$ if there exists a sequence of standard Kronecker tableaux $\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_d \in \mathrm{Std}_s^0(\lambda \to \nu)$ such that

$$\mathsf{s} = \mathsf{t}_1 \overset{k_1}{\sim} \mathsf{t}_2,\ \mathsf{t}_2 \overset{k_2}{\sim} \mathsf{t}_3,\ \ldots\ , \mathsf{t}_{d-1} \overset{k_{d-1}}{\sim} \mathsf{t}_d = \mathsf{t}$$

for some $k_1, \ldots, k_{d-1} \in \{1, \ldots, s-1\} \setminus \{|\mu|_c \mid c = 1, \ldots, l-1\}$. We define a tableau of weight μ to be an equivalence class of tableau under $\overset{\mu}{\sim}$, denoted $[\mathsf{t}]_\mu = \{\mathsf{s} \in \mathrm{Std}_s^0(\lambda \to \nu) \mid \mathsf{s} \overset{\mu}{\sim} \mathsf{t}\}$.

(3) We say that a Kronecker tableau, $[\mathsf{t}]_\mu$, of shape $\lambda \to \nu$ and weight μ is semistandard if, for all $1 \leqslant c \leqslant l$, the skew partitions $\mathsf{t}(|\mu|_c) \setminus (\mathsf{t}(|\mu|_{c-1}) \cap \mathsf{t}(|\mu|_c))$ and $\mathsf{t}(|\mu|_{c-1}) \setminus (\mathsf{t}(|\mu|_{c-1}) \cap \mathsf{t}(|\mu|_c))$ have no two boxes in the same column.
We denote the set of all semistandard Kronecker tableaux of shape $\lambda \to \nu$ and weight μ by $\mathrm{SStd}_s^0(\lambda \to \nu, \mu)$.

Remark 4.2. Note that if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu)$ with $\mathsf{s} \in [\mathsf{t}]_\mu$ then $\mathsf{s}(|\mu|_c) = \mathsf{t}(|\mu|_c)$ for all $1 \leqslant c \leqslant l$. So part (3) is independent of the choice of representative in $[\mathsf{t}]_\mu$ and hence the notion of semistandard Kronecker tableau is well-defined.

Remark 4.3. If (λ, ν, μ) is a co-Pieri triple, it follows from Lemma 3.18 that for any $\mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu)$ the class $[\mathsf{t}]_\mu$ is a semistandard Kronecker tableau.

Remark 4.4. If (λ, ν, μ) is a triple of maximal depth then $\mathrm{SStd}_s^0(\lambda \to \nu, \mu)$ coincide with the classical notion of semistandard Young tableaux of shape $\lambda \to \nu$ and weight μ (and similarly for the non-semistandard tableaux of a given weight).

To represent these semistandard Kronecker tableaux graphically, we will add 'frames' corresponding to the composition μ on the set of paths $\mathrm{Std}_s^0(\lambda \to \nu)$ in the branching graph. For $\mathsf{t} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, \ldots, -\varepsilon_{i_s}, +\varepsilon_{j_s})$ we say that the integral step $(-\varepsilon_{i_k}, +\varepsilon_{j_k})$ belongs to the cth frame if $|\mu|_{c-1} < k \leqslant |\mu|_c$. Thus for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_s^0(\lambda \to \nu)$ we have that $\mathsf{s} \overset{\mu}{\sim} \mathsf{t}$ if and only if s is obtained from t by permuting integral steps within each frame.

Example 4.5. Let $\lambda = (4,2)$, $\nu = (5,3,1)$ and $s = 3$. Then (λ, ν, s) is a triple of maximal depth. Take $\mu = (2,1) \vDash 3$. We have three semistandard tableaux of shape $\lambda \to \nu$ and weight μ given by

$$\begin{aligned}
\mathsf{S}_1 &= \{a(2) \circ a(3) \circ a(1)\ ,\ a(3) \circ a(2) \circ a(1)\} \\
\mathsf{S}_2 &= \{a(1) \circ a(3) \circ a(2)\ ,\ a(3) \circ a(1) \circ a(2)\} \\
\mathsf{S}_3 &= \{a(1) \circ a(2) \circ a(3)\ ,\ a(2) \circ a(1) \circ a(3)\}.
\end{aligned}$$

They are depicted in Figure 15 and ordered so that one can compare them directly with the tableaux in Example 1.8.

Example 4.6. Let $\lambda = (7)$, $\nu = (6)$ and $s = 6$. Then $(\lambda, \nu, 6)$ is a co-Pieri triple. We have $|\mathrm{SStd}_6^0(\lambda \to \nu, (6))| = 3$ and a representative for each of these semistandard tableaux is given by

$$\begin{aligned}
\mathsf{t}_1 &= r(1) \circ r(1) \circ r(1) \circ d(1) \circ a(1) \circ a(1) \\
\mathsf{t}_2 &= r(1) \circ r(1) \circ d(1) \circ d(1) \circ d(1) \circ a(1) \\
\mathsf{t}_3 &= r(1) \circ d(1) \circ d(1) \circ d(1) \circ d(1) \circ d(1)
\end{aligned}$$

We have $|\mathrm{SStd}_6^0(\lambda \to \nu, (3,2,1))| = 27$. To see this, observe that $[\mathsf{t}_1]_{(6)}$ and $[\mathsf{t}_2]_{(6)}$ each splits into 12 semistandard Kronecker tableaux of weight $(3,2,1)$, whereas $[\mathsf{t}_3]_{(6)}$ splits into 3 semistandard Kronecker tableaux of weight $(3,2,1)$.

Theorem 4.7. *Let (λ, ν, s) be a co-Pieri triple and $\mu \vdash s$. Then we define*

$$\varphi_{\mathsf{T}}(u_{\mathsf{t}^\mu}) = \sum_{\mathsf{s} \in \mathsf{T}} u_{\mathsf{s}}.$$

for $\mathsf{T} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)$. We have that

$$\{\varphi_{\mathsf{T}} \mid \mathsf{T} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)\}$$

is a $\mathbb{Z}$-basis for $\mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\lambda \to \nu))$.

Proof. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\lambda \to \nu)) \cong \mathrm{Hom}_{\mathfrak{S}_\mu}(\mathbb{Q}, \Delta_s^0(\lambda \to \nu)\downarrow_{\mathfrak{S}_\mu})$$

It is clear from equation (3.4) and Remarks 4.2 and 4.3 that $\Delta_s^0(\lambda \to \nu)\downarrow_{\mathfrak{S}_\mu}$ decomposes as

$$\Delta_s^0(\lambda \to \nu)\downarrow_{\mathfrak{S}_\mu} = \bigoplus_{\mathsf{T} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)} V(\mathsf{T})$$

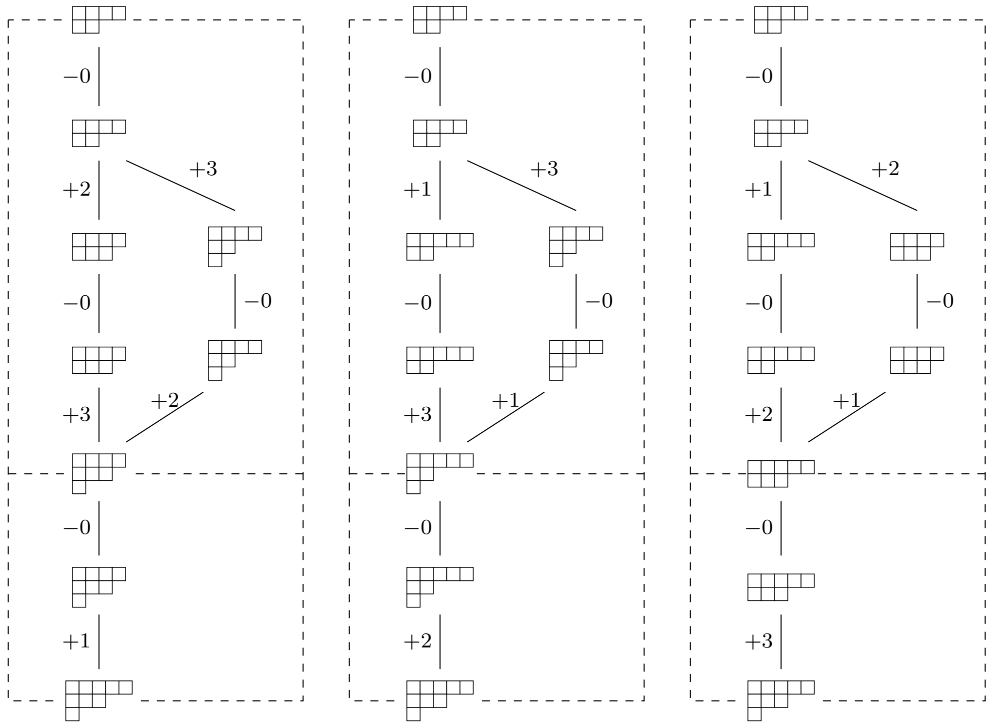


FIGURE 15. The three elements of $\mathrm{SStd}_3^0((4,2) \to (5,3,1),(2,1))$. These tableaux are ordered to facilitate comparison with Figures 4 and 7.

where $V(\mathsf{T}) = \mathrm{Span}_{\mathbb{Q}}\{m_{\mathsf{t}} + \mathsf{DR}(\lambda \to \nu) \mid [\mathsf{t}]_\mu = \mathsf{T}\}$. Moreover, each $V(\mathsf{T})$ is itself a permutation module of the form $\mathbb{Q}\uparrow_{\mathfrak{S}_\tau}^{\mathfrak{S}_\mu}$ for some composition $\tau \vDash s$ which is a refinement of μ. Thus we have that $\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_\mu}(\mathbb{Q}, V(\mathsf{T})) = 1$ for each $\mathsf{T} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)$ and the result follows. □

Example 4.8. Let $\lambda = (8,5,3)$, $\nu = (6,5,3,2)$ and $s = 3$. Then $(\lambda, \nu, 3)$ is a co-Pieri triple. We have that $|\mathrm{SStd}_3^0(\lambda \to \nu, (3))| = 6$. A representative for each of these semistandard tableaux is given as follows,

$$\begin{array}{ccc} d(1)\circ m{\downarrow}(1,4)\circ m{\downarrow}(1,4) & d(2)\circ m{\downarrow}(1,4)\circ m{\downarrow}(1,4) & m{\downarrow}(1,2)\circ m{\downarrow}(1,4)\circ m{\downarrow}(2,4) \\ d(3)\circ m{\downarrow}(1,4)\circ m{\downarrow}(1,4) & r(1)\circ m{\downarrow}(1,4)\circ a(4) & m{\downarrow}(1,3)\circ m{\downarrow}(1,4)\circ m{\downarrow}(3,4). \end{array}$$

The semistandard tableau corresponding to the first of these tableaux is depicted in Figure 16. We have that $|\mathrm{SStd}_3^0(\lambda \to \nu, (2,1))| = 15$. Two examples of such tableaux are depicted in Figure 16.

5. LATTICED KRONECKER TABLEAUX

In this section we prove the main result of the paper, namely we find a combinatorial description for

$$\overline{g}(\lambda, \nu, \mu) = \dim \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\lambda \to \nu))$$

for all co-Pieri triples (λ, ν, s) and all $\mu \vdash s$ which naturally extends the Littlewood–Richardson rule.

In the previous section we saw that the semistandard Kronecker tableaux of shape $\lambda \to \nu$ and weight μ index a basis for $\mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta_s^0(\lambda \to \nu))$. We will now find which of these index a basis for $\mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\lambda \to \nu))$. We follow James' approach [20] and extend his notion of latticed semistandard tableaux.

We start with any standard tableau $\mathsf{s} \in \mathrm{Std}_s^0(\lambda \to \nu)$ and any $\mu = (\mu_1, \mu_2, \dots, \mu_l) \vDash s$. Write

$$\mathsf{s} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, -\varepsilon_{i_2}, +\varepsilon_{j_2}, \dots, -\varepsilon_{i_s}, +\varepsilon_{j_s}).$$

Recall from the previous section that, to each integral step $(-\varepsilon_{i_k}, +\varepsilon_{j_k})$ in s, we associate its frame c, that is the unique positive integer such that

$$|\mu|_{c-1} < k \leqslant |\mu|_c.$$

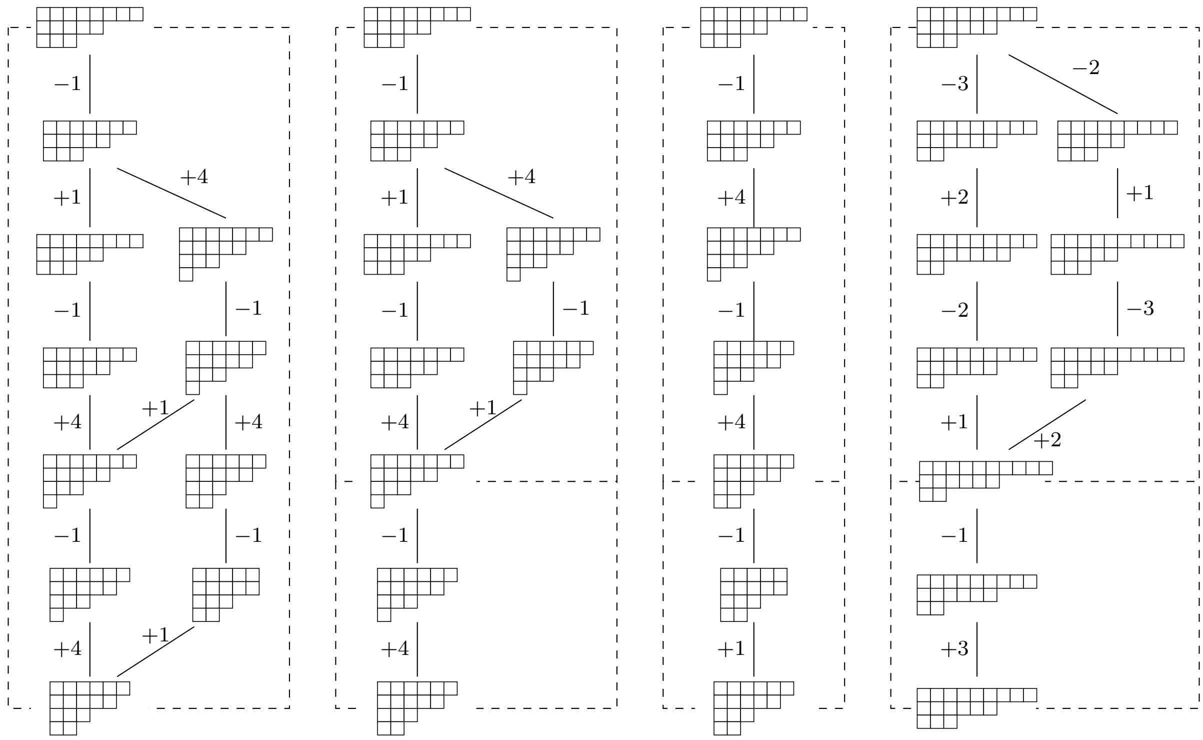


FIGURE 16. Three semistandard Kronecker tableaux of shape $(8,5,3) \to (6,5,3,2)$ and one of shape $(9,6,3) \to (9,6,3)$. The leftmost is of weight (3) and the latter three are of weight $(2,1)$.

Now we encode the integral steps of s and their frames in a $2 \times s$ array, which we will denote by $\omega_\mu(\mathsf{s})$, and refer to as the μ-reverse reading word of s as follows. The first row of $\omega_\mu(\mathsf{s})$ contains all the integral steps of s and the second row contains their corresponding frames. We order the columns of $\omega_\mu(\mathsf{s})$ increasingly using the ordering on integral steps given in Definition 2.5 (and we place a vertical lines between any two integral steps which are not equal). For two equal integral steps, we order the columns so that the frame numbers are weakly decreasing (and so between any two vertical lines, the entries in in the second row are weakly decreasing).

Note that if $\mathsf{t} \in [\mathsf{s}]_\mu$ then $\omega_\mu(\mathsf{t}) = \omega_\mu(\mathsf{s})$. So it makes sense to define the reverse reading word, $\omega(\mathsf{S})$, of a semistandard Kronecker tableau $\mathsf{S} \in \mathrm{SStd}^0_s(\lambda \to \nu, \mu)$ to be the $2 \times s$ array $\omega(\mathsf{S}) := \omega_\mu(\mathsf{s})$ for any $\mathsf{s} \in \mathsf{S}$. We will write $\omega(\mathsf{S}) = (\omega_1(\mathsf{S}), \omega_2(\mathsf{S}))$ where $\omega_1(\mathsf{S})$ (respectively $\omega_2(\mathsf{S})$) is the first (respectively second) row of the array $\omega(\mathsf{S})$. Note that $\omega_2(\mathsf{S})$ is a sequence of type μ, that is a sequence of positive integers such that i appears precisely μ_i times, for all $i \geqslant 1$.

Example 5.1. We begin with an example of a triple of maximal depth. Let $\nu = (9,8,6,3)$, $\lambda = (6,4,3)$ and $s = 13$. Let $\mathsf{s} \in \mathrm{Std}^0_s(\lambda \to \nu)$ be the path

$$a(1) \circ a(1) \circ a(4) \circ a(4) \circ a(4) \circ a(1) \circ a(2) \circ a(2) \circ a(2) \circ a(3) \circ a(2) \circ a(3) \circ a(3).$$

Let $\mu = (5,5,3)$, then in classical notation, the semistandard tableau $\mathsf{S} = [\mathsf{s}]_\mu$ is the leftmost tableaux depicted in Figure 5. The reverse reading word of S is as follows:

$$\left(\begin{array}{ccc|cccc|ccc|ccc} a(1) & a(1) & a(1) & a(2) & a(2) & a(2) & a(2) & a(3) & a(3) & a(3) & a(4) & a(4) & a(4) \\ 2 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \end{array}\right).$$

Compare the second row of the above array with the corresponding word given in Examples 1.3 and 1.5.

Remark 5.2. Let (λ, ν, μ) be of maximal depth and $\mathsf{S} \in \mathrm{SStd}^0_s(\lambda \to \nu, \mu)$. The second row of the reverse reading word of S coincides with the classical reverse reading word given in Definition 1.4.

Definition 5.3. Given a finite sequence of positive integers we define the quality (good/bad) of each term as follows.

(1) All 1's are good.
(2) An $i+1$ is good if and only if the number of previous good i's is strictly greater than the number of previous good $i+1$'s.

A sequence of positive integers is called a lattice word if every term in the sequence is good.

Definition 5.4. For $\mathsf{S} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)$ we say that its reverse reading word $\omega(\mathsf{S})$ is a lattice word if $\omega_2(\mathsf{S})$ is a lattice word. We define $\mathrm{Latt}_s^0(\lambda \to \nu, \mu)$ to be the set of all $\mathsf{S} \in \mathrm{SStd}_s^0(\lambda \to \nu, \mu)$ such that $\omega(\mathsf{S})$ is a lattice word.

Example 5.5. Continuing from Example 5.1, the quality of each term (or step) in the reverse reading word of S is as follows

$$\left(\begin{array}{ccc|cccc|ccc|ccc} a(1) & a(1) & a(1) & a(2) & a(2) & a(2) & a(2) & a(3) & a(3) & a(3) & a(4) & a(4) & a(4) \\ \times & \checkmark & \checkmark & \times & \checkmark & \checkmark & \times & \checkmark & \checkmark & \times & \checkmark & \checkmark & \checkmark \\ 2 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 & 2 & 1 & 1 & 1 \end{array}\right).$$

We have indicated good steps with a $\checkmark$ and each bad step with a $\times$. We see that $\mathsf{S} \notin \mathrm{Latt}_s^0((6,4,3) \to (9,8,6,3)(5,5,3))$.

Example 5.6. Of the three semistandard Kronecker tableaux depicted in Figure 15, the reverse reading words of the final two are lattice words, whereas the first one is not.

Example 5.7. Of the two elements of $\mathrm{SStd}_3((8,5,3) \to (6,5,3,2),(2,1))$ depicted in Figure 16, the reverse reading word of the former is a lattice word, whereas the latter is not.

Example 5.8. We continue with Example 4.6. So we take $\lambda = (7), \nu = (6)$ and $s = 6$. Let $\mathsf{S} \in \mathrm{SStd}_6^0(\lambda \to \nu, \mu)$ for any $\mu \vdash 6$. Then $\omega_1(\mathsf{S})$ must be one of the following

$$(r(1)\, r(1)\, r(1)\, d(1)\, a(1)\, a(1)), \quad (r(1)\, r(1)\, d(1)\, d(1)\, d(1)\, a(1)) \quad \text{or} \quad (r(1)\, d(1)\, d(1)\, d(1)\, d(1)\, d(1)).$$

It is easy to check that for $\mu = (3,2,1)$ we have $\mathsf{S} \in \mathrm{Latt}_6^0(\lambda \to \nu, \mu)$ if and only if $\omega(\mathsf{S})$ is one of the following

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 2 & 1 & 3 \end{array}\right).$$

Thus $|\mathrm{Latt}_6^0(\lambda \to \nu, \mu)| = 2$. Similarly, for $\tau = (4,2)$ we have that $\mathsf{S} \in \mathrm{Latt}_6^0(\lambda \to \nu, \tau)$ if and only if $\omega(\mathsf{S})$ is one of the following

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}\right) \quad , \quad \left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 1 \end{array}\right),$$
$$\left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 1 & 1 & 2 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 2 & 1 & 1 \end{array}\right).$$

So we get $|\mathrm{Latt}_6^0(\lambda \to \nu, \tau)| = 4$.

Theorem 5.9. *For any co-Pieri triple (λ, ν, s) and any $\mu \vdash s$ we have that*

$$\overline{g}(\lambda, \nu, \mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\lambda \to \nu)) = |\mathrm{Latt}_s^0(\lambda \to \nu, \mu)|.$$

In the rest of this section we will prove this result. The main technique we will use is James' pairs of partitions method which describes how to 'turn bad steps into good ones'.

Definition 5.10. Let $\mu \vDash s$ and let $\mu^\sharp \in \mathscr{P}_{\leqslant s}$ be such that $\mu_c^\sharp \leqslant \mu_c$, for all $c \geqslant 1$. Then $(\mu^\sharp, \mu)$ is called a pair of partitions for s.

We record a pair of partitions diagrammatically by drawing the Young diagram for μ and filling all boxes corresponding to $\mu^\sharp$ with a $\times$, for example we have that $(\mu^\sharp, \mu) = ((2^2,1),(2^4))$ is represented as in the leftmost diagram in Figure 17.

Definition 5.11. Let $(\mu^\sharp, \mu)$ be a pair of partitions of s. We denote by $s(\mu)$ the set of all sequences of type μ and by $s(\mu^\sharp, \mu) \subseteq s(\mu)$ the set of all sequences of type μ having at least $\mu_i^\sharp$ good i's for all i.

By definition we have

$$s(\emptyset,\mu)=s((\mu_1),\mu)=s(\mu)$$

and if $\tau^\sharp\subseteq\mu^\sharp$ then

$$s(\mu^\sharp,\mu)\subseteq s(\tau^\sharp,\mu).$$

Definition 5.12. Let $(\mu,\mu^\sharp)$ be a pair of partitions for s and let $1<c\leqslant\ell(\mu)$ be the smallest integer such that $\mu^\sharp_c<\mu_c$.

- We let $r_c^{\mu_c-\mu^\sharp_c}(\mu)$ denote the composition of s obtained by removing the $\mu_c-\mu^\sharp_c$ boxes at the end of row c and adding them at the end of row $c-1$.
- We let $a_c(\mu^\sharp)$ denote the partition obtained by adding a single box to the end of $\mu^\sharp_c$ if the result is a partition. If the result is not a partition, then set $(\mu,a_c(\mu^\sharp))=(\varnothing,\varnothing)$.

Example 5.13. For example, let $(\mu^\sharp,\mu)=((2^2,1),(2^4))$. Some of the pairs of partitions obtained by applying the moves in Definition 5.12 to $(\mu^\sharp,\mu)$ are depicted in Figure 17.

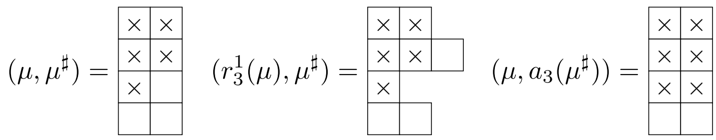


FIGURE 17. Examples for Definition 5.12

Definition 5.14. Fix $\mu\vdash s$ and define a plane binary tree $\mathscr{T}(\mu)=(V(\mathscr{T}(\mu)),E)$ with vertices labelled by pairs of partitions. Its root is labelled by $((\mu_1),\mu)$ and given a vertex, $(\tau^\sharp,\tau)\in V(\mathscr{T}(\mu))$ its descendants are labelled by the pairs of partitions

$$(\tau^\sharp,r_c^{\tau_c-\tau^\sharp_c}(\tau))\qquad\text{and}\qquad(a_c(\tau^\sharp),\tau)$$

where $1<c\leqslant\ell(\mu)$ is minimal such that and $\tau^\sharp_c<\tau_c$. If there is no such $1<c\leqslant\ell(\mu)$, then $\tau^\sharp=\tau$ and (τ,τ) is a terminal vertex (also called a leaf). Note that we identify the labels $(\tau^\sharp,\tau)$ and $(\eta^\sharp,\tau)$ if $\tau^\sharp$ and $\eta^\sharp$ only differ in the first row and usually choose to write the label $(\tau^\sharp,\tau)$ with $\tau^\sharp_1=\tau_1$.

We decorate the edges of the tree with the appropriate operators, r_c^k and a_c, for $k\geqslant1$ and $c\geqslant2$. We let $V_T(\mathscr{T}(\mu))$ denote the set of terminal vertices in $V(\mathscr{T}(\mu))$ which are not labeled by pairs of the form $(\varnothing,\varnothing)$. Given $\mathsf{t}\in V_T(\mathscr{T}(\mu))$, we associate the ordered sequence of operators, r_t, labelling the edges in the path from the root to the vertex t. A pair of partitions (τ,τ) will not (in general) label a unique terminal vertex (see for example, Figure 18).

Example 5.15. The tree $\mathscr{T}(\mu)$ for $\mu=(3,2,1)$ is given in Figure 18. There are 8 vertices in $V_T(\mathscr{T}(\mu))$. The rightmost terminal vertex is labelled by $((6),(6))$ and it correspondes to the path $r_2r_3r_2^2=r_2^1\circ r_3^1\circ r_2^2$. Note that we write the composition of operators from right to left and write r_j for r_j^1 to simplify the notation. The sequences of operators labelling terminal vertices are as follows,

$$a_3a_2a_2\quad a_2r_3a_2a_2\quad r_2r_3a_2a_2\quad a_3r_2a_2\quad a_2r_3r_2a_2\quad r_2r_3r_2a_2\quad a_2r_3r_2^2\quad r_2r_3r_2^2$$

where each of the paths above can be identified from left to right with the terminal nodes in the graph in Figure 18.

James proved the following result, see [21, Theorem 15.14].

Theorem 5.16. *Let $(\mu^\sharp,\mu)$ be a pair of partitions of s and let $c>1$ be minimal such that $\mu^\sharp_c<\mu_c$. There is a bijection*

$$R_c\,:\,s(\mu^\sharp,\mu)\setminus s((a_c(\mu^\sharp),\mu))\to s(\mu^\sharp,r_c^{\mu_c-\mu^\sharp_c}(\mu))$$

defined by changing all bad c's into $c-1$'s.

The next lemma shows that we can extend this bijection to sets of semistandard Kronecker tableaux for co-Pieri triples. The corresponding result for triples of maximal depth is given in [21](16.3 Lemma).

Define $\mathrm{SStd}^0_s(\lambda\to\nu,(\mu^\sharp,\mu))\subseteq\mathrm{SStd}^0_s(\lambda\to\nu,\mu)$ to be the subset of all semistandard Kronecker tableaux S whose reverse reading word satisfies $\omega_2(\mathsf{S})\in s(\mu^\sharp,\mu)$.

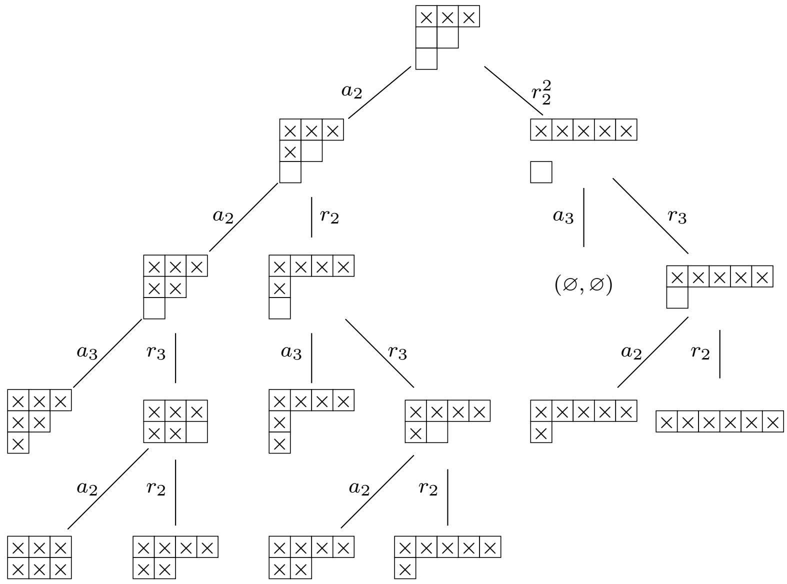


FIGURE 18. The tree, $\mathscr{T}(\mu)$, for $\mu = (3, 2, 1)$.

Lemma 5.17. *Let (λ, ν, s) be a co-Pieri triple and let $(\mu^\sharp, \mu)$ be a pair of partitions of s. Take $c > 1$ to be minimal such that $\mu^\sharp_c < \mu_c$. The map*

$$\mathscr{R}_c : \mathrm{SStd}^0_s(\lambda \to \nu, (\mu^\sharp, \mu)) \setminus \mathrm{SStd}^0_s(\lambda \to \nu, (a_c(\mu^\sharp), \mu)) \to \mathrm{SStd}^0_s(\lambda \to \nu, (\mu^\sharp, r_c^{\mu_c - \mu^\sharp_c}(\mu))$$

defined by taking

$$\omega_1(\mathscr{R}_c(\mathsf{S})) = \omega_1(\mathsf{S}) \quad \textit{and} \quad \omega_2(\mathscr{R}_c(\mathsf{S})) = R_c(\omega_2(\mathsf{S}))$$

for all $\mathsf{S} \in \mathrm{SStd}^0_s(\lambda \to \nu, (\mu^\sharp, \mu)) \setminus \mathrm{SStd}^0_s(\lambda \to \nu, (a_c(\mu^\sharp), \mu))$ (where the map R_c is given in Theorem 5.16) is a bijection.

Proof. Note that each semistandard Kronecker tableau S is completely determined by the multisets $X_i(\mathsf{S})$ containing the integral steps in frame i for each i. Hence, the reverse reading word $\omega(\mathsf{S})$ completely determines S. Moreover, as $\mathsf{t}_{k \leftrightarrow k+1} \in \mathrm{Std}^0_s(\lambda \to \nu)$ for all $\mathsf{t} \in \mathrm{Std}^0_s(\lambda \to \nu)$ and all k, if we move some integral steps from one frame of S to another, the result will still be a semistandard tableau of the same shape and the appropriate weight. So, using Theorem 5.16, the only thing we need to prove the bijection is that $(\omega_1(\mathsf{S}), R_c(\omega_2(\mathsf{S}))$ is the reverse reading word of a semistandard tableau if and only if so is $(\omega_1(\mathsf{S}), \omega_2(\mathsf{S}))$.

Write $\omega_1(\mathsf{S}) = (x_1, x_2, \ldots, x_s)$ where the x_i's are integral steps, $\omega_2(\mathsf{S}) = (u_1, u_2, \ldots, u_s)$ and $R_c(\omega_2(\mathsf{S})) = (v_1, v_2, \ldots, v_s)$. We need to show that for $x_j = x_{j+1}$ we have $u_j \geqslant u_{j+1}$ if and only if $v_j \geqslant v_{j+1}$. Assume first that $u_j \geqslant u_{j+1}$ and $v_j < v_{j+1}$. By definition of the map R_c we must have $u_j = u_{j+1} = c$, $v_j = c - 1$ and $v_{j+1} = c$. This means that u_j is a bad c and u_{j+1} is a good c but this is impossible by Definition 5.3.

Conversely, assume that $v_j \geqslant v_{j+1}$ and $u_j < u_{j+1}$. By definition of R_c we must have $u_j = c - 1$, $u_{j+1} = c$ and $v_j = v_{j-1} = c - 1$. This means that u_{j+1} is a bad c but it is preceeded by $u_j = c - 1$ so u_{j+1} has to be a good c by Definition 5.3. So again this case cannot occur. $\square$

Starting at the root vertex of $\mathscr{T}(\mu)$ and working our way down the edges, Lemma 5.17 allows us to partition the set $\mathrm{SStd}^0_s(\lambda \to \nu, \mu)$ into subsets corresponding to $\mathrm{Latt}^0_s(\lambda \to \nu, \tau)$ for each terminal vertex labelled by (τ, τ) for $\tau \vdash s$. The next lemma describes the terminal vertices of the $\mathscr{T}(\mu)$.

Lemma 5.18. *Let $\mu, \tau \vdash s$. There is a bijective correspondence between the set of terminal vertices in $\mathscr{T}(\mu)$ labelled by (τ, τ) and the set $\mathrm{SStd}_s(\tau, \mu)$ of semistandard Young tableaux of shape τ and weight μ.*

Proof. For this proof, it is easier to view the set $\mathrm{SStd}_s(\tau, \mu)$ in the classical way, as Young diagrams of shape τ with boxes filled with μ_1 1's, μ_2 2's , ... For each edge a_c and $r_c^{\tau_c - \tau^\sharp_c}$ in the tree $\mathscr{T}(\mu)$, we define corresponding maps

$$a_c : \mathrm{SStd}_s(\tau,\mu) \to \mathrm{SStd}_s(\tau,\mu)\ :\ \mathsf{T} \mapsto a_c(\mathsf{T}) = \mathsf{T},$$
$$r_c^{\tau_c-\tau_c^\sharp} : \mathrm{SStd}_s(\tau,\mu) \to \mathrm{SStd}_s(r_c^{\tau_c-\tau_c^\sharp}(\tau),\mu)\ :\ \mathsf{T} \mapsto r_c^{\tau_c-\tau_c^\sharp}(\mathsf{T})$$

where $r_c^{\tau_c-\tau_c^\sharp}(\mathsf{T})$ is obtained from T by moving the last $\tau_c - \tau_c^\sharp$ boxes at the end of row c to the end of row $c-1$ together with their content.

Now each terminal vertex in $\mathscr{T}(\mu)$ correspond to a unique path $\mathfrak{t}$ starting at the root vertex and ending at a vertex labelled by (τ,τ) for some $\tau \vdash s$. Let T^μ be the unique element in $\mathrm{SStd}_s(\mu,\mu)$ and denote by $r_\mathfrak{t}(\mathsf{T}^\mu)$ the tableau obtained by applying the operators along the edges of $\mathfrak{t}$ to T^μ. We claim that the map $\mathfrak{t} \mapsto r_\mathfrak{t}(\mathsf{T}^\mu)$ for each terminal vertex labelled by (τ,τ) gives a bijection between these terminal vertices and $\mathrm{SStd}_s(\tau,\mu)$.

As the operator $r_\mathfrak{t}$ moves up the boxes of content 2 first, then the boxes of content 3, then 4, and so on, it is clear that the result will be a semistandard tableau of shape τ and weight μ, and moreover, different paths will lead to different semistandard tableaux.

It remains to show that this map is surjective. First note that if $\mathrm{SStd}_s(\tau,\mu) \neq \emptyset$ then $\tau \trianglerighteq \mu$. Now let $\mathsf{T} \in \mathrm{SStd}_s(\tau,\mu)$ for some $\tau \rhd \mu$. Assume that T has precisely k_d^c boxes of content c in row d. (Note that of $k_d^c \neq 0$ then $d \leqslant c$.) For each $2 \leqslant c \leqslant \ell(\mu)$ define

$$r^{(c)} = r_2^{k_1^c} \circ \ldots \circ (a_{c-2})^{k_{c-2}^c} \circ r_{c-1}^{\sum_{d=1}^{c-2} k_d^c} \circ (a_c)^{k_{c-1}^c} \circ r_c^{\sum_{d=1}^{c-1} k_d^c} \circ (a_c)^{k_c^c}.$$

By construction, we have $r^{(\ell(\mu))} \ldots r^{(3)} r^{(2)}(\mathsf{T}^\mu) = \mathsf{T}$ and $r^{(\ell(\mu))} \ldots r^{(3)} r^{(2)}$ is a path in $\mathscr{T}(\mu)$ starting at the root vertex and ending at a vertex labelled with (τ,τ). Thus the map is surjective as required. □

Example 5.19. Given $\mu = (3,2,1)$, we have that the sequences

$$a_3a_2a_2 \quad a_2r_3a_2a_2 \quad r_2r_3a_2a_2 \quad a_3r_2a_2 \quad a_2r_3r_2a_2 \quad r_2r_3r_2a_2 \quad a_2r_3r_2^2 \quad r_2r_3r_2^2 \tag{5.1}$$

label the terminal vertices in $\mathscr{T}(\mu)$. Applying these operators to T^μ we obtain all semistandard Young tableaux of weight μ. This procedure is illustrated in Figure 19.

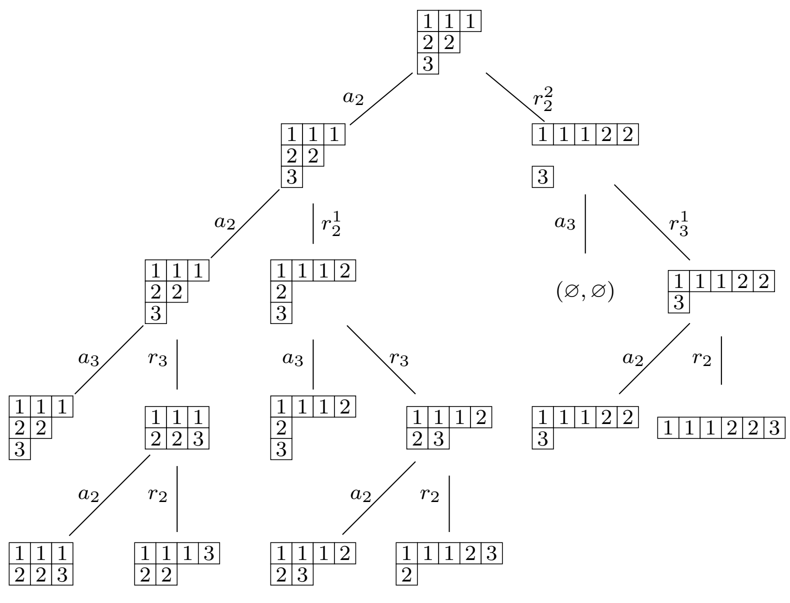


FIGURE 19. The set of terminal vertices of this graph gives precisely the set of all semistandard Young tableaux of weight $(3,2,1)$.

Corollary 5.20. *Let (λ,ν,s) be a co-Pieri triple and $\mu \vdash s$. There is one-to-one correspondence*

$$\mathrm{SStd}_s^0(\lambda \to \nu,\mu) \overset{1-1}{\longleftrightarrow} \bigsqcup_{\tau \vdash s} \mathrm{SStd}_s(\tau,\mu) \times \mathrm{Latt}_s^0(\lambda \to \nu,\tau).$$

Proof. By repeated applications of Lemma 5.17, we have a bijection between $\mathrm{SStd}^0_s(\lambda \to \nu, \mu)$ and the disjoint union over all terminal vertices of $\mathscr{T}(\mu)$ of the sets $\mathrm{SStd}^0_s(\lambda \to \nu, (\tau,\tau))$, where (τ,τ) is the label of the corresponding terminal vertex. Now, by Lemma 5.18, we have that for each $\tau \vdash s$, the number of terminal vertices labelled by (τ,τ) is precisely the cardinality of $\mathrm{SStd}_s(\tau,\mu)$. Moreover, by definition we have that $\mathrm{SStd}^0_s(\lambda \to \nu, (\tau,\tau)) = \mathrm{Latt}^0_s(\lambda \to \nu, \tau)$. Hence the result follows. □

Example 5.21. Let $\lambda = (7)$, $\nu = (6)$, $\mu = (3,2,1)$ and $\tau = (4,2)$. We have that

$$|\mathrm{Latt}^0_6(\lambda \to \nu, \tau) \times \mathrm{SStd}_s(\tau,\mu)| = 4 \times 2 = 8$$

and the tableaux are listed explicitly in Examples 5.8 and 5.19. We shall now list the 8 elements of $\mathrm{SStd}^0_6(\lambda \to \nu, \mu)$ which correspond to these pairs of tableaux under the bijection given in Corollary 5.20. The two terminal vertices labelled by (τ,τ) are determined by the paths $r_2r_3a_2a_2$ and $a_2r_3r_2a_2$.

First consider the path $r_2r_3a_2a_2$. Using Lemma 5.17 we apply $\mathscr{R}_3^{-1} \circ \mathscr{R}_2^{-1}$ to the tableaux in $\mathrm{Latt}^0_6(\lambda \to \nu, \tau)$ to get

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}\right) \xrightarrow{\mathscr{R}_3^{-1}\circ\mathscr{R}_2^{-1}} \left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 3 & 1 & 1 & 1 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 1 \end{array}\right) \xrightarrow{\mathscr{R}_3^{-1}\circ\mathscr{R}_2^{-1}} \left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 3 & 1 & 1 & 2 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 1 & 1 & 2 \end{array}\right) \xrightarrow{\mathscr{R}_3^{-1}\circ\mathscr{R}_2^{-1}} \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 3 & 1 & 2 & 1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\mathscr{R}_3^{-1}\circ\mathscr{R}_2^{-1}} \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 3 & 2 & 2 & 1 \end{array}\right)$$

Now consider the path $a_2r_3r_2a_2$. Using Lemma 5.17 we apply $\mathscr{R}_2^{-1} \circ \mathscr{R}_3^{-1}$ to the tableaux in $\mathrm{Latt}^0_6(\lambda \to \nu, \tau)$ to get the following four elements of $\mathrm{SStd}^0_6(\lambda \to \nu, \mu)$.

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 1 & 2 & 2 \end{array}\right) \xrightarrow{\mathscr{R}_2^{-1}\circ\mathscr{R}_3^{-1}} \left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 2 & 1 & 1 & 1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 1 \end{array}\right) \xrightarrow{\mathscr{R}_2^{-1}\circ\mathscr{R}_3^{-1}} \left(\begin{array}{ccc|c|cc} r(1) & r(1) & r(1) & d(1) & a(1) & a(1) \\ 2 & 1 & 1 & 3 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 1 & 1 & 2 \end{array}\right) \xrightarrow{\mathscr{R}_2^{-1}\circ\mathscr{R}_3^{-1}} \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 2 & 1 & 3 & 1 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 1 & 1 & 2 & 2 & 1 & 1 \end{array}\right) \xrightarrow{\mathscr{R}_2^{-1}\circ\mathscr{R}_3^{-1}} \left(\begin{array}{cc|ccc|c} r(1) & r(1) & d(1) & d(1) & d(1) & a(1) \\ 2 & 1 & 3 & 2 & 1 & 1 \end{array}\right)$$

We are now ready to prove our main theorem.

Proof of Theorem 5.9. Recall that

$$\overline{g}(\lambda,\nu,\mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta^0_s(\lambda \to \nu)).$$

We prove the result by downwards induction on μ (using the dominance order $\trianglerighteq$). If μ is maximal then $\mu = (s)$ and $\mathsf{S}(\mu) = \mathsf{M}(\mu)$. Moreover $\mathrm{Latt}^0_s(\lambda \to \nu, \mu) = \mathrm{SStd}^0_s(\lambda \to \nu, \mu)$. Thus the result follows from Theorem 4.7. We now assume that the result holds for all partitions $\tau \triangleright \mu$. We have

$$\mathsf{M}(\mu) = \bigoplus_{\tau \trianglerighteq s} |\mathrm{SStd}_s(\tau,\mu)| \mathsf{S}(\tau). \tag{5.2}$$

By induction we have

$$\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\tau), \Delta^0_s(\lambda \to \nu)) = |\mathrm{Latt}^0_s(\lambda \to \nu, \tau)| \quad \forall \tau \triangleright \mu. \tag{5.3}$$

By Theorem 4.7, (5.2) and (5.3) we have

$$\begin{aligned} |\mathrm{SStd}^0_s(\lambda \to \nu, \mu)| &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{M}(\mu), \Delta^0_s(\lambda \to \nu)) \\ &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta^0_s(\lambda \to \nu)) + \sum_{\tau \triangleright \mu} |\mathrm{SStd}_s(\tau,\mu)| \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\tau), \Delta^0_s(\lambda \to \nu)) \\ &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta^0_s(\lambda \to \nu)) + \sum_{\tau \triangleright \mu} |\mathrm{SStd}_s(\tau,\mu)|\, |\mathrm{Latt}^0_s(\lambda \to \nu, \tau)|. \end{aligned}$$

Comparing this equality with Corollary 5.20 and noting that $|\mathrm{SStd}_s(\mu,\mu)| = 1$ we get

$$\overline{g}(\lambda,\nu,\mu) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{S}_s}(\mathsf{S}(\mu), \Delta_s^0(\lambda \to \nu)) = |\mathrm{Latt}_s^0(\lambda \to \nu, \mu)|$$

as required. □

6. Examples

In this section we provide several illustrative examples of how to calculate Kronecker coefficients in terms of latticed Kronecker tableaux. As a warm up exercise, we first consider the decomposition of tensor products of the form $\mathsf{S}(\lambda_{[n]}) \otimes \mathsf{S}(n-1,1)$. These coefficients are trivial to calculate but they provided our initial motivation for this paper and they illustrate some of the basic ideas very well. We have

$$\begin{aligned} g(\nu_{[n]}, \lambda_{[n]}, (n-1,1)) &= \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_n}(\mathsf{S}(\lambda_{[n]}) \otimes \mathsf{S}(n-1,1), \mathsf{S}(\nu_{[n]}))) \\ &= \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_1}(\mathsf{S}((1)), \Delta_1^0(\lambda \to \nu))) \\ &= \dim_{\mathbb{Q}}(\mathrm{Hom}_{\mathfrak{S}_1}(\mathsf{M}((1)), \Delta_1^0(\lambda \to \nu))) \\ &= |\mathrm{SStd}_1^0(\lambda \to \nu, (1))|. \end{aligned}$$

Note that as $s = 1$ we have $\mathrm{SStd}_1^0(\lambda \to \nu, (1)) = \mathrm{Std}_1^0(\lambda \to \nu)$. Moreover, we have $\mathrm{Std}_1^0(\lambda \to \nu) = \mathrm{Std}_1(\lambda \to \nu)$ unless $\lambda = \nu$, in which case we have $\mathrm{Std}_1^0(\lambda \to \lambda) = \mathrm{Std}_1(\lambda \to \lambda) \setminus \{(-\varepsilon_0, +\varepsilon_0)\}$. The coefficient $g(\nu_{[n]}, \lambda_{[n]}, (n-1,1))$ is therefore equal to the number of paths of length 1 from λ to ν for $\lambda \neq \nu$ and is equal to one fewer for $\lambda = \nu$. In the former (respectively latter) case the number of such paths is equal to 1 (respectively equal to the number of removable nodes of λ). Compare with [40, Exercise 7.81].

Example 6.1. For example, the coefficients stabilise for $n \geqslant 7$ and we have that

$$\begin{aligned} \mathbf{S}(n-3,2,1) \otimes \mathbf{S}(n-1,1) =& \mathbf{S}(n-2,2) \oplus \mathbf{S}(n-2,1^2) \oplus \mathbf{S}(n-3,3) \oplus 2\mathbf{S}(n-3,2,1) \oplus \mathbf{S}(n-3,1^3) \\ & \oplus \mathbf{S}(n-4,3,1) \oplus \mathbf{S}(n-4,2^2) \oplus \mathbf{S}(n-4,2,1^2). \end{aligned}$$

The only coefficient not equal to 0 or 1 is $g((n-3,2,1),(n-3,2,1),(n-1,1)) = 2$ for $n \geqslant 7$. See Figure 20 for the paths from $(2,1) \in \mathcal{Y}_3$ to points in $\mathcal{Y}_4$.

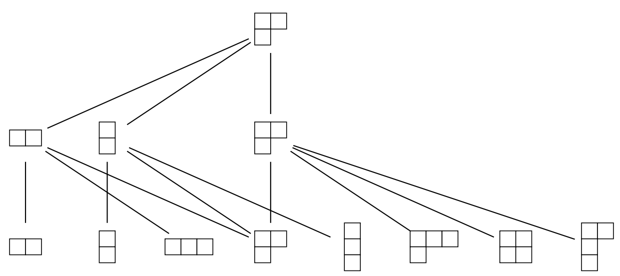


Figure 20. Paths of degree 1 beginning at $(2,1)$ in $\mathcal{Y}_3$.

We now revisit some of the earlier examples in the paper.

Example 6.2. Consider the rightmost example in Figure 2 from the introduction. We have that $((5,3,3),(7,5,1,1),\mu)$ is a co-Pieri triple for $\mu \vdash 5$. We have that

$$g((n-11,5,3,3),(n-14,7,5,1,1),(n-5,2,2,1)) = \overline{g}((5,3,3),(7,5,1,1),(2,2,1)) = 11$$

for all $n \geqslant 21$ and an example of an element of $\mathrm{Latt}_5^0((7,5,1,1) \to (5,3,3),(2,2,1))$ is depicted in Figure 2.

Example 6.3. We have that $((6,5,3,2),(12,8,5,3),\mu)$ is a co-Pieri triple for $\mu \vdash 3$. Some of the corresponding semistandard and latticed tableaux are depicted in Figure 16 and discussed in Example 5.7. We have that

$$\overline{g}((6,5,3,2),(12,8,5,3),(3)) = 6,$$
$$\overline{g}((6,5,3,2),(12,8,5,3),(2,1)) = 9,$$
$$\overline{g}((6,5,3,2),(12,8,5,3),(1^3)) = 3.$$

The six latticed tableaux of weight (3) are given in Example 4.8. We leave constructing those of weight $(2,1)$ and (1^3) as an exercise for the reader.

Example 6.4. We have that $((9,6,3),(9,6,3),(2,1))$ is a co-Pieri triple and that

$$\overline{g}((9,6,3),(9,6,3),(2,1)) = \mathrm{Latt}_3^0(((9,6,3) \to (9,6,3),(2,1))) = 60.$$

The dedicated reader might wish to attempt this calculation themselves once they have digested the other examples in this section. The rightmost tableau in Figure 16 is an example of a latticed tableau for this triple.

6.1. **Kronecker coefficients labelled by two 2-row partitions.** In this section we provide examples of our tableaux combinatorics for coefficients $g(\lambda_{[n]},\nu_{[n]},\mu_{[n]})$ in which $\lambda_{[n]}$ and $\nu_{[n]}$ are two-part partitions but $\mu_{[n]}$ is arbitrary. These coefficients have been described in many ways and received the attention of many authors [2, 36, 37, 6, 9, 27]; Hilbert series related to these coefficients have been linked to problems in quantum information theory [26, 17]. The advantage of our description over previous work is that it covers these coefficients as a simple example in a far broader class of Kronecker coefficients (including the Littlewood–Richardson coefficients).

The following proposition is well known from the interpretation of the Kronecker coefficients in the setting of the representations of the general linear groups. We include a simple proof as it illustrates some basic properties of latticed Kronecker tableaux.

Proposition 6.5. *If $\lambda_{[n]}$ and $\nu_{[n]}$ are 2-part partitions and $g(\lambda_{[n]},\nu_{[n]},\mu_{[n]}) \neq 0$, then $\ell(\mu_{[n]}) \leqslant 4$.*

Proof. First note that $g(\lambda_{[n]},\nu_{[n]},\mu_{[n]}) \neq 0$ implies that $g(\lambda,\nu,\mu) = |\mathrm{Latt}_s^0(\lambda \to \nu,\mu)| \neq 0$. Now the only possible steps in semistandard Kronecker tableaux in $\mathrm{Latt}_s^0(\lambda \to \nu,\mu)$ are $r(1)$, $d(1)$, or $a(1)$ and the ordering on these steps is $r(1) < d(1) < a(1)$. Now by definition of a lattice word, for any $\mathsf{S} \in \mathrm{Latt}_s^0(\lambda \to \nu,\mu)$, the frame number of a step of type $r(1)$ in S is equal to 1, the frame number of a step of type $d(1)$ is less or equal to 2 and the frame number of a step of type $a(1)$ is less or equal to 3. Thus if $\mathrm{Latt}_s^0(\lambda \to \nu,\mu) \neq \emptyset$ then $\ell(\mu) \leqslant 3$ and hence $\ell(\mu_{[n]}) \leqslant 4$ as required. □

Proposition 6.6. *Let $\lambda_{[n]}$ and $\nu_{[n]}$ be 2-part partitions. Let $\mu_{[n]}$ be an arbitrary partition. Then we have that*

$$g(\lambda_{[n]},\nu_{[n]},\mu_{[n]}) = \sum_{i=0}^{3}(-1)^i|\mathrm{Latt}_{s_i}^0(\lambda \to \nu,\mu^{(i)})|$$

where $\mu^{(0)} = \mu$ and for $i \geqslant 1$ the partition $\mu^{(i)}$ is obtained from $\mu^{(i-1)}$ by adding a single row of boxes in the ith row, the last of which having content $n - |\mu^{(i-1)}|$, and $s_i = |\mu^{(i)}|$.

Proof. By [6, Theorem 3] (and see [4, Theorem 3.7] for the partition algebra theoretic interpretation and proof) we can write

$$g(\lambda_{[n]},\nu_{[n]},\mu_{[n]}) = \sum_{i \geqslant 0}(-1)^i\overline{g}(\lambda,\nu,\mu^{(i)}).$$

Using Proposition 6.5 we have that if $\ell(\mu^{(i)}) > 3$ then $\overline{g}(\lambda,\nu,\mu^{(i)}) = 0$. Now the result follows from Theorem 5.9. □

Remark 6.7. It is proved in [6, Theorem 3] that the last term in the alternating sum given in Proposition 6.6 is zero, so in fact we only have three non-zero terms.

Remark 6.8. Note also that if $|\mu^{(i)}| > |\lambda| + |\nu|$ then $\mathrm{Latt}^0_{s_i}(\lambda \to \nu, \mu^{(i)}) = \emptyset$. Thus as n gets larger the sum in Proposition 6.6 has fewer than 4 terms. In fact when $n > |\lambda| + |\nu| + \mu_1 - 1$ then we have $|\mu^{(1)}| > |\lambda| + |\nu|$ and so (letting $s = |\mu|$) we have that

$$g(\lambda_{[n]}, \nu_{[n]}, \mu_{[n]}) = |\mathrm{Latt}^0_s(\lambda \to \nu, \mu)|.$$

Example 6.9. Let $\lambda = (7)$ and $\nu = (6)$ and $\mu = (4, 3, 1)$. Then $\omega_1(\mathsf{S})$ must be one of the following

$$(r(1)\, r(1)\, r(1) \mid d(1)\, d(1)\, d(1) \mid a(1)\, a(1)) \quad (r(1)\, r(1)\, r(1)\, r(1) \mid d(1) \mid a(1)\, a(1)\, a(1)).$$

Is is easy to check that $\mathsf{S} \in \mathrm{Latt}^0_8(\lambda \to \nu, \mu)$ if and only if $\omega(\mathsf{S})$ is one of the following

$$\left(\begin{array}{ccc|ccc|cc} r(1) & r(1) & r(1) & d(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 2 & 3 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|cc} r(1) & r(1) & r(1) & d(1) & d(1) & d(1) & a(1) & a(1) \\ 1 & 1 & 1 & 2 & 2 & 1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c|ccc} r(1) & r(1) & r(1) & r(1) & d(1) & a(1) & a(1) & a(1) \\ 1 & 1 & 1 & 1 & 2 & 3 & 2 & 2 \end{array}\right)$$

Therefore $g((n-7, 7), (n-6, 6), (n-8, 4, 3, 1)) = 3$ for $n \geqslant 15$. We leave it as an exercise for the reader to verify that these semistandard Kronecker tableaux are orbits of size 12, 3, and 1 respectively.

6.2. **A Kronecker product labelled by two three-row partitions.** We now consider the next simplest case: namely a pair of 3-row partitions. Let $\lambda = (6, 1)$ and $\nu = (4, 3)$, we have $|\mathrm{SStd}^0_3(\lambda \to \nu, (3))| = |\mathrm{Latt}^0_3(\lambda \to \nu, (3))| = 3$. The corresponding reading words are as follows,

$$\left(\begin{array}{c|cc} d(1) & m{\downarrow}(1,2) & m{\downarrow}(1,2) \\ 1 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|cc} d(2) & m{\downarrow}(1,2) & m{\downarrow}(1,2) \\ 1 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|c|c} r(1) & m{\downarrow}(1,2) & a(2) \\ 1 & 1 & 1 \end{array}\right)$$

It is easy to check that any $\mathsf{S} \in \mathrm{SStd}^0_3(\lambda \to \nu, (2, 1))$ must have $\omega_1(\mathsf{S})$ as follows,

$$(d(1) \mid m{\downarrow}(1,2)\, m{\downarrow}(1,2)) \qquad (d(2) \mid m{\downarrow}(1,2)\, m{\downarrow}(1,2)) \qquad (r(1) \mid m{\downarrow}(1,2) \mid a(2))$$

and $|\mathrm{SStd}^0_3(\lambda \to \nu, (2, 1))| = 7$. We have $\mathsf{S} \in \mathrm{Latt}^0_3(\lambda \to \nu, (2, 1))$ if and only if $\omega(\mathsf{S})$ is one of the following,

$$\left(\begin{array}{c|cc} d(1) & m{\downarrow}(1,2) & m{\downarrow}(1,2) \\ 1 & 2 & 1 \end{array}\right) \quad \left(\begin{array}{c|c|c} r(1) & m{\downarrow}(1,2) & a(2) \\ 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|cc} d(2) & m{\downarrow}(1,2) & m{\downarrow}(1,2) \\ 1 & 2 & 1 \end{array}\right) \quad \left(\begin{array}{c|c|c} r(1) & m{\downarrow}(1,2) & a(2) \\ 1 & 1 & 2 \end{array}\right)$$

We have that $|\mathrm{SStd}^0_3(\lambda \to \nu, (1^3))| = 12$. The unique element $\mathsf{S} \in \mathrm{Latt}^0_3(\lambda \to \nu, (1^3))$ has $\omega(\mathsf{S})$ equal to

$$\left(\begin{array}{c|c|c} r(1) & m{\downarrow}(1,2) & a(2) \\ 1 & 2 & 3 \end{array}\right)$$

We therefore conclude that

$$\overline{g}((6,1), (4,3), (3)) = 3 \qquad \overline{g}((6,1), (4,3), (2,1)) = 4 \qquad \overline{g}((6,1), (4,3), (1^3)) = 1.$$

The Kronecker coefficients quickly stabilise in this case, for example

$$g((6^2, 1), (6, 4, 3), (10, 3)) = 3 \qquad g((7, 6, 1), (7, 4, 3), (11, 3)) = 4$$

and $g((n-7, 6, 1), (n-7, 4, 3), (n-3, 3)) = 4$ for $n \geqslant 14$.

6.3. **A larger example.** Let $\lambda = (6,2)$, $\nu = (7,4)$. We have that (λ, ν, s) is a co-Pieri triple for $s \leqslant 5$. Let $s = 4$ and $\mu \vdash s$. Given $\mathsf{S} \in \mathrm{SStd}_4^0(\lambda \to \nu, \mu)$, we have that $\omega_1(\mathsf{S})$ is equal to one of

$$(d(1) \mid a(1) \mid a(2)\, a(2)) \quad (d(2) \mid a(1) \mid a(2)\, a(2)) \quad (m{\downarrow}(1,2) \mid a(1)\, a(1) \mid a(2)) \quad (m{\uparrow}(2,1) \mid a(2)\, a(2)\, a(2)).$$

We now consider the semistandard and latticed tableaux for each weight μ for $\mu \vdash 4$. We have that $|\mathrm{SStd}_4^0(\lambda \to \nu), (4))| = |\mathrm{Latt}_4^0(\lambda \to \nu, (4))| = 4$. The corresponding $\omega_1(\mathsf{S})$ are as follows:

$$\left(\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|cc|c} m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|c|cc} d(2) & a(1) & a(2) & a(2) \\ 1 & 1 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|ccc} m{\uparrow}(2,1) & a(2) & a(2) & a(2) \\ 1 & 1 & 1 & 1 \end{array}\right).$$

Given $\mathsf{S} \in \mathrm{Latt}_4^0(\lambda \to \nu, (3,1))$, we have that $\omega_1(\mathsf{S})$ is one of the following,

$$\left(\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 1 \end{array}\right) \quad \left(\begin{array}{c|cc|c} m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 2 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|c|cc} d(2) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 1 \end{array}\right) \quad \left(\begin{array}{c|cc|c} m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 1 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{c|c|cc} d(2) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|ccc} m{\uparrow}(2,1) & a(2) & a(2) & a(2) \\ 1 & 2 & 1 & 1 \end{array}\right)$$

Given $\mathsf{S} \in \mathrm{Latt}_4^0(\lambda \to \nu, (2,2))$, we have that $\omega_1(\mathsf{S})$ is one of the following,

$$\left(\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 2 \end{array}\right) \quad \left(\begin{array}{c|cc|c} m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 2 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{c|c|cc} d(2) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 2 \end{array}\right)$$

Given $\mathsf{S} \in \mathrm{Latt}_4^0(\lambda \to \nu, (2,1^2))$, we have that $\omega_1(\mathsf{S})$ is one of the following,

$$\left(\begin{array}{c|c|cc} d(1) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{c|cc|c} m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 2 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{c|c|cc} d(2) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 3 \end{array}\right)$$

Finally, we have that $|\mathrm{Latt}_4^0(\lambda \to \nu, (1^4))| = 0$ and therefore

$$\begin{array}{lll} \overline{g}((6,2),(7,4),(4)) = 4 & \overline{g}((6,2),(7,4),(2,2)) = 3 & \overline{g}((6,2),(7,4),(1^4)) = 0 \\ \overline{g}((6,2),(7,4),(3,1)) = 7 & \overline{g}((6,2),(7,4),(2,1^2)) = 3 & \end{array}.$$

We do not calculate all the coefficients $\overline{g}(\lambda,\nu,\mu)$ for $\mu \vdash 5$ and instead only calculate the $\mu = (2^2,1)$ case. Given $\mathsf{S} \in \mathrm{Latt}_5^0(\lambda \to \nu, (2^2,1))$, we have that $\omega(\mathsf{S})$ is one of the following,

$$\left(\begin{array}{c|cc|cc} r(1) & a(1) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 3 & 2 \end{array}\right) \quad \left(\begin{array}{cc|c|cc} d(1) & d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{c|c|cc|c} d(1) & m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 1 & 2 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{c|c|cc|c} d(1) & m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 2 & 3 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{c|c|c|cc} d(2) & d(1) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 3 & 2 \end{array}\right) \quad \left(\begin{array}{c|c|cc|c} d(2) & m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 2 & 3 & 1 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|c|cc} d(2) & d(2) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 3 & 2 \end{array}\right) \quad \left(\begin{array}{c|c|c|cc} m{\uparrow}(2,1) & m{\downarrow}(1,2) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{c|c|cc|c} d(2) & m{\downarrow}(1,2) & a(1) & a(1) & a(2) \\ 1 & 1 & 2 & 2 & 3 \end{array}\right) \quad \left(\begin{array}{c|c|c|cc} m{\uparrow}(2,1) & m{\downarrow}(1,2) & a(1) & a(2) & a(2) \\ 1 & 1 & 2 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{c|c|c|cc} d(2) & d(1) & a(1) & a(2) & a(2) \\ 1 & 2 & 1 & 3 & 2 \end{array}\right)$$

and therefore $\overline{g}(\lambda,\nu,(2,2,1)) = 11$.

Example 6.10. We now consider an example which is not a co-Pieri triple. We let $\lambda = \nu = (1^2)$. We have that $\mathsf{DR}_2(\Delta_2(\lambda \to \nu))$ is 5-dimensional and is isomorphic to $\Delta_2(1) \oplus \Delta_2(\varnothing)$. The former summand is spanned by the basis elements indexed by the Kronecker tableaux

$$d(2) \circ d(2) \qquad d(0) \circ d(2) \qquad d(2) \circ d(0)$$

and the latter summand is spanned by the basis elements indexed by the Kronecker tableaux

$$a(3) \circ r(3) \qquad d(0) \circ d(0).$$

The quotient $\Delta_2^0(\lambda \to \nu)$ decomposes as a direct sum of two transitive permutation modules

$$\mathbb{Q}\{u_{\mathsf{t}} \mid \mathsf{t} = m\!\uparrow\!(2,1) \circ m\!\downarrow\!(1,2)\} \oplus \mathbb{Q}\{u_{\mathsf{s}} \mid \mathsf{s} \in \{a(1) \circ r(1), r(2) \circ a(2)\}\}.$$

Note that $\mathsf{t}_{1\leftrightarrow 2}$ is not a standard Kronecker tableau and hence we cannot use the results of this paper to understand $\Delta_2^0(\lambda \to \nu)$. However, one can see that the former summand is isomorphic to $\Delta_2(2)$ via the isomorphism $\Delta_2(2) \cong \Delta_2(1^2) \otimes \Delta_2(1^2)$. The latter summand is isomorphic to $\Delta_2(2) \oplus \Delta_2(1^2)$.

Appendix A. The action of the partition algebra on skew cell modules

This section is dedicated to the proof of Theorem 3.3. This is essentially an extensive book keeping exercise, which begins with a few simple but important observations concerning the branching coefficients of Definition 2.7 which are, in turn, given as products of the diagrammatic elements $s_{k,l}, e_k^{(l)}$ and $e_{k+\frac12}^{(l)}$ defined in Definition 2.6. The first step in our book keeping is as follows: Fix $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $1 \leqslant k \leqslant r-1$. First note that we can factorise u_{t} as follows,

$$u_{\mathsf{t}} = u_{\mathsf{t}[k+1,r]} u_{\mathsf{t}[k-1,k+1]} u_{\mathsf{t}[0,k-1]}.$$

Now as $u_{\mathsf{t}[0,k-1]} \in P_{k-1}(n)$, it commutes with $s_{k,k+1}$ and so we have

$$u_{\mathsf{t}} = u_{\mathsf{t}[k+1,r]} u_{\mathsf{t}[k-1,k+1]} s_{k,k+1} u_{\mathsf{t}[0,k-1]}. \tag{A.1}$$

So let us first consider $u_{\mathsf{t}[k-1,k+1]}$. We fix the following notation.

$$\mathsf{t}[k-1,k+1] \;=\; \mathsf{t}(k-1) \xrightarrow{-t} \mathsf{t}(k-\tfrac12) \xrightarrow{+u} \mathsf{t}(k) \xrightarrow{-v} \mathsf{t}(k+\tfrac12) \xrightarrow{+w} \mathsf{t}(k+1) \tag{A.2}$$

with

$$\mathsf{t}(k-1) = (\alpha, a) \quad \mathsf{t}(k-\tfrac12) = (\beta, b) \quad \mathsf{t}(k) = (\gamma, c) \quad \mathsf{t}(k+\tfrac12) = (\delta, d) \quad \mathsf{t}(k+1) = (\zeta, z).$$

As in Definition 3.2, if $u = v > 0$ we define $\mathsf{s} := e_k(\mathsf{t})$ by $\mathsf{s}(l) = \mathsf{t}(l)$ for $l \neq k$ and

$$\mathsf{s}(k-\tfrac12) \xrightarrow{+L} \mathsf{s}(k) \xrightarrow{-L} \mathsf{s}(k+\tfrac12) \tag{A.3}$$

where $L = \ell(\mathsf{t}(k-\frac12))+1$. If $\mathsf{t}(k-\frac12) \neq \mathsf{t}(k+\frac12)$, then $\mathsf{e}_k(\mathsf{t})$ is undefined. For the purposes of book-keeping we now introduce some notation. Given ν a partition and $u, w \geqslant 0$ we set

$$m_{\nu-\varepsilon_w \to \nu} = \sum_{i=0}^{\nu_w - 1} s_{|\nu|_w - i, |\nu|_w} \qquad m_{\nu,u,w} = \begin{cases} m_{\nu-\varepsilon_w\to\nu} m_{\nu-\varepsilon_u\to\nu} & \text{if } u \neq w \\ m_{\nu-\varepsilon_w\to\nu} m_{\nu-2\varepsilon_w\to\nu-\varepsilon_w} & \text{if } u = w. \end{cases}$$

(Note that $m_{\nu,u,w} = m_{\nu,w,u}$ and that $m_{\nu\to\nu} = 1$ following the conventions of Definition 2.7.) By Definition 2.7, we have

$$u_{\mathsf{t}[k-1,k+1]} = e_{k+1}^{(z)} m_{\delta\to\zeta} s_{|\zeta|_w, |\zeta|} e_{k+\frac12}^{(d)} s_{|\gamma|, |\gamma|_v} e_k^{(c)} m_{\beta\to\gamma} s_{|\gamma|_u, |\gamma|} e_{k-\frac12}^{(b)} s_{|\alpha|, |\alpha|_t}. \tag{$\star$}$$

In order to prove Theorem 3.3, we need to multiply the element in equation ($\star$) by the Coxeter generator $s_{k,k+1}$ and re-express this in terms of branching coefficients. The proof is not conceptually difficult but requires numerous concatenation of diagrams and careful book keeping. To help the reader follow the argument, we collect some relations between the elements $s_{i,j}$, $e_k^{(l)}$ and $e_{k+\frac12}^{(l)}$ in the following lemma.

Lemma A.1. *We have the following commutation and almost-commutation rules in the partition algebra:*

(1) (a) $s_{i,j} e_k^{(l)} = e_k^{(l)} s_{i,j}$ *if* $i, j \leqslant k-l$ *or* $i, j \geqslant k+1$. (b) $s_{i,j} e_{k+\frac12}^{(l)} = e_{k+\frac12}^{(l)} s_{i,j}$ *if* $i, j \leqslant k-l$ *or* $i, j \geqslant k+2$.

(2) $s_{i,j} s_{l,k} = s_{l,k} s_{i,j}$ *if* $i, j < \min\{l, k\}$ *or* $i, j > \max\{l, k\}$.

(3) For $k < i < j < l$ *we have* (a) $s_{l,k} s_{i,j} = s_{i-1,j-1} s_{l,k}$ *and* (b) $s_{k,l} s_{i,j} = s_{i+1,j+1} s_{k,l}$.

(4) $s_{k,j} s_{i,j} = s_{i,j-1} s_{k,j-1}$ *for* $i < j < k$.

(5) $e_{k+\frac12}^{(l)} s_{k,k+1} = e_{k+\frac12}^{(l)}$.

(6) $e_{k+\frac12}^{(l+1)} e_k^{(l)} e_{k+\frac12}^{(l)} = e_{k+\frac12}^{(l+1)}$

Proof. (1) and (2) are obvious from the diagrams of these elements given in Definition 2.6. For (3)(a) note that this is equivalent to showing that $s_{l,k}s_{i,j}s_{l,k}^{-1} = s_{l,k}s_{i,j}s_{k,l} = s_{i-1,j-1}$. This now follows by concatenating the diagrams as in Figure 21. The case (3)(b) is similar. The case (4) can be seen by multiplying the diagrams as in Figure 22. Case (5) follows from the fact that k and $k+1$ are in the same block in $e^{(l)}_{k+\frac{1}{2}}$. Finally, (6) can also be seen by concatenating the diagrams as in Figure 23. □

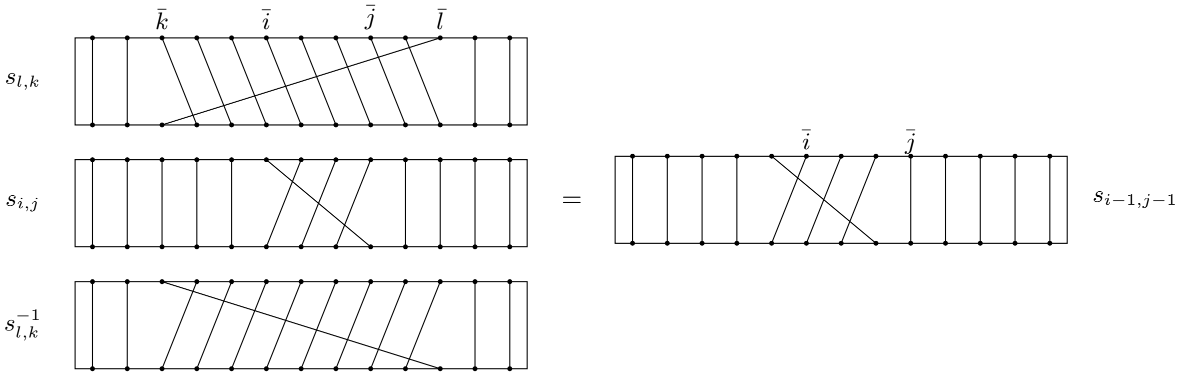


FIGURE 21. The product $s_{l,k}s_{i,j}s_{l,k}^{-1} = s_{i-1,j-1}$

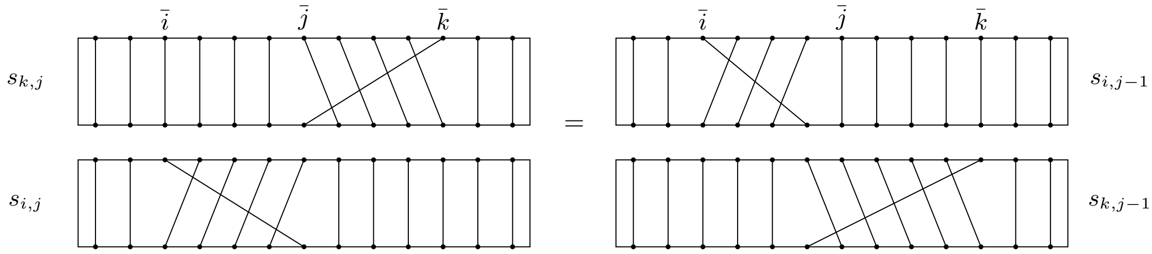


FIGURE 22. The product $s_{k,j}s_{i,j} = s_{i,j-1}s_{k,j-1}$

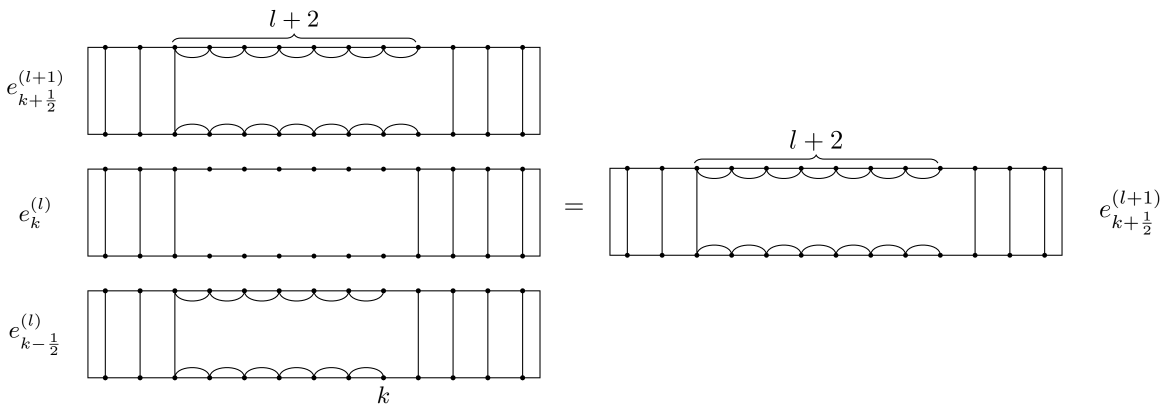


FIGURE 23. The product $e^{(l+1)}_{k+\frac{1}{2}}e^{(l)}_{k}e^{(l)}_{k-\frac{1}{2}} = e^{(l+1)}_{k+\frac{1}{2}}$

Proposition A.2. *For* t *as in equation* (A.2)*, we have*

$$u_{\mathsf{t}[k-1,k+1]} = m_{\zeta,u,w}P_k(\mathsf{t}) + (1-\delta_{u,0})\delta_{u,v}u_{\mathsf{s}[k-1,k+1]}$$

where $\delta_{u,v}$ *and* $\delta_{u,0}$ *are the usual Kronecker deltas ,* $\mathsf{s} = e_k(\mathsf{t})$ *as in equation* (A.3) *and*

$$P_k(\mathsf{t}) = \begin{cases} e^{(z)}_{k+1}s_{|\zeta|_w,|\zeta|}e^{(d)}_{k+\frac{1}{2}}s_{|\gamma|,|\gamma|_u-1}e^{(c)}_k s_{|\gamma|_u,|\gamma|}e^{(b)}_{k-\frac{1}{2}}s_{|\alpha|,|\alpha|_t} & \textit{if } u=v>0 \\ e^{(z)}_{k+1}s_{|\zeta|_w,|\zeta|}e^{(d)}_{k+\frac{1}{2}}s_{|\gamma|,|\gamma|_v}e^{(c)}_k s_{|\gamma|_u,|\gamma|}e^{(b)}_{k-\frac{1}{2}}s_{|\alpha|,|\alpha|_t} & \textit{otherwise.} \end{cases}$$

Proof. By definition 2.7, we have

$$u_{\mathsf{t}[k-1,k+1]} = e^{(z)}_{k+1}m_{\delta\to\zeta}s_{|\zeta|_w,|\zeta|}e^{(d)}_{k+\frac{1}{2}}s_{|\gamma|,|\gamma|_v}e^{(c)}_k m_{\beta\to\gamma}s_{|\gamma|_u,|\gamma|}e^{(b)}_{k-\frac{1}{2}}s_{|\alpha|,|\alpha|_t}.$$

Claim A. If $k \geqslant 1$, $(\lambda, l) \in \mathcal{Y}_k$ and $(\mu, m) \to (\lambda, l)$ is an edge in $\mathcal{Y}$ then we have

- $e^{(l)}_k m_{\mu\to\lambda} = m_{\mu\to\lambda}e^{(l)}_k$,
- $e^{(l)}_k s_{|\lambda|_a,|\lambda|} = s_{|\lambda|_a,|\lambda|}e^{(l)}_k$ for any $a \geqslant 0$.

We have that $|\lambda| = k - l$ so Claim A follows from Lemma A.1(1).

Claim B. We have that

$$s_{|\gamma|,|\gamma|_v}m_{\beta\to\gamma} = \begin{cases} m_{\delta-\varepsilon_u\to\delta}s_{|\gamma|,|\gamma|_u-1} + s_{|\gamma|,|\gamma|_u} & \text{if } u=v>0, \\ m_{\delta-\varepsilon_u\to\delta}s_{|\gamma|,|\gamma|_v} & \text{otherwise.} \end{cases}$$

(We note that $\beta = \gamma - \varepsilon_u$.) If $v = 0$, then $s_{|\gamma|,|\gamma|_v} = 1$ and $\delta = \gamma$ and so the result holds trivially. If $u = 0$, then $m_{\beta\to\gamma} = 1 = m_{\delta-\varepsilon_u\to\delta}$ and so the result also holds trivially. We now assume that $u, v > 0$. If $u < v$ then $\gamma_u = \delta_u$ and $|\delta|_u = |\gamma|_u < |\gamma|_v$ and so

$$s_{|\gamma|,|\gamma|_v}m_{\gamma-\varepsilon_u\to\gamma} = m_{\gamma-\varepsilon_u\to\gamma}s_{|\gamma|,|\gamma|_v} = m_{\delta-\varepsilon_u\to\gamma}s_{|\gamma|,|\gamma|_v},$$

by Lemma A.1(1). If $v < u$ then $|\gamma|_v < |\gamma|_u - i \leqslant |\gamma|$ for all $0 \leqslant i \leqslant \gamma_u - 1$ and so

$$s_{|\gamma|,|\gamma|_v}m_{\gamma-\varepsilon_u\to\gamma} = s_{|\gamma|,|\gamma|_v}\sum_{i=0}^{\gamma_u-1}s_{|\gamma|_u-i,|\gamma|_u} = \left(\sum_{i=0}^{\gamma_u-1}s_{|\gamma|_u-i-1,|\gamma|_u-1}\right)s_{|\gamma|,|\gamma|_v} = \left(\sum_{i=0}^{\delta_u-1}s_{|\delta|_u-i,|\delta|_u}\right)s_{|\gamma|,|\gamma|_v}$$

where the second equality follows from Lemma A.1(3)(a) and the third equality follows as $\delta_u = \gamma_u$ and $|\delta|_u = |\gamma|_u - 1$. Now the final term is equal to $m_{\delta-\varepsilon_u\to\delta}s_{|\gamma|,|\gamma|_v}$ by definition. Finally if $u = v > 0$ then

$$s_{|\gamma|,|\gamma|_v}m_{\beta\to\gamma} = s_{|\gamma|,|\gamma|_u}m_{\gamma-\varepsilon_u\to\gamma} = s_{|\gamma|,|\gamma|_u}\sum_{i=0}^{\gamma_u-1}s_{|\gamma|_u-i,|\gamma|_u} = s_{|\gamma|,|\gamma|_u}\left(1 + \sum_{i=1}^{\gamma_u-1}s_{|\gamma|_u-i,|\gamma|_u}\right).$$

Expanding out the brackets and shifting the indices, we obtain

$$\begin{aligned} s_{|\gamma|,|\gamma|_u} + \sum_{i=1}^{\gamma_u-1}s_{|\gamma|,|\gamma|_u}s_{|\gamma|_u-i,|\gamma|_u} &= s_{|\gamma|,|\gamma|_u} + \sum_{i=1}^{\gamma_u-1}s_{|\gamma|_u-i,|\gamma|_u-1}s_{|\gamma|,|\gamma|_u-1} \quad \text{by Lemma A.1(4)} \\ &= s_{|\gamma|,|\gamma|_u} + \sum_{i=0}^{\gamma_u-2}s_{|\gamma|_u-1-i,|\gamma|_u-1}s_{|\gamma|,|\gamma|_u-1} \\ &= s_{|\gamma|,|\gamma|_u} + \sum_{i=1}^{\delta_u-1}s_{|\delta|_u-i,|\delta|_u}s_{|\gamma|,|\gamma|_u-1} \\ &= m_{\delta-\varepsilon_u\to\delta}s_{|\gamma|,|\gamma|_u-1} + s_{|\gamma|,|\gamma|_u} \end{aligned}$$

where the penultimate equality follows as $|\gamma|_u - 1 = |\delta|_u$ and $\gamma_u - 2 = \delta_u - 1$. Therefore Claim B follows.

Claim C. We have that

$$s_{|\zeta|_w,|\zeta|}m_{\delta-\varepsilon_u\to\delta} = \begin{cases} m_{\zeta-\varepsilon_u\to\zeta}s_{|\zeta|_w,|\zeta|} & \text{if } w \neq u \\ m_{\zeta-2\varepsilon_w\to\zeta-\varepsilon_w}s_{|\zeta|_w,|\zeta|} & \text{otherwise.} \end{cases}$$

If $u = 0$ or $w = 0$ the result holds trivially. We assume $u, w > 0$. If $u < w$ then $|\delta|_u = |\zeta|_u < |\zeta|_w$ so we get

$$s_{|\zeta|_w,|\zeta|}m_{\delta-\varepsilon_u\to\delta} = m_{\delta-\varepsilon_u\to\delta}s_{|\zeta|_w,|\zeta|} = m_{\zeta-\varepsilon_u\to\zeta}s_{|\zeta|_w,|\zeta|},$$

using Lemma A.1(2). If $u > w$ then $|\delta|_u = |\zeta|_u - 1 \geqslant |\zeta|_w$ and so we get

$$s_{|\zeta|_w,|\zeta|} m_{\delta-\varepsilon_u\to\delta} = s_{|\zeta|_w,|\zeta|} \sum_{i=0}^{\delta_u-1} s_{|\delta|_u-i,|\delta|_u} = \sum_{i=0}^{\delta_u-1} s_{|\delta|_u-i+1,|\delta|_u+1} s_{|\zeta|_w,|\zeta|} = \sum_{i=0}^{\zeta_u-1} s_{|\zeta|_u-i,|\zeta|_u} s_{|\zeta|_w,|\zeta|}$$

where the second equality follows from Lemma A.1(3)(b). Now the final term is equal to $m_{\zeta-\varepsilon_u\to\zeta} s_{|\zeta|_w,|\zeta|}$, as required. Finally, if $u = w > 0$ then $|\zeta|_w = |\zeta|_u = |\delta|_u + 1$ and using Lemma A.1(1) we get

$$s_{|\zeta|_w,|\zeta|} m_{\delta-\varepsilon_u\to\delta} = m_{\delta-\varepsilon_u\to\delta} s_{|\zeta|_w,|\zeta|} = m_{\zeta-2\varepsilon_w\to\zeta-\varepsilon_w} s_{|\zeta|_w,|\zeta|},$$

as required. Therefore Claim C follows.

Applying Claim A and Claim B (and noting that $s_{|\gamma|,|\gamma|_u} s_{|\gamma|_u,|\gamma|} = 1$) we deduce that

$$u_{\mathsf{t}[k-1,k+1]} = \begin{cases} m_{\delta\to\zeta} e^{(z)}_{k+1} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} m_{\delta-\varepsilon_u\to\delta} s_{|\gamma|,|\gamma|_u-1} e^{(c)}_{k} s_{|\gamma|_u,|\gamma|} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \\ \quad + e^{(z)}_{k+1} m_{\delta\to\zeta} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} e^{(c)}_{k} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \text{if } u = v > 0 \\ m_{\delta\to\zeta} e^{(z)}_{k+1} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} m_{\delta-\varepsilon_u\to\delta} s_{|\gamma|,|\gamma|_v} e^{(c)}_{k} s_{|\gamma|_u,|\gamma|} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \text{otherwise.} \end{cases}$$

Applying Claim A and Claim C to the above equation, we deduce that

$$u_{\mathsf{t}[k-1,k+1]} = \begin{cases} m_{\zeta,u,w} e^{(z)}_{k+1} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} s_{|\gamma|,|\gamma|_u-1} e^{(c)}_{k} s_{|\gamma|_u,|\gamma|} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \\ \quad + e^{(z)}_{k+1} m_{\zeta-\varepsilon_w\to\zeta} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} e^{(c)}_{k} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \text{if } u = v > 0 \\ m_{\zeta,u,w} e^{(z)}_{k+1} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} s_{|\gamma|,|\gamma|_v} e^{(c)}_{k} s_{|\gamma|_u,|\gamma|} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t} & \text{otherwise.} \end{cases}$$

Finally, note that

$$u_{\mathsf{s}[k-1,k+1]} = e^{(z)}_{k+1} m_{\zeta-\varepsilon_w\to\zeta} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} s_{|\beta+\varepsilon_L|,|\beta+\varepsilon_L|} e^{(c)}_{k} m_{\beta\to\beta+\varepsilon_L} s_{|\beta+\varepsilon_L|,|\beta+\varepsilon_L|} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t}$$

and as $s_{|\beta+\varepsilon_L|,|\beta+\varepsilon_L|} = 1 = m_{\beta\to\beta+\varepsilon_L}$ we have

$$u_{\mathsf{s}[k-1,k+1]} = e^{(z)}_{k+1} m_{\zeta-\varepsilon_w\to\zeta} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} e^{(c)}_{k} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t}. \tag{A.4}$$

This completes the proof of Proposition A.2. □

Using Proposition A.2 and equation (A.1) we have

$$u_{\mathsf{t}} s_{k,k+1} = u_{\mathsf{t}[k+1,r]} m_{\zeta,u,w} P_k(\mathsf{t}) s_{k,k+1} u_{\mathsf{t}[0,k-1]} + (1-\delta_{u,0})\delta_{u,v} u_{\mathsf{t}[k+1,r]} u_{\mathsf{s}[k-1,k+1]} s_{k,k+1} u_{\mathsf{t}[0,k-1]}.$$

Lemma A.3. *For* $\mathsf{s} = e_k(\mathsf{t})$ *as in equation* (A.3)*, we have that* $u_{\mathsf{s}[k-1,k+1]} s_{k,k+1} = u_{\mathsf{s}[k-1,k+1]}$.

Proof. As we have seen in equation (A.4),

$$u_{\mathsf{s}[k-1,k+1]} = e^{(z)}_{k+1} m_{\zeta-\varepsilon_w\to\zeta} s_{|\zeta|_w,|\zeta|} e^{(d)}_{k+\frac12} e^{(c)}_{k} e^{(b)}_{k-\frac12} s_{|\alpha|,|\alpha|_t},$$

with $b = c$ and $d = b+1$. Now $s_{|\alpha|,|\alpha|_t} \in P_{k-1}(n)$ and so it commutes with $s_{k,k+1}$. Moreover, we have

$$e^{(b+1)}_{k+\frac12} e^{(b)}_{k} e^{(b)}_{k-\frac12} = e^{(b+1)}_{k+\frac12}$$

and $e^{(b+1)}_{k+\frac12} s_{k,k+1} = e^{(b+1)}_{k+\frac12}$ by Lemma A.3(5) and (6). Hence $u_{\mathsf{s}[k-1,k+1]} s_{k,k+1} = u_{\mathsf{s}[k-1,k+1]}$ as required. □

Applying Lemma A.3 and noting that $u_{\mathsf{t}[k+1,r]} u_{\mathsf{s}[k-1,k+1]} u_{\mathsf{t}[0,k-1]} = u_{\mathsf{s}}$ we get

$$u_{\mathsf{t}} s_{k,k+1} = u_{\mathsf{t}[k+1,r]} m_{\zeta,u,w} P_k(\mathsf{t}) s_{k,k+1} u_{\mathsf{t}[0,k-1]} + (1-\delta_{u,0})\delta_{u,v} u_{\mathsf{s}}. \tag{A.5}$$

It remains to consider the first term in this sum. Note that $P_k(\mathsf{t})$ is a single partition diagram and so we should, in theory, be able to describe both this set-partition and the set-partition $P_k(\mathsf{t}) s_{k,k+1}$. This calculation can, however, be much simplified by making the following observation. Using [15], we have

$$u_{\mathsf{t}[0,k-1]} = c_{\mathsf{t}(k-1)} d^*_{\mathsf{t}[0,k-1]}$$

where $c_{\mathsf{t}(k-1)} = e^{(a)}_{k-1} \sum_{\sigma\in\mathfrak{S}_\alpha} \sigma \in P_{k-1}(n)$. So the first term in the sum equation (A.5) can be rewritten as follows,

$$\begin{aligned} u_{\mathsf{t}[k+1,r]} m_{\zeta,u,w} P_k(\mathsf{t}) s_{k,k+1} u_{\mathsf{t}[0,k-1]} &= u_{\mathsf{t}[k+1,r]} m_{\zeta,u,w} P_k(\mathsf{t}) s_{k,k+1} e^{(a)}_{k-1} \textstyle\sum_{\sigma\in\mathfrak{S}_\alpha} \sigma d^*_{\mathsf{t}[0,k-1]} \\ &= u_{\mathsf{t}[k+1,r]} m_{\zeta,u,w} \left(P_k(\mathsf{t}) e^{(a)}_{k-1}\right) s_{k,k+1} \textstyle\sum_{\sigma\in\mathfrak{S}_\alpha} \sigma d^*_{\mathsf{t}[0,k-1]} \end{aligned} \tag{A.6}$$

using Lemma A.1(1). Now $P_k(\mathsf{t})e_k^{(a)}$ is also a single partition diagram and can be described (more simply than $P_k(\mathsf{t})$) as follows.

Definition A.4. Let $S = \{S_1, S_2, \dots, S_j\}$ be a set of pairwise disjoint subsets of

$$\{1, \dots, k+1, \bar{1}, \dots, \overline{k+1}\}$$

such that there is a bijection between the barred and unbarred elements of

$$\{1, \dots, k+1, \bar{1}, \dots, \overline{k+1}\} \setminus (S_1 \cup S_2 \cup \cdots \cup S_j).$$

Write

$$\{1, \dots, k+1, \bar{1}, \dots, \overline{k+1}\} \setminus (S_1 \cup S_2 \cup \cdots \cup S_j) = \{i_1 < i_2 \cdots < i_\ell\} \cup \{\bar{j}_1 < \bar{j}_2 < \cdots < \bar{j}_\ell\}.$$

We define $\widehat{S} \in P_{k+1}(n)$ to be the set partition

$$\widehat{S} = S \cup \bigcup_{1 \leqslant m \leqslant \ell} \{\{i_m, \bar{j}_m\}\}.$$

In other words, $\widehat{S}$ contains the blocks $S_1, S_2, \dots S_j$ and determines an order preserving bijection between the barred and unbarred elements of $\{1, \dots, k+1, \bar{1}, \dots, \overline{k+1}\} \setminus (S_1 \cup S_2 \cup \cdots \cup S_j)$.

Example A.5. Let $k+1 = 10$ and

$$S = \{\{4, 9, \bar{6}\}, \{6, 10, \bar{4}\}, \{\bar{9}\}, \{\overline{10}\}\},$$

then

$$\widehat{S} = \{\{4, 9, \bar{6}\}, \{6, 10, \bar{4}\}, \{\bar{9}\}, \{\overline{10}\}, \{1, \bar{1}\}, \{2, \bar{2}\}, \{3, \bar{3}\}, \{5, \bar{5}\}, \{7, \bar{7}\}, \{8, \bar{8}\}\}.$$

Proposition A.6. *We have that*

$$P_k(\mathsf{t})e_{k-1}^{(a)} = \widehat{S_k(\mathsf{t})}$$

where $S_k(\mathsf{t})$ is the set of pairwise disjoint subsets of $\{1, \dots, k+1, \bar{1}, \dots, \overline{k+1}\}$ obtained by omitting all occurrences of 0 and $\bar{0}$ from

$$\Big\{\{|\alpha|_t, k, \overline{|\zeta - \delta_{w,u}\varepsilon_u|_u}\}, \{|\alpha - \delta_{t,v}\varepsilon_v|_v, k+1, \overline{|\zeta|_w}\}, \{k-1-i\}_{0 \leqslant i \leqslant a-1}, \{\overline{k+1-j}\}_{0 \leqslant j \leqslant z-1}\Big\}.$$

Here again, the proof of this proposition is not conceptually difficult but it involves carefully keeping track of the blocks of the set partitions when concatenating diagrams. To help the reader, we start with an example to illustrate the proposition and also provide pictures of the concatenations of diagrams involved at each stage in the proof. Having an explicit description of the element $P_k(\mathsf{t})e_{k-1}^{(a)}$ as a set partition will allow us to understand its product with $s_{k,k+1}$ in the next proposition, and hence conclude the proof of Theorem 3.3.

Example A.7. Let $k+1 = 14$. Let t be any tableau such that

$$\mathsf{t}(12) = (4, 2, 1^2) \xrightarrow{-1} (3, 2, 1^2) \xrightarrow{+2} (3^2, 1^2) \xrightarrow{-2} (3, 2, 1^2) \xrightarrow{+1} (4, 2, 1^2) = \mathsf{t}(14).$$

So that $\alpha = \zeta = (4, 2, 1^2)$, $\beta = \delta = (3, 2, 1^2)$, $\gamma = (3^2, 1^2)$ (so that $t = w = 1$ and $u = v = 2$). Then $\widehat{S}_{13}(\mathsf{t}) = \widehat{S}$ from Example A.5.

Proof of Proposition A.6. By the definition of $P_k(\mathsf{t})$ given in Proposition A.2, we have that

$$P_k(\mathsf{t})e_{k-1}^{(a)} = e_{k+1}^{(z)}\left(s_{|\zeta|_w,|\zeta|}e_{k+\frac{1}{2}}^{(d)}s_{|\gamma|,x}\right)e_k^{(c)}\left(s_{|\gamma|_u,|\gamma|}e_{k-\frac{1}{2}}^{(b)}s_{|\alpha|,|\alpha|_t}\right)e_{k-1}^{(a)}$$

where

$$x = \begin{cases} |\gamma|_u - 1 & \text{if } u = v > 0 \\ |\gamma|_v & \text{otherwise.} \end{cases}$$

By concatenating diagrams, it is easy to see that

$$s_{|\zeta|_w,|\zeta|}e_{k+\frac{1}{2}}^{(d)}s_{|\gamma|,x} = \widehat{S_{k+1}}$$

where

$$S_{k+1} = \Big\{\{k+1, k, \dots, k-d+2, \overline{k+1}, \bar{k}, \dots, \overline{k-d+2}, \overline{|\zeta|_w}, x\}\Big\} \tag{A.7}$$

if $v, w > 0$. If $w = 0$, S_{k+1} is obtained by replacing $\overline{|\zeta|_w}$ with $\overline{k-d+1}$ in equation (A.7) above. If $v = 0$, S_{k+1} is obtained by replacing x with $k-d+1$ in equation (A.7) above. Similarly, we have

$$s_{|\gamma|_u,|\gamma|} e^{(b)}_{k-\frac{1}{2}} s_{|\alpha|,|\alpha|_t} = \widehat{S_{k-1}} \tag{A.8}$$

where

$$S_{k-1} = \left\{ \{k, k-1, \ldots, k-b+1, \overline{k}, \overline{k-1}, \ldots, \overline{k-b+1}, \overline{|\gamma|_u}, |\alpha|_t\} \right\}$$

if $u, t > 0$. If $u = 0$, then S_{k-1} is obtained by replacing $\overline{|\gamma|_u}$ by $\overline{k-b}$ in equation (A.8) above. If $t = 0$, then S_{k-1} is obtained by replacing $\overline{|\alpha|_t}$ by $k-b$ in equation (A.8) above. Now we have

$$P_k(\mathsf{t}) e^{(a)}_{k-1} = e^{(z)}_{k+1} \widehat{S_{k+1}} e^{(c)}_k \widehat{S_{k-1}} e^{(a)}_{k-1},$$

which for $u, v, t, w \neq 0$ can be represented by the concatenation of diagrams of the form depicted in Figure 24, below. This diagram is meant to be seen as a generic example of such a concatenation of diagrams; however, it can also be seen to be the diagram obtained from the path t in Example A.5.

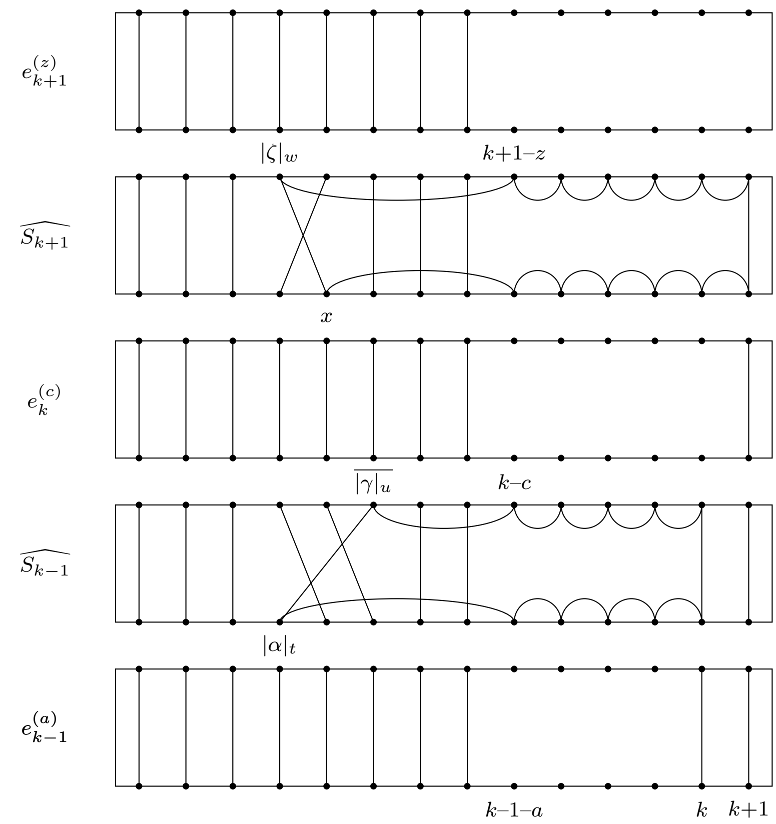


FIGURE 24. An example of the product $P_k(\mathsf{t}) e^{(a)}_{k-1} = e^{(z)}_{k+1} \widehat{S_{k+1}} e^{(c)}_k \widehat{S_{k-1}} e^{(a)}_{k-1}$.

For $u, v, t, w \neq 0$ the result would follow if we can show that

(1) $\{\overline{x}, |\alpha - \delta_{t,v}\varepsilon_v|_v\}$ is a block of $\widehat{S_{k-1}}$;
(2) $\{|\gamma|_u, \overline{|\zeta - \delta_{u,w}\varepsilon_u|_u}\}$ is a block of $\widehat{S_{k+1}}$.

To prove (1), note that $\alpha - \varepsilon_t = \gamma - \varepsilon_u$ and the propagating lines in $\widehat{S_{k-1}}$ give a bijection between the nodes of these two partitions (reading along successive rows starting with the top row). So for $v \neq u$ we have that $\{\overline{|\gamma|_v}, |\alpha|_v\}$ is a block of $\widehat{S_{k-1}}$ unless $v = t$, in which case $\{\overline{|\gamma|_v}, |\alpha|_v - 1\}$ is a block of $\widehat{S_{k-1}}$. Similarly, $\{\overline{|\gamma|_u - 1}, |\alpha|_u\}$ is a a block of $\widehat{S_{k-1}}$ unless $u = t$, in which case $|\gamma|_u = |\alpha|_u$ and $\{\overline{|\gamma|_u - 1}, |\alpha|_u - 1\}$ is a block of $\widehat{S_{k-1}}$.

The proof of (2) follows similarly by noting that $\zeta - \varepsilon_w = \gamma - \varepsilon_v$ and that the propagating lines in $\widehat{S_{k+1}}$ give a bijection between the nodes of these partitions. So we have that $\{|\gamma|_u, \overline{|\zeta|_u}\}$ is a block of $\widehat{S_{k+1}}$ unless $u = w$, in which case $\{|\gamma|_u, \overline{|\zeta|_u - 1}\}$ is a block of $\widehat{S_{k+1}}$. For $u = v$, note that $\{|\gamma|_u, \overline{|\gamma|_u}\}$ is a block of $\widehat{S_{k+1}}$ unless $|\gamma|_u \leqslant |\zeta|_w$, in which case $\{|\gamma|_u, \overline{|\gamma|_u - 1}\}$ is a block of $\widehat{S_{k+1}}$. If $w < u$ then $|\zeta|_w = |\gamma|_w - 1 < |\gamma|_u$ and $|\gamma|_u = |\zeta|_u$ so $\{|\gamma|_u, \overline{|\zeta|_u}\}$ is a block of $\widehat{S_{k+1}}$, as required. If $u < w$ then

$|\gamma|_u - 1 = |\zeta|_u < |\zeta|_w$ so $|\gamma|_u \leqslant |\zeta|_w$ and $\{|\gamma|_u, \overline{|\zeta|_u}\}$ is a block of $\widehat{S_{k+1}}$, as required. Finally, if $u = w$ then $\gamma = \zeta$ and $|\gamma|_u = |\zeta|_u = |\zeta|_w$ and $\{|\gamma|_u, \overline{|\zeta|_u - 1}\}$ is a block of $\widehat{S_{k+1}}$, as required. This completes the proof for $t, u, v, w \neq 0$.

We now consider the cases in which some of t, u, v, w are equal to zero. We treat these as degenerate versions of the above.

Let $w = 0$. This is the simplest degenerate case to describe, however the other cases only differ by superficial book-keeping. If $w = 0$, then $z = d + 1$ and $\gamma - \varepsilon_v = \zeta$. We replace the top two diagrams in Figure 24 by the two diagrams in Figure 25 (which establish the bijection between the nodes of $\gamma - \varepsilon_v$ and ζ). The values of a, b, c, d, $|\alpha|_t$, $|\gamma|_u$ and x go through unchanged. Thus the block containing $k + 1$ in $\widehat{S_k(\mathsf{t})}$ collapses to $\{|\alpha - \delta_{t,v}\varepsilon_v|_v, k + 1\}$ as required.

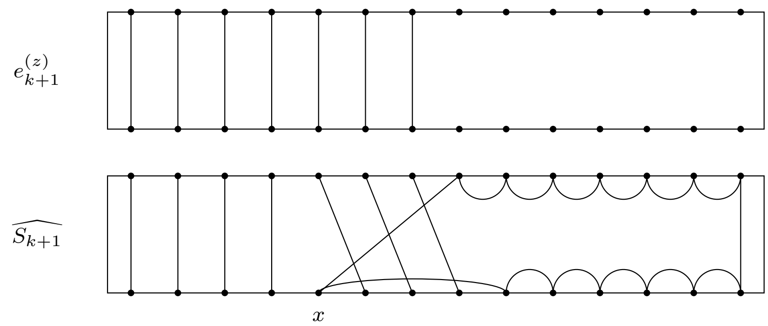


FIGURE 25. The $w = 0$ case.

If $v = 0$, then $c = d$ and $\gamma = \zeta - \varepsilon_w$. We replace $\widehat{S_{k+1}} e_k^{(c)}$ in Figure 24 by the two diagrams in Figure 26 (which establish the bijection between γ and $\zeta - \varepsilon_w$). The values of a, b, c, $|\alpha|_t$, and $|\gamma|_u$ go through unchanged and so the bottom two diagrams of Figure 24 go through unchanged. Therefore the block containing $k + 1$ collapses to $\{k + 1, \overline{|\zeta|_w}\}$ as required. The value of $|\zeta|_w$ will either decrease by 1 (if $w \geqslant v$) or go through unchanged (if $w < t$ as in the case depicted in Figure 26). This results in the necessary superficial edits to the propagating lines in $\widehat{S_{k+1}}$ in order to obtain the required bijection between the nodes of γ and $\zeta - \varepsilon_w$; hence all the blocks of $\widehat{S_k(\mathsf{t})}$ which do not contain $k + 1$ remain unchanged.

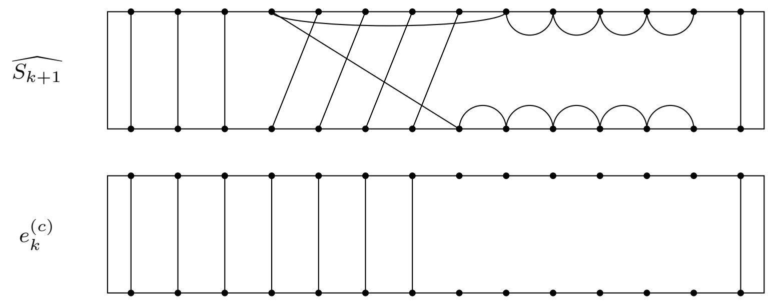


FIGURE 26. The $v = 0$ case.

Similarly if $t = 0$, then $a = b$ and $\alpha = \gamma - \varepsilon_u$. We replace the bottom two diagrams in Figure 24 by the two diagrams in Figure 27 which establish the bijection between α and $\gamma - \varepsilon_u$. Thus the block containing k in $\widehat{S_k(\mathsf{t})}$ collapses to $\{k, |\zeta - \delta_{w,u}\varepsilon_u|_u\}$ as required. As above, one can verify that all other blocks of $S_k(\mathsf{t})$ remain the same, as required.

Finally, if $u = 0$ then $c = b + 1$ and we have $e_k^{(c)} \widehat{S_{k-1}}$ is given by the leftmost diagram in Figure 28. Arguing as above, the block of $S_k(\mathsf{t})$ containing k collapses to $\{|\alpha|_t, k\}$ and all other blocks in $S_k(\mathsf{t})$ remain the same, as required. $\square$

Proposition A.8. *Assume that* $\mathsf{t}_{k \leftrightarrow k+1}$ *exists. Then we have*

$$m_{\zeta,u,w} P_k(\mathsf{t}) e_{k-1}^{(a)} s_{k,k+1} \sum_{\sigma \in \mathfrak{S}_\alpha} \sigma = m_{\zeta,u,w} P_k(\mathsf{t}_{k \leftrightarrow k+1}) e_{k-1}^{(a)} \sum_{\sigma \in \mathfrak{S}_\alpha} \sigma.$$

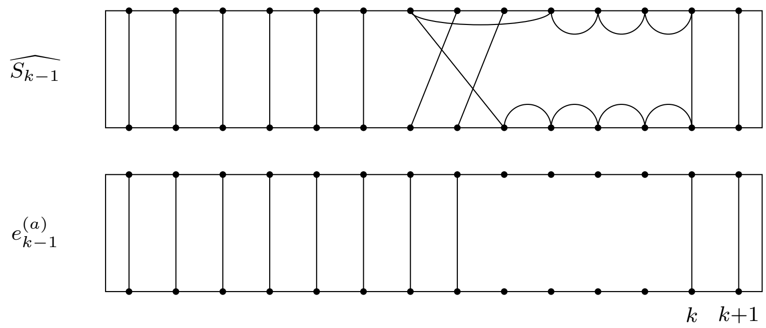


FIGURE 27. The $t = 0$ case.

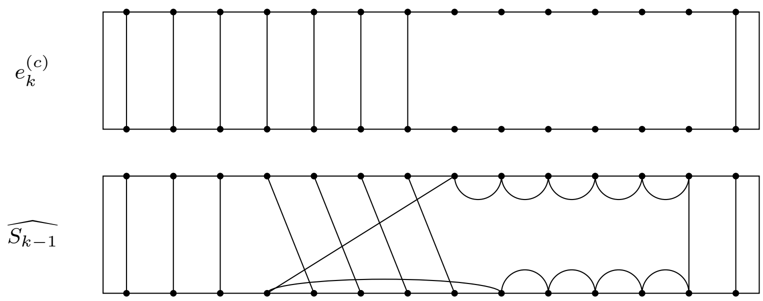


FIGURE 28. The $u = 0$ case.

Proof. Using Proposition A.6 and the fact that $s_{k,k+1}$ swaps k and $k+1$, we have

$$P_k(\mathsf{t})e^{(a)}_{k-1}s_{k,k+1} = \widehat{S_k(\mathsf{t})}s_{k,k+1} = \widehat{S'_k(\mathsf{t})}$$

where $S'_k(\mathsf{t})$ is obtained by omitting all occurrences of 0 and $\overline{0}$ from

$$\Big\{\{|\alpha|_t, k+1, \overline{|\zeta-\delta_{w,u}\varepsilon_u|_u}\}, \{|\alpha-\delta_{t,v}\varepsilon_v|_v, k, \overline{|\zeta|_w}\}, \{k-1-i\}_{0\leqslant i\leqslant a-1}, \{\overline{k+1-j}\}_{0\leqslant j\leqslant z-1}\Big\}.$$

Now, we observe (simply by definition) that $S_k(\mathsf{t}_{k\leftrightarrow k+1})$ is obtained by omitting all occurrences of 0 and $\overline{0}$ from

$$\Big\{\{|\alpha|_v, k+1, \overline{|\zeta-\delta_{w,u}\varepsilon_w|_w}\}, \{|\alpha-\delta_{t,v}\varepsilon_t|_t, k, \overline{|\zeta|_u}\}, \{k-1-i\}_{0\leqslant i\leqslant a-1}, \{\overline{k+1-j}\}_{0\leqslant j\leqslant z-1}\Big\}.$$

So we get

$$\widehat{S'_k(\mathsf{t})} = s_{|\zeta-\delta_{u,w}\varepsilon_w|_w,|\zeta|_w}\widehat{S_k(\mathsf{t}_{k\leftrightarrow k+1})}s_{|\alpha-\delta_{t,v}\varepsilon_t|_t,|\alpha|_t}.$$

If $t \neq v$, then $s_{|\alpha-\delta_{t,v}\varepsilon_t|_t,|\alpha|_t} = 1$ and if $t = v$, then

$$s_{|\alpha-\varepsilon_t|_t,|\alpha|_t}\sum_{\sigma\in\mathfrak{S}_\alpha}\sigma = \sum_{\sigma\in\mathfrak{S}_\alpha}\sigma,$$

as required. If $u \neq w$ then $s_{|\zeta-\varepsilon_w|_w,|\zeta|_w} = 1$. Finally, if $u = w$, then

$$m_{\zeta,w,w} = \sum_{1\leqslant j<i\leqslant|\zeta|_w} s_{|\zeta|_w-j,|\zeta|_w}s_{|\zeta-\varepsilon_w|_w-i,|\zeta-\varepsilon_w|_w}\big(1+s_{|\zeta-\varepsilon_w|_w,|\zeta|_w}\big).$$

Clearly, we have that

$$\big(1+s_{|\zeta-\varepsilon_w|_w,|\zeta|_w}\big)s_{|\zeta-\varepsilon_w|_w,|\zeta|_w} = \big(1+s_{|\zeta-\varepsilon_w|_w,|\zeta|_w}\big)$$

and therefore $(m_{\zeta,w,w})s_{|\zeta-\varepsilon_w|_w,|\zeta|_w} = m_{\zeta,w,w}$. The result follows. □

Finally, we let $\mathsf{t}' := \mathsf{t}_{k\leftrightarrow k+1}$ and $\mathsf{s}' = e_k(\mathsf{t}')$. Combining equation (A.5) and (A.6) and Proposition A.8, we get

$$\begin{aligned}
u_{\mathsf{t}}s_{k,k+1} &= u_{\mathsf{t}[k+1,r]}m_{\zeta,u,w}P_k(\mathsf{t}')e^{(a)}_{k-1}(\textstyle\sum_{\sigma\in\mathfrak{S}_\alpha}\sigma)d^*_{\mathsf{t}[0,k-1]} + (1-\delta_{u,0})\delta_{u,v}u_{\mathsf{s}}\\
&= u_{\mathsf{t}[k+1,r]}m_{\zeta,u,w}P_k(\mathsf{t}')u_{\mathsf{t}[0,k-1]} + (1-\delta_{u,0})\delta_{u,v}u_{\mathsf{s}}\\
&= u_{\mathsf{t}[k+1,r]}\Big(u_{\mathsf{t}'_{[k-1,k+1]}} - (1-\delta_{w,0})\delta_{w,t}u_{\mathsf{s}'[k-1,k+1]}\Big)u_{\mathsf{t}[0,k-1]} + (1-\delta_{u,0})\delta_{u,v})u_{\mathsf{s}}\\
&= u_{\mathsf{t}'} + (1-\delta_{u,0})\delta_{u,v}u_{\mathsf{s}} - (1-\delta_{w,0})\delta_{w,t}u_{\mathsf{s}'}
\end{aligned}$$

which completes the proof of Theorem 3.3.

Appendix B. The Dvir radical

We now set about proving Proposition 3.7, namely that certain paths will always label basis elements of the Dvir radical.

Definition B.1. For $s \in \mathbb{Z}_{\geqslant 0}$ and $(\lambda,\nu)\in\mathscr{P}_{r-s}\times\mathscr{P}_{\leqslant r}$ we define

$$\mathrm{DR}^0\text{-}\mathrm{Std}_s(\lambda\to\nu) = \{\mathsf{t}\in\mathrm{Std}_s(\lambda\to\nu) \mid \sharp\{\text{integral steps of the form } (-\varepsilon_0,+\varepsilon_0) \text{ in } \mathsf{t}\}\geqslant 1\},$$

and for $i\geqslant 1$, we define

$$\mathrm{DR}^i\text{-}\mathrm{Std}_s(\lambda\to\nu) = \{\mathsf{t}\in\mathrm{Std}_s(\lambda\to\nu) \mid \sharp\{\text{steps of the form } -\varepsilon_i \text{ in } \mathsf{t}\} > \lambda_i\}.$$

and we set $\mathrm{DR}\text{-}\mathrm{Std}_s(\lambda\to\nu) = \bigcup_{i\geqslant 0}\mathrm{DR}^i\text{-}\mathrm{Std}_s(\lambda\to\nu)$.

Note that for $i\geqslant 1$ we can also define $\mathrm{DR}^i\text{-}\mathrm{Std}_s(\lambda\to\nu)$ as

$$\mathrm{DR}^i\text{-}\mathrm{Std}_s(\lambda\to\nu) = \{\mathsf{t}\in\mathrm{Std}_s(\lambda\to\nu) \mid \sharp\{\text{steps of the form } +\varepsilon_i \text{ in } \mathsf{t}\} > \nu_i\}.$$

This follows from the fact that $\lambda_i - \sharp\{\text{steps of the form } -\varepsilon_i \text{ in } \mathsf{t}\} + \sharp\{\text{steps of the form } +\varepsilon_i \text{ in } \mathsf{t}\} = \nu_i$.

We can write $u_{\mathsf{t}^\lambda\circ\mathsf{t}}$ as a sum of partition diagrams in $P_r(n)$. In order to prove the above proposition we need to understand some properties of the diagrams that can occur in this sum.

Lemma B.2. *Let* $\mathsf{t} = (-\varepsilon_{i_1},+\varepsilon_{j_1},\dots,-\varepsilon_{i_r},+\varepsilon_{j_r})\in\mathrm{Std}_r(\nu)$. *Write* $u_{\mathsf{t}} = u_{\mathsf{t}[r-1,r]}u_{\mathsf{t}[0,r-1]}$ *where* $\mathsf{t}[0,r-1]\in\mathrm{Std}_{r-1}(\nu')$ *with* $\mathsf{t}(r-1)=\nu'$. *We have that*

(*i*) *if* $i_r, j_r\neq 0$ *then* $u_{\mathsf{t}[r-1,r]} = \sum_{k=0}^{\nu_{j_r}-1} d_k$ *with* d_k *as in the first diagram in Figure 29.*
(*ii*) *if* $i_r=0$, $j_r\neq 0$ *then* $u_{\mathsf{t}[r-1,r]} = \sum_{k=0}^{\nu_{j_r}-1} d_k$ *with* d_k *as in the second diagram in Figure 29.*
(*iii*) *if* $i_r\neq 0$, $j_r=0$ *then* $u_{\mathsf{t}[r-1,r]} = d_0$ *as in the third diagram in Figure 29.*
(*iv*) *if* $i_r=j_r=0$ *then* $u_{\mathsf{t}[r-1,r]} = d_0 = e^{(1)}_r$ *depicted in Figure 10.*

Proof. By definition, we have

$$\begin{aligned}
u_{\mathsf{t}[r-1,r]} &= u_{\mathsf{t}(r-\frac12)\to\mathsf{t}(r)}u_{\mathsf{t}(r-1)\to\mathsf{t}(r-\frac12)}\\
&= \sum_{k=0}^{\nu_{j_r}-1} e^{(r-|\nu|)}_r s_{|\nu|_{j_r}-k,|\nu|_{j_r}} s_{|\nu|_{j_r},|\nu|} e^{(r-1-|\mathsf{t}(r-\frac12)|)}_{r-\frac12} s_{|\nu'|,|\nu'|_{i_r}}\\
&= \sum_{k=0}^{\nu_{j_r}-1} e^{(r-|\nu|)}_r s_{|\nu|_{j_r}-k,|\nu|} e^{(r-1-|\mathsf{t}(r-\frac12)|)}_{r-\frac12} s_{|\nu'|,|\nu'|_{i_r}}.
\end{aligned}$$

The result follows by concatenating the four diagrams in each case. $\square$

Remark B.3. Note that in each of cases (*i*) to (*iv*) of Lemma B.2, the diagrams in Figures 10 and 29 provide the natural bijection between the nodes of $\nu-(j_r,\nu_{j_r}-k)$ and the nodes of $\nu'-(i_r,\nu_{i_r})$.

Lemma B.4. *Let* $\mathsf{t} = (-\varepsilon_{i_1},+\varepsilon_{j_1},\dots,-\varepsilon_{i_r},+\varepsilon_{j_r})\in\mathrm{Std}_r(\nu)$. *Write*

$$u_{\mathsf{t}} = \sum_d \alpha_{d,\mathsf{t}}d \tag{B.1}$$

with $\alpha_{d,\mathsf{t}}\in\mathbb{Z}_{>0}$ *and* d *partition diagrams in* $P_r(n)$. *Then, for any* d *appearing in this sum, we have*

(1) *the northern nodes* $\{\overline{r}\}$, $\{\overline{r-1}\},\dots,\{\overline{r-|\nu|}\}$ *are singleton blocks of* d;
(2) *for each* $1\leqslant i\leqslant \ell(\nu)$, *any northern nodes in the set* $\{\overline{|\nu|_{i-1}+1},\overline{|\nu|_{i-1}+2},\dots,\overline{|\nu|_i}\}$ *is connected to some southern node* k *satisfying* $j_k=i$.

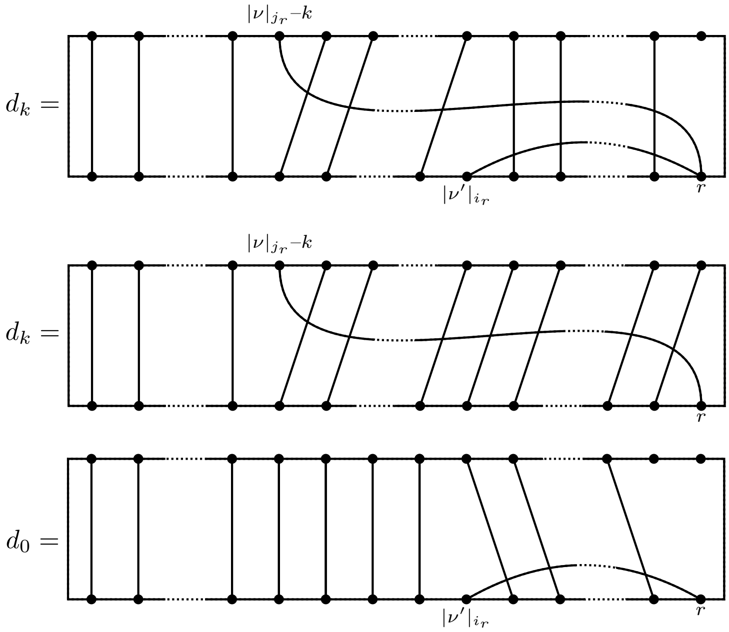


FIGURE 29. The diagrams d_k of parts (i) to (iii) of Lemma B.2 respectively. The first diagram is drawn under the assumption that $|\nu|_{j_r} - k < |\nu'|_{i_r}$, the cases $|\nu|_{j_r} - k = |\nu'|_{i_r}$ and $|\nu|_{j_r} - k > |\nu'|_{i_r}$ are similar.

Proof. Part (1) follows directly from the fact that $u_{\mathsf{t}} = e_r^{(r-|\nu|)}x$ for some $x \in P_r(n)$. We prove (2) by induction on r. If $r = 1$, then either $\mathsf{t} = (-\varepsilon_0, +\varepsilon_0)$ or $\mathsf{t} = (-\varepsilon_0, +\varepsilon_1)$. In the first case, there is nothing to prove. In the second case, we have that $\overline{1}$ is connected to 1, which satisfies $j_1 = 1$, as required. Now assume the result holds for $r-1$. Write $u_{\mathsf{t}} = u_{\mathsf{t}[r-1,r]}u_{\mathsf{t}'}$ where $\mathsf{t}' = \mathsf{t}[0, r-1] \in \mathrm{Std}_{r-1}(\nu')$ with $\mathsf{t}(r-1) = \nu'$. By induction, we write

$$u_{\mathsf{t}'} = \sum_{d'} \alpha_{d',\mathsf{t}'} d'$$

with $\alpha_{d',\mathsf{t}'} \in \mathbb{Z}_{>0}$. For any d' appearing in this sum and any $1 \leqslant k \leqslant \ell(\nu')$, we have that any northern node in the set

$$\{\overline{|\nu'|_{k-1}+1}, \overline{|\nu'|_{k-1}+2}, \ldots, \overline{|\nu'|_k}\}$$

is connected to some southern nodes l satisfying $j_l = i$. Now any diagram, d, appearing in equation (B.1) is of the form $d = d_k d'$ (for cases (i) and (ii)) or $d_0 d'$ (for cases (iii) and (iv)) as in Lemma B.2. If d_0 is as in case (iii) and (iv), then the diagram d_0 provides the natural bijection between the nodes of ν and $\nu' - \varepsilon_{i_r}$ and the result follows. If d_k is as in case (i) or (ii) we must show that $\overline{|\nu|_{j_r} - i}$ is connected to a southern nodes of the required form. (That any other northern node in d_k is connected to a southern node of the required form is immediate, as in cases (iii) and (iv) above.) Now, as $\{\overline{r}, r\}$ is a block of d', we have that $\overline{|\nu|_{j_r} - i}$ is connected to r in $d_k d' = d$ as required. □

Lemma B.5. *Let* $\mathsf{t} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, \ldots, -\varepsilon_{i_r}, +\varepsilon_{j_r}) \in \mathrm{Std}_r(\nu)$. *Write*

$$u_{\mathsf{t}} = \sum_{d} \alpha_{d,\mathsf{t}} d \tag{B.2}$$

with $\alpha_{d,\mathsf{t}} \in \mathbb{Z}_{>0}$ *and* d *partition diagrams in* $P_r(n)$. *For any diagram* d *appearing in this sum and any* $1 \leqslant k \leqslant r$, *we have that*

(a) if $i_k = j_k = 0$ *then the southern node* k *in* d *is a singleton;*
(b) if $i_k \neq 0$ *then the southern node* k *in* d *is connected to a southern node* $l < k$ *with* $j_l = i_k$.

Proof. We prove this lemma by induction on r. If $r = 1$ then $k = 1$ and $i_k = 0$. The only path to consider is $\mathsf{t} = (-\varepsilon_0, +\varepsilon_0)$. In this case we have $u_{\mathsf{t}} = d$ with $d = \{\{\bar{1}\}, \{1\}\}$, so the result holds.

We shall ssume that the result holds for $r-1$ and prove it for r. As in Lemma B.4, we write $u_{\mathsf{t}} = u_{\mathsf{t}[r-1,r]}u_{\mathsf{t}'}$ with $\mathsf{t}' = \mathsf{t}[0, r-1] \in \mathrm{Std}_{r-1}(\nu')$ and $\nu' = \mathsf{t}'(r-1)$. Write

$$u_{\mathsf{t}'} = \sum_{d'} \alpha_{d',\mathsf{t}'} d'$$

with $\alpha_{d',t'} \in \mathbb{Z}_{>0}$ and d' a partition diagram in $P_{r-1}(n) \subset P_r(n)$. By induction, the result holds for all d' in this sum and all $1 \leqslant k \leqslant r-1$. As any diagram d appearing in equation (B.2) has the form $d_k d'$ where the d_k's are given in Lemma B.2, we have that the result holds for d and any $1 \leqslant k \leqslant r-1$. It remains to prove it for $k = r$.

For part (a), note that d_0 is as in Lemma B.2(iv). Now using Lemma B.4(1) we know that $\{\overline{r-1}\}$, $\{\overline{r-2}\}, \ldots, \{\overline{r-1-|\nu'|}\}$ are all singleton blocks in d'. As $\{\bar{r}, r\}$ is a block in d' we deduce that $\{r\}$ is a singleton block in $d = d_0 d'$.

For part (b), note that d_k is as in Lemma B.2(i) or (iii). Thus the southern nodes r and $|\nu'|_{i_r}$ are connected in d_k. But now, using Lemma B.4(2) we have that in d' the northern node $\overline{|\nu'|_{i_r}}$ is connected to some southern node $k \leqslant r-1$ with $j_k = i_r$. Moreover, $\{\bar{r}, r\}$ is a block of d'. Concatenating d_k with d' we deduce that in d the node r is connected to some $k < r$ with $j_k = i_r$ as required. □

Proof of Proposition 3.7. Recall that $\mathsf{DR}_s(\lambda \to \nu) = \Delta_s(\lambda \to \nu) P_s(n) p_r P_s(n)$. So if $m + P_r^{\rhd(\lambda\to\nu)}(n) \in \Delta_s(\lambda \to \nu)$ and $m \in P_r(n) p_r P_s(n)$ then $m + P_r^{\rhd(\lambda\to\nu)}(n) \in \mathsf{DR}_s(\lambda \to \nu)$. Now $P_r(n)p_r P_s(n)$ is spanned by all partition diagrams in $P_r(n)$ having at most $s-1$ distinct blocks containing both an element of the set $\{r-s+1, \ldots r\}$ and an element of the set $\{\bar{1}, \ldots, \bar{r}, 1, \ldots, r-s\}$. We claim that $u_{\mathsf{t}^\lambda \circ \mathsf{t}}$ is a sum of such diagrams for any $\mathsf{t} \in \text{DR-Std}_s(\lambda \to \nu)$. Thus $u_{\mathsf{t}^\lambda \circ \mathsf{t}} \in P_r(n) p_r P_s(n)$ as required.

We now set about proving this claim. Write $\mathsf{t}^\lambda \circ \mathsf{t} = (-\varepsilon_{i_1}, +\varepsilon_{j_1}, \ldots, -\varepsilon_{i_r}, +\varepsilon_{j_r})$ and

$$u_{\mathsf{t}^\lambda \circ \mathsf{t}} = \sum_d \alpha_{\mathsf{t},d}\, d \tag{B.3}$$

with $\alpha_{\mathsf{t},d} \in \mathbb{Z}_{>0}$ and d a partition diagram in $P_r(n)$. First suppose that $\mathsf{t} \in \mathrm{DR}^0\text{-Std}_s(\lambda \to \nu)$. Then there exists $k \geqslant r-s+1$ such that the k-th integral step of $\mathsf{t}^\lambda \circ \mathsf{t}$ has the form $(-\varepsilon_0, +\varepsilon_0)$. Using Lemma B.5(a), we deduce that k is a singleton in *any* diagram d appearing in equation (B.3) and hence $d \in P_r(n) p_r P_s(n)$.

Now suppose that $\mathsf{t} \in \mathrm{DR}^x\text{-Std}_s(\lambda \to \nu)$ for some $x > 0$. Then $M = \{k \mid k \geqslant r-s+1 \,\text{and}\, i_k = x\}$ satisfies $|M| > \lambda_x$. By Lemma B.5(b) for any $k \in M$ and any diagram d appearing in equation (B.3), we have that the southern node k is connected to a southern node $l < k$ satisfying $j_l = x$. Now, by definition of t^λ, there are precisely λ_x such l with $l \leqslant r-s$. We conclude that there must be at least one $k \in M$ such that the southern node k in d is connected to a southern node from the set $\{r-s+1, \ldots, r\}$. This proves that $d \in P_r(n) p_r P_s(n)$ as required. □

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