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A metric version of Poincaré's theorem concerning biholomorphic inequivalence of domains

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Abstract

We show that if $Y_j \subset \mathbb{C}^{n_j}$ is a bounded strongly convex domain with C^3-boundary for $j = 1, \ldots, q$, and $X_j \subset \mathbb{C}^{m_j}$ is a bounded convex domain for $j = 1, \ldots, p$, then the product domain $\prod_{j=1}^{p} X_j \subset \mathbb{C}^m$ cannot be isometrically embedded into $\prod_{j=1}^{q} Y_j \subset \mathbb{C}^n$ under the Kobayashi distance, if $p > q$. This result generalises Poincaré's theorem which says that there is no biholomorphic map from the polydisc onto the Euclidean ball in $\mathbb{C}^n$ for $n \geq 2$.

The method of proof only relies on the metric geometry of the spaces and will be derived from a result for products of proper geodesic metric spaces with the sup-metric. In fact, the main goal of the paper is to establish a general criterion, in terms of certain asymptotic geometric properties of the individual metric spaces, that yields an obstruction for the existence of an isometric embedding between product metric spaces.

Keywords: Product metric spaces, Product domains, Kobayashi distance, isometric embeddings, metric compactification, Busemann points, detour distance

Subject Classification: Primary 32F45; Secondary 51F99

1 Introduction

Numerous theorems in several complex variables are instances of results in metric geometry. In this paper we shall see that a classic theorem due to Poincaré [21], which says that there is no biholomorphic map from the polydisc Δ^n onto the (open) Euclidean ball B_n in $\mathbb{C}^n$ if $n \geq 2$, is a case in point. In fact, it is known [18, 27, 28] that there exists no surjective Kobayashi distance isometry of Δ^n onto B_n. More generally one may wonder when it is possible to isometrically embed a product domain $\prod_{j=1}^{p} X_j \subset \mathbb{C}^m$ into another product domain $\prod_{j=1}^{q} Y_j \subset \mathbb{C}^n$ under the Kobayashi distance. In this paper we show the following result.

Theorem 1.1. *Suppose that $X_j \subset \mathbb{C}^{m_j}$ is a bounded convex domain for $j = 1, \ldots, p$, and $Y_j \subset \mathbb{C}^{n_j}$ is a bounded strongly convex domain with C^3-boundary for $j = 1, \ldots, q$.. If $p > q$, then there is no isometric embedding of $\prod_{j=1}^{p} X_j$ into $\prod_{j=1}^{q} Y_j$ under the Kobayashi distance.*

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Note that Poincaré's theorem is a special case where $p = n \geq 2$ and $q = 1$, as the boundary of the Euclidean ball is smooth. The case where $\sum_j m_j = \sum_j n_j$ and the isometry is surjective was analysed by Zwonek [27, Theorem 2.2.5] who used different methods.

A key property of the Kobayashi distance is the product property, see [12, Theorem 3.1.9]. Indeed, if $X_j \subset \mathbb{C}^{m_j}$ is a bounded convex domain for $j = 1, \dots, p$, then the Kobayashi distance, k_X, on the product domain $X := \prod_{j=1}^{p} X_j$ satisfies

$$k_X(w, z) = \max_{j=1,\dots,p} k_{X_j}(w_j, z_j) \quad \text{for all } w = (w_1, \dots, w_p), z = (z_1, \dots, z_p) \in X.$$

In view of the product property it natural to consider product metric spaces with the sup-metric. Given metric spaces (M_j, d_j), $j = 1, \dots, p$, the *product metric space* $(\prod_{j=1}^{p} M_j, d_\infty)$ is given by

$$d_\infty(x, y) := \max_j d_j(x_j, y_j) \quad \text{for } x = (x_1, \dots, x_p), y = (y_1, \dots, y_p) \in \textstyle\prod_{j=1}^{p} M_j,$$

In this general context it is interesting to understand when one can isometrically embed a product metric space into another one. The main goal of this paper is to establish a general criterion, in terms of certain asymptotic geometric properties of the individual metric spaces, that yields an obstruction for the existence of an isometric embedding between product metric spaces, and to show how this criterion can be used to derive Theorem 1.1.

The key concepts from metric geometry involved are: the horofunction boundary of proper geodesic metric spaces, the Busemann points, and the detour distance, δ, on the set of Busemann points, which will all be recalled in the next section. Our main result is the following.

Theorem 1.2. *Suppose that (M_j, d_j) is a proper geodesic space containing an almost geodesic sequence for $j = 1, \dots, p$, and (N_j, ρ_j) is a proper geodesic metric space such that all its horofunctions are Busemann points, and $\delta(h_j, h'_j) = \infty$ for all $h_j \neq h'_j$ Busemann points of (N_j, ρ_j), for $j = 1, \dots, q$. If $p > q$, then there exists no isometric embedding of $(\prod_{j=1}^{p} M_j, d_\infty)$ into $(\prod_{j=1}^{q} N_j, d_\infty)$.*

The assumptions that each horofunction is a Busemann point and that any two distinct Busemann points lie at infinite detour distance from each other is a type of regularity condition on the asymptotic geometry of the space, which is satisfied by numerous metric spaces, such as finite dimensional normed spaces with smooth norms [24], Hilbert geometries on bounded strictly convex domains with C^1-boundary [25], and, as we shall see in Lemma 3.3, Kobayashi metric spaces (D, k_D), where $D \subset \mathbb{C}^n$ is a bounded strongly convex domain with C^3-boundary.

It turns out that the parts of the horofunction boundary and the detour cost in product metric spaces have a special structure that is closely linked to a quotient space of $(\mathbb{R}^n, 2\|\cdot\|_\infty)$, where $\|x\|_\infty = \max_j |x_j|$. More precisely, if we let $\mathrm{Sp}(\mathbf{1}) := \{\lambda(1, \dots, 1) \in \mathbb{R}^n : \lambda \in \mathbb{R}\}$, then the quotient space $\mathbb{R}^n/\mathrm{Sp}(\mathbf{1})$ with respect to $2\|\cdot\|_\infty$ has the *variation norm* as the quotient norm, which is given by

$$\|\overline{x}\|_{\mathrm{var}} := \max_j x_j + \max_j(-x_j) \quad \text{for } \overline{x} \in \mathbb{R}^n/\mathrm{Sp}(\mathbf{1}), \tag{1.1}$$

see [15, Section 4]. It is known, e.g., [14, Proposition 2.2.4], that $(\mathbb{R}^n/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ is isometric to the Hilbert metric space on the open $(n-1)$-dimensional simplex.

We show in Theorem 2.10 that if, for $j = 1, \dots, q$, we have that (N_j, ρ_j) is a proper geodesic metric space such that all its horofunctions are Busemann points, and $\delta(h_j, h'_j) = \infty$ for all $h_j \neq h'_j$ Busemann points of (N_j, ρ_j), then each part of $(\prod_{j=1}^{q} N_j, d_\infty)$ is isometric to $(\mathbb{R}^n/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ for some $1 \leq n \leq q$.

The work in this paper has links to work by Bracci and Gaussier [6] who studied the interaction between topological properties and the metric geometry of hyperbolic complex spaces. It is also worth mentioning that various other aspects of the metric geometry of product metric spaces have been studied in context of Teichmüller space in [8, 19].

2 The metric compactification of product spaces

In our set-up we will follow the terminology in [11], which contains further references and background on the metric compactification.

Let (M,d) be a metric space, and let $\mathbb{R}^M$ be the space of all real functions on M equipped with the topology of pointwise convergence. Fix $b\in M$, which is called the *basepoint*. Let $\mathrm{Lip}_b^1(M)$ denote the set of all functions $h\in\mathbb{R}^M$ such that $h(b)=0$ and h is 1-Lipschitz, i.e., $|h(x)-h(y)|\leq d(x,y)$ for all $x,y\in M$. Then $\mathrm{Lip}_b^1(M)$ is a closed subset of $\mathbb{R}^M$. Moreover, as

$$|h(x)|=|h(x)-h(b)|\leq d(x,b)$$

for all $h\in\mathrm{Lip}_b^1(M)$ and $x\in M$, we get that $\mathrm{Lip}_b^1(M)\subseteq[-d(x,b),d(x,b)]^M$, which is compact by Tychonoff's theorem. Thus, $\mathrm{Lip}_b^1(M)$ is a compact subset of $\mathbb{R}^M$.

Now for $y\in M$ consider the real valued function

$$h_y(z):=d(z,y)-d(b,y)\quad\text{with } z\in M.$$

Then $h_y(b)=0$ and $|h_y(z)-h_y(w)|=|d(z,y)-d(w,y)|\leq d(z,w)$. Thus, $h_y\in\mathrm{Lip}_b^1(M)$ for all $y\in M$. The closure of $\{h_y\colon y\in M\}$ is called the *metric compactification of* M, and is denoted $\overline{M}^h$. The boundary $\partial\overline{M}^h:=\overline{M}^h\setminus\{h_y\colon y\in M\}$ is called the *horofunction boundary of* M, and its elements are called *horofunctions*. Given a horofunction h and $r\in\mathbb{R}$ the set $\mathcal{H}(h,r):=\{x\in M\colon h(x)<r\}$ is a called a *horosphere* or *horoball*.

We will assume that the metric space (M,d) is *proper*, meaning that all closed balls are compact. Such metric spaces are separable, since every compact metric space is separable. It is known that if (M,d) is separable, then the topology of pointwise convergence on $\mathrm{Lip}_b^1(M)$ is metrizable, and hence each horofunction is the limit of a sequence of functions (h_{y^n}) with $y^n\in M$ for all $n\geq 1$. In general, however, horofunctions are limits of nets (h_{y^α}) with $y^\alpha\in M$ for all $\alpha\in A$.

A curve $\gamma\colon I\to(M,d)$, where I is a possibly unbounded interval in $\mathbb{R}$, is called a *geodesic path* if

$$d(\gamma(s),\gamma(t))=|s-t|\quad\text{for all } s,t\in I.$$

The metric space (M,d) is said to be a *geodesic space* if for each $x,y\in M$ there exists a geodesic path $\gamma\colon[a,b]\to M$ with $\gamma(a)=x$ and $\gamma(b)=y$. A prove of the following well known fact can be found in [13, Lemma 2.1].

Lemma 2.1. *If (M,d) is a proper geodesic metric space, then $h\in\partial\overline{M}^h$ if and only if there exists a sequence (y^n) in M such that $h_{y^n}\to h$ and $d(y^n,b)\to\infty$ as $n\to\infty$.*

A sequence (y^n) in (M,d) is called an *almost geodesic sequence* if $d(y^n,y^0)\to\infty$ as $n\to\infty$, and for each $\varepsilon>0$ there exists $N\geq 0$ such that

$$d(y^m,y^k)+d(y^k,y^0)-d(y^m,y^0)<\varepsilon\quad\text{for all } m\geq k\geq N.$$

The notion of an almost geodesic sequence goes back to Rieffel [22] and was further developed in [4, 16, 23, 24]. In particular, any almost geodesic sequence yields a horofunction, as the following lemma shows.

Lemma 2.2. *Let (M,d) be a proper geodesic metric space. If (y^n) is an almost sequence in M, then*

$$h(x) = \lim_{n\to\infty} d(x,y^n) - d(b,y^n)$$

exists for all $x \in M$ and $h \in \partial\overline{M}^h$.

Proof. Note that for all $\varepsilon > 0$ there exists $N \geq 0$ such that for all $m \geq k \geq N$ we have that

$$d(x,y^m) - d(y^0,y^m) - (d(x,y^k) - d(y^0,y^k)) \leq d(y^m,y^k) + d(y^0,y^k) - d(y^0,y^m) < \varepsilon$$

and

$$d(x,y^k) - d(y^0,y^k) - (d(x,y^m) - d(y^0,y^m)) \geq -d(y^m,y^k) + d(y^0,y^m) - d(y^0,y^k) > -\varepsilon,$$

which shows that $\lim_{n\to\infty} d(x,y^n) - d(y^0,y^n)$ exists for each $x \in M$. This implies that

$$h(x) = \lim_{n\to\infty} d(x,y^n) - d(b,y^n) = \lim_{n\to\infty} d(x,y^n) - d(y^0,y^n) - (d(b,y^n) + d(y^0,y^n))$$

exists for all $x \in M$. It now follows from Lemma 2.1 that $h \in \partial\overline{M}^h$. $\square$

Given a proper geodesic metric space (M,d), a horofunction $h \in \overline{M}^h$ is called a *Busemann point* if there exists an almost geodesic sequence (y^n) in M such that $h(x) = \lim_{n\to\infty} d(x,y^n) - d(b,y^n)$ for all $x \in M$. We denote the collection of all Busemann points by $\mathcal{B}_M$.

It is known that a product metric space $(\prod_{j=1}^p M_j, d_\infty)$, where

$$d_\infty(x,y) = \max_j d_j(x_j,y_j) \quad \text{for } x = (x_1,\dots,x_p), y = (y_1,\dots,y_p) \in \textstyle\prod_{j=1}^p M_j,$$

is a proper geodesic metric space, if each (M_j,d_j) is a proper geodesic metric space, see for instance [20, Proposition 2.6.6]. Moreover, we have the following general fact concerning the horofunctions of product metric spaces.

Theorem 2.3. *For $j = 1,\dots,p$ let (M_j,d_j) be proper geodesic metric spaces. If h is a horofunction of $(\prod_{j=1}^p M_j, d_\infty)$ with basepoint $b = (b_1,\dots,b_p)$, then there exist $J \subseteq \{1,\dots,p\}$ non-empty, a horofunction h_j of (M_j,d_j) with respect to basepoint b_j for $j \in J$, and $\alpha \in \mathbb{R}^J$ with $\min_{j\in J} \alpha_j = 0$ such that h is of the form,*

$$h(x) = \max_{j\in J} h_j(x_j) - \alpha_j \quad \text{for } x = (x_1,\dots,x_p) \in \textstyle\prod_{j=1}^p M_j. \tag{2.1}$$

Moreover, there exists a sequence (y^n) in $\prod_{j=1}^p M_j$ with (h_{y^n}) converging to h such that $(h_{y_j^n})$ converges to h_j for $j \in J$, $d_\infty(y^n,b) - d_j(y_j^n,b_j) \to \infty$ for $j \notin J$, and $d_\infty(y^n,b) - d_j(y_j^n,b_j) \to \alpha_j$ for $j \in J$.

Proof. Let (y^n) be a sequence in $\prod_{j=1}^p M_j$ such that (h_{y^n}) converges to a horofunction h. So $h(x) = \lim_{n\to\infty} d_\infty(x,y^n) - d_\infty(b,y^n)$ for all $x \in \prod_{j=1}^p M_j$. As the product metric space is a proper geodesic metric space, it follows from Lemma 2.1 that $d_\infty(b,y^n) \to \infty$ as $n \to \infty$. Write $y^n := (y_1^n,\dots,y_p^n)$ and let $\alpha_j^n := d_\infty(b,y^n) - d_j(b_j,y_j^n) \geq 0$ for all $j = 1,\dots,p$ and $n \geq 0$.

We may assume, after taking a subsequence, that $h_{y_j^n}(\cdot) := d_j(\cdot,y_j^n) - d_j(b_j,y_j^n)$ converges to $h_j \in \overline{M_j}^h$ and $\alpha_j^n \to \alpha_j \in [0,\infty]$ for all $j \in \{1,\dots,p\}$, and $\alpha_{j_0}^n = 0$ for all $n \geq 0$ for some $j_0 \in \{1,\dots,p\}$. Let $J := \{j \colon \alpha_j < \infty\}$ and note that $j_0 \in J$. So,

$$h(x) = \lim_{n\to\infty} d_\infty(x,y^n) - d_\infty(b,y^n) = \lim_{n\to\infty} \max_j (d_j(x_j,y_j^n) - d_j(b_j,y_j^n) - \alpha_j^n) = \max_{j\in J} h_j(x_j) - \alpha_j.$$

To complete the proof note that $\alpha_j < \infty$ implies that $d_j(b_j,y_j^n) \to \infty$, and hence by Lemma 2.1 we find that h_j is a horofunction of (M_j,d_j) with basepoint b_j. $\square$

For convenience we introduce the following terminology.

Definition 2.4. We call a pair $(h,(y^n))$, where h is a horofunction of $(\prod_{j=1}^p M_j, d_\infty)$ and (y^n) is a sequence in $\prod_{j=1}^p M_j$ a *canonical pair* if they satisfy the properties of Theorem 2.3.

The following notion will be useful in the sequel. A path $\gamma\colon [0,\infty)\to(M,d)$ is a called an *almost geodesic ray* if $d(\gamma(t),\gamma(0))\to\infty$, and for each $\varepsilon>0$ there exists $T\geq 0$ such that

$$d(\gamma(t),\gamma(s))+d(\gamma(s),\gamma(0))-d(\gamma(t),\gamma(0))<\varepsilon \quad \text{for all } t\geq s\geq T.$$

Let (y^n) be an almost geodesic sequence in a geodesic metric space (M,d), and assume that

$$d(y^n,y^0)<d(y^{n+1},y^0) \quad \text{for all } n\geq 0. \tag{2.2}$$

For simplicity we write $\Delta_n := d(y^n,y^0)$ and we let $\gamma_n\colon [0,d(y^{n+1},y^n)]\to(M,d)$ be a geodesic path connecting y^n and y^{n+1}, i.e., $\gamma_n(0)=y^n$ and $\gamma_n(d(y^{n+1},y^n))=y^{n+1}$. for all $n\geq 0$.

We write $I_n := [\Delta_n,\Delta_{n+1}]$ and let $\bar\gamma_n\colon I_n\to(M,d)$ be the affine reparametrisation of γ_n given by

$$\bar\gamma_n(t) := \gamma_n\left(\frac{d(y^{n+1},y^n)}{\Delta_{n+1}-\Delta_n}(t-\Delta_n)\right) \quad \text{for all } t\in I_n.$$

We call the path $\bar\gamma\colon[0,\infty)\to(M,d)$ given by

$$\bar\gamma(t) := \bar\gamma_n(t) \quad \text{for } t\in I_n$$

a *ray induced by* (y^n). Note that $\bar\gamma$ is well defined for all $t\geq 0$ by (2.2).

Lemma 2.5. *If (y^n) is an almost geodesic sequence in a geodesic metric space (M,d) satisfying (2.2), then each ray, $\bar\gamma$, induced by (y^n) satisfies:*

(i) $\bar\gamma$ is an almost geodesic ray,

(ii) the map $t\mapsto d(\bar\gamma(t),\bar\gamma(0))$ is continuous on $[0,\infty)$.

Proof. We first show that for each $\varepsilon>0$ there exists $T\geq 0$ such that

$$d(\bar\gamma(t),y^n)+d(y^n,y^0)-d(\bar\gamma(t),y^0)<\varepsilon \quad \text{for all } t\geq T \text{ and } n\geq 0 \text{ with } t\in I_n. \tag{2.3}$$

To get this inequality just note that there exists $N\geq 0$ such that

$$\begin{aligned} d(\bar\gamma(t),y^n)+d(y^n,y^0)-d(\bar\gamma(t),y^0) &= d(y^{n+1},\bar\gamma(t))+d(\bar\gamma(t),y^n)+d(y^n,y^0)\\ &\quad -d(\bar\gamma(t),y^0)-d(y^{n+1},\bar\gamma(t))\\ &\leq d(y^{n+1},y^n)+d(y^n,y^0)-d(y^{n+1},y^0)<\varepsilon, \end{aligned}$$

for all $n\geq N$, as (y^n) is an almost geodesic sequence. So we can take $T=\Delta_n$.

We need to show that for each $\varepsilon>0$ there exists $T\geq 0$ such that

$$d(\bar\gamma(t),\bar\gamma(s))+d(\bar\gamma(s),\bar\gamma(0))-d(\bar\gamma(t),\bar\gamma(0))<\varepsilon \quad \text{for all } t\geq s\geq T.$$

Suppose that $t > s$ are such that $t \in I_n$ and $s \in I_k$ with $n > k$. Then by using (2.3) we know that for all n and k large,

$$\begin{aligned} d(\bar{\gamma}(t),\bar{\gamma}(s)) + d(\bar{\gamma}(s),\bar{\gamma}(0)) - d(\bar{\gamma}(t),\bar{\gamma}(0)) &\leq d(\bar{\gamma}(t),\bar{\gamma}(s)) + d(\bar{\gamma}(s),y^k) + d(y^k,y^0) \\ &\qquad -d(\bar{\gamma}(t),y^0) \\ &\leq d(\bar{\gamma}(t),y^n) + d(y^n,\bar{\gamma}(s)) + d(\bar{\gamma}(s),y^k) \\ &\qquad +d(y^k,y^0) - d(\bar{\gamma}(t),y^0) \\ &< -d(y^n,y^0) + d(y^n,\bar{\gamma}(s)) + d(\bar{\gamma}(s),y^k) \\ &\qquad +d(y^k,y^0) + \varepsilon \\ &\leq -d(y^n,y^0) + d(y^n,y^{k+1}) + d(y^{k+1},\bar{\gamma}(s)) \\ &\qquad +d(\bar{\gamma}(s),y^k) + d(y^k,y^0) + \varepsilon \\ &= -d(y^n,y^0) + d(y^n,y^{k+1}) + d(y^{k+1},y^k) \\ &\qquad +d(y^k,y^0) + \varepsilon \\ &< -d(y^n,y^0) + d(y^n,y^{k+1}) \\ &\qquad +d(y^{k+1},y^0) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Finally suppose that $t \geq s$ are such that $t, s \in I_n$. Then for all $n \geq 0$ large we have that

$$\begin{aligned} d(\bar{\gamma}(t),\bar{\gamma}(s)) + d(\bar{\gamma}(s),\bar{\gamma}(0)) - d(\bar{\gamma}(t),\bar{\gamma}(0)) &= d(\bar{\gamma}(t),y^n) - d(y^n,\bar{\gamma}(s)) + d(\bar{\gamma}(s),\bar{\gamma}(0)) \\ &\qquad -d(\bar{\gamma}(t),\bar{\gamma}(0)) \\ &\leq d(\bar{\gamma}(t),y^n) + d(y^n,y^0) - d(\bar{\gamma}(t),y^0) < \varepsilon. \end{aligned}$$

To prove the second assertion we note that the affine map

$$t \mapsto \frac{d(y^{n+1},y^n)}{\Delta_{n+1} - \Delta_n}(t - \Delta_n)$$

is a continuous map from I_n onto $[0, d(y^{n_1}, y^n)]$, and the map $\gamma_n \colon [0, d(y^{n+1}, y^n)] \to (M, d)$ is continuous, as γ_n is a geodesic. Thus, the map $t \mapsto d(\bar{\gamma}(t), \bar{\gamma}(0))$ is continuous on the interior of the interval I_n for all $n \geq 0$. To get continuity at the endpoints we simply note that for all $n \geq 0$,

$$\lim_{t \to \Delta_n^-} d(\bar{\gamma}(t),\bar{\gamma}(0)) = d(y^n,\bar{\gamma}(0)) = \lim_{t \to \Delta_n^+} d(\bar{\gamma}(t),\bar{\gamma}(0)),$$

which completes the proof. □

Lemma 2.6. *If (y^n) is an almost geodesic sequence in a geodesic metric space (M, d) satisfying (2.2) and $\bar{\gamma}$ is a ray induced by (y^n), then for each sequence (β^n) in $[0, \infty)$ with $\beta^{n+1} > \beta^n$ for all $n \geq 0$ there exists sequence (t^n) in $[0, \infty)$ with $t^{n+1} > t^n$ for all $n \geq 0$ such that $d(\bar{\gamma}(t^n), \bar{\gamma}(0)) = \beta^n$ for all $n \geq 0$.*

Proof. Let $\Delta_n = d(y^n, y^0)$ and $I_n := [\Delta_n, \Delta_{n+1}]$ for $n \geq 0$. As $d(y^n, y^0) \to \infty$, we know there exists $n_0 \geq 0$ such that

$$\Delta_{n_0} \leq \beta^0 \leq \Delta_{n_0+1}.$$

Now take n_0 as small as possible. By Lemma 2.5(ii) we know that there exists $t^0 \in I_{n_0}$ such that $d(\bar{\gamma}(t^0), \bar{\gamma}(0)) = \beta^0$ by the intermediate value theorem. For $\beta^1 > \beta^0$ we know there exists $n_1 \geq n_0$ such that $\beta^1 \in I_{n_1}$ and $n_1 \geq n_0$ is as small as possible. If $n_1 = n_0$, then there exists $t^1 > t^0$ with $t^1 \in I_{n_0}$ such that $d(\bar{\gamma}(t^1), \bar{\gamma}(0)) = \beta^1$, as $d(\bar{\gamma}(t^0), \bar{\gamma}(0)) = \beta^0 < \beta^1 \leq d(y^{n_0+1}, \bar{\gamma}(0))$. If $n_1 > n_0$, then there exists $t^1 \in I_{n_1}$ such that $d(\bar{\gamma}(t^1), \bar{\gamma}(0)) = \beta^1$, as $d(y^{n_1}, \bar{\gamma}(0)) \leq \beta^1 \leq d(y^{n_1+1}, \bar{\gamma}(0))$. Repeating this argument yields the desired sequence (t^n). □

2.1 Detour distance

Suppose that (M, d) is a proper geodesic metric space. Given two horofunctions $h_1, h_2 \in \partial\overline{M}^h$ and sequence (z^n) and (w^n) such that $h_{z^n} \to h_1$ and $h_{w^m} \to h_2$ the *detour cost* is defined by

$$H(h_1, h_2) := \lim_{n\to\infty} d(b, z^n) + \lim_{m\to\infty} d(z^n, w^m) - d(b, w^m) = \lim_{n\to\infty} d(b, z^n) + h_2(z^n).$$

and the *detour distance* is given by

$$\delta(h_1, h_2) := H(h_1, h_2) + H(h_2, h_1).$$

Note that for all $m, n \geq 0$ we have that

$$d(b, z^n) + d(z^n, w^m) - d(b, w^m) \geq 0,$$

so that $H(h_1, h_2) \geq 0$ for all $h_1, h_2 \in \partial\overline{M}^h$. It is, however, possible for $H(h_1, h_2)$ to be infinite. It can be shown, see [16, Section 3] or [23, Section 2] that the detour distance is independent of the basepoint.

The detour distance was introduced in [4] and has been exploited and further developed in [16, 23]. It is known, see for instance [16, Section 3] or [23, Section 2], that on $\mathcal{B}_M \subseteq \partial\overline{M}^h$ the detour distance is symmetric, satisfies the triangle inequality, and $\delta(h_1, h_2) = 0$ if and only if $h_1 = h_2$. This yields a partition of $\mathcal{B}_M$ into equivalence classes, where $h_!$ and h_2 are said to be equivalent if $\delta(h_1, h_2) < \infty$. The equivalence class of h will be denoted by $\mathcal{P}(h)$. Thus, the set of Busemann points, $\mathcal{B}_M$, is the disjoint union of metric spaces under the detour distance, which are called *parts* of $\mathcal{B}_M$.

Isometric embeddings between metric spaces can be extended to the parts of the metric spaces as detour distance isometries. Indeed, suppose that $\varphi\colon (M, d) \to (N, \rho)$ is an *isometric embedding*, i.e., $\rho(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in M$. (Note that φ need not be onto.) If h is a Busemann point of (M, d) with basepoint b and (z^n) is an almost geodesic sequence such that (h_{z^n}) converges to h, then (u^n), with $u^n := \varphi(z^n)$ for $n \geq 0$, is an almost geodesic sequence in (N, ρ), and hence (h_{u^n}) converges to a Busemann point, say $\varphi(h)$, of (N, ρ) with basepoint $\varphi(b)$.

We note that $\varphi(h)$ is independent of the almost geodesic sequence (z^n). To see this let (w^n) be another almost geodesic such that (h_{w^n}) converges to h. Write $v^n := \varphi(w^n)$ for $n \geq 0$ and let $\varphi(h)'$ be the limit of (h_{v^n}). Then

$$\begin{aligned} H(h, h) &= \lim_{n\to\infty} d(w^n, b) + \lim_{m\to\infty} d(w^n, z^m) - d(b, z^m) \\ &= \lim_{n\to\infty} \rho(v^n, \varphi(b)) + \lim_{m\to\infty} \rho(v^n, u^m) - \rho(\varphi(b), u^m) \\ &= H(\varphi(h)', \varphi(h)). \end{aligned}$$

Likewise, $H(\varphi(h), \varphi(h)') = H(h, h)$, and we deduce that $\delta(\varphi(h)', \varphi(h)) = H(\varphi(h)', \varphi(h)) + H(\varphi(h), \varphi(h)') = \delta(h, h) = 0$, which shows that $\varphi(h)' = \varphi(h)$, as $\varphi(h)'$ and $\varphi(h)$ are Busemann points. Thus, there exists a well defined map $\Phi\colon \mathcal{B}_M \to \mathcal{B}_N$ given by $\Phi(h) := \varphi(h)$.

Lemma 2.7. *If $\varphi\colon (M,d) \to (N,\rho)$ is an isometric embedding, then $\Phi(\mathcal{P}(h)) \subseteq \mathcal{P}(\varphi(h))$ for all Busemann points h of (M,d) and*

$$\delta(h',h) = \delta(\Phi(h'),\Phi(h)) \quad \textit{for all } h,h' \in \mathcal{B}_M.$$

Proof. Let (z^n) and (w^n) be almost geodesic sequences such that (h_{z^n}) converges to h and (h_{w^n}) converges to h' in (M,d) with basepoint b. Then

$$\begin{aligned} H(h',h) &= \lim_{n\to\infty} d(w^n,b) + \lim_{m\to\infty} d(w^n,z^m) - d(b,z^m) \\ &= \lim_{n\to\infty} \rho(v^n,\varphi(b)) + \lim_{m\to\infty} \rho(v^n,u^m) - \rho(\varphi(b),u^m) \\ &= H(\varphi(h)',\varphi(h)). \end{aligned}$$

Likewise, $H(h,h') = H(\varphi(h),\varphi(h)')$, so that $\delta(h',h) = \delta(\Phi(h'),\Phi(h))$, which completes the proof. □

It could happen that all parts consist of a single Busemann point, but there are also natural instances where there are nontrivial parts. In case of products of metric spaces coming from proper geodesic metric spaces, it turns out that the parts and the detour distance have a special structure that is linked to the quotient space, $(\mathbb{R}^n/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ given in (1.1).

Proposition 2.8. *If, for $j = 1,\dots,p$, (M_j,d_j) is proper geodesic metric spaces with almost geodesic sequence (y_j^n) and corresponding Busemann point h_j with basepoint y_j^0, and $J \subseteq \{1,\dots,p\}$ is non-empty, then the following assertions hold:*

(i) For $\alpha \in \mathbb{R}^J$ with $\min_{j\in J}\alpha_j = 0$ there exists a canonical pair $(h,(z^n))$ such that (z^n) is an almost geodesic sequence and h is a Busemann point of $(\prod_{j=1}^p M_j, d_\infty)$ with basepoint $y^0 = (y_1^0,\dots,y_p^0)$ of the form,

$$h(x) = \max_{j\in J} h_j(x_j) - \alpha_j, \quad \textit{for } x \in \textstyle\prod_{j=1}^p M_j. \tag{2.4}$$

(ii) If $\beta \in \mathbb{R}^J$ with $\min_{j\in J}\beta_j = 0$ and $(h',(w^n))$ is a canonical pair such that (w^n) is an almost geodesic sequence and h' is a Busemann point of $(\prod_{j=1}^p M_j, d_\infty)$ with basepoint $y^0 = (y_1^0,\dots,y_p^0)$ of the form,

$$h'(x) = \max_{j\in J} h_j(x_j) - \beta_j, \quad \textit{for } x \in \textstyle\prod_{j=1}^p M_j,$$

then $\delta(h,h') = \|\alpha-\beta\|_{\mathrm{var}}$.

(iii) For h as in (2.4) the part $(\mathcal{P}(h),\delta)$ contains an isometric copy of $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$.

Proof. We know there exists an almost geodesic sequence (y_j^n) in (M_j,d_j) such that $h_{y_j^n} \to h_j$ as $n\to\infty$. for each $j \in J$. As $d_j(y_j^n,b_j) \to \infty$ by Lemma 2.1 we can take a subsequence and assume that $d_j(y_j^{n+1},y_j^0) > d_j(y_j^n,y_j^0) > \alpha_j$ for all $n \geq 1$. Let $\bar{\gamma}_j$ be a ray induced by (y_j^n).

For $j \in J$ we get from Lemma 2.6 a sequence (t_j^n) in $[0,\infty)$ with $t_j^0 = 0$ and

$$d_j(\gamma_j(t_j^n),y_j^0) = (\max_{i\in J} d_i(y_i^n,y_i^0)) - \alpha_j \geq 0 \quad \text{for all } n \geq 1.$$

Let $z^0 := (y_1^0,\dots,y_p^0)$ and for $n \geq 1$ define $z^n = (z_1^n,\dots,z_p^n) \in \prod_{j=1}^p M_j$ by $z_j^n := \bar{\gamma}_j(t_j^n)$ if $j \in J$, and $z_j^n := y_j^0$.

As $\min_{j\in J}\alpha_j = 0$, we have for all $j\in J$ and $n\geq 1$ by construction that

$$d_\infty(z^n,z^0) = \max_{i\in J} d_i(y_i^n,y_i^0) = d_j(z_j^n,z_j^0)+\alpha_j.$$

Moreover, it follows from Lemma 2.5 that (z_j^n) is an almost geodesic sequence for all $j\in J$.

We claim that (z^n) is an almost geodesic sequence in $(\prod_{j=1}^p M_j, d_\infty)$. Indeed, note that for all $n\geq k\geq 0$ we have that

$$d_\infty(z^n,z^k)+d_\infty(z^k,z^0)-d_\infty(z^n,z^0) = d_j(z_j^n,z_j^k)+d_\infty(z^k,z^0)-d_\infty(z^n,z^0)$$

for some $j=j(n,k)\in J$, as $d_j(z_j^n,z_j^k)=0$ for all $j\notin J$. As J is finite, we find for all $n\geq k$ large that

$$d_\infty(z^n,z^k)+d_\infty(z^k,z^0)-d_\infty(z^n,z^0) = d_j(z_j^n,z_j^k)+d_j(z_j^k,z_j^0)+\alpha_j-d_j(z_j^k,z_j^0)-\alpha_j<\varepsilon.$$

Also for $n\geq 0$ large and $x\in\prod_{j=1}^p M_j$ we have that

$$h_{z^n}(x) = \max_{j\in J}(d_j(x_j,z_j^n)-d_\infty(z^n,z^0)) = \max_{j\in J}(d_j(x_j,z_j^n)-d_j(z_j^n,z_j^0)-\alpha_j).$$

Letting $n\to\infty$ gives

$$h(x) = \max_{j\in J} h_j(x_j)-\alpha_j \quad \text{for all } x\in \textstyle\prod_{j=1}^p M_j$$

and shows that h is a Busemann point with basepoint $y^0=(y_1^0,\ldots,y_p^0)$. This completes the proof of assertion (i).

To prove the second assertion note that if $(h',(w^n))$ is a canonical pair as in part (ii), then

$$\begin{aligned}
\lim_{n\to\infty} d_\infty(w^n,y^0)+h(w^n) &= \lim_{n\to\infty} d_\infty(w^n,y^0)+\max_{j\in J}(h_j(w_j^n)-\alpha_j)\\
&= \max_{j\in J}\big(\lim_{n\to\infty} d_\infty(w^n,y^0)+h_j(w_j^n)-\alpha_j)\\
&= \max_{j\in J}\big(\lim_{n\to\infty} d_j w_j^n,y_j^0)+\beta_j+h_j(w_j^n)-\alpha_j)\\
&= \max_{j\in J}(H(h_j,h_j)+\beta_j-\alpha_j)\\
&= \max_{j\in J}(\beta_j-\alpha_j).
\end{aligned}$$

Interchanging the roles of h and h', we find that

$$\delta(h',h) = H(h',h)+H(h,h') = \max_{j\in J}(\beta_j-\alpha_j)+\max_{j\in J}(\alpha_j-\beta_j) = \|\alpha-\beta\|_{\mathrm{var}}.$$

The final assertion is a direct consequence of the previous two, as $(S,\|\cdot\|_{\mathrm{var}})$ with $S:=\{\alpha\in\mathbb{R}^J\colon \min_{j\in J}\alpha_j=0\}$ is isometric to $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}),\|\cdot\|_{\mathrm{var}})$. $\square$

It is interesting to understand when a part $(\mathcal{P}(h),\delta)$ is isometric to $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}),\|\cdot\|_{\mathrm{var}})$. The following proposition will be useful in the analysis of this problem.

Proposition 2.9. *Suppose, for $j=1,\ldots,q$, that (N_j,ρ_j) is a proper geodesic metric space such that all horofunctions are Busemann points, and $\delta(h_j,h_j')=\infty$ for every $h_j\neq h_j'$ Busemann points of (N_j,ρ_j). If $(h,(z^n))$ and $(h',(w^n))$ are canonical pairs of $(\prod_{j=1}^q N_j,d_\infty)$ with basepoint b such that*

$$h(x)=\max_{j\in J} h_j(x_j)-\alpha_j \quad \textit{and} \quad h'(x)=\max_{j\in J'} h_j'(x_j)-\beta_j,$$

then $\delta(h,h')=\infty$ whenever $J\neq J'$, or, $h_k\neq h_k'$ for some $k\in J\cap J'$.

Proof. Suppose that $J \neq J'$ and $k \in J$, but $k \notin J'$. As (z^n) and (w^n) are canonical sequences converging to h and h', repsectively, we know that

$$d_\infty(z^n, b) - d_k(z^n_k, b_k) \to \alpha_k \quad \text{and} \quad d_\infty(w^n, b) - d_k(w^n_k, b_k) \to \infty, \quad \text{as } n \to \infty.$$

This implies that

$$\begin{aligned}\lim_{m\to\infty} d_\infty(w^n, z^m) - d_\infty(b, z^m) &= \lim_{m\to\infty} d_\infty(w^n, z^m) - d_k(b_k, z^m_k) - \alpha_k \\ &\geq \lim_{m\to\infty} d_k(w^n_k, z^m_k) - d_k(b_k, z^m_k) - \alpha_k \\ &\geq -d_k(w^n_k, b_k) - \alpha_k,\end{aligned}$$

so that

$$\lim_{n\to\infty} d_\infty(w^n, b) + \lim_{m\to\infty} d_\infty(w^n, z^m) - d_\infty(b, z^m) \geq \lim_{n\to\infty} d_\infty(w^n, b) - d_k(w^n_k, b_k) - \alpha_k = \infty.$$

Thus, $H(h', h) = \infty$ and hence $\delta(h', h) = \infty$.

Now suppose that $h_k \neq h'_k$ for some $k \in J \cap J'$. By assumption we know that $\delta(h'_k, h_k) = \infty$. Note that

$$\lim_{n\to\infty} d_\infty(w^n, b) + h(w^n) = \lim_{n\to\infty} d_\infty(w^n, b) + \max_{j\in J} h_j(w^n_j) - \alpha_j \geq \liminf_{n\to\infty} d_k(w^n_k, b_k) + h_k(w^n_k) - \alpha_k,$$

which shows that $H(h', h) \geq H(h'_k, h_k)$, as $\alpha_k \geq 0$. Interchanging the roles of h and h' we also get that $H(h, h') \geq H(h_k, h'_k)$, and hence $\delta(h', h) \geq \delta(h'_k, h_k) = \infty$. □

Theorem 2.10. *If, for $j = 1, \dots, q$, (N_j, ρ_j) is a proper geodesic metric space such that all horofunctions are Busemann points, and $\delta(h_j, h'_j) = \infty$ for all $h_j \neq h'_j$ Busemann points of (N_j, ρ_j), then every horofunction of $(\prod_{j=1}^q N_j, d_\infty)$ is a Busemann point. Moreover, if $(h, (z^n))$ is a canonical pair of $(\prod_{j=1}^q N_j, d_\infty)$ with*

$$h(x) = \max_{j\in J} h_j(x_j) - \alpha_j \quad \text{for } x \in \textstyle\prod_{j=1}^q N_j,$$

then $(\mathcal{P}(h), \delta)$ is isometric to $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$.

Proof. Let h be a horofunction of $(\prod_{j=1}^q N_j, d_\infty)$ with respect to basepoint $b = (b_1, \dots, b_q)$. By Theorem 2.3 we know that h is of the form

$$h(x) = \max_{j\in J} h_j(x_j) - \alpha_j \quad \text{for } x \in \textstyle\prod_{j=1}^q N_j,$$

and h_j is a horofunction of (N_j, ρ_j) with respect to basepoint b_j for each $j \in J$. As each horofunction of (N_j, ρ_j), is a Busemann point, there exists an almost geodesic sequence (y^n_j) such that $(h_{y^n_j})$ converges to h_j with basepoint b_j.

For $j \notin J$ let $y^0_j = b_j$ and define $y^0 := (y^0_1, \dots, y^0_q)$. Let h_{j,y^0_j} be the Busemann point obtained by changing the base point of h_j to y^0_j, so $h_{j,y^0_j}(x_j) := h_j(x_j) - h_j(y^0_j)$. Now note that if we change the basepoint for h to y^0, we get the Busemann point

$$\begin{aligned}h_{y^0}(x) &:= h(x) - h(y^0) \\ &= \max_{j\in J} h_j(x_j) - \alpha_j - \max_{i\in J} (h_i(y^0_i) - \alpha_i) \\ &= \max_{j\in J} (h_{j,y^0_j}(x_j) + h_j(y^0_j) - \alpha_j - \max_{i\in J} (h_i(y^0_i) - \alpha_i)) \\ &= \max_{j\in J} h_{j,y^0_j}(x_j) - \gamma_j,\end{aligned}$$

where $\gamma_j := \max_{i\in J}(h_i(y_i^0) - \alpha_i) - (h_j(y_j^0) - \alpha_j) \geq 0$ for $j \in J$ and $\min_{j\in J}\gamma_j = 0$. It now follows from Proposition 2.8 that h_{y^0} is a Busemann point of $(\prod_{j=1}^q N_j, d_\infty)$ with respect to basepoint y^0, and hence h is a Busemann point $(\prod_{j=1}^q N_j, d_\infty)$ with respect to basepoint b.

To prove the second assertion we note that $(\mathcal{P}(h), \delta)$ is isometric to $(\mathcal{P}(h_{y^0}), \delta)$, since δ is independent of the basepoint. If h' is a Busemann point of of $(\prod_{j=1}^q N_j, d_\infty)$ with respect to basepoint y^0, then by Theorem 2.3 we know that there exists a canonical pair $(h', (w^n))$ and h' is of the form

$$h'(x) = \max_{j\in J'} h'_j(x_j) - \beta_j, \quad \text{for } x \in \textstyle\prod_{j=1}^q N_j. \tag{2.5}$$

If $J \neq J'$, or, $J = J'$ and $h_k \neq h'_k$ for some $k \in J$, we know by Proposition 2.9 that $\delta(h, h') = \infty$. On the other hand, if $J = J'$ and $h_j = h'_j$ for all $j \in J$, then it follows from Proposition 2.8(i) that $\delta(h.h') = \|\alpha - \beta\|_{\mathrm{var}}$. Moreover, it follows from that Proposition 2.8(i) that for each $\beta \in \mathbb{R}^J$ with $\min_{j\in J}\beta_j = 0$ there exists a canonical pair $(h', (w^n))$ such that h' is as above, and hence $\mathcal{P}(h_{y^0})$ consists of all h' of the form (2.5), where $\min_{j\in J}\beta_j = 0$. So if we let $S := (\beta \in \mathbb{R}^J\colon \min_{j\in J}\beta_j = 0\}$, then $(\mathcal{P}(h_{y^0}), \delta)$ is isometric to $(S, \|\cdot\|_{\mathrm{var}})$, which in turn is isometric to the quotient space $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$. □

An elementary example is the product space $(\mathbb{R}^n, d_\infty)$ where $d_\infty(x, y) = \max_j |x_j - y_j|$. It is easy to verify that $(\mathbb{R}, |\cdot|)$ with basepoint 0 has only two horofunctions, namely $h_+\colon x \mapsto x$ and $h_-\colon x \mapsto -x$, both of which are Busemann points and $\delta(h_+, h_-) = \infty$. So, in this case we see that the horofunctions h of $(\mathbb{R}^n, d_\infty)$ are all Busemann points and of the form,

$$h(x) = \max_{j\in J} \pm x_j - \alpha_j,$$

for some $J \subseteq \{1, \ldots, n\}$ non-empty and $\alpha \in \mathbb{R}^J$ with $\min_{j\in J}\alpha_j = 0$, where the sign is fixed for each $j \in J$, see also [9, Theorem 5.2]. Moreover, $(\mathcal{P}(h), \delta)$ is isometric to $(\mathbb{R}^J/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$.

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. As each (M_j, d_j) contains an almost geodesic sequence for $j = 1, \ldots, p$, we know from Proposition 2.8(i) there exists a canonical pair $(h, (z^n))$ with h a Busemann point of $(\prod_{j=1}^p M_j, d_\infty)$ of the form $h(x) = \max_{j=1,\ldots,p} h_j(x_j)$ for $x \in \prod_{j=1}^p M_j$. Moreover, it follows from the third part of the same proposition that $(\mathcal{P}(h), \delta)$ contains an isometric copy of $(\mathbb{R}^p/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$.

Now suppose, for the sake of contradiction, that there exists an isometric embedding $\varphi\colon (\prod_{j=1}^p M_j, d_\infty) \to (\prod_{j=1}^q N_j, d_\infty)$. Then it follows from Lemma 2.7 that the restriction of Φ to $\mathcal{P}(h)$ yields an isometric embedding of $(\mathcal{P}(h), \delta)$ into $(\mathcal{P}(\Phi(h)), \delta)$. By theorem 2.3 there exists a sequence (y^n) such that (h_{y^n}) converges to $\Phi(h)$. It now follows from Theorem 2.10 that $(\mathcal{P}(\Phi(h)), \delta)$ is isometric to $(\mathbb{R}^k/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ for some $k \in \{1, \ldots, q\}$. As $(\mathcal{P}(h), \delta)$ contains an isometric copy of $(\mathbb{R}^p/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$, Φ yields an isometric embedding of $(\mathbb{R}^p/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ into $(\mathbb{R}^k/\mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$ with $k < p$, which contradicts Brouwer's invariance of domains theorem [7]. □

3 Product domains in $\mathbb{C}^n$

Before we show how we can use Theorem 1.2 to derive Theorem 1.1, we first recall some basic facts concerning the Kobayashi distance, see [12, Chapter 4] for more details. On the disc,

$\Delta := \{z \in \mathbb{C}\colon |z| < 1\}$, the *hyperbolic distance* is given by

$$\rho(z,w) := \log \frac{1 + \left|\frac{w-z}{1-\bar{z}w}\right|}{1 - \left|\frac{w-z}{1-\bar{z}w}\right|} = 2\tanh^{-1}\left(1 - \frac{(1-|w|^2)(1-|z|^2)}{|1-w\bar{z}|^2}\right)^{1/2} \quad \text{for } z,w \in \Delta.$$

Given a convex domain $D \subseteq \mathbb{C}^n$ the *Kobayashi distance* is given by

$$k_D(z,w) := \inf\{\rho(\zeta,\eta)\colon \; \exists f\colon \Delta \to D \text{ holomorphic with } f(\zeta) = z \text{ and } f(\eta) = w\}.$$

for all $z, w \in D$. This identity is due to Lempert [17], who also showed that on bounded convex domains the Kobayashi distance coincides with the *Caratheodory distance*, which is given by

$$c_D(z,w) := \sup_f \rho(f(z), f(w)) \quad \text{for all } z,w \in D,$$

where the sup is taken over all holomorphic maps $f\colon D \to \Delta$.

It is known, see [1, Proposition 2.3.10], that if $D \subset \mathbb{C}^n$ is bounded convex domain, then (D, k_D) is a proper metric space, whose topology coincides with the usual topology on $\mathbb{C}^n$. Moreover, (D, k_D) is a geodesic metric space containing geodesics rays, see [1, Theorem 2.6.19] or [12, Theorem 4.8.6].

In the case of the Euclidean ball $B^n := \{(z_1,\ldots,z_n) \in \mathbb{C}^n\colon \|z\|^2 < 1\}$, where $\|z\|^2 = \sum_i |z_i|^2$, the Kobayashi distance has an explicit formula:

$$k_{B^n}(z,w) = 2\tanh^{-1}\left(1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-\langle z,w\rangle|^2}\right)^{1/2}$$

for all $z, w \in B^n$, see [1, Chapters 2.2 and 2.3].

On the other hand, on the polydisc $\Delta^n := \{(z_1,\ldots,z_n) \in \mathbb{C}^n\colon \max_i |z_i| < 1\}$ the Kobayashi distance satisfies

$$k_{\Delta^n}(z,w) = \max_i \rho(z_i,w_i) \quad \text{for all } w = (w_1,\ldots,w_n), z = (z_1,\ldots,z_n) \in \Delta^n,$$

by the product property, see [12, Theorem 3.1.9].

To determine the horofunctions (B^n, k_{B^n}), with basepoint $b = 0$, it suffices to consider limits of sequences (h_{w_n}), where $w_n \to \xi \in \partial B^n$ in norm. As

$$k_{B^n}(z,w_n) = \log \frac{\left(|1-\langle z,w_n\rangle| + (|1-\langle z,w_n\rangle|^2 - (1-\|z\|^2)(1-\|w_n\|^2))^{1/2}\right)^2}{(1-\|z\|^2)(1-\|w_n\|^2)},$$

and

$$k_{B^n}(0,w_n) = \log \frac{1+\|w_n\|}{1-\|w_n\|},$$

it follows that

$$\begin{aligned} h(z) &= \lim_{n\to\infty} k_{B^n}(z,w_n) - k_{B^n}(0,w_n) \\ &= \log \frac{(|1-\langle z,\xi\rangle| + |1-\langle z,\xi\rangle|)^2}{(1-\|z\|^2)(1+\|\xi\|^2)} \\ &= \log \frac{|1-\langle z,\xi\rangle|^2}{1-\|z\|^2}. \end{aligned}$$

for all $z \in B^n$. Thus, if we write

$$h_\xi(z) := \log \frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} \quad \text{for all } z \in B^n, \tag{3.1}$$

then we obtained $\partial \overline{B^n}^h = \{h_\xi \colon \xi \in \partial B^n\}$, see also [10, Remark 3.1] and [3, Lemma 2.28]. Moreover, each h_ξ is a Busemann point, as it is the limit induced by the geodesic ray $t \mapsto \frac{e^t - 1}{e^t + 1}\xi$, for $0 \le t < \infty$.

Corollary 3.1. *If h_ξ and h_η are distinct horofunctions of (B^n, k_{B^n}), then $\delta(h_\xi, h_\eta) = \infty$.*

Proof. If $\xi \neq \eta$ in ∂B^n, then

$$\lim_{z \to \eta} k_{B^n}(z, 0) + h_\xi(z) = \lim_{z \to \eta} \log \frac{1 + \|z\|}{1 - \|z\|} + \log \frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} = \infty,$$

which implies that $\delta(h_\xi, h_\eta) = \infty$. □

Note that if $n = 1$ we recover the well-known expression for the horofunctions of the hyperbolic distance on Δ:

$$h_\xi(z) = \log \frac{|1 - z\bar{\xi}|^2}{1 - |z|^2} = \log \frac{|\xi - z|^2}{1 - |z|^2} \quad \text{for all } z \in \Delta,$$

Combining (3.1) with Theorems 2.3 and 2.10 we get the following.

Corollary 3.2. *For $B^{n_1} \times \cdots \times B^{n_q}$ the Kobayashi distance horofunctions with basepoint $b = 0$ are precisely the functions of the form,*

$$h(z) = \max_{j \in J} \left(\log \frac{|1 - \langle z_j, \xi_j \rangle|^2}{1 - \|z_j\|^2} - \alpha_j \right),$$

where $J \subseteq \{1, \ldots, q\}$ non-empty, $\xi_j \in \partial B^{n_j}$ for $j \in J$, and $\min_{j \in J} \alpha_j = 0$. Moreover, each horofunction is a Busemann point, and $(\mathcal{P}(h), \delta)$ is isometric to $(\mathbb{R}^J / \mathrm{Sp}(\mathbf{1}), \|\cdot\|_{\mathrm{var}})$.

Corollary 3.2 should be compared with [1, Proposition 2.4.12].

Lemma 3.3. *If $D \subset \mathbb{C}^n$ is a bounded strongly convex domain with C^3-boundary, then each horofunction of (D, k_D) is a Busemann point and $\delta(h, h') = \infty$ for each $h \neq h'$ in $\partial \overline{D}^h$.*

Proof. Let $h \neq h'$ be horofunctions. As (D, k_D) is a proper geodesic metric space, we know there exists sequences (w_n) and (z_n) in D such that $h_{w_n} \to h$ and $h_{z_n} \to h'$ as $n \to \infty$. By taking a further subsequence we may assume that $w_n \to \xi \in \partial D$ and $z_n \to \eta \in \partial D$, since D has a compact norm closure and h and h' are horofunctions.

We claim that $\xi \neq \eta$. To prove this we need the assumption that $D \subset \mathbb{C}$ is bounded strongly convex domain with C^3-boundary and use results by Abate [2] concerning the so-called small and large horospheres. These are defined as follows: for $R > 0$ the *small horosphere* with center $\zeta \in \partial D$ (and basepoint $b \in D$) is given by

$$\mathcal{E}(\zeta, R) := \left\{ x \in D \colon \limsup_{z \to \zeta} k_D(x, z) - k_D(b, z) < \frac{1}{2} \log R \right\}$$

and the *large horosphere* with center $\zeta \in \partial D$ (and basepoint $b \in D$) is given by

$$\mathcal{F}(\zeta, R) := \left\{x \in D\colon \liminf_{z\to\zeta} k_D(x,z) - k_D(b,z) < \frac{1}{2}\log R\right\}.$$

We note that the horoballs,

$$\mathcal{H}(h, \frac{1}{2}\log R) = \left\{x \in D\colon \lim_{n\to\infty} k_D(x,w_n) - k_D(b,w_n) < \frac{1}{2}\log R\right\}$$

and

$$\mathcal{H}(h', \frac{1}{2}\log R) = \left\{x \in D\colon \lim_{n\to\infty} k_D(x,z_n) - k_D(b,z_n) < \frac{1}{2}\log R\right\}$$

satisfy

$$\mathcal{E}(\xi, R) \subseteq \mathcal{H}(h, \frac{1}{2}\log R) \subseteq \mathcal{F}(\xi, R) \quad \text{and} \quad \mathcal{E}(\eta, R) \subseteq \mathcal{H}(h', \frac{1}{2}\log R) \subseteq \mathcal{F}(\eta, R).$$

It follows from [1, Theorem 2.6.47] (see also [2]) that $\mathcal{E}(\xi,R) = \mathcal{H}(h, \frac{1}{2}\log R) = \mathcal{F}(\xi,R)$ and $\mathcal{E}(\eta,R) = \mathcal{H}(h', \frac{1}{2}\log R) = \mathcal{F}(\eta,R)$, as D strongly convex and has C^3-boundary. Thus, if $\xi = \eta$, then $h = h'$, since the horoballs, $\mathcal{H}(h,r)$ and $\mathcal{H}(h',r)$ for $r \in \mathbb{R}$, completely determine the horofunctions. This shows that $\xi \neq \eta$.

On the other hand, if (w^n) converges to $\xi \in \partial D$, then by taking a subsequence we may assume that (h_{w^n}) converges to a horofunction h_ξ, and the previous claim shows that h_ξ is unique. It follows that there is a one-to-one correspondence between the horofunctions of (D, k_D) and $\xi \in \partial D$. The fact that each horofunction is a Busemann point follows from [1, Theorem 2.6.45], which implies that for each $\xi \in \partial D$ there exists a unique geodesic ray $\gamma\colon [0,\infty) \to D$ such that $\gamma(0) = b$ and $\lim_{t\to\infty} \gamma(t) = \xi$, if $D \subset \mathbb{C}$ is bounded strongly convex domain with C^3-boundary.

To show the second assertion note that, as D is strongly convex, D is strictly convex, i.e., for each $\nu \neq \mu$ in ∂D the open straight line segment $(\nu,\mu) \subset D$. Thus $\partial D \cap \mathrm{cl}(\mathcal{H}(h,r)) = \{\xi\}$ and $\partial D \cap \mathrm{cl}(\mathcal{H}(h',r)) = \{\eta\}$ for all $r \in \mathbb{R}$, since the horoballs $\mathcal{H}(h,r)$ and $\mathcal{H}(h',r)$ are convex. Hence there exists a neighbourhood $W \subset \mathbb{C}^n$ of η such that $W \cap \mathrm{cl}(\mathcal{H}(h,0)) = \emptyset$. We deduce that

$$H(h',h) = \lim_{k\to\infty} k_D(w_k, b) + h(w_k) \geq \lim_{k\to\infty} k_D(w_k,b) = \infty,$$

since $h(w_k) \geq 0$ for all k large. This implies that $\delta(h,h') = \infty$. □

The proof of Theorem 1.1 is now elementary.

Proof of Theorem 1.1. If $X_j \subset \mathbb{C}^{m_j}$ is a bounded convex domain, then (X_j, k_{X_j}) is proper geodesic metric space which contains a geodesic ray by [1, Theorem 2.6.19]. Moveover, if $Y_j \subset \mathbb{C}^{n_j}$ is a bounded strongly convex domain with C^3-boundary, then by Lemma 3.3 all the horofunctions of (Y_j, k_{Y_j}) are Busemann points and any two distinct Busemann points have infinite detour distance. So, Theorem 1.2 applies and gives the desired result. □

Remark 3.4. I am grateful to Andrew Zimmer for sharing the following observations with me. In the case where $q = 1$, Theorem 1.1 can be strengthened and proved in a variety of other ways. Indeed, it was shown by Balogh and Bonk [5] that the Kobayashi distance is Gromov hyperbolic on a strongly pseudo-convex domains with C^2-boundary, but the Kobayashi distance on a product domain is clearly not Gromov hyperbolic. This immediately implies Theorem

1.1 for $q = 1$ in the more general case where the image domain is strongly pseudo-convex and has C^2-boundary.

In fact, if $q = 1$ there exists a further strengthening of Theorem 1.1 which only requires the image domain to be strictly convex by using a local argument. The isometric embedding is a locally Lipschitz map with respect to the Euclidean norm, and hence differentiable almost everywhere by Rademacher's theorem. This implies that the embedding is also an isometric embedding under the Kobayashi infinitesimal metric. On strictly convex domains, the unit balls in the tangent spaces are strictly convex and in product domains they are not, which yields a contradiction.

Finally, for holomorphic isometric embeddings and $q = 1$, Theorem 1.1 can be extended to the case where the image domain is convex with $C^{1,\alpha}$-boundary, see [26, Theorem 2.22].

It would be interesting to understand if the regularity conditions on the domains Y_j in Theorem 1.1 can be relaxed. In particular one may speculate that it sufficient to assume that each domain Y_j is strictly convex and has a C^1-boundary.

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