Three computational approaches to weakly nonlocal Poisson brackets

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Abstract

We compare three different ways of checking the Jacobi identity for weakly nonlocal Poisson brackets using the theory of distributions, of pseudodifferential operators and of Poisson vertex algebras, respectively. We show that the three approaches lead to similar computations and same results.

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Contents

1 Introduction

An autonomous system of evolutionary PDEs

$$u^i_t = f^i(u^j, u^j_x, u^j_{xx}, \ldots), \qquad i, j = 1, \ldots, n \tag{1}$$

in two independent variables t, x and n dependent variables (u^j) is said to be *Hamiltonian* with respect to a *local Hamiltonian structure* if it can be written as

$$u^i_t = P^{ij}\frac{\delta H}{\delta u^i}, \tag{2}$$

where

$$H = \int h(u^j, u^j_x, u^j_{xx}, \dots)\, dx$$

is a local functional called *the Hamiltonian functional*, $\delta/\delta u^i$ are the variational derivatives and

$$P^{ij} = \sum_{\sigma \geq 0} P^{ij\sigma}(u^k, u^k_x, u^k_{xx}, \dots)\partial_x^\sigma$$

is a Hamiltonian operator. This means that the bilinear map on the space of local functionals defined by

$$\{F, G\}_P = \int \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j}\, dx, \tag{3}$$

satisfies the following properties

- skew-symmetry: $\{G, F\}_P = -\{F, G\}_P$.
- Jacobi identity: $\{\{F, G\}, H\} + \{\{H, F\}, G\} + \{\{G, H\}, F\} = 0$.

We point out that in this infinite dimensional framework the Leibniz property cannot be required since the product of local functionals is not a local functional. If the Hamiltonian operator satisfies the conditions above the local functional $\{F, G\}_P$ is called *the Poisson bracket of F and G*. The above definitions were proposed at the end of the '60 in order to mimick the widely known finite-dimensional Hamiltonian formalism for systems of ODEs and to introduce the notion of integrability for Hamiltonian PDEs by analogy.

Famous examples of Hamiltonian evolutionary PDEs come from the theory of solitons. The prototype of such equations is the Korteweg–de Vries (KdV) equation, that was shown to be a completely integrable Hamiltonian system in [40, 25]. We refer to the book [33] and to [16] for a general introduction to this subject and the books [17, 32] for an account of the role played by Hamiltonian formalism in the study of evolutionary PDEs integrable via the inverse scattering transform (see also [1]).

In the case of first order quasilinear systems of evolutionary PDEs (systems of hydrodynamic type)

$$u^i_t = V^i_j(u^k)u^j_x, \qquad i = 1, \dots, n, \tag{4}$$

the relevant class of Hamiltonian operators was introduced by Dubrovin and Novikov in [14]. Since the right hand side of the system (4) are differential polynomials of degree 1 it is natural to consider homogeneous differential operators of the same degree, i.e. operators of the form

$$P^{ij} = g^{ij}(u^h)\partial_x - g^{il}(u^h)\Gamma^j_{lk}(u^h)u^k_x. \tag{5}$$

Assuming that g is non degenerate and imposing the skew symmetry and the Jacobi identity, Dubrovin and Novikov proved that g^{ij} must be the contravariant components of a flat (pseudo)-metric and Γ^j_{lk} the Christoffel symbols of the associated Levi-Civita connection. Hamiltonian operators of the form (8) are called local Hamiltonian operators of hydrodynamic type.

The Hamiltonian formalism can be extended to nonlocal brackets defined by pseudo-differential operators. We refer to [38, 39] for a list of equations admitting nonlocal Poisson brackets.

In this paper we focus on the class of weakly nonlocal Hamilonian operators introduced in [29]. They are Hamiltonian operators of the form

$$P^{ij} = \text{local differential operator} + \sum_\alpha c^\alpha w^i_\alpha \partial_x^{-1} w^j_\alpha, \tag{6}$$

where $w^i_\alpha = w^i_\alpha(u^j, u^j_x, u^j_{xx}, \ldots)$ and c^α are constants. The operator ∂_x^{-1} is defined as

$$\partial_x^{-1} = \frac{1}{2}\int_{-\infty}^{x} dy - \frac{1}{2}\int_{x}^{+\infty} dy. \tag{7}$$

Due to the presence of the nonlocal 'tail' the Poisson bracket of two local functionals in general is not a local functional. For this reason a rigorous definition of the associated Poisson bracket requires a suitable extension of the space of allowed functionals (see [34] for a detailed discussion of this point).

The first examples of weakly nonlocal Hamiltonian operator appeared in [36] in the study of Krichever–Novikov equation:

$$P = u_x \partial_x^{-1} u_x.$$

Multi component generalizations of this operator have been studied in [31, 22]. Further examples of weakly nonlocal Poisson brackets arise in the study of evolutionary systems of PDEs like KdV equation, the AKNS equation, Nonlinear Schrödinger equation, the Sine-Gordon and the Liouville equations written in laboratory coordinates [4, 28, 38, 39].

In the case of systems of hydrodynamic type the nonlocal extension of Dubrovin-Novikov Hamiltonian operators was introduced by Ferapontov and

Mokhov in [18] in the special case of metrics of constant curvature, and further generalized by Ferapontov in [19, 20]. The Ferapontov class is defined by operators P of the form

$$P^{ij} = g^{ij}\partial_x - g^{il}\Gamma^j_{lk}u^k_x + \sum_\alpha c^\alpha w^i_{\alpha k}u^k_x\partial_x^{-1}w^j_{\alpha h}u^h_x, \tag{8}$$

where c^α are constants and other coefficients are functions of the field variables (u^i); g is assumed to be non degenerate. Like in the local case the conditions coming from skew symmetry and Jacobi identity have a nice geometric interpretation. For instance, considering for simplicity a nonlocal tail containing a single term, one obtains the conditions

$$g^{ij} = g^{ji}, \tag{9a}$$
$$g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k, \tag{9b}$$
$$g^{is}\Gamma^{jk}_s = g^{js}\Gamma^{ik}_s, \tag{9c}$$
$$g^{is}w^j_s = g^{js}w^i_s \tag{9d}$$
$$\nabla_i w^j_k = \nabla_k w^j_i, \tag{9e}$$
$$R^{ij}_{kh} = w^i_k w^j_h - w^j_k w^i_h. \tag{9f}$$

where ∇ is the linear connection with Christoffel symbols Γ^k_{ij}, $\Gamma^{ij}_k = -g^{il}\Gamma^j_{lk}$ and $R^{ij}_{kh} = g^{is}R^j_{skh}$ is the Riemannian curvature. The above conditions, first obtained in [19] (for details of computations see [20, 34]), admit the following interpretation: the first three equations appear also in the local case and allow us to regard the functions g^{ij} as the contravariant components of a (pseudo)-euclidean metric and Γ^{ij}_k as the Christoffel symbols of the corresponding Levi-Civita connection, while the remaining equations coincide with the classical Gauss-Peterson-Mainardi-Codazzi equations for submanifolds with a flat normal connection in (pseudo)-Euclidean space. The (pseudo)-metric g and the affinor w can be identified with the induced metric and the Weingarten operator respectively.

Many examples of systems of PDEs that admit nonlocal Hamiltonian operators of the type (8) have been found so far: besides the simplest examples of the AKNS system and the Nonlinear Schrödinger equation [38, 39], we recall the Riemann invariant forms of the shallow water equation and of the chromatography equation [20, 21].

Hamiltonian operators of the form (8) are called weakly nonlocal Hamiltonian operators of hydrodynamic type.

It was conjectured in [20] that every diagonalizable first order quasilinear system of PDEs of the form (4) which fulfills an integrability property (semi-Hamiltonianity, that implies the existence of infinitely many generalized symmetries) is Hamiltonian with respect to a suitable weakly nonlocal Poisson bracket of hydrodynamic type (8) (with possibly an infinite sum in the nonlocal tail). A strategy to prove this conjecture based on inverse scattering techniques was proposed by V.E. Zakharov in [41]. In the case of first order quasilinear systems obtained as reductions of dispersionless KP and 2D Toda hierachies an explicit formula of the weakly nonlocal Poisson bracket in terms of the conformal maps defining the reductions was found in [22] and [5] respectively.

Since the dispersionless limit of a large class of evolutionary systems of PDEs consists in a system of first order quasilinear PDEs, it is natural to expect that weakly nonlocal Poisson brackets of hydrodynamic type and their deformations will play an important role in their description. This notwithstanding, the study of weakly nonlocal Poisson brackets has been quite limited so far, especially if compared to local Poisson brackets of hydrodynamic type and their dispersive deformations. The main reason is probably the much higher computational difficulties with respect to the local case.

In the literature one can find (at least) three approaches to the Hamiltonian formalism for PDEs:

1. the approach with distributions [14, 15, 16];

2. the approach with differential operators [3, 10, 33];

3. a new algebraic approach based on Poisson Vertex Algebras, introduced in [2] for local Poisson brackets and later extended to nonlocal Poisson brackets in [9].

The aim of this work is to illustrate an algorithmic procedure to compute the Jacobi identity for weakly nonlocal Poisson brackets in the three formalisms above. We hope in this way to make it accessible to the widest possible audience, ranging from theoretical physicsts to pure mathematicians.

In the case of distributions, the algorithm has been introduced in [28] in order to study the bi-Hamiltonian structure of the Liouville and sine-Gordon PDEs. In the case of differential operators, the algorithm is shown here for the first time thanks to the explicit correspondence between the languages of distributions and differential operators. In the case of Poisson

Vertex Algebra, the algorithm is obtained observing that the infinitely many conditions coming from Jacobi identity appearing in [9] reduce to a finite set (at least) in the case of weakly nonlocal operators. A nontrivial application of this procedure can be found in [7].

In all cases, the algorithm consists in the reduction of the Jacobi identity for weakly nonlocal operators to a canonical form: this is practically achieved by means of identities between distributions, or integration by parts, or algebraic manipulations. The dictionary between the three formalisms shows that there is a bijective correspondence between the canonical forms in the three formalism, and that the computations that are performed in order to reduce the Jacobi identity to the canonical form are the same (6).

In order to illustrate the algorithm and the correspondence between the different formalisms we will consider the case of weakly nonlocal Poisson brackets of hydrodynamic type.

The paper is organized as follows. In Sections $2, 3$ and 4 we explain the algorithm to check Jacobi identity in the three formalisms and we write a sort of dictionary between the three approaches. The remaining sections are devoted to illustrate the algorithm in the case of weakly nonlocal Poisson brackets of hydrodynamic type. We consider the case where the nonlocal tail contains a single term but the computations can be performed in the same way in the general case.

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2 Jacobi identity and distributions

Here we briefly introduce weakly nonlocal Poisson brackets as distributions and describe the algorithm for bringing the Jacobi identity to a reduced canonical form.

2.1 The Jacobi identity

Following [29], we consider weakly nonlocal Poisson brackets of the form

$$\{u^i(x), u^j(y)\}_P = \sum_{k\geq 0} B^{ij}_k(u^h, u^h_\sigma)\delta^{(k)}(x-y) \\ + e^\alpha w^i_\alpha(u^k, u^k_\sigma)\nu(x-y)w^j_\alpha(u^k, u^k_\sigma) \quad (10)$$

where $\nu(x-y) = \frac{1}{2}\,\mathrm{sgn}(x-y)$.

The Jacobi identity

$$\{u^i(x), u^j(y)\}_P, u^k(z)\}_P + \{u^k(z), u^i(x)\}_P, u^j(y)\}_P \\ + \{u^j(y), u^k(z)\}_P, u^i(x)\}_P = 0 \quad (11)$$

can be written as [16]

$$J^{ijk}_{xyz} = \frac{\partial P^{ij}_{x,y}}{\partial u^l_\sigma(x)}\partial^\sigma_x P^{lk}_{x,z} + \frac{\partial P^{ij}_{x,y}}{\partial u^l_\sigma(y)}\partial^\sigma_y P^{lk}_{y,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^l_\sigma(z)}\partial^\sigma_z P^{lj}_{z,y} + \\ \frac{\partial P^{ki}_{z,x}}{\partial u^l_\sigma(x)}\partial^\sigma_x P^{lj}_{x,y} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_\sigma(y)}\partial^\sigma_y P^{li}_{y,x} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_\sigma(z)}\partial^\sigma_z P^{li}_{z,x} = 0, \quad (12)$$

where $P^{ij}_{x,y} = \{u^i(x), u^j(y)\}_P$. The vanishing of the distribution J^{ijk}_{xyz} means that for any choice of the test functions $p_i(x), q_j(y), r_k(z)$ the triple integral

$$\iiint J^{ijk}_{xyz} p_i(x) q_j(y) r_k(z)\, dxdydz \quad (13)$$

should vanish.

2.2 The algorithm

Following [28], we present a procedure to collect together all terms which are related by a distributional identity. We call the result of this procedure the *reduced form* of the Jacobi identity.

1. Using the identity

$$\nu(z-y)\delta(z-x) = \nu(x-y)\delta(x-z) \tag{14}$$

and its two obvious analogues obtained by a cyclic permutation of the variables, together with their differential consequences, we can eliminate all terms containing $\nu(z-y)\delta^{(n)}(z-x)$, $\nu(y-x)\delta^{(n)}(y-z)$, $\nu(x-z)\delta^{(n)}(x-y)$ producing nonlocal terms containing $\nu(x-y)\delta^{(n)}(x-z)$, $\nu(z-x)\delta^{(n)}(z-y)$, $\nu(y-z)\delta^{(n)}(y-x)$ and additional local terms.

2. Using the identity

$$f(z)\delta^{(n)}(x-z) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\delta^{(n-k)}(x-z), \tag{15}$$

(and its cyclic permutations) we can eliminate the dependence on z in the coefficients of the terms containing $\nu(x-y)\delta^{(n)}(x-z)$, the dependence on y in the coefficients of the terms containing $\nu(z-x)\delta^{(n)}(z-y)$ and the dependence on x in the coefficients of the terms containing $\nu(y-z)\delta^{(n)}(y-x)$. After the first two steps the nonlocal part of J^{ijk}_{xyz} has the form

$$\begin{aligned} a_1(x,y,z)\nu(x-y)\nu(x-z) + \mathrm{cyclic}(x,y,z) \\ + \sum_{n\geq 0} b_n(x,y)\nu(x-y)\delta^{(n)}(x-z) + \mathrm{cyclic}(x,y,z). \end{aligned} \tag{16}$$

3. The local part of J^{ijk}_{xyz} (which contain also some additional terms coming from the nonlocal part) can be treated as usual and reduced to the form

$$\sum_{m,n} e_{mn}(x)\delta^{(m)}(x-y)\delta^{(n)}(x-z) \tag{17}$$

using the identities (and their differential consequences)

$$\delta(z-x)\delta(z-y) = \delta(y-x)\delta(y-z) = \delta(x-y)\delta(x-z) \tag{18}$$

and the identities (15).

It is easy to check that no further simplifications are possible. We will see later that the fulfillment of the Jacobi identity turns out to be equivalent to the vanishing of each coefficient in the reduced form.

3 Jacobi identity and differential operators

3.1 The Jacobi identity

The conditions under which the bracket (3) is a Poisson bracket can be written as requirements on the differential operator P (6). We recall that the operator P is a variational bivector [3, 10, 33], hence it is defined up to total divergencies. We consider Poisson brackets defined by differential operators of the form $P^{ij} = B^{ij\sigma}\partial_\sigma + e^\alpha w^i_\alpha \partial_x^{-1} w^j_\alpha$ (6). Then, it is well-known that

- the skew-symmetry of $\{,\}_J$ is equivalent to the formal skew-adjointness of P, $P^* = -P$;
- the Jacobi identity for $\{,\}_P$ is equivalent to the vanishing of the *Schouten bracket* $[P, P] = 0$.

Note that the Schouten bracket of two variational bivectors is a variational three-vector, *i.e.*, it is a skew-symmetric differential operator with three arguments whose value is defined up to total divergencies.

In coordinates, the formal adjoint P^* of the operator P is

$$P^*(\psi)^j = (-1)^{|\sigma|}\partial_\sigma(B^{ij\sigma}\psi_i) - e^\alpha w^j_\alpha \partial_x^{-1}(w^i_\alpha \psi_i); \tag{19}$$

here and in what follows $\psi = (\psi_i)$ is a covector of differential functions $\psi_i = \psi_i(u^j, u^j_x, u^j_{xx}, \ldots)$ We stress that the non-local summand of weakly nonlocal operators is skew-adjoint by construction: we have $(\partial_x^{-1})^* = -\partial_x^{-1}$.

Let us denote by $\ell_{P,\psi}(\varphi)$ the linearization of the (coefficients of the) operator P. We have the following coordinate expressions:

$$\begin{aligned}\ell_{P,\psi}(\varphi)^i = \frac{\partial B^{ij\sigma}}{\partial u^k_\tau}\partial_\sigma \psi^1_j \partial_\tau \varphi^k + e^\alpha \frac{\partial w^i_\alpha}{\partial u^k_\tau}\partial_\tau \varphi^k \partial_x^{-1}(w^j_\alpha \psi_j)& \\ + e^\alpha w^i_\alpha \partial_x^{-1}\left(\frac{\partial w^j_\alpha}{\partial u^k_\tau}\partial_\tau \varphi^k \psi_j\right),&\end{aligned} \tag{20}$$

where we used (106) and the fact that ∂_x^{-1} commutes with linearization. Then, we have the following expression for the Schouten bracket:

$$[P, P](\psi^1, \psi^2, \psi^3) = 2[\ell_{P,\psi^1}(P(\psi^2))(\psi^3) + \mathrm{cyclic}(\psi^1, \psi^2, \psi^3)], \tag{21}$$

where square brackets denote the fact that the expression is calculated up to total divergencies. We observe that the expression of the Schouten bracket of two operators can be written in different ways, which differ up to total divergencies. In the Appendix we wrote two more expressions that are more commonly used in the formalism of differential operators, together with a proof of their equivalence.

3.2 Dictionary: distributions and differential operators

Here we present a dictionary between the language of operators and the language of distributions for the reader's convenience. The calculus with distributions is defined in [16, Subsect. 2.3].

The following notation for a local multivector coincide:

$$P = B^{i_1 i_2 \cdots i_k}_{\sigma_2 \cdots \sigma_k}(u^i(x^1), u^i_\sigma(x^1))\,\delta^{(\sigma_2)}(x^1 - x^2)\cdots\delta^{(\sigma_k)}(x^1 - x^k), \tag{22}$$

$$P = \int B^{i_1 i_2 \cdots i_k}_{\sigma_2 \cdots \sigma_k}\psi^1_{i_1}\partial^{\sigma_2}_{i_2}\psi^2\cdots\partial^{\sigma_k}_{i_k}\psi^k\,dx. \tag{23}$$

In particular the value of the multivector in the distributional notation is obtained by evaluating it on test vector functions of the arguments x^1, ..., x^k. The above correspondence can easily be extended between the nonlocal multivectors (10) and (6). Then, it is clear that the expressions (12) and (21) coincide up to the evaluation on test vector functions.

3.3 The algorithm

The result of the Schouten bracket $[P, P]$ (21) is a three-vector and has the following coordinate expression:

$$[P,P](\psi^1,\psi^2,\psi^3) = T(\psi^1,\psi^2,\psi^3) = \int T^{i_1 i_2 \sigma_2 i_3 \sigma_3}\psi^1_{i_1}\partial_{\sigma_2}\psi^2_{i_2}\partial_{\sigma_3}\psi^3_{i_1}\,dx \tag{24}$$

$T^{i_1 i_2 \sigma_2 i_3 \sigma_3}$ is defined up to total divergencies: this means that three-vectors of the type $\partial_x\left(T^{i_1\sigma_1 i_2\sigma_2 i_3\sigma_3}\partial_{\sigma_1}\psi^1_{i_1}\partial_{\sigma_2}\psi^2_{i_2}\partial_{\sigma_3}\psi^3_{i_1}\right)$ are zero. It immediately follows that a local three-vector which is of order zero in one of its arguments is zero if and only if its coefficients are zero.

The algorithm in Section 2.2 translates into the language of differential operators as follows. Let us introduce the notation

$$\tilde{\psi}^a_\alpha = \partial_x^{-1}(w^i_\alpha\psi^a_i), \tag{25}$$

where a refers to the particular argument of the operator. Then, the vector functions ψ^1, ψ^2, ψ^3 play the role of test vector functions of the variables x, y, z in the language of distributions.

1. The first step in Section 2.2 is not needed in the differential operator formalism, as it boils down to a change in the variable of integration (and its differential consequences).

2. The second step aims at bringing the nonlocal part of the three-vector in the reduced form (16). To this aim, we remark that the reduced form of the distributions implies that there is no distribution of the type $\nu(x-y)$ acting on two vector test functions. This means effecting the following substitution (up to total divergencies)

$$e^\alpha w^i_\alpha \partial_x^{-1}\left(\frac{\partial w^j_\alpha}{\partial u^k_\tau}\partial_\tau(B^{kp\sigma}\partial_\sigma\psi^b_p + e^\alpha w^k_\alpha \tilde{\psi}^b_\alpha)^k\psi^c_j\right)\psi^a_i = \\ - e^\alpha\tilde{\psi}^a_\alpha\left(\frac{\partial w^j_\alpha}{\partial u^k_\tau}\partial_\tau(B^{kp\sigma}\partial_\sigma\psi^b_p + e^\alpha w^k_\alpha \tilde{\psi}^b_\alpha)^k\psi^c_j\right) \quad (26)$$

After such a substitution, we observe that the generic summands of (20) are of three types:

$$C^{\alpha\beta k}\tilde{\psi}^a_\alpha\tilde{\psi}^b_\beta\psi^c_k, \quad (27)$$

$$C^{\alpha kj\sigma}\tilde{\psi}^a_\alpha\partial_\sigma(\psi^b_j)\psi^c_k, \quad (28)$$

$$C^{kj\sigma i\tau}\partial_\tau(\psi^a_i)\partial_\sigma(\psi^b_j)\psi^c_k, \quad (29)$$

where C's are functions of (u^i, u^i_σ). The reduced form of the three-vector in the formalism of differential operators amounts at bringing the operator to a canonical form where the arguments ψ^a, ψ^b, ψ^c are a fixed sequence of integers (say, 1, 2, 3) or its cyclic permutations (in the previous example, 3, 1, 2 and 2, 3, 1). This task can always be achieved by integration by parts that will produce the required terms plus extra terms.

3. The third step of the algorithm amounts at bringing the local part into a reduced form. This is achieved with the usual procedure of integrating by parts the three-vector with respect to one distinguished argument (say ψ^3) in such a way that the result will be of order zero in that argument.

4 Jacobi identity and Poisson Vertex Algebras

4.1 The Jacobi identity

Following [9] we introduce the notion of (nonlocal) Poisson vertex algebra.

Definition 1. A (nonlocal) Poisson vertex algebra (PVA) is a differential algebra $(\mathcal{A}, \partial)$ endowed with a derivation ∂ and a bilinear operation $\{\cdot_\lambda\cdot\}\colon \mathcal{A}\otimes\mathcal{A}\to\mathbb{R}((\lambda^{-1}))\otimes\mathcal{A}$ called a (nonlocal) λ-*bracket*, satisfying the following set of properties:

1. $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$ (left sesquilinearity),
2. $\{f_\lambda \partial g\} = (\lambda+\partial)\{f_\lambda g\}$ (right sesquilinearity),
3. $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$ (left Leibniz property),
4. $\{fg_\lambda h\} = \{f_{\lambda+\partial}h\}g + \{g_{\lambda+\partial}h\}f$ (right Leibniz property),
5. $\{g_\lambda f\} = -{}_{\to}\{f_{-\lambda-g}g\}$ (PVA skew-symmetry),
6. $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu}h\}$ (PVA-Jacobi identity).

In the notation for the bracket, the symbol separating the two arguments is the formal parameter of the expansion. We denote

$$\{f_\lambda g\} = \sum_{s\le S} C_s(f,g)\lambda^s,$$

with $C_s(f,g)\in\mathcal{A}$; the argument signals that each of the coefficients of the expansion depends on the two elements f and g in $\mathcal{A}$. Such an expansion is bounded by $0\le s\le S$ for *local* PVAs and is not bounded from below for *nonlocal* PVAs.

The special notation used on the RHS of Property 4 is to be understood as

$$\{f_{\lambda+\partial}g\}h = \sum_{s\le S} C_s(f,g)(\lambda+\partial)^s h = \sum_{s,t}\binom{s}{t}C_s(f,g)\partial^t h\,\lambda^{s-t}.$$

Similarly, the RHS of Property 5 (the skewsymmetry) reads

$${}_{\to}\{f_{-\lambda-\partial}g\} = \sum_s (-\lambda-\partial)^s C_s(f,g).$$

For a nonlocal λ bracket, the three terms of PVA-Jacobi identity do not necessarily belong to the same space, because of the double infinite expansion of the brackets (in terms of (λ, μ), (μ, λ) and $(\lambda, \lambda + \mu)$, respectively). A bracket is said to be *admissible* if all the three terms can be (not uniquely) expanded as

$$\{f_\lambda \{g_\mu h\}\} = \sum_{m \leq M} \sum_{n \leq N} \sum_{p \leq 0} a_{m,n,p} \lambda^m \mu^n (\lambda + \mu)^p.$$

Only admissible brackets can define a nonlocal PVA. We denote the space where the PVA-Jacobi identity of admissible brackets takes values by $V_{\lambda,\mu}$. This space can be decomposed by the total degree d in $(\lambda, \mu, \lambda + \mu)$; elements of each homogeneous component $V^{(d)}_{\lambda,\mu}$ can be *uniquely* expressed in the basis [11]

$$\begin{aligned} &\lambda^i \mu^{d-i} && i \in \mathbb{Z}, \\ &\lambda^{d+i} (\lambda + \mu)^{-i} && i = \{1, 2, \ldots\}. \end{aligned}$$

This filtration in the total degree d and the subsequent choice of a basis plays a crucial role in obtaining the normal form for the PVA-Jacobi identity.

The main result used to perform most of the computations is the so called *master formula*. Under the hypothesis that the differential algebra $\mathcal{A}$ is generated by the elements (u^i), the λ-bracket between any two elements of $\mathcal{A}$ is explicitly given by

$$\{f_\lambda g\} = \frac{\partial g}{\partial u^j_\sigma} (\lambda + \partial)^\sigma \{u^i_{\lambda+\partial} u^j\} (-\lambda - \partial)^\tau \frac{\partial f}{\partial u^i_\tau} \tag{30}$$

Thus, the structure of a PVA is defined by the matrix of the λ brackets between the generators $\{u^i_\lambda u^j\} = P^{ji}(\lambda)$.

In the nonlocal case, expressions such as $(\lambda + \partial)^p$ for $p < 0$ arise in $P^{ji}(\lambda + \partial)$ from the master formula (30). In such cases, the rigorous approach – working for any kind of nonlocality – is to expand the negative powers of $(\lambda + \partial)$ as

$$(\lambda + \partial)^{-p} = \sum_{k \geq 0} \binom{-p}{k} \lambda^{-p-k} \partial^k, \qquad p > 0 \tag{31}$$

In the weakly nonlocal case this can be avoided, relying only on Properties (1)-(4) of the lambda bracket. More details on this will be provided in Section 7.2.

4.2 Dictionary: Poisson Vertex Algebras and differential operators

The connection between the theory of PVA and Hamiltonian operator is given by Theorem 4.8 in [11, pag. 261]. In short, there is a 1-1 correspondence between λ-brackets of a (nonlocal) PVA and (pseudo)differential Hamiltonian operators; the entries of the matrix $P^{ji}(\lambda)$ correspond to the differential operator P^{ij} (6) after the formal replacement of λ by ∂.

More precisely, the equivalence between the expression of the Poisson bracket (3) and the expression of a λ-bracket according with the master formula (30) is:

$$\begin{aligned}\{F,G\}_J &= \int \frac{\delta f}{\delta u^i} P^{ij\sigma}\partial_\sigma \frac{\delta g}{\delta u^j}\,dx \\ &= \int \frac{\partial g}{\partial u^i_\sigma}\partial_\sigma\left(P^{ij\tau}\partial_\tau(-\partial)^\epsilon \frac{\partial f}{\partial u^j_\epsilon}\right)dx = \int \{f_\lambda g\}\big|_{\lambda=0}\,dx, \end{aligned} \tag{32}$$

using (30). The PVA-Jacobi identity for a triple of generators (u^i, u^j, u^k) can also be expressed by means of differential operators. First of all, we compute the PVA-Jacobi identity using the master formula; we have

$$\{u^i_\lambda\{u^j_\mu u^k\}\} = \frac{\partial P^{kj}(\mu)}{\partial u^l_\sigma}(\lambda+\partial)^\sigma P^{li}(\lambda) \tag{33}$$

$$\{u^j_\mu\{u^i_\lambda u^k\}\} = \frac{\partial P^{ki}(\lambda)}{\partial u^l_\sigma}(\mu+\partial)^\sigma P^{lj}(\mu) \tag{34}$$

$$\{\{u^i_\lambda u^j\}_{\lambda+\mu} u^k\} = P^{kl}(\lambda+\mu+\partial)(-\lambda-\mu-\partial)^\sigma \frac{\partial P^{ji}(\lambda)}{\partial u^l_\sigma} \tag{35}$$

The PVA-Jacobi identity is $J^{ijk}_{\lambda,\mu}(P,P) = (33) - (34) - (35) = 0$. We evaluate the expression on three covectors $\psi^1_i\psi^2_j\psi^3_k$, and regard each power of λ as derivations acting on ψ^1, and each power of μ as derivations acting on ψ^2. Then, the three summands correspond to

$$\langle \psi^3, \ell_{P,\psi^1}(P\psi^2)\rangle, \tag{36}$$

$$\langle \psi^3, \ell_{P,\psi^2}(P\psi^1)\rangle, \tag{37}$$

$$\langle \psi^3, P\,\ell^*_{P,\psi^1}(\psi^2)\rangle, \tag{38}$$

respectively, and the PVA-Jacobi identity is the vanishing of the Schouten bracket $[P,P]$ in the form of (110).

4.3 The algorithm

For the local case, the expression of the PVA-Jacobi identity is a polynomial in λ and μ, and the vanishing of the coefficients of $\lambda^p\mu^q$ corresponds to the vanishing of the coefficients for $\partial^p(\psi^1_i)\partial^q(\psi^2_j)\psi^3_k$.

In the nonlocal case, the PVA-Jacobi identity is a Laurent series in λ^{-1}, μ^{-1} and $(\lambda+\mu)^{-1}$ living in the space $V_{\lambda,\mu}$ defined in Section 4.1: in the weakly nonlocal case, these coefficients come, respectively, from the expansion of $(\lambda+\partial)^{-1}$, $(\mu+\partial)^{-1}$, $(\lambda+\mu+\partial)^{-1}$.

From the computation of the PVA-Jacobi identity we obtain seven types of terms including one or two nonlocal factors, together with the pure local terms; each of them corresponds to the types of summands in the three-vector of the Schouten identity in (24), as detailed in (27) and following. They are

1. $A^{ijk}\lambda^p\mu^q$ with $p,q\geq 0$, corresponding to $\partial^p(\psi^1_i)\partial^q(\psi^2_j)\psi^3_k$;
2. $w^k(\lambda+\mu+\partial)^{-1}A^{ij}\lambda^p$ with $p\geq 0$, corresponding to $\partial^p(\psi^1_i)\psi^2_j\tilde{\psi}^3$;
3. $w^k(\lambda+\mu+\partial)^{-1}A^{ij}\mu^p$ with $p>0$, corresponding to $\psi^1_i\partial^p(\psi^2_j)\tilde{\psi}^3$;
4. $[(\lambda+\partial)^{-1}w^i]A^{jk}\mu^p$ with $p\geq 0$, corresponding to $\tilde{\psi}^1\partial^p(\psi^2_j)\psi^3_k$;
5. $[(\mu+\partial)^{-1}w^j]A^{ki}\lambda^p$ with $p\geq 0$, corresponding to $\partial^p(\psi^1_i)\tilde{\psi}^2\psi^3_k$;
6. $w^k(\lambda+\mu+\partial)^{-1}A^j(\lambda+\partial)^{-1}w^i$, corresponding to $\tilde{\psi}^1\psi^2_j\tilde{\psi}^3$;
7. $w^k(\lambda+\mu+\partial)^{-1}A^i(\mu+\partial)^{-1}w^j$, corresponding to $\psi^1_i\tilde{\psi}^2\tilde{\psi}^3$;
8. $[(\lambda+\partial)^{-1}w^i]A^k[(\mu+\partial)^{-1}w^j]$, corresponding to $\tilde{\psi}^1\tilde{\psi}^2\psi^3_k$.

The square brackets denote that the differential operators obtained by the expansion of the pseudodifferential operator do not act outside them.

Note that the expansion of the terms 3 and 7 is not expressed in the basis for $V_{\lambda,\mu}$ we have chosen; on the other hand, terms 3 and 5 do not correspond to the choice of coefficients for the normalization algorithm of the previous Sections (when one takes the cyclic ordering $\tilde{\psi}^a\partial^p(\psi^b)\psi^c$).

We give a different treatment of the terms including at most one nonlocal expression and of the ones with two: in the first case, we bring them to a form whose expansion is automatically expressed in our chosen basis for $V_{\lambda,\mu}$; in the second case, we show that the vanishing of the term 7, together with the other ones, is equivalent to the vanishing of the corresponding terms in the expansion on the basis.

Finally, we comment on the equivalence between the vanishing of the PVA-Jacobi identity on our chosen basis and as a result of the normalization algorithm of the previous Sections.

Proposition 2. *The terms of type $w^k(\lambda+\mu+\partial)^{-1}A^{ij}\mu^p$ can be brought to a combination of terms of type $A^{ijk}\lambda^p\mu^q$ with $p,q\geq 0$ and $w^k(\lambda+\mu+\partial)^{-1}A^{ij}\lambda^p$, reducing the PVA-Jacobi identity to the expansion of seven terms.*

Proof. From the expansion $(\lambda+\mu+\partial)^p=\sum_{l=0}^p\binom{p}{l}\mu^{p-l}(\lambda+\partial)^l$ we can rewrite a term of the form $w^k(\lambda+\mu+\partial)^{-1}A^{ij}\mu^p$ as

$$w^k(\lambda+\mu+\partial)^{-1}\left[(\lambda+\mu+\partial)^pA^{ij}-\sum_{l=0}^{p-1}\binom{p}{l}\mu^l(\lambda+\partial)^{p-l}A^{ij}\right],$$

which gives

$$w^k(\lambda+\mu+\partial)^{p-1}A^{ij}-w^k(\lambda+\mu+\partial)^{-1}\left[\sum_{l=0}^{p-1}\binom{p}{l}\mu^l(\lambda+\partial)^{p-l}A^{ij}\right].$$

The expression in the square bracket has top degree $p-1$ in μ. Repeating the operation we obtain only local terms or terms of the type 2. □

Theorem 3. *The PVA-Jacobi identity, expressed using terms of the type* 1, 2, 4 – 8 *as above, can always be expressed in the space $V^{(d)}_{\lambda,\mu}$. This latter expression vanishes if and only if the former does.*

Proof. Expressing the PVA-Jacobi identity in the space $V^{(d)}_{\lambda\,\mu}$, for all for $d\leq D$, means expanding it on the basis $\lambda^p\mu^{d-p}$, $p\in\mathbb{Z}$ and $(\lambda+\mu)^{-p}\lambda^{d+p}$, $p>0$. Terms of type 1 do not need to be expanded, as they are already expressed in the basis for $V^{(d)}_{\lambda,\mu}$, $d\geq 0$.

For the types with one nonlocal term, namely 2, 4 and 5 in the previous list, the expansions of the pseudodifferential operators give the series

$$\sum_{t\geq 0}(-1)^t w^k(\lambda+\mu)^{-t-1}\lambda^p\partial^tA^{ij} \tag{39}$$

$$\sum_{t\geq 0}(-1)^t A^{jk}\mu^p\lambda^{-t-1}\partial^t w^i \tag{40}$$

$$\sum_{t\geq 0}(-1)^t A^{ki}\lambda^p\mu^{-t-1}\partial^t w^j \tag{41}$$

which are in our chosen basis of $V^{(d)}_{\lambda,\mu}$, for $d \leq p-1$. The vanishing of the $t=0$ term in the expansion is a sufficient and necessary condition for the vanishing of the whole series: all the subsequent terms in the expansion vanish if the first one does, and it must vanish because that is the only one in $V^{(p-1)}_{\lambda,\mu}$ containing the factor $(\lambda+\mu)^{-1}$ (resp. λ^{-1} and μ^{-1}). It is hence enough to check (or impose) the vanishing of the coefficients A or w. However, the vanishing of the w terms coincides with the dropping of the nonlocal part of the λ bracket, so the condition is only on A's.

A similar point can be made for the types 6 and 8 with the double nonlocality: their expansion is expressed in our chosen basis and starts, respectively, with $(\lambda+\mu)^{-1}\lambda^{-1}$ and $\lambda^{-1}\mu^{-1}$ in $V^{(-2)}_{\lambda,\mu}$. The expansion of the term 7 starts with $A^i w^k w^j (\lambda+\mu)^{-1}\mu^{-1}$, which is not an element in the basis of $V^{(-2)}_{\lambda,\mu}$. However, this is a term we can rearrange as an infinite series

$$A^i w^k w^j \left(\lambda^{-1}\mu^{-1} - (\lambda+\mu)^{-2} + \sum_{m>0} c_m (\lambda+\mu)^{-2-m}\lambda^m \right)$$

for some fixed constants c_m.

Note that elements in $V^{(-2)}_{\lambda,\mu}$ could be obtained by the expansions (for $t=1$) of the previous terms with only one nonlocality. However, the vanishing of the elements in $V^{(-1)}_{\lambda,\mu}$ implies their vanishing, too, and hence we can focus on the terms arising from the expansion of double nonlocalities only.

It is straightforward to see that we get only one expression in front of $(\lambda+\mu)^{-1}\lambda^{-1}$ (from type 6) and $(\lambda+\mu)^{-2}$ (from our rearrangement of type 8); on the other hand, there could be two sources of terms of the form $\lambda^{-1}\mu^{-1}$. The vanishing of either A or w for all i, j, k in the first two cases is a necessary and sufficient condition; once that this has been imposed or checked, the only surviving class of terms of the form $\lambda^{-1}\mu^{-1}$ comes from the expansion of 7.

Since the vanishing of w is equivalent to simply dropping the nonlocal term of the operator, the condition we need to consider is only the vanishing of the expressions A's. □

Remark 4. The above theorem has two important consequences.

1. This algorithm always yields a divergence-free form of the Jacobi identity; this means that the Jacobi identity holds if and only if the coefficient of the Laurent series in the spaces $V^{(d)}_{\lambda,\mu}$ vanish.

2. There is no need to expand in Laurent series: indeed, the expansion is always ruled by the zeroth-order coefficients, which are just the coefficients of the terms 2, 4 – 8.

Remark 5. Writing the PVA-Jacobi identity on our chosen basis of $V^{(d)}_{\lambda,\mu}$ for $d \geq -2$ yields a different result than the one obtained with the algorithm described in Section 3.3. For the terms with one nonlocality, indeed, the PVA-Jacobi identity produces the coefficients corresponding to $\partial^p(\psi^1_i)\psi^2_j\tilde{\psi}^3$, $\partial^p(\psi^2_j)\psi^3_k\tilde{\psi}^1$ and $\partial^p(\psi^1_i)\psi^3_j\tilde{\psi}^2$, while the latter is replaced by $\partial^p(\psi^3_i)\psi^1_j\tilde{\psi}^2$ in Section 3.3.

Nevertheless, the sets of condition given by the vanishing of the coefficients in front of the terms obtained with the two different algorithms are equivalent. Let us demonstrate it assuming that the terms of type 5 are

$$A_2\,\lambda^2(\mu+\partial)^{-1}w + A_1\,\lambda(\mu+\partial)^{-1}w + A_0\,(\mu+\partial)^{-1}w, \tag{42}$$

corresponding to

$$A_2\,\tilde{\psi}^2\partial^2(\psi^1)\psi^3 + A_1\tilde{\psi}^2\partial(\psi^1)\psi^3 + A_0\,\tilde{\psi}^2\psi^1\psi^3. \tag{43}$$

This latest expression is equivalent, up to total derivatives, to

$$A_2\tilde{\psi}^2\psi^1\partial^2(\psi^3) + (2\partial A_2 - A_1)\,\tilde{\psi}^2\psi^1\partial(\psi^3) + \left(A_0 + \partial^2 A_2 - \partial A_1\right)\tilde{\psi}^2\psi^1\psi^3 + \text{local terms}. \tag{44}$$

The vanishing of expression (44) at top degree implies the vanishing of the lower degree coefficients, being hence equivalent to the vanishing of (42).

The same result can be obtained in the framework of Poisson vertex algebras introducing the symbol $\nu = -\lambda - \mu - \partial$, representing derivations acting on ψ^3 [12, Section 4.1].

5 Weakly nonlocal PBHT and distributions

5.1 Calculation of the Jacobi identity

In this section we will consider, as an example, weakly nonlocal Poisson bracket of hydrodynamic type, of the form (8). In the language of distribution it has the form

$$P^{ij}_{x,y} = g^{ij}(\mathbf{u}(x))\delta'_{xy} + \Gamma^{ij}_k(\mathbf{u}(x))u^k_x\delta_{xy} + w^i_s(\mathbf{u}(x))u^s_x\nu_{xy}w^j_t(\mathbf{u}(y))u^t_y \tag{45}$$

(we will use only one 'tail summand' to make calculations simpler) where $\delta_{xy} = \delta(x-y)$ e $\nu_{xy} = \nu(x-y)$. We assume g to be non degenerate. In what follows, an index after a comma denotes a partial derivative with respect to the corresponding field variable, *e.g.* $g^{ij}_{,k} = \partial g^{ij}/\partial u^k$.

From the skew-symmetry the two conditions (9a), (9b) follow, namely: $g^{ij} = g^{ji}$ and $g^{ij}_{,k} = \Gamma^{ij}_k + \Gamma^{ji}_k$. We apply now the reducing procedure explained in Section 2.2. Since P^{ij}_{xy} depend only on $u(x)$ and u_x each sum in (12) contains only two terms. The Jacobi identity can be rewritten as

$$\begin{aligned}\frac{\partial P^{ij}_{x,y}}{\partial u^l(x)}P^{lk}_{x,z} + \frac{\partial P^{ij}_{x,y}}{\partial u^l(y)}P^{lk}_{y,z} + \frac{\partial P^{ki}_{z,x}}{\partial u^l(z)}P^{lj}_{z,y} + \frac{\partial P^{ki}_{z,x}}{\partial u^l(x)}P^{lj}_{x,y} + \\ + \frac{\partial P^{jk}_{y,z}}{\partial u^l(y)}P^{li}_{y,x} + \frac{\partial P^{jk}_{y,z}}{\partial u^l(z)}P^{li}_{z,x} + \frac{\partial P^{ij}_{x,y}}{\partial u^l_x}\partial_x P^{lk}_{x,z} + \frac{\partial P^{ij}_{x,y}}{\partial u^l_y}\partial_y P^{lk}_{y,z} + \\ \frac{\partial P^{ki}_{z,x}}{\partial u^l_z}\partial_z P^{lj}_{z,y} + \frac{\partial P^{ki}_{z,x}}{\partial u^l_x}\partial_x P^{lj}_{x,y} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_y}\partial_y P^{li}_{y,x} + \frac{\partial P^{jk}_{y,z}}{\partial u^l_z}\partial_z P^{li}_{z,x} = 0 \quad (46)\end{aligned}$$

5.2 Calculation of the reduced form

The first summand in (46) is

$$\begin{aligned}\frac{\partial P^{ij}_{x,y}}{\partial u^l(x)}P^{lk}_{x,z} = \left(g^{ij}_{,l}\delta'_{xy} + \Gamma^{ij}_{s,l}u^s_x\delta_{xy} + w^i_{s,l}u^s_x\nu_{xy}w^j_t\, u^t_y\right)\cdot \\ \cdot\left(g^{lk}(x)\delta'_{xz} + \Gamma^{lk}_t u^t_x\delta_{xz} + w^l_s u^s_x\nu_{xz}w^k_t u^t_z\right) \quad (47)\end{aligned}$$

The coefficients of the reduced form are listed below.

- The coefficient of $\delta'_{xy}\delta'_{xz}$ is $g^{lk}g^{ij}_{,l}$.
- The coefficient of $\nu_{xy}\nu_{xz}$ is $w^i_{s,l}w^l_m u^m_x u^s_x w^j_t u^t_y w^k_n u^n_z$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $g^{ij}_{,l}\Gamma^{lk}_t u^t_x$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $g^{lk}\Gamma^{ij}_{s,l}u^s_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is $\Gamma^{ij}_{s,l}\Gamma^{lk}_t u^t_x u^s_x - g^{ij}_{,l}w^l_s u^s_x w^k_t u^t_x$.
- The coefficient of $\delta_{yx}\nu_{yz}$ is $-\partial_y(g^{ij}_{,l}w^l_s u^s_y)w^k_t u^t_z + \Gamma^{ij}_{s,l}w^l_r u^r_x u^s_x w^k_t u^t_z$.
- The coefficient of $\delta_{xz}\nu_{xy}$ is $w^i_{s,l}\Gamma^{lk}_r u^r_x u^s_x w^j_t u^t_y$.

- The coefficient of $\delta'_{yx}\nu_{yz}$ is $-g^{ij}_{,l} w^l_s u^s_y w^k_t u^t_z$.
- The coefficient of $\delta'_{xz}\nu_{xy}$ is $g^{lk} w^i_{s,l} u^s_x w^j_t u^t_y$.

The second summand in (46) is

$$\frac{\partial P^{ij}_{x,y}}{\partial u^l(y)} P^{lk}_{y,z} = w^i_s u^s_x \nu_{xy} g^{lk} w^j_{t,l} u^t_y \delta'_{yz} + \\ + w^i_s u^s_x \nu_{xy} w^j_{t,l} \Gamma^{lk}_m u^m_y u^t_y \delta_{yz} + w^i_s u^s_x \nu_{xy} w^j_{t,l} w^l_m u^m_y u^t_y \nu_{yz} w^k_n u^n_z. \quad (48)$$

The coefficients of the reduced form are listed below:

- The coefficient of $\delta_{xy}\delta_{xz}$ is $w^i_s u^s_x g^{lk} w^j_{t,l} u^t_x$.
- The coefficient of $\nu_{xz}\delta'_{zy}$ is $-w^i_s u^s_x g^{lk} w^j_{t,l} u^t_z$.
- The coefficient of $\nu_{xz}\delta_{zy}$ is $-w^i_s u^s_x \partial_z \left(g^{lk} w^j_{t,l} u^t_z \right) + w^i_s u^s_x w^j_{t,l} \Gamma^{lk}_m u^m_y u^t_y$.
- The coefficient of $\nu_{xy}\nu_{yz}$ is $w^i_s u^s_x w^j_{t,l} w^l_m u^m_y u^t_y w^k_n u^n_z$.

The third summand in (46) is

$$\frac{\partial P^{ki}_{z,x}}{\partial u^l(z)} P^{lj}_{z,y} = g^{lj} g^{ki}_{,l} \delta'_{zx} \delta'_{zy} + g^{ki}_{,l} \Gamma^{lj}_t u^t_z \delta'_{zx} \delta_{zy} + g^{lj} \Gamma^{ki}_{s,l} u^s_z \delta_{zx} \delta'_{zy} \\ + \Gamma^{ki}_{s,l} \Gamma^{lj}_t u^t_z u^s_z \delta_{zx} \delta_{zy} + g^{ki}_{,l} w^l_s u^s_z \delta'_{zx} \nu_{zy} w^j_t u^t_y + \Gamma^{ki}_{s,l} w^l_r u^r_z u^s_z \delta_{zx} \nu_{zy} w^j_t u^t_y \\ + g^{lj} w^k_{s,l} u^s_z \delta'_{zy} \nu_{zx} w^i_t u^t_x + w^k_{s,l} \Gamma^{lj}_t u^t_z u^s_z \delta_{zy} \nu_{zx} w^i_t u^t_x \\ + w^k_{s,l} w^l_m u^m_z u^s_z \nu_{zx} w^i_t u^t_x \nu_{zy} w^j_n u^n_y. \quad (49)$$

The coefficients of the reduced form are listed below.

- The coefficient of $\delta''_{xy}\delta_{xz}$ is $-g^{lj} g^{ki}_{,l}$.
- The coefficient of $\delta'_{xy}\delta'_{xz}$ is $-g^{lj} g^{ki}_{,l}$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $-\partial_x \left(g^{lj} g^{ki}_{,l} \right) - g^{ki}_{,l} \Gamma^{lj}_t u^t_x + g^{lj} \Gamma^{ki}_{s,l} u^s_x$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $-g^{ki}_{,l} \Gamma^{lj}_t u^t_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is $-\partial_x \left(g^{ki}_{,l} \Gamma^{lj}_t u^t_x \right) + \Gamma^{ki}_{s,l} \Gamma^{lj}_t u^t_x u^s_x - g^{ki}_{,l} w^l_s u^s_x w^j_t u^t_x$.

- The coefficient of $\delta_{xz}\nu_{xy}$ is $-\partial_x\left(g^{ki}_{,l}w^l_s u^s_x\right)w^j_t u^t_y + \Gamma^{ki}_{s,l}w^l_r u^r_x u^s_x w^j_t u^t_y$.
- The coefficient of $\delta_{zy}\nu_{zx}$ is $w^k_{s,l}\Gamma^{lj}_t u^t_z u^s_z w^i_t u^t_x$.
- The coefficient of $\delta'_{zy}\nu_{zx}$ is $g^{lj}w^k_{s,l}u^s_z w^i_t u^t_x$.
- The coefficient of $\delta'_{xz}\nu_{xy}$ is $-g^{ki}_{,l}w^l_s u^s_x w^j_t u^t_y$.
- The coefficient of $\nu_{zx}\nu_{zy}$ is $w^k_{s,l}w^l_m u^m_z u^s_z w^i_t u^t_x w^j_n u^n_y$.

The fourth summand in (46) is

$$\frac{\partial P^{ki}_{z,x}}{\partial u^l(x)}P^{lj}_{x,y} = w^k_s u^s_z \nu_{zx} g^{lj} w^i_{t,l} u^t_x \delta'_{xy} \\ + w^k_s u^s_z \nu_{zx} w^i_{t,l}\Gamma^{lj}_m u^m_x u^t_x \delta_{xy} + w^k_s u^s_z \nu_{zx} w^i_{t,l} w^l_m u^m_x u^t_x \nu_{xy} w^j_n u^n_y. \quad (50)$$

The coefficients of the reduced form are listed below:

- The coefficient of $\delta_{xy}\delta_{xz}$ is $w^k_s u^s_x g^{lj} w^i_{t,l} u^t_x$.
- The coefficient of $\nu_{zy}\delta'_{yx}$ is $-w^k_s u^s_z g^{lj} w^i_{t,l} u^t_y$.
- The coefficient of $\nu_{zy}\delta_{xy}$ is $-w^k_s u^s_z \partial_y\left(g^{lj} w^i_{t,l} u^t_y\right) - w^k_s u^s_z w^i_{t,l}\Gamma^{lj}_m u^m_x u^t_x$.
- The coefficient of $\nu_{zx}\nu_{xy}$ is $w^k_s u^s_z w^i_{t,l} w^l_m u^m_x u^t_x w^j_n u^n_y$.

The fifth summand in (46) is

$$\frac{\partial P^{jk}_{y,z}}{\partial u^l(y)}P^{li}_{y,x} = g^{li}g^{jk}_{,l}\delta'_{yz}\delta'_{yx} + g^{jk}_{,l}\Gamma^{li}_t u^t_y \delta'_{yz}\delta_{yx} + g^{li}\Gamma^{jk}_{s,l}u^s_y \delta_{yz}\delta'_{yx} \\ + \Gamma^{jk}_{s,l}\Gamma^{li}_t u^t_y u^s_y \delta_{yz}\delta_{yx} + g^{jk}_{,l}w^l_s u^s_y \delta'_{yz}\nu_{yx} w^i_t u^t_x + \Gamma^{jk}_{s,l}w^l_s u^s_y u^s_y \delta_{yz}\nu_{yx} w^i_t u^t_x \\ + g^{li}w^j_{s,l}u^s_y \delta'_{yx}\nu_{yz} w^k_t u^t_z + +w^j_{s,l}\Gamma^{li}_t u^t_y u^s_y \delta_{yx}\nu_{yz} w^k_t u^t_z \\ + w^j_{s,l}w^l_m u^m_y u^s_y \nu_{yz} w^k_t u^t_z \nu_{yx} w^i_n u^n_x. \quad (51)$$

The coefficients of the reduced form are

- The coefficient of $\delta'_{xy}\delta'_{xz}$ is $-g^{li}g^{jk}_{,l}$.
- The coefficient of $\delta_{xy}\delta''_{xz}$ is $-g^{li}g^{jk}_{,l}$.

- The coefficient of $\delta_{xy}\delta'_{xz}$ is $-\partial_x\left(g^{li}g^{jk}_{,l}\right)+g^{jk}_{,l}\Gamma^{li}_{t}u^t_y-g^{li}\Gamma^{jk}_{s,l}u^s_x$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $-g^{li}\Gamma^{jk}_{s,l}u^s_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is $-\partial_x\left(g^{li}\Gamma^{jk}_{s,l}u^s_x\right)+\Gamma^{jk}_{s,l}\Gamma^{li}_{t}u^t_x u^s_x-g^{jk}_{,l}w^l_s u^s_x w^i_t u^t_x$.
- The coefficient of $\delta_{zy}\nu_{zx}$ is $-\partial_z\left(g^{jk}_{,l}w^l_s u^s_z\right)w^i_t u^t_x+\Gamma^{jk}_{s,l}w^l_s u^s_y u^s_y w^i_t u^t_x$.
- The coefficient of $\delta'_{zy}\nu_{zx}$ is $-g^{jk}_{,l}w^l_s u^s_z w^i_t u^t_x$.
- The coefficient of $\delta'_{yx}\nu_{yz}$ is $g^{li}w^j_{s,l}u^s_y w^k_t u^t_z$.
- The coefficient of $\nu_{yz}\nu_{yx}$ is $w^j_{s,l}w^l_m u^m_y u^s_y w^k_t u^t_z w^i_n u^n_x$.

The sixth summand in (46) is

$$\frac{\partial P^{jk}_{y,z}}{\partial u^l(z)}P^{li}_{z,x}=w^j_s u^s_y\nu_{yz}g^{li}w^k_{t,l}u^t_z\delta'_{zx}\\+w^j_s u^s_y\nu_{yz}w^k_{t,l}\Gamma^{li}_{m}u^m_z u^t_z\delta_{zx}+w^j_s u^s_y\nu_{yz}w^k_{t,l}w^l_m u^m_z u^t_z\nu_{zx}w^i_n u^n_x. \quad (52)$$

The coefficients of the reduced form are listed below

- The coefficient of $\delta_{xy}\delta_{xz}$ is $w^j_s u^s_x g^{li}w^k_{t,l}u^t_x$.
- The coefficient of $\nu_{xy}\delta'_{xz}$ is $w^j_s u^s_y g^{li}w^k_{t,l}u^t_x$.
- The coefficient of $\nu_{xy}\delta_{xz}$ is $+w^j_s u^s_y\partial_x\left(g^{li}w^k_{t,l}u^t_x\right)-w^j_s u^s_y w^k_{t,l}\Gamma^{li}_{m}u^m_x u^t_x$.
- The coefficient of $\nu_{yz}\nu_{zx}$ is $+w^j_s u^s_y w^k_{t,l}w^l_m u^m_z u^t_z w^i_n u^n_x$.

The seventh summand in (46) is

$$\frac{\partial P^{ij}_{x,y}}{\partial u^l_x}\partial_x P^{lk}_{x,z}=\\\left(\Gamma^{ij}_{l}\delta_{xy}+w^i_l\nu_{xy}w^j_t\,u^t_y\right)\partial_x\left(g^{lk}(x)\delta'_{xz}+\Gamma^{lk}_{t}u^t_x\delta_{xz}+w^l_s u^s_x\nu_{xz}w^k_t u^t_z\right). \quad (53)$$

The coefficients of the reduced form are listed below:

- The coefficient of $\delta_{xy}\delta''_{xz}$ is $g^{lk}\Gamma^{ij}_{l}$.

- The coefficient of $\delta''_{xz}\nu_{xy}$ is $w^i_l g^{lk} w^j_r\, u^r_y$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $\Gamma^{ij}_l \Gamma^{lk}_t u^t_x + \Gamma^{ij}_l g^{lk}_{,s} u^s_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is $\Gamma^{ij}_l \Gamma^{lk}_{t,s} u^s_x u^t_x + \Gamma^{ij}_l \Gamma^{lk}_t u^t_{xx} + \Gamma^{ij}_l w^l_s u^s_x w^k_t u^t_x$.
- The coefficient of $\delta_{yx}\nu_{yz}$ is $\Gamma^{ij}_l w^l_{s,m} u^m_x u^s_x w^k_t u^t_z + \Gamma^{ij}_l w^l_s u^s_{xx} w^k_t u^t_z$.
- The coefficient of $\delta_{xz}\nu_{xy}$ is

$$w^i_l \Gamma^{lk}_{t,s} u^s_x u^t_x w^j_r\, u^r_y + w^i_l \Gamma^{lk}_t u^t_{xx} w^j_r u^r_y + w^i_l w^l_s u^s_x w^k_t u^t_x w^j_r\, u^r_y.$$

- The coefficient of $\delta'_{xz}\nu_{xy}$ is $w^i_l g^{lk}_{,s} u^s_x w^j_r\, u^r_y + w^i_l \Gamma^{lk}_t u^t_x w^j_r\, u^r_y$.
- The coefficient of $\nu_{xz}\nu_{xy}$ is $w^i_l w^l_{s,m} u^m_x u^s_x w^k_t u^t_z w^j_r\, u^r_y + w^i_l w^l_s u^s_{xx} w^k_t u^t_z w^j_r\, u^r_y$.

The eighth summand in (46) is

$$\frac{\partial P^{ij}_{x,y}}{\partial u^l_y} \partial_y P^{lk}_{y,z} = w^i_s u^s_x \nu_{xy} w^j_l \partial_y \left(g^{lk}\delta'_{yz} + \Gamma^{lk}_t u^t_y \delta_{yz} + w^l_s u^s_y \nu_{yz} w^k_t u^t_z \right).$$

The coefficients of the reduced form are listed below:

- The coefficient of $\delta_{xy}\delta_{xz}$ is $w^i_s u^s_x w^j_l g^{lk}_{,m} u^m_x - w^i_s u^s_x \partial_x (w^j_l g^{lk}) + w^i_s u^s_x w^j_l \Gamma^{lk}_t u^t_x$.
- The coefficient of $\nu_{xz}\delta'_{zy}$ is

$$-w^i_s u^s_x w^j_l g^{lk}_{,m} u^m_z + 2 w^i_s u^s_x \partial_z (w^j_l g^{lk}) - w^i_s u^s_x w^j_l \Gamma^{lk}_t u^t_z.$$

- The coefficient of $\nu_{xz}\delta''_{zy}$ is $w^i_s u^s_x w^j_l g^{lk}$.
- The coefficient of $\nu_{xz}\delta_{zy}$ is

$$\begin{aligned} w^i_s u^s_x \big(\partial^2_z (w^j_l g^{lk}) - \partial_z \left(w^j_l g^{lk}_{,m} u^m_z \right) + w^j_l \Gamma^{lk}_{t,m} u^m_z u^t_z \\ + w^j_l \Gamma^{lk}_t u^t_{zz} \delta_{zy} - \partial_z (w^j_l \Gamma^{lk}_t u^t_z) + w^j_l w^l_r u^r_z w^k_t u^t_z \big). \end{aligned}$$

- The coefficient of $\nu_{xy}\nu_{yz}$ is $w^i_s u^s_x w^j_l w^l_{r,m} u^m_y u^r_y w^k_t u^t_z + w^i_s u^s_x w^j_l w^l_r u^r_{yy} w^k_t u^t_z$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $w^i_s u^s_x w^j_l g^{lk}$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $-w^i_s u^s_x w^j_l g^{lk}$.

The ninth summand in (46) is

$$\begin{aligned}\frac{\partial P^{ki}_{z,x}}{\partial u^l_z}\partial_z P^{lj}_{z,y} = \Gamma^{ki}_l \frac{\partial g^{lj}}{\partial u^s} u^s_z \delta_{zx}\delta'_{zy} + g^{lj}\Gamma^{ki}_l \delta_{zx}\delta''_{zy} \\
+ \Gamma^{ki}_l \Gamma^{lj}_{t,s} u^s_z u^t_z \delta_{zx}\delta_{zy} + \Gamma^{ki}_l \Gamma^{lj}_t u^t_{zz}\delta_{zx}\delta_{zy} + \Gamma^{ki}_l \Gamma^{lj}_t u^t_z \delta_{zx}\delta'_{zy} \\
+ \Gamma^{ki}_l w^l_{s,m} u^m_z u^s_z \delta_{zx}\nu_{zy} w^j_t u^t_y + \Gamma^{ki}_l w^l_s u^s_{zz}\delta_{zx}\nu_{zy} w^j_t u^t_y \\
+ \Gamma^{ki}_l w^l_s u^s_z \delta_{zx}\delta_{zy} w^k_t u^t_y + w^k_l g^{lj}_{,s} u^s_z \delta'_{zy}\nu_{zx} w^i_r\, u^r_x \\
+ w^k_l g^{lj}\delta''_{zy}\nu_{zx} w^i_r\, u^r_x + w^k_l \Gamma^{lj}_{t,s} u^s_z u^t_z \delta_{zy}\nu_{zx} w^j_r\, u^r_x \\
+ w^k_l \Gamma^{lj}_t u^t_{zz}\delta_{zy}\nu_{zx} w^i_r\, u^r_x + w^k_l \Gamma^{lj}_t u^t_z \delta'_{zy}\nu_{zx} w^i_r\, u^r_x \\
+ w^k_l w^l_{s,m} u^m_z u^s_z \nu_{zy} w^j_t u^t_y \nu_{zx} w^i_r\, u^r_x + w^k_l w^l_s u^s_{zz}\nu_{zy} w^j_t u^t_y \nu_{zx} w^i_r\, u^r_x \\
+ w^k_l w^l_s u^s_z \delta_{zy} w^j_t u^t_y \nu_{zx} w^i_r\, u^r_x.\end{aligned} \tag{54}$$

The coefficients of the reduced form are listed below

- The coefficient of $\delta''_{xy}\delta_{xz}$ is $g^{lj}\Gamma^{ki}_l$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $\Gamma^{ki}_l g^{lj}_{,s} u^s_x + \Gamma^{ki}_l \Gamma^{lj}_t u^t_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is $\Gamma^{ki}_l \Gamma^{lj}_{t,s} u^s_x u^t_x + \Gamma^{ki}_l \Gamma^{lj}_t u^t_{xx} + \Gamma^{ki}_l w^l_s u^s_x w^j_t u^t_x$.
- The coefficient of $\delta_{xz}\nu_{xy}$ is $\left(\Gamma^{ki}_l w^l_{s,m} u^m_x u^s_x + \Gamma^{ki}_l w^l_s u^s_{xx}\right) w^j_t u^t_y$.
- The coefficient of $\delta_{zy}\nu_{zx}$ is $\left(w^k_l \Gamma^{lj}_t u^t_{zz} + w^k_l \Gamma^{lj}_{t,s} u^s_z u^t_z + w^k_l w^l_s u^s_z w^j_t u^t_z\right) w^j_r\, u^r_x$.
- The coefficient of $\delta'_{zy}\nu_{zx}$ is $w^k_l g^{lj}_{,s} u^s_z w^i_r\, u^r_x + w^k_l \Gamma^{lj}_t u^t_z w^i_r\, u^r_x$.
- The coefficient of $\nu_{zy}\nu_{zx}$ is $w^k_l w^l_{s,m} u^m_z u^s_z w^j_t u^t_y w^i_r\, u^r_x + w^k_l w^l_s u^s_{zz} w^j_t u^t_y w^i_r\, u^r_x$.
- The coefficient of $\delta''_{zy}\nu_{zx}$ is $w^k_l g^{lj} w^i_r\, u^r_x$.

The tenth summand in (46) is

$$\begin{aligned}\frac{\partial P^{ki}_{z,x}}{\partial u^l_x}\partial_x P^{lj}_{x,y} = w^k_s u^s_z \nu_{zx} w^i_l g^{lj}_{,m} u^m_x \delta'_{xy} + w^k_s u^s_z \nu_{zx} w^i_l g^{lj}\delta''_{xy} \\
+ w^k_s u^s_z \nu_{zx} w^i_l \Gamma^{lj}_{t,m} u^m_x u^t_x \delta_{xy} + w^k_s u^s_z \nu_{zx} w^i_l \Gamma^{lj}_t u^t_{xx}\delta_{xy} + w^k_s u^s_z \nu_{zx} w^i_l \Gamma^{lj}_t u^t_x \delta'_{xy} \\
+ w^k_s u^s_z \nu_{zx} w^i_l w^l_{r,m} u^m_x u^r_x \nu_{xy} w^j_t u^t_y + w^k_s u^s_z \nu_{zx} w^i_l w^l_r u^r_{xx}\nu_{xy} w^j_t u^t_y \\
+ w^k_s u^s_z \nu_{zx} w^i_l w^l_r u^r_x \delta_{xy} w^j_t u^t_y.\end{aligned} \tag{55}$$

The coefficients of the reduced form are listed below:

- The coefficient of $\delta_{xy}\delta_{xz}$ is $w^k_s u^s_x w^i_l g^{lj}_{,m} u^m_x + w^k_s u^s_x w^i_l \Gamma^{lj}_t u^t_x + \partial_x(w^k_s u^s_x) w^i_l g^{lj}$.
- The coefficient of $\nu_{yz}\delta'_{yx}$ is $w^k_s u^s_z w^i_l g^{lj}_{,m} u^m_y + w^k_s u^s_z w^i_l \Gamma^{lj}_t u^t_y - 2w^k_s u^s_z \partial_y(w^i_l g^{lj})$.
- The coefficient of $\nu_{zy}\delta_{yx}$ is

$$-w^k_s u^s_z\Big(\partial_y(w^i_l g^{lj}_{,m} u^m_y) + \partial^2_y(w^i_l g^{lj}) + w^i_l \Gamma^{lj}_{t,m} u^m_y u^t_y + w^i_l \Gamma^{lj}_t u^t_{yy} - \partial_y(w^i_l \Gamma^{lj}_t u^t_y) + w^i_l w^l_r u^r_y w^j_t u^t_y\Big).$$

- The coefficient of $\nu_{zx}\nu_{xy}$ is $w^k_s u^s_z w^i_l w^l_{r,m} u^m_x u^r_x w^j_t u^t_y + w^k_s u^s_z w^i_l w^l_r u^r_{xx} w^j_t u^t_y$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $2w^k_s u^s_x w^i_l g^{lj}$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $w^k_s u^s_x w^i_l g^{lj}$.
- The coefficient of $\nu_{zy}\delta''_{yx}$ is $w^k_s u^s_z w^i_l g^{lj}$.

The eleventh summand in (46) is

$$\begin{aligned}\frac{\partial P^{jk}_{y,z}}{\partial u^l_y}\partial_y P^{li}_{y,x} &= \Gamma^{jk}_l g^{li}_{,s} u^s_y \delta_{yz}\delta'_{yx} + g^{li}\Gamma^{jk}_l \delta_{yz}\delta''_{yx} + \Gamma^{jk}_l \Gamma^{li}_{t,s} u^s_y u^t_y \delta_{yz}\delta_{yx} \\ &+ \Gamma^{jk}_l \Gamma^{li}_t u^t_{yy}\delta_{yz}\delta_{yx} + \Gamma^{jk}_l \Gamma^{li}_t u^t_y \delta_{yz}\delta'_{yx} + \Gamma^{jk}_l w^l_{s,m} u^m_y u^s_y \delta_{yz}\nu_{yx} w^i_t u^t_x \\ &+ \Gamma^{jk}_l w^l_s u^s_{yy}\delta_{yz}\nu_{yx} w^i_t u^t_x + \Gamma^{jk}_l w^l_s u^s_y \delta_{yz}\delta_{yx} w^i_t u^t_x + w^j_l g^{li}_{,s} u^s_y \delta'_{yx}\nu_{yz} w^k_r u^r_z \\ &+ w^j_l g^{li}\delta''_{yx}\nu_{yz} w^k_r u^r_z + w^j_l \Gamma^{li}_{t,s} u^s_y u^t_y \delta_{yx}\nu_{yz} w^k_r u^r_z + w^j_l \Gamma^{li}_t u^t_{yy}\delta_{yx}\nu_{yz} w^k_r u^r_z \\ &+ w^j_l \Gamma^{li}_t u^t_y \delta'_{yx}\nu_{yz} w^k_r u^r_z + w^j_l w^l_{s,m} u^m_y u^s_y \nu_{yx} w^i_t u^t_x \nu_{yz} w^k_r u^r_z \\ &+ w^j_l w^l_s u^s_{yy}\nu_{yx} w^i_t u^t_x \nu_{yz} w^k_r u^r_z + w^j_l w^l_s u^s_y \delta_{yx} w^i_t u^t_x \nu_{yz} w^k_r u^r_z.\end{aligned} \tag{56}$$

The coefficients of the reduced form are listed below

- The coefficient of $\delta_{xy}\delta'_{xz}$ is $-\Gamma^{jk}_l g^{li}_{,s} u^s_x + 2\partial_x(g^{li}\Gamma^{jk}_l) - \Gamma^{jk}_l \Gamma^{li}_t u^t_x$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $2\partial_x(g^{li}\Gamma^{jk}_l) - \Gamma^{jk}_l g^{li}_{,s} u^s_x - \Gamma^{jk}_l \Gamma^{li}_t u^t_x$.
- The coefficient of $\delta_{xy}\delta_{xz}$ is

$$-\partial_x(\Gamma^{jk}_l g^{li}_{,s} u^s_x) + \partial^2_x(g^{li}\Gamma^{jk}_l) + \Gamma^{jk}_l \Gamma^{li}_{t,s} u^s_x u^t_x + \Gamma^{jk}_l \Gamma^{li}_t u^t_{xx} - \partial_x(\Gamma^{jk}_l \Gamma^{li}_t u^t_x) + \Gamma^{jk}_l w^l_s u^s_x w^i_t u^t_x.$$

- The coefficient of $\delta_{xy}\delta''_{xz}$ is $g^{li}\Gamma^{jk}_l$.
- The coefficient of $\delta'_{xy}\delta'_{xz}$ is $2g^{li}\Gamma^{jk}_l$.
- The coefficient of $\delta''_{xy}\delta_{xz}$ is $g^{li}\Gamma^{jk}_l$.
- The coefficient of $\delta_{yz}\nu_{zx}$ is $(\Gamma^{jk}_l w^l_{s,m}u^m_z u^s_z + \Gamma^{jk}_l w^l_s u^s_{zz})w^i_t u^t_x$.
- The coefficient of $\delta_{yx}\nu_{yz}$ is $(w^j_l \Gamma^{li}_{t,s}u^s_y u^t_y + w^j_l\Gamma^{li}_t u^t_{yy} + w^j_l w^l_s u^s_y w^i_t u^t_y)w^k_r\, u^r_z$.
- The coefficient of $\delta'_{yx}\nu_{yz}$ is $w^j_l g^{li}_{,s}u^s_y w^k_r\, u^r_z + w^j_l\Gamma^{li}_t u^t_y w^k_r\, u^r_z$.
- The coefficient of $\nu_{yx}\nu_{yz}$ is $w^j_l w^l_{s,m}u^m_y u^s_y w^i_t u^t_x w^k_r\, u^r_z + w^j_l w^l_s u^s_{yy} w^i_t u^t_x w^k_r\, u^r_z$.
- The coefficient of $\delta''_{yx}\nu_{yz}$ is $w^j_l g^{li} w^k_r\, u^r_z$.

The twelfth (and last) summand in (46) is

$$\begin{aligned}\frac{\partial P^{jk}_{y,z}}{\partial u^l_z}\partial_z P^{li}_{z,x} &= w^j_s u^s_y \nu_{yz} w^k_l g^{li}_{,m}u^m_z \delta'_{zx} + w^j_s u^s_y \nu_{yz} w^k_l g^{li}\delta''_{zx} \\ &+ w^j_s u^s_y \nu_{yz} w^k_l \Gamma^{li}_{t,m}u^m_z u^t_z \delta_{zx} + w^j_s u^s_y \nu_{yz} w^k_l \Gamma^{li}_t u^t_{zz}\delta_{zx} + w^j_s u^s_y \nu_{yz} w^k_l \Gamma^{li}_t u^t_z \delta'_{zx} \\ &+ w^j_s u^s_y \nu_{yz} w^k_l w^l_{r,m}u^m_z u^r_z \nu_{zx} w^i_t u^t_x + w^j_s u^s_y \nu_{yz} w^k_l w^l_r u^r_{zz}\nu_{zx} w^i_t u^t_x \\ &+ w^j_s u^s_y \nu_{yz} w^k_l w^l_r u^r_z \delta_{zx} w^i_t u^t_x. \quad (57)\end{aligned}$$

The coefficients of the reduced form are listed below

- The coefficient of $\delta_{xy}\delta_{xz}$ is
$$w^j_s u^s_x w^k_l g^{li}_{,m}u^m_x - \partial_x(w^j_s u^s_x)w^k_l g^{li} + w^j_s u^s_x w^k_l \Gamma^{li}_t u^t_x - 2w^j_s u^s_x \partial_x(w^k_l g^{li}).$$
- The coefficient of $\nu_{yx}\delta''_{xz}$ is $w^j_s u^s_y w^k_l g^{li}$.
- The coefficient of $\nu_{yx}\delta'_{xz}$ is $2w^j_s u^s_y \partial_x(w^k_l g^{li}) - w^j_s u^s_y w^k_l \Gamma^{li}_t u^t_x - w^j_s u^s_y w^k_l g^{li}_{,m}u^m_x$.
- The coefficient of $\delta_{xy}\delta'_{xz}$ is $-2w^j_s u^s_x w^k_l g^{li}$.
- The coefficient of $\delta'_{xy}\delta_{xz}$ is $-w^j_s u^s_x w^k_l g^{li}$.
- The coefficient of $\nu_{yx}\delta_{xz}$ is
$$w^j_s u^s_y\Big(w^k_l \Gamma^{li}_{t,m}u^m_x u^t_x + w^k_l\Gamma^{li}_t u^t_{xx} + \partial^2_x(w^k_l g^{li}) - \partial_x(w^k_l\Gamma^{li}_t u^t_x) + w^k_l w^l_r u^r_x w^i_t u^t_x - \partial_x(w^k_l g^{li}_{,m}u^m_x)\Big).$$
- The coefficient $\nu_{yz}\nu_{zx}$ is $w^j_s u^s_y w^k_l w^l_{r,m}u^m_z u^r_z w^i_t u^t_x + w^j_s u^s_y w^k_l w^l_r u^r_{zz} w^i_t u^t_x$.

5.3 The conditions

Collecting all similar terms we get the following conditions

- The coefficients of $\delta''_{xy}\delta_{xz}$, $\delta'_{xy}\delta'_{xz}$ and $\delta_{xy}\delta''_{xz}$ vanish iff (9c) holds, namely $g^{li}\Gamma^{jk}_l = g^{lj}\Gamma^{ik}_l$. Combining this condition with skew-symmetry of the bracket (9b) we obtain that $\Gamma^i_{jk} = -g_{jl}\Gamma^{li}_k$ are the Christoffel symbols of the Levi-Civita connection of g.

- The coefficients of products of step functions vanish.

- The coefficients of $\delta''_{xz}\nu_{xy}$, $\delta''_{zy}\nu_{zx}$ and $\delta''_{yx}\nu_{yz}$ vanish iff (9d) holds, namely $g_{ik}w^k_j = g_{jk}w^k_i$.

- Using the above conditions the coefficient of $u^s_{xx}\delta_{xy}\delta_{xz}$ can be written as

$$g^{li}\Gamma^{jk}_{l,s} - g^{li}\Gamma^{jk}_{s,l} + \Gamma^{ij}_l\Gamma^{lk}_s - \Gamma^{ik}_l\Gamma^{lj}_s + g^{li}(w^k_s w^j_l - w^j_s w^k_l) = R^{ijk}_s + g^{li}(w^k_s w^j_l - w^j_s w^k_l), \quad (58)$$

  where R^{ijk}_s is Riemann tensor (in upper indices). This yields the condition (9f).

- The coefficient of $u^t_{xx}\nu_{xy}\delta_{xz}$ (up to a common factor) is

$$g^{li}(w^k_{t,l} + \Gamma^k_{lm}w^m_t - w^k_{l,t} - \Gamma^k_{mt}w^m_l) = g^{li}(\nabla_l w^k_t - \nabla_t w^k_l), \quad (59)$$

  which yields the condition (9e).

- The coefficient of $\delta'_{xy}\delta_{xz}$ is a linear combination of the coefficient (58) repeated two times.

- The coefficients of $u^r_x u^s_x\delta_{xy}\delta_{xz}$ vanish using the x-derivative of the condition (58). Indeed, the coefficient reduces to

$$\begin{aligned}&(\Gamma^{ij}_{s,l} - \Gamma^{ij}_{l,s} + w^i_s w^j_l)\Gamma^{lk}_t + (\Gamma^{ki}_{s,l} - \Gamma^{ki}_{l,s} + w^k_s w^i_l)\Gamma^{lj}_t\\ &+ (\Gamma^{jk}_{s,l} - \Gamma^{jk}_{l,s} + w^j_s w^k_l)\Gamma^{li}_t + w^k_s g^{lj}(w^i_{t,l} - w^i_{l,t} + \Gamma^i_{lm}w^m_t)\\ &+ w^i_s g^{lk}(w^j_{t,l} - w^j_{l,t} + \Gamma^j_{lm}w^m_t) + w^j_s g^{li}(w^k_{t,l} - w^k_{l,t} + \Gamma^k_{lm}w^m_t).\end{aligned}$$

Using the condition (9e) we obtain

$$\begin{aligned}&(\Gamma^{ij}_{s,l}-\Gamma^{ij}_{l,s}+w^i_sw^j_l)\Gamma^{lk}_t+(\Gamma^{ki}_{s,l}-\Gamma^{ki}_{l,s}+w^k_sw^i_l)\Gamma^{lj}_t\\&+(\Gamma^{jk}_{s,l}-\Gamma^{jk}_{l,s}+w^j_sw^k_l)\Gamma^{li}_t+w^k_sg^{lm}(\Gamma^i_{tm}w^j_l)\\&+w^i_sg^{lm}(\Gamma^j_{tm}w^k_l)+w^j_sg^{lm}(\Gamma^k_{tm}w^i_l).\end{aligned}$$

Using again (58) we obtain

$$\begin{aligned}&(\Gamma^{ij}_{s,l}-\Gamma^{ij}_{l,s}+w^i_sw^j_l-w^j_sw^i_l)\Gamma^{lk}_t+(\Gamma^{ki}_{s,l}-\Gamma^{ki}_{l,s}+w^k_sw^i_l-w^i_sw^k_l)\Gamma^{lj}_t+\\&(\Gamma^{jk}_{s,l}-\Gamma^{jk}_{l,s}+w^j_sw^k_l-w^k_sw^j_l)\Gamma^{li}_t=\Gamma^{lk}_t(-\Gamma^i_{lm}\Gamma^{mj}_s+\Gamma^j_{lm}\Gamma^{mi}_s)+\\&\Gamma^{lj}_t(-\Gamma^k_{lm}\Gamma^{mi}_s+\Gamma^i_{lm}\Gamma^{mk}_s)+\Gamma^{lj}_t(-\Gamma^j_{lm}\Gamma^{mk}_s+\Gamma^k_{lm}\Gamma^{mj}_s)=0\end{aligned}$$

- The coefficient of $u^t_yu^r_xu^s_x\nu_{xy}\delta_{xz}$ vanishes due to the previous conditions. Indeed

$$\begin{aligned}&u^s_xu^r_xw^j_tu^t_y\Big(w^l_s(\Gamma^{ki}_{r,l}-\Gamma^{ki}_{l,r}+w^i_lw^k_r-w^k_lw^i_r)-(\Gamma^{ki}_l+\Gamma^{ik}_l)\partial_rw^l_s\\&+(\Gamma^{li}_r+\Gamma^{il}_r)(w^k_{s,l}-w^k_{l,s})-\Gamma^{ik}_{l,r}w^l_s+g^{li}(w^k_{s,l}-w^k_{l,s})_{,r}\\&+\Gamma^{li}_r(w^k_{l,s}-w^k_{s,l})+w^i_l\Gamma^{lk}_{s,r}+\Gamma^{ki}_lw^l_{s,r}+\Gamma^{lk}_sw^i_{r,l}\Big)\\=&u^s_xu^r_xw^j_tu^t_y\Big(w^l_s(\Gamma^{ki}_{r,l}-\Gamma^{ki}_{l,r}+w^i_lw^k_r-w^k_lw^i_r)-(\Gamma^{ki}_l+\Gamma^{ik}_l)w^l_{s,r}+\\&-\Gamma^{ik}_{l,r}w^l_s+(\Gamma^{ik}_lw^l_s-\Gamma^{lk}_sw^i_l)_{,r}+\Gamma^{li}_r(w^k_{l,s}-w^k_{s,l})\\&+w^i_l\Gamma^{lk}_{s,r}+\Gamma^{ki}_lw^l_{s,r}+\Gamma^{lk}_sw^i_{r,l}\Big)\\=&u^s_xu^r_xw^j_tu^t_y\Big(w^m_s(\Gamma^k_{rl}\Gamma^{li}_m-\Gamma^i_{rl}\Gamma^{lk}_m)-\Gamma^{lk}_sw^i_{l,r}\\&+\Gamma^{li}_r(\Gamma^k_{lm}w^m_s-\Gamma^k_{ms}w^m_l)+\Gamma^{lk}_sw^i_{r,l}\Big)\\=&u^s_xu^r_xw^j_tu^t_y\Big(w^m_s\Gamma^k_{rl}\Gamma^{li}_m-\Gamma^{lk}_s(w^i_{l,r}+\Gamma^i_{mr}w^m_l)+\Gamma^{lk}_sw^i_{r,l}\Big)=0.\end{aligned}$$

Similar computations hold for the coefficients of $\nu_{yz}\delta_{yx}$ and $\nu_{xz}\delta_{zy}$.

- The coefficient of $u^s_yu^t_x\delta'_{xz}\nu_{xy}$ is

$$\begin{aligned}&w^j_t\Big(g^{lk}w^i_{s,l}-g^{ki}_{,l}w^l_s+g^{li}w^k_{s,l}+w^i_lg^{lk}_{,s}\\&\qquad+w^i_l\Gamma^{lk}_s-2(w^k_lg^{li})_{,s}+w^k_l\Gamma^{li}_s+w^k_lg^{li}_{,s}\Big)\\&\qquad\quad=u^s_xu^t_y\Big(g^{lk}(\nabla_lw^i_s-\nabla_sw^i_l)+g^{li}(\nabla_lw^k_s-\nabla_sw^k_l)\Big),\end{aligned}$$

which vanishes upon (9e).

6 Weakly nonlocal PBHT and differential operators

Here we will just show the main steps of the algorithm in Section 3.3.

We assume that

$$P = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x + w^i_k u^k_x \partial_x^{-1} w^j_h u^h_x \tag{60}$$

where $\det(g^{ij}) \neq 0$ and $\epsilon_\alpha \in \mathbb{R}$.

We will compute the conditions of Hamiltonianity of the operator P of the type (60) using formula (84) and the Algorithm 3.3. Let us set:

$$P = L + N \quad \text{where} \quad L = g^{ij}\partial_x + \Gamma^{ij}_k u^k_x, \quad N = w^i_k u^k_x \partial_x^{-1} w^j_h u^h_x. \tag{61}$$

The conditions of skew-adjointness are obvious.

6.1 Calculation of the Jacobi identity

From now on we will assume L to be skew-adjoint (N is skew-adjoint by construction).

Lemma 6. *We have*

$$\begin{aligned}\frac{1}{2}[P,P] =& \frac{1}{2}[L,L] + [L,N] + \frac{1}{2}[N,N] \\ =& [\ell_{L,\psi^1}(L(\psi^2))(\psi^3) + \ell_{L,\psi^1}(N(\psi^2))(\psi^3) + \\ & \ell_{N,\psi^1}(L(\psi^2))(\psi^3) + \ell_{N,\psi^1}(N(\psi^2))(\psi^3) + cyclic(\psi^1,\psi^2,\psi^3)]\end{aligned} \tag{62}$$

We begin by computing the linearization of L and N. Let us introduce the new non-local scalar functions

$$\tilde{\psi}^k = \partial_x^{-1}(w^i_l u^l_x \psi^k_i), \qquad k = 1,2,3. \tag{63}$$

Lemma 7. *The linearization of L and N have the following expressions:*

$$\ell_{L,\psi^1}(\varphi)^i = \Big(g^{ij}_{,k}\partial_x\psi^1_j + \Gamma^{ij}_{h,k}u^h_x\psi^1_j\Big)\varphi^k + \Gamma^{ij}_h\psi^1_j\partial_x\varphi^h \tag{64}$$

$$\begin{aligned}\ell_{N,\psi^1}(\varphi)^i =& (w^i_{k,l}u^k_x\varphi^l + w^i_k\partial_x\varphi^k)\tilde{\psi}^1 \\ & + w^i_k u^k_x \partial_x^{-1}\big((w^j_{h,l}u^h_x\varphi^l + w^j_l\partial_x\varphi^l)\psi^1_j\big)\end{aligned} \tag{65}$$

We compute the first summand of (62):

$$\begin{aligned}\ell_{L,\psi^1}(L(\psi^2))(\psi^3) &= g^{ij}_{,k}g^{kp}\partial_x\psi^1_j\partial_x\psi^2_p\psi^3_i + g^{ij}_{,k}\Gamma^{kp}_h u^h_x\partial_x\psi^1_j\psi^2_p\psi^3_i \\ &+ \Big(\Gamma^{ij}_{h,k}u^h_x g^{kp} + \Gamma^{ij}_h\partial_x(g^{hp}) + \Gamma^{ij}_h\Gamma^{hp}_k u^k_x\Big)\psi^1_j\partial_x\psi^2_p\psi^3_i \\ &+ \Gamma^{ij}_h g^{hp}\psi^1_j\partial^2_x\psi^2_p\psi^3_i + \Big(\Gamma^{ij}_{h,k}u^h_x\Gamma^{kp}_h u^h_x + \Gamma^{ij}_h\partial_x(\Gamma^{hp}_k u^k_x)\Big)\psi^1_j\psi^2_p\psi^3_i.\end{aligned} \tag{66}$$

We compute the second summand of (62):

$$\begin{aligned}\ell_{L,\psi^1}(N(\psi^2))(\psi^3) &= g^{ij}_{,k}w^k_m u^m_x\partial_x\psi^1_j\tilde{\psi}^2\psi^3_i \\ &+ \Big(\Gamma^{ij}_{h,k}u^h_x w^k_m u^m_x + \Gamma^{ij}_k\partial_x(w^k_h u^h_x)\Big)\psi^1_j\tilde{\psi}^2\psi^3_i + \Gamma^{ij}_k w^k_h u^h_x w^p_l u^l_x\psi^1_j\psi^2_p\psi^3_i.\end{aligned} \tag{67}$$

We compute the third summand of (62). Here we integrate non-local expressions by parts in order to concentrate integrals in expressions of the form (63):

$$\begin{aligned}\ell_{N,\psi^1}(L(\psi^2))(\psi^3) &= (w^i_{l,k}u^l_x g^{kp} + w^i_k\partial_x(g^{kp}) + w^i_k\Gamma^{kp}_m u^m_x)\tilde{\psi}^1\partial_x\psi^2_p\psi^3_i \\ &+ (w^i_{l,k}u^l_x\Gamma^{kp}_m u^m_x + w^i_k\partial_x(\Gamma^{kp}_m u^m_x))\tilde{\psi}^1\psi^2_p\psi^3_i + w^i_k g^{kp}\tilde{\psi}^1\partial^2_x\psi^2_p\psi^3_i \\ &+ (-w^j_{h,k}u^h_x g^{kp} - w^j_k\partial_x(g^{kp}) - w^j_k\Gamma^{kp}_m u^m_x)\psi^1_j\partial_x\psi^2_p\tilde{\psi}^3 \\ &+ (-w^j_{h,k}u^h_x\Gamma^{kp}_m u^m_x - w^j_k\partial_x(\Gamma^{kp}_m u^m_x))\psi^1_j\psi^2_p\tilde{\psi}^3 - w^j_k g^{kp}\psi^1_j\partial^2_x\psi^2_p\tilde{\psi}^3.\end{aligned} \tag{68}$$

We compute the fourth summand of (62). Here we integrate non-local expressions by parts in order to concentrate integrals in expressions of the form (63):

$$\begin{aligned}\ell_{N,\psi^1}(N(\psi^2))(\psi^3) &= (w^i_{k,l}u^k_x w^l_h u^h_x + w^i_k\partial_x(w^k_m u^m_x))\tilde{\psi}^1\tilde{\psi}^2\psi^3_i \\ &+ w^i_k w^k_m u^m_x w^p_h u^h_x\tilde{\psi}^1\psi^2_p\psi^3_i + (-w^j_{h,l}u^h_x w^l_k u^k_x - w^j_l\partial_x(w^l_m u^m_x))\psi^1_j\tilde{\psi}^2\tilde{\psi}^3 \\ &- w^j_l w^l_m u^m_x w^p_h u^h_x\psi^1_j\psi^2_p\tilde{\psi}^3.\end{aligned} \tag{69}$$

The three-vector (62) can be written, after adding the cyclically permuted summands, as the sum $T_l + T_n$, where T_l is the local part and T_n is the non-local part.

6.2 Calculation of the reduced form

Now, we fix the indices 1, 2, 3 and we bring the nonlocal part to the normal form with respect to the three ordered indices. This means that the terms which are quadratic in the nonlocal expressions (63) shall be preserved, while terms which are linear in the nonlocal expressions should be brought to one of the following forms by integrating by parts:

$$\tilde{\psi}^1\partial_x^k\psi_p^2\psi_i^3, \quad \tilde{\psi}^2\partial_x^k\psi_p^3\psi_i^1, \quad \tilde{\psi}^3\partial_x^k\psi_p^1\psi_i^2. \tag{70}$$

For example, $g^{ij}_{,k}w^k_m u^m_x\tilde{\psi}^2\psi^3_i\partial_x\psi^1_j$ must be replaced by $-\partial_x\big(g^{ij}_{,k}w^k_m u^m_x\tilde{\psi}^2\psi^3_i\big)\psi^1_j$ (of course, up to a total divergence). After the above computational step we can write the final form of the nonlocal part of the three-vector. Note that T_n acquired some local terms at the end of the first step of the algorithm. We introduce the notation $T_n = T_N + T_{nL}$, where T_N is the non-local part and T_{nL} is the local part of T_n after the first step of the algorithm. We have, after collecting like terms:

$$\begin{aligned} T_N = &\\ &\Big(-\partial_x\big(g^{ij}_{,k}w^k_m u^m_x\big) + \Gamma^{ij}_{h,k}u^h_x w^k_m u^m_x + \Gamma^{ij}_k\partial_x(w^k_h u^h_x)\\ &\quad + w^j_{l,k}u^l_x\Gamma^{ki}_m u^m_x + w^j_k\partial_x(\Gamma^{ki}_m u^m_x)\Big)\\ &\quad - \partial_x\big(- w^i_{h,k}u^h_x g^{kj} - w^i_k\partial_x(g^{kj}) - w^i_k\Gamma^{kj}_m u^m_x\big) \qquad (71)\\ &\quad - w^i_{h,k}u^h_x\Gamma^{kj}_m u^m_x - w^i_k\partial_x(\Gamma^{kj}_m u^m_x) + \partial_x^2\big(- w^i_k g^{kj}\big)\\ &\quad + w^j_k w^k_m u^m_x w^i_h u^h_x - w^i_l w^l_m u^m_x w^j_h u^h_x\Big)\tilde{\psi}^2\psi^3_i\psi^1_j\\ &\Big(- g^{ij}_{,k}w^k_m u^m_x + w^j_{l,k}u^l_x g^{ki} + w^j_k\partial_x(g^{ki}) + w^j_k\Gamma^{ki}_m u^m_x\\ &\quad + w^i_{h,k}u^h_x g^{kj} + w^i_k\partial_x(g^{kj}) + w^i_k\Gamma^{kj}_m u^m_x \qquad (72)\\ &\quad + 2\partial_x\big(- w^i_k g^{kj}\big)\Big)\tilde{\psi}^2\partial_x\psi^3_i\psi^1_j\\ &+ (w^j_k g^{ki} - w^i_k g^{kj})\tilde{\psi}^2\partial_x^2\psi^3_i\psi^1_j \qquad (73) \end{aligned}$$

plus a cyclic permutation of the above terms. Moreover:

$$T_{nL} =$$

$$
\begin{aligned}
&\Big(- g^{ij}_{,k} w^k_m u^m_x w^p_l u^l_x - g^{jp}_{,k} w^k_m u^m_x w^i_l u^l_x - g^{pi}_{,k} w^k_m u^m_x w^j_l u^l_x \\
&\qquad + (w^j_{h,k} u^h_x g^{kp} + w^j_k \partial_x (g^{kp}) + w^j_k \Gamma^{kp}_m u^m_x) w^i_l u^l_x \\
&\qquad - 2\partial_x \big(w^j_k g^{kp} \big) w^i_l u^l_x - w^j_k g^{kp} \partial_x (w^i_l u^l_x) \\
&\qquad + (w^p_{h,k} u^h_x g^{ki} + w^p_k \partial_x (g^{ki}) + w^p_k \Gamma^{ki}_m u^m_x) w^j_l u^l_x \\
&\qquad + 2\partial_x \big(- w^p_k g^{ki} \big) w^j_l u^l_x + (-w^p_k g^{ki}) \partial_x (w^j_l u^l_x) \\
&\qquad + (w^i_{h,k} u^h_x g^{kj} + w^i_k \partial_x (g^{kj}) + w^i_k \Gamma^{kj}_m u^m_x) w^p_l u^l_x \\
&\qquad + (-w^i_k g^{kj}) \partial_x (w^p_l u^l_x) + 2\partial_x \big(- w^i_k g^{kj} \big) w^p_l u^l_x \Big) \\
&\qquad \psi^1_j \psi^2_p \psi^3_i
\end{aligned} \tag{74}
$$

$$
\big(- 2 w^j_k g^{kp} w^i_l u^l_x - w^p_k g^{ki} w^j_l u^l_x \big) \partial_x \psi^1_j \psi^2_p \psi^3_i \tag{75}
$$

plus a cyclic permutation of the last summand.

Let us introduce the notation $T_L = T_l + T_{nL}$. We shall bring each summand of T_L to the canonical form

$$
c^{jpi} \partial^k_x \psi^1_j \partial^h_x \psi^2_p \psi^3_i, \tag{76}
$$

where c^{jpi} are coefficient functions, using integration by parts on summands that contain $\partial^l_x \psi^3_i$ with $l > 0$. We have:

$$
\begin{aligned}
T_L = &\\
&\Big(g^{ij}_{,k} g^{kp} - g^{jp}_{,k} g^{ki} + 2\Gamma^{jp}_h g^{hi} - g^{pi}_{,k} g^{kj} \Big) \partial_x \psi^1_j \partial_x \psi^2_p \psi^3_i \\
&+ \Big(g^{ij}_{,k} \Gamma^{kp}_h u^h_x - \Gamma^{jp}_{h,k} u^h_x g^{ki} - \Gamma^{jp}_h \partial_x (g^{hi}) - \Gamma^{jp}_h \Gamma^{hi}_k u^k_x \\
&\qquad + 2\partial_x \big(\Gamma^{jp}_h g^{hi} \big) - \partial_x \big(g^{pi}_{,k} g^{kj} \big) - g^{pi}_{,k} \Gamma^{kj}_h u^h_x + \Gamma^{pi}_{h,k} u^h_x g^{kj} \\
&\qquad + \Gamma^{pi}_h \partial_x (g^{hj}) + \Gamma^{pi}_h \Gamma^{hj}_k u^k_x \\
&\qquad - 2 w^j_k g^{kp} w^i_l u^l_x - w^p_k g^{ki} w^j_l u^l_x + w^j_k g^{kp} w^i_l u^l_x + 2 w^i_k g^{kj} w^p_l u^l_x \Big) \\
&\qquad \partial_x \psi^1_j \psi^2_p \psi^3_i \\
&+ \Big(\Gamma^{ij}_{h,k} u^h_x g^{kp} + \Gamma^{ij}_h \partial_x (g^{hp}) + \Gamma^{ij}_h \Gamma^{hp}_k u^k_x - \partial_x \Big(g^{jp}_{,k} g^{ki} \Big) \\
&\qquad + g^{jp}_{,k} \Gamma^{ki}_h u^h_x - \Gamma^{jp}_{h,k} u^h_x g^{ki} - \Gamma^{jp}_h \partial_x (g^{hi}) - \Gamma^{jp}_h \Gamma^{hi}_k u^k_x \\
&\qquad + 2\partial_x \big(\Gamma^{jp}_h g^{hi} \big) - g^{pi}_{,k} \Gamma^{kj}_h u^h_x \\
&\qquad + w^j_k g^{kp} w^i_l u^l_x + 2 w^i_k g^{kj} w^p_l u^l_x - 2 w^p_k g^{ki} w^j_l u^i_x - w^i_k g^{kj} w^p_l u^l_x \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\psi^1_j \partial_x \psi^2_p \psi^3_i \\
&+ \Big(\Gamma^{ij}_h g^{hp} - g^{jp}_{,k} g^{ki} + \Gamma^{jp}_h g^{hi}\Big) \psi^1_j \partial^2_x \psi^2_p \psi^3_i \\
&+ \Big(\Gamma^{ij}_{h,k} u^h_x \Gamma^{kp}_l u^l_x + \Gamma^{ij}_h \partial_x (\Gamma^{hp}_k u^k_x) \\
&\quad - \partial_x \big(\Gamma^{jp}_{h,k} u^h_x g^{ki} + \Gamma^{jp}_h \partial_x (g^{hi}) + \Gamma^{jp}_h \Gamma^{hi}_k u^k_x\big) + \partial^2_x \big(\Gamma^{jp}_h g^{hi}\big) \\
&\quad + \Gamma^{jp}_{h,k} u^h_x \Gamma^{ki}_l u^l_x + \Gamma^{jp}_h \partial_x (\Gamma^{hi}_k u^k_x) - \partial_x \big(g^{pi}_{,k} \Gamma^{kj}_h u^h_x\big) \\
&\quad + \Gamma^{pi}_{h,k} u^h_x \Gamma^{kj}_l u^l_x + \Gamma^{pi}_h \partial_x (\Gamma^{hj}_k u^k_x) \\
&\quad + \Gamma^{ij}_k w^k_h u^h_x w^p_l u^l_x + \Gamma^{jp}_k w^k_h u^h_x w^i_l u^l_x + \Gamma^{pi}_k w^k_h u^h_x w^j_l u^l_x \\
&\quad - g^{ij}_{,k} w^k_m u^m_x w^p_l u^l_x - g^{jp}_{,k} w^k_m u^m_x w^i_l u^l_x - g^{pi}_{,k} w^k_m u^m_x w^j_l u^l_x \\
&\quad + (w^j_{h,k} u^h_x g^{kp} + w^j_k \partial_x (g^{kp}) + w^j_k \Gamma^{kp}_m u^m_x) w^i_l u^l_x \\
&\quad - 2\partial_x \big(w^j_k g^{kp}\big) w^i_l u^l_x - w^j_k g^{kp} \partial_x (w^i_l u^l_x) \\
&\quad + (w^p_{h,k} u^h_x g^{ki} + w^p_k \partial_x (g^{ki}) + w^p_k \Gamma^{ki}_m u^m_x) w^j_l u^l_x \\
&\quad + 2\partial_x \big(-w^p_k g^{ki}\big) w^j_l u^l_x + (-w^p_k g^{ki}) \partial_x (w^j_l u^l_x) \\
&\quad + (w^i_{h,k} u^h_x g^{kj} + w^i_k \partial_x (g^{kj}) + w^i_k \Gamma^{kj}_m u^m_x) w^p_l u^l_x \\
&\quad + (-w^i_k g^{kj}) \partial_x (w^p_l u^l_x) + 2\partial_x \big(-w^i_k g^{kj}\big) w^p_l u^l_x \\
&\quad - \partial_x \Big(-w^j_k g^{kp} w^i_l u^l_x - 2 w^i_k g^{kj} w^p_l u^l_x\Big) \\
&\psi^1_j \psi^2_p \psi^3_i \\
&+ \Big(\Gamma^{jp}_h g^{hi} - g^{pi}_{,k} g^{kj} + \Gamma^{pi}_h g^{hj}\Big) \partial^2_x \psi^1_j \psi^2_p \psi^3_i
\end{aligned}
$$

6.3 The conditions

The vanishing of coefficients of the 3-vector T yields the conditions on P to be Hamiltonian. Below we list the basic elements of T and the conditions that arise from their coefficients. We assume the condition of skew-adjointness of P.

$\psi^1_j \partial^2_x \psi^2_p \psi^3_i$: the coefficient is

$$\Gamma^{ij}_h g^{hp} - g^{jp}_{,k} g^{ki} + \Gamma^{jp}_h g^{hi} \tag{77}$$

and corresponds to the coefficient of $\delta''_{xy}\delta_{xz}$ and similar terms in Section 5.3. Its vanishing is equivalent to the condition $\Gamma^{jp}_h g^{hi} = \Gamma^{ip}_h g^{hj}$.

$\partial_x\psi^1_j\partial_x\psi^2_p\psi^3_i$**:** the coefficient vanish on account of the above condition.

$\tilde{\psi}^1\partial^2_x\psi^2_p\psi^3_i$**:** the coefficient is

$$w^i_k g^{kp} - w^p_k g^{ki} \tag{78}$$

and corresponds to the coefficients of $\delta''_{xz}\nu_{xy}$ and similar terms in Section 5.3.

$\psi^1_j\psi^2_p\psi^3_i$**:** This coefficient is a differential polynomial; the coefficient of u^k_{xx} reduces to

$$(\Gamma^{jp}_{h,k} - \Gamma^{jp}_{k,h})g^{hi} + \Gamma^{ij}_h\Gamma^{hp}_k - \Gamma^{ip}_h\Gamma^{hj}_k + g^{hi}(w^j_h w^p_k - w^p_h w^j_k) \tag{79}$$

using (78). This corresponds to the coefficient of $u^k_{xx}\delta_{xy}\delta_{xz}$ in Section 5.3.

$\tilde{\psi}^2\psi^3_i\psi^1_j$**:** This coefficient is a differential polynomial; the coefficient of u^m_{xx} reduces to

$$-g^{ij}_{,k}w^k_m + \Gamma^{ij}_k w^k_m + w^j_k\Gamma^{ki}_m + w^i_{m,k}g^{kj} - w^i_{k,m}g^{kj}, \tag{80}$$

and corresponds to the coefficient of $u^m_{xx}\nu_{xy}\delta_{xz}$ in Section 5.3. The coefficient is equal to $g^{kj}(\nabla_k w^i_m - \nabla_m w^i_k)$.

The correspondence between the coefficients of the three-vector in the language of operators and of distributions extends to all remaining terms; there is no need to repeat the computation that shows that all other coefficients vanish on account of the above conditions.

7 Weakly nonlocal PBHT and Poisson Vertex Algebras

In this section we will use the master formula to compute the skewsymmetry condition and the PVA-Jacobi identity for the λ bracket

$$\{u^i{}_\lambda u^j\}_P = g^{ji}\lambda + \Gamma^{ji}_s u^s_x + w^j_m u^m_x(\lambda+\partial)^{-1}w^i_n u^n_x, \tag{81}$$

corresponding to the weakly non-local Hamiltonian operator (61). As before, we split the operator in the local and nonlocal parts

$$\{u^i{}_\lambda u^j\}_L = g^{ji}\lambda + \Gamma^{ji}_s u^s_x, \tag{82}$$

$$\{u^i{}_\lambda u^j\}_N = w^j_m u^m_x(\lambda+\partial)^{-1}w^i_n u^n_x. \tag{83}$$

Enforcing Property 5 of Section 4.1 on the two λ brackets (82) and (83) gives the conditions for the corresponding operator to be skewsymmetric.

Indeed, for the local part, we have

$$\{u^i_\lambda u^j\}^L = g^{ji}\lambda + \Gamma^{ji}_s u^s_x = -_{\to}\{u^j_{-\lambda-\partial} u^i\}^L = g^{ij}\lambda + \partial_s g^{ij} u^s_x - \Gamma^{ij}_s u^s_x, \quad (84)$$

which implies the conditions (9a), (9b), and it is easy to prove that the nonlocal part is skewsymmetric by construction.

7.1 Computations with the master formula

For convenience, we split the PVA-Jacobi identity on the generators – defined in Section 4.2 – in the four parts

$$J^{ijk}_{\lambda,\mu}(P,P) = J^{ijk}_{\lambda,\mu}(L,L) + J^{ijk}_{\lambda,\mu}(N,N) + J^{ijk}_{\lambda,\mu}(N,L) + J^{ijk}_{\lambda,\mu}(L,N),$$

where the last two terms correspond to the Schouten bracket $[L,N]$.

The purely local part $J^{ijk}_{\lambda,\mu}(L,L)$ is a straightforward application of the master formula:

$$\begin{aligned}\{u^i_\lambda\{u^j_\mu u^k\}^L\}^L &= g^{li}\partial_l g^{kj}\lambda\mu + \partial_l g^{kj}\Gamma^{li}_s u^s_x\mu + g^{li}\partial_l\Gamma^{kj}_s u^s_x\lambda \\ &\quad + \Gamma^{li}_s\partial_l\Gamma^{kj}_t u^s_x u^t_x + g^{li}\Gamma^{kj}_l\lambda^2 + \partial_s g^{li}\Gamma^{kj}_l u^s_x\lambda \\ &\quad + \Gamma^{kj}_l\Gamma^{li}_s u^s_x\lambda + \Gamma^{kj}_l\partial_s\Gamma^{li}_t u^t_x u^s_x + \Gamma^{kj}_l\Gamma^{li}_s u^s_{xx}\end{aligned} \quad (85)$$

$$\begin{aligned}\{u^j_\mu\{u^i_\mu u^k\}^L\}^L &= g^{lj}\partial_l g^{ki}\lambda\mu + \partial_l g^{ki}\Gamma^{lj}_s u^s_x\lambda + g^{lj}\partial_l\Gamma^{ki}_s u^s_x\mu \\ &\quad + \Gamma^{lj}_s\partial_l\Gamma^{ki}_t u^s_x u^t_x + g^{lj}\Gamma^{ki}_l\mu^2 + \partial_s g^{lj}\Gamma^{ki}_l u^s_x\mu \\ &\quad + \Gamma^{ki}_l\Gamma^{lj}_s u^s_x\mu + \Gamma^{ki}_l\partial_s\Gamma^{lj}_t u^t_x u^s_x + \Gamma^{ki}_l\Gamma^{lj}_s u^s_{xx}\end{aligned} \quad (86)$$

$$\begin{aligned}\{\{u^i_\lambda u^j\}^L_{\lambda+\mu} u^k\}^L &= g^{kl}\partial_l g^{ji}\lambda^2 + g^{kl}\partial_l g^{ji}\lambda\mu + g^{kl}\partial_{sl} g^{ji} u^s_x\lambda \\ &\quad + g^{kl}\partial_l\Gamma^{ji}_s u^s_x\lambda + g^{kl}\partial_l\Gamma^{ji}_s u^s_x\mu + g^{kl}\partial_{sl}\Gamma^{ji}_t u^s_x u^t_x \\ &\quad + g^{kl}\partial_l\Gamma^{ji}_s u^s_{xx} - g^{kl}\Gamma^{ji}_l\lambda^2 - g^{kl}\Gamma^{ji}_l\mu^2 \\ &\quad - 2g^{kl}\Gamma^{ji}_l\lambda\mu - 2g^{kl}\partial_s\Gamma^{ji}_l u^s_x\lambda - 2g^{kl}\partial_s\Gamma^{ji}_l u^s_x\mu \\ &\quad - g^{kl}\partial_s\Gamma^{ji}_l u^s_{xx} - g^{kl}\partial_{st}\Gamma^{ji}_l u^s_x u^t_x + \Gamma^{kl}_s\partial_l g^{ji} u^s_x\lambda \\ &\quad + \Gamma^{kl}_s\partial_l\Gamma^{ji}_t u^s_x u^t_x - \Gamma^{kl}_s\Gamma^{ji}_l u^s_x\lambda - \Gamma^{kl}_s\Gamma^{ji}_l u^s_x\mu \\ &\quad - \Gamma^{kl}_s\partial_t\Gamma^{ji}_l u^s_x u^t_x\end{aligned} \quad (87)$$

where all the monomials are of the form $\lambda^p\mu^q$ with $p,q \geq 0$.

Computing the expressions with nonlocal terms is more complicated. However, it is possible to rely on Leibniz's and sesquilinearity properties of the λ brackets to split the problem into smaller chunks. The basic observation, that can be proved by expanding the $(\lambda+\partial)^{-1}$ expression, is that

$$\{f_\mu(\lambda+\partial)^{-1}g\} = (\lambda+\mu+\partial)^{-1}\{f_\mu g\}.$$

Let us start with $J^{ijk}_{\lambda,\mu}(N,L)$. Combining this with the left and right Leibniz properties we get

$$\begin{aligned}
\{u^i_\lambda\{u^j_\mu u^k\}^N\}^L &= \{u^i_\lambda w^k_m u^m_x(\mu+\partial)^{-1}w^j_n u^n_x\}^L\\
&= \{u^i_\lambda w^k_m u^m_x\}^L[(\mu+\partial)^{-1}w^j_n u^n_x] + w^k_m u^m_x\{u^i_\lambda(\mu+\partial)^{-1}w^j_n u^n_x\}^L\\
&= \{u^i_\lambda w^k_m u^m_x\}^L[(\mu+\partial)^{-1}w^j_n u^n_x] + w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\{u^i_\lambda w^j_n u^n_x\}^L\\
&= [(\mu+\partial)^{-1}w^j_n u^n_x]\left(g^{li}w^k_l\lambda^2\right.\\
&\qquad + g^{li}\partial_l w^k_s u^s_x\lambda + w^k_l\partial_s g^{li}u^s_x\lambda + w^k_l\Gamma^{li}_s u^s_x\lambda\\
&\qquad \left.+ \Gamma^{li}_s\partial_l w^k_t u^s_x u^t_x + w^k_l\partial_t\Gamma^{li}_s u^s_x u^t_x + w^k_l\Gamma^{li}_s u^s_{xx}\right)\\
&\quad + w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\left(w^j_l g^{li}\lambda^2\right.\\
&\qquad + g^{li}\partial_l w^j_s u^s_x\lambda + w^j_l\partial_s g^{li}u^s_x\lambda + w^j_l\Gamma^{li}_s u^s_x\lambda\\
&\qquad \left.+\partial_l w^j_s\Gamma^{li}_t u^s_x u^t_x + w^j_l\partial_t\Gamma^{li}_s u^s_x u^t_x + w^j_l\Gamma^{li}_s u^s_{xx}\right),
\end{aligned}\tag{88}$$

$$\begin{aligned}
\{u^j_\lambda\{u^i_\lambda u^k\}^N\}^L &= [(\lambda+\partial)^{-1}w^i_n u^n_x]\left(g^{lj}w^k_l\mu^2\right.\\
&\qquad + g^{lj}\partial_l w^k_s u^s_x\mu + w^k_l\partial_s g^{lj}u^s_x\mu + w^k_l\Gamma^{lj}_s u^s_x\mu\\
&\qquad \left.+\Gamma^{lj}_s\partial_l w^k_t u^s_x u^t_x + w^k_l\partial_t\Gamma^{lj}_s u^s_x u^t_x + w^k_l\Gamma^{lj}_s u^s_{xx}\right)\\
&\quad + w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\Big(\partial_l w^i_s\Gamma^{lj}_t u^s_x u^t_x\\
&\qquad +g^{lj}\partial_l w^i_s u^s_x\mu + w^i_l(\mu+\partial)\left(g^{lj}\mu+\Gamma^{lj}_s u^s_x\right)\Big),
\end{aligned}\tag{89}$$

$$\begin{aligned}
\{\{u^i_\lambda u^j\}^N_{\lambda+\mu}u^k\}^L &= \{w^j_m u^m_x(\lambda+\partial)^{-1}w^i_n u^n_{x\,\lambda+\mu}u^k\}^L\\
&= \{w^j_m u^m_x{}_{\lambda+\mu+\partial}u^k\}^L(\lambda+\partial)^{-1}w^i_n u^n_x\\
&\quad + \{w^i_n u^n_{x\lambda+\mu+\partial}u^k\}^L(\cancel{\lambda}-\cancel{\lambda}-\mu-\partial)^{-1}w^j_m u^m_x\\
&= g^{kl}(\lambda+\mu+\partial)\partial_l w^j_m u^m_x(\lambda+\partial)^{-1}w^i_n u^n_x\\
&\quad + \Gamma^{kl}_s u^s_x\partial_l w^j_t u^t_x(\lambda+\partial)^{-1}w^i_n u^n_x\\
&\quad - g^{kl}(\lambda+\mu+\partial)^2 w^j_l(\lambda+\partial)^{-1}w^i_n u^n_x\\
&\quad - \Gamma^{kl}_s u^s_x(\lambda+\mu+\partial)w^j_l(\lambda+\partial)^{-1}w^i_n u^n_x\\
&\quad - \Big(i\leftrightarrow j,\lambda\leftrightarrow\mu\Big)
\end{aligned}$$

$$
\begin{aligned}
&= g^{kl}\partial_l w^j_s u^s_x u^t_x - g^{kl} w^j_l w^i_s u^s_x \lambda - g^{kl} w^j_l w^i_s u^s_x \mu \\
&\quad - g^{kl}\partial(w^j_l w^i_s u^s_x) - g^{kl} w^j_l w^i_s u^s_x \mu - g^{kl} w^j_l \mu^2 [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad - g^{kl}\partial_s w^j_l u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] - g^{kl}\partial_s w^j_l w^i_t u^s_x u^t_x \\
&\quad - g^{kl}\partial_s w^j_l u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] - g^{kl}\partial^2 w^j_l [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad - \Gamma^{kl}_s w^j_l w^i_t u^s_x u^t_x + g^{kl}\partial_l w^j_s u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad + g^{kl}\partial(\partial_l w^j_m u^m_x)[(\lambda+\partial)^{-1} w^i_n u^n_x] + \Gamma^{kl}_s \partial_l w^j_t u^s_x u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad - \Gamma^{kl}_s w^j_l u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] - \Gamma^{kl}_s \partial_t w^j_l u^s_x u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&- \Big(i \leftrightarrow j, \lambda \leftrightarrow \mu\Big).
\end{aligned}
\tag{90}
$$

The notation $\big(i \leftrightarrow j, \lambda \leftrightarrow \mu\big)$ we used in Equation (90) means that all the terms are to be replaced with the ones obtained by switching the corresponding indices and parameters. The expressions of the form $[(\lambda+\partial)^{-1}A]$ enclosed within square brackets denote terms on which derivation operators "from outside" do not act and containing derivations which do not act "on the outside".

The computation of the terms of $J^{ijk}_{\lambda,\mu}(L,N)$ is straightforward for the first two addends

$$
\begin{aligned}
\{u^i_\lambda \{u^j_\mu u^k\}^L\}^N &= \{u^i_\lambda g^{kj}\}^N \mu + \{u^i_\lambda \Gamma^{kj}_s u^s_x\}^N \\
&= \partial_l g^{kj} w^l_s u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] + \partial_l \Gamma^{kj}_s u^s_x w^l_t u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad + \Gamma^{kj}(\lambda+\partial) w^l_s u^s_x (\lambda+\partial)^{-1} w^i_n u^n_x \\
&= \partial_l g^{kj} w^l_s u^s_x \mu [(\lambda+\partial)^{-1} w^i_n u^n_x] + \partial_l \Gamma^{kj}_s u^s_x w^l_t u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad + \Gamma^{kj}_l w^l_s w^i_t u^s_x u^t_x + \Gamma^{kj}_l \partial_t w^l_s u^s_x u^t [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad + \Gamma^{kj}_l w^l_s u^s_{xx} [(\lambda+\partial)^{-1} w^i_n u^n_x],
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
\{u^j_\mu \{u^i_\lambda u^k\}^L\}^N &= \{u^j_\mu g^{ki}\}^N \lambda + \{u^j_\mu \Gamma^{ki}_s u^s_x\}^N \\
&= \partial_l g^{ki} w^l_s u^s_x \lambda [(\mu+\partial)^{-1} w^j_n u^n_x] + \partial_l \Gamma^{ki}_s u^s_x w^l_t u^t_x [(\mu+\partial)^{-1} w^j_n u^n_x] \\
&\quad + \Gamma^{ki}_l w^l_s w^j_t u^s_x u^t_x + \Gamma^{ki}_l \partial_t w^l_s u^s_x u^t [(\mu+\partial)^{-1} w^j_n u^n_x] \\
&\quad + \Gamma^{ki}_l w^l_s u^s_{xx} [(\mu+\partial)^{-1} w^j_n u^n_x].
\end{aligned}
\tag{92}
$$

In the computation of the third one, we exploit the identity $(A+\partial)^{-1}B(A+$

$\partial)C = (A+\partial)^{-1}(A+\partial)BC - (A+\partial)^{-1}[\partial B]C$

$$
\begin{aligned}
&\{\{u^i_\lambda u^j\}^L_{\lambda+\mu} u^k\}^N = \{g^{ji}_{\lambda+\mu} u^k\}^N \lambda + \{\Gamma^{ji}_s u^s_{x\,\lambda+\mu} u^k\} \\
&\quad = w^k_n u^n_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x \partial_l g^{ji} \lambda + w^k_n u^n_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x \partial_l \Gamma^{ji}_t u^t_x \\
&\qquad - w^k_n u^n_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x (\lambda+\mu+\partial)\Gamma^{ji}_l \\
&\quad = -\Gamma^{ji}_l w^k_s w^l_t u^s_x u^t_x \\
&\qquad + w^k_n u^n_x (\lambda+\mu+\partial)^{-1} \big(\partial_l g^{ji} w^l_s u^s_x \lambda + \partial_l \Gamma^{ji}_s w^l_t u^s_x u^t_x \\
&\qquad\quad + \partial_s w^l_t \Gamma^{ji}_l u^t_x u^s_x + w^l_s \Gamma^{ji}_l u^s_{xx} \big).
\end{aligned} \tag{93}
$$

We compute now the expression for $J^{ijk}_{\lambda,\mu}(N,N)$. We have

$$
\begin{aligned}
&\{u^i_\lambda \{u^j_\mu u^k\}^N\}^N = \{u^i_\lambda w^k_m u^m_x (\mu+\partial)^{-1} w^j_n u^n_x\}^N \\
&\quad = w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \{u^i_\lambda w^j_n u^n_x\}^N + [(\mu+\partial)^{-1} w^j_n u^n_x] \{u^i_\lambda w^k_m u^m_x\}^N \\
&\quad = w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(\partial_l w^j_s u^s_x w^l_t u^t_x (\lambda+\partial)^{-1} w^i_n u^n_x \\
&\qquad\quad + w^j_l (\lambda+\partial) w^l_t u^t_x (\lambda+\partial)^{-1} w^i_n u^n_x \big) \\
&\qquad + [(\mu+\partial)^{-1} w^j_n u^n_x] \partial_l w^k_s u^s_x w^l_t u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\qquad + [(\mu+\partial)^{-1} w^j_n u^n_x] w^k_l (\lambda+\partial) w^l_s u^s_x (\lambda+\partial)^{-1} w^i_n u^n_x \\
&\quad = w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(w^j_l w^l_s w^i_t u^t_x u^s_x \big) + \\
&\quad + \quad w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(\big(\partial_l w^j_s w^l_t u^s_x u^t_x + w^j_l \partial_s w^l_t u^s_x u^t_x \\
&\qquad\quad + w^j_l w^l_s u^s_{xx} \big) (\lambda+\partial)^{-1} w^i_n u^n_x \big) + [(\mu+\partial)^{-1} w^j_n u^n_x] w^k_l w^l_s w^i_t u^s_x u^t_x \\
&\qquad + [(\mu+\partial)^{-1} w^j_n u^n_x][(\lambda+\partial)^{-1} w^i_n u^n_x] \big(\partial_l w^k_s w^l_t u^s_x u^t_x \\
&\qquad\quad + w^k_l \partial_t w^l_s u^s_x u^t + w^k_l w^l_s u^s_{xx} \big)
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
&\{u^j_\mu \{u^i_\lambda u^k\}^N\}^N = w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(w^i_l w^l_s w^j_t u^t_x u^s_x \big) + \\
&\quad + \quad w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(\big(\partial_l w^i_s w^l_t u^s_x u^t_x + w^i_l \partial_s w^l_t u^s_x u^t_x \\
&\qquad\quad + w^i_l w^l_s u^s_{xx} \big) (\mu+\partial)^{-1} w^j_n u^n_x \big) + [(\lambda+\partial)^{-1} w^i_n u^n_x] w^k_l w^l_s w^j_t u^s_x u^t_x \\
&\qquad + [(\mu+\partial)^{-1} w^j_n u^n_x][(\lambda+\partial)^{-1} w^i_n u^n_x] \big(\partial_l w^k_s w^l_t u^s_x u^t_x \\
&\qquad\quad + w^k_l \partial_t w^l_s u^s_x u^t + w^k_l w^l_s u^s_{xx} \big)
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
\{\{u^i_\mu u^j\}^N_{\lambda+\mu} u^k\}^N &= \{w^j_m u^m_x (\lambda+\partial)^{-1} w^i_n u^n_x{}_{\lambda+\mu} u^k\}^N \\
&= \{w^j_m u^m_x{}_{\lambda+\mu+\partial} u^k\}^N (\lambda+\partial)^{-1} w^i_n u^n_x \\
&\quad + \{(\lambda+\partial)^{-1} w^i_n u^n_x{}_{\lambda+\mu+\partial} u^k\}^N w^j_m u^m_x \\
&= \{w^j_m u^m_x{}_{\lambda+\mu+\partial} u^k\}^N (\lambda+\partial)^{-1} w^i_n u^n_x \\
&\quad - \{w^i_n u^n_x{}_{\lambda+\mu+\partial} u^k\}^N (\mu+\partial)^{-1} w^j_m u^m_x \\
&= w^k_m u^m_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x \partial_l w^j_t u^t_x (\lambda+\partial)^{-1} w^i_n u^n_x \\
&\quad - w^k_m u^m_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x (\lambda+\mu+\partial) w^j_l (\lambda+\partial)^{-1} w^i_n u^n_t \\
&\quad - w^k_m u^m_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x \partial_l w^i_t u^t_x (\mu+\partial)^{-1} w^j_n u^n_x \\
&\quad + w^k_m u^m_x (\lambda+\mu+\partial)^{-1} w^l_s u^s_x (\lambda+\mu+\partial) w^i_l (\mu+\partial)^{-1} w^j_n u^n_t \\
&= w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(\big(w^l_s u^s_x \partial_l w^j_t u^t_x + \partial_t w^l_s u^s_x u^t_x w^j_l + \\
&\qquad + w^l_s u^s_{xx} w^j_l \big) (\lambda+\partial)^{-1} w^i_n u^n_x \big) \\
&\quad - w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \big(\big(w^l_s u^s_x \partial_l w^i_t u^t_x + \partial_t w^l_s u^s_x u^t_x w^i_l + \\
&\qquad + w^l_s u^s_{xx} w^i_l \big) (\mu+\partial)^{-1} w^j_n u^n_x \big) \\
&\quad - w^k_s w^l_t w^j_l u^s_x u^t_x [(\lambda+\partial)^{-1} w^i_n u^n_x] \\
&\quad + w^k_s w^l_t w^i_l u^s_x u^t_x [(\mu+\partial)^{-1} w^j_n u^n_x].
\end{aligned} \tag{96}
$$

Note that in the last passage we have used the same identity as in Equation (93) to simplify the terms of the form $(\lambda+\mu+\partial)^{-1} A (\lambda+\mu+\partial) B$.

7.2 Projection onto the basis

The Jacobi identity lives in the previously defined space $V_{\lambda,\mu}$. The partial results of our computation are not all expressed in such a form that, after the expansion of the nonlocal terms, will produce elements on the basis $\lambda^p \mu^{d-p}$ and $(\lambda+\mu)^{-q} \lambda^{d+q}$ for $p \in \mathbb{Z}$, $q > 0$ for all $d \in \mathbb{Z}$.

All the double nonlocal terms in $J^{ijk}_{\lambda,\mu}(N,N)$ cancel out, leaving with a simplified expression

$$
\begin{aligned}
J^{ijk}_{\lambda,\mu}(N,N) &= w^k_m u^m_x (\lambda+\mu+\partial)^{-1} \left(w^j_l w^l_s w^i_t u^t_x u^s_x - w^i_l w^l_s w^j_t u^t_x \right) \\
&\quad + [(\mu+\partial)^{-1} w^j_n u^n_x] \left(w^k_l w^l_s w^i_t u^s_x u^t_x - w^k_s w^l_t w^i_l u^s_x u^t_x \right) \\
&\quad + [(\lambda+\partial)^{-1} w^i_n u^n_x] \left(-w^k_l w^l_s w^j_t u^s_x u^t_x + w^k_s w^l_t w^j_l u^s_x u^t_x \right).
\end{aligned} \tag{97}
$$

There are terms which would not expand in the chosen basis in the last line of Equation (89). We have

$$\begin{aligned}
&w^k_m u^m_x(\lambda+\mu+\partial)^{-1}(g^{lj}\partial_l w^i_s u^s_x \mu) = \\
&\quad= w^k_m u^m_x(\lambda+\mu+\partial)^{-1}((\lambda+\mu+\partial)g^{lj}\partial_l w^i_s u^s_x - (\lambda+\partial)g^{lj}\partial_l w^i_s u^s_x) \\
&\quad= g^{lj}\partial_l w^i_s w^k_t u^s_x u^t_x - w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\big(g^{lj}\partial_l w^i_s u^s_x\lambda + \partial_t g^{lj}\partial_l w^i_s u^s_x u^t_x \\
&\qquad + g^{lj}\partial_{tl} w^i_s u^s_x u^t_x + g^{lj}\partial_l w^i_s u^s_{xx}\big)
\end{aligned} \tag{98}$$

and

$$\begin{aligned}
&w^k_m u^m_x(\lambda+\mu+\partial)^{-1} w^i_l(\mu+\partial)(g^{lj}\mu + \Gamma^{lj}_s u^s_x) = \\
&\quad= w^k_m u^m_x(\lambda+\mu+\partial)^{-1}(\lambda+\mu+\partial)w^i_l\left(g^{lj}\mu + \Gamma^{lj}_s u^s_x\right) \\
&\qquad - w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\left(\lambda w^i_l(g^{lj}\mu + \Gamma^{lj}_s u^s_x) + \partial_t w^i_l u^t_x(g^{lj}\mu + \Gamma^{lj}_s u^s_x)\right) \\
&\quad= g^{lj} w^i_l w^k_s u^s_x \mu + \Gamma^{lj}_s w^k_t u^s_x u^t_x - w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\left(w^i_l \Gamma^{lj}_s u^s_z \lambda + \partial_t w^i_l \Gamma^{lj}_s u^s_x u^t_x\right) \\
&\qquad - w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\big((\lambda+\mu+\partial)(g^{lj} w^i_l \lambda + g^{lj}\partial_t w^i_l u^t_x) \\
&\qquad\quad -(\lambda+\partial)(w^i_l g^{lj}\lambda + g^{lj}\partial_t w^i_l u^t_x)\big) \\
&\quad= g^{lj} w^i_l w^k_s u^s_x \mu + \Gamma^{lj}_s w^k_t u^s_x u^t_x - g^{lj} w^i_l w^k_s u^s_x \lambda - g^{lj}\partial_s w^i_l w^k_t u^s_x u^t_x \\
&\qquad + w^k_m u^m_x(\lambda+\mu+\partial)^{-1}\big(g^{lj} w^i_l \lambda^2 + \partial_s g^{lj} w^i_l u^s_x \lambda + 2g^{lj}\partial_s w^i_l u^s_x \lambda \\
&\qquad\quad -\Gamma^{lj}_s w^i_l u^s_x \lambda + g^{lj}\partial_{st} w^i_l u^s_x u^t_x + \partial_s g^{lj}\partial_t w^i_l u^s_x u^t_x - \Gamma^{lj}_s \partial_t w^i_l u^s_x u^t_x \\
&\qquad\quad + g^{lj}\partial_s w^i_l u^s_{xx}\big)\,.
\end{aligned} \tag{99}$$

The full PVA-Jacobi identity can be then obtained in terms of the previously computed expressions, provided that we replace the last line in Equation (89) with the expression (98) + (99). The full form of $J^{ijk}_{\lambda,\mu}$ is then

$$\begin{aligned}
J^{ijk}_{\lambda,\mu}(P,P) = J^{ijk}_{\lambda,\mu}(L,L) + J^{ijk}_{\lambda,\mu}(L,N) + J^{ijk}_{\lambda,\mu}(N,L) + J^{ijk}_{\lambda,\mu}(N,N) \\
= ((85)-(86)-(87)) \\
+ ((91)-(92)-(93)) + ((88)-(89)-(90)) \\
+ ((94)-(95)-(96))\,.
\end{aligned}$$

7.3 The conditions

Assuming the skewsymmetry of the bracket P, the PVA-Jacobi equation is symmetric for cyclic permutations of $(i,\lambda),(j,\mu),(k,\nu=-\lambda-\mu-\partial)$, and it is fulfilled if and only if all the coefficients in the basis of $V_{\lambda,\mu}$ we have chosen vanish. We report here the coefficients corresponding to the elements of Section 6.3, under the condition of skewsymmetry for the bracket.

- The coefficient of λ^2 is

$$g^{li}\Gamma^{kj}_l - g^{kl}g^{ji}_{,l} + g^{kl}\Gamma^{ji}_l \tag{100}$$

whose vanishing, given the skewsymmetry of the bracket, is equivalent to $g^{il}\Gamma^{kj}_l = g^{kl}\Gamma^{ij}_l$.

- The coefficient of $\lambda\mu$ would be

$$g^{il}g^{kj}_{,l} - g^{lj}g^{ki}_{,l} - g^{kl}g^{ji}_{,l} + 2g^{kl}\Gamma^{ji}_l \tag{101}$$

which vanishes on account of the previous condition.

- The coefficient of $((\mu+\partial)^{-1}w^j_n u^n_x)\,\lambda^2$ is

$$g^{il}w^k_l - g^{kl}w^i_l \tag{102}$$

- The expression that is obtained when $\lambda^0\mu^0 = 1$ is a differential polynomial. The coefficient multiplying u^s_{xx} is

$$g^{kl}\left(\partial_s\Gamma^{ji}_l - \partial_l\Gamma^{ji}_s\right) + \Gamma^{kj}_l\Gamma^{li}_s - \Gamma^{ki}_l\Gamma^{lj}_s + g^{kl}\left(w^j_l w^i_s - w^i_l w^j_s\right), \tag{103}$$

where the two summands with ww come from (90).

- The expression $((\mu+\partial)^{-1}w^j_n u^n_x)$ is a differential polynomial. The coefficient multiplying u^s_{xx} is

$$w^k_l\Gamma^{li}_s - w^l_s\Gamma^{ki}_l + g^{kl}\left(\partial_l w^i_s - \partial_s w^i_l\right) \tag{104}$$

which is equal to Equation (80) (after the exchange of the free indices $(j,m) \leftrightarrow (k,s)$).

8 Concluding remarks

In this paper we have considered three different approaches to the problem of verifying Jacobi identity for weakly nonlocal Poisson brackets of hydrodynamic type, and we have showed their equivalence. While the equivalence between the formalism based on distributions and the one based on (pseudo)-differential operators is quite straightforward, the equivalence between them and the formalism of Poisson vertex algebras is more subtle and requires additional work (Proposition 2 and Theorem 3). The final result is an algorithmic procedure for each different case. It is also clear that the procedure can be used to check the compatibility of two different Poisson brackets.

We point out that the algorithm presented in this paper can be easily programmed in a computer algebra system generalizing existing packages for local structures (see for instance the package CDE [37] that is available for REDUCE [35] or the package MasterPVA [8] that is available for Wolfram Mathematica). We plan to do this in a future work. We also remark that the ideas outlined in the present work could be extended to weakly nonlocal symplectic operators [30] and to recursion operators.

A comprehensive differential-geometric theory of nonlocal integrability operators (i.e., Hamiltonian operators, symplectic operators and recursion operators for symmetries and conserved quantities) for partial differential equations, including weakly nonlocal operators, already exist [24], but it does not include Schouten brackets, the formulation of the symplectic property and the formulation of the hereditary property for recursion operators (the variational Nijenhuis bracket). However, this seems to be a possible goal and at the moment it is in development, see [23] for latest advances.

To conclude let us mention that one of the main challenge in the theory of integrable systems is the problem of classification of Hamiltonian integrable PDEs. The results obtained so far concern deformations of local bi-Hamiltonian structures of hydrodynamic type (see [6, 13, 27] and references therein). We hope that the results of the present paper will contribute to the study of the weakly nonlocal case.

9 Appendix: three different recipes for the Jacobi identity

In this section we will show that the expression of the Jacobi identity can be written in three different ways up to total divergencies.

We recall that the formal adjoint is defined by the equality

$$[\langle A(\psi^1_i), \psi^2\rangle - \langle \psi^1_i, A^*(\psi^2)\rangle] = 0, \tag{105}$$

where $\langle , \rangle$ is the pairing between vectors and covectors and square brackets mean that the result is an equivalence class up to total divergencies. In what follows we will need the standard facts [3, 26]:

$$\ell_{\Delta(\psi)}(\varphi) = \ell_{\Delta,\psi}(\varphi) + \Delta \circ \ell_\psi(\varphi), \tag{106}$$

and

$$\mathcal{E}(\langle \psi, \varphi \rangle) = \ell^*_\psi(\varphi) + \ell^*_\varphi(\psi), \tag{107}$$

where $\mathcal{E}$ is the Euler–Lagrange operator. If $P^* = -P$, then it is easy to prove [26] that

$$\ell^*_{P,\psi^1}(\psi^2) = \ell^*_{P^*,\psi^2}(\psi^1) = -\ell^*_{P,\psi^2}(\psi^1). \tag{108}$$

Theorem 8. *Let P, Q be skew-adjoint variational bivectors. Then, the following formulae for the Schouten bracket coincide up to total divergencies:*

$$\begin{aligned}[P,Q](\psi^1,\psi^2,\psi^3) =& \langle \ell_{P,\psi^1}(Q(\psi^2)), \psi^3\rangle + \mathit{cyclic}(\psi^1,\psi^2,\psi^3)\\ &+ \langle \ell_{Q,\psi^1}(P(\psi^2)), \psi^3\rangle + \mathit{cyclic}(\psi^1,\psi^2,\psi^3)\end{aligned} \tag{109}$$

$$\begin{aligned}[P,Q](\psi^1,\psi^2,\psi^3) =& \langle \ell_{P,\psi^1}(Q(\psi^2)), \psi^3\rangle - \langle \ell_{P,\psi^2}(Q(\psi^1)), \psi^3\rangle\\ &+ \langle \ell_{Q,\psi^1}(P(\psi^2)), \psi^3\rangle - \langle \ell_{Q,\psi^2}(P(\psi^1)), \psi^3\rangle\\ &- \langle P(\ell^*_{Q,\psi^2}(\psi^1)), \psi^3\rangle - \langle Q(\ell^*_{P,\psi^2}(\psi^1)), \psi^3\rangle\end{aligned} \tag{110}$$

$$\begin{aligned}[P,Q](\psi^1,\psi^2,\psi^3) =& \langle P(\mathcal{E}(\langle Q(\psi^1), \psi^2\rangle), \psi^3\rangle + \mathit{cyclic}(\psi^1,\psi^2,\psi^3)\\ &+ \langle Q(\mathcal{E}(\langle P(\psi^1), \psi^2\rangle), \psi^3\rangle + \mathit{cyclic}(\psi^1,\psi^2,\psi^3)\end{aligned} \tag{111}$$

Proof. The equivalence between (109) and (110) is given by the following formulae (all equalities are up to total divergencies!):

$$\begin{aligned}\langle \ell_{P,\psi^2}(Q(\psi^1)), \psi^3\rangle =& \langle Q(\psi^1), \ell^*_{P,\psi^2}(\psi^3)\rangle\\ =& \langle Q(\psi^1), \ell^*_{P^*,\psi^3}(\psi^2)\rangle\\ =& -\langle Q(\psi^1), \ell^*_{P,\psi^3}(\psi^2)\rangle\\ =& -\langle \ell_{P,\psi^3}(Q(\psi^1)), \psi^2\rangle\\ -\langle Q(\ell^*_{P,\psi^2}(\psi^1)), \psi^3\rangle =& -\langle \ell^*_{P,\psi^2}(\psi^1), Q^*(\psi^3)\rangle\\ =& \langle \ell^*_{P,\psi^2}(\psi^1), Q(\psi^3)\rangle\\ =& \langle \psi^1, \ell_{P,\psi^2}(Q(\psi^3))\rangle\end{aligned}$$

The equivalence between (109) and (111) is given by the following formulae:

$$\langle P(\mathcal{E}(Q(\psi^1)(\psi^2)), \psi^3\rangle = -\langle \mathcal{E}(Q(\psi^1)(\psi^2)), P(\psi^3)\rangle$$

$$
\begin{aligned}
&= - \langle \ell^*_{Q(\psi^1)}(\psi^2)) + \ell^*_{\psi^2}(Q(\psi^1)), P(\psi^3)\rangle \\
&= - \langle \psi^2, \ell_{Q,\psi^1}(P(\psi^3))\rangle - \langle \psi^2, Q(\ell_{\psi^1}(P(\psi^3)))\rangle + \langle \psi^1, Q(\ell_{\psi^2}(P(\psi^3)))\rangle \\
&= \langle \psi^1, \ell_{Q,\psi^2}(P(\psi^3))\rangle - \langle \psi^2, Q(\ell_{\psi^1}(P(\psi^3)))\rangle + \langle \psi^1, Q(\ell_{\psi^2}(P(\psi^3)))\rangle \\
&= \langle \psi^1, \ell_{Q,\psi^2}(P(\psi^3))\rangle + \langle Q(\psi^2), \ell_{\psi^1}(P(\psi^3))\rangle - \langle Q(\psi^1), \ell_{\psi^2}(P(\psi^3))\rangle
\end{aligned}
$$

In the cyclic sum in (111) all terms of the form $\langle Q(\psi^i), \ell_{\psi^j}(P(\psi^k))$ cancel if the computation is restricted to covector-valued densities ψ that lie in the image of the Euler–Lagrange operator $\mathcal{E}$: $\psi = \mathcal{E}(F)$, where $F = \int f\, dx$. In that case, we have $\ell_\psi = \ell^*_\psi$. The proof is completed by the remark that multivector identities hold true in general even if they are proved on the image of $\mathcal{E}$ only [26]. □

Remark 9. The expression (111) was used in [20] in order to check the Jacobi identity. The expressions (109) and (110) are more commonly used (see e.g. [3, 10, 33]). In particular, the expression (109) is the formula that we use throughout this paper. The three expressions do not exhaust all possibilities; see the above references for more exotic expressions of the Jacobi identity.

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