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Deaño, Alfredo and Eijssvoegel, Bruno and Román, Pablo (2019) Ladder relations for a class of matrix valued orthogonal polynomials. *Communications in Mathematical Physics*. ISSN 0010-3616. (Submitted)

DOI

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LADDER RELATIONS FOR A CLASS OF MATRIX VALUED ORTHOGONAL POLYNOMIALS

ALFREDO DEAÑO, BRUNO EIJSVOOGEL, AND PABLO ROMÁN

ABSTRACT. In this paper we study algebraic and differential relations for matrix valued orthogonal polynomials (MVOPs) defined on \mathbb{R} . Using recent results by Casper and Yakimov, we investigate MVOPs with respect to a matrix weight of the form $W(x) = e^{-v(x)}e^{xA}Te^{xA^*}$, where v is a scalar polynomial of even degree with positive leading coefficient and A and T are constant matrices. We obtain ladder operators, discrete string equations for the recurrence coefficients and multi-time Toda equations for deformations with respect to parameters in the weight, and we show that the Lie algebra generated by the ladder operators is finite dimensional. Hermite-type matrix valued weights are studied in detail: in this case the weight is characterized by the ladder operators, and the Lie algebra generated by them can be extended to a Lie algebra that is isomorphic to the standard Harmonic oscillator algebra. Freud-type matrix weights are also discussed. Finally, we establish the link between these ladder relations and those considered previously by A. Durán and M. Ismail.

1. INTRODUCTION

Matrix valued orthogonal polynomials (MVOPs) were introduced by Krein in the 1940's and they appear in different areas of mathematics and mathematical physics, including spectral theory [22], scattering theory [21], tiling problems [14], integrable systems [6, 2, 28] and stochastic processes [24, 12, 13]. There is also a fruitful interaction between harmonic analysis of matrix valued functions on compact symmetric pairs and matrix valued orthogonal polynomials. The first example of such an interaction is a family of matrix valued orthogonal polynomials related with the spherical functions of the compact symmetric pair $(\mathrm{SU}(3), \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1)))$, which appeared in [23]. Inspired by [37], the case of $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$ gave a direct approach [34, 35] leading to a general set-up in the context of multiplicity free pairs [27, 41]. In this context, some properties of the orthogonal polynomials such as orthogonality, recurrence relations and differential equations are understood in terms of the representation theory of the corresponding symmetric spaces, see also [1] for the quantum group case and [36] for multivariable matrix orthogonal polynomials.

The interpretation of matrix valued orthogonal polynomials in terms of the representation theory of a certain symmetric pair is typically only for a limited (discrete) number of the parameters involved. It is then necessary to develop analytic tools to extend to a general set of parameters. In this context, shift operators for matrix valued orthogonal polynomials turned out to be very useful [7, 8, 33, 31, 28].

In the last two decades, there has been significant progress in understanding how the differential and algebraic properties of the classical scalar orthogonal polynomials can be extended to the matrix valued setting. A. Durán and M. Ismail [19] introduced first order lowering and raising operators for MVOPs, and these results were rederived later on using the Riemann-Hilbert formulation by Grünbaum and coauthors [25]. There is also extensive work on orthogonal polynomial solutions of matrix valued differential equations of second order from an analytic point of view, we refer the reader for instance to [15, 18, 16, 17]. Recently, Casper and Yakimov [9] proposed a general framework to investigate the matrix Bochner problem, that is, the classification of $N \times N$ weight matrices $W(x)$ whose associated MVOPs are eigenfunctions of a second order differential operator.

The purpose of this paper is to apply the setup of [9] to MVOPs defined on the real line, in particular for exponential-type weights.

Given $N \in \mathbb{N}$, we denote by $M_N(\mathbb{C})$ the space of all $N \times N$ matrices with complex entries. Let $W: \mathbb{R} \rightarrow M_N(\mathbb{C})$ be a positive definite matrix weight supported in the (possibly infinite) interval $[a, b]$. For $M_N(\mathbb{C})$ -valued functions H, G , we define the matrix valued inner product

$$(1.1) \quad \langle H, G \rangle = \int_a^b H(y)W(y)G(y)^* dy \in M_N(\mathbb{C}).$$

Using standard arguments it can be shown that there exists a unique sequence $(P(x, n))_n$ of monic matrix valued orthogonal polynomials (MVOPs) with respect to W , in the following sense:

$$\langle P(x, n), P(x, m) \rangle = \mathcal{H}(n)\delta_{n,m},$$

where the squared norm $\mathcal{H}(n)$ is a positive definite matrix, see for instance [11, 26]. As a direct consequence of orthogonality, the polynomials $P(x, n)$ satisfy the following three-term recurrence relation

$$(1.2) \quad xP(x, n) = P(x, n+1) + B(n)P(x, n) + C(n)P(x, n-1),$$

where $B(n), C(n) \in M_N(\mathbb{C})$. Note that these matrix coefficients multiply the MVOPs from the left. From the orthogonality relations, we also obtain that

$$B(n) = X(n) - X(n+1), \quad C(n) = \mathcal{H}(n)\mathcal{H}(n-1)^{-1},$$

where $X(n)$ is the one-but-leading coefficient of $P(x, n)$, i.e. $P(x, n) = x^n + x^{n-1}X(n) + \dots$.

The structure of this paper is the following: in Section 2, following the approach of Casper and Yakimov in [9], we discuss differential and difference operators for these MVOPs. In this noncommutative setting operators can act both from the right and from the left. We consider two isomorphic algebras of operators acting on MVOPs, one algebra of matrix valued differential operators acting from the right, $\mathcal{F}_R(P)$, and a second algebra of matrix valued discrete operators acting from the left, $\mathcal{F}_L(P)$. In this construction, a differential operator $\mathcal{D} \in \mathcal{F}_R(P)$ acts naturally on the variable of the MVOPs, whereas a difference operator $M \in \mathcal{F}_L(P)$ acts on its degree.

The approach proposed in [9] is particularly explicit in the case of exponential weights defined on the real line; these weights are studied in Section 3 and written in the form $W(x) = e^{-v(x)}e^{xA}Te^{xA^*}$, with $x \in \mathbb{R}$, where the potential v is an even polynomial with positive leading coefficient and A and T are constant matrices. In this case, the differential operator $\mathcal{D} = \partial_x + A$ has a simple adjoint $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$, with respect to the matrix valued inner product given by W . The actions of \mathcal{D} and \mathcal{D}^\dagger on the MVOPs are

$$(P \cdot \mathcal{D})(x, n) = P'(x, n) + P(x, n)A, \quad (P \cdot \mathcal{D}^\dagger)(x, n) = -P'(x, n) - P(x, n)A - v'(x)P(x, n),$$

which will imply that $\mathcal{D}, \mathcal{D}^\dagger \in \mathcal{F}_R(P)$. Our first result states that $\mathcal{D}, \mathcal{D}^\dagger$ induce ladder relations:

$$P \cdot \mathcal{D}(x, n) = \sum_{j=-k+1}^0 A_j(n)P(x, n+j), \quad P \cdot \mathcal{D}^\dagger(x, n) = \sum_{j=0}^{k-1} \tilde{A}_j(n)P(x, n+j),$$

where $k = \deg v$, with some matrix coefficients $A_j(n)$ and $\tilde{A}_j(n)$. These operators are closely related to the creation and annihilation operators given in [19], with the advantage that \mathcal{D} and \mathcal{D}^\dagger are each other's adjoint. This property is crucial to show that the Lie algebra generated by the operators \mathcal{D} and \mathcal{D}^\dagger is finite dimensional and it is isomorphic to the algebra generated by the ladder operators for the scalar weight $w(x) = e^{-v(x)}$, see for instance [10], [29, Chapter 3]. From the ladder relations, we obtain nonlinear algebraic equations for the coefficients of the recurrence relation (1.2). In the literature these identities are often called discrete (or Freud) string equations. We include two examples: Hermite-type weights with $v(x) = x^2 + tx$ and $t \in \mathbb{R}$, and Freud-type weights with $v(x) = x^4 + tx^2$, and in this last case the discrete string equations are in fact a matrix analogue of the discrete Painlevé I equation [40]. We remark that this kind of identity, which is very relevant in integrable systems, is obtained here as a result of the relation between the two Fourier algebras of operators and in particular from the fact that $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ are isomorphic.

Section 4 is devoted to the detailed study of Hermite-type matrix valued weights. In this setting, we show that the ladder relations in fact characterize the matrix valued weight. We establish the

link between the Lie algebra generated by \mathcal{D} and \mathcal{D}^\dagger and the standard Harmonic oscillator algebra. An important feature of this case is that the matrix valued Hermite polynomials are related to a degenerate Hamiltonian which is a composition of the quantum harmonic oscillator with a non-relativistic spin Hamiltonian. This generalizes the well known property of scalar Hermite polynomials being related to the Schrödinger Hamiltonian.

Complementing the previous results, in Section 5, we investigate similar identities of differential and algebraic type for a deformation of the matrix weight with respect to extra parameters. Examples include the non-Abelian Toda and Langmuir lattice equations. For the particular case of a multi-time Toda deformation, we give a Lax pair formulation, analogous to [29, (2.8.5)] for the scalar case.

In the appendix we establish the link between the ladder relations obtained with this methodology and the ladder operators previously considered by A. Durán and M. Ismail in [19].

Acknowledgements. The authors would like to thank Erik Koelink and Mourad E. H. Ismail for fruitful discussions about the content and scope of this paper. Bruno Eijsvoegel thanks Koen Reijnders, John van de Wetering and Walter Van Assche for useful discussions as well.

The support of an Erasmus+ travel grant and EPSRC grant ‘‘Painlevé equations: analytical properties and numerical computation’’, reference EP/P026532/1 is gratefully acknowledged. The work of Pablo Román was supported a FONCyT grant PICT 2014-3452 and by SeCyTUNC.

2. PRELIMINARIES

In this section we introduce the left and right Fourier algebras related to the sequence of monic MVOPs, following a recent work of Casper and Yakimov [9]. For this, we view the sequence $P(x, n)$ as a function $P : \mathbb{C} \times \mathbb{N}_0 \rightarrow M_N(\mathbb{C})$. It is, therefore, natural to consider the space of functions

$$\mathcal{P} = \{Q : \mathbb{C} \times \mathbb{N}_0 \rightarrow M_N(\mathbb{C}) : Q(x, n) \text{ is rational in } x \text{ for fixed } n\}.$$

A differential operator of the form

$$(2.1) \quad \mathcal{D} = \sum_{j=0}^n \partial_x^j F_j(x),$$

where $F_j : \mathbb{C} \rightarrow M_N(\mathbb{C})$ is a rational function of x , acts on an element $Q \in \mathcal{P}$ from the right by

$$(Q \cdot \mathcal{D})(x, n) = \sum_{j=0}^n (\partial_x^j Q)(x, n) F_j(x) = \sum_{j=0}^n \frac{d^j Q}{dx^j}(x, n) F_j(x).$$

We denote the algebra of all differential operators of the form (2.1) by \mathcal{M}_N . Now we consider a left action on \mathcal{P} by discrete operators. For $j \in \mathbb{Z}$, let δ^j be the discrete operator which acts on a sequence $A : \mathbb{N}_0 \rightarrow M_N(\mathbb{C})$ by

$$(\delta^j \cdot A)(n) = A(n + j),$$

where we take the value of a sequence at a negative integer to be equal to 0. A discrete operator

$$(2.2) \quad M = \sum_{j=-\ell}^k A_j(n) \delta^j,$$

where $A_{-\ell}, \dots, A_k$ are sequences, acts on elements of \mathcal{P} from the left by

$$(M \cdot Q)(x, n) = \sum_{j=-\ell}^k A_j(n) (\delta^j \cdot Q)(x, n) = \sum_{j=-\ell}^k A_j(n) Q(x, n + j).$$

We denote the algebra of all discrete operators of the form (2.2) by \mathcal{N}_N , and we adapt the construction given in [9, Definition 2.20] to our setting:

Definition 2.1. *For the sequence $(P(x, n))_n$ of MVOPs we define:*

$$(2.3) \quad \begin{aligned} \mathcal{F}_L(P) &= \{M \in \mathcal{N}_N : \exists \mathcal{D} \in \mathcal{M}_N, M \cdot P = P \cdot \mathcal{D}\} \subset \mathcal{N}_N, \\ \mathcal{F}_R(P) &= \{\mathcal{D} \in \mathcal{M}_N : \exists M \in \mathcal{N}_N, M \cdot P = P \cdot \mathcal{D}\} \subset \mathcal{M}_N. \end{aligned}$$

Using these Fourier algebras, we prove the following uniqueness result:

Lemma 2.2. *Given $\mathcal{D} \in \mathcal{F}_R(P)$, there exists a unique $M \in \mathcal{F}_L(P)$ such that $M \cdot P = P \cdot \mathcal{D}$. Conversely, given $M \in \mathcal{F}_L(P)$, there exists a unique $\mathcal{D} \in \mathcal{F}_R(P)$ such that $M \cdot P = P \cdot \mathcal{D}$.*

Proof. Let us assume that there exist $M_1, M_2 \in \mathcal{F}_L(P)$ such that

$$(M_1 \cdot P)(x, n) = (P \cdot \mathcal{D})(x, n), \quad (M_2 \cdot P)(x, n) = (P \cdot \mathcal{D})(x, n),$$

then $((M_1 - M_2) \cdot P)(x, n) = 0$. Suppose that $M_1 - M_2$ has the following expression

$$(2.4) \quad ((M_1 - M_2) \cdot P)(x, n) = \sum_{j=-\ell}^k A_j(n) P(x, n + j).$$

By taking the leading coefficient of (2.4) we obtain that $A_k(n) = 0$. Proceeding recursively we conclude that $A_j(n) = 0$ for all $j = -\ell, \dots, k$. The converse is proven in a similar way. \square

It follows directly from the definition that the elements of $\mathcal{F}_L(P)$ are related to the elements of $\mathcal{F}_R(P)$. Lemma 2.2 shows that the map

$$\varphi: \mathcal{F}_L(P) \rightarrow \mathcal{F}_R(P), \quad \text{defined by} \quad M \cdot P = P \cdot \varphi(M),$$

is in fact a bijection.

Remark 2.3. For $M_1, M_2 \in \mathcal{F}_L(P)$ we have that

$$(2.5) \quad M_1 M_2 \cdot P = M_1 \cdot P \cdot \varphi(M_2) = P \cdot \varphi(M_1) \varphi(M_2),$$

which implies that $M_1 M_2 \in \mathcal{F}_L(P)$. Therefore the linear space $\mathcal{F}_L(P)$ is a subalgebra of \mathcal{N}_N . A similar computation shows that $\mathcal{F}_R(P)$ is an algebra. We shall refer to $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ as the left and right Fourier algebras respectively.

Now it follows from (2.5) that $M_1 M_2 \cdot P = P \cdot \varphi(M_1) \varphi(M_2)$ for all $M_1, M_2 \in \mathcal{F}_L(P)$. On the other hand, by the definition of φ , we have that $M_1 M_2 \cdot P = P \cdot \varphi(M_1 M_2)$ and, since φ is bijective, we conclude that φ is an isomorphism of algebras.

Remark 2.4. We can write the three-term recurrence relation (1.2) as

$$xP = P \cdot x = L \cdot P, \quad \text{where} \quad L = \delta + \alpha(n) + \beta(n)\delta^{-1}.$$

Therefore $x \in \mathcal{F}_R$, $L \in \mathcal{F}_L$ and $\varphi(L) = x$. Moreover, for every polynomial $v \in \mathbb{C}[x]$, we have

$$P \cdot v(x) = P \cdot v(\varphi(L)) = v(L) \cdot P.$$

The main result from [9] that we use in this paper is the existence of an adjoint operation \dagger in the Fourier algebras $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$, see [9, §3.1]. In order to introduce the adjoint on $\mathcal{F}_L(P)$, we first note that the algebra of discrete operators \mathcal{N}_N has a $*$ -operation given by

$$(2.6) \quad \left(\sum_{j=-\ell}^k A_j(n) \delta^j \right)^* = \sum_{j=-\ell}^k A_j(n-j)^* \delta^{-j},$$

where $A_j(n-j)^*$ denotes the conjugate transpose of $A_j(n-j)$. The adjoint of $M \in \mathcal{N}_N$ is

$$(2.7) \quad M^\dagger = \mathcal{H}(n) M^* \mathcal{H}(n)^{-1},$$

where the squared norm $\mathcal{H}(n)$ is viewed as a sequence. The following relation holds:

$$\langle (M \cdot P)(x, n), P(x, m) \rangle = \langle P(x, n), (M^\dagger \cdot P)(x, m) \rangle.$$

On the other hand, we say that \mathcal{D}^\dagger is the adjoint of an operator $\mathcal{D} \in \mathcal{F}_R(P)$ if

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P, Q \cdot \mathcal{D}^\dagger \rangle,$$

for all $P, Q \in M_N(\mathbb{C})[x]$. It follows from [9, Theorem 3.7] that $\varphi(M^\dagger) = \varphi(M)^\dagger$ for all $M \in \mathcal{F}_L(P)$, so that the Fourier algebras $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ are closed under the the adjoint operation \dagger .

Definition 2.5. Given a pair (M, \mathcal{D}) with $M \in \mathcal{F}_L(P)$ and $\mathcal{D} \in \mathcal{F}_R(P)$, a relation of the form

$$M \cdot P = P \cdot \mathcal{D}, \quad \text{where} \quad M = \sum_{j=-\ell}^k A_j(n) \delta^j.$$

is called a ladder relation. If the operator M only contains nonpositive (nonnegative) powers of δ , we say that it is a lowering (raising) relation.

Observe that if a pair (M, \mathcal{D}) gives a raising relation, then it follows from (2.7) and (2.6) that $(M^\dagger, \mathcal{D}^\dagger)$ gives a lowering relation and viceversa.

3. LADDER RELATIONS FOR EXPONENTIAL WEIGHTS

In this section we investigate the existence of lowering and raising relations for a class of matrix valued weights. Assume that $(a, b) = (-\infty, \infty)$ and that

$$(3.1) \quad W(x) = e^{-v(x)} e^{xA} T e^{xA^*}, \quad v(x) = x^k + v_{k-1} x^{k-1} + \cdots + v_0,$$

where k is even, A and T are constant matrices and T is invertible and self-adjoint. In the approach of [19, 10], the lowering and raising operators depend on both the degree of the polynomials and the variable x . In our case, we split the dependence on the variables x and n . We start with the matrix valued differential operator

$$(3.2) \quad \mathcal{D} = \partial_x + A,$$

and we compute the following explicit form of the adjoint:

Proposition 3.1. The adjoint of the differential operator (3.2) with respect to W is

$$\mathcal{D}^\dagger = -\partial_x - A + v'(x).$$

Proof. For polynomials $P, Q \in M_N(\mathbb{C})[x]$, we have from (1.1) and (3.2) that

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P', Q \rangle + \langle PA, Q \rangle.$$

Integrating $\langle P', Q \rangle$ by parts and using that W is invertible and self-adjoint, we obtain

$$(3.3) \quad \langle P', Q \rangle = -\langle P, Q' \rangle - \langle P, QW'W^{-1} \rangle = \langle P, -Q' - QW'W^{-1} \rangle.$$

Observe that the boundary terms on the right hand side of (3.3) vanish because of the exponential decay of the matrix weight at $\pm\infty$. Since $\langle PA, Q \rangle = \langle P, QWA^*W^{-1} \rangle$, we have

$$\langle P \cdot \mathcal{D}, Q \rangle = \langle P, -Q' - QW'W^{-1} + QWA^*W^{-1} \rangle.$$

Therefore $\mathcal{D}^\dagger = -\partial_x - W'(x)W(x)^{-1} + W(x)A^*W(x)^{-1}$. Using the explicit expression for W we complete the proof of the proposition. \square

Remark 3.2. For a given polynomial q , we denote by $(q(L))_j(n)$ the coefficient of the difference operator $q(L)$ of order j in δ . In other words, we have

$$(3.4) \quad q(L) = \sum_{j=-\deg q}^{\deg q} (q(L))_j(n) \delta^j.$$

The calculation of $(q(L))_j(n)$ can be carried out following the scheme shown in Figure 1: $(q(L))_j(n)$ is equal to the sum over all possible paths from $P(x, n)$ to $P(x, n+j)$ in $\deg q$ steps, where in each path we multiply the coefficients corresponding to each arrow.

For example if $q(x) = x^3$ we have $q(L) = L^3$, and in order to compute $(L^3)_{-1}(n)$ we have a total of six paths from $P(x, n)$ to $P(x, n-1)$ in three steps:

$$(L^3)_{-1}(n) = C(n)B(n-1)^2 + B(n)C(n)B(n-1) + B(n)^2C(n) \\ + C(n+1)C(n) + C(n)C(n-1) + C(n)^2.$$

$$\cdots \xrightarrow[\leftarrow]{C(n+2)} P(x, n+1) \xrightarrow[\leftarrow]{C(n+1)} P(x, n) \xrightarrow[\leftarrow]{C(n)} P(x, n-1) \xrightarrow[\leftarrow]{C(n-1)} \cdots$$

$\begin{array}{ccc} B(n+1) & B(n) & B(n-1) \\ \curvearrowright & \curvearrowright & \curvearrowright \\ \downarrow & \downarrow & \downarrow \end{array}$

FIGURE 1. Scheme for the calculation of $(q(L))_j(n)$.

Proposition 3.3. *Let W be a matrix weight as in (3.1). Then the monic polynomials $P(x, n)$ satisfy the lowering relation*

$$P \cdot \mathcal{D} = M \cdot P, \quad \mathcal{D} = \partial_x + A, \quad M = \sum_{j=-k+1}^0 A_j(n) \delta^j,$$

where $k = \deg v(x)$ and, using the notation (3.4), we have

$$A_0(n) = A, \quad A_j(n) = (v'(L))_j(n).$$

Proof. Notice that $(P \cdot \partial_x)(x, n)$ is a polynomial of degree $n - 1$. Furthermore $(P \cdot \mathcal{D})(x, n)$ is a polynomial of degree n with A as its leading coefficient. Therefore

$$(P \cdot \mathcal{D})(x, n) = \sum_{j=-n}^0 A_j(n) (\delta^j \cdot P)(x, n), \quad A_0(n) = A.$$

Moreover for $j < 0$, we have

$$\begin{aligned} A_j(n) &= \langle P \cdot \mathcal{D}, \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1} = \langle P, \delta^j \cdot P \cdot \mathcal{D}^\dagger \rangle \mathcal{H}(n-j)^{-1} \\ &= \langle P, \delta^j \cdot P \cdot v'(x) \rangle \mathcal{H}(n-j)^{-1} = \langle P \cdot v'(x), \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1} \\ &= \langle v'(L) \cdot P, \delta^j \cdot P \rangle \mathcal{H}(n-j)^{-1}, \end{aligned}$$

where we have used that $\langle P, \delta^j \cdot P \cdot \mathcal{D} \rangle = 0$ for $j < 0$ in the third equality, and the fact that $v'(x)$ is a scalar function in the fourth equality. Using (3.4) we get

$$A_j(n) = (v'(L))_{-j}(n).$$

In order to complete the proof we note that $(v'(L))_{-j}(n) = 0$ for all $j \geq k$. \square

Corollary 3.4. *Let W and \mathcal{D} be as in Proposition 3.3, then $\mathcal{D} \in \mathcal{F}_R(P)$.*

Proof. This is an immediate consequence of Definition 2.1 and Proposition 3.3. \square

This is a refinement for exponential weights of the general results in [25] where the authors use a Riemann-Hilbert formulation for matrix orthogonal polynomials. In particular, we give an elementary proof for the lowering relation and obtain the exact degree of the lowering operator M .

It follows from the discussion in Section 2 that there exists a unique $M^\dagger \in \mathcal{F}_L(P)$ such that

$$(3.5) \quad M \cdot P = P \cdot \mathcal{D} \quad \text{and} \quad M^\dagger \cdot P = P \cdot \mathcal{D}^\dagger.$$

Using the explicit expressions of \mathcal{D} and \mathcal{D}^\dagger , we find that

$$(3.6) \quad [\mathcal{D}^\dagger, \mathcal{D}] = v''(x), \quad \mathcal{D} + \mathcal{D}^\dagger = v'(x).$$

The second equation of (3.6) and Remark 2.4 imply that we can write

$$(3.7) \quad \varphi^{-1}(\mathcal{D} + \mathcal{D}^\dagger) = M + M^\dagger = v'(L).$$

explicitly in terms of the difference operator coming from the three-term recurrence relation. Using the explicit formula for \mathcal{D} in Proposition 3.3 and the definition of M^\dagger in (2.7), we verify

$$(3.8) \quad (v'(L))_0(n) = (\varphi^{-1}(\mathcal{D} + \mathcal{D}^\dagger))_0(n) = (M + M^\dagger)_0(n) = A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1},$$

using the notation (3.4) again.

Moreover we can use (3.5) to arrive at

$$[M^\dagger, M] \cdot P = P \cdot [\mathcal{D}^\dagger, \mathcal{D}] = P \cdot v''(x) = v''(L) \cdot P.$$

as well as similar formulas for higher order commutators. Note that each element in a Fourier algebra corresponds to a discrete-differential relation for the monic orthogonal polynomials. In particular, there is a distinguished subalgebra of $\mathcal{F}_R(P)$, namely the Lie algebra \mathfrak{g} generated by \mathcal{D} and \mathcal{D}^\dagger . In the following theorem we prove that \mathfrak{g} is independent of the matrix A and is isomorphic to the Lie algebra corresponding to the scalar case $A = 0$, $N = 1$ which was studied in [10, Theorem 3.1].

Theorem 3.5. *The differential operators \mathcal{D} and \mathcal{D}^\dagger generate a Lie algebra \mathfrak{g} of dimension $k + 1$.*

Proof. Let $v^{(j)}$ be the j -th derivative of v . Using that $v^{(j)}$ is scalar, and so it commutes with the matrix A , we first observe that $[\mathcal{D}, v^{(j)}] = -v^{(j+1)}$. Then for any $M_N(\mathbb{C})$ -valued smooth function F we have

$$F \cdot [\mathcal{D}, v^{(j)}] = F \cdot \mathcal{D}v^{(j)} - (Fv^{(j)}) \cdot \mathcal{D} = -v^{(j+1)}F.$$

Since $\mathcal{D}^\dagger = -\mathcal{D} + v'(x)$, we obtain that the Lie algebra generated by \mathcal{D} and \mathcal{D}^\dagger is generated by $\{\mathcal{D}, v'(x), \dots, v^{(k)}\}$, and is, therefore, $(k + 1)$ -dimensional. \square

From the previous results, we obtain nonlinear relations for the coefficients of the three-term recurrence relation. These identities can be seen as a non-Abelian analogue of the discrete string or Freud equations, see for instance [4, §4.1.1.5].

Theorem 3.6. *Let W be a matrix weight with monic orthogonal polynomials $P(x, n)$ such that*

$$\mathcal{D} = \partial_x + A \in \mathcal{F}_R(P), \quad \text{and} \quad \mathcal{D}^\dagger = -\mathcal{D} + v'(x),$$

for some polynomial $v(x)$ of degree k . Then the coefficients of the three term recurrence relation for $P(x, n)$ satisfy the following commutation relations

$$(3.9) \quad \begin{aligned} [B(n), A] &= I + (v'(L))_{-1}(n) - (v'(L))_{-1}(n+1), \\ [C(n), A] &= C(n)(v'(L))_0(n-1) - (v'(L))_0(n)C(n). \end{aligned}$$

Proof. Consider

$$(P \cdot \mathcal{D})(x, n) = (M \cdot P)(x, n).$$

In particular the coefficient of x^{n-1} gives

$$(3.10) \quad nI + X(n)A = AX(n) + A_{-1}(n).$$

Taking the difference of (3.10) for $n + 1$ and n we obtain

$$[B(n), A] - I = A_{-1}(n) - A_{-1}(n+1),$$

which together with Proposition 3.3 gives the first desired result.

For the second commutation relation we take (3.8) with n replaced by $n - 1$ and multiplied by $C(n)$ from the left, and we subtract from it (3.8) with parameter n and multiplied by $C(n)$ from the right. The result follows after cancellation of $\mathcal{H}(n)A^*\mathcal{H}(n-1)^{-1}$ terms. \square

Remark 3.7. Theorem 3.6 does not require the weight to be of exponential type as in (3.1). In the next section, however, we prove that for a polynomial v of degree two, the ladder relations (3.9), together with the moment of order zero, determine the weight to be of Hermite type.

Example 3.8. (*Hermite-type weight*) In the case of a Hermite-type weight with a Toda deformation, namely a matrix weight of the form (3.1) with $v(x) = x^2 + tx$, we have the operators

$$(3.11) \quad \mathcal{D} = \partial_x + A, \quad \mathcal{D}^\dagger = -\partial_x - A + 2x + t.$$

The Lie algebra generated by \mathcal{D} and \mathcal{D}^\dagger is 3-dimensional and we have the following relations

$$(3.12) \quad \mathcal{D} + \mathcal{D}^\dagger = 2x + t, \quad [\mathcal{D}^\dagger, \mathcal{D}] = 2.$$

Using Proposition 3.3 with $v'(x) = 2x + t$, we obtain $A_{-1}(n) = 2C(n) = 2\mathcal{H}(n)\mathcal{H}(n-1)^{-1}$. Moreover

$$(3.13) \quad M = A + 2C(n)\delta^{-1}, \quad M^\dagger = 2\delta + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1} = 2\delta + 2B(n) - A + tI.$$

The discrete string equations from Theorem 3.6 are

$$(3.14) \quad [B(n), A] = 2(C(n) - C(n+1)) + I, \quad [C(n), A] = 2(C(n)B(n-1) - B(n)C(n)).$$

Example 3.9. (*Freud type weight*) For a quartic even potential $v(x) = x^4 + tx^2$ we have the operators:

$$\mathcal{D} = \partial_x + A, \quad \mathcal{D}^\dagger = -\partial_x - A + 4x^3 + 2tx.$$

The following relations hold true:

$$[\mathcal{D}^\dagger, \mathcal{D}] = 12x^2 + 2t, \quad [[\mathcal{D}^\dagger, \mathcal{D}], \mathcal{D}] = 24x, \quad [[[\mathcal{D}^\dagger, \mathcal{D}], \mathcal{D}], \mathcal{D}] = 24.$$

The relation $P \cdot \mathcal{D} = M \cdot P$ in Proposition 3.3 is written explicitly as follows:

$$P'(x, n) + P(x, n)A = (A + A_{-1}(n)\delta^{-1} + A_{-2}(n)\delta^{-2} + A_{-3}(n)\delta^{-3})P(x, n).$$

where the coefficients are computed using that $A_{-j}(n) = (v'(L))_j(n)$ and the scheme in Figure 1:

$$\begin{aligned} A_{-1}(n) &= 4\left(C(n)C(n-1) + C(n)^2 + C(n+1)C(n) + B(n)^2C(n)\right. \\ &\quad \left.+ B(n)C(n)C(n-1) + C(n)B(n-1)^2\right) + 2tC(n) \\ A_{-2}(n) &= 4(B(n)C(n)C(n-1) + C(n)B(n-1)C(n-1) + C(n)C(n-1)B(n-2)), \\ A_{-3}(n) &= 4C(n)C(n-1)C(n-2). \end{aligned}$$

Furthermore, we use Theorem 3.6 to compute the discrete string equations

$$\begin{aligned} [C(n), A] &= C(n)(v'(L))_0(n-1) - (v'(L))_0(n)C(n), \\ [B(n), A] &= I + (v'(L))_{-1}(n) - (v'(L))_{-1}(n+1). \end{aligned}$$

If we replace $(v'(L))_0(n) = A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1}$ in the first equation, we obtain a trivial identity, however in terms of the coefficients of the recurrence relation we have

$$(v'(L))_0(n) = B(n)(C(n) + C(n+1)) + (C(n) + C(n+1) + B(n)^2 + 2t)B(n),$$

which implies the identity

$$\begin{aligned} [C(n), A] &= C(n)B(n-1)(C(n-1) + C(n)) \\ &\quad + C(n)(C(n-1) + C(n) + B(n-1)^2 + 2t)B(n-1) \\ &\quad - B(n)(C(n) + C(n+1))C(n) - (C(n) + C(n+1) + B(n)^2 + 2t)B(n)C(n). \end{aligned}$$

On the other hand, we can sum the second identity from 0 to $n-1$, to obtain

$$\sum_{k=0}^{n-1} [B(k), A] = n + (v'(L))_{-1}(0) - (v'(L))_{-1}(n) = n - (v'(L))_{-1}(n),$$

since $C(0) = 0$. Also, because $B(n) = X(n) - X(n+1)$, in terms of the subleading coefficient of $P(x, n)$, we obtain (3.10). If $A = 0$ then the weight is even and therefore $X(n) = 0$. In this case (3.10) reduces to the discrete Painlevé I equation, see e.g. [40, §1.2.2].

4. HERMITE-TYPE WEIGHTS

In this section we investigate further properties of Hermite-type matrix weights. First we will show that an arbitrary matrix weight W having operators \mathcal{D} and \mathcal{D}^\dagger as in Theorem 3.6 and with a specific moment of order zero is equivalent to a Hermite-type weight. Later we investigate a particular case of the matrix A that leads to the Harmonic oscillator algebra and a link with a quantum mechanical composite system.

4.1. Characterization of Hermite-type weights with a ladder relation. The proof of this characterization follows the lines of the main result in [5], where the authors discuss a scalar Freud weight.

Theorem 4.1. *Let \widetilde{W} be a matrix weight, supported on \mathbb{R} , and let $(P(x, n))_n$ be the sequence of monic orthogonal polynomials. Let A be a matrix such that*

$$\mathcal{D} = \partial_x + A \in \mathcal{F}_R(P), \quad \mathcal{D}^\dagger = -\mathcal{D} + 2x + t, \quad \text{and} \quad \mathcal{H}(0) = \int_{-\infty}^{\infty} e^{-x^2 - xt} e^{xA} T e^{xA^*} dx.$$

Then \widetilde{W} is equivalent to the Hermite-type weight

$$(4.1) \quad W(x) = e^{-x^2 - xt} e^{xA} T e^{xA^*}.$$

Proof. In this proof, we first show that the string equations determine uniquely, up to the zeroth moment, the coefficients of the recurrence relation for the monic MVOPs with respect to \widetilde{W} . Then the theorem follows by showing that the matrix weight (4.1) corresponds to a determinate moment problem, in the sense of [3].

We recall from Remark 3.7 that the string equations in Theorem 3.6 hold for the case of the matrix weight \widetilde{W} . We need to write the string equations in terms of the squared norms. Using the isomorphism φ^{-1} and (3.12) we verify that $2L = M + M^\dagger - t$ and using (3.13) we obtain

$$(4.2) \quad 2B(n) = A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1} - tI.$$

If we replace (4.2) in the first identity of (3.14), we obtain

$$C(n) - C(n+1) = \frac{1}{4} [\mathcal{H}(n)A^*\mathcal{H}(n)^{-1}, A] - \frac{1}{2}I.$$

Since $C(n) = \mathcal{H}(n)\mathcal{H}(n-1)^{-1}$ for all $n > 0$, we get

$$\mathcal{H}(n+1) = \frac{1}{2}\mathcal{H}(n) + \mathcal{H}(n)\mathcal{H}(n-1)^{-1}\mathcal{H}(n) - \frac{1}{4}\mathcal{H}(n)A^*\mathcal{H}(n)^{-1}A\mathcal{H}(n) + \frac{1}{4}A\mathcal{H}(n)A^*, \quad n > 0.$$

Moreover, $C(0) = 0$ gives the initial condition

$$\mathcal{H}(1) = \frac{1}{2}\mathcal{H}(0) - \frac{1}{4}\mathcal{H}(0)A^*\mathcal{H}(0)^{-1}A\mathcal{H}(0) + \frac{1}{4}A\mathcal{H}(0)A^*.$$

Therefore, the squared norms $\mathcal{H}(n)$ for $n > 0$ are completely determined by the choice of $\mathcal{H}(0)$. Furthermore, using the identities $C(n) = \mathcal{H}(n)\mathcal{H}(n-1)^{-1}$ and $2B(n) = A + \mathcal{H}(n)A^*\mathcal{H}(n)^{-1}$, we find that the coefficients of the recurrence relation and, thus, the monic orthogonal polynomials are completely determined as well.

Finally we need to prove that the moment problem for the Hermite weight W has a unique solution. By [3, Theorem 3.6], it suffices to show that the diagonal entries of the matrix valued measure $W(x)dx$ are determinate. We follow the approach given by Freud in [20, Theorem 5.1. and 5.2]: We observe that

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{\beta|x|} W(x)_{i,i} dx \right| &\leq \left\| \int_{-\infty}^{\infty} e^{\beta|x|} e^{-x^2 - xt} e^{xA} T e^{xA^*} dx \right\|_1 \\ &\leq \int_{-\infty}^{\infty} e^{\beta|x|} e^{-x^2 - xt} e^{x\|A\|_1} \|T\|_1 e^{x\|A^*\|_1} dx \leq M < \infty, \end{aligned}$$

for any $\beta > 0$. Therefore by [20, Theorem 5.2] the diagonal measures $W(x)_{i,i}dx$ are determinate and so is W . \square

4.2. The harmonic oscillator algebra. In the special case of a Hermite-type weight of the form (3.1) with $v(x) = x^2$ and $A = \sum_{j=2}^N \mu_j E_{j,j-1}$, for some coefficients μ_j , there exists a second order differential operator D which is symmetric with respect to W and so it has the orthogonal polynomials as eigenfunctions, see for instance [18, 31]. The nilpotent matrix considered in [18] is upper triangular instead of lower triangular as in this paper so, in the rest of the section we will

refer to [31]. It follows from [31, Proposition 3.5] that the monic orthogonal polynomials $P(x, n)$ with respect to W satisfy

$$(4.3) \quad (P \cdot D)(x, n) = \Gamma(n) \cdot P(x, n),$$

where D and $\Gamma(n)$ are given by

$$D = -\frac{1}{2}\partial_x^2 + \partial_x(xI - A) - \frac{1}{2}(A^2 + I) + J, \quad \Gamma(n) = nI + J - \frac{1}{2}(A^2 + I),$$

and J is the diagonal matrix defined by $J_{i,i} = i$. The expressions that we have are not identical to those in [31] because our weight W is related to theirs by a lower triangular matrix $L(0)$:

$$\widetilde{W} = L(0)WL(0)^*, \quad (L(0))_{k,\ell} = \frac{(-1)^{(k-\ell)/2}(k-\ell)!!}{(k-\ell)!}$$

when $k - \ell$ is non-negative and even. Note that the matrices J and A satisfy the commutation relation $[J, A] = A$. By Definition 2.3, we have that $D \in \mathcal{F}_R(P)$, $\Gamma \in \mathcal{F}_L(P)$ and $\varphi(\Gamma) = D$. We consider the operators \mathcal{D} and \mathcal{D}^\dagger from (3.11) with $t = 0$ and normalized in the following way:

$$\widetilde{\mathcal{D}} = 2^{-\frac{1}{2}}(\partial_x + A), \quad \widetilde{\mathcal{D}}^\dagger = 2^{-\frac{1}{2}}(-\partial_x - A + 2xI).$$

Using the explicit expressions of the differential operators, we easily verify that

$$(4.4) \quad [D, \widetilde{\mathcal{D}}] = \widetilde{\mathcal{D}}, \quad [D, \widetilde{\mathcal{D}}^\dagger] = -\widetilde{\mathcal{D}}^\dagger, \quad [\widetilde{\mathcal{D}}, \widetilde{\mathcal{D}}^\dagger] = -1.$$

Therefore, D , $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}^\dagger$ and the identity operator generate a four dimensional Lie algebra called the *harmonic oscillator algebra* which we denote by \mathfrak{h} . Note that since D , $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}^\dagger$ are elements of $\mathcal{F}_R(P)$, we have that $\mathfrak{h} \subset \mathcal{F}_R(P)$. Moreover the Lie algebra \mathfrak{g} in Theorem 3.5 is a three dimensional ideal in the Lie algebra \mathfrak{h} . The isomorphism φ immediately gives an isomorphic subalgebra $\varphi^{-1}(\mathfrak{h}) \subset \mathcal{F}_L(P)$.

With the identification

$$\widetilde{\mathcal{D}} \longleftrightarrow \mathcal{J}^+, \quad \widetilde{\mathcal{D}}^\dagger \longleftrightarrow \mathcal{J}^-, \quad D \longleftrightarrow \mathcal{J}^3, \quad I \longleftrightarrow \mathcal{E},$$

we find that \mathfrak{h} is isomorphic to the four dimensional Lie algebra $\mathcal{G}(a, b)$ given in [38, §2.5] with parameters $a = 0$ and $b = 1$, see also [39, Chapter 10, (1.1)]. The Casimir operator is given by

$$\mathcal{C} = D - \widetilde{\mathcal{D}}^\dagger \widetilde{\mathcal{D}} + I,$$

and commutes with $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}^\dagger$ and D . This follows directly from the commutation relations (4.4). Moreover, $\mathcal{C} \in \mathcal{F}_R(P)$ and \mathcal{C} is self-adjoint, i.e. $\mathcal{C}^\dagger = \mathcal{C}$. Using the explicit expressions of the operators, we obtain that \mathcal{C} is the zeroth order operator

$$\mathcal{C} = J - \frac{1}{2}I - xA.$$

Furthermore, if we denote $\widetilde{M} = 2^{-\frac{1}{2}}M$ and $\widetilde{M}^\dagger = 2^{-\frac{1}{2}}M^\dagger$, using (3.13) we obtain

$$\begin{aligned} \varphi^{-1}(\mathcal{C}) &= \Gamma(n) - \widetilde{M}^\dagger \widetilde{M} + I \\ &= -A\delta + J + \left(n + \frac{1}{2}\right)I - 2C(n+1) - B(n)A + (AC(n) - 2B(n)C(n))\delta^{-1}, \end{aligned}$$

which gives the following relation for the monic orthogonal polynomials:

$$\begin{aligned} -AP(x, n+1) + \left(J + \left(n + \frac{1}{2}\right)I - 2C(n+1) - B(n)A\right)P(x, n) \\ + (AC(n) - 2B(n)C(n))P(x, n-1) = P(x, n) \left(J - \frac{1}{2} - xA\right). \end{aligned}$$

Observe that the Casimir operator $\varphi^{-1}(\mathcal{C})$ of the Lie algebra $\varphi^{-1}(\mathfrak{h})$ is a second order difference operator having the sequence of monic MVOPs as eigenfunctions with a non-diagonal eigenvalue acting on $P(x, n)$ from the right.

The fact that the operator \mathcal{C} commutes with $\tilde{\mathcal{D}}$, $\tilde{\mathcal{D}}^\dagger$ and D is translated via the isomorphism φ into the following relations

$$[\varphi^{-1}(\mathcal{C}), \tilde{M}] = 0, \quad [\varphi^{-1}(\mathcal{C}), \tilde{M}^\dagger] = 0, \quad [\varphi^{-1}(\mathcal{C}), \Gamma(n)] = 0.$$

Writing the last equation explicitly, we obtain two new commutation relations for the coefficients of the recurrence relation:

$$\begin{aligned} A^2 + \left[2\mathcal{C}(n+1) + B(n)A, J - \frac{1}{2}A^2 \right] &= 0, \\ 2B(n)\mathcal{C}(n) - AC(n) - \left[2B(n)\mathcal{C}(n) - AC(n), J - \frac{1}{2}A^2 \right] &= 0. \end{aligned}$$

Note that these commutation relations involve the diagonal matrix J which does not appear in the matrix valued string equations.

4.2.1. *Vector-valued eigenfunctions of D .* For every $\lambda \in \mathbb{C}$ we define V_λ to be the vector space of all vector-valued eigenfunctions of the differential operator D which are in $L^2(W)$, i.e.

$$V_\lambda = \left\{ F : \mathbb{R} \rightarrow \mathbb{C}^N : F \cdot D = \lambda F, \quad \int_{-\infty}^{\infty} F(x)W(x)F(x)^* dx < \infty \right\}.$$

Note that from (4.3), the k -th row of $P(x, n)$ is an eigenfunction of D with eigenvalue $n+k$ and, thus, it is in V_{n+k} . Since \mathcal{C} commutes with D , we have that $F \cdot \mathcal{C}^\ell \in V_\lambda$ for all $F \in V_\lambda$ and $\ell \in \mathbb{N}$. From the first commutation relation in (4.4), we obtain that

$$(\lambda - 1)F \cdot \tilde{\mathcal{D}} = F \cdot \tilde{\mathcal{D}}D, \quad \text{for } F \in V_\lambda,$$

in such a way that $\tilde{\mathcal{D}} : V_\lambda \rightarrow V_{\lambda-1}$. Similarly we verify that $\tilde{\mathcal{D}}^\dagger : V_\lambda \rightarrow V_{\lambda+1}$. The operators D , \mathcal{D} , \mathcal{D}^\dagger and \mathcal{C} act on the vector spaces V_λ in the following way:

$$D : V_\lambda \rightarrow V_\lambda, \quad \tilde{\mathcal{D}} : V_\lambda \rightarrow V_{\lambda-1}, \quad \tilde{\mathcal{D}}^\dagger : V_\lambda \rightarrow V_{\lambda+1}, \quad \mathcal{C} : V_\lambda \rightarrow V_\lambda.$$

In this sense, $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\dagger$ are lowering and raising operators respectively.

If we conjugate the differential operators D , $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}^\dagger$ by $\Phi(x) = e^{-\frac{x^2}{2}} e^{xA}$, we obtain

$$\begin{aligned} E &= \Phi(x)^{-1} \tilde{\mathcal{D}} \Phi(x) = \frac{1}{\sqrt{2}} (\partial_x + x) I, & E^\dagger &= \Phi(x)^{-1} \tilde{\mathcal{D}}^\dagger \Phi(x) = \frac{1}{\sqrt{2}} (-\partial_x + x) I, \\ H &= \Phi(x)^{-1} D \Phi(x) = hI + \Delta, & h &= -\frac{1}{2} \partial_{xx} + \frac{1}{2} x^2, & \Delta &= \Phi(x)^{-1} \mathcal{C} \Phi(x) = J - \frac{1}{2} I, \end{aligned}$$

In this way the second order differential operator H splits into a scalar x -dependent part h plus a constant diagonal matrix Δ . Moreover, note that for $N=1$, the H reduces to the operator h which is precisely the Hamiltonian for a spinless particle subject to a harmonic potential as in [39, Chapter 10, (1.1)] and the operators E and E^\dagger are, respectively, the operators J^- and J^+ of [39, Chapter 10].

For $\beta \in \mathbb{C}$, we consider the following vector space of eigenfunctions of H .

$$\tilde{V}_\beta = \left\{ G : \mathbb{R} \rightarrow \mathbb{C}^N : G \cdot H = \beta G, \quad \int_{-\infty}^{\infty} G(x)TG(x)^* dx < \infty \right\}.$$

Observe that for every $\beta \in \mathbb{C}$, the function

$$\eta : \tilde{V}_\beta \rightarrow V_\beta, \quad G(x) \mapsto G(x)\Phi(x)^{-1},$$

is an isomorphism of vector spaces. The operators H , E , E^\dagger and Δ act on the vector spaces \tilde{V}_β in the following way:

$$H : \tilde{V}_\beta \rightarrow \tilde{V}_\beta, \quad E : \tilde{V}_\beta \rightarrow \tilde{V}_{\beta-1}, \quad E^\dagger : \tilde{V}_\beta \rightarrow \tilde{V}_{\beta+1}, \quad \Delta : \tilde{V}_\beta \rightarrow \tilde{V}_\beta.$$

In order to find the eigenfunctions of H , we treat the operators h and Δ separately. The scalar operator h is the well-known Schrödinger operator corresponding to the quantum harmonic

oscillator. Its square-integrable eigenfunctions are the Hermite functions

$$\psi_n(x) = \frac{e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x), \quad \psi_n(x)h = \left(n + \frac{1}{2}\right) \psi_n(x),$$

where $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$ are the scalar Hermite polynomials. On the other hand, the left eigenvectors of the matrix Δ are exactly the canonical vectors e_k , $k = 1, \dots, N$. Therefore

$$(4.5) \quad \{\psi_{\lambda-k}(x)e_k\}_{k=1}^{\min(\lambda, N)}$$

is an orthogonal basis for V_λ for $\lambda \in \mathbb{N}_{>0}$. We can however construct a different orthogonal basis using the MVOPs. The row vectors

$$(4.6) \quad q_{\lambda-k}^{(k)}(x) = e_k \cdot Q(x, \lambda - k), \quad \text{with} \quad Q(x, n) := P(x, n)\Phi(x),$$

is also an orthogonal basis of V_λ .

Remark 4.2. We can interpret the differential operator H as a Hamiltonian of a quantum mechanical composite system. See [32, §11.8] for a treatment of composite quantum systems.

The first part of the system would be the harmonic oscillator with h as its Hamiltonian. The second part would be a static spin $s = (N - 1)/2$ particle in a homogeneous magnetic field. Δ corresponds to the Hamiltonian of the second system if we tune the magnetic field strength¹ such that the spacing of energy levels of the two separate systems, is equal. The combined system could then be seen as a particle with spin s and without electrical charge, that is constrained to move in one direction subject to a quadratic potential with an additional (tuned) magnetic field in a perpendicular direction.

Then the basis in eq. (4.5) corresponds to an orthogonal basis of *separable states* and the basis in eq. (4.6), to an orthogonal basis of non-separable (i.e. entangled) states.

5. DEFORMATION OF THE WEIGHT AND MULTI-TIME TODA LATTICE

In this section, we consider an arbitrary matrix weight of the form

$$(5.1) \quad W(x) = e^{-v(x,t)} \widetilde{W}(x),$$

where $v(x; t)$ is a polynomial of even degree with positive leading coefficient depending smoothly on a parameter $t \geq 0$. In the following theorem we study the effect of differentiating the recurrence coefficients with respect to t , an idea that is natural when one considers orthogonal polynomials in the context of integrable systems.

Theorem 5.1. *If we denote by \cdot the derivative with respect to t , then the recurrence coefficients in (1.2) satisfy the following deformation equations:*

$$(5.2) \quad \begin{aligned} \cdot B(n) &= (\cdot v(L))_{-1}(n) - (\cdot v(L))_{-1}(n+1) \\ \cdot C(n) &= (\cdot v(L))_{-2}(n) - (\cdot v(L))_{-2}(n+1) + (\cdot v(L))_{-1}(n)B(n-1) - B(n)(\cdot v(L))_{-1}(n), \end{aligned}$$

where we use the same notation as in Remark 3.4.

Proof. Let $P(x, n)$ be the monic orthogonal polynomials with $W(x)$. Since $\langle P(x, n), P(x, m) \rangle = 0$ for $n > m$, we have

$$0 = \frac{\partial}{\partial t} \langle P(x, n), P(x, m) \rangle = \langle \cdot P(x, n), P(x, m) \rangle - \langle P(x, n) \cdot \cdot v(x), P(x, m) \rangle,$$

and then we can expand

$$\cdot P(x, n) = \sum_{m=0}^{n-1} \langle P(x, n) \cdot \cdot v(x), P(x, m) \rangle \mathcal{H}(m)^{-1} P(x, m) = \sum_{m=0}^{n-1} (\cdot v(L))_{m-n}(n) P(x, m),$$

¹Equivalently one could tune the quartic potential of the harmonic oscillator.

where we use the notation in Remark 3.4 for $(\cdot v(L))_k(n)$. On the other hand, if we differentiate the three-term recurrence relation (1.2) with respect to t , we obtain

$$\begin{aligned}
(5.3) \quad x \cdot P(x, n) &= \cdot P(x, n+1) + B(n) \cdot P(x, n) + C(n) \cdot P(x, n-1) \\
&\quad + \cdot B(n) P(x, n) + \cdot C(n) P(x, n-1) \\
&= \sum_{m=0}^n (\cdot v(L))_{m-n-1} (n+1) P(x, m) + B(n) \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) P(x, m) \\
&\quad + C(n) \sum_{m=0}^{n-2} (\cdot v(L))_{m-n+1} (n-1) P(x, m) + \cdot B(n) P(x, n) + \cdot C(n) P(x, n-1),
\end{aligned}$$

while on the left hand side we get

$$\begin{aligned}
(5.4) \quad x \cdot P(x, n) &= \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) P(x, m+1) + \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) B(m) P(x, m) \\
&\quad + \sum_{m=1}^{n-1} (\cdot v(L))_{m-n} (n) C(m) P(x, m-1).
\end{aligned}$$

Combining (5.3) and (5.4), isolating the derivatives of the recurrence coefficients, we get

$$\begin{aligned}
\cdot B(n) P(x, n) + \cdot C(n) P(x, n-1) &= \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) P(x, m+1) \\
&\quad + \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) B(m) P(x, m) + \sum_{m=1}^{n-1} (\cdot v(L))_{m-n} (n) C(m) P(x, m-1) \\
&\quad - \sum_{m=0}^n (\cdot v(L))_{m-n-1} (n+1) P(x, m) - B(n) \sum_{m=0}^{n-1} (\cdot v(L))_{m-n} (n) P(x, m) \\
&\quad \quad \quad - C(n) \sum_{m=0}^{n-2} (\cdot v(L))_{m-n+1} (n-1) P(x, m).
\end{aligned}$$

Comparing coefficients of $P(x, n)$ and $P(x, n-1)$ we obtain (5.2) for $\cdot B(n)$ and $\cdot C(n)$. \square

Example 5.2. If $\cdot v(x) = x$, we obtain the non-Abelian Toda lattice

$$\cdot B(n) = C(n) - C(n+1), \quad \cdot C(n) = C(n)B(n-1) - B(n)C(n).$$

Note that for $v(x) = x^2 + xt$ the relations (3.14) give

$$2 \cdot B(n) = [B(n), A] - I, \quad 2 \cdot C(n) = [C(n), A].$$

Example 5.3. If $\cdot v(x) = x^2$, we obtain the non-Abelian Langmuir lattice

$$\begin{aligned}
\cdot B(n) &= B(n)C(n) - B(n+1)C(n+1) + C(n)B(n-1) - C(n+1)B(n), \\
\cdot C(n) &= C(n)C(n-1) - C(n+1)C(n) + C(n)B(n-1)^2 - B(n)^2C(n).
\end{aligned}$$

5.1. Multi-time Toda lattice equations. Let W be a weight of the form (5.1) with a multi-time Toda deformation, namely a polynomial v of the form

$$v(x, \vec{t}) = v(x, t_1, \dots, t_k) = \sum_{j=1}^k t_j x^j.$$

If we denote by \cdot the derivative with respect to t_j , then we have $\cdot v(L) = L^j$. Theorem 5.1 gives the expressions for the derivatives of the recurrence coefficients, but if j is large, then the coefficients $(\cdot v'(L))_{-1, -2}$ can be difficult to compute, and a much more convenient formulation is given as a Lax pair.

In the spirit of [29, §2.8], we identify the operator L with the block tridiagonal matrix with block entries (L_{nm}) , $L_{n, n+1} = I$, $L_{n, n} = B(n)$, $L_{n, n-1} = C(n)$, and $L_{n, m} = 0I$ if $|n-m| \geq 2$. For

a $N \times N$ -block semi-infinite matrix $S = (S_{nm})$, we define S_+ as the matrix obtained by replacing all the $N \times N$ blocks of S below the main diagonal by zero. Analogously, we let S_- to be the matrix obtained by replacing all the $N \times N$ blocks above the subdiagonal by zero.

Then, we have the following result:

Theorem 5.4. *For $j = 1, \dots, k-1$, we have*

$$.L = [L, (L^j)_+] = -[L, (L^j)_-].$$

Proof. We first observe that

$$[L, (L^j)_+] + [L, (L^j)_-] = [L, (L^j)_+ + (L^j)_-] = [L, L^j] = 0,$$

which proves the second equality. Using that $.B(n) = (.L)_{n,n}$ and $.C(n) = (.L)_{n,n-1}$, we will complete the proof by showing that $[L, (L^j)_+]_{n,n}$ equals the right hand side of the first equation in (5.2), that $[L, (L^j)_+]_{n,n-1}$ equals the right hand side of the second equation of (5.2) and that $[L, (L^j)_+]_{n,m} = 0$ otherwise.

Note that $(.v(L))_m(n) = (L^j)_{n,n+m}$ for any indices m, n so the first equation in (5.2) reads

$$.B(n) = (L^j)_{n,n-1} - (L^j)_{n+1,n}, \quad n \geq 1.$$

On the other hand, bearing in mind that L is block tridiagonal and L_-^j is lower triangular with zeros on the diagonal, we have

$$[L, (L^j)_-]_{n,n} = L_{n,n+1}(L^j)_{n+1,n} - (L^j)_{n,n-1}L_{n-1,n} = (L^j)_{n+1,n} - (L^j)_{n,n-1},$$

since $L_{n,n+1} = I$ for any $n \geq 0$, which proves the result for the main diagonal. The second equation in (5.2) is

$$.C(n) = (L^j)_{n,n-2} - (L^j)_{n+1,n-1} + (L^j)_{n,n-1}B(n-1) - B(n)(L^j)_{n,n-1}, \quad n \geq 2.$$

We also have

$$\begin{aligned} [L, (L^j)_-]_{n,n-1} &= L_{n,n}(L^j)_{n,n-1} + L_{n,n+1}(L^j)_{n+1,n-1} - (L^j)_{n,n-1}L_{n-1,n-1} - (L^j)_{n,n-2}L_{n-2,n-1} \\ &= B(n)(L^j)_{n,n-1} + (L^j)_{n+1,n-1} - (L^j)_{n,n-1}B(n-1) - (L^j)_{n,n-2}, \end{aligned}$$

which proves the result for the first subdiagonal.

Finally, if $k \geq n+1$ we repeat the calculation using $[L, (L^j)_-]_{n,k}$, which gives 0, consistently with $(.L)_{n,k}$, and if $k \leq n-2$, we compute $[L, (L^j)_+]_{n,k}$, which gives 0 on both sides again. \square

APPENDIX A. COMPARISON OF LADDER OPERATORS

In this article we have taken a different approach to ladder operators for matrix valued orthogonal polynomials than for example the one in [19]. Their approach is inspired by [10] and [30] for the scalar orthogonal polynomials. This appendix is meant to compare our approach with theirs for our class of weight functions, i.e. the exponential weights in (3.1).

The ladder relations for exponential weights are stated in Proposition 3.3:

$$(A.1) \quad P'(x, n) = \sum_{j=1}^{k-1} A_{-j}(n)P(x, n-j) + [A, P(x, n)],$$

for monic polynomials P . For exponential weights on the real line, we have the following identity:

$$(A.2) \quad W'(x) = -W(x)V(x), \quad V(x) = v'(x)I - A^* - \rho(x),$$

where $\rho(x) = W^{-1}(x)AW(x)$.

The ladder relation given in [19] for exponential weights reads

$$(A.3) \quad P'(x, n) = F(x, n)P(x, n) - E(x, n)P(x, n-1),$$

where the coefficients are

$$(A.4) \quad \begin{aligned} E(x, n)\mathcal{H}(n-1) &= - \int_{\mathbb{R}} P(y, n)W(y) \frac{V(x) - V(y)}{x - y} P(y, n)^* dy, \\ F(x, n)\mathcal{H}(n-1) &= - \int_{\mathbb{R}} P(y, n)W(y) \frac{V(x) - V(y)}{x - y} P(y, n-1)^* dy. \end{aligned}$$

These identities are obtained in the following way: we expand the derivative of $P(x, n)$ in the basis of MVOPs, with coefficients multiplying on the left:

$$\begin{aligned} P'(x, n) &= \sum_{k=0}^{n-1} \langle P'(x, n), P(x, k) \rangle \mathcal{H}(k)^{-1} P(x, k) \\ &= \sum_{k=0}^{n-1} \left(\int_{\mathbb{R}} P'(y, n)W(y)P(y, k)^* dy \right) \mathcal{H}(k)^{-1} P(x, k) \\ &= \int_{\mathbb{R}} P'(n, y)W(y) \left(\sum_{k=0}^{n-1} P(y, k)^* \mathcal{H}(k)^{-1} P(x, k) \right) dy. \end{aligned}$$

We integrate by parts, and the boundary terms vanish because of the decay of $W(x)$ at $\pm\infty$. This, together with (A.2), gives

$$\begin{aligned} P'_n(x) &= - \int_{\mathbb{R}} P(n, y)W(y)(-V(y)) \sum_{k=0}^{n-1} P(y, k)^* \mathcal{H}(k)^{-1} P(x, k) dy \\ &= - \int_{\mathbb{R}} P(n, y)W(y)(V(x) - V(y)) \sum_{k=0}^{n-1} P(y, k)^* \mathcal{H}(k)^{-1} P(x, k) dy, \end{aligned}$$

where we have used the fact that the integral with $-V(x)$ vanishes by orthogonality. If we now apply the Christoffel-Darboux formula

$$(A.5) \quad (x-y) \sum_{k=0}^{n-1} P(y, k)^* \mathcal{H}(k)^{-1} P(x, k) = P(y, n-1)^* \mathcal{H}(n-1)^{-1} P(x, n) - P(y, n)^* \mathcal{H}(n-1)^{-1} P(x, n-1),$$

we obtain the formulas (A.4) for the coefficients $E(x, n)$ and $F(x, n)$. Furthermore, using the formula for $V(x)$ in (A.2), we can write

$$(A.6) \quad \begin{aligned} F(x, n)\mathcal{H}(n-1) &= - \int_{\mathbb{R}} P(y, n)W(y) \left[\frac{v'(x) - v'(y)}{x - y} - \frac{\rho(x) - \rho(y)}{x - y} \right] P(y, n-1)^* dy \\ E(x, n)\mathcal{H}(n-1) &= - \int_{\mathbb{R}} P(y, n)W(y) \left[\frac{v'(x) - v'(y)}{x - y} - \frac{\rho(x) - \rho(y)}{x - y} \right] P(y, n)^* dy. \end{aligned}$$

On the other hand, by direct computation using the fact that $P(x, n)$ is monic and (A.2), we have

$$(A.7) \quad \begin{aligned} P(x, n)A - AP(x, n) &= \sum_{k=0}^{n-1} \langle P_n A, P_k \rangle \mathcal{H}(k)^{-1} P_k(x) \\ &= \sum_{k=0}^{n-1} \left(\int_{\mathbb{R}} P(y, n)W(y) (\rho(y) - \rho(x)) P(y, k)^* dy \right) \mathcal{H}(k)^{-1} P(x, k). \end{aligned}$$

Therefore, applying (A.5) again, we obtain

$$\begin{aligned} P(x, n)A - AP(x, n) &= \left(\int_{\mathbb{R}} P(y, n)W(y) \frac{\rho(y) - \rho(x)}{x - y} P(y, n-1)^* dy \right) \mathcal{H}(n-1)^{-1} P(x, n) \\ &\quad + \left(\int_{\mathbb{R}} P(y, n)W(y) \frac{\rho(y) - \rho(x)}{x - y} P(y, n)^* dy \right) \mathcal{H}(n-1)^{-1} P(x, n-1). \end{aligned}$$

Comparing this last equation with (A.6), we find a relation between the two ladder operators, since

$$\begin{aligned} F(x, n)\mathcal{H}(n-1) + E(x, n)\mathcal{H}(n-1) &= P(x, n)A - AP(x, n) \\ &\quad - \int_{\mathbb{R}} P(y, n)W(y) \frac{v'(x) - v'(y)}{x - y} P(y, n-1)^* dy \\ &\quad - \int_{\mathbb{R}} P(y, n)W(y) \frac{v'(x) - v'(y)}{x - y} P(y, n)^* dy. \end{aligned}$$

REFERENCES

- [1] Noud Aldenhoven, Erik Koelink, and Pablo Román. Matrix-valued orthogonal polynomials related to the quantum analogue of $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$. *Ramanujan J.*, 43(2):243–311, 2017.
- [2] Gerardo Ariznabarreta and Manuel Mañas. Matrix orthogonal Laurent polynomials on the unit circle and Toda type integrable systems. *Adv. Math.*, 264:396–463, 2014.
- [3] C. Berg. The matrix moment problem. In A.J.P.L. Branquinho and A. P. Foulquié Moreno, editors, *Coimbra Lecture Notes On Orthogonal Polynomials*, pages 1–58. Nova Science, 2008.
- [4] Pavel M. Bleher. Lectures on random matrix models: the Riemann-Hilbert approach. In *Random matrices, random processes and integrable systems*, CRM Ser. Math. Phys., pages 251–349. Springer, New York, 2011.
- [5] Stanford Bonan and Paul Nevai. Orthogonal polynomials and their derivatives. I. *J. Approx. Theory*, 40(2):134–147, 1984.
- [6] Mattia Cafasso and Manuel D. de la Iglesia. Non-commutative Painlevé equations and Hermite-type matrix orthogonal polynomials. *Comm. Math. Phys.*, 326(2):559–583, 2014.
- [7] M. J. Cantero, L. Moral, and L. Velázquez. Differential properties of matrix orthogonal polynomials. *J. Concr. Appl. Math.*, 3(3):313–334, 2005.
- [8] M. J. Cantero, L. Moral, and L. Velázquez. Matrix orthogonal polynomials whose derivatives are also orthogonal. *J. Approx. Theory*, 146(2):174–211, 2007.
- [9] W.R. Casper and M. Yakimov. The matrix Bochner problem. *preprint, arXiv:1803.04405*, 2018.
- [10] Yang Chen and Mourad E. H. Ismail. Ladder operators and differential equations for orthogonal polynomials. *J. Phys. A*, 30(22):7817–7829, 1997.
- [11] David Damanik, Alexander Pushnitski, and Barry Simon. The analytic theory of matrix orthogonal polynomials. *Surv. Approx. Theory*, 4:1–85, 2008.
- [12] Manuel D. de la Iglesia. Spectral methods for bivariate Markov processes with diffusion and discrete components and a variant of the Wright-Fisher model. *J. Math. Anal. Appl.*, 393(1):239–255, 2012.
- [13] Manuel D. de la Iglesia and Pablo Román. Some bivariate stochastic models arising from group representation theory. *Stochastic Process. Appl.*, 128(10):3300–3326, 2018.
- [14] Maurice Duits and Arno B.J. Kuijlaars. The two periodic aztec diamond and matrix valued orthogonal polynomials. to appear in *Journal of the European Mathematical Society*, 2017.
- [15] Antonio J. Durán. Matrix inner product having a matrix symmetric second order differential operator. *Rocky Mountain J. Math.*, 27(2):585–600, 1997.
- [16] Antonio J. Durán. Rodrigues’ formulas for orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not. IMRN*, 2010(5):824–855, 2010.
- [17] Antonio J. Durán and Manuel D. de la Iglesia. Second-order differential operators having several families of orthogonal matrix polynomials as eigenfunctions. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 084, 24, 2008.
- [18] Antonio J. Durán and F. Alberto Grünbaum. Orthogonal matrix polynomials satisfying second-order differential equations. *Int. Math. Res. Not.*, 2004(10):461–484, 2004.
- [19] Antonio J. Durán and Mourad E. H. Ismail. Differential coefficients of orthogonal matrix polynomials. *J. Comput. Appl. Math.*, 190(1-2):424–436, 2006.
- [20] G. Freud. *Orthogonal Polynomials*. Pergamon Press, 1971.
- [21] J. S. Geronimo. Scattering theory and matrix orthogonal polynomials on the real line. *Circuits Systems Signal Process.*, 1(3-4):471–495, 1982.
- [22] Wolter Groenevelt, Mourad E. H. Ismail, and Erik Koelink. Spectral decomposition and matrix-valued orthogonal polynomials. *Adv. Math.*, 244:91–105, 2013.
- [23] F. A. Grünbaum, I. Pacharoni, and J. Tirao. Matrix valued spherical functions associated to the complex projective plane. *J. Funct. Anal.*, 188(2):350–441, 2002.
- [24] F. Alberto Grünbaum and Manuel D. de la Iglesia. Matrix valued orthogonal polynomials arising from group representation theory and a family of quasi-birth-and-death processes. *SIAM J. Matrix Anal. Appl.*, 30(2):741–761, 2008.
- [25] F. Alberto Grünbaum, Manuel D. de la Iglesia, and Andrei Martínez-Finkelshtein. Properties of matrix orthogonal polynomials via their Riemann-Hilbert characterization. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 7(098):31 pages, 2011.
- [26] F. Alberto Grünbaum and Juan Tirao. The algebra of differential operators associated to a weight matrix. *Integral Equations Operator Theory*, 58(4):449–475, 2007.

- [27] Gert Heckman and Maarten van Pruijssen. Matrix valued orthogonal polynomials for Gelfand pairs of rank one. *Tohoku Math. J. (2)*, 68(3):407–437, 2016.
- [28] Mourad Ismail, Erik Koelink, and Pablo Román. Burchnell-type identities for matrix-valued orthogonal polynomials. *To appear in Advances in applied mathematics*, 2019.
- [29] Mourad E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey.
- [30] Mourad E. H. Ismail and Jet Wimp. On differential equations for orthogonal polynomials. *Methods Appl. Anal.*, 5(4):439–452, 1998.
- [31] Mourad E.H. Ismail, Erik Koelink, and Pablo Román. Generalized Burchnell-type identities for orthogonal polynomials and expansions. *Symmetry, Integrability and Geometry: Methods and Applications*, 14(072):24 pages, July 2018.
- [32] Josef M. Jauch. *Foundations of quantum mechanics*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1968.
- [33] Erik Koelink, Ana M. de los Ríos, and Pablo Román. Matrix-valued Gegenbauer-type polynomials. *Constr. Approx.*, 46(3):459–487, 2017.
- [34] Erik Koelink, Maarten van Pruijssen, and Pablo Román. Matrix-valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$. *Int. Math. Res. Not. IMRN*, 2012(24):5673–5730, 2012.
- [35] Erik Koelink, Maarten van Pruijssen, and Pablo Román. Matrix-valued orthogonal polynomials related to $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag})$, II. *Publ. Res. Inst. Math. Sci.*, 49(2):271–312, 2013.
- [36] Erik Koelink, Maarten van Pruijssen, and Pablo Román. Matrix elements of irreducible representations of $\mathrm{SU}(n+1) \times \mathrm{SU}(n+1)$ and multivariable matrix-valued orthogonal polynomials. *preprint, arXiv:1706.01927*, 2017.
- [37] Tom H. Koornwinder. Matrix elements of irreducible representations of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and vector-valued orthogonal polynomials. *SIAM J. Math. Anal.*, 16(3):602–613, 1985.
- [38] Willard Miller, Jr. *Lie theory and special functions*. Mathematics in Science and Engineering, Vol. 43. Academic Press, New York-London, 1968.
- [39] Willard Miller, Jr. *Symmetry groups and their applications*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 50.
- [40] Walter Van Assche. *Orthogonal polynomials and Painlevé equations*, volume 27 of *Australian Mathematical Society Lecture Series*. Cambridge University Press, Cambridge, 2018.
- [41] Maarten van Pruijssen and Pablo Román. Matrix valued classical pairs related to compact Gelfand pairs of rank one. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 10(113):28 pages, 2014.

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