INTEGRABLE AND NON-INTEGRABLE EQUATIONS WITH PEAKED SOLITON SOLUTIONS

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By

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To Grandma - for always giving me somewhere I could call home.

Abstract

This thesis explores a number of nonlinear PDEs that have peaked soliton solutions, to apply reductions to such PDEs and solve the resultant equations.

Chapter 1 provides a brief history of peakon equations, where they come from and the different viewpoints of various authors. The rest of the chapter is then devoted to detailing the mathematical tools that will be used throughout the rest of the thesis.

Chapter 2 concerns a coupling of two integrable peakon equations, namely the Popowicz system, which itself is not integrable. The 2-peakon dynamics are studied, and an explicit solution to the 2-peakon dynamics is given alongside some features of the interaction.

In chapter 3 a reduction from two integrable peakon equations with quadratic nonlinearity to the third Painlevé equation is given. Bäcklund transformations and solutions for the Painlevé equations are expressed, and then used to find solutions of the original PDEs. A general peakon family, the b-family, is also explored, giving a more general result.

Chapter 4 examines two peakon equations with cubic nonlinearity, and their reductions to Painlevé equations. A link is shown between these cubic nonlinear peakon equations and the quadratic nonlinear equations in chapter 3.

Chapter 5 has conclusions and outlook in the area.

Contents

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Chapter 1

Introduction

This thesis forms a study of a group of partial differential equations which all share a certain type of solution. These solutions are a type of soliton solution, which are waves that keep their structure after interactions, and continue for long time periods without changing shape or velocity. Solitons have a physical manifestation, noted by the famous John Scott Russell [91]:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and

beautiful phenomenon which I have called the Wave of Translation. "

This observation and subsequent experiments by Russell, led to the discovery by Korteweg and de-Vries of the famous KdV equation [66]

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

which is a shallow water equation and has soliton solutions, consolidating what Russell witnessed.

The paper of Camassa and Holm [16], where they derived the equation

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \tag{2}$$

as a shallow water equation model, now known as the Camassa-Holm (CH) equation. Though the very first mention of this Partial Differential Equation (PDE) was made instead by Fuchssteiner and Fokas [42, 37], who studied the recursion operator, it took Camassa and Holm's paper to really start the ongoing interest in this area. In Camassa and Holm's paper they showed the equation was integrable and also discovering what they coined peaked soliton solutions or 'peakons', in the dispersionless $\kappa = 0$ case.

In the original Fokas and Fuchssteiner paper there was a slight error with some coefficients which meant that they could have written down a new hierarchy if it had been correct, as noted in [43]. The equation was also derived by Dai [29] as a model for nonlinear waves in hyperelastic rods, and Busuioc [15] when looking at non-Newtonian fluids.

The CH equation has physical origins, as it can be derived from the incompressible Euler equations as a new approximation to shallow water theory [25, 34, 35]. It emits smooth fluid-like solutions for $\kappa > 0$ [85, 84, 74, 55]. This thesis is interested in the special case of $\kappa = 0$, and when referring to CH this is the version of (2) to which we refer. It must be noted that one must be careful to define in what sense peakons are solutions - see subsection 1.2.5 below.

Beals and Sattinger [11, 12] wrote down the explicit solution for an arbitrary number of peakons, whereas CH [16] did only 1- and 2- peakon solutions. Other works have covered orbital stability [26], peakon scattering [23], peakon solutions [46, 89] and global solutions [24].

In Camassa and Holm's paper the derivation was from a physical representation, designed to develop a new shallow water equation. However there is some controversy regarding the physical derivation. A more thorough derivation was conducted by Dullin, Gottwald and Holm [35] but Bhatt and Mikhailov [13] disagree on the physical aspects. Actually the derivation produces a 1-parameter family (b-family)

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{3}$$

originally introduced by Degasperis-Holm-Hone [31]. However, Constantin-Lannes [25] have a different derivation of b-family, with a different interpretation: unlike Dullin-Gottwald-Holm [34], $u(x,t)$ does not represent the amplitude of the surface wave (as in Korteweg de Vries), but rather one component of the fluid velocity at a particular depth below the surface, with the depth determined by the parameter b.

There exist integrable and also non-integrable PDEs that have these peakon solutions, and they all possess a rich structure. However there is not a 'one size fits all' definition of integrability, as noted in the books of Zakharov [100] and Grammaticos [48]. This introduction will define what we mean by integrability and the mathematical constructs we will use in the rest of the thesis.

1.1 Background

In this introduction we introduce several concepts that run through the entire thesis, giving an overview of many of the tools available when studying integrable systems. In doing so we also review the Camassa-Holm literature, using the

Camassa-Holm equation as a key example. One reason that the CH equation has been so widely studied is partly its richness of mathematical features. It is completely integrable, having been found to have a Lax pair it is also bi-Hamiltonian and has an infinite number of conserved quantities.

Before we define what we mean by integrable, we need to explain some of the tools we shall be using. From the Poisson brackets we can define the Hamiltonian formalism, and what that looks like for Camassa-Holm. From there we then explain what a peakon is, and what is meant by a weak solution. Using Camassa-Holm peakons as the running example, we show how they are solutions of the PDE and also a travelling wave reduction.

1.2 Hamiltonian Systems

Here we define Hamiltonian systems, and describe various notions of integrability that are applicable in different contexts. This section will set up the structure behind the Hamiltonian systems of which the peakon PDEs are a part of. There are several books that contain a thorough introduction to integrable systems; see [8, 81, 9, 38].

1.2.1 Poisson Structure

Definition 1.2.1. Poisson Structure: A Poisson structure on a manifold M is a bilinear bracket $\{,\}$ on the space of functions $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

1. Skew symmetry: $\{F, G\} = -\{G, F\}$
2. Leibniz rule: $\{F, GH\} = \{F, G\} + \{F, H\}$
3. Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$

On a d-dimensional manifold M with coordinates $x = (x_i)$, the bracket of two functions F, G is given in terms of the brackets between the coordinates by

$$\{ F, G \} = \sum_{i,j=1}^{d} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \{ x_i, x_j \} \tag{4}$$

In these coordinates the Poisson tensor P is given by the matrix with entries $P_{ij} = \{ x_i, x_j \}$. The formula for a canonical bracket on $\mathbb{R}^{2n} = (p, q)$ is a special case where $d = 2n$ and the coordinates are divided into positions $q = (q_i)$ and momenta $p = (p_i)$ for $i = 1, \ldots, n$, is given by

$$\{F, G\} = \sum_i \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial G}{\partial p_i} \frac{\partial F}{\partial q_i} \tag{5}$$

where $\{F, G\}$ is a Poisson bracket, and

$$\{q_i, q_j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{p_i, q_j\} = \delta_{ij}. \tag{6}$$

1.2.2 Hamilton's Equations

To define Hamilton's equations we first define a Lagrangian system [8], with Lagrange's equations $\dot{p} = \frac{\partial L}{\partial q}$, where $p = \frac{\partial L}{\partial \dot{q}}$, with a given lagrangian function $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

The Hamiltonian H is a function that represents the legendre transform (transforming functions on a vector space to functions on a dual space) of the Lagrangian function. This can be written as

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t) \tag{7}$$

Hamilton's equations in terms of general coordinates (x_i) are given by

$$\dot{x}_i = \{x_i, H\} \tag{8}$$

for $i = 1, .., d$.

Definition 1.2.2. Phase space [8] The $2n$-dimensional space with coordinates $p_1, ..., p_n$, $q_1, ..., q_n$ is called phase space.

Any function $F(x)$ on phase space M satisfies

$$\dot{F} = \{F, H\} \tag{9}$$

and in the canonical case with $d = 2n$ they take the form

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{10}$$

for $i = 1, \ldots, n$;

Definition 1.2.3. Integral of motion: A function $K(x) \in C^\infty(M)$ that is constant on trajectories of Hamilton's equations is called an integral of motion. Due to (9), it satisfies

$$\{K, H\} = 0, \tag{11}$$

or in other words K is in involution (Poisson commutes) with H. Therefore $K(q, p)$ commutes with $H(q, p)$.

Definition 1.2.4. Casimir function: A function $C \in C^\infty(M)$ on a Poisson manifold M is called a Casimir function if $\{C, F\} = 0 \ \forall \ F \in C^\infty(M)$.

Casimirs can be fixed and used to reduce the order of a system of ODEs, we do this in chapter 2. We define, as in [94]:

Definition 1.2.5. Complete integrability (Poisson case): Suppose that the Poisson tensor (which maps 1-forms to vector fields) is of constant rank $2n$ on a dense open subset of a Poisson manifold M of dimension d, and that the algebra of Casimir functions is maximal, i.e. it contains $d - 2n$ independent functions. A Hamiltonian system on M is said to be completely integrable if it admits $d - n$ independent functions (including the Hamiltonian H) which are in involution.

In the canonical case $d = 2n$ the rank is maximal and there are no Casimirs, so only n conserved quantities in involution are needed.

Theorem 1.2.1. Liouville's Theorem: The solution of the equations of motion of a completely integrable system is obtained by quadratures.

This essentially means we only need to calculate a finite amount of integrals and explicit solutions can be found, this theorem also applies when dealing with Poisson manifolds as in Definition 5 defined above.

1.2.3 Infinite-dimensional Hamiltonian Systems

PDEs in 1+1 dimensions (space x + time t), like KdV and CH, can be formulated as Hamiltonian systems in infinite dimensions. To do this for a PDE in terms of $u(x,t)$ requires a Poisson bracket on functionals of u (rather than functions). So for two functionals F, G the bracket is given in terms of a skew-symmetric operator $\mathbf{B}$ by

$$\{F,G\} = \left\langle \frac{\delta F}{\delta u}, \mathbf{B}\frac{\delta G}{\delta u} \right\rangle = \int \frac{\delta F}{\delta u} u_t dx = \int \frac{\delta F}{\delta u}\mathbf{B}\frac{\delta G}{\delta u} dx \tag{12}$$

with $\langle , \rangle$ denoting the L^2 pairing on $\mathbb{R}^d$. If F is a local functional, given by

$$F = \int \mathcal{F} dx \tag{13}$$

where $\mathcal{F}$ is a function of u and its derivatives, then the variational (Fréchet) derivative of F can be rewritten in terms of the Euler operator acting on $\mathcal{F}$:

$$\frac{\delta F}{\delta u} = \frac{\partial \mathcal{F}}{\partial u} - \partial\left(\frac{\partial \mathcal{F}}{\partial u_x}\right) + ... \tag{14}$$

The bracket $\{,\}$ is automatically skew-symmetric whenever the operator $\mathbf{B}$ is, but is also required to satisfy the Jacobi identity (but one doesn't bother with the Leibniz rule in infinite dimensions). The domain of integration in $\int$: this is the whole real line in the case that u and its derivatives vanish at spatial infinity, or e.g. $-\pi$ to π in the case that u is periodic in x with period 2π.

Then given a functional $H[u]$, Hamilton's equations are

$$u_t = \mathbf{B}\frac{\delta H}{\delta u}. \tag{15}$$

Writing CH in Hamiltonian form, with

$$\mathbf{B} = -(m\partial_x + \partial_x m), \tag{16}$$

where $m = u - u_{xx}$ is the fluid momentum variable [16], that satisfies

$$m_t = \mathbf{B}\frac{\delta H}{\delta m} \tag{17}$$

with

$$H = \int \frac{1}{2}(u_x^2 + u^2)dx, \tag{18}$$

as

$$\frac{\delta H}{\delta m} = (1 - \partial_x^2)^{-1}\frac{\delta H}{\delta u}. \tag{19}$$

The fluid momentum m was obtained via a Legendre transformation as the variational derivative of the Lagrangian $L[u]$, with u being the fluid velocity.

1.2.4 Bi-Hamiltonian Systems

If we can write a system in terms of two distinct Hamiltonians and Hamiltonian operators

$$u_t = B_0\frac{\delta H}{\delta u} = B_1\frac{\delta G}{\delta u} \tag{20}$$

and also the sum (or any linear combination) of B_0 and B_1 is a Hamiltonian operator then we call the system bi-Hamiltonian. Now if

$$B_0 + B_1 \tag{21}$$

is also Hamiltonian, then we can define a recursion operator

$$R = B_1 B_0^{-1} \tag{22}$$

that can generate an infinite hierarchy of flows. This means that by showing a system is bi-Hamiltonian, with an infinite hierarchy of local flows that commute, we can infer that the system is integrable.

Definition 1.2.6. Bi-Hamiltonian System [81] A pair of skew-adjoint $q \times q$ matrix differential operators B_0 and B_1 is said to form a Hamiltonian pair if every linear combination $aB_0 + bB_1$, $a, b \in \mathbb{R}$, is a Hamiltonian operator. A system of evolution equations is a bi-Hamiltonian system if it can be written in the form (20) where B_0, B_1 form a Hamiltonian pair.

Writing CH in terms of the momentum variable m

$$m_t + um_x + 2u_x m = 0, \qquad m = u - u_{xx} \tag{23}$$

we can write down

Example 1.2.1. Bi-Hamiltonian form of Camassa-Holm

$$m_t = B_0 \frac{\delta H_1}{\delta m} = B_1 \frac{\delta H_2}{\delta m}, \tag{24}$$

$$B_0 = -m\partial_x - \partial_x m, \qquad H_1 = \frac{1}{2}\int (u^2 + u_x^2)dx, \tag{25}$$

$$B_1 = -\partial_x - \partial_x^3, \qquad H_2 = \frac{1}{2}\int (u^3 + uu_x^2)dx. \tag{26}$$

1.2.5 Weak Solutions

A weak solution for an ODE or PDE is a solution which may only satisfy the equation under some very specific circumstances. The solutions may not exist for all derivatives, however we can say that is a solution in this weak sense if the cirucumstances are satisfied.

Peakons are weak solutions of the PDE's that have them as solutions, by this we mean that they are valid only for the weak formulation of the PDE as the derivatives are discontinuous. The weak formulation for the whole b-family (3) was studied in [58], given by

$$E \equiv (1 - \partial_x^2)u_t + (b + 1 - \partial_x^2)\partial_x \left(\frac{1}{2}u^2\right) + \partial_x \left(\frac{3-b}{2}u_x^2\right) = 0 \tag{27}$$

so for all time t

$$\int E\psi \ dx = 0. \tag{28}$$

$\psi(x)$ is any smooth function on the real line with compact support, meaning that it is enough to do the integration over any finite interval where ψ has support (though an infinite interval would still be valid), and the derivatives in (27) are to be interpreted as weak derivatives, in the sense to be explained below.

Here we write down the case for $b = 2$ the Camassa-Holm example, and how to derive it. To verify the weak formulation, given by (27) with $b = 2$ and (28) for all ψ, we multiply by some test function ,$\psi(x,t)$, and move the derivatives onto that rather than the equations dependent variable $u(x,t)$.

$$\int (u_t - u_{xxt} + u_x(u - u_{xx}) + (u(u - u_{xx}))_x)\psi dx = 0 \tag{29}$$

To get this into the weak form, we shall consider the terms seperately and integrate by parts to move the derivatives greater than one. So we have

$$\int (u_t - u_{xxt})\psi dx = \int (\psi - \psi_{xx})u_t dx \tag{30}$$

and

$$\int u_x(u - u_{xx})\psi dx = \int (uu_x\psi + \frac{1}{2}u_x^2\psi_x)dx \tag{31}$$

and

$$\int ((u - u_{xx}))u_x\psi dx = -\int (u^2 + u_x^2)\psi_x + uu_x\psi_{xx}dx \tag{32}$$

So the weak formulation from (29) is:

$$\int (-(uu_x - u_t)\psi_{xx} - (u^2 + \frac{1}{2}u_x^2)\psi_x + (uu_x + u_t)\psi)dx = 0. \tag{33}$$

The weak formulation is equivalent to requiring (33).

1.3 Peakons

As we briefly mentioned, peakons are like non-smooth solitons. Different peakon equations can have different peakon solutions, for example CH and Novikov's equation, but here we shall use CH to introduce their main properties. Camassa-Holm has peakons of the form

$$u(x,t) = ce^{-|x-ct|}, \tag{34}$$

where the wave speed c is proportional to its amplitude. The constant c can also be negative and, if this is the case, we call the solutions 'antipeakons' and they move from right to left unlike peakons which move from left to right. These are

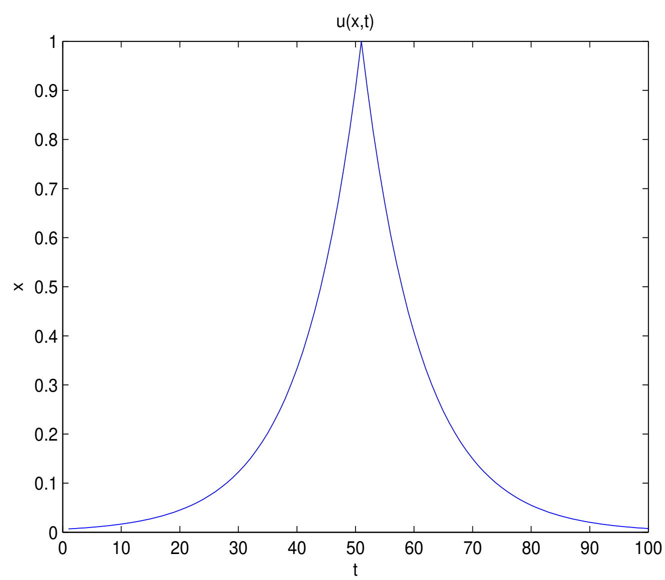


Figure 1: Single peakon, with c=1

known as weak solutions [70], as not all higher order derivatives of u that appear in (2) exist everwhere. We can see from (34) and Figure 1 that the first derivative of u is discontinuous at the peak $x = ct$.

To verify that the the single peakon solution (34) is a weak solution, we can substitute into (27) to find

$$m = (1 - \partial_x^2)u = 2c\delta(x - ct) \tag{35}$$

and then

$$(1 - \partial_x^2)u_t = -2c^2\delta'(x - ct) \tag{36}$$

from taking the t derivative of the distribution (35). Noting that δ is the dirac delta function

$$\delta(x) = \begin{cases} +\infty, x = 0 \\ 0, x \neq 0 \end{cases} \tag{37}$$

and

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{38}$$

so

$$\int e^{-|x|}(\psi - \psi_{xx})dx = 2\psi(0) \tag{39}$$

noting that $\frac{1}{2}e^{-|x|}$ is the Green's function of the Helmholtz operator $1 - \partial_x^2$, therefore

$$(1 - \partial_x^2)e^{-|x|} = 2\delta(x) \tag{40}$$

Then we also need

$$(1 - \frac{1}{4}\partial_x^2)u^2 = c^2 e^{-2|x-ct|} = c^2\delta(x - ct) \tag{41}$$

as $\delta(kx) = \frac{1}{|k|}\delta(x)$. So we have

$$E = -2c^2\delta'(x - ct) + (3 - \partial_x^2)\partial_x(\frac{1}{2}c^2 e^{-2|x-ct|}) + \partial_x(\frac{1}{2}c^2 e^{-2|x-ct|}), \tag{42}$$

$$= -2c^2\delta'(x - ct) + 2\partial_x(1 - \frac{1}{4}\partial_x^2)u^2 = 0 \tag{43}$$

using (41) and hence verifying that a single peakon solution is also a weak solution.

To check if single peakons are weak solution of the travelling wave reduction, similar technique is employed as with the PDE, multiply by a suitable test function and integrate by parts. A single peakon is given as follows

$$u(x,t) = ce^{-|x-ct|}, \tag{44}$$

Starting with the travelling wave ansatz [17]

$$u(x,t) = \phi(x - ct) \tag{45}$$

with c the wavespeed, take $\zeta = x - ct$ and substitute into (2) we find

$$-c(\phi' - \phi''') + 3\phi\phi' = 2\phi'\phi'' + \phi\phi''' \tag{46}$$

Integrating this once

$$-c(\phi - \phi'') + \frac{3}{2}\phi^2 = \phi\phi'' + \frac{1}{2}\phi'^2 + a \tag{47}$$

Multiply by $2\phi'$ and integrate again

$$(\phi - c)(\phi^2 - \phi'^2) = a\phi + b \tag{48}$$

with $a \in \mathbb{R}$ is a constant. Camassa and Holm guessed the form of the peakon from looking at (48) and requiring $a = b = 0$ for solutions vanishing at infinity; and the single peakon is a solution that satisfies (48) everywhere except at the peak $\zeta = 0$, where $\phi = c$.

1.3.1 Multi-peakons

Multi-peakons have been studied extensively, and have proven to be an interesting area of research which we touched upon at the start of this chapter. They were initially discovered in Camassa and Holm's original paper [16] and are a linear superposition of N singlepeakons [12, 52]

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|} \tag{49}$$

with time dependent amplitudes p_j and speeds q_j. The derivatives of which are

$$u_x = -\sum_{j=1}^{N} p_j \operatorname{sgn}(x-q_j) e^{-|x-q_j|}, \tag{50}$$

$$u_{xx} = \sum_{j=1}^{N} p_j (e^{-|x-q_j|} - 2\delta(x-q_j)) \tag{51}$$

This is a completely integrable system, Camassa and Holm found a complete set of conserved quantities using a Lax pair and Beals et al [12]. described them in more detail. The weak formulation (27) and (28) leads to the equations

$$\dot{q}_i = \sum_{j=1}^{N} p_i e^{-|q_i-q_j|}, \tag{52}$$

$$\dot{p}_i = \sum_{j=1}^{N} p_i p_j \operatorname{sgn}(q_i-q_j) e^{-|q_i-q_j|} \tag{53}$$

by integrating against test functions ψ with support at $x = q_i$. Both (52) and (53) take the canonical form (10) which has the Hamiltonian function

$$H = \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j e^{-|q_i-q_j|} \tag{54}$$

1.3.2 2-peakon

Writing the CH 2-peakon example here as a canonical Hamiltonian system, and a special case of (49)

$$u(x,t) = \sum_{i=1}^{2} p_i(t) e^{-|x-q_i(t)|}. \tag{55}$$

for $N = 2$ (52) and (53) leads to

$$\dot{p}_1 = p_1 p_2 \,\mathrm{sgn}(q_1 - q_2) e^{-|q_1 - q_2|}, \tag{56}$$

$$\dot{p}_2 = p_1 p_2 \,\mathrm{sgn}(q_2 - q_1) e^{-|q_1 - q_2|}, \tag{57}$$

$$\dot{q}_1 = p_1 + p_2 e^{-|q_1 - q_2|}, \tag{58}$$

$$\dot{q}_2 = p_1 e^{|q_2 - q_1|} + p_2. \tag{59}$$

and also substituting for the Hamiltonian (54)

$$H = \frac{1}{2}(p_1^2 + p_2^2) + p_1 p_2 e^{-|q_1 - q_2|} \tag{60}$$

Using the 2-peakon evolution equations with the following change of variables:

$$P = p_1 + p_2 \qquad Q = q_1 + q_2, \tag{61}$$

$$p = p_1 - p_2 \qquad q = q_1 - q_2 \tag{62}$$

The exact solution of 2-peakon equations found by Camassa and Holm (also with Hyman in more detail [17]) is given by

$$q_1(t) = c_1 t + \frac{1}{2}\log[4\gamma(c_1 - c_2)^2] - \log[\gamma c^{(c_1 - c_2)t} + 4c_1^2], \tag{63}$$

$$q_2(t) = c_2 t - \frac{1}{2}\log[4\gamma(c_1 - c_2)^2] + \log[\gamma c^{(c_1 - c_2)t} + 4c_2^2] \tag{64}$$

where γ is an arbitrary integration constant. As is convention we set the leftmost peak with position, q_1, in this case it has asymptotic velocity c_1, and the rightmost peak q_2 with asymptotic velocity c_2. There is also freedom to shift q_1, q_2 by another constant x_0, which has been set to zero. More precisely, the phase shift is the difference between the phases of the peakons at $\rightarrow -\infty$ and $t \rightarrow +\infty$ before and after interaction. This is a characteristic feature of solitons e.g. it is known for KdV solitons. The fast soliton is given by

$$\Delta q_t \equiv q_2(+\infty) - q_1(-\infty) \tag{65}$$

and ends up to the right of the slower one

$$\Delta q_s \equiv q_1(+\infty) - q_2(-\infty) \tag{66}$$

so the shifts are given by

$$\Delta q_t = \log\left[\frac{c_1^2}{(c_1 - c_2)^2}\right], \qquad \Delta q_s = \log\left[\frac{(c_1 - c_2)^2}{c_2^2}\right] \tag{67}$$

Asymptotically this corresponds to

$$q \equiv \mp(c_1 - c_2)t + \text{const} \qquad \text{as } t \to \pm\infty \tag{68}$$

Plotting the peaks of two interacting peakons with initial data

$$c_1 = 0.2, \qquad c_2 = 0.3, \tag{69}$$

and

$$p_1(0) = 2, \qquad p_2(0) = 3, \qquad q_1(0) = 0, \qquad q_2(0) = -30, \tag{70}$$

as in Figure 2 we can see the shift of trajectories after interaction.

1.4 Lax Pairs

A Lax pair [9] is made up of two matrices L, M that can be used to write the Hamiltonian evolution equations (10) as

$$\frac{dL}{dt} \equiv \dot{L} = [M, L] \tag{71}$$

where $[M, L] = ML - LM$ is the commutator of the matrices M and L.

Simply finding a Lax pair [68] is a very good sign of integrability. Lax looked at linear operators which gave a compatible equation (Lax equation) with the

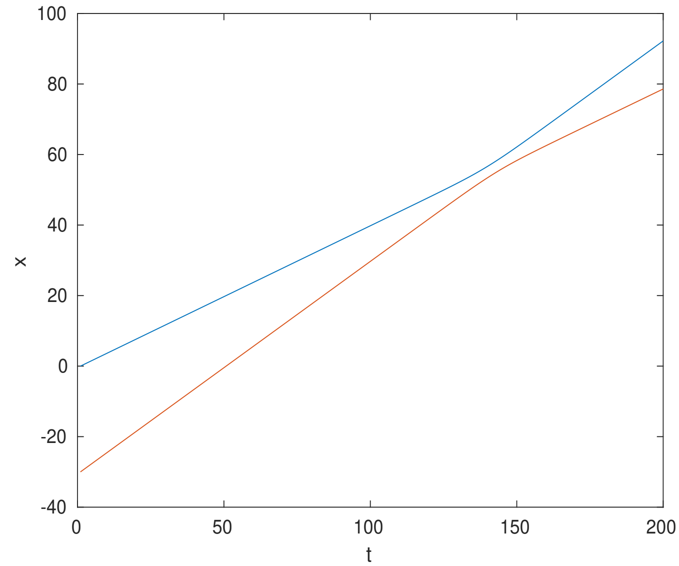


Figure 2: A spacetime plot of two interacting peakons with a shift with initial data (69),(70)

Schrödinger operator

$$L = \partial_x^2 + u \tag{72}$$

written

$$L_t = [P, L] = (PL - LP) \tag{73}$$

for some pair of operators or matrices L, P. We can write Lax pairs in a matrix form such as

$$\Psi_t = A\Psi, \qquad \Psi_x = B\Psi \tag{74}$$

with A, B matrices, and $\Psi = \binom{\psi_1}{\psi_2}$ if the matrices are 2×2. They must satisfy what's known as the zero curvature condition

$$A_t - B_x + [A, B] = 0 \tag{75}$$

which is the compatibility condition of the linear system, and is essential to the notion of a Lax pair.

Lax pairs can also be written in scalar form; the scalar form of Camassa-Holm Lax pair is

$$\psi_{xx} = \left(\frac{1}{4} + \lambda(m+\kappa)\right)\psi, \tag{76}$$

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{u_x}{2}\psi. \tag{77}$$

and the matrix form of the Lax pair:

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda(\kappa + m) & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{2}u_x & \frac{1}{2\lambda} - u \\ \frac{1}{2}(\kappa + \frac{1}{4\lambda} + \frac{1}{2}u) - \lambda u(\kappa + m) & -\frac{1}{2}u_x \end{pmatrix}$$

the matrices A,B, must satisfy (75).

1.5 Painlevé Equations

Paul Painlevé and co-workers derived a classification of second order ODEs of the form

$$\frac{d^2w}{dz^2} = F\left(\frac{dw}{dz}, w, z\right) \tag{78}$$

where F is rational in $\frac{dW}{dZ}$ and W, and analytic in Z, such that they are free of movable critical points. Painlevé and others obtained a list of (approximately) fifty different types of equations, all but six of which could be reduced to equations for previously known functions, i.e. linear special functions or elliptic functions.

The remaining six equations, known as Painlevé $I{-}VI$, and commonly denoted

P_I-P_{VI}, are given as follows:

$$
\begin{aligned}
w'' &= 6w^2 + z, \qquad P_I \\
w'' &= 2w^3 + zw + \alpha, \qquad P_{II} \\
w'' &= \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \qquad P_{III} \\
w'' &= \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \qquad P_{IV} \\
w'' &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \qquad P_V \\
w'' &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right), \qquad P_{VI}
\end{aligned}
$$

where $'$ denotes differentiation with respect to z, and α, β, γ, δ are constant parameters.

A connection between Painlevé equations and integrable PDEs was initially made by Ablowitz and Segur [6] who applied a scaling similarity reduction of mKdV which resulted in a P_{II} equation, see subsection 1.7.1 for details. According to Ablowitz, Ramani and Segur's famous conjecture [1, 5, 2],

Conjecture 1.6. ARS Conjecture Every ODE which arises as a reduction of a completely integrable PDE is of Painlevé type (perhaps after a transformation of variables.

By finding such relationships between PDEs and ODEs via reduction, we are able to obtain new solutions of the PDEs. Also discovering that we can pass from integrable PDEs to Painlevé in two different ways such as

$$
\begin{array}{ccc}
\text{Integrable PDE} & \xrightarrow{\text{Sim. Red}} & \text{3rd Order Equation} \\
\Big\downarrow{\scriptstyle \text{Rec.Trans}} & & \Big\downarrow{\scriptstyle \text{Hodograph}} \\
\text{Hierarchy} & \xrightarrow{\text{Sim. Red}} & \text{Painlevé}
\end{array}
$$

- see chapters 3 and 4 below.

The field of Painlevé equations has been widely studied, the main references that are used in this thesis for Bäcklund transformations are Milne [77] and Gromak [49]. However it would be amiss to not acknowledge Okamoto's work [80] who contributed greatly to the area and also Forrester and Witte [41] who extended the work of Okamoto.

1.7 Symmetry Reductions and Transformations

To solve nonlinear partial differential equations we are able to use a geometric approach to exploit symmetry properties that allow us to garner multiple solutions [82]. To clarify, when discussing symmetries of an equation we mean that if we have an initial solution we are able to find further solutions by applying said symmetry. An algorithmic approach was developed by Lie [8] to find continuous symmetries of differential equations, this technique does not capture all the symmetries but in most cases will provide most. Another method of finding these symmetries is simply by inspection and this will likely find the most important symmetries of the differential equation.

Earlier we used a travelling wave reduction to derive the Camassa Holm peakon equation, this is a type of translation symmetry. There are a number of different symmetry reductions, but the ones we shall discuss here are similarity reductions which are scaling symmetries. Similarity reductions can reduce a system by 1 independent variable, e.g. the Camassa-Holm PDE with $u(x,t)$ reduces to an ODE of type $U(z)$. Reciprocal transformations do not reduce the order, they do however transform conservation laws into conservation laws.

Definition 1.7.1. Conservation law If we are able to write an equation in the following form:

$$\partial_t A(u) + \partial_x B(u) = 0 \tag{79}$$

we call it a conservation law.

1.7.1 Similarity Reductions

The main motivation to employ these techniques is to utilise the work by Hone [55] on the associated Camassa-Holm (aCH) equation

$$p_t = p^2 f_x, \qquad f = \frac{p}{4}(\log p)_{xt} - \frac{p^2}{2}, \tag{80}$$

where the author used scaling similarity reductions to transform the aCH equation to one of the Painlevé equations. Also (80) is also equivalent to (238) in chapter 3.

Similarity reductions [87] are primarily used to simplify a PDE, if said PDE is linear then there are lots of techniques available. Applying a similarity reduction to a PDE reduces the equations independent variables by 1, so if the PDE has n independent variables our new PDE will have $n-1$. To illustrate a scaling similarity reduction [21] we give an example below. For a known example, KdV (1) has the following scaling similarity reduction

$$u(x,t) = \frac{U(z)}{(3t)^{\frac{2}{3}}}, \qquad z = x(3t)^{-\frac{1}{3}}. \tag{81}$$

The one-parameter Lie group of scaling symmetries:

$$x \to \lambda t, \qquad t \to \lambda^3 t, \qquad u \to \lambda^2 u(\lambda x, \lambda^3 t) \tag{82}$$

under which the given form of solution is invariant. Substituting (81) into (1) we find that U satisfies

$$U''' + 6U'U - zU' - 2U = 0 \tag{83}$$

Shift $U = V + \frac{1}{2}$

$$V''' + 6VV' + V + 2zV' = 0, \tag{84}$$

integrate with respect to z

$$V'' - \frac{V'^2}{2V} + 2V^2 + zV - \frac{\alpha}{V} = 0, \tag{85}$$

which is P_{34} as given by Ince [62].

Definition 1.7.2. Miura map The Miura map transforms solutions from KdV to mKdV

$$u = v_x - v^2 \tag{86}$$

with u the KdV variable and v the mKdV variable.

Applying a Miura map, $V = w' - w^2$ we find the second Painlevé equation, referred to as P_{II}

$$w'' = 2w^2 + zw + \alpha + \frac{1}{2}. \tag{87}$$

KdV also has a reduction to the first Painlevé equation, known as P_I.

1.7.2 Reciprocal Transformations

Definition 1.7.3. Hodograph A Hodograph transformation essentially interchanges the roles of the dependent and independent variables of an equation. Reciprocal transformations can be viewed as a particular type of hodograph transformation, but hodographs are defined for ODEs as well as PDEs.

Proposition 1.7.1. Reciprocal transformations transform conservation laws into conservation laws [90].

Integrating (79) with respect to x provided $u \to 0$ when $x \to \pm\infty$

$$\int A dx \tag{88}$$

New independent variables are introduced, and interestingly these reciprocal transformations seem to form a natural link between the peakon equations and soliton emitting hierarchies.

$$dX = Adx + Bdt \qquad\qquad dT = Cdx + Ddt \tag{89}$$

Then we have

$$d^2X = 0 \Rightarrow A_t = B_x \ , \tag{90}$$

$$d^2T = 0 \Rightarrow C_t = D_x \tag{91}$$

Conversely given any pair of conservation laws (90) and (91) you can define a reciprocal transformation. These transformations are used in chapter 3 and 4. For KdV, we can re-write as

$$u_t = (u_{xx} + 3u^2)_x \tag{92}$$

the reciprocal transformation of which is

$$dX = udx + (u_{xx} + 3u^2)dt, \qquad dT = dt \tag{93}$$

so the transformed derivatives are

$$\frac{dX}{dt} = u_{xx} + 3u^2, \qquad \frac{dX}{dx} = u, \qquad \frac{dT}{dt} = 1, \qquad \frac{dT}{dx} = 0. \tag{94}$$

Writing the derivatives of (92) in terms of X and T we are able to write it down as

$$(u^{-1})_T = -(u^{-1}(u_{XX}u^2 + u_X^2 + 3u^2))_X \tag{95}$$

which is in the same form as (79) and hence another conservation law thereby statisfying proposition 1.7.1.

1.7.3 Bäcklund Transformations

Bäcklund transformations are an incredibly useful mathematical tool that allows us to find a list of new solutions from a known solution. Using these transformations we are able to use a solution of a PDE to find a sequence of solutions for another PDE, which may or may not be the same. They are key features of soliton equations, like KdV, and are also applicable to the peakon equations here. Originally developed as a tool in differential geometry, it was Bianchi [14] who developed the idea constructing surfaces going from one to another. A reciprocal transformation and a Miura transformation are examples of Bäcklund transformations.

A classical transformation is one from Miura [78]. The well known example gives a transformation which maps KdV to mKdV, substituting

$$u = v_x - v^2 \tag{96}$$

into (1), where u is a solution to KdV, we find

$$(2v + \frac{\partial}{\partial x})(v_t - 6v^2 v_x + v_{xxx}) = 0 \tag{97}$$

therefore if v satisfies the mKdV equation

$$v_t = -v_{xxx} + 6v^2 v_x \tag{98}$$

then v also gives a solution to (97). However given a solution of KdV does not necessarily mean that the Miura transformation will give an equation of mKdV.

There are also auto-Bäcklund transformations, which unlike Bäcklund transformations, produce a set of solutions to the original equation. The auto-Bäcklund transformation for KdV was discovered by Wahlquist and Estabrook [97].

1.8 Integrable Hierarchies

In this thesis we shall frequently be referring to the fact that a number of these peakon PDEs are related to various hierarchies. For example Camassa-Holm is related to the negative flow of the KdV hierarchy [55]. An integrable hierarchy such as this is a system of commuting flows, coming from integrals of motion which are involution (i.e. for any pairs of flows the Poisson bracket vanishes). These flows are obtained recursively, and for the examples given below we give the recursion operator $\mathcal{R}$.

1.8.1 KdV Hierarchy

The KdV hierarchy was first constructed by Lax [69] who discovered it via a recursive approach. It was then further developed by Gelfand and Dikii [44].

Using the KdV hierarchy as an example, we can think of this particular hierarchy as an infinite sequence of PDEs that starts with KdV. We are able to use the operator

$$\mathcal{R}^n = -\partial_x^3 - 4u\partial_x - 2u_x \tag{99}$$

to produce an infinite number of PDEs by applying

$$u_{t_n} = \mathcal{R}^n u_x \tag{100}$$

recursively. The operator, $\mathcal{R} = B_1 B_0^{-1}$, is constructed from the two Hamiltonian operators

$$B_0 = \partial_x, \qquad B_1 = \partial_x^3 + 4u\partial_x + 2u_x. \tag{101}$$

Fokas and Fuchssteiner found all the flows (100) for $n = 0, 1, 2, ..$ commute with each other. The positive hierarchy consisting of the equations (100) for $n \geq 0$ is well-known, but the negative hierarchy, corresponding to $n < 0$, is not. The first interest in the negative hierarchy was due to Verosky [96] who wrote down the following

$$v_t = w_x, \qquad w_{xxx} + 4vw_x + 2v_x w = 0. \tag{102}$$

Interestingly, Fuchssteiner found a relationship between this negative flow (102) and Camassa-Holm (2), which we will make use of in chapter 3.

1.8.2 Other Hierarchies

We make use of other hierarchies in subsequent chapters, and shall refer back to here when making the connections.

The fifth-order Kaup-Kuperschmidt [65, 67] equation

$$u_t - 10u_{xxx}u - 25u_{xx}u_x - 20u^2u_x - u_{xxxxx} = 0 \tag{103}$$

is the first equation in the Kaup-Kuperschmidt hierarchy with Lax operator [40]

$$L = \partial_x^3 + 2u\partial_x + u_x \tag{104}$$

The fifth-order Sawada-Kotera equation [93] is given by

$$u_t + 45u^2u_x + 15(u_xu_{xx} + uu_{xxx}) + u_{xxxxx} = 0 \tag{105}$$

is the first equation in the Sawada-Kotera hierarchy with the Lax operator [40]

$$L = \partial_x^3 + u\partial_x \tag{106}$$

up to a suitable scaling for u. Sawada-Kotera and Kaup-Kuperschmidt are both are related to each other [39], Gordoa discusses in depth the Sawada-Kotera hierarchy in [47].

1.9 Summary

Here we have given an overview of the main concepts we need for the thesis, this however only just touches the large body of work that makes up integrable systems. The peakon equations we have studied are interesting not only for their integrable properties, but also have gained much interest for their physical representation and how they can help further knowledge of solitary waves. Understanding other properties of these equations will only add to their appeal to other aspects of the mathematical community.

Camassa-Holm is part of a wider group of equations which emit these peakon solutions, we shall call this the b-family, though only one other member of this family is known to be integrable. This one-parameter family of partial differential equations (PDEs), given by (3) can also be written in terms of m

$$m_t + um_x + bu_x m = 0, \qquad m = u - u_{xx} \tag{107}$$

with our variable b. When $b = 2$ the equation is equivalent to CH, when $b = 3$ it gives this other integrable equation called the Degasperis-Procesi (DP) equation [32].

In the second chapter we study a coupled PDE first presented by Popowicz [86], which is a coupling between

$$m_t + um_x + 2u_x m = 0, \qquad \text{Camassa-Holm} \tag{108}$$

and

$$m_t + um_x + 3u_x m = 0, \qquad \text{Degasperis-Procesi} \tag{109}$$

with $m = u - u_{xx}$ in both cases. Popowicz gave the coupled system

$$m_t + m_x(2u + v) + 3m(2u_x + v_x) = 0, \qquad m = u - u_{xx}, \tag{110}$$

$$n_t + n_x(2u + v) + 2n(2u_x + v_x) = 0, \qquad n = v - v_{xx}. \tag{111}$$

Initially we study the 2-peakon interaction, solving for the phase shift numerically and analytically. Over the last few years, there have been many authors looking for these kind of coupled peakon equations, starting with Falqui [36] who studied a generalization of the Camassa-Holm equation it has lead to a number of other studies, including [18, 95, 98, 59].

In the third chapter we review a known reduction of Camassa-Holm to P_{III}, providing further details and solutions. Additionally we use an alternative method to achieve the same result, and then use both methods to find reductions of Degasperis-Procesi and the b-family.

Chapter 4 continues in a similar vein to chapter 3, but discussing peakon equations with cubic nonlinearity. We also derive reductions of these cubically nonlinear PDEs to Painlevé equations, and discuss and make use of connections with the peakon equations with quadratic nonlinearity.

In the last chapter we conclude by briefly discussing the outlook of the thesis, with some possible directions for future work.

Overall we find interesting properties of interacting peakons from the Popowicz system, and chapter 3 and 4 provide a near algorithmic approach to finding Painlevé equations for some quadratic and cubic peakon equations. For the integrable peakon equations, the latter result satisfying conjecture 1.6 (ARS conjecture).

Chapter 2

The Popowicz System

In this chapter we consider a coupled Hamiltonian system of partial differential equations (PDEs) derived by Popowicz, which has reductions to both the Camassa-Holm and the Degasperis-Procesi equations. It was shown by Hone and Irle [61] that the Popowicz system admits N-peakon solutions, whose dynamics is described by a $3N$ dimensional Hamiltonian system, which is Liouville integrable when $N = 1$ and $N = 2$, but they also gave arguments to suggest that the full system of PDEs is not integrable. The main aim of this chapter is to perform an explicit integration of the equations of motion for $N = 2$, thereby describing the interaction of two peakons. We comment on possible implications for the case $N > 2$. The main results of this chapter have been published [10].

2.1 The Popowicz System and its Reductions

Popowicz [86] introduced the following coupled system of PDEs:

$$\begin{aligned} m_t + m_x(2u+v) + 3m(2u_x+v_x) &= 0, \qquad m = u - u_{xx} \\ n_t + n_x(2u+v) + 2n(2u_x+v_x) &= 0, \qquad n = v - v_{xx} \end{aligned} \tag{112}$$

derived from a Hamiltonian operator with three fields by use of a Dirac reduction, Popowicz showed that the above system can be written in Hamiltonian form, that is

$$\begin{pmatrix} m_t \\ n_t \end{pmatrix} = \mathcal{Z} \begin{pmatrix} \frac{\delta H_0}{\delta m} \\ \frac{\delta H}{\delta n} \end{pmatrix}$$

with Hamiltonian

$$H_0 = \int (m+n)dx, \tag{113}$$

and the non-local operator

$$\mathcal{Z} = -\begin{pmatrix} 9m^{\frac{2}{3}}\partial_x m^{\frac{1}{3}}\mathcal{L}^{-1}m^{\frac{1}{3}}\partial_x m^{\frac{2}{3}} & 6m^{\frac{2}{3}}\partial_x m^{\frac{1}{3}}\mathcal{L}^{-1}n^{\frac{1}{2}}\partial_x n^{\frac{1}{2}} \\ 6n^{\frac{1}{2}}\partial_x n^{\frac{1}{2}}\mathcal{L}^{-1}m^{\frac{1}{3}}\partial_x m^{\frac{2}{3}} & 4n^{\frac{1}{2}}\partial_x n^{\frac{1}{2}}\mathcal{L}^{-1}n^{\frac{1}{2}}\partial_x n^{\frac{1}{2}} \end{pmatrix}$$

where $\mathcal{L} = \partial_x(1-\partial_x^2)$. The system given by (112) can be regarded as a coupling between the Camassa-Holm and Degasperis-Procesi equations since by setting $u=0$ (and therefore $m=0$) it reduces to the Camassa-Holm equation (108), and setting $v=0$ (and therefore $n=0$) it reduces to the Degasperis-Procesi equation (109). The original paper of Popowicz states three independent conserved quantities and suggests that because of this the system is quite possibly integrable. In addition to H_0 there is also the following Hamiltonian operators

$$H_1 = \int (nm^{-\frac{2}{3}})^{\lambda} m^{\frac{1}{3}} dx, \tag{114}$$

$$H_2 = \int (-9n_x^2 n^{-2} m^{-\frac{1}{3}} + 12 n_x m_x n^{-1} m^{-\frac{4}{3}} - 4m_x^2 m^{-\frac{7}{3}})(nm^{-\frac{2}{3}})^{\lambda} dx \tag{115}$$

with λ an arbitrary constant. However he did not find a Lax pair for the system nor a recursion operator from a bi-Hamiltonian formulation to back this up. Further work on this system by Hone and Irle [61], provides further evidence for this system to be non-integrable through Painlevé analysis. Hone and Irle also showed that the Popowicz system admits weak solutions given by a superposition of N

peaks (peakons), given by

$$u(x,t) = \sum_{j=1}^{N} a_j(t) e^{-|x-q_j(t)|}, \tag{116}$$

$$v(x,t) = \sum_{j=1}^{N} b_j(t) e^{-|x-q_j(t)|}. \tag{117}$$

We now will describe further the N-peakon solutions and give details of some of the dynamics of these Popowicz peakons, which behave differently to those of Camassa-Holm and Degasperis-Procesi.

2.2 N-peakon Solutions

The ODEs for peakons in the Popowicz system as derived by Hone and Irle [57] were stated without proof as

Theorem 2.2.1. With the formulation, the Popowicz system admits N-peakon solutions of the form (116), where the amplitudes a_j, b_j and positions q_j satisfy the dynamical system

$$\dot{a}_j = 2a_j \sum_{k=1}^{N} (2a_k + b_k)\,\mathrm{sgn}(q_j - q_k) e^{-|q_j - q_k|}, \tag{118}$$

$$\dot{b}_j = b_j \sum_{k=1}^{N} (2a_k + b_k)\,\mathrm{sgn}(q_j - q_k) e^{-|q_j - q_k|}, \tag{119}$$

$$\dot{q}_j = \sum_{k=1}^{N} (2a_k + b_k) e^{-|q_j - q_k|}. \tag{120}$$

for $j = 1, .., n$. These ODE's are in Hamiltonian form,

$$\dot{a}_j = \{a_j, H\}, \qquad \dot{b}_j = \{b_j, H\}, \qquad \dot{q}_j = \{q_j, H\}, \tag{121}$$

with the non-canonical Poisson bracket

$$\{a_j, a_k\} = 2a_j a_k \operatorname{sgn}(q_j - q_k)e^{-|q_j - q_k|}, \tag{122}$$

$$\{b_j, b_k\} = \frac{1}{2} b_j b_k \operatorname{sgn}(q_j - q_k)e^{-|q_j - q_k|}, \tag{123}$$

$$\{q_j, q_k\} = \frac{1}{2} \operatorname{sgn}(q_j - q_k)(1 - e^{-|q_j - q_k|}), \tag{124}$$

$$\{q_j, a_k\} = a_k e^{-|q_j - q_k|}, \tag{125}$$

$$\{q_j, b_k\} = \frac{1}{2} b_k e^{-|q_j - q_k|}, \tag{126}$$

$$\{a_j, b_k\} = a_j b_k \operatorname{sgn}(q_j - q_k)e^{-|q_j - q_k|}. \tag{127}$$

where the Hamiltonian is

$$H = 2\sum_{k=1}^{N}(a_k + b_k) = 2\sum_{k=1}^{N}(c_k b_k^2 + b_k) \tag{128}$$

with the Casimirs

$$c_j = \frac{a_j}{b_j^2}, \qquad \text{for} \qquad b_j \neq 0, \tag{129}$$

for $j = 1..N$ which are N conserved quantities. There is also an additional quantity, J, for the case $N = 2$

Lemma 2.3. [10] In addition to the Hamiltonian, H, and the Casimirs, c_j, with $j = 1, ..N$, the ODEs (118) for the peakons in the Popowicz system admit the first integral

$$J = \left(\prod_{j=1}^{N} b_j\right) \prod_{k=1}^{N-1} (1 - e^{-|q_k - q_{k+1}|}). \tag{130}$$

Proof. Taking the logarithm of (130) and differentiating gives

$$\frac{d}{dt}\log J = \sum_{j=1}^{N} \frac{d}{dt}\log b_j + \sum_{k=1}^{N-1} \frac{(\dot{q}_k - \dot{q}_{k+1}) \operatorname{sgn}(q_k - q_{k+1})E_{k,k+1}}{1 - E_{k,k+1}}, \tag{131}$$

where we have introduced the convenient notation

$$E_{j,k} = E_{k,j} = e^{-|q_j - q_k|}. \tag{132}$$

Substituting for the time derivatives from (118) yields

$$\frac{d}{dt}\log J = \sum_{j,k=1}^{N}(2a_k + b_k)\,\mathrm{sgn}(q_j - q_k)E_{jk} + \tag{133}$$

$$\sum_{k=1}^{N-1}\sum_{l=1}^{N}(2a_l + b_l)\,\mathrm{sgn}(q_k - q_{k+1})\frac{(E_{k,l} - E_{k+1,l})E_{k,k+1}}{1 - E_{k,k+1}}, \tag{134}$$

$$= \sum_{k=1}^{N}(2a_k + b_k)S_k, \tag{135}$$

where, upon taking the ordering $q_1 < q_2 < ... < q_N$ without loss of generality,

$$S_k = -\sum_{j=1}^{k-1}E_{jk} + \sum_{j=k+1}^{N}E_{jk} - \sum_{l=1}^{N-1}\frac{(E_{l,k} - E_{l+1,k})E_{l,l+1}}{1 - E_{l,l+1}} \tag{136}$$

Then the properties of the exponential, together with the assumed ordering of the peakons, produce the identity

$$\frac{(E_{l,k} - E_{l+1,k})E_{l,l+1}}{1 - E_{l,l+1}} = \begin{cases} -E_{l,k}, for 1 \leq l \leq k; \\ E_{l+1,k}, for k \leq l \leq N-1 \end{cases} \tag{137}$$

Thus $S_k = 0$ for all k, and the result follows. □

A trivial example for $N = 1$, the single peakon case, is represented by the ODE system

$$u(x,t) = a_1 e^{-|x-kt-x_0|}, \qquad v(x,t) = b_1 e^{-|x-kt-x_0|}, \tag{138}$$

with $k = 2a + b$ and a_1, b_1 are arbitrary constants. Using (118) to solve for the amplitudes a, b and position q

$$\dot{a}_1 = 0, \qquad \dot{b}_1 = 0, \qquad \dot{q}_1 = 2a_1 + b_1 \tag{139}$$

therefore we have $a_1 = a$ and $b_1 = b$, with both a and b constants. Also $q_1 = (2a + b)(t - t_0)$ with t_0 arbitary. The signs of a and b correspond to u and v

becoming peakons (positive amplitude) or anti-peakons (negative amplitude). If either u or v vanishes, then we are left with a single peakon for Camassa-Holm or Degasperis-Procesi.

Apart from this trivial example, it is interesting to study the dynamics of N-peakon solutions. Numerical studies of the b-family [54, 53] and other peakon equations show that even non-integrable PDE's can yield some stable multi-peakon solutions. Currently it is not known as to why this can happen.

2.4 Integrating the 2-peakon Equations

For $N = 2$ we find ourselves with six equations to solve for the 2-peakon case, the Hamiltonian system has two Casimirs (c_1 and c_2) and two further conserved quantities (H and J) in involution;

Corollary 2.4.1. The Hamiltonian system (118) is Liouville integrable

Therefore by Liouville's theorem it can be integrated by quadratures. We reduce the number of equations needed to carry out the integration by fixing the Casimir (129) to be constant and eliminating the a_j's.

$$\dot{b}_j = b_j \sum_{k=1}^{2} (2c_k b_k^2 + b_k) \operatorname{sgn}(q_j - q_k) e^{-|q_j - q_k|} \tag{140}$$

$$\dot{q}_j = \sum_{k=1}^{2} (2c_k b_k^2 + b_k) e^{-|q_j - q_k|} \tag{141}$$

Having fixed the Casimirs c_1, c_2 to be constant as in (140) above, the 2-peakon dynamics can be found by solving the following four equations

$$\dot{b}_1 = b_1 (2c_2 b_2^2 + b_2) \operatorname{sgn}(q_1 - q_2) e^{-|q_1 - q_2|}, \tag{142}$$

$$\dot{b}_2 = b_2 (2c_1 b_1^2 + b_1) \operatorname{sgn}(q_2 - q_1) e^{-|q_1 - q_2|}, \tag{143}$$

$$\dot{q}_1 = (2c_1 b_1^2 + b_1) + (2c_2 b_2^2 + b_2) e^{-|q_1 - q_2|}, \tag{144}$$

$$\dot{q}_2 = (2c_1 b_1^2 + b_1) e^{-|q_1 - q_2|} + (2c_2 b_2^2 + b_2), \tag{145}$$

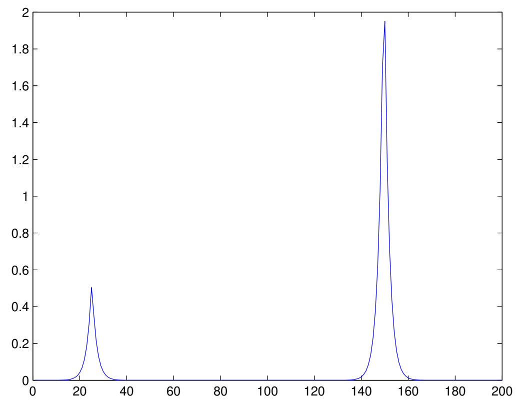


Figure 3: 2-peakon case, with x the horizontal axis

and then the amplitudes a_i are related by $a_i = k_i b_i^2$. Solving the system of ODEs numerically using Matlab's inbuilt `ode45`, Runge Kutta integrator we can clearly see two peakons in Figure 3. They behave in the same way as Camass-Holm peakons, the larger, faster peakon can overtake a shorter, slower peakon without loss of shape after interacting. By fixing the Hamiltonian H, H can now be interpreted as an ellipse in the b_1, b_2 plane, which we write as

$$H = 2b_1(2c_1b_1 + 1) + 2b_2(2c_2b_2 + 1) = \text{constant}. \tag{146}$$

This means that we can specify it parametrically in terms of an angle $\theta \in (-\pi, \pi]$ and b_1 and b_2 are given by

$$b_1 = \frac{\lambda}{\sqrt{c_1}} \sin\theta - \frac{1}{2c_1}, \qquad b_2 = \frac{\lambda}{\sqrt{c_2}} \cos\theta - \frac{1}{2c_2}. \tag{147}$$

where

$$\lambda^2 = \frac{1}{2}\left(H + \frac{1}{2c_1} + \frac{1}{2c_2}\right). \tag{148}$$

Here we are only concerned with peakons (rather than anti-peakons), therefore a_j and b_j must be positive for $j = 1, 2$, which also implies that $H > 0$ and c_1, $c_2 > 0$. Certain ratios of $\dot{b}_1$ and $\dot{b}_2$ are multiples of $\dot{\theta}$. Setting $c_j = k_j^2$ for $j = 1, 2$, and

$q = q_1 - q_2$ we find

$$\dot{b_1} = \frac{\lambda}{k_1}\dot{\theta}\cos\theta = -b_1 b_2(2k_2^2 b_2 + 1)e^{-|q|} \tag{149}$$

$$\dot{b_2} = -\frac{\lambda}{k_2}\dot{\theta}\sin\theta = b_1 b_2(2k_1^2 b_1 + 1)e^{-|q|} \tag{150}$$

More easily seen

$$\frac{\dot{b_1}}{2k_2^2 b_2 + 1} = \frac{\dot{b_2}}{2k_1^2 b_1 + 1} = b_1 b_2 e^{-|q|} \tag{151}$$

Using the formula for an additional conserved quantity, J,

$$J = b_1 b_2(1 - e^{-|q|}) \tag{152}$$

as well as the assumption that $q < 0$, we find

$$\dot{\theta} = -2k_1 k_2 b_1 b_2 e^{q} = 2k_1 k_2 (J - b_1 b_2). \tag{153}$$

Now we have reduced our 2-peakon system to quadratures (as is guaranteed by Liouville's theorem [8]) and can now calculate θ. Writing the b_i's in their parametric forms in terms of θ leads to an autonomous equation for θ alone, namely

$$\dot{\theta} = 2\left(Jk_1 k_2 - \frac{1}{4k_1 k_2} + \frac{\lambda}{2k_1}\cos\theta + \frac{\lambda}{2k_2}\sin\theta - \lambda^2 \sin\theta\cos\theta\right) \equiv f(\theta). \tag{154}$$

We can show this by performing the quadrature

$$\int \frac{d\theta}{f(\theta)} = \int dt = t + const \tag{155}$$

and then b_1 and b_2 are now specified in functions of t by (149) and (150). Hence $q(t)$ can be found from the quantity J (152)

$$q = \log\left(1 - \frac{J}{b_1 b_2}\right) \tag{156}$$

(noting that $b_1, b_2 > J$ must hold by the initial assumption on $\mathrm{sgn}(q)$). Following the work of Camassa, Holm and Hyman [16, 17], and presented in section 1.3.2, we set

$$Q = q_1 + q_2. \tag{157}$$

Then equations (149) and (150) become

$$\dot{q} = (2(c_1 b_1^2 - c_2 b_2^2) + b_1 - b_2)(1 - e^{-|q|}), \tag{158}$$

$$\dot{Q} = (2(c_1 b_1^2 + c_2 b_2^2) + b_1 + b_2)(1 + e^{-|q|}) \tag{159}$$

Now that the right hand side of (159) is specified as functions of t, an additional quadrature with respect to t yields $Q = Q(t)$.

To complete the integration explicitly it is convenient to make use of the standard T-substitution, which shall convert the trigonometric expressions such as those in (154) into rational functions of the variable

$$T = \tan\frac{\theta}{2} \tag{160}$$

via

$$\sin\theta = \frac{2T}{1+T^2}, \qquad \cos\theta = \frac{1-T^2}{1+T^2}, \qquad d\theta = \frac{2}{1+T^2}dT. \tag{161}$$

For b_i's in terms of T we have:

$$b_1 = \frac{2\lambda T}{k_1(1+T^2)} - \frac{1}{2k_1^2}, \qquad b_2 = \frac{\lambda(1-T^2)}{k_2(1+T^2)} - \frac{1}{2k_2^2}. \tag{162}$$

and using our relation for T above

$$\frac{dT}{dt} = \frac{P(T)}{1+T^2} = F(T) \tag{163}$$

with the quartic polynomial

$$P(T) = \left(Jk_1k_2 - \frac{1}{4k_1k_2}\right)(1+T^2)^2 + \frac{\lambda}{2k_1}(1-T^4) + \frac{\lambda T}{k_2}(1+T^2) - 2\lambda^2 T(1-T^2). \tag{164}$$

In order to compute the equations of motion we need to employ partial fraction decomposition. We recall that for a rational function $\frac{R(T)}{S(T)}$ with $degR < degS = m$ has a partial fraction decomposition of the form

$$\frac{R(T)}{S(T)} = \sum_{k=1}^{m} \frac{r_k}{T - T_k} \tag{165}$$

where T_k are the roots of $S(T)$ in the denominator (assumed simple) and r_k is the residue at the poles of the rational function at $T = T_k$. This is an example of a Mittag-Leffler expansion in complex analysis [4]. Applying this result to (163), we find

$$\frac{1}{F(T)} = \frac{T^2+1}{P(T)} = K^{-1} \sum_{j=1} \frac{(T_j^2+1)e_j}{T - T_j} \tag{166}$$

We can factorize $P(T)$ in the following way

$$P(T) = K \prod_{j=1}^{4} (T - T_j), \qquad K = Jk_1k_2 - \frac{1}{4k_1k_2} - \frac{\lambda}{2k_1} \tag{167}$$

and

$$e_j = \prod_{1 \le k \le 4, k \ne j} (T_j - T_k)^{-1} \tag{168}$$

The general solution of (163) is given implicitily by

$$K^{-1} \sum_{j=1}^{4} (1 + T_j^2) \log(T - T_j) = t + \text{const} \tag{169}$$

The solution above is valid for complex values of T (and t), the constant of integration should also be allowed to be complex. We, however, are only interested in real values for $T = \tan(\frac{\theta}{2})$ for real t, in the case where the coefficients of $P(T)$ are all real. The different combination of real and complex roots may require the solution to be specified in different forms. For example, if the four roots of T_j are all real, then for real T the solution can be written as

$$K^{-1}\sum_{j=1}^{4}(T_j^2+1)e_j\log|T-T_j| = t-t_0 \tag{170}$$

with real constant of integration t_0. If $P(T)$ has two real roots and a complex conjugate pair, then for real T two of the logarithms in (169) can be combined into an arctangent.

For the second quadrature to find $Q(T)$ from (159), it is convenient to write

$$\dot{Q} = \frac{dQ}{dT}\dot{T} \tag{171}$$

Replace the right hand side of (159) by the corresponding equations in terms of T we can then obtain $Q(T)$ by integrating in terms of T instead of t:

$$\frac{dQ}{dT} = 2\lambda R(T)\left(\frac{1}{P(T)}-\frac{1}{\hat{P}(T)}\right) \tag{172}$$

Given as two additional polynomials, one quadratic and the other quartic, namely

$$R(T) = \lambda(T^2+1) - \frac{T}{k_1} + \frac{T^2-1}{2k_2} \tag{173}$$

and

$$\hat{P}(T) = Jk_1k_2(T^2+1)^2 - P(T). \tag{174}$$

Then writing

$$\hat{P}(T) = \hat{K}\prod_{j=1}^{4}(T-\hat{T}_j) \tag{175}$$

the general solution of (159) can be written in terms of $T = T(t)$ as

$$Q = 2\lambda\left(K^{-1}\sum_{j=1}^{6}R(T_j)e_j^*\log(T-T_j) - \hat{K}^{-1}\sum_{j=1}^{6}R(\hat{T}_j)\hat{e}_j^*\log(T-\hat{T}_j)\right) + \text{const}, \tag{176}$$

with

$$e_j^* = \prod_{1\le k\le 4, k\ne j} (T_j - T_k)^{-1}, \tag{177}$$

In the next section we will use these explicit formulae to describe the asymptotic behaviour of the two peakons as $t \to \pm\infty$.

2.5 Asymptotics

Analysing the asymptotics of the 2-peakon problem, we find that the behavior of the peakons in this system of Popowicz is qualitatively similar to those in the Camassa-Holm and Degasperis-Procesi case in that at extreme times, either relatively large or small, they behave as we expect, the two peaks are well separated and travel with constant velocity and amplitude. However, at the point of interaction, Popowicz peakons exchange varying amounts of velocity and amplitude during the interaction. This results in the pair of peakon velocities being different before and after interaction. In contrast the peakons in the Camassa-Holm and Degasperis-Procesi equations switch velocities and amplitudes asymptotically. This results in a phase shift of the peakons, the Camassa-Holm shift is explicitily given in [17] and is here in section 1.3.2.

To observe the behavior of the peaks over long time scales, we use (163), as the asymptotic form is controlled by this ODE for T. This equation has fixed points at the roots of $F(T)$, these are found at the roots of the quartic, $P(T)$. Near to a fixed point, the local behaviour is

$$T \sim T_k + A_k e^{F'(T_k)t}, \tag{178}$$

where A_k is a constant, and we have

$$F'(T_k) = \frac{K}{(T_k^2+1)e_k} \tag{179}$$

compared with the coefficients in (169). Taking some initial data at $t = 0$ for b_1,

b_2, k_1 and k_2 we are able to find an initial point on the ellipse H from (146) in the (b_1, b_2) plane. Therefore we can also find an initial angle $\theta(0)$ and also $T(0) = \tan\left(\frac{\theta(0)}{2}\right)$ from the standard T-substitution. $T(0)$ lies between two adjacent real roots, T_+ and T_-,

$$T_+ < T(0) < T_- \tag{180}$$

with the asymptotic behavior given by

$$T \sim T_\pm \pm e^{F'(T_\pm)t+\delta_\pm} \tag{181}$$

as $t \to \pm\infty$ and $\delta_\pm$ depends on the terms in (170) that are regular at $T = T_\pm$, as well as the integration constant. So if there are four real roots, then $\delta_\pm$ depends on the constant t_0 in (170). The roots $T_\pm$ are both fixed points of (163), with T_+ stable and T_- unstable so

$$F'(T_+) < 0 < F'(T_-) \tag{182}$$

We can find the asymptotic forms for the b_i's from (162)

$$b1 \to \frac{2\lambda T_\pm}{k_1(1+T_\pm^2)} - \frac{1}{2k_1^2}, \qquad b_2 \to \frac{\lambda(1-T_\pm^2)}{k_2(1+T_\pm^2)} - \frac{1}{2k_2^2}, \qquad \text{as } t \to \pm\infty \tag{183}$$

when the two peakons are in the field $v(x,t)$. To find the corresponding amplitudes for the field $u(x,t)$ we use the Casimir that was fixed at the start,

$$a_j = c_j b_j^2 = k_j^2 b_j^2. \tag{184}$$

The asymptotic behavior for the positions q_1 and q_2 are however more complicated. We have two equations for the difference $q = q_1 - q_2$, (158) and (156), therefore

$$q = \log(-T) - \log(k_1 k_2 b_1 b_2 (1 + T^2)), \tag{185}$$

as $t \to \pm\infty$, and using (181)

$$q \sim F'(T_\pm)t + \delta_\pm + \log(\mp F'(T_\pm) - \log(k_1 k_2 J(1 + T_\pm^2)), \tag{186}$$

and the fact that $b_1 b_2 \to J$ as $|t| \to \infty$. So for $Q = q_1 + q_2$ we find from (176) that near the roots $T = T_\pm$, Q is

$$Q \sim \frac{2\lambda e_\pm R(T_\pm)F'(T_\pm)}{K} t + \text{const} \tag{187}$$

and substituting in for $F'(T_\pm)$

$$Q \sim \frac{2\lambda R(T_\pm)}{1 + T_\pm^2} t + \text{const as } t \to \infty \tag{188}$$

where the constant depends on $\delta_\pm$, as well as the other terms and the arbitrary constant of integration in (176).

2.6 Numerical Results

We will apply three choices of initial conditions to the system (142) and explore what has been discussed in the chapter. The first example is possibly not the best to demonstrate the peakon dynamics, however it is a good choice of initial conditions in the sense that the solutions do not numerically 'blow up'. This does happen when we shift any of these initial conditions slightly, this is not unexpected given the numerical challenges with other peakon equations [19].

Unlike Example 1, the latter two examples are able to demonstrate more definitively 2-peakon dynamics such as asymptotic switching and also some new behaviour not observed with other integrable peakon equations.

2.6.1 Example 1

Take the initial values

$$q_1(0) = -30, \qquad q_2(0) = 0, \qquad b_1(0) = 3, \qquad b_2(0) = 2, \tag{189}$$

and Casimirs

$$c_1 = 0.3, \qquad c_2 = 0.2 \tag{190}$$

which fixes

$$H = 16.999999999999375, \qquad J = 5.999999999999439, \qquad \lambda = 3.253203549323808 \tag{191}$$

this example makes it quite difficult to differentiate J from $b_1(0)b_2(0)$. However we found this to be a stable choice of initial conditions, the problem was surprisingly unstable and was extremely sensitive to small changes in values.

The roots of $P(T)$ in this case are

$$T_+ = 0.3142320479, \qquad T_- = 0.4854161155, \qquad 11.27226037, \qquad -0.7888302327. \tag{192}$$

As the peakons have been taken so far apart, the value of $T(0) \approx 0.4854161$ is nearly indistinguishable from T_-. Now we calculate

$$\frac{P'(T_\pm)}{(T_\pm^2 + 1)} \tag{193}$$

and

$$\frac{2\lambda R(T_\pm)}{1 + T_\pm^2} \tag{194}$$

which we use to can calculate q (186) and Q (188)

$$q \sim -4.746989214..t, \qquad Q \sim 11.80243976..t, \qquad t \to +\infty \tag{195}$$

and

$$q \sim 4.799999018..t, \qquad Q \sim 11.99999999..t, \qquad t \to -\infty \tag{196}$$

As the seperation of the peaks was significant at the start, it doesn't really show the interaction very well. What we find is the interaction looks like a phase shift as in Figure 4, however it has not shifted. For switching to occur, we would require

$$\frac{2\lambda R(T_+)}{1+T_+^2} = \frac{2\lambda R(T_-)}{1+T_-^2} \tag{197}$$

and

$$\frac{P'(T_+)}{T_+^2+1} = -\frac{P'(T_-)}{T_-^2+1} \tag{198}$$

which means the aymptotic values of Q would have to be the same, and the asymptotic values of q would require a change of sign between $\pm\infty$. For comparison in the Camassa-Holm case, the asymptotic velocity of Q is c_1+c_2 at both $\pm\infty$, while the asymptotic velocity of q switches between $\pm(c_1-c_2)$, as they differ in the first decimal place they have not switched.

The asymptotic positions of the peakons are

$$q_1 = \frac{(Q+q)}{2}, \qquad q_2 = \frac{(Q-q)}{2} \tag{199}$$

which for this example are

$$q_1 = 3.516273810, \qquad q_2 = 8.286165950, \qquad t \to \infty \tag{200}$$

and

$$q_1 = 8.400000, \qquad q_2 = 3.600000, \qquad t \to -\infty \tag{201}$$

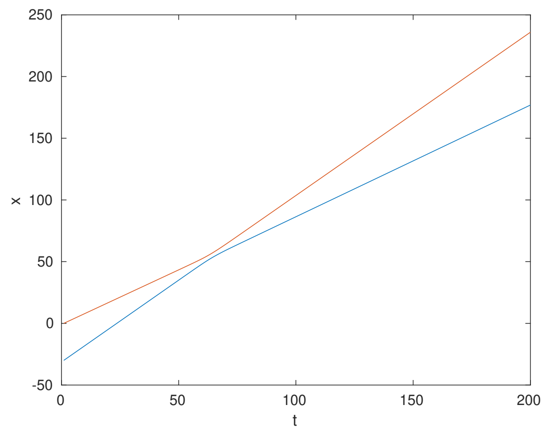


Figure 4: Two interacting peakons, using initial conditions in Example 1

2.6.2 Example 2

Apart from the example already in the chapter, here are some more numerical choices which have been chosen to illustrate the solution. By choosing 3 roots of the quartic, the fourth root is fixed, and all the coefficients are also fixed up to rescaling. Note that scaling k_1, k_2 by the same amount is equivalent to scaling b_j and t; so we can always fix $k_1 = 1$ if necessary. The roots being

$$T = -111/152, 3/10, 1/2, 8. \tag{202}$$

Fixing $k_1 = 1 = c_1$, this corresponds to

$$k_2 = \frac{496}{593}, \tag{203}$$

so $c_2 = \left(\frac{496}{593}\right)^2$, and the other parameters appearing in $P(T)$ are

$$\lambda = \frac{35425}{22816}, \qquad J = \frac{298710111}{1008604096}, \tag{204}$$

with the value of (half) the Hamiltonian being

$$\frac{H}{2} = \frac{58672865}{32535616}. \tag{205}$$

The corresponding initial data is

$$b_1(0) = \frac{94417}{165416}, \qquad b_2(0) = \frac{103298821}{164092672}, \tag{206}$$

and

$$q_1(0) = \log\left(\frac{2890857125}{16447158149}\right) \qquad q_2(0) = 0. \tag{207}$$

Then we can consider a solution which corresponds to starting with

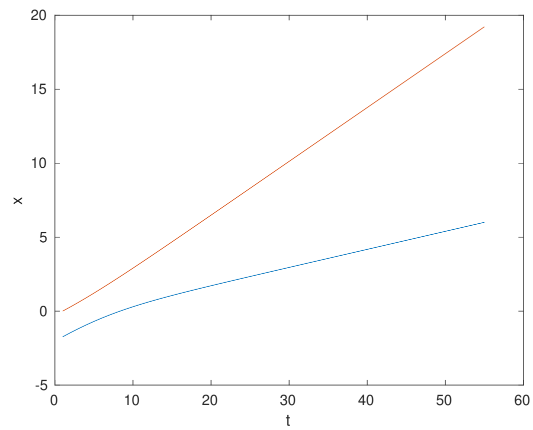


Figure 5: Two interacting peakons, using initial conditions in Example 2

$$\frac{3}{10} < T(0) < \frac{1}{2}, \tag{208}$$

which will move between the asymptotic values

$$(b_1, b_2) = \left(\frac{4233}{5704}, \frac{70567}{176824}\right) \tag{209}$$

as $t \to -\infty$ and

$$(b_1, b_2) = \left(\frac{2023}{5704}, \frac{147657}{176824}\right) \tag{210}$$

as $t \to +\infty$, using (162). The asymptotic velocities of the two peakons are

$$c_1 = \frac{9862125}{16267808}, \qquad c_2 = \frac{7364175}{4066952}, \qquad t \to +\infty \tag{211}$$

and

$$c_2 = \frac{29990805}{16267808}, \qquad c_2 = \frac{2529345}{4066952}, \qquad t \to -\infty. \tag{212}$$

The peakons therefore only approximately exchange asymptotic velocities.

2.6.3 Example 3

Take $k_1 = 1, k_2 = \frac{357080}{117173}$, so $c_1 = 1$ and $c_2 = (\frac{357080}{117173})^2$. We wish to fix initial values so that

$$\frac{H}{2} = \frac{131169407947}{11475551376} \tag{213}$$

and

$$J = \frac{12691383169465}{68294831422368}, \tag{214}$$

which gives

$$\lambda = \frac{7330681}{2142480} \tag{215}$$

and

$$Jk_1k_2 - \frac{1}{4k_1k_2} = \frac{18524630887}{38251837920} \tag{216}$$

(which is the coefficient in front of the $(T^2+1)^2$ in (164)). This means that the quartic polynomial $P(T)$ has roots at

$$T = -\frac{1909}{1920}, \qquad \frac{1}{10}, \qquad \frac{9}{10}, \qquad 20. \tag{217}$$

Suppose we wish to take a solution which moves between the unstable fixed point of the ODE for T at $T = T_- = \frac{1}{10}$ and the stable fixed point at $T = T_+ = -\frac{1909}{1920}$. We choose initial values for b_1, b_2 to be the same at $t = 0$, so

$$b_1(0) = b_2(0) = B. \tag{218}$$

The definition of H at $t = 0$ gives

$$\frac{H}{2} = (k_1^2 + k_2^2)B^2 + 2B, \tag{219}$$

and solving this quadratic gives $B = -1.155790328029969, 0.9613703913703853$; we need to take the negative root to give $T(0) = -0.09672794535064579$, which lies between T_+ and T_-. The definition of J at $t = 0$ gives

$$J = B^2(1 - e^{-|q(0)|}) \tag{220}$$

hence

$$|q(0)| = 0.1497902737164921 \tag{221}$$

is the initial separation. Taking the convention $q(0) < 0$, we can choose

$$q_1(0) = -0.1497902737164921, \qquad q_2(0) = 0. \tag{222}$$

The exact asymptotic values of b_j as $t \to +\infty$ are

$$b_1^+ = -\frac{70015}{17854}, \qquad b_2^+ = -\frac{181266631}{3825183792}, \tag{223}$$

and as $t \to -\infty$ they are

$$b_1^- = \frac{19019}{107124}, \qquad b_2^- = \frac{667300235}{637530632}. \tag{224}$$

However, note that while this is corresponds to an exact solution of the ODE system, the amplitudes b_1 and b_2 must both change sign in order to go from (-,-) to the (+,+) quadrant in the (b_1, b_2) plane. Thus such a solution is not realistic, as once either b_1 or b_2 reaches zero, it should stay there; in other words a 2-peakon solution can collapse into a single peakon in a finite amount of time. This is a new phenomenon compared with integrable peakon equations like Camassa-Holm and Degasperis-Procesi.

2.7 Summary

In this chapter we have explored the 2-peakon interaction, writing down the explicit solution and also exploring some of the dynamics such as 2-peakon collapse. The peakon interactions are particularly interesting, in future studying the peakon/anti-peakon interactions could yield some other results. To do this we may have to relax some of the rules on the signs of the roots.

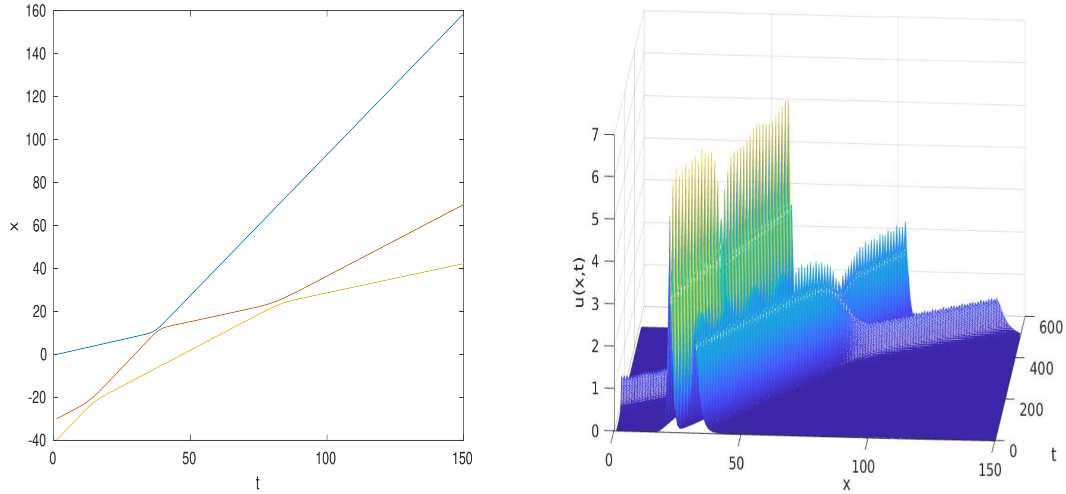
Investigating the Popowicz PDE system numerically poses well known challenges for standard integration schemes [19], but such a study may yield some interesting peakon solutions. Another question is to look at $N > 2$ peakon ODEs and explore the numerics further. Here we give a short introduction to what we expect with the 3-peakon dynamics, given the ODE system

$$
\begin{aligned}
\dot{b_1} &= b_1((2c_2b_2^2 + b_2)\operatorname{sgn}(q_1 - q_2)e^{-|q_1-q_2|} + (2c_3b_3^2 + b_3)\operatorname{sgn}(q_1 - q_3)e^{-|q_1-q_3|}),\\
\dot{b_2} &= b_2((2c_1b_1^2 + b_1)\operatorname{sgn}(q_2 - q_1)e^{-|q_2-q_1|} + (2c_3b_3^2 + b_3)\operatorname{sgn}(q_2 - q_3)e^{-|q_2-q_3|}),\\
\dot{b_3} &= b_3((2c_1b_1^2 + b_1)\operatorname{sgn}(q_3 - q_1)e^{-|q_3-q_1|} + (2c_2b_2^2 + b_2)\operatorname{sgn}(q_3 - q_2)e^{-|q_3-q_2|}),\\
\dot{q_1} &= (2c_1b_1^2 + b_1) + (2c_2b_2^2 + b_2)e^{-|q_1-q_2|} + (2c_3b_3^2 + b_3)e^{-|q_1-q_3|},\\
\dot{q_2} &= (2c_1b_1^2 + b_1)e^{-|q_2-q_1|} + (2c_2b_2^2 + b_2) + (2c_3b_3^2 + b_3)e^{-|q_2-q_3|},\\
\dot{q_3} &= (2c_1b_1^2 + b_1)e^{-|q_3-q_1|} + (2c_2b_2^2 + b_2)e^{-|q_3-q_2|} + (2c_3b_3^2 + b_3),
\end{aligned}
$$

the ODE's were solved using the in built integrator `RKF45` in Matlab using the same initial conditions as in Example 1 for the first two peakons. The third peakon is described initially by

$$
c_3 = 0.4, \qquad b_3(0) = 4, \qquad q_3(0) = -40. \tag{225}
$$

which we can then plot as in Figure 6 and can see that they behave in a similar manner to the 2-peakon case, or at least superficially. Further investigation of these 3-peakons could be similar to the work of Parker [85] who looked at the three



(a) Trajectories (b) Surface plot

Figure 6: Three interacting peakons

soliton solution from the associated Camassa-Holm equation using Matsuno's bilinear method [73]

$$f(y,t) = 1 + \sum_{i=1}^{3} e^{\theta_i} + \sum_{i<j}^{3} A_{ij} e^{\theta_i+\theta_j} + A_{12}A_{13}A_{23}e^{\theta_1+\theta_2+\theta_3} \tag{226}$$

with $\theta_i = p_i y + \omega_i t + \sigma_i \tau + \eta_i$ and $i = 1, 2, 3$ are the usual phase variables. Setting $i < j$ means we get three pairs $(1, 2)$, $(1, 3)$ and $(2, 3)$. It may be possible to apply a similar technique to these Popowicz peakons and essentially study the three peakon problem by looking at pairs of peakons.

Chapter 3

From peakon equations to Painlevé equations

In this chapter we reduce two integrable PDEs, Camassa-Holm and Degasperis-Procesi to particular instances of the third Painlevé equation (P_{III}), that is

$$\frac{d^2W}{dZ^2} = \frac{1}{W}\left(\frac{dW}{dZ}\right)^2 - \frac{1}{Z}\frac{dW}{dZ} + \frac{1}{Z}(\alpha W^2 + \beta) + \gamma W^3 + \frac{\delta}{W}. \qquad (227)$$

We then derive explicit solutions of these Painlevé equations both algebraic and special function solutions. We also show how applying similar reduction techniques give us particular solutions of the more general b-family. It is known that Camassa-Holm, after a reciprocal transformation, is related to the first negative flow of the Korteweg-de-Vries hierarchy. In [55] it was also found that from the reciprocally transformed equation a scaling similarity reduction could be applied resulting in a form of P_{III} but the details were not given. Solutions of the resulting P_{III} equation were briefly discussed in [56], we explore to a greater depth our solutions and give in terms of the original coordinates. We also go further by applying a similarity reduction first and then a hodograph transformation which results in the same version of P_{III} and find solutions.

Similarly the reciprocal transformation for Degasperis-Procesi is known [31], but in this case the reduction to P_{III} was not. Again we apply a similarity reduction followed by a hodograph transformation, and show that this results in the same P_{III} equation as is obtained by applying the similarity reduction to the reciprocally transformed equation.

Interestingly the two forms of P_{III} related to Camassa-Holm and Degasperis-Procesi are different, and when we look at solutions of these Painlevé equations we find they are of different 'types' according to Gromak [49]. We also consider a one-parameter family of PDEs labelled by a parameter b, which although we don't find an exact Painlevé reduction for the general b, it does reduce to the respective P_{III}'s for Camassa-Holm and Degasperis-Procesi.

3.1 Reductions of Camassa-Holm

This diagram helps set the structure of this section, noting that the corresponding subsections are represented by the various arrows. It is like a commutative diagram, so by following the arrows from Camassa-Holm in two different ways leads to the same result. The work shown in 3.1.1 is well known, and 3.1.2 was stated

$$\begin{array}{ccc}
\text{Camassa-Holm} & \xrightarrow[\text{3.1.3}]{\text{Sim. Red}} & \text{3rd Order Equation} \\
{\scriptstyle 3.1.1}\Big\downarrow{\scriptstyle \text{Rec.Trans}} & & {\scriptstyle 3.1.4}\Big\downarrow{\scriptstyle \text{Hodograph}} \\
\text{negative KdV} & \xrightarrow[\text{3.1.2}]{\text{Sim. Red}} & \text{2nd Order Equation, } P_{III}
\end{array}$$

Figure 7: Camassa-Holm reductions

in [55] but exact details were not given. However, sections 3.1.3 and 3.1.4 are new contributions to the area thereby giving the following result

Theorem 3.1.1. The Camassa-Holm equation can be written as a form of the third Painlevé equation, via two seperate reduction schemes.

3.1.1 From Camassa-Holm to the First Negative Flow of KdV

Here we address the vertical arrows on the left of the diagram in Figure 7 above, to show the following:

Proposition 3.1.1. [60, 75] The Camassa-Holm equation, via a reciprocal transformation, can be written as the first negative flow of KdV.

From now onwards, we use what is sometimes known as the m form of Camassa-Holm given by

$$m_t = -2mu_x - um_x, \qquad\qquad m = u - u_{xx}. \tag{228}$$

Proof. Making a change to a new dependent variable p, given by

$$p^2 = u - u_{xx}, \tag{229}$$

and substituting into (228) gives

$$p_t = -(up)_x, \tag{230}$$

which is a conservation law, and can be used to define a reciprocal transformation as in (89). Therefore a reciprocal transformation for Camassa-Holm can be defined by

$$dX = pdx - updt, \qquad\qquad dT = dt \tag{231}$$

we can then extract the following information for the derivatives

$$\frac{\partial X}{\partial t} = -up, \qquad \frac{\partial X}{\partial x} = p, \qquad \frac{\partial T}{\partial t} = 1, \qquad \frac{\partial T}{\partial x} = 0. \tag{232}$$

Using this information we can substitute into the components of the conservation law

$$\frac{\partial p}{\partial t} = -up\frac{\partial p}{\partial X} + \frac{\partial p}{\partial T}, \qquad -\frac{\partial (up)}{\partial x} = -p\frac{\partial (up)}{\partial X}. \tag{233}$$

Therefore we have

$$(p^{-1})_T = u_X \tag{234}$$

so the reciprocal transformation (231) maps the conservation law (230) to the conservation law (234). However this is incomplete as it doesn't describe the relationship between $u(X,T)$ and $p(X,T)$: (228) is a system of two equations, the second of which gives

$$p^2 = u - u_{xx} = u - p\frac{\partial}{\partial X}\left(p\frac{\partial}{\partial X}u\right) \tag{235}$$

since $\frac{\partial}{\partial x} = p\frac{\partial}{\partial X}$ by the chain rule (this is the general rule for transforming x to X derivatives, of which (231) is a special case). Then replacing $\frac{\partial}{\partial X}u$ by $(p^{-1})_T$ in (235) we get the following:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1}{p}\frac{\partial p}{\partial T}\right) = \frac{\partial}{\partial X}\frac{\partial X}{\partial x}\left(-\frac{1}{p}\frac{\partial p}{\partial T}\right) = -p_{XT} + \frac{p_X p_T}{p}. \tag{236}$$

Thus using the fact that $u = p^2 + u_{xx}$, we get the relation

$$u = p^2 - p_{XT} + \frac{p_X p_T}{p}. \tag{237}$$

Rewriting the conservation law (234) in terms of p alone gives

$$(p^{-1})_T = (p^2 - p(\log p)_{XT})_X \tag{238}$$

These two equations (237)and (238) are equivalent to the first negative flow of the KdV hierarchy (see section 1.8 for details on the KdV hierarchy), showing that Proposition 3.1.1 is true. □

The positive hierarchy can be written

$$V_{t_n} = R^{n-1}V_X, \qquad R = \partial_X^2 + 4V + 2V_X\partial_X^{-1}, \qquad n = 1, 2, 3, ... \tag{239}$$

which is why (238) can be though of as a negative flow. If we have

$$V_T = -2p_X \tag{240}$$

then it is equivalent to (238) when

$$V = -\frac{p_{XX}}{2p} + \frac{p_X^2}{4p^2} - \frac{1}{4p^2}, \tag{241}$$

Then as a consequence we have

$$RV_T = (\partial_X^2 + 2V + V_X\partial_X^{-1})V_T = 0. \tag{242}$$

3.1.2 Reduction of First Negative Flow of KdV to P_{III}

Following on from Proposition 3.1.1, we now explore the bottom arrow in figure 7 and propose the following:

Proposition 3.1.2. A similarity reduction of the first negative flow of KdV (237) and (238) results in the third Painlevé equation.

Proof. To reduce the reciprocally transformed system, (237) and (238), to that of P_{III}, we take the following similarity variables to make the reduction:

$$p = T^{-\frac{1}{2}}P(Z), \qquad u = T^{-1}U(Z), \qquad Z = XT^{1/2}. \tag{243}$$

The scaling reduction of the third order equation (238) implies that if $p = T^{-\frac{1}{2}}P(Z)$ with Z as given then u must be of the form $T^{-1}U(Z)$, by (237). This is a scaling similarity reduction Therefore (243) with (237) implies

$$U = P^2 - \frac{1}{2}ZP'' - \frac{1}{2}P' + \frac{ZP'^2}{2P} \tag{244}$$

Although (238) reduces to a 3rd order ODE for $P(Z)$, we will not write this out, as can integrate once to get a second order ODE. The easiest way to integrate is to use the conservation law (234), which implies that

$$\frac{1}{2}\frac{d}{dZ}(ZP^{-1}) = \frac{dU}{dZ} \tag{245}$$

holds for the similarity reduction. We can now integrate (245) to obtain

$$U = \frac{1}{2}\frac{Z}{P} - a. \tag{246}$$

with a an integration constant. We can now compare our two expressions for U, provided by (244) and (246) to get the following case of P_{III} [62]:

$$\frac{d^2P}{dZ^2} = \frac{1}{P}\left(\frac{dP}{dZ}\right)^2 - \frac{1}{Z}\frac{dP}{dZ} + \frac{1}{Z}(2P^2 + 2a) - \frac{1}{P}. \tag{247}$$

This has the following parameter values compared with the general P_{III} in (227):

$$\alpha = 2, \qquad \beta = 2a, \qquad \gamma = 0, \qquad \delta = -1. \tag{248}$$

This shows that Proposition 3.1.2 is true, which agrees with the work of Hone in [55]. □

3.1.3 From Camassa-Holm to 3rd Order ODE

Proposition 3.1.3. Applying a similarity reduction to Camassa-Holm reduces the PDE to a 3rd order ODE

Proof. Using the Camassa-Holm conservation law (230) and (229) we now make a direct application of a scaling similarity reduction. We will be making an abuse of notation here, using P and U to refer to functions of two different variables: Z which we had before, and a new similarity variable z given by

$$z = x + c \log t. \tag{249}$$

which is introduced to replace the variables x and t. Specifically, it is invariant to the following choice of scaling similarity transformation:

$$p = t^{\mu} P(z), \qquad\qquad u = t^{\nu} U(z). \tag{250}$$

For the case specific to Camassa-Holm we can use the ansatz (250) to find μ and ν, by substituting into the right side of (228)

$$p^2 = u - u_{xx} \Rightarrow t^{2\mu} P^2 = t^{\nu}(U - \ddot{U}) \tag{251}$$

where $\dot{} = \frac{d}{dz}$. This implies

$$\nu = 2\mu, \tag{252}$$

and therefore

$$P^2 = U - \ddot{U} \tag{253}$$

and substituting (250) into (230)

$$(t^{\mu} P)_t + (t^{3\mu} U P)_x = 0 \tag{254}$$

Equations (250) to (254) can also be obtained from the Lie symmetry approach which is described in 1.7. To find μ, need to lose x and t derivatives and find an equation depending only on z:

$$\mu t^{\mu-1} P + c t^{\mu-1} \dot{P} + t^{3\mu} (\dot{UP}) = 0 \qquad \Rightarrow \qquad \mu - 1 = 3\mu, \tag{255}$$

To remove the t dependence we solve (252) and (255)

$$\mu = -\frac{1}{2}, \qquad \nu = -1. \tag{256}$$

Substituting (256) into (255) and rearranging we find

$$\frac{d}{dz}[(U+c)P] = \frac{1}{2}P. \tag{257}$$

Then the 3rd order system for U, P just consists of (253) with (257), and then by differentiating (253) and substituting in for P^2 and $P\dot{P}$ we get a third order ODE for $U(z)$, that is

$$\left(\dot{U} - \frac{1}{2}\right)(U - \ddot{U}) - \frac{1}{2}(c+U)(\dot{U} - \dddot{U}) = 0. \tag{258}$$

□

3.1.4 From the 3rd Order ODE to P_{III}

This subsection shall describe a hodograph transformation that shall show the following:

Proposition 3.1.4. The 3rd order ODE (258) that arises from a scaling similarity reduction of Camassa-Holm can be transformed to the third Painlevé equation.

Proof. Starting with (257) and then using

$$dZ = Pdz \tag{259}$$

gives

$$\frac{d}{dZ}((U+c)P) = \frac{1}{2}, \tag{260}$$

which integrates to

$$(U+c)P = \frac{1}{2}Z, \tag{261}$$

where we have ignored an integration constant (absorbed into Z). Then obtaining P_{III} is as shown in subsection 3.1.2. From 3.1.2 we get

$$dZ = T^{\frac{1}{2}}dX + \frac{1}{2}ZT^{-1}dT, \tag{262}$$

$$= T^{\frac{1}{2}}(T^{-\frac{1}{2}}Pdx - T^{-\frac{3}{2}}PUdt) + \frac{1}{2}Zd\log t, \tag{263}$$

where we used (231). Then

$$dZ = Pdx - (PU - \frac{1}{2}Z)d\log t, \tag{264}$$

$$= P(dx + c\ d\log t), \tag{265}$$

where we used (261), hence (259). We have an arbitary constant c from the similarity variable (249) which we can identify as the integration constant a found in the previous reduction.

From $P^2 = U - \ddot{U}$, with $\dot{U} = \frac{dU}{dz}$

$$U = P^2 + \left(P\frac{d}{dZ}\right)^2 U \tag{266}$$

Rearranging (261) for U and substituting into above

$$U = P^2 + P\frac{d}{dZ}\left(\frac{1}{2} - \frac{ZP'}{2P}\right) \tag{267}$$

noting that this is the same as (244). Now we have two equations for U, and equating the expressions from (261) and (267) gives

$$\frac{d^2P}{dZ^2} = \frac{1}{P}\left(\frac{dP}{dZ}\right)^2 - \frac{1}{Z}\frac{dP}{dZ} + \frac{1}{Z}(2P^2 + 2c) - \frac{1}{P} \tag{268}$$

which is the same case of P_{III} as before, namely (247) if we identify $c = a$. □

3.1.5 Bäcklund Transformation and Solutions

The general solution of P_{III} is transcendental, and cannot be written in terms of more elementary transcendental functions (solutions of linear equations or elliptic functions) [3]. However, for some special parameter values, Painlevé equations (apart from P_I) do have some particular solutions which are rational, algebraic or given by some classical special functions (like Airy, Hermite, Bessel). Painlevé equations (apart from P_I) also admit Bäcklund transformations (BTs), which map one solutions to another solution with different parameter values. We are going to use BTs for P_{III} to obtain a sequence of rational solutions.

Now we find some solutions of the Painlevé equation (247), which falls into the $\gamma = 0$ and $\alpha\beta \neq 0$ parameter category, using the methods in [77, 49]. To use the same notation as the literature, we set $P = w$ so (247) becomes

$$w'' = \frac{(w')^2}{w} - \frac{w'}{Z} + \frac{1}{Z}(2w^2 + 2c) - \frac{1}{w} \tag{269}$$

We consider a pair of first order differential equations of the form

$$w' = a_0(Z) + a_1(Z)w + a_2(Z)w^2 + a_3(Z)w^2 v, \tag{270}$$

$$v' = b_0(Z) + b_1(Z)w + b_2(Z)v + b_3(Z)wv + b_4(Z)wv^2, \tag{271}$$

where w has to satisfy P_{III} and v is an auxiliary function. For that, the following conditions need to be imposed

$$a_0^2 = 1, \qquad a_1' = -\frac{1}{Z}a_1, \tag{272}$$

$$a_2' + a_1 a_2 + b_0 a_3 = \frac{1}{Z}(2 - a_2), \qquad a_3' + a_1 a_3 + b_2 a_3 = -\frac{1}{Z}a_3, \tag{273}$$

$$a_0' - a_0 a_1 = \frac{1}{Z}(2c - a_0), \qquad a_2^2 + b_1 a_3 = 0, \tag{274}$$

$$b_3 + 2a_2 = 0, \qquad b_4 = -a_3 \tag{275}$$

As $\delta \neq 0$ without loss of generality we may assume $a_2 = 0$ and $a_3 = 1$, solving the above we have the following system of first order differential equations

$$Zw' = \epsilon Z + (1 - 2c\epsilon)w + Zw^2 v, \tag{276}$$

$$Zv' = 2 - (2 - 2c\epsilon)v - Zwv^2 \tag{277}$$

with $\epsilon = \pm 1$ from the relation with a_0. Rearranging the second equation for w

$$w = \frac{v'}{v^2} + \frac{2}{Zv^2} - \frac{(2 - 2c\epsilon)}{Zv} \tag{278}$$

and substituting into second equation (277) for w

$$v'' = \frac{v'^2}{v} - \frac{v'}{z} - \epsilon v^2 + \frac{4(1 - c\epsilon)}{Z^2} - \frac{4}{Z^2 v} \tag{279}$$

Define the following function by making the substitution $v = \frac{2\tilde{w}}{Z}$

$$\tilde{w}(Z, \tilde{\beta}) = \frac{1}{2} w^{-2}(Z - \epsilon Z w' + (\epsilon - 2c)w) \tag{280}$$

with $\tilde{\beta} = \beta - 2\epsilon$, and we now have a BT for P_{III}.

Starting with a simple seed solution with $c = 0$, given by $P = w = (\frac{1}{2}Z)^{\frac{1}{3}}$, and then applying to the BT (280) this will always produce solutions rational in $Z^{\frac{1}{3}}$ [20]. As the solutions have to be rational in $Z^{\frac{1}{3}}$ we make the substitution $Z = \tau^3$ into (269).

$$w'' = \frac{w'^2}{w} - \frac{w'}{\tau} + 9\tau \left(2w^2 + 2c - \frac{\tau^3}{w} \right) \tag{281}$$

where $w' = \frac{dw}{d\tau}$. The choice $c = i\epsilon$ gives a sequence of algebraic solutions w_i, satisfying

$$w_{i+1} = \frac{1}{2w_i^2} \left(\tau^3 - \frac{1}{3} \epsilon \tau \frac{dw_i}{d\tau} + (\epsilon - 2c) w_i \right), \tag{282}$$

and then the asymptotic behavior for large τ is

$$\frac{1}{2}\lambda^{-2}\tau^{-2}(\tau^3 + O(\tau)), \tag{283}$$

$$= \frac{\tau}{2\lambda^2} + O(\tau^{-1}) = \lambda\tau \tag{284}$$

for all i, $w_i(\tau) \sim \lambda\tau$ as $\tau \to \infty$. The seed solution is $w_0 = \lambda\tau$ with $\lambda^3 = \frac{1}{2}$ by applying the BT recursively, we find a sequence of algebraic solutions, w_i, for $i = 1, 2, 3...$ and $c = -i\epsilon$

$$c = -\epsilon, \qquad w_1 = \frac{2\epsilon\lambda + 3\tau^2}{6\lambda^2\tau} \tag{285}$$

$$c = -2\epsilon, \qquad w_2 = \frac{(9\lambda\tau^5 + 10\tau + 24\lambda^2\tau^3\epsilon)}{(2\lambda\epsilon + 3\tau^2)^2} \tag{286}$$

$$c = -3\epsilon, \qquad w_3 = \frac{(243\tau^{10} + 1782\lambda\epsilon\tau^8 + 5400\lambda^2\tau^6 + 3960\epsilon\tau^4 + 2520\lambda\tau^2 + 560\lambda^2\epsilon)}{6\tau(24\lambda^2\epsilon\tau^2 + 9\lambda\tau^4 + 10)^2} \tag{287}$$

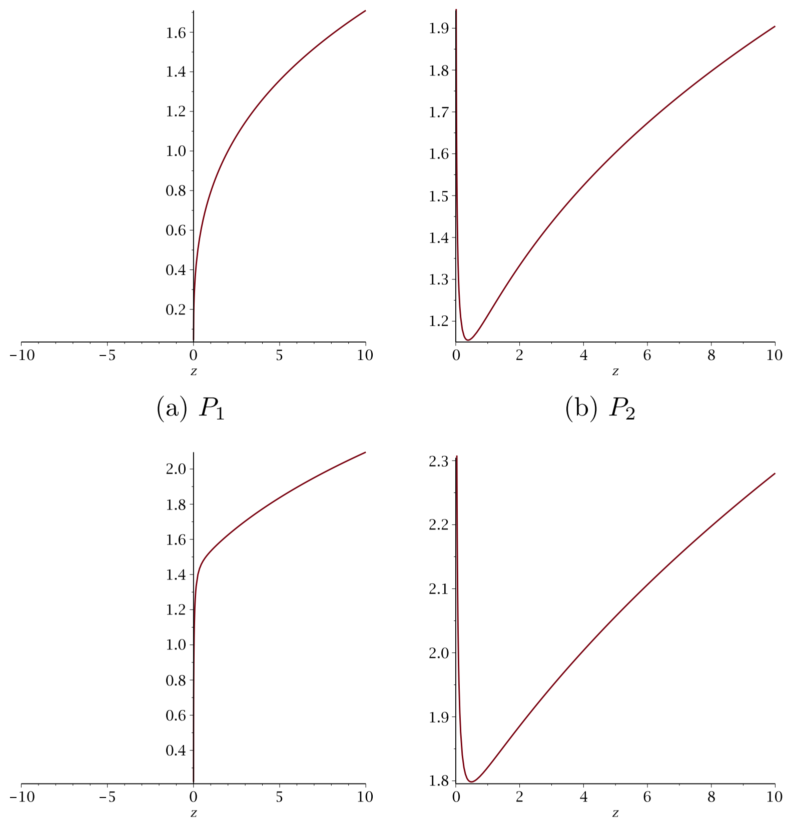
Plotting these with respect to Z with $\epsilon = 1$ as in Figure 8. Conversley at $\epsilon = -1$ as in Figure 9 we can see two asymptotes, one vertical and the other horizontal. With Camassa-Holm being quite difficult to solve numerically, it is interesting to note that we are able to use these Painlevé solutions to find solutions in terms of Camassa-Holm's original coordinates $u(x, t)$. We are able to do this by working back though the calculations, applying the same transformations. However for most of the solutions we are only able to write them down implicitly, in these case we are able to plot the solutions parametrically.

The seed solution for $c = 0$ gives

$$P(Z) = \lambda Z^{\frac{1}{3}}, \qquad U(Z) = \frac{Z^{\frac{2}{3}}}{2\lambda} \tag{288}$$

which satisfy (227) and (246), so

$$p(X,T) = \lambda X^{\frac{1}{3}}T^{-\frac{1}{3}} \qquad u(X,T) = \frac{X^{\frac{2}{3}}}{2\lambda T^{\frac{2}{3}}} \tag{289}$$



(a) P_1 (b) P_2

(c) P_3 (d) P_4

Figure 8: First four algebraic solutions plotted against Z, where $P_i(Z) = w_i(Z^{\frac{1}{3}})$

are corresponding solutions of (237) and (238). To find the solution in terms of the original coordinates (x, t) we apply the hodograph transformation (259)

$$\int \frac{1}{P(Z)} dZ = \int dz \qquad Z = (2^{-\frac{2}{3}} \frac{2}{3} z)^{\frac{3}{2}} \tag{290}$$

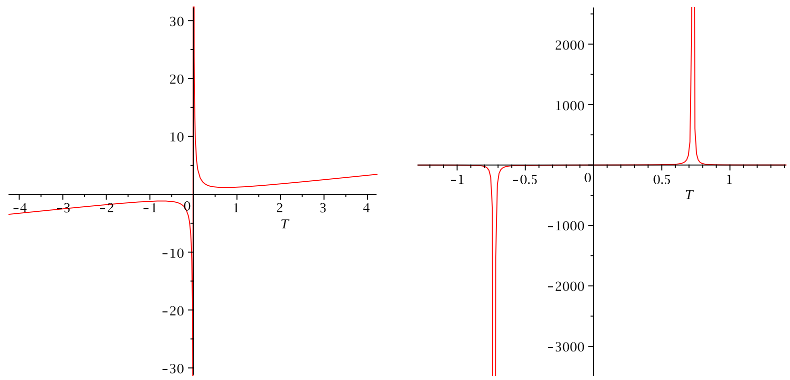
Substituting Z into (288)

$$p(x,t) = \frac{1}{\sqrt{3}} \left(\frac{x}{t} \right)^{\frac{1}{2}}, \qquad u(x,t) = \frac{x}{3t} \tag{291}$$

so

$$p^2 = \frac{1}{3} \left(\frac{x}{t} \right) = u - u_{xx} \tag{292}$$

This is a nice example, but unusual in the fact we are able to explicitly write z as a function of Z. We can also write the solution of w_1 in terms of Z to get various



(a) P_1 (b) P_2

Figure 9: First two algebraic solutions plotted against Z, where $P_i(Z) = w_i(Z^{\frac{1}{3}})$

equations in X and T. For example the solution (286) gives

$$p(X,T) = \frac{(2\epsilon\lambda + 3X^{\frac{2}{3}}T^{\frac{1}{3}})}{6\lambda^2 X^{\frac{1}{3}}T^{\frac{2}{3}}} \tag{293}$$

$$u(X,T) = \frac{(6\lambda X^{\frac{2}{3}}\epsilon T^{\frac{2}{3}} + 4\lambda^2 T^{\frac{1}{3}} + 3TX^{\frac{4}{3}})}{2T^{\frac{4}{3}}\lambda(2\epsilon\lambda + 3X^{\frac{2}{3}}T^{\frac{1}{3}})} \tag{294}$$

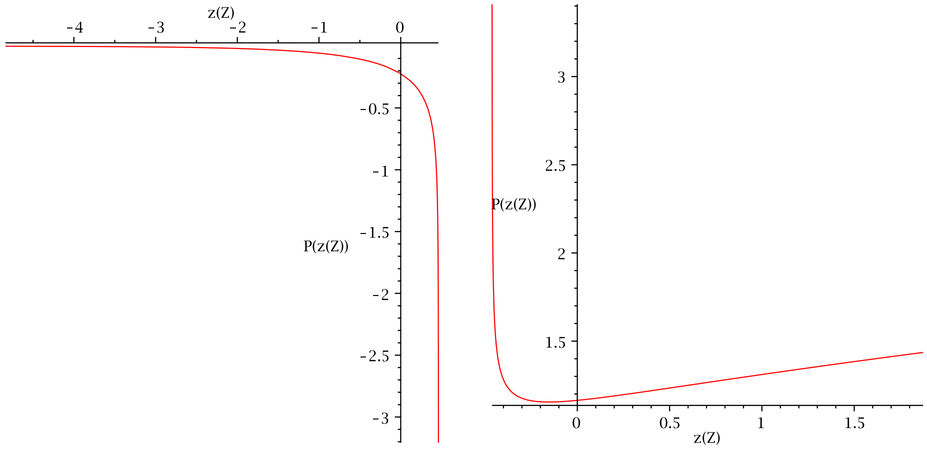
To find the corresponding solutions of Camassa-Holm, we write them down implicitly. So for w_2

$$z = \int \frac{1}{w_2(Z)}dZ = \int \frac{3\tau^2}{P(\tau^3)}d\tau \tag{295}$$

Below we have a table of results for both w_2 and w_3 and write down z explicitly.

w_i	c	z
w_2	1	$3\lambda^2 Z^{\frac{2}{3}} + 2\lambda^3 \ln(-3Z^{\frac{2}{3}} + 2\lambda)$,
	-1	$3\lambda^2 Z^{\frac{2}{3}} - 2\lambda^3 \ln(3Z^{\frac{2}{3}} + 2\lambda)$,
w_3	1	$-2\lambda^2 Z^{\frac{2}{3}} + \frac{3}{4}\lambda Z^{\frac{4}{3}} + \frac{5}{3}\ln(2\lambda - 3Z^{\frac{2}{3}}) - \frac{10}{3}\ln(2\lambda - 3Z^{\frac{2}{3}})\lambda^3 + \frac{10}{3}\frac{\lambda}{2\lambda - 3Z^{\frac{2}{3}}} - \frac{4\lambda^4}{2\lambda - 3Z^{\frac{2}{3}}}$
	-1	$2\lambda^2 Z^{\frac{2}{3}} + \frac{3}{4}\lambda Z^{\frac{4}{3}} + \frac{5}{3}\ln(2\lambda - 3Z^{\frac{2}{3}}) - \frac{10}{3}\ln(2\lambda - 3Z^{\frac{2}{3}})\lambda^3 + \frac{10}{3}\frac{\lambda}{2\lambda - 3Z^{\frac{2}{3}}} - \frac{4\lambda^4}{2\lambda - 3Z^{\frac{2}{3}}}$

Here we plot some solutions of $P(z(Z))$



(a) $P_2(z(Z))$ with $c = 1$ (b) $P_2(z(Z))$ with $c = -1$

Figure 10: Two parametric plots of P_2

3.2 Reduction of the Degasperis-Procesi Equation

In the previous section we were able to transform, via reductions, the Camassa-Holm equation to the third Painlevé equation. Here we do something similar for the Degasperis-Procesi equation [32, 31]

$$u_t - u_{xxt} + 4uu_x - 3u_xu_{xx} - uu_{xxx} = 0 \tag{296}$$

to reduce it to a particular case of P_{III}, but different from the case found above. The Lax pair for Degasperis-Procesi was discovered by [31]

$$\Psi_{xxx} = \Psi_x + \lambda m\Psi, \tag{297}$$

$$\Psi_t = \lambda^{-1}\Psi_{xx} - u\Psi_x + u_x\Psi, \tag{298}$$

and is important when looking at the relationship between the PDE and Kaup-Kuperschmidt hierarchy.

Theorem 3.2.1. The Degasperis-Procesi equation can be transformed to the third Painlevé equation via two different reduction schemes.

Introducing the diagram for the section as before, we shall follow the arrows in the same order as in the Camassa-Holm case.

$$\begin{array}{ccc} \text{Degasperis-Procesi} & \xrightarrow[\text{Sim. Red}]{3.2.3} & \text{3rd Order Equation} \\ {\scriptstyle 3.2.1}\Big\downarrow{\scriptstyle \text{Rec.Trans}} & & {\scriptstyle 3.2.4}\Big\downarrow{\scriptstyle \text{Hodograph}} \\ \text{Negative flow KK} & \xrightarrow[\text{Sim.Red}]{3.2.2} & \text{2nd Order Equation, } P_{III} \end{array}$$

Figure 11: Degasperis-Procesi reductions

3.2.1 Degasperis-Procesi to Negative Kaup-Kuperschmidt

Proposition 3.2.1. Degasperis-Procesi, via a reciprocal transformation, can be written as the negative flow of the Kaup-Kuperschmidt hierarchy.

The connection between the Degasperis-Procesi equation (296) and the negative flow of the Kaup-Kuperschmidt hierarchy via a reciprocal transformation has been known since [31]. We now show how they are related here.

Proof. Introducing a momentum variable m in the same way as for Camassa-Holm, we may write (296) as the system

$$m_t + um_x + 3mu_x = 0, \qquad m = u - u_{xx}. \tag{299}$$

Then making a change of variable

$$p^3 = u - u_{xx} \tag{300}$$

and substituting (300) into (299) gives

$$p_t = -(up)_x. \tag{301}$$

This is a conservation law which we can use to define a reciprocal transformation. Therefore a reciprocal transformation for Degasperis-Procesi is given by

$$dX = pdx - updt, \qquad\qquad dT = dt, \tag{302}$$

and we note that it is in the same form as the Camassa-Holm reciprocal transformation (231). Using this we write the conservation law (301) in terms of X and T, to find

$$(p^{-1})_T = u_X \tag{303}$$

and as in the Camassa-Holm case, the reciprocal transformation (302) maps the conservation law (301) to the conservation law (303). A relationship between p and u is found from the second equation of (299), giving

$$p^3 = u - u_{xx} = u - p\frac{\partial}{\partial X}\left(p\frac{\partial}{\partial X}u\right) \tag{304}$$

Replacing u_X by $(p^{-1})_T$ in the above produces

$$u = p^3 - p_{XT} + \frac{p_X p_T}{p} \tag{305}$$

Then eliminating u to rewrite (303) in terms of p alone gives

$$(p^{-1})_T = -(p^3 + p(\log p)_{XT})_X. \tag{306}$$

If we have

$$V_T = \frac{3}{4}(p^2)_X \tag{307}$$

then it is equivalent to (306) when

$$V = \frac{p_X^2 - 2pp_{XX} - 1}{4p^2} \tag{308}$$

The Lax pair for this system is the reciprocally transformed Lax pair (297), spatial component given by

$$\Psi_{XXX} + 4V\Psi_X + (2V_X - \lambda)\Psi = 0 \tag{309}$$

applying the transformation to the time component (298) results in

$$\Psi_T + \lambda^{-1}(p^2\Psi_{XX} - pp_X\Psi_X - (pp_{XX} - p_X^2 + \frac{2}{3})\Psi) = 0 \tag{310}$$

showing that (309) and (310) is equivalent to the first negative flow of Kaup-Kuperschmidt. Differentiating (308) by X we have

$$p_{XXX} + 2V_X p + 4V p_X = 0 \tag{311}$$

which is equivalent to (309) and also is the Lax operator for Kaup-Kuperschmidt.

□

3.2.2 Reduction of Kaup-Kuperschmidt to P_{III}

Here we focus on the bottom arrow of the diagram, implementing the same technique as for Camassa-Holm.

Proposition 3.2.2. By an appropriate similarity reduction, the first negative flow of Kaup-Kuperschmidt can be transformed to the third Painlevé equation.

Proof. To reduce the reciprocally transformed system to P_{III} we take the following variables to make the reduction:

$$p = T^{-\frac{1}{3}}P(Z), \qquad u = T^{-1}U(Z), \qquad Z = XT^{\frac{1}{3}} \tag{312}$$

This is another scaling similarity reduction, which from (305) produces

$$U = P^3 - \frac{1}{3}ZP'' - \frac{1}{3}P' + \frac{ZP'^2}{3P}. \tag{313}$$

Similarly as before for Camassa-Holm, (306) reduces to a third order ODE for $P(Z)$, which we can integrate once to get a second order ODE. The conservation law (303) reduces to

$$\frac{1}{3}\frac{d}{dZ}(ZP^{-1}) = \frac{dU}{dZ}, \tag{314}$$

and integrating gives

$$U = \frac{1}{3}\frac{Z}{P} - c \tag{315}$$

with c an integration constant. Now unlike for Camassa-Holm, we do not arrive at the standard form of P_{III} straight away. Comparing the two formulae (313) and (315) for U we get

$$P'' = \frac{P'^2}{P} - \frac{P'}{Z} + \frac{3P^3}{Z} - \frac{1}{P} + \frac{3c}{Z}. \tag{316}$$

To write as the normal form of P_{III} we apply a change of variables

$$Z = \zeta^m, \qquad P(Z) = \zeta^n \pi(\zeta) \tag{317}$$

Substituting into (316)

$$\frac{1}{m^2}\frac{d}{d\zeta}\zeta\left(\frac{d}{d\zeta}\log\pi\right) = 3\zeta^{2n+m-1}\pi^2 + \frac{\zeta^{2m-2n-1}}{\pi^2} - \frac{3c\zeta^{m-n-1}}{\pi} \tag{318}$$

We require that the above equation (318) should take the form

$$\frac{d}{d\zeta}\zeta\left(\frac{d}{d\zeta}\log\pi\right) = \alpha\pi + \frac{\beta}{\pi} + \zeta\left(\gamma\pi^2 + \frac{\delta}{\pi^2}\right). \tag{319}$$

Then we find

$$2m - 2n - 1 = 1, \qquad m - n - 1 = 0, \qquad 2n + m - 1 = 0 \tag{320}$$

so $m = \frac{4}{3}$ and $n = \frac{1}{3}$. Therefore the change of variables we need to apply are the following

$$Z = \zeta^{\frac{4}{3}}, \qquad P(Z) = \zeta^{\frac{1}{3}}\pi(\zeta). \tag{321}$$

Substituting m and n into (318) reveals the following 2nd order ODE

$$\pi_{\zeta\zeta} = \frac{\pi_\zeta^2}{\pi} - \frac{\pi_\zeta}{\zeta} + \frac{16c}{3\zeta} + \frac{16\pi^3}{3} - \frac{16}{9\pi} \tag{322}$$

related to the standard form of P_{III} (227), by

$$\alpha = 0, \qquad \beta = \frac{16c}{3}, \qquad \gamma = \frac{16}{3}, \qquad \delta = -\frac{16}{9} \tag{323}$$

showing Proposition 3.2.2 to be true. □

3.2.3 Degasperis-Procesi to 3rd order ODE

Having shown one method of transforming Degasperis-Procesi to P_{III}, we now apply similar techniques but in reverse order as in the diagram for the section.

Proposition 3.2.3. Applying a similarity reduction to Degasperis-Procesi reduces the PDE to a 3rd order ODE

Proof. Applying a similarity reduction to the Degasperis-Procesi conservation law (301) with (300), again misusing notation with the similarity variable

$$z = x + c \log t \tag{324}$$

and scaling similarity reduction

$$p = t^{-\frac{1}{3}} P(z), \qquad u = t^{-1} U(z), \tag{325}$$

from (300) we obtain

$$P^3 = U - \ddot{U} \tag{326}$$

where dot represents $\frac{d}{dz}$. Substituting the reduction (325) into the conservation law (301) gives

$$\frac{d}{dz}[(U + c)P] = \frac{1}{3}P \tag{327}$$

Using $P^3 = U - \ddot{U}$ to eliminate P^3 and $P^2\dot{P}$ gives a third order equation in $U(z)$.

$$\left(\dot{U} - \frac{1}{3}\right)(U - \ddot{U}) + \frac{1}{3}(c + U)(\dot{U} - \dddot{U}) = 0 \tag{328}$$

□

3.2.4 3rd Order ODE to P_{III}

Similarly to the case of Camassa-Holm, we find equations for U from (326) and (327).

Proposition 3.2.4. The 3rd order ODE (328) that arises as a reduction of Degasperis-Procesi can be written as a third Painlevé equation by means of a hodograph transformation.

Proof. Using the hodograph transformation

$$dZ = Pdz \tag{329}$$

we note

$$\ddot{U} = \left(P\frac{d}{dZ}\right)^2 U, \qquad P^3 = U - \left(P\frac{d}{dZ}\right)^2 U, \tag{330}$$

Rearranging for U gives

$$U = P^3 + \left(P\frac{d}{dZ}\right)^2 U, \tag{331}$$

So now (327) gives

$$U = \frac{Z}{3P} - c, \tag{332}$$

and comparing with (331) we find

$$\frac{Z}{3P} - c = P^3 - \frac{1}{3}P\frac{d}{dZ}\left(Z\frac{d}{dZ}\log P\right). \tag{333}$$

This gives us

$$\frac{d}{dZ}\left(Z\frac{d}{dZ}\log P\right) = -\frac{Z}{P^2} + \frac{3c}{P} + 3P^2, \tag{334}$$

which is equivalent to (316). Applying the same change of variables as above we get P_{III}. □

3.2.5 Bäcklund Transformations and Solutions

The category of solutions for this version of P_{III} has a fixed value of $\alpha = 0$ with a free parameter β. Using the work of Gromak [49] and Milne [77]. The general solution we will be applying comes from Milne [77], and is given here

$$\tilde{\pi} = \pi\left(1 + \frac{(2 + \beta(-\delta)^{-\frac{1}{2}} + \alpha\gamma^{-\frac{1}{2}})\pi}{\zeta\pi_\zeta + \gamma^{\frac{1}{2}}\zeta\pi^2 + (-\delta)^{\frac{1}{2}}\zeta - (1 + \beta(-\delta)^{-\frac{1}{2}})\pi}\right) \tag{335}$$

Incrementing $(\alpha, \beta, \gamma, \delta)$ in the following manner

$$\tilde{\alpha} = -(2 + \beta(-\delta)^{-\frac{1}{2}})\gamma^{\frac{1}{2}}, \qquad \tilde{\beta} = -(2 + \alpha\gamma^{-\frac{1}{2}})(-\delta)^{\frac{1}{2}}, \tag{336}$$

$$\tilde{\gamma}^{\frac{1}{2}} = \gamma^{\frac{1}{2}}, \qquad (-\tilde{\delta})^{\frac{1}{2}} = (-\delta)^{\frac{1}{2}} \tag{337}$$

For the Degasperis-Procesi P_{III}, due to the square root terms in γ and δ, the transformation (335) results in the following four transformations:

$$T_1 : \tilde{\pi}_1 = \frac{\pi(12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 + 16\zeta + 12\pi + 3\sqrt{3}\alpha\pi)}{12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 + 16\zeta - 12\pi - 9\pi\beta}, \tag{338}$$

$$\tilde{\alpha}_1 = -\frac{1}{\sqrt{3}}(8+3\beta), \qquad \tilde{\beta}_1 = -\frac{1}{3}(8+\sqrt{3}\alpha) \tag{339}$$

$$T_2 : \tilde{\pi}_2 = \frac{\pi(-12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 - 16\zeta - 12\pi + 3\sqrt{3}\alpha\pi)}{-12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 - 16\zeta + 12\pi + 9\pi\beta}, \tag{340}$$

$$\tilde{\alpha}_2 = \frac{1}{\sqrt{3}}(8+3\beta), \qquad \tilde{\beta}_2 = \frac{1}{3}(-8+\sqrt{3}\alpha) \tag{341}$$

$$T_3 : \tilde{\pi}_3 = \frac{\pi(12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 - 16\zeta + 12\pi + 3\sqrt{3}\alpha\pi)}{12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 - 16\zeta - 12\pi + 9\pi\beta}, \tag{342}$$

$$\tilde{\alpha}_3 = \frac{1}{\sqrt{3}}(-8+3\beta), \qquad \tilde{\beta}_3 = \frac{1}{3}(8+\sqrt{3}\alpha) \tag{343}$$

$$T_4 : \tilde{\pi}_4 = \frac{\pi(-12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 + 16\zeta - 12\pi + 3\sqrt{3}\alpha\pi)}{-12\zeta\pi_\zeta + 16\sqrt{3}\zeta\pi^2 + 16\zeta + 12\pi - 9\pi\beta}, \tag{344}$$

$$\tilde{\alpha}_4 = \frac{1}{\sqrt{3}}(8-3\beta), \qquad \tilde{\beta}_4 = \frac{1}{3}(8-\sqrt{3}\alpha). \tag{345}$$

We are interested in fixing α, as for the Degasperis-Procesi Painlevé equation $\alpha = 0$ and β is a free variable. To do this, we must apply, in the correct order, a sequence of transformations. To achieve this, under the sequence $T_3T_4T_1T_2$, we find α and β are transformed in the following manner:

$$(\alpha,\beta) \to \left(\frac{1}{\sqrt{3}}(8+3\beta), \frac{1}{3}(-8+\sqrt{3}\alpha)\right) \to \left(-\alpha, -\frac{1}{3}(16+3\beta)\right) \tag{346}$$

$$\to \left(\frac{1}{\sqrt{3}}(24+3\beta), \frac{1}{3}(8+\sqrt{3}\alpha)\right) \to \left(\alpha, \beta + \frac{32}{3}\right) \tag{347}$$

These transformations are particularly messy, however Gromak wrote down a scaling transformation T_5 which meant for any P_{III} equation where ϕ is a solution, then

$$T_5(\sigma_1,\sigma_2) : \phi \mapsto \tilde{\phi}(Z, \alpha\sigma_1\sigma_2, \beta\sigma_1^{-1}\sigma_2, \gamma\sigma_1^2\sigma_2^2, \delta\sigma_1^{-2}\sigma_2^2, \tag{348}$$

$$:= \sigma^{-1}\phi(\sigma_2 Z, \alpha, \beta, \gamma, \delta) \tag{349}$$

with

$$\sigma_1 = \frac{1}{\sigma_2\sqrt{\gamma}}, \qquad \sigma_2 = \left(-\frac{1}{\gamma\delta}\right)^{\frac{1}{4}} \tag{350}$$

is also a solution. We are able to use transformation T_5 to find additional solutions based upon an initial application of the transformation sequence to a seed solution.

3.2.6 Rational Solutions

To use the above transformations to find a sequence of solutions, we must first find a seed solution. For this we set $c = 0$ which implies $\beta = 0$, and we find

$$\pi_0 = \left(\frac{1}{3}\right)^{\frac{1}{4}} \tag{351}$$

which satisfies (322) for $(\alpha, \beta, \gamma, \delta) = (0, 0, \frac{16}{3}, -\frac{16}{9})$.

$$\pi_1\left(\frac{8}{\sqrt{3}}, -\frac{8}{3}\right) = -\frac{1}{3}3^{\frac{3}{4}}, \tag{352}$$

$$\pi_2\left(0, -\frac{16}{3}\right) = -\frac{3^{\frac{3}{4}}(-8\zeta + 3^{\frac{7}{4}})}{3(-8\zeta + 3^{\frac{3}{4}})}, \tag{353}$$

$$\pi_3\left(\frac{24}{\sqrt{3}}, \frac{8}{3}\right) = -\frac{(72\zeta\sqrt{3}\zeta + 512\zeta^3 - 192\zeta^2 3^{\frac{3}{4}} + 27(3^{\frac{1}{4}}))(-8\zeta + 3(3^{\frac{3}{4}})3^{\frac{3}{4}}}{3(-8\zeta + 3^{\frac{3}{4}})(648\zeta\sqrt{3} + 512\zeta^3 - 576\zeta^2(3^{\frac{3}{4}}) - 135(3^{\frac{1}{4}}))}, \tag{354}$$

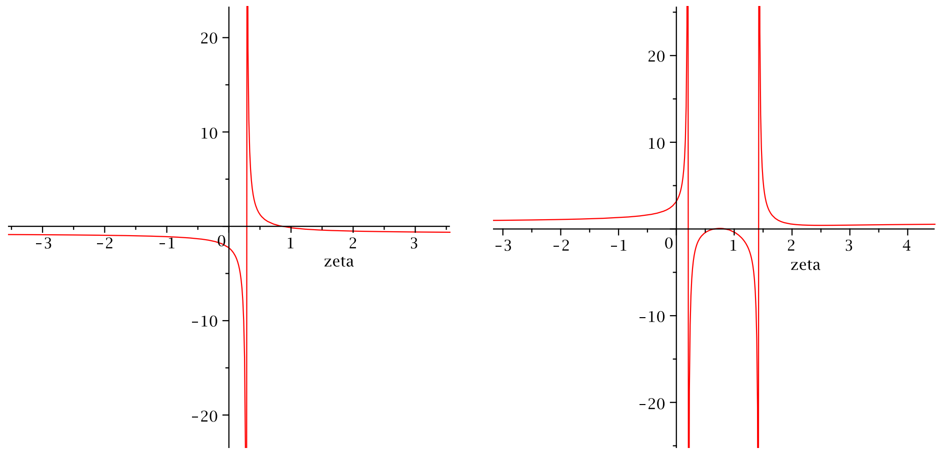
Plotting the solutions at π_2 and π_4 in Figure 12 Plotting the solutions at $\pi_6(0, -16)$ and $\pi_8(0, \frac{64}{3})$ in Figure 14. We also have a set of solutions for $\gamma = 1$ and $\delta = -1$ which are directly related to the solutions above by using the scaling transformation (348). Assume $\pi_0 = 1$ at $(\alpha, \beta) = (0, 0)$, applying the transformations as above we get the following solutions

$$\pi_1(\zeta; 2, -2, 1, -1) = -1, \tag{355}$$

$$\pi_2(\zeta; 0, -4, 1, -1) = -\frac{2\zeta - 3}{2x - 1}, \tag{356}$$

$$\pi_3(\zeta; 6, 2, 1, -1) = -\frac{-48\zeta^3 + 48\zeta^2 + 16\zeta^4 - 12\zeta - 9}{(2\zeta - 1)(-36\zeta^2 + 54\zeta + 8\zeta^3 - 15)}, \tag{357}$$

$$\pi_4(\zeta; 0, 8, 1, -1) = \frac{16\zeta^4 - 144\zeta^3 + 480\zeta^2 - 660\zeta + 315}{(2\zeta - 5)(-36\zeta^2 + 54\zeta + 8\zeta^3 - 15)} \tag{358}$$



(a) π_2 (b) π_4

Figure 12: Two rational solutions of π

Solution at $\pi_8 = (0, 16)$

$$\frac{AB}{CD} \tag{359}$$

with

$$A =(1024x^{10} - 46080x^9 + 933120x^8 - 11128320x^7 + 86002560x^6 - 447068160x^5 \tag{360}$$

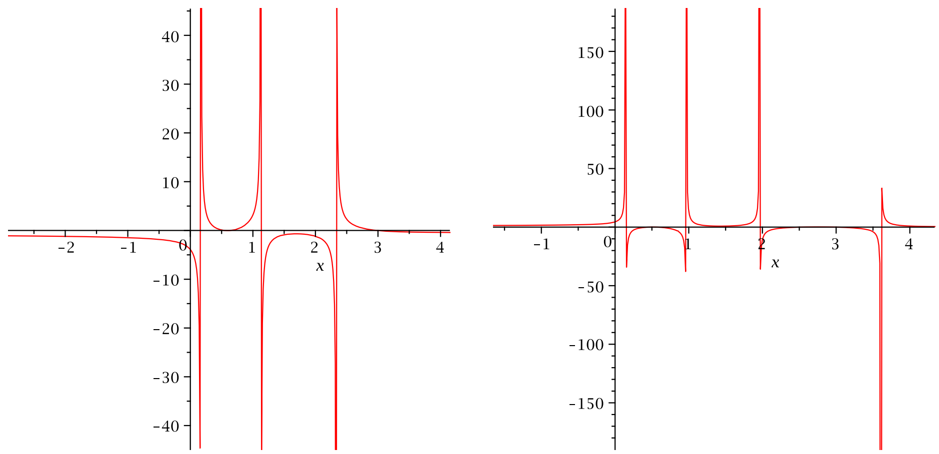
$$+ 1571724000x^4 - 3657376800x^3 + 5327021700x^2 - 4305401100x + 1404728325) \tag{361}$$

$$B =(64x^6 - 1344x^5 + 11760x^4 - 53760x^3 + 132300x^2 - 162540x + 72765) \tag{362}$$

$$C =(64x^6 - 1728x^5 + 19440x^4 - 115200x^3 + 374220x^2 - 632700x + 405405) \tag{363}$$

$$D =(1024x^{10} - 35840x^9 + 564480x^8 - 5214720x^7 + 30952320x^6 - 121927680x^5 \tag{364}$$

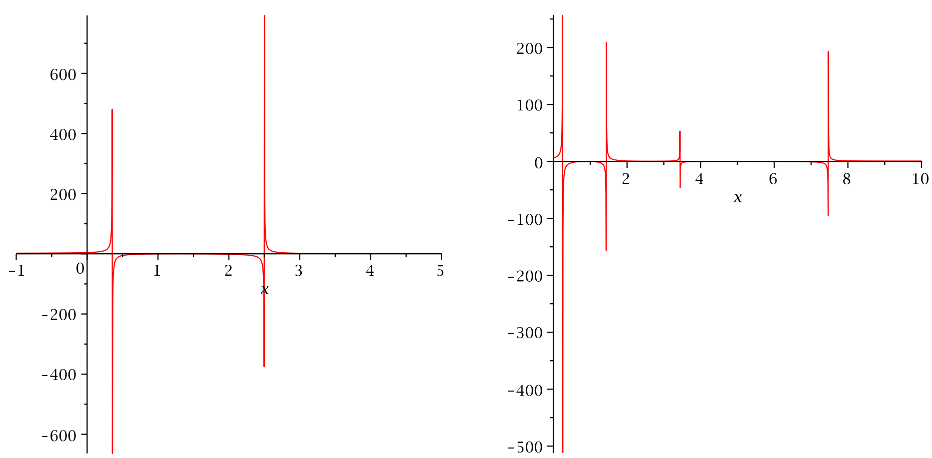
$$+ 318578400x^4 - 535096800x^3 + 537005700x^2 - 275051700x + 42567525) \tag{365}$$



(a) π_6 (b) π_8

Figure 13: Two rational solutions of π

Plotting the solutions at π_4 and π_8 All these figures seem to show vertical and



(a) π_4 (b) π_8

Figure 14: Two rational solutions of π

horizontal asymptotes.

3.2.7 Special Function Solutions

So unlike the Camassa-Holm P_{III} we also have special function solutions for Degasperis-Procesi P_{III}, due to (322) satisfying the one-parameter family condition,

$$2 + \alpha\gamma^{-\frac{1}{2}} + \beta(-\delta)^{-\frac{1}{2}} = 0. \tag{366}$$

This is only satisfied when $\beta = \mp\frac{8}{3}$, we are then able to write P_{III} as the following Riccati equation

$$\pi_\zeta = -\gamma^{\frac{1}{2}}\pi^2 - (\alpha\gamma^{-\frac{1}{2}} + 1)\frac{\pi}{\zeta} - (-\delta)^{\frac{1}{2}}, \tag{367}$$

which for (322) is

$$\pi_\zeta = \mp\bigg(\frac{4}{\sqrt{3}}\bigg)\pi^2 - \frac{\pi}{\zeta}\mp\bigg(\frac{4}{3}\bigg). \tag{368}$$

It is well known that the riccati equation (367) can be linearised via

$$\pi = \gamma^{-\frac{1}{2}}\frac{\phi_\zeta(\zeta)}{\phi(\zeta)} = \pm\frac{\sqrt{3}\phi_\zeta(\zeta)}{4\phi(\zeta)} \tag{369}$$

which gives

$$\phi_{\zeta\zeta} + \frac{\phi_\zeta}{\zeta} + \bigg(\pm\frac{4}{\sqrt{3}}\bigg)\bigg(\pm\frac{4}{3}\bigg)\phi = 0 \tag{370}$$

With the choice of signs for γ and δ we have the choice of four seed solutions,

$$\pi_0 = \gamma^{-\frac{1}{2}}\frac{\phi_\zeta}{\phi} \tag{371}$$

as we also have a choice of sign for β. The first seed solution is

$$\pi_0^1 = \frac{\sqrt{3}\phi_\zeta}{4\phi}, \tag{372}$$

which satisfies P_{III} at $(0, \pm\frac{8}{3}, \frac{16}{3}, -\frac{16}{9})$ and the second

$$\pi_0^2 = -\frac{\sqrt{3}\phi_\zeta}{4\phi}, \tag{373}$$

which also satisfies P_{III} at $(0, \pm\frac{8}{3}, \frac{16}{3}, -\frac{16}{9})$. Taking (372), and writing in terms of Bessel functions we find

$$\pi_0^1 = -\frac{AJ_1(\psi) + BY_1(\psi)}{3^{\frac{1}{4}}(AJ_0(\psi) + BY_0(\psi))} \tag{374}$$

where $\psi = \frac{4}{3^{\frac{3}{4}}}\zeta$. J and Y represent Bessel functions of the first and second kind respectively [33]. Unlike the rational solutions we will not be following the same order of transformations, if we apply T_2 to (374) we simply recover the same seed solution once again. First we apply T_3 in the same manner as before, and find

$$\pi_1^1 = \frac{2\zeta(J_1(\psi) + Y_1(\psi))}{3^{\frac{1}{4}}(-2J_0(\psi)\zeta - 2Y_0(\psi)\zeta + 3^{\frac{3}{4}}Y_1(\psi) + 3^{\frac{3}{4}}J_1(\psi))} \tag{375}$$

which satisfies P_{III} for $(\alpha, \beta, \gamma, \delta) = (-\frac{16}{\sqrt{3}}, \frac{8}{3}, \frac{16}{3}, \frac{16}{9})$. To find another solution to (322) we apply transformation T_4 to recover the case $\alpha = 0$

$$\pi_1^2 = \frac{A}{B} \tag{376}$$

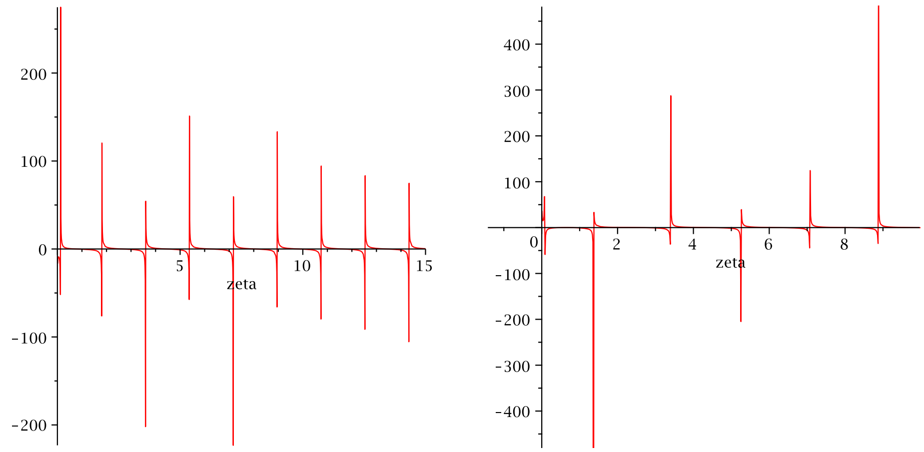
with

$$\begin{aligned} A =& (6\sqrt{3}J_1Y_1 - 8\zeta^2(J_0Y_0 + J_1Y_1) - 4\zeta^2(J_0 + Y_1^2 + Y_0^2 + J_1) + 3\sqrt{3}(J_1^2 + Y_1^2))(J_1 + Y_1) \\ B =& (-2\zeta(J_0 + Y_0) + 3^{\frac{3}{4}}(Y_1 + J_1))(-4\zeta(J_0Y_0 + J_1Y_1) + 3^{\frac{3}{4}}(J_1Y_0 + Y_0Y_1 + J_0J_1 + J_0Y_1) \\ &- 2\zeta(J_0^2 + Y_0^2 + Y_1^2 + J_1^2)) \end{aligned}$$

which is a solution to (322) with $(\alpha, \beta, \gamma, \delta) = (0, 8, \frac{16}{3}, \frac{16}{9})$. Plotting both π_1^1 and π_1^2

3.2.8 Original Coordinates

Having found solutions for the Painlevé equation, we now wish to find solutions to the original Degasperis-Procesi. This is achieved by applying the inverse of the transformations that were required to find P_{III}. Starting with the rational seed



(a) π_1^1 (b) π_1^2

Figure 15: Two special function solutions of π that satisfy Degasperis-Procesi P_{III}

solution (3.2.6)

$$P_0(\zeta) = \zeta^{\frac{1}{3}}\left(\frac{1}{3}\right)^{\frac{1}{4}}, \tag{377}$$

using the change of variable

$$P_0(Z) = Z^{\frac{1}{4}}\left(\frac{1}{3}\right)^{\frac{1}{4}}, \qquad U_0(Z) = \frac{1}{3}Z^{\frac{3}{4}}3^{\frac{1}{4}} \tag{378}$$

and then from

$$\int Z^{-\frac{1}{4}}\left(\frac{1}{3}\right)^{-\frac{1}{4}} dZ = \int dz \tag{379}$$

we know

$$Z = \frac{4^{\frac{2}{3}}}{16}(3^{\frac{3}{4}}z)^{\frac{4}{3}} \tag{380}$$

which leaves us with

$$u(x,t) = \frac{x}{4t} \tag{381}$$

as a solution to the original Degasperis-Procesi PDE. Applying the same method to the second valid solution π_2, we cannot write down an explicit solution but we can write down an implicit relation. The solution in terms of $P(Z)$ is

$$P(Z) = Z^{\frac{1}{4}}\left(- \frac{3^{\frac{3}{4}}(-8Z^{\frac{3}{4}} + 3^{\frac{7}{4}})}{3(-8Z^{\frac{3}{4}} + 3^{\frac{3}{4}})}\right) \tag{382}$$

integrating

$$-\frac{4}{3}Z^{\frac{3}{4}}3^{\frac{1}{4}} - \ln(-8Z^{\frac{3}{4}} + 3^{\frac{7}{3}}) = z - c \tag{383}$$

where the constant $c = -1$, as

$$\beta = -\frac{16}{3} = \frac{16c}{3}. \tag{384}$$

we can use (383) to write down implicit solutions to the PDE.

3.3 The b-family of Equations

$$\begin{array}{ccc} b\text{-family equation in (x,t)} & \xrightarrow[\text{Sim.Red}]{3.3.3} & \text{3rd order eqn in } z \\ {\scriptstyle 3.3.1}\downarrow{\scriptstyle \text{Rec.Trans}} & & {\scriptstyle 3.3.4}\downarrow{\scriptstyle \text{Hodograph}} \\ \text{Equation in (X,T)} & \xrightarrow[\text{Sim.Red}]{3.3.2} & \text{2nd order equation in } Z \end{array}$$

Figure 16: b-family reductions

In [31] what is known as the b-family was introduced

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx} \tag{385}$$

where $b = 2$ is equivalent to Camassa-Holm and $b = 3$ equivalent to Degasperis-Procesi. These are the only integrable members of this family as shown in [76, 31]. Numerical solutions of the b-family have been studied in [54] for a variety of b values, which itself behaves as a bifurcation parameter. The bifurcation behavior was studied in Holm and Staley [53], exploring the changes in stability of its travelling wave solutions. They found that when $b > 1$ the peakons and their interactions were stable and exhibited all the usual soliton behaviours. When $0 \leq$

$b < 1$ they found ramp/cliff solutions, like those found in Burgers equation. Lastly when $b < -1$ they discovered the b-family exhibited leftward moving structures. Studies into the blow-up phenomena have also been studied by Zhou [101].

As the whole family is not integrable it is unlikely we will find an exact P_{III} equation like in the previous sections. However we are able to find an 'almost' P_{III} equation, which as a second order ODE we are able to solve numerically in a far easier manner than the b-family itself.

We can rewrite (385) in the form

$$m_t + um_x + bu_x m = 0, \qquad m = u - u_{xx}, \tag{386}$$

similar to the Camassa-Holm and Degasperis-Procesi cases before. We can also write this in conservation form as

$$(m^{\frac{1}{b}})_t = -(um^{\frac{1}{b}})_x \tag{387}$$

and taking a new variable p given by

$$p = m^{\frac{1}{b}} \tag{388}$$

we can construct a reciprocal transformation [43] for the b-family.

3.3.1 From the b-family to a PDE in (X, T)

Substituting the new variable p into (387) we find

$$p_t = -(up)_x, \tag{389}$$

therefore we can take the reciprocal transformation

$$dX = p\ dx - up\ dt, \qquad dT = dt. \tag{390}$$

Using the transformation (390) we substitute into the second part of (386) to find

$$u = p^b - p_{XT} + \frac{p_X p_T}{p} \tag{391}$$

and from the conservation law (389)

$$(p^{-1})_T = u_X. \tag{392}$$

We then combine both (391) and (392) to find

$$(p^{-1})_T = bp^{b-1}p_X - (p(\log p)_{XT})_X. \tag{393}$$

By substituting in $b = 2$ we find (238) for Camassa-Holm and with $b = 3$ we have (306) for Degasperis-Procesi.

3.3.2 From a PDE in (X, T) to a 2nd Order ODE

We now apply a similarity reduction to take the reciprocally transformed equation (393) to an equation that is 'nearly' P_{III}.

$$u = T^{-1}U(Z), \qquad p = T^{-\frac{1}{b}}P(Z), \qquad Z = XT^{\frac{1}{b}} \tag{394}$$

Using (392) and the similarity reduction (394) we calculate for U the equation

$$\frac{1}{b}\frac{d}{dZ}(ZP^{-1}) = U' \tag{395}$$

Integrating (395) once we have

$$U = \frac{1}{b}ZP^{-1} - a \tag{396}$$

with a an integration constant. Applying the reduction to (391) gives

$$P^b = U - \left(P\frac{\partial}{\partial Z}\right)^2 U \tag{397}$$

Then rearranging (397) for U, we compare with (396) to find a second order equation in terms of P, that is

$$\frac{d^2P}{dZ^2} = \frac{1}{P}\left(\frac{dP}{dZ}\right)^2 - \frac{1}{Z}\frac{dP}{dZ} + \frac{1}{Z}(bP^b + ab) - \frac{1}{P}. \tag{398}$$

Comparing to the third Painlevé equation (227) we note it is very similar, and also aligns with the P_{III} equations found in the Camassa-Holm case and Degasperis-Procesi cases. However, if the power of P inside the bracket in (398) is not 2 or 3 then it cannot be transformed to P_{III}. Indeed according to the classification in Ince [62] it fails the Painlevé test unless $b = 2, 3$.

3.3.3 b-family to 3rd Order ODE

The similarity reduction for the general b-family is as follows:

$$p = t^{-\frac{1}{b}}P(z), \qquad u = t^{-1}U(z), \qquad z = x + c\log t \tag{399}$$

Applying these to (386) we find

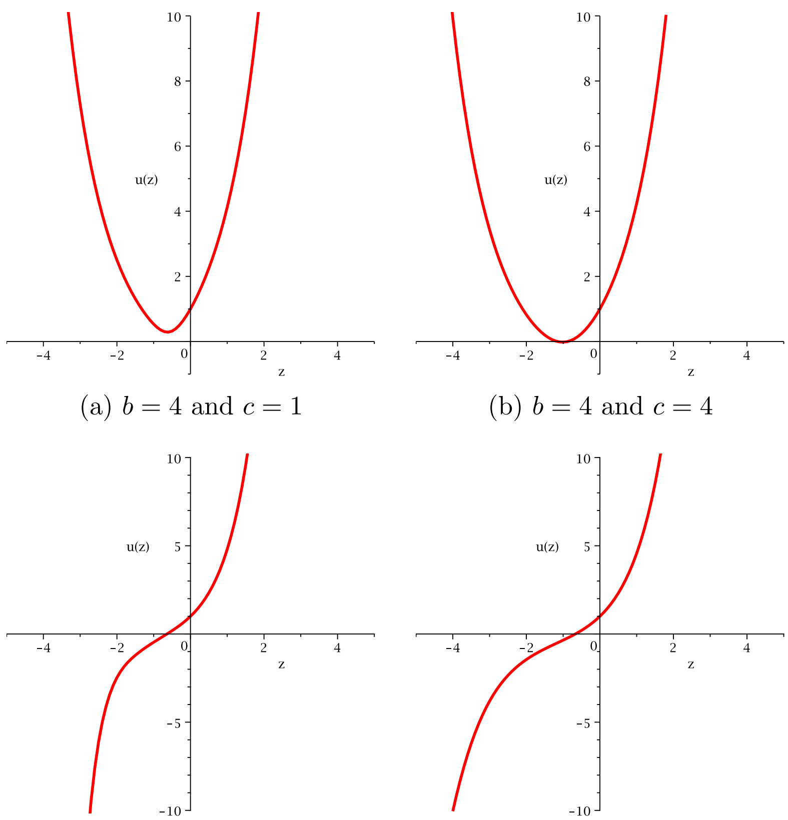
$$P^b = U - \ddot{U}, \qquad \frac{d}{dz}[(U+c)P] = \frac{1}{b}P \tag{400}$$

where dot $= \frac{d}{dz}$, both of which can be used to find a 3rd order ODE

$$(\dot{U} - \frac{1}{b})(U - \ddot{U}) + \frac{1}{b}(c+U)(\dot{U} - \dddot{U}) = 0 \tag{401}$$

We plot some solutions in Figure 17 for $U(z)$ for some of the b values when the b-family equation is non-integrable, which otherwise would be difficult to recover.

We use Maples in-built Runge-Kutta 45 ODE integrator for the numerics. We



(a) $b = 4$ and $c = 1$

(b) $b = 4$ and $c = 4$

(c) $b = -1$ and $c = 1$

(d) $b = -1$ and $c = 4$

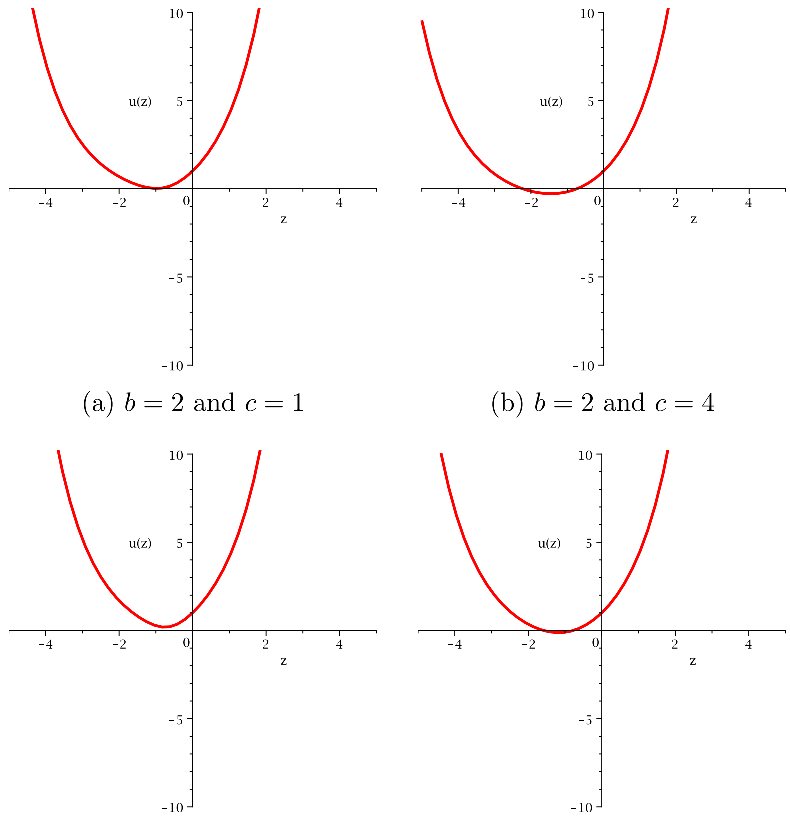
Figure 17: Solving the general equation (401) numerically for non-integrable b values, $[U(0), U'(0), U''(0)] = [1, 2, 2]$

also can plot for the integrable cases, $b = 2, 3$ and compare. For the two cases plotted in Figure 18 they are all have quadratic curves, like the $b = 4$ case. The case of $b = -1$ interestingly is cubic in shape.

3.3.4 3rd Order ODE to 2nd Order ODE

Similarly to both Camassa-Holm and Degasperis-Procesi we take

$$dZ = Pdz \tag{402}$$



(a) $b = 2$ and $c = 1$

(b) $b = 2$ and $c = 4$

(c) $b = 3$ and $c = 1$

(d) $b = 3$ and $c = 4$

Figure 18: Solving the general equation (401) numerically for integrable b values, $[U(0), U'(0), U''(0)] = [1, 2, 2]$

and applying to the first part of (400)

$$U = P^b + P\frac{d}{dZ}\left(\frac{1}{b} - \frac{Z\dot{P}}{bP}\right) \tag{403}$$

simililarly the second part results in

$$\frac{d}{dZ}((U + c)P) = \frac{1}{b} \tag{404}$$

integrating (404) gives a second equation in terms of U. Equating this and (403)

$$U = \frac{Z}{bP} - c = P^b - \frac{1}{b}P\frac{d}{dZ}(Z\frac{d}{dZ}\log P) \tag{405}$$

and rearranging in terms of P we arrive at (398). Using the same b values which we plotted in Figure 17 and Figure 18 we shall now plot for the second oder ODE (398).

3.3.5 Solutions

Even though (398) is not in Painlevé form for all b, it does have a particular analytic solution which is valid for any b, namely

$$P = \lambda Z^{\frac{1}{b+1}} \tag{406}$$

which leads to the solution

$$u(x,t) = \frac{x}{(b+1)t} \tag{407}$$

of the b-family. This is the Burgers "ramp" part of the "ramp-cliff" solutions found in Holm and Staley [54].

3.4 Summary

In this chapter we have explored the reductions of Camassa-Holm and Degasperis-Procesi to the third Painlevé equation, P_{III}. Using the extensive literature on P_{III} we have written down algebraic and special function solutions. We've also reduced the b-family to a 'nearly' Painlevé form that is valid for the integrable case of Camassa-Holm and Degasperis-Procesi. Having found the Painlevé equations, both Camassa-Holm and Degasperis-Procesi satisfy the ARS conjecture. But more interestingly is the fact we have found an almost algorithmic method in reducing integrable peakon equations to their Painlevé equivalents.

There are many works on the peakon solutions of both Camassa-Holm and Degasperis-Procesi, and many on the difficult problems of numerically integrating these peakon equations [52, 53, 99, 22]. This chapter has explored solutions of the P_{III} ODEs which provide solutions of the original PDEs. The reciprocal transformations that relate these PDEs to other integrable hierarchies are known, however the step to reduce these transformed PDEs to P_{III} were not, other than in the case of Camassa-Holm. Additionally, applying the similarity reduction to the peakon PDE and then a hodograph transformation to find the same P_{III} equation is new.

Though these peakon equations are notoriously difficult to integrate numerically, here we have been able to take the well studied solutions of P_{III} to derive implicit exact solutions of the PDEs in their original coordinates, which could be used to test the accuracy of numerical schemes.

Chapter 4

Reductions of peakon equations with cubic nonlinearity

4.1 Introduction

In this chapter we will be applying similar reductions to those in chapter 3, but for the modified Camassa-Holm (mCH) and Novikov equations. Unlike Camassa-Holm and Degasperis-Procesi they are cubic nonlinear, and require more changes of variables to get into the Painlevé form we are after.

Modified Camassa-Holm has been discovered separately by Fokas [37], Fuchssteiner [43], Olver and Rosenau [83] and Qiao [88]. So in the literature you can find the same equation being referred to as FORQ and Qiao's equation, although they were derived in different ways by different authors. So, for example, Qiao used the two-dimensional Euler equations with the aim of finding a new equation for fluids, whereas the other authors used the method of tri-Hamiltonian duality with modified KdV to try and find new integrable nonlinear differential equations from the hierarchy.

We are also interested in finding the links between Camassa-Holm and mCH, as well as between Degasperis-Procesi and Novikov. This hypothesis comes from the knowledge that Camassa-Holm is related to the negative flow of KdV, and mCH is related to the negative flow of modified KdV and both hierarchies are connected by a Miura transformation.

$$\begin{array}{ccc} \text{CH} & \longleftrightarrow & \text{negative KdV} \\ & & \updownarrow \text{ Miura transformation} \\ \text{mCH} & \longleftrightarrow & \text{negative mKdV} \end{array}$$

Same is true for Degasperis-Procesi which is related to the negative flow of Kaup-Kupershmidt and Novikov's equation which is related to negative flow of Sawada-Kotera, and again both hierarchies have an explicit connection.

$$\begin{array}{ccc} \text{DP} & \longleftrightarrow & \text{negative KK} \\ & & \updownarrow \text{ Miura transformation via a fifth order equation} \\ \text{Nov} & \longleftrightarrow & \text{negative SK} \end{array}$$

The relationship of these two sets of hierarchies [30] indicates that the PDEs have a map between each other as well.

4.2 Reductions of modified Camassa-Holm

As before we introduce a diagram that represents the reductions and transformations in this section. We are interested in the mCH equation [88, 37, 50, 72]

$$\begin{array}{ccc} \text{mCH} & \xrightarrow[\text{Sim. Red}]{4.2.3} & \text{3rd Order Equation} \\ {\scriptstyle 4.2.1}\downarrow{\scriptstyle \text{Rec.Trans}} & & {\scriptstyle 4.2.4}\downarrow{\scriptstyle \text{Hodograph}} \\ \text{Negative flow mKdV} & \xrightarrow[\text{Sim.Red}]{4.2.2} & \text{2nd Order Equation} \end{array}$$

Figure 19: Qiao reductions

as it is integrable and has peakon solutions, and we wish to investigate both the reduction to Painlevé type and also its Lax pair. It was given as a completely new integrable water wave equation [88] by Qiao,but we investigate whether it can be transformed to any of the more well known systems. In fact it is stated that this new equation can be reduced from the two-dimensional Euler equation [88]. The equation itself is given as:

$$u_t - u_{xxt} + 3u^2 u_x - u_x^3 = (4u - 2u_{xx})u_x u_{xx} + (u^2 - u_x^2)u_{xxx}, \tag{408}$$

which can be written more simply in a similar manner to Camassa-Holm:

$$m_t + m_x(u^2 - u_x^2) + 2m^2 u_x = 0, \qquad m = u - u_{xx}. \tag{409}$$

Qiao found that the equation has so called 'W-shaped' peakons which are unlike other solutions that are found in peakon equations. In [88] Qiao has the following solution for u:

$$u(\zeta) = A\left(\frac{5}{3} - (3z+2)\left(z - \sqrt{z^2 - \frac{4}{9}}\right)\right), \tag{410}$$

$$z = \cosh\left(\frac{|\zeta|}{2} - \ln(2)\right) - \frac{1}{3}, \tag{411}$$

$$\zeta = x - \frac{11}{3}A^2 t. \tag{412}$$

This solution has the shape shown in Figure 20

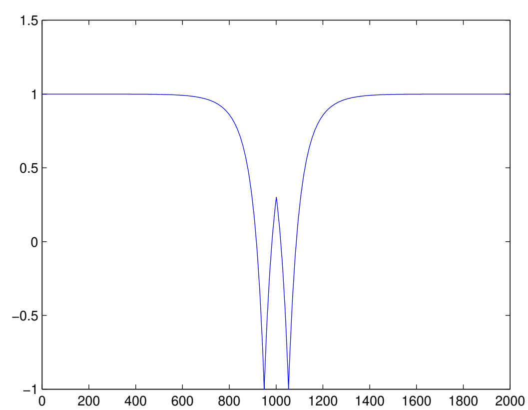


Figure 20: W-Shaped Peakon

Before we study the reductions, it will be useful to show how the Lax pair of mCH can help see the connection between the PDE and the KdV hierarchy. The

Lax pair for mCH is as follows

$$\Psi_x = U\Psi, \qquad\qquad \Psi_t = V\Psi, \tag{413}$$

with

$$U = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ -\frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix}$$

$$V = \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}m\lambda(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}m\lambda(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}$$

which is needed for the proof in the following subsection.

4.2.1 From mCH to Negative mKdV

Addressing the vertical left arrows in Figure 19 we show the following result:

Proposition 4.2.1. [60] The modified Camassa-Holm equation can be transformed via a reciprocal transformation to the negative flow of modified KdV.

This is achieved by performing similar calculations found in Chapter 3, but as we see in this section requires an additional variable to get to the mKDV hierarchy.

Proof. Writing the mCH equation (409) in conservation form gives the following

$$m_t + ((u^2 - u_x^2)m)_x = 0. \tag{414}$$

Simplifying (414) further by introducing an additional variable, f,

$$m_t = -(mf)_x, \tag{415}$$

with

$$f = u^2 - u_x^2, \tag{416}$$

and its derivative

$$f_x = 2mu_x \tag{417}$$

we can note that (415) is comparible to the conservation laws found in both the Camassa-Holm (230) and Degasperis-Procesi cases (301). From (415) we are able to read off the reciprocal transformation

$$dX = \frac{m}{2}dx - \frac{1}{2}mf\,dt, \qquad\qquad dT = dt. \tag{418}$$

Using (418) we find

$$\frac{\partial X}{\partial x} = \frac{m}{2}, \qquad \frac{\partial X}{\partial t} = -\frac{1}{2}mf, \qquad \frac{\partial T}{\partial x} = 0, \qquad \frac{\partial T}{\partial t} = 1 \tag{419}$$

which allows us to transform the variables x, t to X, T. Substituting (419) into (415)

$$m_T + \frac{1}{2}m^2 f_X = 0 \implies (m^{-1})_T = \frac{1}{2}f_X, \tag{420}$$

which can be considered as the 'basic' conservation law. We also have f_x (417) which is transformed via the reciprocal transformation as:

$$f_X = 2mu_X. \tag{421}$$

Using the identity (420) and (421) we find

$$(m^{-2})_T = 2u_X. \tag{422}$$

which is also a conservation law. By transforming all the x derivatives to X derivatives, the equation from the second half of (409) becomes

$$m = u - \frac{m}{2}\frac{\partial}{\partial X}\left(\frac{m}{2}u_X\right) = u - \frac{1}{8}mf_{XX} \tag{423}$$

Then the reciprocal transformation of (409) as given by Hone and Wang [60] is the system made of (423) and (422), although in the paper of Hone and Wang there is a typo with a minus sign appearing in equation (422).

Using the reciprocal transformation to find the derivatives for u together with (422) gives

$$\frac{\partial u}{\partial x} = \frac{m}{2} u_X = -\frac{m_T}{2m^2}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{m_X m_T}{2m^2} - \frac{m_{TX}}{4m} \tag{424}$$

Using $m = u - u_{xx}$ we have

$$m = u - \frac{m_X m_T}{2m^2} + \frac{m_{XT}}{4m}. \tag{425}$$

We are now able to substitute into the original system (409) in terms of the new independent variables (X, T). Rearranging for u and substitute into our conservation law (422)

$$m_T = -m^3 \left(m_X + \frac{3m_{XT} m_X}{4m^2} + \frac{m_T m_{XX}}{2m^2} - \frac{m_T m_X^2}{m^3} - \frac{m_{XXT}}{4m} \right) \tag{426}$$

and also for the additional identity for f

$$f = u^2 - \frac{1}{4} m^2 u_X^2 \tag{427}$$

which we require for the following section. The reciprocally transformed Lax equation obtained from (413) is

$$\Psi_{1XX} - \left(\frac{m_X}{m^2} + \frac{1}{m^2} \right) \Psi_1 = -\lambda^2 \Psi_1. \tag{428}$$

Then introducing $v = m^{-1}$ as in [60] gives

$$\Psi_{XX} + (v_X - v^2)\Psi = -\lambda^2 \Psi, \tag{429}$$

noting that $v_X - v^2$ is the formula for the Miura transformation from modified

KdV to KdV. The T part of the Lax pair is

$$\Psi_T = -\frac{1}{\lambda^2}\left(a\Psi_X - \frac{1}{2}a_X\Psi\right), \qquad a = u - \frac{1}{4}f_X. \tag{430}$$

If we have

$$V_T = 4p_X, \tag{431}$$

which is a rescaled version of negative KdV (240), then it is equivalent to (426) when

$$V = -\frac{p_{XX}}{2p} + \frac{p_X^2}{4p^2} - \frac{1}{4p^2}, \tag{432}$$

If p satisfies the negative KdV equation (431)therefore

$$m = p^{-1} \tag{433}$$

satisfies (426) with

$$v = -\frac{1}{2}\left(\frac{p_X}{p} + \frac{1}{p}\right) \tag{434}$$

□

4.2.2 Negative mKdV to 2nd Order ODE

We have shown in 4.2.1 that mCH is indeed related to the first negative flow of mKdV via a reciprocal transformation. We now apply a similarity reduction to the identities described earlier, the conservation law (420) and f (427) more precisely. This enables us to write mCH as a 2nd order ODE which, unlike Camassa-Holm, is not automatically of Painlevé type.

Using a scaling similarity reduction to get an equation in Z

$$m = T^{-\frac{1}{2}}M(Z), \qquad u = T^{-\frac{1}{2}}U(Z), \qquad f = T^{-1}F(Z), \qquad Z = XT^{\frac{1}{2}} \tag{435}$$

Applying the reduction to the conservation law (420) gives

$$(T^{\frac{1}{2}}M^{-1})_T = \frac{1}{2}\left(T^{-1}F\right)_X \tag{436}$$

and we can write (436) in terms of Z

$$F' = -M^{-1} - \frac{ZM'}{M^2} = \frac{d}{dZ}\left(\frac{Z}{M}\right) \tag{437}$$

Integrating (437) with respect to Z

$$F = \frac{Z}{M} - a \tag{438}$$

where a is an integration constant. This is again very similar to what we found in the previous chapter, and applying the RT to (421) we have an additional variable

$$F' = 2MU'. \tag{439}$$

Writing $f = u^2 - u_x^2$ in terms of F, U and M, using the relation (439)

$$F = U^2 - \frac{1}{4}M^2U'^2 = U^2 - \frac{1}{16}F'^2 \tag{440}$$

we can use this and the following relation to eliminate U to find an equation dependent on M and F only. Substituting the scaling similarity reduction into the latter part of (423)

$$U = M + \frac{1}{8}MF'' = M(1 + \frac{1}{8}F'') \tag{441}$$

Combining both (441) and (440) we find a 2nd order 2nd degree equation [27]

$$\left(M + \frac{1}{8}MF''\right)^2 = \frac{1}{16}F'^2 + F. \tag{442}$$

In terms of the KdV variable

$$V = M^{-1} \tag{443}$$

(442) can be written as

$$F''^2 + 16F'' - 4V^2(F'^2 + 16F) + 64 = 0 \tag{444}$$

which is a 2nd order 2nd degree ODE.

4.2.3 mCH to 3rd Order ODE

Using the modified Camassa-Holm conservation law (415) and (416) we now make a direct application of a scaling similarity reduction. As before making an abuse of the notation, introducing the similarity variable z given by

$$z = x + c\log t. \tag{445}$$

To find the scaling similarity reduction we make the following ansatz

$$m = t^{\mu}M(z), \qquad u = t^{\xi}U(z) \qquad f = t^{\nu}F(z), \tag{446}$$

substituting into (415) to obtain

$$\mu t^{\mu-1}M + t^{\mu-1}c\dot{M} + t^{\mu+\nu}(\dot{M}F + M\dot{F}) \tag{447}$$

where $\dot{} = \frac{d}{dz}$. Equating powers of t to find values of μ and ν that eliminate t, we require a three equations in total. From (447) we have the first relation

$$\mu - 1 = \mu + \nu. \tag{448}$$

Then (416), in terms of the scaling similarity reduction is

$$t^\nu F' = t^{2\xi}(U^2 - \dot{U}^2) \tag{449}$$

with

$$\nu = 2\xi \tag{450}$$

the second relation. Lastly, from (417) we find

$$t^\nu \dot{F} = 2t^{\mu+\xi} M\dot{U} \tag{451}$$

after applying the similarity reduction and hence the third and final relation we require

$$\nu = \mu + \xi. \tag{452}$$

Therefore we have $\mu = -\frac{1}{2}$ and $\nu = -1$. So the conservation law (415) becomes

$$\frac{d}{dz}((c+F)M) = \frac{1}{2}M. \tag{453}$$

and also

$$F = U^2 - \dot{U}^2 \tag{454}$$

the derivative of which is equivalent to

$$\dot{F} = 2M\dot{U}. \tag{455}$$

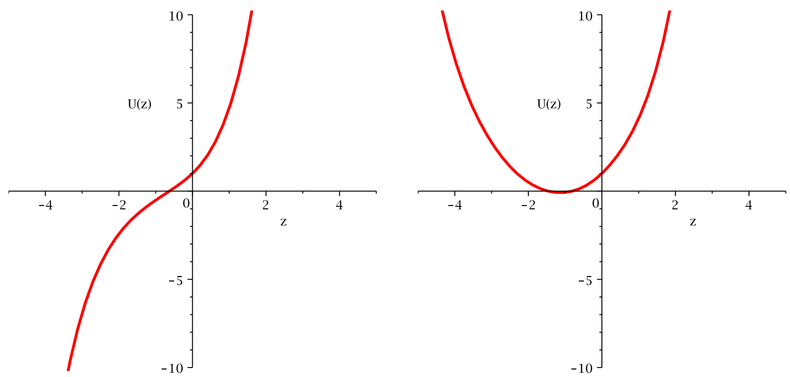
Also

$$M = U - \ddot{U} \tag{456}$$

Then the third order system for U is given by (453) and (456)

$$2\dot{U}(U-\ddot{U})(U-\ddot{U}) + (c+U-\dot{U}^2)(\dot{U}-\dddot{U}) - \frac{1}{2}(U-\ddot{U}) = 0 \tag{457}$$

which we can plot for particular values of c. As we would expect there are singularities, but as you can see in Figure 21 for $c = 1$ and $c = 5$ with the given initial conditions we have avoided them.



(a) $c = 1$ (b) $c = 5$

Figure 21: Solving the general equation (457) numerically with initial conditions $[U(0), U'(0), U''(0)] = [1, 2, 2]$

4.2.4 From the 3rd Order ODE to 2nd Order ODE

We apply a transformation to reduce the order of the 3rd order ODE (457) by one, as Painlevé equations are 2nd order.

Proposition 4.2.2. Applying a hodograph transformation to the third order ODE results in the same 2nd order ODE (444) as is obtained from applying a reciprocal transformation and then a similarity reduction to the modified Camassa-Holm equation.

Proof. The hodograph transformation we use is the following

$$dZ = \frac{M}{2} dz, \tag{458}$$

the calculation of which is nearly identical to that in (262). Applying the hodograph (458) to the transformed conservation law (453) results in

$$\frac{d}{dZ}((c+F)M) = 1, \tag{459}$$

and integrating with respect to Z

$$c + F = \frac{Z}{M}. \tag{460}$$

We note that (455) with derivatives in terms of Z becomes

$$F' = 2MU' \tag{461}$$

and using this and the relation for F (454) we find

$$F = U^2 - \left(\frac{M}{2}U'\right)^2 = U^2 - \frac{1}{16}F'^2 \tag{462}$$

Finally the relation for $M = U - U''$ becomes

$$U = M + \frac{1}{8}MF'', \tag{463}$$

which gives an equation in terms of U to substitute into (462) and we find the 2nd order equation (444) as before, verifying proposition 4.2.2. □

4.2.5 Connection with Camassa-Holm P_{III} Reduction

We have found the same second order ODE for modified Camassa-Holm twice, as we know of the link between the modified KdV hierarchy and the KdV hierarchy we propose the following:

Proposition 4.2.3. The 2nd order ODE (444) that arises from reductions of modified Camassa-Holm can be transformed via a Miura transformation to the Camassa-Holm P_{III} (247).

Proof. We will now write (444) in terms of negative KdV variable P. Writing (438) in terms of V we calculate the derivatives of F as functions of Z and V:

$$F = ZV - a, \qquad F' = V + ZV', \qquad F'' = 2V' + ZV''. \tag{464}$$

We know V from the Miura transformation (434) in terms of Z, that is

$$V = -\frac{1}{2}(\Lambda' + P^{-1}), \tag{465}$$

with

$$\Lambda' = \frac{P'}{P}, \tag{466}$$

being the logarithmic derivative. Form of P_{III} for the negative KdV reduction as in chapter 3 written in terms of the second logarithmic derivative

$$\Lambda'' = \frac{1}{Z}\left(-\frac{P'}{P} + \alpha P + \frac{\beta}{P} + \frac{\delta Z}{P^2}\right) \tag{467}$$

To write F and its derivatives in terms of p we need to write the derivatives of V in terms of p

$$V' = \frac{-\alpha P^3 + \Lambda' P^2 + (\Lambda' Z - \beta)P - \delta Z}{2ZP} \tag{468}$$

and

$$V'' = -\frac{\Lambda'^2}{2P} + \frac{\Lambda'}{2Z}\left(-\alpha P + \frac{\beta}{P} + \frac{2Z\delta}{P^2} - \frac{1}{P} - \frac{2}{Z}\right) + \frac{\alpha}{Z}\left(\frac{1}{2} + \frac{P}{Z}\right) + \frac{\beta}{ZP}\left(\frac{1}{2P} + \frac{1}{Z}\right) + \frac{\delta}{2P^2}\left(\frac{1}{P} + \frac{1}{Z}\right). \tag{469}$$

Substituting for V' (468) and V'' (469) for the derivatives of F (464)

$$F = -\frac{\Lambda' Z}{2} - \frac{Z}{2P} - a, \tag{470}$$

$$F' = -\frac{\alpha P}{2} + \frac{\Lambda' Z}{2P} - \frac{\beta}{2P} - \frac{\delta Z}{2P^2} - \frac{1}{2P}, \tag{471}$$

$$F'' = -\frac{\Lambda'^2 Z}{2P} + \left(-\frac{\alpha P}{2} + \frac{\beta}{2P} + \frac{\delta Z}{P^2} + \frac{1}{2P}\right)\Lambda' + \frac{\alpha}{2}\frac{\beta}{2P^2} + \frac{\delta Z}{2P^3} - \frac{\delta}{2P^2}. \tag{472}$$

Expanding (444) in powers of the log derivative of P

$$A\Lambda'^3 + B\Lambda'^2 + C\Lambda' + D = 0 \tag{473}$$

with

$$A = (8+\alpha)Z - \frac{Z^2}{2P^3}(1+\delta), \tag{474}$$

and removing the leading order terms by fixing $\alpha = -8$ and $\delta = -1$ we find

$$8(\beta + 2a + 1)\Lambda'^2 + \frac{16}{P}(\beta + 2a + 1)\Lambda' + \frac{8}{P^2}(\beta + 2a + 1) = 0 \tag{475}$$

then to satisfy the CH P_{III} we choose,

$$\beta = -2a - 1. \tag{476}$$

Substitute in the values for α, β, δ into (467)

$$\frac{d^2P}{dZ^2} = \frac{1}{P}\left(\frac{dP}{dZ}\right)^2 - \frac{1}{Z}\frac{dP}{dZ} + \frac{1}{Z}(-8P^2 - (2a-1)) - \frac{1}{P} \tag{477}$$

which is the same form of P_{III} as in the Camassa-Holm case with

$$\alpha = -8, \qquad \beta = -2a - 1, \qquad \gamma = 0, \qquad \delta = -1 \tag{478}$$

By applying the miura transformation to the second order ODE of mCH (444), the same form of P_{III} for Camassa-Holm has been found verifying proposition 4.2.3.

We can simplify (477) in a similar manner to the Degasperis-Procesi case in section 3.2.5 where we used a scaling transformation to simplify the resulting equations. We use the same transformation as before

$$T_5(\sigma_1, \sigma_2) : \phi \mapsto \tilde{\phi}(Z, \alpha\sigma_1\sigma_2, \beta\sigma_1^{-1}\sigma_2, \gamma\sigma_1^2\sigma_2^2, \delta\sigma_1^{-2}\sigma_2^2, \tag{479}$$

$$:= \sigma^{-1}\phi(\sigma_2 Z, \alpha, \beta, \gamma, \delta) \tag{480}$$

but with a different σ_1 and σ_2

$$\sigma_1 = \frac{1}{\sigma_2 \alpha}, \qquad \sigma_2 = \left(-\frac{1}{\alpha^2 \delta} \right)^{\frac{1}{4}}. \tag{481}$$

Using this transformation with (477) and we get

$$\frac{d^2 P}{dZ^2} = \frac{1}{P} \left(\frac{dP}{dZ} \right)^2 - \frac{1}{Z} \frac{dP}{dZ} + \frac{1}{Z} (P^2 + (2a+1)) - \frac{1}{P} \tag{482}$$

with

$$\alpha = 1, \qquad \beta = 2a + 1, \qquad \gamma = 0, \qquad \delta = -1. \tag{483}$$

Applying the same transformation to the Camassa-Holm P_{III} (247)

$$\frac{d^2 P}{dZ^2} = \frac{1}{P} \left(\frac{dP}{dZ} \right)^2 - \frac{1}{Z} \frac{dP}{dZ} + \frac{1}{Z} (P^2 + 2a) - \frac{1}{P} \tag{484}$$

with

$$\alpha = 1, \qquad \beta = 2a, \qquad \gamma = 0, \qquad \delta = -1. \tag{485}$$

Therefore by shifting the β in the modified Camassa-Holm case we will find the same solutions as for Camassa-Holm. □

4.3 Reductions of Novikov's Equation

Novikov's equation [79]

$$u_t - u_{xxt} + 4u^2 u_x = 3u u_x u_{xx} + u^2 u_{xxx} \tag{486}$$

was discovered when looking at a classification of PDEs with infinitely many symmetries. It has cubic nonlinearity and interestingly unlike the quadratic nonlinear peakon equations we've discussed, the peakons and anti-peakons both travel to the right [64], which is also true for mCH. A relationship between Novikov's equation and Degasperis-Procesi was studied in [63] using the Lax pair to examine the link between the Sawada-Kotera and Kaup-Kuperschmidt hierarchies. This Resulted in an implicit relationship between Novikov and Degasperis-Procesi. We show that there is also a link between the Painlevé reductions of Degasperis-Procesi and Novikov's equation.

Here we demonstrate how P_V

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (487)$$

can be found from the Novikov PDE. As with mCH we shall briefly describe the Lax equations for Novikov which helps understand where the relationship to Sawada-Kotera appears. The Lax pair is

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \quad (488)$$

with U and V given by

$$U = \begin{pmatrix} 0 & \lambda m & 1 \\ 0 & 0 & \lambda m \\ 1 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{3}\lambda^{-2} - uu_x & \lambda^{-1}u_x - \lambda u^2 m & u_x^2 \\ \lambda^{-1}u & -\frac{2}{3}\lambda^{-2} & -\lambda^{-1}u_x - \lambda u^2 m \\ -u^2 & \lambda^{-1}u & \frac{1}{3}\lambda^{-2} + uu_x \end{pmatrix}$$

Using the x-part of the matrix Lax pair to find the scalar equation

$$\psi_{xxx} - \frac{2\psi_{xx}m_x}{m} + \frac{2\psi_x m_x^2}{m^2} - \frac{\psi_x m_{xx}}{m} = m^2\lambda^2\psi + \psi_x, \quad (489)$$

and the corresponding t-part is

$$\psi_t = \frac{u\psi_{xx}}{\lambda m} - \frac{u_x\psi_x}{\lambda m} - \frac{um_x\psi_x}{\lambda m^2} - u^2\psi_x \quad (490)$$

The scalar Lax pair can demonstrate the relationship between Novikov's equation and the Sawada-Kotera hierarchy [61]. This is explicitly given in the following subsection, after reciprocally transforming the Lax pair.

The following diagram shows the two ways in which the PDE can be transformed to the fifth Painlevé equation.

$$\begin{array}{ccc}
\text{Novikov} & \xrightarrow[4.3.3]{\text{Sim. Red}} & \text{3rd Order Equation} \\
{\scriptstyle 4.3.1}\big\downarrow{\scriptstyle \text{Rec.Trans}} & & {\scriptstyle 4.3.4}\big\downarrow{\scriptstyle \text{Hodograph}} \\
\text{negative Sawada-Kotera} & \xrightarrow[4.3.2]{\text{Sim.Red}} & \text{2nd Order Equation, } P_V
\end{array}$$

Figure 22: Novikov reductions

Theorem 4.3.1. Novikov's equation can be written as the fifth Painlevé equation after transformations.

4.3.1 Novikov to Negative Sawada-Kotera

The left vertical arrow in Figure 22 implies a relationship between Novikov's equation and the Sawada-Kotera hierarchy, as noted by Hone and Wang:

Proposition 4.3.1. [60] The Novikov equation, via a reciprocal transformation, can be written as a negative flow of Sawada-Kotera

Proof. Re-writing the Novikov equation in terms of a "momentum" variable m simplifies the form of the PDE:

$$m_t + m_x u^2 + 3muu_x = 0, \qquad\qquad m = u - u_{xx} \tag{491}$$

upon making the substitution $p^{\frac{3}{2}} = m$ in (491) we can rewrite it in conservation form as

$$p_t + (pu^2)_x = 0. \tag{492}$$

This now allows us to define the reciprocal transformation which we will use to find a new conservation law in terms of the new independent variables (X, T).

$$dX = pdx - pu^2 dt, \qquad dT = dt \tag{493}$$

Substituting for the derivatives in (492) produces

$$\left(\frac{1}{p}\right)_T = (u^2)_X. \tag{494}$$

We need to use the second part of (491) to find an additional relationship between $u(X, T)$ and $p(X, T)$ which in its reciprocally transformed state is the following:

$$u = p^{\frac{3}{2}} + p(p_X u_X + p u_{XX}). \tag{495}$$

To understand that (495) and (494) really are part of the Sawada-Kotera hierarchy we look back at the Lax pair and give the following proof for proposition 4.3.1:

Proof. Taking the spatial (489) scalar equation, we can apply the reciprocal transformation (493) which gives us

$$\Psi_{XXX} + G\Psi_X = \lambda^2\Psi = \lambda^2\Psi, \qquad G = -\frac{p_{XX}}{2p} + \frac{p_X^2}{4p^2} - \frac{1}{p^2}. \tag{496}$$

As noted in [60] the first equation in (496) the third order operator $\partial_{XXX} + G\partial_X$ is the standard operator for the Sawada-Kotera hierarchy. □

The same equation is found if we make a change of variable to (495). So setting

$$v = up^{\frac{1}{3}} \tag{497}$$

we have

$$v_{XX} - \left(\frac{p_{XX}}{2p} - \frac{p_X^2}{4p^2} + \frac{1}{p^2}\right)v + 1 = 0 \tag{498}$$

and the T part is

$$\Psi_T = \frac{1}{\lambda^2}(v\Psi_{XX} - v_X\Psi_X) - \frac{2}{3\lambda^2}\Psi \tag{499}$$

□

4.3.2 Negative Flow of Sawada Kotera to P_V

Proposition 4.3.2. The negative flow of Sawada-Kotera that arises as a reduction from Novikov's equation can be further reduced to the fifth Painlevé equation.

Proof. To reduce the equation to being dependent on only one independent variable, we introduce a scaling similarity reduction

$$u = T^{-\frac{1}{2}}U(Z), \qquad p = T^{-\frac{1}{3}}P(Z), \qquad Z = XT^{\frac{1}{3}} \tag{500}$$

Applying the reduction to (494)

$$\frac{d}{dZ}\left(\frac{Z}{3P}\right) = 2UU' \tag{501}$$

To deal with the fractional power of p we introduce two new variables Q and V inspired by the change of variable needed to find the relationship to the Sawada-Kotera hierarchy

$$P = Q^2, \qquad V = UQ. \tag{502}$$

Applying the gauge transformation and our new variables to (495)

$$P^{\frac{3}{2}} = U - (P\frac{d}{dZ})^2 U, \tag{503}$$

becomes

$$Q^4 = V - Q^3(QV'' - VQ'') \tag{504}$$

with (502), where $' = \frac{d}{dZ}$. Integrating (501)

$$\frac{1}{3}(ZP^{-1}) = U^2 + a, \tag{505}$$

gives

$$Q^2 = \frac{1}{a}(\frac{1}{3}Z - V^2) \tag{506}$$

Eliminating Q^2, $(Q^2)'$, $(Q^2)''$ in terms of V and rearranging for V gives the second order equation

$$(3V^2 - Z)V'' = 3VV'^2 - \frac{3V^2}{Z}V' - \frac{1}{Z}\bigg(-9V^4 + 6ZV^2 - \frac{V}{4} + 9Va^2 - Z^2\bigg) \tag{507}$$

we require yet another change of variables to shift the poles at

$$V = \pm\sqrt{\frac{Z}{3}} \tag{508}$$

using

$$V = \zeta\bigg(\frac{1+w}{1-w}\bigg), \qquad z = 3\zeta^2, \qquad d\zeta = \pm\frac{1}{6\zeta}dZ \tag{509}$$

in terms of $w_{\zeta\zeta}$

$$w_{\zeta\zeta} = \bigg(\frac{1}{2w} + \frac{1}{w-1}\bigg)w_\zeta^2 - \frac{w_\zeta}{\zeta} + \frac{9a^2}{2z^2}(w-1)^2\bigg(w - \frac{1}{w}\bigg) - 72\zeta w \tag{510}$$

In a similar vein to DP, to get in the correct P_V form we apply another transformation

$$\zeta = \eta^{\frac{1}{3}}, \qquad d\zeta = \frac{1}{3}\eta^{-\frac{2}{3}} \tag{511}$$

which gives the following form of P_V:

$$w_{\eta\eta} = \bigg(\frac{1}{2w} + \frac{1}{w-1}\bigg)w_\eta^2 - \frac{w_\eta}{\eta} + \frac{(w-1)^2}{\eta^2}\frac{a^2}{2}\bigg(w - \frac{1}{w}\bigg) - \frac{8w}{\eta}, \tag{512}$$

corresponding to (487) with parameter values

$$\alpha = \frac{a^2}{2}, \qquad \beta = -\frac{a^2}{2}, \qquad \gamma = -8, \qquad \delta = 0, \tag{513}$$

hence satisfying proposition 4.3.2. □

4.3.3 From Novikov to 3rd Order ODE

The latter two subsections detailed one way to reduce Novikov's equation to P_V, these next two will apply similar reductions in reverse to show an alternative method in achieving the same result. This is detailed in Figure 22 given at the start of this section, the reduction here to a 3rd order ODE is the top arrow on that diagram.

The Novikov equation has the conservation law

$$(m^{\frac{2}{3}})_t + (m^{\frac{2}{3}}u^2)_x = 0, \qquad\qquad m = u - u_{xx}. \tag{514}$$

Setting $m = p^{\frac{3}{2}}$ simplifies the powers, we then look for a scaling similarity reduction of the form

$$p = t^{\mu}P(z), \qquad u = t^{\nu}U(z), \qquad z = x + c\log t. \tag{515}$$

To find μ and ν we substitute (515) into the first part of the conservation law (514) which produces

$$\mu t^{\mu-1}P + t^{\mu-1}c\dot{P} + t^{4\mu}(\dot{P}U + 2PU\dot{U}) = 0, \tag{516}$$

and balancing powers of t gives

$$\mu - 1 = 4\mu. \tag{517}$$

Solving (517) for μ results in $\mu = -\frac{1}{3}$. We require an additional equation to find ν, so we use the equation for m from (514) and again use the substitutions (515) to find

$$t^{\frac{3}{2}\mu}P^{\frac{3}{2}} = t^{\nu}(U - \ddot{U}). \tag{518}$$

Balancing the powers of t we get

$$\nu = \frac{3}{2}\mu, \tag{519}$$

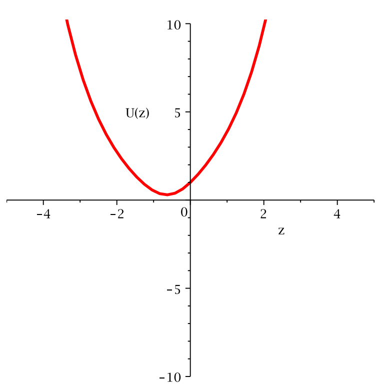
which means $\nu = -\frac{1}{2}$. Using the results for μ and ν, we substitute for the conservation law (514) and rearrange to find

$$\frac{d}{dz}[(U^2 + c)P] = \frac{1}{3}P. \tag{520}$$

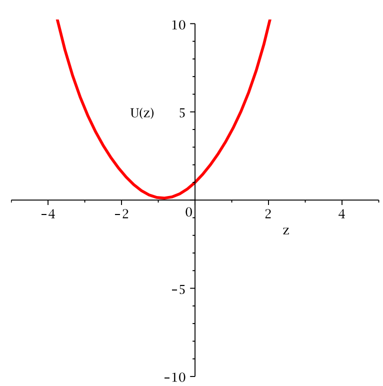
To find an equation dependent on U only, we substitute $P^{\frac{3}{2}} = U - \ddot{U}$ into (520) and differentiate with resepect to z to get the 3rd order ODE

$$\left(2U\dot{U} - \frac{1}{2}\right)(U - \ddot{U}) - \frac{2}{3}(U^2 + c)\dddot{U} = 0. \tag{521}$$

Using Maple's inbuilt ODE integrator we are able to plot solutions of (521) for different values of c



(a) $c = 1$



(b) $c = 5$

Figure 23: Solving the general equation (521) numerically with initial conditions $[U(0), U'(0), U''(0)] = [1, 2, 2]$

4.3.4 3rd Order ODE to P_V

Proposition 4.3.3. The 3rd order ODE that arises from a scaling similarity reduction of Novikov can be written as a version of P_V. The Painlevé equation found is the same found when applying a reciprocal transformation and then a similarity reduction to Novikov's equation.

Proof. Taking (520) we apply the hodograph

$$dZ = Pdz, \tag{522}$$

to find

$$\frac{d}{dZ}[(U^2 + c)P] = \frac{1}{3}. \tag{523}$$

Integrating (523) with respect to Z results in

$$U^2 + c = \frac{Z}{3P}, \tag{524}$$

which we note is the same as (505). To find another equation in terms of U^2, we use the relation

$$P^{\frac{3}{2}} = U - \ddot{U}, \tag{525}$$

with $\dot{U} = \frac{dU}{dz}$ gives

$$P^3 = U - \left(P\frac{d}{dZ}\right)^2 U, \tag{526}$$

and hence we are in a position to apply the same substitutions as our previous example which gets us back to P_V and satisfying proposition 4.3.3. □

4.3.5 Solutions

Here we find some solutions to P_V in the form (512), additionally we use the solutions from the Degasperis-Procesi P_{III} case to find solutions of (512). There is not a general relationship between P_{III} and P_V, however for a special case where

$\gamma \neq 0$ and $\delta = 0$ this is possible [49]. To simplify the solutions we use the scaled solutions from DP, that is the ones that satisfy $(\alpha, \beta, -1, 1)$.

Writing (512) in the same form as Gromak

$$w_{\zeta\zeta} = \frac{3w-1}{2w(w-1)}w_\zeta^2 - \frac{w_\zeta}{\zeta} + \frac{(w-1)^2}{\zeta^2}(aw + \frac{b}{w}) + \frac{c}{\zeta}w, \tag{527}$$

with

$$a = \frac{A^2}{2}, \qquad b = -\frac{A^2}{2}, \qquad c = -8 \tag{528}$$

with $a =$ constant. As in Gromak we can take $c^2 = 1$ without loss of generality and make the substitution $\zeta^2 = 2\tau$

$$u'' = \frac{3u-1}{2u(u-1)}u'^2 - \frac{u'}{z} + \frac{4(u-1)^2}{z^2}(au + \frac{b}{u}) + 2cu \tag{529}$$

Rational solutions exist if either of the following are satisfied

$$a = \frac{(2n-1)^2}{8} \qquad n \in \mathbb{N} \tag{530}$$

or

$$a \neq 0, \qquad b = -\frac{(2n-1)^2}{8}, \qquad n \in \mathbb{N} \tag{531}$$

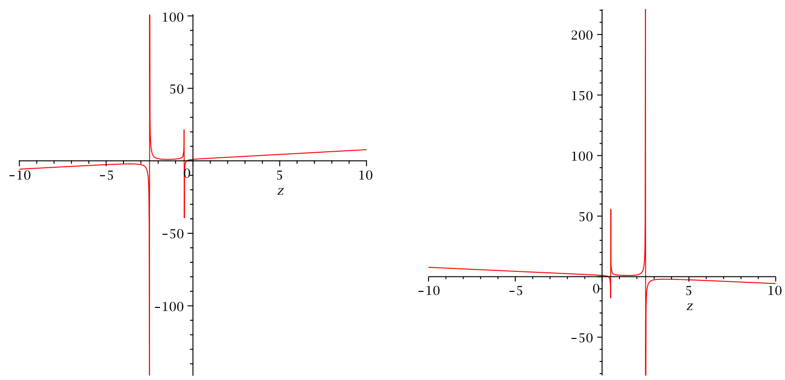
Taking (530) by rearranging as a quadratic for n and solving we find

$$n = \frac{1}{2} \pm A \tag{532}$$

which is satisfied. We can find rational solutions of (529) using our free variable A to satisfy relationships between the variables a, b, c.

a	b	c	d	α
$\frac{1}{8}$	$-\frac{1}{8}$	-1	$1 \pm 2z$	$\alpha = \pm$
$\frac{9}{8}$	$-\frac{9}{8}$	-1	$\frac{8z^3+36z^2+54z+15}{3(4z^2+12z+5)}$	5
		-1	$\frac{8z^3-36z^2+54z-15}{-3(4z^2-12z+5)}$	-1

Plotting some of these solutions



(a) u_1 (b) u_2

Figure 24: First two algebraic solutions

4.4 Summary

This chapter has shown how modified Camassa-Holm and Novikov can be related to Painlevé equations. Not only this but we have found a relationship between Degasperis-Procesi and Novikov. A Miura type map has been found between both these equations previously, but no direct solution comparisons were found and neither was the Painlevé reduction. As with chapter 3, this only goes to bolster the ARS conjecture that integrable PDE's have a reduction to Painlevé equations.

We have employed similar techniques to find Painlevé equations from peakon PDEs with cubic nonlinearity as we did with those that have quadratic nonlinearity. For the equations studied, it has been a sound algorithmic approach which gives an indication that a similar method could be applied to other related peakon equations.

Chapter 5

Conclusions and future work

This thesis has explored several different peakon equations, both integrable and non-integrable. In chapter 2 we study the Popowicz system, presenting the solution for the 2-peakon dynamics and exploring some of the features of the interaction. In chapter 3 we transform via reductions Camassa-Holm and Degasperis-Procesi to P_{III}, and then give explicit solutions for the PDEs. Chapter 4 we apply similar reductions to modified Camassa-Holm and Novikov's equation, and explore links from these cubic nonlinear PDEs to the quadratic nonlinear PDEs of the previous section.

Peakon equations have existed for more than 20 years, and not only have they been of interest to the integrable systems community, but they have also encouraged novel numerical schemes due to the difficulty of solving these PDEs numerically. Most recently there has seen a huge surge in papers developing new families of equations [92, 51, 7, 28], much like the b-family, but encapsulating more of the integrable equations in a single family. Also there is an increased interest in multi-component systems [98], such as the Popowicz system, that lend itself to studying the dynamics of multi-peakons and stability problems. An example is the Geng-Xue equation [45]

$$m_t + 3u_x v m + u v m_x = 0, \qquad m = u - u_{xx} \tag{533}$$

$$n_t + 3v_x u n + u v n_x = 0, \qquad n = v - v_{xx} \tag{534}$$

which has interesting dynamics of peakons and shock-peakons [71]. Here we discuss some future work that relates some new families of equations to some of the results discussed earlier in this thesis.

5.1 ab-family

The ab-family [51] is the cubic equivalent of the b-family, with a, b any real numbers, being given by

$$u_t - u_{txx} + (b+1)u^2 u_x - 3au_x^3 - (6a+b)uu_x u_{xx} + 6au_x u_{xx}^2 - u^2 u_{xxx} + 3au_x^2 u_{xxx} = 0. \tag{535}$$

We can write it in terms of $m = u - u_{xx}$ as

$$m_t + (u^2 - 3au_x^2)m_x + u_x((b-6a)u + m)m = 0. \tag{536}$$

Setting $b = 2$ can put (536) into a conserved form

$$m_t + ((u^2 - 3au_x^2)m)_x = 0 \tag{537}$$

by setting $a = \frac{1}{3}$ we get the conservation law for mCH. For Novikov, $a = 0$ and $b = 3$. Under these specific parameters, this local form of the ab equation conserves the H^1 norm, as noted by Himonas et.al [51]. They also derived peakon travelling wave solutions for the whole ab family.

5.2 abc-family

Previously we have discussed the b-family which we know exhibits quadratic nonlinearity, we have also seen the Novikov equation which has cubic nonlinearity. Anco et.al [7] have discovered a family which encapsulates these quadratic and cubic nonlinear equations found in the previous chapters of this thesis. This 4 parameter family, which we shall call the abc-family is the following,

$$u_t - u_{txx} + au^p u_x - bu^{p-1}u_x u_{xx} - cu^p u_{xxx} = 0. \tag{538}$$

So for example the (p, a, b, c) constants for the various equations are

(p, a, b, c)	Equation
$(1, 3, 2, 1)$	Camassa-Holm
$(1, 4, 3, 1)$	Degasperis-Procesi
$(2, 4, 3, 1)$	Novikov

where p determines the nonlinearity in these cases. As we have done previously, we write (538)

$$m_t + bu_x u^{p-1}m + cu^p m_x = 0, \qquad m = u - u_{xx}, \tag{539}$$

which loses the dependence on a but we shall look at that below. Using (539) as a starting point we can now write down this in conservation form

$$(m^{\frac{p}{b}})_t + (u^p m^{\frac{p}{b}})_x = 0. \tag{540}$$

This is a simpler case, but sets the scene for further exploits as studied in chapters 3 and 4. We are able to find another conservation law from (538) but instead keeps the variable a in play. We give it here as

$$(m^{\alpha})_t + (u^{\gamma} m^{\alpha})_x = 0, \tag{541}$$

which fixes the relationship between p, a, b and c as the following

$$p = \gamma, \qquad a = \frac{\gamma}{\alpha} + 1, \qquad b = a - 1, \qquad c = 1. \tag{542}$$

Setting $g = m^\alpha$, this conservation law leads to a reciprocal transformation, namely

$$dX = gdx - u^p gdt, \qquad dT = dt \tag{543}$$

Another example is the $abc - k$ family, which not only contains the integrable equations found in the previous abc family, but also modified Camassa-Holm (though it is called the Fokas-Olver-Rosenau-Qiao equation in the reference) [51].

$$u_t - u_{xxt} + (b+1)u^k u_x + (3k - 9a - b - 2c)u^{k-2}u_x^3 + (2c - 3k)u^{k-1}u_x u_{xx} \tag{544}$$

$$+ 6au^{k-2}u_x u_{xx}^2 - u^k u_{xxx} + 3au^{k-2}u_x^2 u_{xxx} = 0 \tag{545}$$

These other families of equations admit peakon solutions and conservation laws, and are amenable to the same techniques as in chapters 3 and 4, namely reciprocal transformations and similarity reductions.

Appendix A

Relationship between Degasperis-Procesi and Novikov Solutions

There exists a well known relationship between the solutions of the third and fifth Painleve equations [49]. Given $\alpha\delta \neq 0$ we can assume $\gamma = 1$ and $\delta = -1$. So if we let $w = w(z; \alpha, \beta, \gamma, \delta)$ be a solution of P_{III}, and $v = \frac{dw}{dz} - \epsilon w^2 + (\frac{(1-\epsilon\alpha)w}{z})$. Then $\tilde{w}(\zeta, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = \frac{v-1}{v+1}$ and $z = \sqrt{2\zeta}$ satisfies P_V with $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = (\frac{(\beta-\epsilon\alpha+2)^2}{32}, \frac{(\beta+\epsilon\alpha-2)^2}{32}, -\epsilon, 0)$.

We have $\delta = 0$ for Novikov's P_V (512), we can then write it as a special case of P_{III}. For clarity we rename the variables for P_V $(\alpha, \beta, \gamma, \delta) = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$.

To write P_V solutions in terms of P_{III} $(\alpha, \beta, \gamma, \delta)$ we take

$$\tilde{a} = \frac{1}{32}(\beta - \alpha\epsilon + 2)^2, \qquad \tilde{b} = -\frac{1}{32}(\beta + \alpha\epsilon - 2)^2, \qquad \tilde{c} = -\epsilon, \qquad \tilde{d} = 0, \quad (546)$$

and

$$R_1 = \pi' - \epsilon\pi^2 - (\alpha\epsilon - 1)\frac{\pi}{\zeta} + 1 \neq 0, \quad (547)$$

with

$$\zeta^2 = 2\rho. \quad (548)$$

Noting $u(\rho)$ a solution of P_V

$$u(\rho_i) = 1 - \frac{2}{R_1(\sqrt{2\rho})}, \tag{549}$$

with $i = 1, 2$. This gives two solutions of P_V from one solution of P_{III}. To find two more solutions, we introduce

$$R_2 = \pi' - \epsilon\pi^2 + \frac{\pi}{\zeta} - 1 \neq 0, \tag{550}$$

with

$$\tilde{a} = \frac{1}{32}(\beta + \alpha\epsilon - 2)^2, \qquad \tilde{b} = -\frac{1}{32}(\beta - \alpha\epsilon + 2)^2, \qquad \tilde{c} = -\epsilon, \qquad \tilde{d} = 0. \tag{551}$$

So the other two solutions come from

$$u(\rho_j) = 1 + \frac{2}{R_2(\sqrt{2\rho})} \tag{552}$$

with $j = 3, 4$.

A.0.1 From seed solution

Taking a seed solution for Degasperis-Procesi P_{III}, $\pi_0(0, 0, 1, -1)$, we can find a solution for Novikov's P_V, and $u(\frac{1}{8}, -\frac{1}{8}, -1, 0)$

$$u_1(\rho_1) = 1 - 2\sqrt{2\rho}. \tag{553}$$

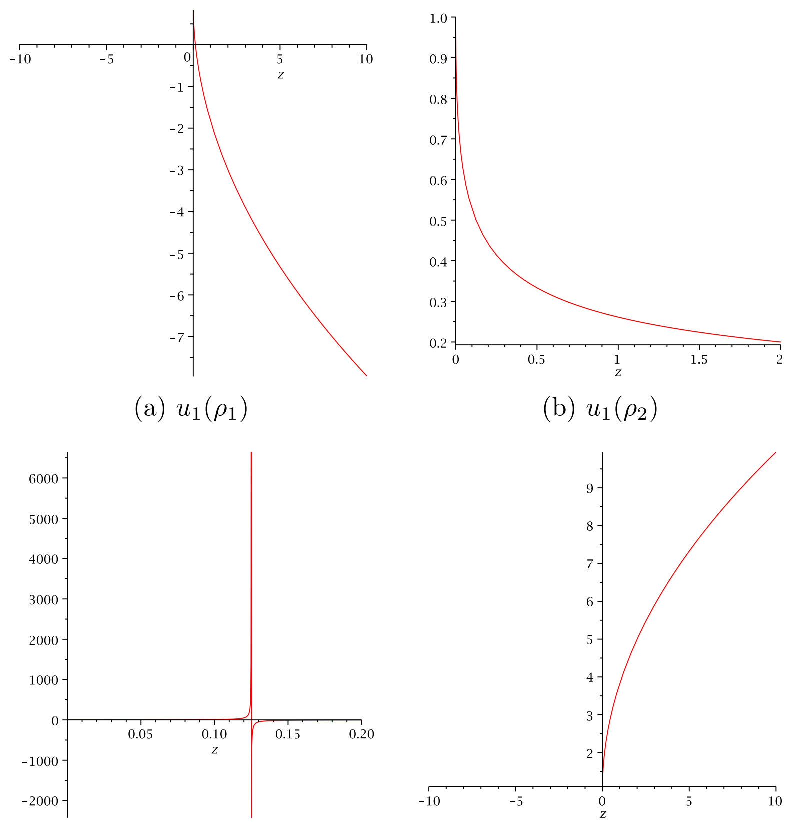
Using the different signs for ϵ we also have $u(\frac{1}{8}, -\frac{1}{8}, 1, 0)$

$$u_1(\rho_2) = \frac{\sqrt{2}}{4\sqrt{\rho} + \sqrt{2}}. \tag{554}$$

Can apply R_2

$$u_1(\rho_3)(\frac{1}{8}, -\frac{1}{8}, -1, 0) = \frac{\sqrt{2}}{\sqrt{2} - 4\sqrt{\rho}}, \tag{555}$$

$$u_1(\rho_4)(\frac{1}{8}, -\frac{1}{8}, 1, 0) = 1 + 2\sqrt{2\rho} \tag{556}$$



(a) $u_1(\rho_1)$ (b) $u_1(\rho_2)$

(c) $u_1(\rho_3)$ (d) $u_1(\rho_4)$

Figure 25: Four solutions from w_2 P_{III} solutions of Degasperis-Procesi, now solutions to Novikov's P_V

A.0.2 2nd P_{III} solution

Here we use the 2nd rational P_{III} solution from Degasperis-Procesi (322)

$$\pi_4(\zeta; 0, 8, 1, -1) = \frac{16\zeta^4 - 144\zeta^3 + 480\zeta^2 - 660\zeta + 315}{(2\zeta - 5)(-36\zeta^2 + 54\zeta + 8\zeta^3 - 15)} \tag{557}$$

to solutions that satisfy Novikov's P_V (512), by finding (549) and (552).

$$u_2(\rho_1)(\frac{25}{8}, -\frac{9}{8}, -1, 0) = \tag{558}$$

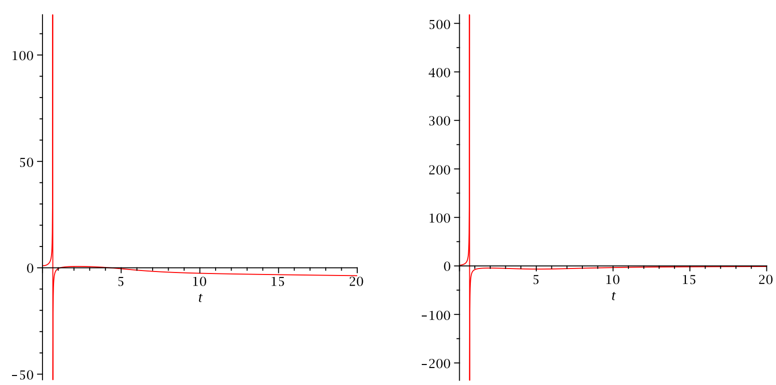
$$\frac{3(-512\rho^3 + 2304\sqrt{2}\rho^{\frac{5}{2}} - 7872\rho^2 + 5952\sqrt{2}\rho^{\frac{3}{2}} - 3960\rho + 180\sqrt{2}\rho^{\frac{1}{2}} + 315)}{1024\sqrt{2}\rho^{\frac{7}{2}} - 10752\rho^3 + 24192\sqrt{2}\rho^{\frac{5}{2}} - 56640\rho^2 + 33840\sqrt{2}\rho^{\frac{3}{2}} - 18360\rho + 990\sqrt{2}\rho^{\frac{1}{2}} + 945}, \tag{559}$$

$$u_2(\rho_2)(\frac{25}{8}, -\frac{9}{8}, 1, 0) = \frac{-120\rho + 150\sqrt{2}\rho^{\frac{1}{2}} - 105 + 16\sqrt{2}\rho^{\frac{3}{2}}}{5(-8\rho + 20\sqrt{2}\rho^{\frac{1}{2}} - 21}, \tag{560}$$

$$u_2(\rho_3)(\frac{9}{8}, -\frac{25}{8}, -1, 0) = \tag{561}$$

$$\frac{1024\sqrt{2}\rho^{\frac{7}{2}} - 10752\rho^3 + 24192\sqrt{2}\rho^{\frac{5}{2}} - 56640\rho^2 + 33840\sqrt{2}\rho^{\frac{3}{2}} - 18360\rho + 990\sqrt{2}\rho^{\frac{1}{2}} + 945}{3(-512\rho^3 + 2304\sqrt{2}\rho^{\frac{5}{2}} - 7872\rho^2 + 5952\sqrt{2}\rho^{\frac{3}{2}} - 3960\rho + 180\sqrt{2}\rho^{\frac{1}{2}} + 315)}, \tag{562}$$

$$u_2(\rho_4)(\frac{9}{8}, -\frac{25}{8}, 1, 0) = \frac{5(-8\rho + 20\sqrt{2}\rho^{\frac{1}{2}} - 21}{-120\rho + 150\sqrt{2}\rho^{\frac{1}{2}} - 105 + 16\sqrt{2}\rho^{\frac{3}{2}}}, \tag{563}$$



(a) $u_2(\rho_3)$ (b) $u_2(\rho_4)$

Figure 26: Two solutions from w_3 P_{III} solutions of Degasperis-Procesi, now solutions to Novikov's P_V

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