ARTICLE TEMPLATE

Zero Dynamics Analysis and Adaptive Tracking Control of Underactuated Multibody Systems with Flexible Links

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ARTICLE HISTORY
Compiled October 10, 2019

ABSTRACT
This paper studies the adaptive tracking control problem for underactuated multibody systems with flexible links in the presence of unknown parameters. A four-body system with input signals acting on the first and fourth bodies is chosen as a benchmark model, which can be decomposed into a control dynamics subsystem and a zero dynamics subsystem. A new and detailed stability analysis is conducted to show that such a zero dynamic system is Lyapunov stable and is partially input-to-state stable under the condition that the speeds of first and fourth bodies are synchronous. The physical meaning of such partial input-to-state stability is clarified. An adaptive controller is developed to ensure such a needed system stabilization condition and thus the boundedness of the desired closed-loop system signal and asymptotic speed tracking of the control dynamics subsystem. Detailed closed-loop system stability and tracking performance analysis are given, in which the tracking errors satisfy the L^1 performance. Extensions to the other multibody systems with flexible links are derived. The developed adaptive controller is applied to a realistic train dynamic model, and simulation results verify the desired system performance.

KEYWORDS
Adaptive control; underactuated multibody systems; trajectory tracking; zero dynamics

1. Introduction

Underactuated mechanical systems have less control inputs than degrees of freedom, which have the advantages of lighter weight, cheaper cost, and less energy (Lai, Wang, Wu, & Cao, 2016). Underactuation is purposely introduced in some systems, such as aircraft, underwater vehicles, and humanoid robots, for which the control problem has attracted much attention (Huang, Wen, Wang, & Song, 2015; Jafari, Mathis, Mukherjee, Khalil, 2016; Lai, Zhang, Wang, & Wu, 2017; Wang, Yang, Shen, Shao, & Wang, 2018; Wu, Luo, Zeng, Li, & Zheng, 2016; Zhang, & Wu, 2015). However, most of the existing results attempt to stabilize only a subset of the system's degrees of

This work was supported in part by the National Natural Science Foundation of China under Grant 61490703, 61922042, 61573180, in part by the USA NSF under Grant ECCS1509704, Qing Lan Project and the Fundamental Research Funds for the Central Universities No. NE2019101.

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freedom, which reduces the complexity of the control problem associated with underactuated mechanical systems (Pucci, Romano, & Nori, 2015). The stability problem of the left subset that has no concern with the system's degrees of freedom, has not been fully studied. Therefore, the study on the control design and stability problem of the underactuated mechanical systems, is of both theoretical challenges and practical importance.

In this paper, the control of a class of underactuated multibody systems, i.e., a four-body system connected by springs and dampers, shown in Fig. 1, will be considered. It is well-known that multibody systems, as a kind of mechanical systems containning several masses connected through joints, can represent many physical systems (Jalon, & Bayo, 1994; Jin, Liu, Zhang, Liu, & Zhao, 2018; Nikravesh, 2007). In recent decades, there have been already some achievements in the multibody systems control area after the tireless effort of researchers. In Rahmani, & Belkheiri (2018), a neural network based adaptive control of flexible multi-link robots in the joint space is presented. In Liu, Tian, & Hu (2012), the dynamics and control of a rigid-flexible multibody system with multiple cylindrical clearance joints are studied via the absolute coordinate based method. In Kim, & Chung (2015), a robust PD control scheme is proposed for flexible joint robots based on a disturbance observer. In He, & Sun (2016), a boundary feedback control is proposed for the flexible robotic manipulator to achieve the desired angular position tracking. Among the existing results, the considered systems are always full-actuated, while the underactuated systems also exist in practice. This has motivated the study of underactuated multibody systems.

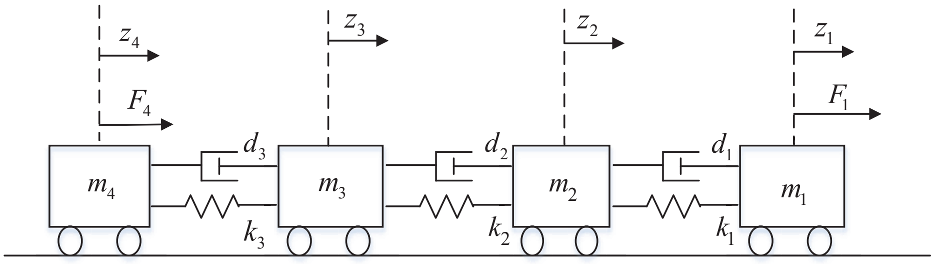

Figure 1. Underactuated four-body system structure

On the other hand, considering the fact that actual values of system parameters are usually not easy to be obtained, and the trajectory tracking problem is an important topic for control systems, the adaptive control of generic systems has received much attention from the control communities (Basin, Panathula, & Shtessel, 2016; Dai, Ren, & Bernstein, 2017; Hovakimyan, Cao, Kharisov, Xargay, & Gregory, 2011; Huang, Wen, Wang, & Jiang, 2013, 2014; Tao, 2003; Wang, Wang, & Shen, 2019). Therefore, an effective adaptive control scheme is to be developed for the underactuated multibody systems to achieve the trajectory tracking and global stability in the presence of unknown system parameters.

Since the considered benchmark model shown in Fig. 1, is an underactuated system, it contains some "internal" behaviors, which are not able to be controlled. The system describing these "internal" behaviors can be defined as *zero dynamics* (Isidori, 1995; Khalil, 2001). The stability performance for the zero dynamics is a key part to ensure the effectiveness of the controller (Hernandez, Castanos, & Fridmana, 2016; Nguyen, Ha, & Lee, 2015; Yue, An, & Sun, 2016). A new type of stability performances called as partially input-to-state stable, is given to solve the stability problem of the zero dynamics of the underactuated multibody systems.

In this paper, an adaptive scheme is proposed to guarantee that the subsystem of the system's degrees of freedom can track the desired trajectories with tracking errors satisfy the L^1 condition, which replaces the traditional obtained L^2 performance in adaptive control design. Another contribution to be highlighted is the detailed and full stability performance being analyzed for the remaining subsystem, which is called the zero dynamics. The stabilization condition is driven and ensured by the proposed adaptive scheme.

The main contributions of this paper cover the following:

(1) There are few research achievements on the adaptive control of underactuated multibody systems in the presence of the unknown parameters. In this paper, the adaptive controller with full-state feedback is applied to control the underactuated multibody systems to fill the gap in this area.
(2) For the adaptive tracking control design, the benchmark 4-body model is decomposed into a control dynamics subsystem and a zero dynamics subsystem, in which the new and detailed stability analysis with the stabilization condition is presented for the zero dynamics subsystem. From the practical application point of view, the physical meaning of such partial input-to-state stability is clarified.
(3) An adaptive controller is designed to achieve the desired tracking performance and ensure the tracking errors satisfy L^1 stabilization condition for the zero dynamics subsystem, in the present of unknown system parameters.
(4) The proposed stability analysis and the adaptive controller design methods are extend to more general multibody systems.

The rest of the paper is organized as follows: Section 2 describes the benchmark four-body system dynamic model and the tracking control problem is formulated. Section 3 focuses on the development of the stabilization condition for the zero dynamics subsystem. Section 4 designs the adaptive tracking controller. Section 5 discusses the extension of the adaptive controller design problem to the other multibody systems. Section 6 includes the simulation study, followed by conclusions in Section VII.

2. System Description and Problem Formulation

A four-body system shown in Fig. 1 is chosen as a benchmark system, which is composed of 4 mass bodies connected by springs and dampers. Based on the Newton's law of motion, the dynamic equation of the i-th $(i = 1, 2, 3, 4)$ body can be described by

$$M_i\ddot{z}_i(t) = F_i(t) + F_{in_{i-1}}(t) - F_{in_i}(t) - F_{r_i}(t), \tag{1}$$

where z_i is the displacement of the i-th body, M_i is the mass of the i-th body, F_i is the external force (control input), F_{in_i} is the restoring force of the spring and damping between the i-th and $(i + 1)$-th bodies, and F_{r_i} is the resistive force due to friction.

The resistive force $F_{r_i}(t)$ is assumed to be linear with respect to the speed v_i, and is described by

$$F_{r_i}(t) = b_{r_i} v_i(t), \tag{2}$$

where $v_i(t)$ is the speed of the i-th body, and b_{r_i} is friction constant.

Due to relatively small displacements, the spring and damping are modelled as a linear function. Then, the restoring force is given as

$$F_{in_i}(t) = k_i(z_i - z_{i+1}) + d_i(\dot{z}_i - \dot{z}_{i+1}), \tag{3}$$

where k_i and d_i are the spring and damping constants; $\dot{z}_i$, $\dot{z}_{i+1}$ and z_i, z_{i+1} are the speed and the displacement of the i-th and $(i+1)$-th bodies, respectively.

The benchmark four-body dynamic model. By direct analysis and from Newton-second law, the motion dynamics of the four-body system with first and fourth bodies having control inputs, are given by

$$M_1\ddot{z}_1(t) = F_1(t) - k_1(z_1(t) - z_2(t)) - d_1(\dot{z}_1(t) - \dot{z}_2(t)) - b_{r_1}\dot{z}_1(t), \tag{4}$$

$$\begin{aligned} M_i\ddot{z}_i(t) = &-k_i(z_i(t) - z_{i+1}(t)) - k_{i-1}(z_i(t) - z_{i-1}(t)) - d_i(\dot{z}_i(t) - \dot{z}_{i+1}(t)) \\ &-d_{i-1}(\dot{z}_i(t) - \dot{z}_{i-1}(t)) - b_{r_i}\dot{z}_i(t), \quad i = 2, 3, \end{aligned} \tag{5}$$

$$M_4\ddot{z}_4(t) = F_4(t) - k_3(z_4(t) - z_3(t)) - d_3(\dot{z}_4(t) - \dot{z}_3(t)) - b_{r_4}\dot{z}_4(t). \tag{6}$$

Set $d_{pq} = \frac{d_p}{M_q}$, $k_{pq} = \frac{k_p}{M_q}$, $b_p = \frac{b_{r_p}}{M_p}$, $m_p = \frac{1}{M_p}$, with $p = 1, 2, 3, 4$ and $q = 1, 2, 3, 4$, and choose $x(t) \in R^8 = [z_1(t), \ \dot{z}_1(t), \ z_2(t), \ \dot{z}_2(t), \ z_3(t), \ \dot{z}_3(t), \ z_4(t), \ \dot{z}_4(t)]^T$. The four-body dynamic equations can be written as

$$\dot{x}(t) = Ax(t) + \begin{bmatrix} 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_4 \end{bmatrix}^T \begin{bmatrix} F_1(t) \\ F_4(t) \end{bmatrix}, \tag{7}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_{11} & -d_{11} - b_1 & k_{11} & d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ k_{12} & d_{12} & -k_{12} - k_{22} & -d_{12} - d_{22} - b_2 & k_{22} & d_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_{23} & d_{23} & -k_{23} - k_{33} & -d_{23} - d_{33} - b_3 & k_{33} & d_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & k_{34} & d_{34} & -k_{34} & -d_{34} - b_4 \end{bmatrix}, \tag{8}$$

with d_{pq}, k_{pq}, b_p, m_p, for $p = 1, 2, 3, 4$ and $q = 1, 2, 3, 4$, being unknown parameters.

Control objective. From the structure of the input matrix, it is clear to see that system (7) is an underactuated system, for which the arbitrary state tracking is not achievable. Here, we choose the speeds of the first and fourth bodies as the controlled variables. When the speeds are non-zero, the displacement states $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ may go to infinity as t goes to infinity. From practical point of view, although the displacement of the first body $z_1(t)$ may be infinity, the displacement errors between adjacent bodies $z_1(t)-z_2(t)$, $z_1(t)-z_4(t)$, and $z_3(t)-z_4(t)$ should be bounded; otherwise the connections between adjacent bodies will be broken.

Based on the analysis above, the control objective of this paper can be summarized as follows: an adaptive controller is to be designed for the multibody system (7) to make $\dot{z}_1(t)$ and $\dot{z}_4(t)$ tracking the same desired speed signal $v_m(t)$, and simultaneously to keep the states $z_1(t) - z_2(t)$, $\dot{z}_2(t)$, $z_1(t) - z_4(t)$, $\dot{z}_3(t)$, and $z_3(t) - z_4(t)$ bounded, in the presence of unknown system parameters d_{pq}, k_{pq} and m_p, for $p, q = 1, 2, 3, 4$.

3. System Zero Dynamics Analysis

For the adaptive control scheme design and stability analysis of the underactuated four-body system (7), we will decompose the considered system (7) into the control dynamics subsystem and zero dynamics subsystem. Then, in this section, the stability analysis for the zero dynamics subsystem will be given in detail, for the first time.

3.1. System Decomposition

To design the tracking controller and analysis the stability property of the system (7), the model (7) should be decomposed, firstly. The basic idea of the decomposition comes from the existing relevant work for nonlinear control systems (Isidori, 1995; Khalil, 2001), which is consistent with the linear control theory.

Based on the state-space form of the system (7), introduce $\xi(t) \triangleq [\xi_1(t), \xi_2(t)]^T = [\dot{z}_1(t), \dot{z}_4(t)]^T$ and $\eta(t) \triangleq [\eta_1(t), \eta_2(t), \eta_3(t), \eta_4(t), \eta_5(t), \eta_6(t)]^T = [z_1(t), z_2(t), \dot{z}_2(t), z_3(t), \dot{z}_3(t), z_4(t)]^T$. Then, system (7) can be decomposed to the control dynamics subsystem:

$$\begin{aligned}\dot{\xi}(t) = &\begin{bmatrix} -d_{11} - b_1 & 0 \\ 0 & -d_{34} - b_4 \end{bmatrix} \xi(t) + \begin{bmatrix} -k_{11} & k_{11} & d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{34} & d_{34} & -k_{34} \end{bmatrix} \eta(t) \\ &+ \begin{bmatrix} m_1 & 0 \\ 0 & m_4 \end{bmatrix} \begin{bmatrix} F_1(t) \\ F_4(t) \end{bmatrix},\end{aligned} \tag{9}$$

and the zero dynamics subsystem

$$\begin{aligned}\dot{\eta}(t) = &\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ k_{12} & -k_{12} - k_{22} & -d_{12} - d_{22} - b_2 & k_{22} & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & k_{23} & d_{23} & -k_{23} - k_{33} & -d_{23} - d_{33} - b_3 & k_{33} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \eta(t) \\ &+ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ d_{12} & 0 \\ 0 & 0 \\ 0 & d_{33} \\ 0 & 1 \end{bmatrix} \xi(t).\end{aligned} \tag{10}$$

From the structure of subsystems (9) and (10), the input signals $F_1(t)$ and $F_4(t)$ influence every state of system (9), while system (10) is decoupled from the input signal.

3.2. Stability of Zero Dynamics

For the subsystem (9), the adaptive control signals $F_1(t)$ and $F_4(t)$ can be designed using the available states $\xi(t)$ and $\eta(t)$, to ensure that the sub-state vector $\xi(t)$ is bounded and tracks an arbitrary desired trajectory. In the subsystem (10), the sub-state vector $\eta(t)$, influenced by the sub-state vector $\xi(t)$, should be bounded to guarantee the stability of system (7) and the effectiveness of the designed control signals $F_1(t)$ and $F_4(t)$. In this subsection, we will present a detailed analysis for the boundedness of the state $\eta(t)$.

Zero dynamics. According to the concept in Isidori (1995), the dynamics of the subsystem (10) correspond to the dynamics describing the "internal" behavior of the system (7), which are called *zero dynamics* driven by $\xi(t)$, the state vector of (9). We will analyze the Lyapunov stability and input-to-state stability of the zero dynamics (10), to ensure that the desired adaptive control performance is guaranteed under the needed stabilization conditions.

State transformation. In (10), the states $\eta_1(t)$, $\eta_2(t)$, $\eta_4(t)$ and $\eta_6(t)$ are the body displacements $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$, which are time-dependent increasing variables, i.e., as t goes to infinity, the states $\eta_1(t)$, $\eta_2(t)$, $\eta_4(t)$ and $\eta_6(t)$ go to infinity, if their corresponding speeds do not go to zero. It means that some of the states in the zero dynamics (10) may not be bounded. In order to deal with this case, an appropriate state transformation should be introduced. The displacement errors between adjoining bodies can and should be bounded; otherwise the connections between adjoining bodies will be broken. Thus, the displacement errors can be employed for the transformation, which makes the corresponding error dynamic system more suitable for the stability analysis of (10).

Introduce the state transformation $\omega(t) = [\omega_1(t),\ \omega_2(t),\ \omega_3(t),\ \omega_4(t),\ \omega_5(t),\ \omega_6(t)]^T = [\eta_1(t),\ \ \eta_1(t) - \eta_2(t),\ \ \eta_3(t),\ \ \eta_1(t) - \eta_6(t),\ \ \eta_5(t),\ \ \eta_4(t) - \eta_6(t)]^T$. The subsystem (10) can be rewritten as

$$\dot{\omega}(t) \triangleq A_1\omega(t) + B_1\xi(t), \tag{11}$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & k_{12}+k_{22} & -d_{12}-d_{22}-b_2 & -k_{22} & d_{22} & k_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_{23} & d_{23} & k_{23} & -d_{23}-d_{33}-b_3 & -k_{23}-k_{33} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ d_{12} & 0 \\ 1 & -1 \\ 0 & d_{33} \\ 0 & -1 \end{bmatrix}. \tag{12}$$

All the elements of the first row and first column of the state distribution matrix A_1 are zero. The state transformation from system (10) to (11) is a nonsingular coordination transformation. Thus, the stability property of (11) is equivalent to that of (10).

Remark 1: Since the system (11) is a linear system, we will introduce a weak input-to-state stability concept for it: the boundedness of $\omega(t)$ for bounded initial conditions $\omega(0)$ and bounded input $\xi(t)$. This will be characterized by the Lyapunov stability of $\dot{\omega} = A_1\omega$ and the bounded-input-bounded-state stability of $\dot{\omega} = A_1\omega + B_1\xi$. We will first establish the desired Lyapunov stability for $\dot{\omega} = A_1\omega$. This means the input-to-state stability study will be based on the study of the bounded-input-bounded-state stability of $\dot{\omega} = A_1\omega + B_1\xi$. Note that an original input-to-state stability concept requires that with the initial condition $\omega(0)$, $\omega(t)$ goes to zero as t goes to infinity (Sontag, 2004). □

Lyapunov stability. Consider the transformed system (11) with $\xi(t)$ as input. Setting $\xi(t) = 0$, the characteristic polynomial of (11) can be calculated as

$$P(\lambda) = \lambda^2(\lambda^4 + a_{13}\lambda^3 + a_{12}\lambda^2 + a_{11}\lambda + a_{10}), \tag{13}$$

where

$$\begin{aligned}
a_{13} &= b_2 + b_3 + d_{12} + d_{22} + d_{23} + d_{33}, &(14)\\
a_{12} &= k_{12} + k_{22} + k_{23} + k_{33} + b_2b_3 + b_3d_{12} + b_2d_{23} + b_3d_{22} + b_2d_{33}\\
&\quad + d_{12}d_{23} + d_{12}d_{33} + d_{22}d_{33}, &(15)\\
a_{11} &= d_{12}k_{23} + d_{23}k_{12} + d_{12}k_{33} + d_{33}k_{12} + d_{22}k_{33} + d_{33}k_{22} + b_3k_{12}\\
&\quad + b_2k_{23} + b_3k_{22} + b_2k_{33}, &(16)\\
a_{10} &= k_{12}k_{23} + k_{12}k_{33} + k_{22}k_{33}. &(17)
\end{aligned}$$

From (13), the matrix A_1 in (11) has two zero eigenvalues, corresponding to the dynamic equations

$$\begin{aligned}
\dot{\omega}_1(t) &= 0, \quad \text{i.e.,} \quad \dot{z}_1(t) = 0, &(18)\\
\dot{\omega}_4(t) &= 0, \quad \text{i.e.,} \quad \dot{z}_1(t) - \dot{z}_4(t) = 0. &(19)
\end{aligned}$$

which implies

$$\begin{aligned}
\omega_1(t) &= z_1(t) = z_1(0), &(20)\\
\omega_4(t) &= z_1(t) - z_4(t) = z_1(0) - z_4(0), \quad \text{i.e.,} \quad z_4(t) = z_4(0). &(21)
\end{aligned}$$

Hence, we can conclude that the zero dynamics (11) is Lyapunov stable, if the other four eigenvalues of A_1 are stable, i.e., the zeros of $\lambda^4 + a_{13}\lambda^3 + a_{12}\lambda^2 + a_{11}\lambda + a_{10}$ have negative real parts.

Since the parameters k_{pq}, d_{pq} and b_p, for $p = 1, 2, 3, 4$ and $q = 1, 2, 3, 4$, are all positive, it follows that $a_{13} > 0$, $a_{12} > 0$, $a_{11} > 0$ and $a_{10} > 0$. According to the Routh criterion, to make the zeros of $\lambda^4 + a_{13}\lambda^3 + a_{12}\lambda^2 + a_{11}\lambda + a_{10}$ have negative real parts, the signs of the first column elements of its Routh Array should not change, which requires

$$b_{11} = \frac{a_{13}a_{12} - a_{11}}{a_{13}} > 0, \quad c_{11} = a_{11} - \frac{a_{13}a_{10}}{b_{11}} > 0. \tag{22}$$

Recalling the definitions of the parameters $k_{22} = \frac{k_2}{M_2}$, $k_{23} = \frac{k_2}{M_3}$, $d_{22} = \frac{d_2}{M_2}$ and $d_{23} = \frac{d_2}{M_3}$, the relationship between the parameters can be obtained as $\frac{k_{22}}{k_{23}} = \frac{d_{22}}{d_{23}}$. Further, using (14)-(16), it is obtained that $b_{11} > 0$ for all positive constant parameters k_{pq}, d_{pq} and b_p. Thus, the following result can be obtained directly.

Lemma 1: *If $c_{11} > 0$ in (22), then the zero dynamic (11) is Lyapunov stable, that is, the solution $\omega(t)$ of $\dot{\omega} = A_1\omega$ is bounded for $\omega(0) \neq 0$.*

More detailed analysis about the condition $c_{11} > 0$ will be given in Section 3.3.

Input-to-state stability. The objective now is to analyze the input-to-state stability property of (11) with $\xi(t)$ as input. Let $\xi(t) \neq 0$, $\omega(0) = 0$ and $\xi(t) = [\xi_1(t), \xi_2(t)]^T$. From (11), we have

$$\dot{\omega}_1(t) = \xi_1(t), \tag{23}$$
$$\dot{\omega}_4(t) = \xi_1(t) - \xi_2(t). \tag{24}$$

As $\xi_1(t)$ and $\omega_1(t)$ represent the speed and displacement of the first body, respectively, the displacement trajectory $\omega_1(t)$ has a desired tracking property, if the speed $\xi_1(t)$ tracks a desired speed trajectory by designing control. Then, the state $\omega_1(t) = z_1(t)$ satisfies the system performance, even if $\lim_{t\to\infty} \omega_1(t) = \infty$. Hence, in the analysis of the bounded-input-bounded-state (bounded-output) stability of (11), the state variable $\omega_1(t)$ is separated from the rest of the state variables in $\omega(t)$ and only the boundedness of the partial state vector $\bar{\omega}(t) = [\omega_2(t), \omega_3(t), \omega_4(t), \omega_5(t), \omega_6(t)]^T$ is considered.

On the other hand, $\omega_4(t) = z_1(t) - z_4(t)$ represents the displacement error between first and fourth bodies, which is bounded under the condition that $\xi_1(t) - \xi_2(t) \in L^1$, that is $\int_0^\infty (\xi_1(t) - \xi_2(t))dt < \infty$, which is the case when $\xi_1(t) = \xi_2(t)$ or $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially, as ensured by a nominal controller, when a specific adaptive controller is used, to be addressed in the next section. So, to analyze the input-to-state stability of the zero dynamics (11), the state $\omega_1(t)$ may be eliminated and the condition $\xi_1(t) = \xi_2(t)$ is considered.

Eliminating the state $\omega_1(t)$ from (11), the following transfer function matrix from ξ to $\bar{\omega}$ can be calculated

$$\bar{G}(s) = \frac{\bar{\omega}(s)}{\xi(s)} \triangleq \frac{1}{P(s)} \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \\ Z_{31}(s) & Z_{32}(s) \\ Z_{41}(s) & Z_{42}(s) \\ Z_{51}(s) & Z_{52}(s) \end{bmatrix}, \tag{25}$$

with

$$P(s) = s(s^4 + a_{13}s^3 + a_{12}s^2 + a_{11}s + a_{10}), \tag{26}$$
$$Z_{11}(s) = s^4 + b_{113}s^3 + b_{112}s^2 + b_{111}s + k_{22}k_{33}, \tag{27}$$
$$Z_{12}(s) = b_{122}s^2 + b_{121}s - k_{22}k_{33}, \tag{28}$$
$$Z_{21}(s) = s(b_{213}s^3 + b_{212}s^2 + b_{211}s + b_{210}), \quad Z_{22}(s) = s(b_{222}s^2 + b_{221}s + b_{220}), \tag{29}$$
$$Z_{31}(s) = s^4 + b_{313}s^3 + b_{312}s^2 + b_{311}s + k_{12}k_{23} + k_{12}k_{33} + k_{22}k_{33}, \tag{30}$$
$$Z_{32}(s) = -s^4 - b_{313}s^3 - b_{312}s^2 - b_{311}s - k_{12}k_{23} - k_{12}k_{33} - k_{22}k_{33}, \tag{31}$$
$$Z_{41}(s) = s(b_{412}s^2 + b_{411}s + b_{410}), \quad Z_{42}(s) = s(b_{423}s^3 + b_{422}s^2 + b_{421}s + b_{420}), \tag{32}$$
$$Z_{51}(s) = b_{512}s^2 + b_{511}s + k_{12}k_{23}, \tag{33}$$
$$Z_{52}(s) = -b_{524}s^4 - b_{523}s^3 - b_{522}s^2 - b_{521}s - k_{12}k_{23}, \tag{34}$$

where a_{ij}, b_{ijk}, $i, j, k = 0, 1, 2, 3, 4$, are the coefficients of the entries of the transfer functions and are all positive, with a_{13}, a_{12}, a_{11} and a_{10} being defined as (14)-(17).

From now on, the input-to-state stability problem of the zero dynamics (11) has been transferred to the bounded-input-bounded-state stability of $\bar{G}(s)$ in (25). From

(25)-(34), it can be seen that under the operating condition $\xi_1(t) = \xi_2(t)$, the zero $s = 0$ of $P(s)$ can be canceled by the zero $s = 0$ in the numerators in $\bar{G}(s)$, that is, $\bar{G}(s) = \frac{N(s)}{\bar{P}(s)}$, $\bar{P}(s) = s^4 + a_{13}s^3 + a_{12}s^2 + a_{11}s + a_{10}$, which implies that the states $\omega_2(t)$, $\omega_3(t)$, $\omega_4(t)$, $\omega_5(t)$ and $\omega_6(t)$ are bounded, if $\xi_1(t) = \xi_2(t)$ is bounded and $\bar{P}(s) = s^4 + a_{13}s^3 + a_{12}s^2 + a_{11}s + a_{10}$ is stable.

Recall that the condition $c_{11} > 0$ in (22) for the zero dynamics (11) to be Lyapunov stable ensures that $\bar{P}(s)$ is stable. Hence, we have the following result.

Lemma 2: *Under the operating condition that $\xi_1(t) = \xi_2(t)$ (i.e., $\dot{z}_1(t) = \dot{z}_4(t)$), the dynamic system $\bar{G}(s)$ defined in (25) is bounded-input-bounded-state (BIBS) stable, if and only if c_{11} defined in (22) is positive.*

Moreover, it can be verified from (11) that for $\xi_1(t) \neq \xi_2(t)$, $\omega_2(s) = \frac{Z_{11}(s)\xi_1(s)+Z_{12}(s)\xi_2(s)}{P(s)}$. Then, for $\xi_1(t) \neq \xi_2(t)$ but $\delta_{12}(t) = \xi_1(t) - \xi_2(t) \in L^1$, that is $\int_0^\infty (\xi_1(t) - \xi_2(t))dt < \infty$, $\omega_2(t)$ is also bounded, as

$$\omega_2(s) = \frac{Z_{11}(s) + Z_{12}(s)}{P(s)}\xi_2(s) + \frac{Z_{11}(s)}{P(s)}\delta_{12}(s), \tag{35}$$

in which the zero $s = 0$ of $P(s)$ is cancelled in $\frac{Z_{11}(s)+Z_{12}(s)}{P(s)}$, $\xi_2(t)$ is bounded, and all poles of $\frac{Z_{11}(s)}{P(s)}$ are stable except for one at $s = 0$ and $\delta_{12}(t)$ is integrable. The same conclusions can be obtained for the states $\omega_4(t)$ and $\omega_6(t)$, that is, $\xi_1(t) - \xi_2(t) \in L^1$ results in the boundedness of $\omega_4(t)$ and $\omega_6(t)$ in (25). Hence, we have:

Corollary 1: *Under the condition that $\xi_1(t) - \xi_2(t) \in L^1$, the partial state vector $\bar{\omega}(t) = [\omega_2(t), \omega_3(t), \omega_4(t), \omega_5(t), \omega_6(t)]^T$ in the zero dynamic (11) is bounded for bounded $\xi(t) = [\xi_1(t), \xi_2(t)]^T$ if and only if c_{11} defined in (22) is positive.*

Summary. From now on, a new partial input-to-state stability property of the multibody system zero dynamics is shown. The zero dynamic system (10) or (11) of the four-body system model (7) is Lyapunov stable (that is, any initial condition in a bounded set leads to a bounded state, for this linear system model case of (10) or (11)). For the input-to-state stability study, there are two special features for the equation (11): one of the zero dynamic system states, $\omega_1(t)$ in (23), is a representative displacement variable, and a speed error (relative speed) signal, $\xi_1(t) - \xi_2(t)$, is acting as the input to the zero dynamic system state variable $\omega_4(t)$ (see (24)). For speed tracking, such a position variable can go infinity and the speed error signal should have certain convergence property in order to ensure the corresponding relative displacement signals, $\omega_2(t)$, $\omega_4(t)$, $\omega_6(t)$ being bounded, to avoid damage to the connecting springs; both are physically meaningful. Hence, the specification of input-to-state stability should be based on such considerations, that the representative displacement variable is allowed to be unbounded, and the relative speed signal as input is taken as $\xi_1(t) - \xi_2(t) = 0$. Then, the input-to-state stability properties can be completely characterized as above. The transfer functions $\frac{Z_{i1}(s)+Z_{i2}(s)}{P(s)}$ in (25), $i = 1, 2, 4, 5$, are all BIBS stable, while $\frac{Z_{31}(s)+Z_{32}(s)}{P(s)} = 0$, which is the result of $\xi_1(t) = \xi_2(t)$ in (11). It should be noted that the desired convergence property of $\xi_1(t) - \xi_2(t)$ is either exponentially convergent in the case of nominal control, or L^1 convergent (integrable) in the case of adaptive control, to be shown in the next section. If $\xi_1(t) - \xi_2(t) \neq 0$ but converges to zero exponentially, or is integrable over time, all state variables except for the representative displacement variable can be shown to remain bounded with bounded $\xi_1(t)$ and $\xi_2(t)$ as inputs, in particular, $\omega_4(t) = \int_0^t (\xi_1(\tau) - \xi_2(\tau))d\tau < \infty$.

3.3. Stability Condition

From Lemma 1 and corollary 1, the key condition to make the zero dynamic system (10) or (11) Lyapunov stable and the system (11) partial BIBS stable, is to ensure $c_{11} > 0$ in (22). With (13)-(17), it can be seen that there are ten parameters in inequality (22), and only one constraint condition $\frac{k_{22}}{k_{23}} = \frac{d_{22}}{d_{23}} = \frac{M_3}{M_2}$ from (7). It is quite complicated to derive the general conditions for these parameters k_{pq}, d_{pq} and b_p, to guarantee $c_{11} > 0$.

We can use the numerical analysis method to check the condition $c_{11} > 0$. From the points of numerical analysis and engineering background, it concludes that c_{11} can always be positive, i.e., the zero dynamic system (10) or (11) is always Lyapunov stable and the states $\omega_2(t)$, $\omega_3(t)$, $\omega_4(t)$, $\omega_5(t)$, and $\omega_6(t)$ of (11) are always BIBS stable, in practice.

If the spring, damping, mass and b_i are the same, it is easy to check the stability of the zero dynamic system (10) via Lemma 1 and Corollary 1. According to (25) and under the case $k \triangleq k_{12} = k_{22} = k_{23} = k_{33}$, $d \triangleq d_{12} = d_{22} = d_{23} = d_{33}$ and $b \triangleq b_1 = b_2 = b_3 = b_4$, we can calculate that $b_{11} > 0$ and $c_{11} > 0$ in (22). Hence, we have the following corollary.

Corollary 2: *Under the condition that $\xi_1(t) - \xi_2(t) \in L^1$, $k_{12} = k_{22} = k_{23} = k_{33}$, $d_{12} = d_{22} = d_{23} = d_{33}$ and $b_1 = b_2 = b_3 = b_4$, the partial state vector $\bar{\omega}(t) = [\omega_2(t), \omega_3(t), \omega_4(t), \omega_5(t), \omega_6(t)]^T$ in the zero dynamic (11) is bounded if $\xi(t) = [\xi_1(t), \xi_2(t)]^T$ is bounded.*

Remark 2: The case that the same kinds of parameters in different bodies in a multibody system have the same values, is a realistic case in many applications. A typical example is the train system, in which the bodies (cars) have the same model. □

We have studied and established the key condition $c_{11} > 0$ for the zero dynamic system to be Lyapunov stable and partial bounded-input-bounded-state stable (see Lemmas 1 and 2, and Corollary 1). Moreover, if the nominal operating condition $\xi_1(t) = \xi_2(t)$ or $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially, or a relaxed operating condition $\xi_1(t) - \xi_2(t) \in L^1$, the states $\omega_2(t)$, $\omega_3(t)$, $\omega_4(t)$, $\omega_5(t)$, and $\omega_6(t)$ (i.e., $z_1(t) - z_2(t)$, $\dot{z}_2(t)$, $z_1(t) - z_4(t)$, $\dot{z}_3(t)$, and $z_3(t) - z_4(t)$) of the zero dynamics (11) can be bounded and $\omega_1(t) = z_1(t)$ satisfies the displacement performance.

4. Adaptive Controller Scheme

A new and detailed study of the zero dynamics of multibody systems has been given in Section 3. According to Corollary 1, we can see that comparing with the condition $\xi_1(t) - \xi_2(t) = 0$ or $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially, $\xi_1(t) - \xi_2(t) \in L^1$ is the weakest condition to ensure the zero dynamic system (10) to satisfy the partial stability, allowing the state $\omega_1(t) = \eta_1(t)$ to go to ∞. Considering the tracking problem for the multibody system, the desired speed tracking of $v_m(t)$ is to be achieved by the designed controller.

An desirable adaptive controller should be designed to ensure the states $\xi_1(t)$ and $\xi_2(t)$ in the control dynamics subsystem (9) to track the signal $v_m(t)$ and the condition in Corollary 1 to be satisfied, i.e., $\lim_{t\to\infty}(\xi_1(t)-v_m(t)) = 0$, $\lim_{t\to\infty}(\xi_2(t)-v_m(t)) = 0$ and $\xi_1(t)-\xi_2(t) \in L^1$. If $\xi_1(t)-v_m(t) \in L^1$ and $\xi_2(t)-v_m(t) \in L^1$, then $\xi_1(t)-\xi_2(t) \in L^1$ is ensured. Thus, the adaptive controller scheme for the system (9) is designed, in the presence of the unknown parameters k_{pq}, d_{pq} and b_p, to satisfy the following properties: $\xi_1(t)-v_m(t) \in L^1$, $\lim_{t\to\infty}(\xi_1(t)-v_m(t)) = 0$, and $\xi_2(t)-v_m(t) \in L^1$, $\lim_{t\to\infty}(\xi_2(t)-v_m(t)) = 0$. In this section, we develop such an adaptive control scheme and give a complete analysis of its stability and tracking properties.

4.1. Adaptive Control Design for $F_1(t)$

From the structure of the system (9), it can be seen that the states $\xi_1(t)$ and $\xi_2(t)$ are decoupled, and thus the control inputs $F_1(t)$ and $F_4(t)$ can be designed separately.

Nominal control. To develop an adaptive control law for the system (9) with unknown parameters, it is needed to derive a nominal control law which can satisfy the matching condition and achieve the tracking of $v_m(t)$ by $\xi_1(t)$, when implemented by the plant parameters. The nominal control law is designed:

$$F_1^*(t)=k_{\xi_1}^*\xi_1(t) + k_{11}^*(\eta_1(t)-\eta_2(t)) - d_{11}^*\eta_3(t) + m_1^*\dot{v}_m(t) + k_{r1}^*v_m(t), \tag{36}$$

with the parameters $k_{\xi_1}^*$, k_{11}^*, d_{11}^*, m_1^* and k_{r1}^*, satisfying:

$$k_{11}=m_1k_{11}^*, \quad d_{11}=m_1d_{11}^*, \quad 1=m_1m_1^*, \tag{37}$$
$$-d_{11}-b_1+m_1k_{\xi_1}^* = -m_1k_{r1}^* < 0. \tag{38}$$

The equations in (37)-(38) are the matching conditions, that is, if the constant parameters k_{11}, d_{11}, m_1, and b_1 are known, then the nominal parameters $k_{\xi_1}^*$, k_{11}^*, d_{11}^*, m_1^*, k_{r1}^* exist to satisfy (37)-(38) and the nominal control law (36) leads the tracking errors $e_{\xi_1}(t) = \xi_1(t)-v_m(t)$ to satisfy

$$\begin{aligned}\dot{e}_{\xi_1}(t)&=(-d_{11}-b_1+m_1k_{\xi_1}^*)\xi_1(t)+m_1k_{r1}^*v_m(t)\\&=-m_1k_{r1}^*e_{\xi_1}(t),\end{aligned} \tag{39}$$

which implies that $e_{\xi_1}(t)$ converges to zero exponentially as $t\to\infty$, due to $-m_1k_{r1}^* < 0$.

Adaptive controller structure. The adaptive state feedback controller structure for $F_1(t)$ is chosen as:

$$\begin{aligned}F_1(t)=&\hat{k}_{\xi_1}(t)\xi_1(t)+\hat{k}_{11}(t)(\eta_1(t)-\eta_2(t))-\hat{d}_{11}(t)\eta_3(t)+\hat{m}_1(t)\dot{v}_m(t)\\&+\hat{k}_{r1}v_m(t)-\alpha_{\xi_1}sgn(\xi_1(t)-v_m(t)),\end{aligned} \tag{40}$$

where $\hat{k}_{\xi_1}(t)$, $\hat{k}_{11}(t)$, $\hat{d}_{11}(t)$, $\hat{m}_1(t)$, and $\hat{k}_{r1}(t)$ are the estimates of the nominal parameters $k_{\xi_1}^*$, k_{11}^*, d_{11}^*, m_1^*, k_{r1}^*, and $\alpha_{\xi_1} > 0$ is a design parameter related to the convergence rate of $\xi_1(t)-v_m(t)$, and sgn is the usual sign function.

Closed-loop adaptive control system. When the constant parameters k_{11}, d_{11}, m_1, and b_1 are unknown, it is required to use the adaptive controller (40) to ensure the stability and tracking of the closed-loop system. To design the adaptive laws for

$\hat{k}_{\xi_1}(t)$, $\hat{k}_{11}(t)$, $\hat{d}_{11}(t)$, $\hat{m}_1(t)$, and $\hat{k}_{r1}(t)$, we define the parameter errors

$$\tilde{k}_{\xi_1}(t) = \hat{k}_{\xi_1}(t) - k^*_{\xi_1}, \quad \tilde{k}_{11}(t) = \hat{k}_{11}(t) - k^*_{11}, \tag{41}$$

$$\tilde{d}_{11}(t) = \hat{d}_{11}(t) - d^*_{11}, \quad \tilde{m}_1(t) = \hat{m}_1(t) - m^*_1, \tag{42}$$

$$\tilde{k}_{r1}(t) = \hat{k}_{r1}(t) - k^*_{r1}, \tag{43}$$

and use the control law (40) and the system (9) under the matching condition (37)-(38), to obtain

$$\begin{aligned}\dot{e}_{\xi_1}(t) = {} & -m_1 k^*_{r1} e_{\xi_1}(t) + \frac{1}{m^*_1}\Big(\tilde{k}_{\xi_1}(t)\xi_1(t) + \tilde{k}_{11}(t)(\eta_1(t) - \eta_2(t)) \\ & -\tilde{d}_{11}(t)\eta_3(t) + \tilde{m}_1(t)\dot{v}_m(t) + \tilde{k}_{r1}(t)v_m(t)\Big) - \alpha_{\xi_1} sgn(e_{\xi_1}(t)). \end{aligned} \tag{44}$$

Adaptive laws. The following parameter update laws are used for the controller (40):

$$\dot{\hat{k}}_{\xi_1}(t) = -\Gamma_{\xi_1}\xi_1(t)e_{\xi_1}(t), \tag{45}$$

$$\dot{\hat{k}}_{11}(t) = -\Gamma_{k_{11}}(\eta_1(t) - \eta_2(t))e_{\xi_1}(t), \tag{46}$$

$$\dot{\hat{d}}_{11}(t) = \Gamma_{d_{11}}\eta_3(t)e_{\xi_1}(t), \tag{47}$$

$$\dot{\hat{m}}_1(t) = -\Gamma_{m_1}\dot{v}_m(t)e_{\xi_1}(t), \tag{48}$$

$$\dot{\hat{k}}_{r1}(t) = -\Gamma_{k_{r1}}v_m(t)e_{\xi_1}(t), \tag{49}$$

where Γ_{ξ_1}, $\Gamma_{k_{11}}$, $\Gamma_{d_{11}}$, Γ_{m_1}, and $\Gamma_{k_{r1}}$ are positive constants.

4.2. Adaptive Control Design for $F_4(t)$

Similar to the design procedure of the adaptive controller $F_1(t)$, the nominal controller $F^*_4(t)$ and adaptive controller $F_4(t)$ are proposed as:

$$\begin{aligned} F^*_4(t) = {} & k^*_{\xi_2}(t)\xi_2(t) - k^*_{34}(t)(\eta_4(t) - \eta_6(t)) - d^*_{34}(t)\eta_5(t) \\ & + m^*_4(t)\dot{v}_m(t) + k^*_{r2}(t)v_m(t), \end{aligned} \tag{50}$$

$$\begin{aligned} F_4(t) = {} & \hat{k}_{\xi_2}(t)\xi_2(t) - \hat{k}_{34}(t)(\eta_4(t) - \eta_6(t)) - \hat{d}_{34}(t)\eta_5(t) + \hat{m}_4(t)\dot{v}_m(t) \\ & + \hat{k}_{r2}(t)v_m(t) - \alpha_{\xi_2} sgn(\xi_2(t) - v_m(t)), \end{aligned} \tag{51}$$

where $\alpha_{\xi_2} > 0$ is a design parameter related to the convergence rate of $\xi_2(t) - v_m(t)$, *sgn* is the sign function, $\hat{k}_{\xi_2}(t)$, $\hat{k}_{34}(t)$, $\hat{d}_{34}(t)$, $\hat{m}_4(t)$, and $\hat{k}_{r2}(t)$ are the time-varying estimates of the nominal controller parameters $k^*_{\xi_2}$, k^*_{34}, d^*_{34}, m^*_4 and k^*_{r2} satisfying:

$$k_{34} = m_4 k^*_{34}, \quad d_{34} = m_4 d^*_{34}, \quad 1 = m_4 m^*_4, \tag{52}$$

$$-d_{34} - b_4 + m_4 k^*_{\xi_2} = -m_4 k^*_{r2} < 0. \tag{53}$$

Then, if the parameters k_{34}, d_{34}, m_4, and b_4 are known, the nominal parameters $k^*_{\xi_2}$, k^*_{34}, d^*_{34}, m^*_4 and k^*_{r2} exist to satisfy (52)-(53) and the nominal control law (50) leads the tracking error $e_{\xi_2}(t) = \xi_2(t) - v_m(t)$ to satisfy

$$\dot{e}_{\xi_2}(t) = -m_4 k_{r2}^* e_{\xi_2}(t), \tag{54}$$

which implies that $e_{\xi_2}(t)$ converges to zero exponentially as $t \to \infty$, due to $-m_4 k_{r2}^* < 0$.

To design the adaptive update law for $\hat{k}_{\xi_2}(t)$, $\hat{k}_{34}(t)$, $\hat{d}_{34}(t)$, $\hat{m}_4(t)$, and $\hat{k}_{r2}(t)$, we define the parameter errors as

$$\tilde{k}_{\xi_2}(t) = \hat{k}_{\xi_2}(t) - k_{\xi_2}^*, \quad \tilde{k}_{34}(t) = \hat{k}_{34}(t) - k_{34}^*, \tag{55}$$

$$\tilde{d}_{34}(t) = \hat{d}_{34}(t) - d_{34}^*, \quad \tilde{m}_4(t) = \hat{m}_4(t) - m_4^*, \tag{56}$$

$$\tilde{k}_{r2}(t) = \hat{k}_{r2}(t) - k_{r2}^*, \tag{57}$$

and use the control law (51) and the system (9) under the matching condition (52)-(53), to obtain

$$\begin{aligned}\dot{e}_{\xi_2}(t) = {} & -m_4 k_{r2}^* e_{\xi_2}(t) + \frac{1}{m_4^*}\Big(\tilde{k}_{\xi_2}(t)\xi_2(t) + \tilde{k}_{34}(t)(\eta_4(t) - \eta_6(t)) \\ & -\tilde{d}_{34}(t)\eta_5(t) + \tilde{m}_4(t)\dot{v}_m(t) + \tilde{k}_{r2}(t)v_m(t)\Big) - \alpha_{\xi_2} sgn(e_{\xi_2}(t)).\end{aligned} \tag{58}$$

The following adaptive laws are designed to update the control parameters of (51):

$$\dot{\hat{k}}_{\xi_2}(t) = -\Gamma_{\xi_2}\xi_2(t)e_{\xi_2}(t), \tag{59}$$

$$\dot{\hat{k}}_{34}(t) = \Gamma_{k_{34}}(\eta_4(t) - \eta_6(t))e_{\xi_2}(t), \tag{60}$$

$$\dot{\hat{d}}_{34}(t) = \Gamma_{d_{34}}\eta_5(t)e_{\xi_2}(t), \tag{61}$$

$$\dot{\hat{m}}_4(t) = -\Gamma_{m_4}\dot{v}_m(t)e_{\xi_2}(t), \tag{62}$$

$$\dot{\hat{k}}_{r2}(t) = -\Gamma_{k_{r2}}v_m(t)e_{\xi_2}(t), \tag{63}$$

where the parameters Γ_{ξ_2}, $\Gamma_{k_{34}}$, $\Gamma_{d_{34}}$, Γ_{m_4}, and $\Gamma_{k_{r2}}$ are positive constants.

4.3. Stability Analysis

In this subsection, the nominal control system performance is to be analyzed firstly, to show that the nominal controllers (36) and (50) can guarantee that the speed tracking is achieved and $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially is ensured, when the system parameters are known. When the system parameters are unknown, the performance analysis for the adaptive control system is to be given as well to show that both the speed tracking and $\xi_1(t) - \xi_2(t) \in L^1$ can be ensured by the proposed adaptive controllers (40) and (51).

Nominal control system performance. For the nominal controllers (36) and (50), it follows from (39) and (54),

$$e_{\xi_1}(t) = e^{-m_1 k_{r1}^* t} e_{\xi_1}(0), \quad e_{\xi_2}(t) = e^{-m_4 k_{r2}^* t} e_{\xi_2}(0), \tag{64}$$

which implies that the nominal controllers $F_1^*(t)$ and $F_4^*(t)$ can make $\lim_{t\to\infty}(\xi_1(t) - v_m(t)) = 0$, $\lim_{t\to\infty}(\xi_2(t) - v_m(t)) = 0$ exponentially, and $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$

exponentially.

With $v_m(t)$ being bounded, $\xi_1(t)$ and $\xi_2(t)$ (i.e., $\dot{z}_1(t)$ and $\dot{z}_4(t)$) are bounded. Using the matching-conditions (37)-(38) and (52)-(53), the nominal controller parameters $k_{\xi_1}^*$, k_{11}^*, d_{11}^*, m_1^*, k_{r1}^*, $k_{\xi_2}^*$, k_{34}^*, d_{34}^*, m_4^* and k_{r2}^* are bounded. Considering the dynamics $\bar{G}(s)$ in (25) and with $\lim_{t\to\infty}(\xi_1(t)-\xi_2(t)) = 0$ exponentially, it has that $\eta_1(t)-\eta_2(t)$, $\eta_3(t)$, $\eta_4(t)-\eta_6(t)$, and $\eta_5(t)$ (i.e., $z_1(t)-z_2(t)$, $\dot{z}_2(t)$, $z_1(t)-z_4(t)$, $\dot{z}_3(t)$, and $z_3(t)-z_4(t)$) are bounded, Then, with the structure of the nominal controllers (36) and (50), the boundedness of $F_1^*(t)$ and $F_4^*(t)$ are ensured.

Further, with Corollary 1, we can have the following result:

Theorem 1: *The nominal controllers (36) and (50), with the parameters satisfying the matching conditions (37)-(38) and (52)-(53), applied to the system (7), guarantee that the corresponding closed-loop state signals $\dot{z}_1(t)$, $z_1(t) - z_2(t)$, $\dot{z}_2(t)$, $z_1(t) - z_4(t)$, $\dot{z}_3(t)$, $z_3(t) - z_4(t)$, $\dot{z}_4(t)$ are bounded, and the speed tracking errors satisfy $\lim_{t\to\infty}(\dot{z}_1(t) - v_m(t)) = 0$, $\lim_{t\to\infty}(\dot{z}_4(t) - v_m(t)) = 0$ exponentially.*

Adaptive control system performance. As the parameters k_{pq}, d_{pq} and b_p of the control dynamics subsystem (9) are unknown, the adaptive controllers (40) and (51) are applied. Consider the following continuous Lyapunov function

$$V = e_{\xi_1}^2 + e_{\xi_2}^2 + \frac{1}{m_1^*}\left(\Gamma_{\xi_1}^{-1}\tilde{k}_{\xi_1}^2 + \Gamma_{k_{11}}^{-1}\tilde{k}_{11}^2 + \Gamma_{d_{11}}^{-1}\tilde{d}_{11}^2 + \Gamma_{m_1}^{-1}\tilde{m}_1^2 + \Gamma_{k_{r1}}^{-1}\tilde{k}_{r1}^2\right)$$
$$+\frac{1}{m_4^*}\left(\Gamma_{\xi_2}^{-1}\tilde{k}_{\xi_2}^2 + \Gamma_{k_{34}}^{-1}\tilde{k}_{34}^2 + \Gamma_{d_{34}}^{-1}\tilde{d}_{34}^2 + \Gamma_{m_4}^{-1}\tilde{m}_4^2 + \Gamma_{k_{r2}}^{-1}\tilde{k}_{r2}^2\right), \quad (65)$$

where e_{ξ_1} and e_{ξ_2} are tracking errors, $\tilde{k}_{\xi_1}$, $\tilde{k}_{11}$, $\tilde{d}_{11}$, $\tilde{m}_1$ and $\tilde{k}_{r1}$ are defined in (41)-(43), $\tilde{k}_{\xi_2}$, $\tilde{k}_{34}$, $\tilde{d}_{34}$, $\tilde{m}_4$ and $\tilde{k}_{r2}$ are defined in (55)-(57).

With the tracking error dynamics in (44) and (58) and the adaptive laws in (45)-(49) and (59)-(63), the time derivative of V becomes

$$\dot{V} = -m_1 k_{r1}^* e_{\xi_1}^2(t) - m_1\alpha_{\xi_1}|e_{\xi_1}(t)| - m_4 k_{r2}^* e_{\xi_2}^2(t) - m_4\alpha_{\xi_2}|e_{\xi_2}(t)| \le 0, \quad (66)$$

which indicates that the closed-loop system consisting of (44)-(49) and (58)-(63) is stable and its solutions are bounded, that is, all the variables $e_{\xi_1}(t)$, $\tilde{k}_{\xi_1}(t)$, $\tilde{k}_{11}(t)$, $\tilde{d}_{11}(t)$, $\tilde{m}_1(t)$, $\tilde{k}_{r1}(t)$, $e_{\xi_2}(t)$, $\tilde{k}_{\xi_2}(t)$, $\tilde{k}_{34}(t)$, $\tilde{d}_{34}(t)$, $\tilde{m}_2(t)$, $\tilde{k}_{r2}(t)$, $\dot{e}_{\xi_1}(t)$ and $\dot{e}_{\xi_2}(t)$ are bounded. Then $\xi_1(t)$, $\xi_2(t)$, $\hat{k}_{\xi_1}(t)$, $\hat{k}_{11}(t)$, $\hat{d}_{11}(t)$, $\hat{m}_1(t)$, $\hat{k}_{r1}(t)$, $\hat{k}_{\xi_2}(t)$, $\hat{k}_{34}(t)$, $\hat{d}_{34}(t)$, $\hat{m}_4(t)$, $\hat{k}_{r2}(t)$ are bounded. (66) also implies $e_{\xi_1}(t) \in L^2$, $e_{\xi_2}(t) \in L^2$, $e_{\xi_1}(t) \in L^1$, $e_{\xi_2}(t) \in L^1$. As $e_{\xi_1}(t) \in L^1$, $e_{\xi_2}(t) \in L^1$, it has $e_{\xi_1}(t) - e_{\xi_2}(t) = \xi_1(t) - \xi_2(t) \in L^1$. With $e_{\xi_1}(t) \in L^2$, $e_{\xi_2}(t) \in L^2$ and Barbălat Lemma, $\lim_{t\to\infty} e_{\xi_1}(t) = 0$ and $\lim_{t\to\infty} e_{\xi_2}(t) = 0$, that is $\lim_{t\to\infty}(\dot{z}_1(t) - v_m(t)) = 0$ and $\lim_{t\to\infty}(\dot{z}_4(t) - v_m(t)) = 0$.

According to Corollary 1 and considering the dynamic $\bar{G}(s)$ in (25) with one pole $s = 0$ and all other poles stable, it has that $\eta_1(t) - \eta_2(t)$, $\eta_3(t)$, $\eta_4(t) - \eta_6(t)$, and $\eta_5(t)$ (i.e., $z_1(t) - z_2(t)$, $\dot{z}_2(t)$, $z_1(t) - z_4(t)$, $\dot{z}_3(t)$, and $z_3(t) - z_4(t)$) are bounded. Then, with the structure of the controllers (40) and (51), the boundedness of $F_1(t)$ and $F_4(t)$ are ensured.

The performance of the adaptive controller can be summarized as follows:

Theorem 2: *The adaptive controllers (40) and (51), with the adaptive laws (45)-(49) and (59)-(63) applied to the system (7), guarantee that the corresponding closed-loop state signals $\dot{z}_1(t)$, $z_1(t) - z_2(t)$, $\dot{z}_2(t)$, $z_1(t) - z_4(t)$, $\dot{z}_3(t)$, $z_3(t) - z_4(t)$, $\dot{z}_4(t)$ are* *bounded, and the tracking errors satisfy* $\lim_{t\to\infty}(\dot{z}_1(t) - v_m(t)) = 0$, $\lim_{t\to\infty}(\dot{z}_4(t) - v_m(t)) = 0$.

For the four-body system (7) with two control inputs acting on first and fourth bodies, the proposed nominal controller (36) and (50) and adaptive controllers (40) and (51), can guarantee partial input-to-state stability and Lyapunov stability of the zero dynamics subsystem, desired tracking performance of the control dynamics subsystem, and the speed error having the exponential convergence under the nominal controller, and L^1 performance under the adaptive controller.

Remark 3: Since the body velocities $\dot{z}_1(t)$ and $\dot{z}_4(t)$ are ensured to track a given desired velocity signal $v_m(t)$, the desired displacement tracking of $\int_0^t v_m(\tau)d\tau$ can be achieved. For a typical multibody system, a high-speed train, the speed tracking and displacement tracking are the main control tasks. This work also clarifies some key technical specifications for the design and analysis of adaptive control schemes for high-speed train control applications in the presence of system parameter uncertainties. □

Remark 4: The adaptive controllers (40) and (51) are different from the nominal controllers (36) and (50), due to $\alpha_{\xi_1} sgn(\xi_1(t) - v_m(t))$ and $\alpha_{\xi_2} sgn(\xi_2(t) - v_m(t))$. As the system parameters are unknown, the adaptive controller (40) and (51) without the terms $\alpha_{\xi_1} sgn(\xi_1(t) - v_m(t))$ and $\alpha_{\xi_2} sgn(\xi_2(t) - v_m(t))$ can only achieve the L^2 convergence of $\xi_1(t) - \xi_2(t)$. The proposed adaptive controllers (40) and (51) can result in the L^1 convergence of $\xi_1(t) - \xi_2(t)$, which satisfies the desired convergence property of $\xi_1(t) - \xi_2(t)$, needed for ensuring the internal stability (of the zero dynamics). □

Remark 5: It is interesting to extend the obtained results to the nonlinear underactuated systems, such as underactuated snake robots (Wang, Yang, Shen, Shao, & Wang, 2018), networked systems (Wang, Wang, & Shen, 2019), ships (Huang, Wen, Wang, & Song, 2015), nonholonomic mobile robot (Huang, Wen, Wang, & Jiang, 2013, 2014), etc., where the analysis of the Lyapunov and partially input-to-state stabilities for the zero dynamics subsystem can be carried out using the similar way as in this paper based on the theories of the nonlinear control systems. □

5. Comparisons and Extensions

A stable adaptive control framework with some key stability properties is developed for the four-body systems with two inputs acting on the 1st and 4th bodies. It is not difficult to use such an adaptive control design method to control a multibody system with each body having an independent input signal, to achieve the desired tracking performance. In this section, we will show that the proposed adaptive control framework can be applied to the general underactuated multibody systems with n bodies and m inputs acting on arbitrary bodies, where n and m can be any positive integers with $m < n$.

5.1. An Alternative Four-body System

To study the adaptive control problem for the general underactuated multibody systems, we start with the four-body system having inputs acting on the other bodies instead of the 1st and 4th bodies as discussed earlier.

The four-body system with two inputs acting on the 2nd and 3rd bodies. We introduce $\xi(t) = [\xi_1(t), \xi_2(t)]^T = [\dot{z}_2(t), \dot{z}_3(t)]^T$ and $\eta(t) =$

$[z_1(t), \dot{z}_1(t), z_2(t), z_3(t), z_4(t),\ \dot{z}_4(t)]^T$, and decouple the original system into a control dynamics subsystem and a zero dynamics subsystem.

Similar to the Lyapunov stability and input-to-state stability analysis in Section 3, we can conclude that the zero dynamics subsystem is Lyapunov stable and $\xi_1(t) = \xi_2(t)$, or $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially, or $\xi_1(t) - \xi_2(t) \in L^1$ can result in that the partial states of the zero dynamics subsystem are bounded.

Nominal and adaptive controllers. Moreover, according to the nominal and adaptive controller design methods proposed in Section 4, for the control dynamics subsystem, we can also design a nominal controller to make the states $\dot{z}_2(t)$, $\dot{z}_3(t)$ track the desired trajectories, and $\lim_{t\to\infty}(\xi_1(t) - \xi_2(t)) = 0$ exponentially, and the corresponding adaptive controller to make the states $\dot{z}_2(t)$, $\dot{z}_3(t)$ track the desired trajectories and $\xi_1(t) - \xi_2(t) \in L^1$. Combining with the stability analysis for the zero dynamics, we can conclude that Theorem 2 is applicable to the four-body system with inputs signals on the 2nd and 3rd bodies.

Other four-body systems. For the other four-body systems with two or three inputs acting on some different bodies, the similar adaptive controller design and stability analysis procedures can be applied as well by similar analysis. It should be noted that for one input ($m = 1$) case, there is only one state as input to the zero dynamics subsystem, and the partial input-to-state stability of the zero dynamics subsystem does not need the operating condition: $\xi_1(t) = \xi_2(t)$ or $\xi_1(t) - \xi_2(t) \in L^1$. Then, the adaptive controller only needs to ensure the state of the control subsystem to track the desired speed trajectory, while the displacement variables are allowed to be unbounded. When $m = 3$, the adaptive controller design and stability analysis procedure for the case that three inputs act on the 1st, 2nd and 4th or the 1st, 3nd and 4th bodies, is similar to the two inputs case. Further, the case that three inputs act on the 1st, 2nd and 3rd or the 2st, 3nd and 4th bodies, is similar to the one input case.

The study above shows that for the four-body systems with any inputs, the proposed stable adaptive control framework can guarantee partial input-to-state stability and Lyapunov stability of the zero dynamics subsystem, desired tracking performance of the control dynamics subsystem, and the speed error having the exponential convergence under the nominal controller, and L^1 performance under the adaptive controller.

5.2. Three or Five-body Systems

In this subsection, we will analyze whether the proposed stable adaptive control framework can be effectively applied to three or five–body systems.

The three-body system. For three-body system, the case of the one input ($m = 1$) is equivalent to the case of the four-body system with one input, which leads to the adaptive controller to ensure the state of the control dynamics subsystem tracking the desired speed trajectory. When there are two inputs ($m = 2$) for a three-body system, acting on the 1st and 2nd bodies or the 2nd and 3rd bodies, only one state is as input to the zero dynamics subsystem. Then, without the operating condition $\xi_1(t) = \xi_2(t)$ or $\xi_1(t) - \xi_2(t) \in L^1$, the zero dynamics subsystem decoupled from inputs is partial input-to-state stable. The adaptive controller only needs to ensure the state of the control dynamics subsystem tracking the desired speed trajectory, while the displacement variables are allowed to be unbounded.

When there are two inputs ($m = 2$) for a three-body system, acting on the 1st and 3rd bodies, the procedures of the stability analysis and adaptive controller design are the same as that for the four-body system with inputs acting on the 1st and 4th bodies. The proposed adaptive controller design scheme with Theorem 1 and 2 can be applied to this system.

We now can conclude that for the three-body systems with any inputs, the proposed stable adaptive control framework can be effective.

The five-body system. For the five-body systems, the case of the one input ($m = 1$) is equivalent to the case of the four-body system with one input.

When there are two or three inputs ($m = 2$ or 3) for a five-body system, the procedures of the stability analysis and adaptive controller that proposed in Section 3 and 4, can be employed. The original five-body system should be decoupled into a control dynamics subsystem and a zero dynamics subsystem, where the zero dynamics subsystem is independent with all the input signals.

Through the Lyapunov and partial input-to-state stability analysis for the zero dynamics subsystem, it can be verified that if the speeds of the bodies that the inputs act on are synchronous, the partial states are bounded, i.e., Lemma 2 and Corollary 1 hold for these five-body system. There are two points needed to be emphasized: (1) The characteristic polynomials of these zero dynamics subsystems may be more complicated, which lead to the Lyapunov and partial input-to-state stability conditions depend on the system parameters. (2) For the three inputs cases ($m = 3$), there are three sub-states $\xi_1(t)$, $\xi_2(t)$ and $\xi_3(t)$ in the virtual input signal $\xi(t)$ of the zero dynamics subsystem. The stabilization condition presented in Lemma 2 should be changed as $\xi_1(t) = \xi_2(t) = \xi_3(t)$, and that in Corollary 1 is $\xi_1(t) - \xi_2(t) \in L^1$ and $\xi_2(t) - \xi_3(t) \in L^1$.

For the control dynamics subsystem, a nominal controller can be designed to make the states track the desired trajectories, and the tracking errors converge to zero exponentially, and the corresponding adaptive control to make the states track the desired trajectories and the tracking errors satisfy the L^1 performance. Thus, the proposed adaptive controller framework can ensure that all the closed-loop state signals are bounded, the tracking errors converge to zero exponentially, and the displacement of the 1st body satisfies the displacement performance.

When there are four inputs ($m = 4$), the five-body system can be considered as a two-body system adding a three-body system. The obtained conclusions can be employed directly.

For the five-body systems, we can conclude that for the five-body systems with any inputs, the proposed stable adaptive control framework is also effective.

5.3. Systems with n Bodies and m Inputs

So far, we have discussed adaptive controller design problems for the four-body, three-body and five-body systems with different inputs. In practice, there are multibody systems with number of bodies and inputs being more than 5. For any multibody system, we can decouple the original system into a control dynamics subsystem and a zero dynamics subsystem, based on the system inputs. For the zero dynamics subsystem, the Lyapunov and partial input-to-state stability can be obtained under certain conditions. A nominal controller can be better designed to ensure the speeds of the control dynamics subsystem track the desired trajectory exponentially, and a needed stabilization condition. The corresponding adaptive controller can be developed to make the speeds of the control dynamics subsystem track the desired trajectory and satisfy the L^1 tracking performance, which guarantees the stabilization condition of the zero dynamics subsystem. The proposed adaptive controller design scheme with the stabilization condition can make the partial states of the closed-loop multibody system bounded, the speeds track the desired trajectories, and the closed-loop system satisfies the desired displacement performance, in the present of the unknown system parameters.

5.4. Summary

In summary, the goal of this paper is to show some fundamental properties of multibody systems and their adaptive control. Such a goal has been achieved, and has been extended to the other multibody systems, leading to a set of new research problems, in particular, specification of stabilization conditions for the zero dynamics subsystem, while the adaptive control framework has been established in this paper. Moreover, this work clarifies some key technical specifications, which can be applied in the design and analysis of adaptive control schemes for high-speed train control in the presence of system parameter uncertainties.

6. Simulation Study

To verify the proposed controller design method, simulation study on a real train model from Chou, Xia, & Kayser (2007) is presented in this section. Here, we consider the two cases:

Case 1 (four-body system): 4 bodies with two inputs acting on 1st and 4th bodies;

Case 2 (five-body system): 5 bodies with two inputs acting on 1st and 5th bodies; for which, the adaptive controllers proposed in (40) and (51) with their adaptive laws (45)-(49) and (59)-(63) are used.

Simulation system. The parameters in the simulation are chosen as:

Case 1 (four-body system): $M_1 = M_4 = 126000$kg, $M_2 = M_3 = 101090$kg, $b_1 = b_2 = b_3 = b_4 = 1.08\times10^{-4}$Ns/(m kg), $k_1 = k_3 = 100\times10^6$N/m, $k_2 = 30\times10^6$N/m, $d_1 = d_3 = 80 \times 10^4$Ns/m, $d_2 = 40 \times 10^4$Ns/m. The initial conditions are chosen as $x(0) = [0 \;\; 0 \;\; -2 \;\; 0 \;\; -4 \;\; 0 \;\; -6 \;\; 0]^T$, and the initial parameter estimates as 95% of their nominal values. The gains of the adaptive laws are chosen as 2.

Case 2 (five-body system): $M_1 = M_5 = 126000$kg, $M_2 = M_3 = M_4 = 101090$kg, $b_1 = b_2 = b_3 = b_4 = b_5 = 1.08 \times 10^{-4}$Ns/(m kg), $k_1 = k_4 = 100 \times 10^6$N/m, $k_2 = k_3 = 30 \times 10^6$N/m, $d_1 = d_4 = 80 \times 10^4$Ns/m, $d_2 = d_3 = 40 \times 10^4$Ns/m. The initial conditions are chosen as $x(0) = 0$, and the initial parameter estimates as 95% of their nominal values. The gains of the adaptive laws are chosen as 0.2.

Simulation results. Fig. 2 shows the simulation results of the speeds for bodies 1 and 4 in case 1 including the plant speed (solid) and desired displacement (dashed), while Fig. 5 shows the simulation results of the speeds for bodies 1 and 5 in case 2, in which the initial value of the desired speed is 5m/s. Figs. 3 and 6 show the speed tracking errors for bodies 1 and 4 in case 1 and bodies 1 and 5 in case 2. From Figs. 3 and 6, it can be seen that the tracking errors are close to 0. There are transit responses due to the adaptive laws and zero dynamics. Fig. 4 shows the displacement error between bodies 1 and 4 ($z_1(t)-z_4(t)$) in case 1, and Fig. 7 shows the displacement error between bodies 1 and 5 ($z_1(t) - z_5(t)$) in case 2. As the velocities of bodies 1 and 4 or 5 are synchronous, the error becomes a constant in steady case, which is in consistence with the real case. The simulation results show that the proposed stable adaptive control framework can achieve the close-loop stability even in the presence of unknown parameters.

It is visible from the simulation results that chattering occurs in Figs. 3, 4, 6 and 7. This is caused by the discontinuous controllers (40) and (51) due to the sign function, which results in a discontinuous right hand side in the dynamical equation (7). In real implementation, the chattering can be reduced or even avoided by using boundary layer method in which the discontinuous sign function is approximated by the continuous saturation function proposed in Burton, & Zinober (1986); Edwards, & Spurgeon (1998); Esfandiari, & Khalil (1991). Furthermore, for some practical system, such as trains, the chattering with small amplitude can be accepted, due to the buffer equipments between cars.

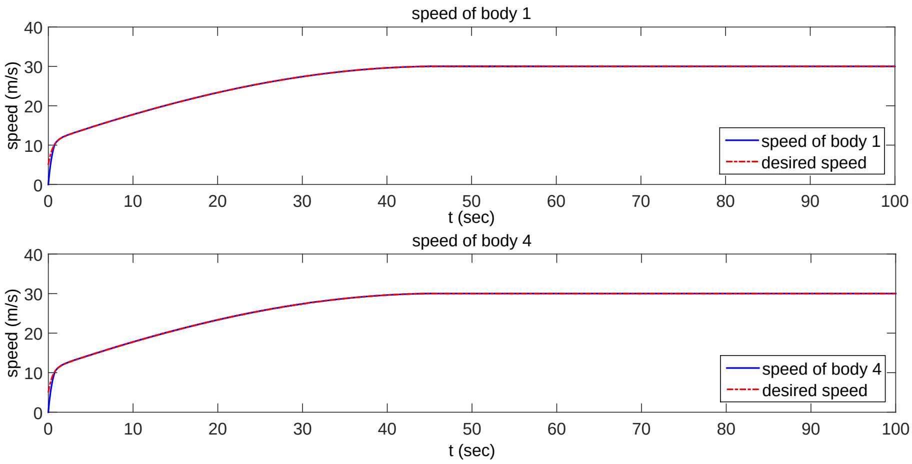

Figure 2. Speeds of bodies 1 and 4 for case 1

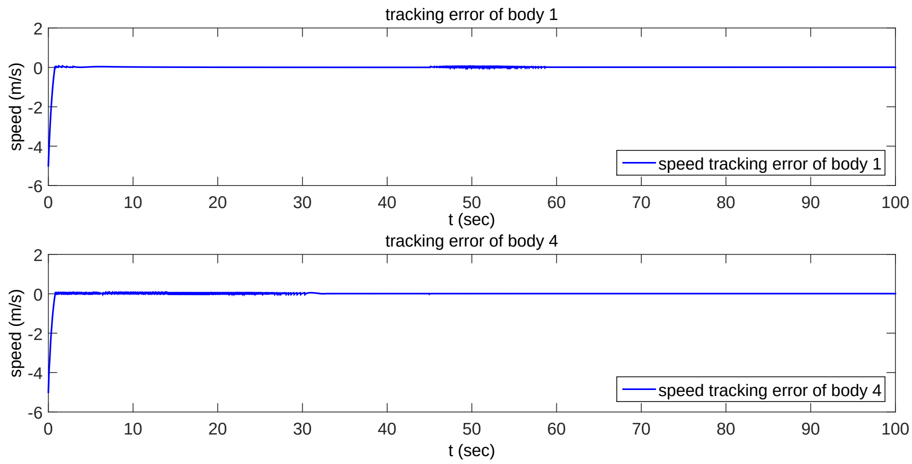

Figure 3. Speed tracking errors of bodies 1 and 4 for case 1

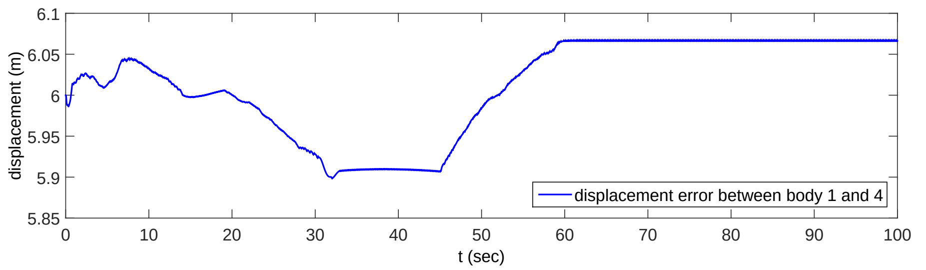

Figure 4. Displacement error between bodies 1 and 4 for case 1

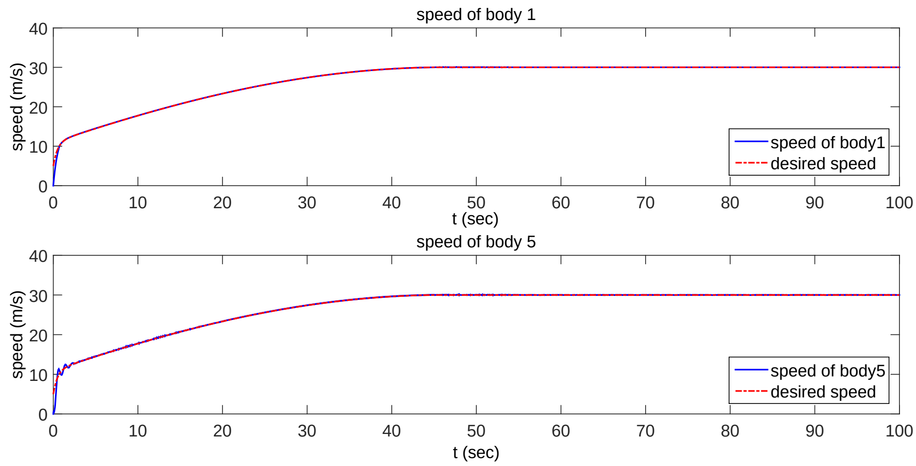

Figure 5. Speeds of bodies 1 and 5 for case 2

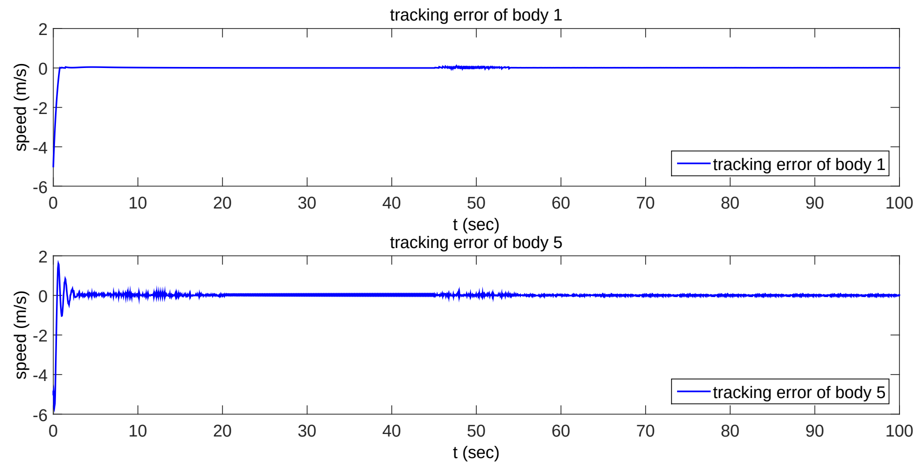

Figure 6. Speed tracking errors of bodies 1 and 5 for case 2

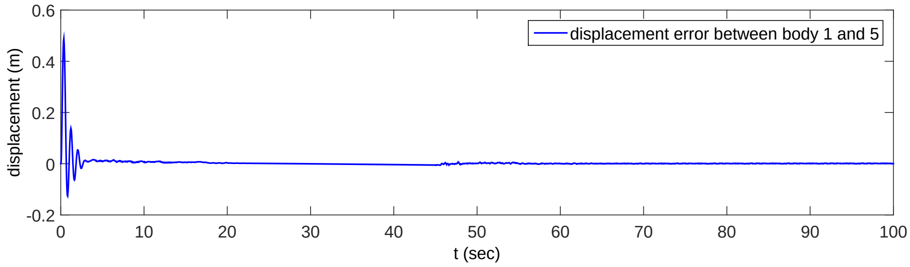

Figure 7. Displacement error between bodies 1 and 5 for case 2

7. Conclusions

In this paper, the adaptive tracking controller design problem has been investigated for underactuated four-body systems even if the parameters are unknown. To design the adaptive tracking controller, the four-body system is decoupled into a control dynamics subsystem and a zero dynamics subsystem. A new and detailed stability analysis is presented to show that the zero dynamic system is Lyapunov stable and partially input-to-state stable under the condition that the speed error of first and fourth bodies belongs to L^1. The adaptive controller is proposed to ensure the needed system stabilization condition, and make the desired closed-loop system signal bounded and asymptotic speed tracking of the control dynamics subsystem. Extensions to other multibody systems have been discussed as well. Simulation results further confirm the obtained theoretical results.

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