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# THE UNIQUENESS OF PLETHYSTIC FACTORISATION

CHRIS BOWMAN AND ROWENA PAGET

## INTRODUCTION

Let  $s_\lambda$  and  $s_\mu$  denote the Schur functions labelled by the partitions  $\lambda$  and  $\mu$ . There are three ways of “multiplying” this pair of functions together in order to obtain a new symmetric function; these are the Littlewood–Richardson, Kronecker, and plethysm products. The primary purpose of this paper is to address the most fundamental question one can ask of such a product: “*does it factorise uniquely?*”. For the Littlewood–Richardson product, this question was answered by Rajan [Raj04]. We solve this question for the most difficult and mysterious of these products, the plethysm product (which we denote  $\circ$ ) as follows.

**Theorem A.** *Let  $\mu, \nu, \pi, \rho$  be arbitrary partitions. If  $s_\nu \circ s_\mu = s_\rho \circ s_\pi$  then either  $\nu = \rho$  and  $\mu = \pi$ ; or we are in one of five exceptional cases,*

$$\begin{aligned} s_{(2,1^2)} \circ s_{(1)} &= s_{(1^2)} \circ s_{(1^2)}, & s_{(3,1)} \circ s_{(1)} &= s_{(1^2)} \circ s_{(2)}, \\ s_{(2,1^2)} \circ s_{(2)} &= s_{(1^2)} \circ s_{(3,1)}, & s_{(2,1^2)} \circ s_{(1^2)} &= s_{(1^2)} \circ s_{(2,1^2)}, \\ s_\nu \circ s_{(1)} &= s_{(1)} \circ s_\nu. \end{aligned}$$

In general, the decomposition of a plethysm product will have very, very many constituents. We ask: “*when is the plethysm product of two Schur functions indecomposable?*”. We prove that in fact such a product is always decomposable, and even inhomogeneous, except for some obvious exceptions. The analogous result for the Kronecker product was obtained by Bessenrodt and Kleshchev [BK99].

**Theorem B.** *Let  $\mu, \nu$  be partitions. The product  $s_\nu \circ s_\mu$  is decomposable and inhomogeneous except in the following exceptional cases:*

$$s_{(1^2)} \circ s_{(1^2)} = s_{(2,1^2)}, \quad s_{(1^2)} \circ s_{(2)} = s_{(3,1)}, \quad s_\nu \circ s_{(1)} = s_\nu = s_{(1)} \circ s_\nu.$$

Understanding and decomposing the Kronecker and plethystic products of pairs of Schur functions was identified by Richard Stanley as two of the most important open problems in algebraic combinatorics [Sta00, Problems 9 & 10]. Almost nothing is known about general constituents of plethysm products; however the maximal terms in the dominance ordering are now well-understood [PW]. Our proof of Theorems A and B proceeds by careful analysis of these maximal terms.

Outside of combinatorics, plethysm products arise naturally in the representation theory of symmetric and general linear groups. In quantum information theory, the positivity of constituents in a plethysm product of two Schur functions is equivalent to the existence of quantum states with certain spectra, margins, and occupation numbers [AK08, BCI11]. Decomposing Kronecker and plethystic products of Schur functions is the central plank of Geometric Complexity Theory, an approach that seeks to settle the P versus NP problem [MS01]; this approach was recently shown to require not only knowledge of the positivity but also precise information on the actual multiplicities of the constituents of the products  $s_\nu \circ s_\mu$  [BIP19].

1. PARTITIONS, SYMMETRIC FUNCTIONS  
AND MAXIMAL TERMS IN PLETHYSM

We define a **composition**  $\lambda \vDash n$  to be a finite sequence of non-negative integers  $(\lambda_1, \lambda_2, \dots)$  whose sum,  $|\lambda| = \lambda_1 + \lambda_2 + \dots$ , equals  $n$ . If the sequence  $(\lambda_1, \lambda_2, \dots)$  is weakly decreasing, we say that  $\lambda$  is a **partition** and write  $\lambda \vdash n$ . Given  $\lambda$  a partition of  $n$ , the **Young diagram** is defined to be the configuration of nodes

$$[\lambda] = \{(r, c) \mid 1 \leq c \leq \lambda_r\}.$$

We say that a partition is **linear** if it consists only of one row, or one column. The conjugate partition,  $\lambda^T$ , is the partition obtained by interchanging the rows and columns of  $\lambda$ . The number of non-zero parts of a partition,  $\lambda$ , is called its **length**,  $\ell(\lambda)$ ; the size of the largest part is called the **width**,  $w(\lambda)$ ; the sum of all the parts of  $\lambda$  is called its size.

Given two partitions  $\lambda$  and  $\mu$ , we let  $\lambda + \mu$  and  $\lambda \sqcup \mu$  denote the partitions obtained by adding the partition horizontally and vertically respectively. In more detail

$$\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \lambda_3 + \mu_3, \dots)$$

and  $\lambda \sqcup \mu$  is the partition whose multiset of parts is the disjoint union of the multisets of parts of  $\lambda$  and  $\mu$ . We have that

$$\lambda \sqcup \mu = (\lambda^T + \mu^T)^T.$$

Finally we remark that, in this paper, the partition  $\lambda \sqcup \mu$  is usually equal to

$$(\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \dots, \mu_{\ell(\mu)}).$$

In other words, we often do not need to reorder the multisets of parts — this is simply because  $\lambda_{\ell(\lambda)} \geq \mu_1$  in most cases.

We now recall the **dominance ordering** on partitions. Let  $\lambda, \mu$  be partitions. We write  $\lambda \trianglerighteq \mu$  if

$$\sum_{1 \leq i \leq k} \lambda_i \geq \sum_{1 \leq i \leq k} \mu_i \text{ for all } k \geq 1.$$

If  $\lambda \trianglerighteq \mu$  and  $\lambda \neq \mu$  we write  $\lambda \triangleright \mu$ . The dominance ordering is a partial ordering on the set of partitions of a given size. This partial order can be refined into a total ordering as follows: we write  $\lambda \succ \mu$  if

$$\lambda_k > \mu_k \text{ for some } k \geq 1 \text{ and } \lambda_i = \mu_i \text{ for all } 1 \leq i \leq k - 1.$$

We refer to  $\succ$  as the **lexicographic ordering**. We now define the **transpose-lexicographic ordering** as follows:

$$\lambda \succ_T \mu \text{ if and only if } \lambda^T \succ \mu^T.$$

We emphasise that this total ordering is not simply the opposite ordering to the lexicographic ordering; minimality with respect to  $\succ$  is not equivalent to maximality with respect to  $\succ_T$ .

Let  $\lambda$  be a partition of  $n$ . A **Young tableau of shape  $\lambda$**  may be defined as a map  $\mathbf{t} : [\lambda] \rightarrow \mathbb{N}$ . Recall that the tableau  $\mathbf{t}$  is **semistandard** if  $\mathbf{t}(r, c - 1) \leq \mathbf{t}(r, c)$  and  $\mathbf{t}(r - 1, c) < \mathbf{t}(r, c)$  for all  $(r, c) \in [\lambda]$ . We let  $\mathbf{t}_k = |\{(r, c) \in [\lambda] \mid \mathbf{t}(r, c) = k\}|$  for  $k \in \mathbb{N}$ . We refer to the composition  $\alpha = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \dots)$  as the **weight** of the tableau  $\mathbf{t}$ . We denote the set of all tableaux of shape  $\lambda$  by  $\text{SStd}_{\mathbb{N}}(\lambda)$ , and the subset of those

having weight  $\alpha$  by  $\text{SStd}_{\mathbb{N}}(\lambda, \alpha)$ . The Schur function  $s_\lambda$ , for  $\lambda$  a partition of  $n$ , may be defined as follows:

$$s_\lambda = \sum_{\alpha \vdash n} |\text{SStd}_{\mathbb{N}}(\lambda, \alpha)| x^\alpha \quad \text{where} \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \dots$$

The plethysm product of two symmetric functions is defined in [Sta99, Chapter 7, A2.6] or [Mac15, Chapter I.8]. The plethysm product of two Schur functions is again a symmetric function and so can be rewritten as a linear combination of Schur functions:

$$s_\nu \circ s_\mu = \sum_{\alpha} p(\nu, \mu, \alpha) s_\alpha$$

such that  $p(\nu, \mu, \alpha) \geq 0$ . We say that the product is **homogeneous** if there is precisely one partition,  $\alpha$ , such that  $p(\nu, \mu, \alpha) > 0$ ; we say that the product is **indecomposable** if, in addition,  $p(\nu, \mu, \alpha) = 1$ . We now recall the role conjugation – often called the  $\omega$  involution – plays in plethysm (see, for example, [Mac15, Ex. 1, Chapter I.8]). For  $\mu \vdash m$ ,  $\nu \vdash n$ , and  $\alpha \vdash mn$  we have that

$$p(\nu, \mu, \alpha) = \begin{cases} p(\nu, \mu^T, \alpha^T) & \text{if } m \text{ is even} \\ p(\nu^T, \mu^T, \alpha^T) & \text{if } m \text{ is odd.} \end{cases} \quad (1.1)$$

Throughout this paper we shall let  $\mu, \nu, \pi, \rho$  be partitions of  $m, n, p$  and  $q$  respectively. In order to keep track of the effect of this conjugation when comparing products  $s_\nu \circ s_\mu$  and  $s_\rho \circ s_\pi$ , we set

$$\nu^M = \begin{cases} \nu & \text{if } m \text{ is even} \\ \nu^T & \text{if } m \text{ is odd} \end{cases} \quad \rho^P = \begin{cases} \rho & \text{if } p \text{ is even} \\ \rho^T & \text{if } p \text{ is odd} \end{cases}$$

Given a total ordering,  $>$ , on partitions we let

$$\max_{>}(s_\nu \circ s_\mu)$$

denote the unique partition,  $\lambda$ , such that  $p(\nu, \mu, \lambda) \neq 0$  and  $p(\nu, \mu, \alpha) = 0$  for all  $\alpha > \lambda$ . We shall use this with both the lexicographic  $>$  and transpose-lexicographic  $>_T$  orderings. By equation (1.1) we have that

$$\max_{>_T}(s_\nu \circ s_\mu) = (\max_{>}(s_{\nu^M} \circ s_{\mu^T}))^T.$$

The following theorems will be incredibly important in our arguments.

**Theorem 1.1** ([PW, Corollary 9.1] and [Iij]). *Let  $\mu, \nu$  be partitions of  $m$  and  $n$  respectively. The unique maximal terms of  $s_\nu \circ s_\mu$  in the lexicographic and transpose lexicographic ordering are as follows :*

$$\max_{>}(s_\nu \circ s_\mu) = (n\mu_1, n\mu_2, \dots, n\mu_{\ell(\mu)-1}, n\mu_{\ell(\mu)} - n + \nu_1, \nu_2, \dots, \nu_{\ell(\nu)}),$$

$$\max_{>_T}(s_\nu \circ s_\mu) = (n\mu_1^T, n\mu_2^T, \dots, n\mu_{\mu_1-1}^T, n\mu_{\mu_1}^T - n + \nu_1^M, \nu_2^M, \dots, \nu_{\ell(\nu^M)}^M)^T.$$

Moreover, we have that

$$p(\nu, \mu, \max_{>}(s_\nu \circ s_\mu)) = 1 = p(\nu, \mu, \max_{>_T}(s_\nu \circ s_\mu)).$$

**Example 1.2.** When  $\mu = (m)$ , Theorem 1.3 shows that

$$\max_{>}(s_\nu \circ s_{(m)}) = (nm - n) + \nu, \quad \max_{>_T}(s_\nu \circ s_{(m)}) = ((n^{m-1}) \sqcup \nu^M)^T.$$

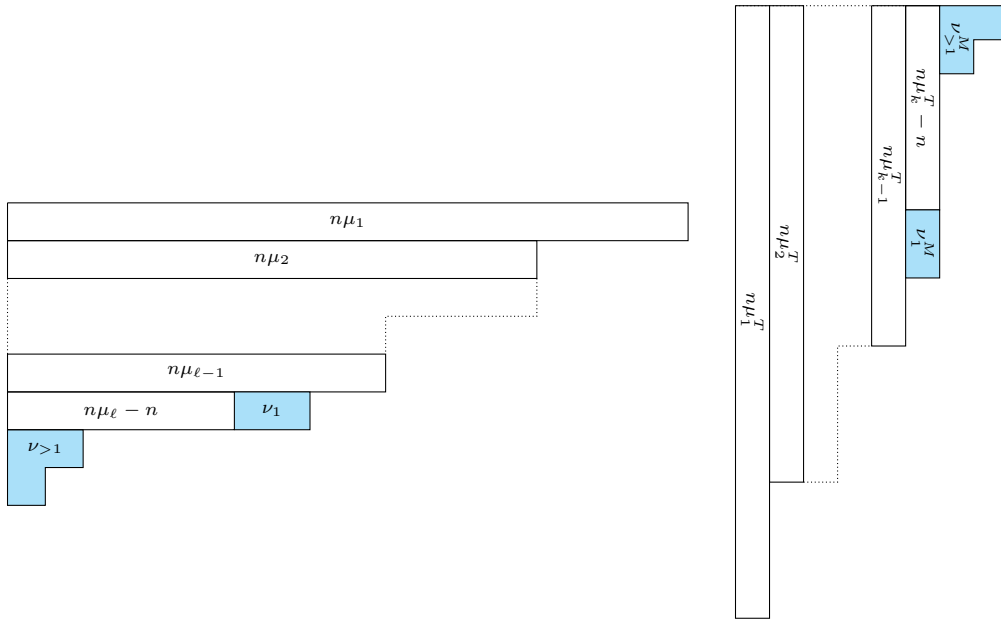


FIGURE 1. Examples of the partitions  $\max_{>}(s_\nu \circ s_\mu)$  and  $\max_{>-T}(s_\nu \circ s_\mu)$  for  $\mu \vdash m$  and  $\nu \vdash n$  with  $\ell(\mu) = \ell$  and  $w(\mu) = k$ .

Sometimes we shall use the dominance ordering  $\triangleright$  to compare the summands of  $s_\nu \circ s_\mu$ , and then there will, in general, be many (incomparable) maximal partitions. To understand these summands, we require some further definitions. We place a lexicographic ordering,  $\prec$ , on the set of semistandard Young tableaux as follows. Let  $S \neq T$  be semistandard  $\mu$ -tableaux, and consider the leftmost column in which  $S$  and  $T$  differ. We write  $S \prec T$  if the greatest entry not appearing in both columns lies in  $T$ . Following [dBPW, Definition 1.4], we define a **plethystic tableau** of shape  $\mu^\nu$  and weight  $\alpha$  to be a map

$$T : [\nu] \rightarrow \text{SStd}_{\mathbb{N}}(\mu)$$

such that the total number of occurrences of  $k$  in the tableau entries of  $T$  is  $\alpha_k$  for each  $k$ . We say that such a tableau is **semistandard** if  $T(r, c - 1) \leq T(r, c)$  and  $T(r - 1, c) \prec T(r, c)$  for all  $(r, c) \in [\nu]$ . We denote the set of all plethystic tableaux of shape  $\mu^\nu$  and weight  $\alpha$  by  $\text{PStd}(\mu^\nu, \alpha)$ .

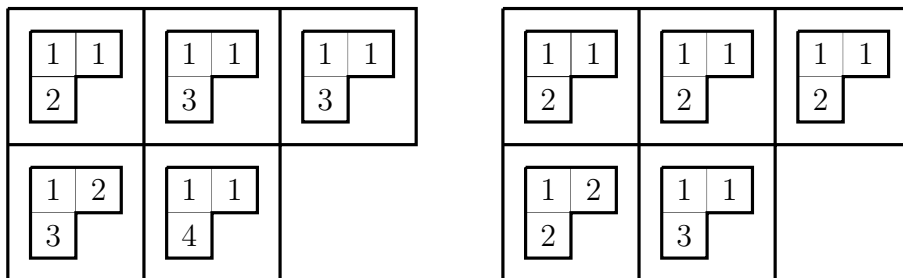


FIGURE 2. Two plethystic tableaux of shape  $(2, 1)^{(3, 2)}$ . The former has weight  $(9, 2, 3, 1)$  and the latter has weight  $(9, 5, 1)$ . The latter is maximal in the dominance ordering; the former is not.

**Theorem 1.3** ([dBPW, Theorem 1.5]). *The maximal partitions  $\alpha$  in the dominance order such that  $s_\alpha$  is a constituent of  $s_\nu \circ s_\mu$  are precisely the maximal weights of the plethystic semistandard tableaux of shape  $\mu^\nu$ . Moreover if  $\alpha$  is such a maximal partition then  $p(\nu, \mu, \alpha)$  is equal to  $|\text{PStd}(\mu^\nu, \alpha)|$ .*

Finally, we recall the one known case in which every term in a plethystic product is both maximal and minimal in the dominance ordering. Given  $\alpha$  a partition of  $n$  with distinct parts, we let  $2[\alpha]$  denote the unique partition of  $2n$  whose leading diagonal hook-lengths are  $2\alpha_1, \dots, 2\alpha_{\ell(\alpha)}$  and whose  $i^{\text{th}}$  row has length  $\alpha_i + i$  for  $1 \leq i \leq \ell(\alpha)$ . (An example follows.) We have the decomposition

$$s_{(1^n)} \circ s_{(2)} = \sum_{\alpha} s_{2[\alpha]}, \tag{1.2}$$

where the sum is over all partitions  $\alpha$  of  $n$  into distinct parts. This decomposition is given in [PW16, Corollary 8.6] and [Mac15, I. 8, Exercise 6(d)]. We observe that for  $n > 2$  this product is never homogeneous (for example  $\alpha = (n)$  and  $\alpha = (n - 1, 1)$  both label summands).

**Example 1.4.** For  $n = 5$  the decomposition obtained is

$$s_{(1^5)} \circ s_{(2)} = s_{2[(3,2)]} + s_{2[(4,1)]} + s_{2[(5)]} = s_{(4^2,2)} + s_{(5,3,1^2)} + s_{(6,1^4)}.$$

We picture these partitions (and the manner in which they are formed) in Figure 3 below. We remark that

$$s_{(1^5)} \circ s_{(1^2)} = s_{(4^2,2)^T} + s_{(5,3,1^2)^T} + s_{(6,1^4)^T} = s_{(3^2,2^2)} + s_{(4,2^2,1^2)} + s_{(5,1^5)}$$

by equation (1.1) simply because  $m = 2$  is even.

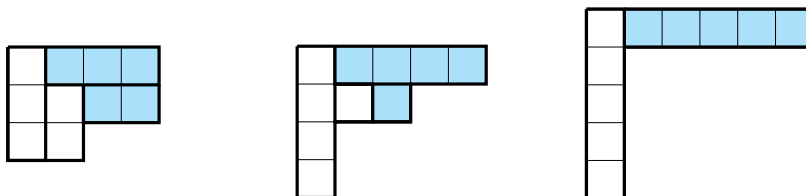


FIGURE 3. The partitions  $2[(3, 2)]$ ,  $2[(4, 1)]$  and  $2[(5)]$  respectively.

## 2. DECOMPOSABILITY AND HOMOGENEITY OF PLETHYSM

In this section, we prove Theorem B of the introduction: namely we classify all decomposable/homogeneous plethystic products of Schur functions. This also serves to remove the homogeneous products from consideration in the proof of Theorem A.

**Theorem 2.1.** *Let  $\mu, \nu$  be partitions of  $m$  and  $n$ , respectively. The product  $s_\nu \circ s_\mu$  is decomposable and inhomogeneous except in the following cases:*

$$s_{(1^2)} \circ s_{(1^2)} = s_{(2,1^2)}, \quad s_{(1^2)} \circ s_{(2)} = s_{(3,1)}, \quad s_\nu \circ s_{(1)} = s_\nu, \quad s_{(1)} \circ s_\mu = s_\mu.$$

*Proof.* That the listed products are homogeneous is obvious. We assume that  $m, n \neq 1$  and

$$\max_{\succ} (s_\nu \circ s_\mu) = \max_{\succ_T} (s_\nu \circ s_\mu). \tag{2.1}$$

We shall show that this implies that  $\nu = (1^2)$  and  $\mu \vdash 2$ . We first assume that  $\mu$  is non-linear, that is  $\mu$  is neither  $(m)$  nor  $(1^m)$ . We set  $k = \ell(\mu)$ . We draw a horizontal line across the Young diagrams of  $\max_{\succ}(s_{\nu} \circ s_{\mu})$  and  $(\max_{\succ}(s_{\nu^M} \circ s_{\mu^T}))^T$  so that the partitions below each of these lines each have strictly fewer than  $n$  nodes in total and are maximal with respect to this property. For  $\max_{\succ}(s_{\nu} \circ s_{\mu})$ , this line is drawn between the  $k^{\text{th}}$  and  $(k+1)^{\text{th}}$  rows (even though the  $(k+1)^{\text{th}}$  row might be zero). For  $(\max_{\succ}(s_{\nu^M} \circ s_{\mu^T}))^T$ , this line is drawn at some point after the  $(n(k-1) + 1)^{\text{th}}$  row. Since  $k < n(k-1) + 1$  for  $n > 1$ , we see that  $\max_{\succ}(s_{\nu} \circ s_{\mu}) \neq (\max_{\succ}(s_{\nu^M} \circ s_{\mu^T}))^T$  as required.

It remains to consider the case that  $\mu$  is linear and we assume (by conjugating if necessary) that  $\mu = (m)$ . Then, as we saw in Example 1.2,

$$\max_{\succ}(s_{\nu} \circ s_{(m)}) = (mn - n) + \nu, \quad (\max_{\succ}(s_{\nu^M} \circ s_{(1^m)}))^T = ((m-1)^n) + (\nu^M)^T.$$

Therefore row  $n$  of  $\max_{\succ}(s_{\nu} \circ s_{(m)})$  has length  $\nu_n$  which is at most 1, and row  $n$  of  $(\max_{\succ}(s_{\nu^M} \circ s_{(1^m)}))^T$  has length at least  $m-1$ . Since we are considering only  $m \geq 2$ , we conclude that  $m = 2$  and  $\nu_n = 1$ , that is  $\nu = (1^n)$ . From the closed formula for the decomposition of  $s_{(1^n)} \circ s_{(2)}$  in equation (1.2), and the resulting decomposition of its plethystic conjugate  $s_{(1^n)} \circ s_{(1^2)}$ , we observe that the product is homogeneous if and only if  $n = 1, 2$ .  $\square$

**Corollary 2.2.** *If  $s_{\nu} \circ s_{(1)} = s_{\rho} \circ s_{\pi}$  or  $s_{(1)} \circ s_{\mu} = s_{\rho} \circ s_{\pi}$  then either:  $\pi = (1^2)$  and  $\rho$  is a partition of 2; or at least one of  $\rho$  or  $\pi$  has size 1.*

Therefore in the remainder of the paper, we can and will assume that none of the indexing partitions in our plethystic products are equal to  $(1) \vdash 1$ .

### 3. UNIQUE FACTORISATION OF PLETHYSM

A quick scan of the diagrams in Figure 1 tells us that the maximal terms in the product under the lexicographic and transpose-lexicographic orderings encode a great deal of information concerning the multiplicands of the product. We might even think that these maximal terms are enough to uniquely determine the multiplicands. In fact, this is not the case (as the following example shows).

**Example 3.1.** Consider the plethysm products

$$s_{(3^3, 2, 1)} \circ s_{(1^2)} \quad \text{and} \quad s_{(2, 1)} \circ s_{(4, 1^4)}.$$

Both have the same maximal terms in the lexicographic and transpose-lexicographic orderings, namely those labelled by  $(12, 3^3, 2, 1)$  and  $(15, 3^2, 2, 1)^T$ . Figures 4 and 5 depict how these two partitions can be seen to be maximal in the lexicographic and transpose-lexicographic orderings using Theorem 1.1.

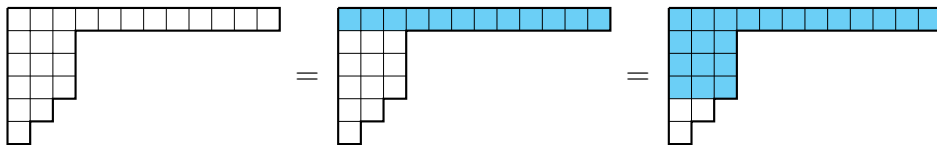


FIGURE 4. Writing  $(12, 3^3, 2, 1)$  as  $\max_{\succ}(s_{(3^3, 2, 1)} \circ s_{(1^2)})$  and  $\max_{\succ}(s_{(2, 1)} \circ s_{(4, 1^4)})$ .

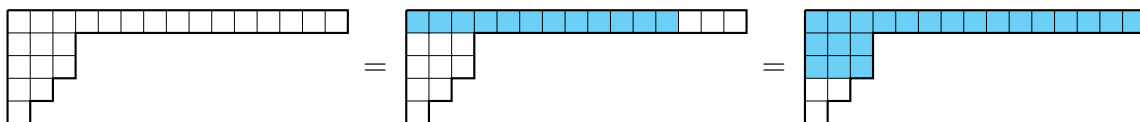


FIGURE 5. Writing  $(15, 3^2, 2, 1)$  as  $\max_{\succ}(s_{(3^3, 2, 1)} \circ s_{(2)})$  and  $\max_{\succ}(s_{(2, 1)} \circ s_{(5, 1^3)})$ .

This puts a scupper on our plans to determine uniqueness solely using maximal terms in the *lexicographic* and *transpose-lexicographic* orderings. Now, we notice that the plethysm products  $s_{(3^3, 2, 1)} \circ s_{(1^2)}$  and  $s_{(2, 1)} \circ s_{(4, 1^4)}$  can still be distinguished by looking at the maximal terms for both products in the *dominance ordering*. For example,  $(11, 4, 4, 3, 2)$  labels a maximal term that appears in  $s_{(3^3, 2, 1)} \circ s_{(1^2)}$  but it is not a maximal term in  $s_{(2, 1)} \circ s_{(4, 1^4)}$ . Similarly,  $(11, 4, 3, 3, 3)$  labels a maximal term in  $s_{(2, 1)} \circ s_{(4, 1^4)}$  but not in  $s_{(3^3, 2, 1)} \circ s_{(1^2)}$ .

Our method of proof will proceed to distinguish plethysm products by first using maximal terms in the lexicographic ordering and only when necessary considering the broader family of terms which are maximal in the dominance ordering. We first consider the case where  $\mu$  consists of a single row.

**Theorem 3.2.** *Let  $\mu, \nu, \pi, \rho$  be partitions of  $m, n, p, q > 1$  respectively. We suppose that  $\mu = (m)$ . If*

$$s_{\nu} \circ s_{\mu} = s_{\rho} \circ s_{\pi}$$

*then either  $\nu = \rho$  and  $\mu = \pi$  or we are in the exceptional case*

$$s_{(2, 1^2)} \circ s_{(2)} = s_{(1^2)} \circ s_{(3, 1)}.$$

*Proof.* From the set-up, we know  $mn = pq$ . We set  $\ell(\pi) = c + 1$  for some  $c \geq 0$ . By assumption, we have that

$$\max_{\succ}(s_{\nu} \circ s_{(m)}) = \max_{\succ}(s_{\rho} \circ s_{\pi}) \quad (3.1)$$

$$\max_{\succ}(s_{\nu^M} \circ s_{(1^m)}) = \max_{\succ}(s_{\rho^P} \circ s_{\pi^T}). \quad (3.2)$$

As a warm-up, we first consider the case where  $\pi$  is linear. If  $\mu = (m)$  and  $\pi = (p)$  then (see Example 1.2) equation (3.2) says that  $(n^{m-1}) \sqcup (\nu^M)^T = (q^{p-1}) \sqcup (\rho^P)^T$ . By comparing widths we deduce that  $q = n$ . This implies  $m = p$  and then  $\nu = \rho$ . Now, suppose that  $\mu = (m)$  and  $\pi = (1^p)$ . Then  $\max_{\succ}(s_{\nu} \circ s_{(m)}) = (nm - n) + \nu$  which, as  $m \geq 2$  and  $\nu$  has size  $n$ , has final column of length 1. For equation (3.1) to hold, the same to be true of  $\max_{\succ}(s_{\rho} \circ s_{(1^p)}) = (q^{p-1}) \sqcup \rho$ ; this implies  $p = 2$ . Similarly, comparing the final columns of  $\max_{\succ}(s_{\nu^M} \circ s_{(1^m)}) = (n^{m-1}) \sqcup \nu^M$  and  $\max_{\succ}(s_{\rho^P} \circ s_{(p)}) = (qp - q) + \rho^P$  also shows that  $m = 2$ . Hence  $n = q$  and we obtain a contradiction from comparing the widths of  $(n) \sqcup \nu^M$  and  $(q) + \rho^M$ .

We now assume that  $\pi$  is non-linear so  $\pi_1 > 1$  and  $c > 0$ . By equation (3.2),

$$(n^{m-1}) \sqcup \nu^M = (q\pi_1^T, q\pi_2^T, \dots, q\pi_{\pi_1-1}^T, q\pi_{\pi_1}^T - q + \rho_1^M, \rho_2^M, \dots). \quad (3.3)$$

Since  $m \geq 2$  and  $\pi_1 > 1$ , it follows that  $n = q\pi_1^T = q(c + 1)$  and, as  $mn = pq$ ,  $p = (1 + c)m$ . If  $\nu^M = (n)$  then the left hand side of equation (3.3) is  $(n^m)$ . Since  $q < n$ , comparing the width in equation (3.3) shows that  $\rho^P = (q)$  and that  $\pi = (m^{c+1})$ . This implies that  $\max_{\succ}(s_{\nu} \circ s_{(m)})$  is a hook partition whereas  $\max_{\succ}(s_{\rho} \circ s_{(m^{c+1})})$  has second row of width at least  $q(m - 1) > 1$ , a contradiction.



Therefore we can assume that  $\nu^M \neq (n)$ . Then equation (3.3) implies that the first  $m - 1$  rows of  $\pi^T$  are all equal to  $n/q = c + 1$  and therefore  $\pi = ((m - 1)^{c+1}) + \pi'$  for some  $\pi' \vdash c + 1$ . In particular,  $\pi_1 - \pi_2 \leq c + 1$ . We now consider equation (3.1): the difference between the first and second rows of  $\max_{\succ}(s_\nu \circ s_\mu)$  is

$$((m - 1)n + \nu_1) - \nu_2$$

whereas the difference between the first and second rows of  $\max_{\succ}(s_\rho \circ s_\pi)$  is less than or equal to  $q \times (\pi_1 - \pi_2 + 1) = n + q$ . Therefore the necessary inequality

$$(m - 1)n + \nu_1 - \nu_2 \leq n + q$$

implies that  $m = 2$  (since  $q < n$ ). For the remainder of the proof  $\mu = (2)$  and  $\pi = (1^{c+1}) + \pi' \vdash 2(c + 1)$  and therefore  $\rho^P = \rho$  and  $\nu^M = \nu$ .

We first consider the case  $c > 1$ . Here we have that  $\ell(\pi) = c + 1 > 2$  and so the difference between the first and second rows of  $\max_{\succ}(s_\rho \circ s_\pi)$  is  $q \times (\pi_1 - \pi_2) = q(\pi'_1 - \pi'_2) \leq q(1 + c) = n$ . On the other hand, for  $\max_{\succ}(s_\nu \circ s_{(m)}) = (n) + \nu$  the difference is at least  $n$ . For equality, we require  $\pi' = (c + 1)$ , that is  $\pi = (c + 1, 1^c)$ . Then equation (3.1) becomes  $(n) + \nu = (q(c + 1) + q, q^{c-1}) \sqcup \rho$  and we find  $\nu = (q^c) \sqcup \rho$ . We now employ the dominance ordering to examine the case

$$\pi = (c + 2, 1^c) \quad \nu = (q^c) \sqcup \rho.$$

A necessary condition for  $\text{PStd}((c + 2, 1^c)^\rho, \alpha) \neq \emptyset$  is that  $\alpha_1 + \alpha_2 \leq q(c + 3)$ . To see this, simply note that if  $\mathbf{S} \in \text{PStd}((c + 2, 1^c)^\rho, \alpha)$ , then

$$\mathbf{S} : [\rho] \rightarrow \text{SStd}_{\mathbb{N}}((c + 2, 1^c))$$

and the maximum number of entries equal to 1 or 2 in a semistandard Young tableaux of shape  $(c + 2, 1^c)$  is equal to  $(c + 2) + 1 = c + 3$  (the sum of the lengths of the first and second rows of  $(c + 2, 1^c)$ ). Thus  $p(\rho, (c + 2, 1^c), \alpha) = 0$  for any  $\alpha$  such that  $\alpha_1 + \alpha_2 > q(c + 3)$  by Theorem 1.3. We shall now construct a plethystic tableau  $\mathbf{S} \in \text{PStd}((2)^{(q^c) \sqcup \rho}, \beta)$  with  $\beta_1 + \beta_2 > q(c + 3)$ . This tableau will either be of maximal possible weight or there exists another plethystic tableau of the same shape but of weight  $\beta' \triangleright \beta$ ; in either case, for a partition for  $\gamma \in \{\beta, \beta'\}$ ,  $0 \neq p((q^c) \sqcup \rho, (2), \gamma)$  whereas  $p(\rho, (c + 2, 1^c), \gamma) = 0$  (by Theorem 1.3), providing us with the necessary contradiction. Let  $\mathbf{T} \in \text{PStd}((2)^{(q^c) \sqcup \rho}, \beta)$  be the plethystic tableau such that

$$\mathbf{T}(a, b) = \begin{cases} \boxed{2} \boxed{2} & \text{if } (a, b) \text{ is the least dominant (lowest) removable node of } (q^c) \sqcup \rho \\ \boxed{1} \boxed{a} & \text{otherwise.} \end{cases}$$

This tableau has weight  $\beta$  with  $\beta_1 = q(c + 2) - 1$  and  $\beta_2 = q + 2$  and so  $\beta_1 + \beta_2 = q(c + 3) + 1$  as required.

Finally, we consider the case  $c = 1$ . Here  $\mu = (2)$  and  $\pi \vdash 2(c + 1) = 4$  is either  $(3, 1)$  or  $(2, 2)$ . In the  $(2^2)$  case, comparing the widths of the partition on the left and right of equation (3.1) we see that  $\nu_1 = 0$ , a contradiction. In the  $(3, 1)$  case, comparison of maximal terms again reveals that  $\nu = (q) \sqcup \rho$ . Now

$$s_\rho \circ s_{(3,1)} = s_\rho \circ (s_{(1^2)} \circ s_{(2)}) = (s_\rho \circ s_{(1^2)}) \circ s_{(2)}.$$

We observe that  $\max_{\succ}(s_\rho \circ s_{(1^2)}) = (q) \sqcup \rho$ , but  $s_\rho \circ s_{(1^2)}$  is decomposable unless  $\rho = (1^2)$  by Theorem 2.1. For  $\rho \neq (1^2)$ , we deduce that  $s_{(q) \sqcup \rho} \circ s_{(2)}$  is properly contained in  $s_\rho \circ s_{(3,1)}$ . Thus we have  $q = 2$ ,  $\rho = (1^2)$  and  $\nu = (2, 1^2)$ , as required.  $\square$

We may conjugate (applying equation (1.1)) to complete the case where  $\mu$  is linear.

**Corollary 3.3.** *Let  $\mu, \nu, \pi, \rho$  be partitions of  $m, n, p, q > 1$  respectively. We suppose that  $\mu = (1^m)$ . If*

$$s_\nu \circ s_\mu = s_\rho \circ s_\pi$$

then either  $\nu = \rho$  and  $\mu = \pi$  or we are in the exceptional case

$$s_{(2,1^2)} \circ s_{(1^2)} = s_{(1^2)} \circ s_{(2,1^2)}.$$

Let  $\mu, \nu, \pi, \rho$  be arbitrary partitions of  $m, n, p, q > 1$  respectively. We now consider what the condition

$$\max_{\succ} (s_\nu \circ s_\mu) = \max_{\succ} (s_\rho \circ s_\pi) \quad (3.4)$$

tells us about this quadruple of partitions. We first suppose that  $\ell(\mu) = \ell(\pi) = k$ . This implies that  $\ell(\nu) = \ell(\rho) = \ell$ , say. Furthermore,

$$\begin{aligned} & (n\mu_1, n\mu_2, \dots, n\mu_{k-1}, n\mu_k - n + \nu_1, \nu_2, \dots, \nu_\ell) \\ &= (q\pi_1, q\pi_2, \dots, q\pi_{k-1}, q\pi_k - q + \rho_1, \rho_2, \dots, \rho_\ell). \end{aligned} \quad (3.5)$$

We set  $d = \gcd(n, q)$ ,  $e = \gcd(m, p)$  and set  $n = n'd$ ,  $q = q'd$ ,  $m = m'e$ ,  $p = p'e$ . Since  $mn = pq$ , we note that  $m'n'ed = p'q'ed$  and so  $m'n' = p'q'$ . Since  $m'$  and  $p'$  are coprime, as are  $n'$  and  $q'$ , it follows that  $m' = q'$  and  $p' = n'$ . Thus

$$m = q'e \quad n = n'd \quad q = q'd \quad p = n'e.$$

From equation (3.5), we observe that  $n\mu_i = \pi_i q$  implies  $n'\mu_i = q'\pi_i$ , and so we can set  $\alpha_i := \frac{\mu_i}{q'} = \frac{\pi_i}{n'} \in \mathbb{N}$  for all  $1 \leq i \leq k-1$ . Now,  $\mu \vdash m = q'e$  and so the final row length satisfies

$$\mu_k = q'e - \sum_{i=1}^{k-1} q'\alpha_i = q' \underbrace{\left( e - \sum_{i=1}^{k-1} \alpha_i \right)}_{\alpha_k}.$$

We have a partition  $(\alpha_1, \dots, \alpha_k) \vdash e$  with  $q'\alpha = \mu$ , and, in a similar fashion, we deduce that  $n'\alpha = \pi$ . Without loss of generality, we now assume that  $n \geq q$ . We plug in our equalities  $\pi = n'\alpha$  and  $\mu = q'\alpha$  back into equation (3.5) and to show that

$$\rho_i = \nu_i \text{ for } i \geq 2 \text{ and } \nu_1 = (n - q) + \rho_1.$$

We immediately obtain the following corollary.

**Corollary 3.4.** *Let  $\mu, \nu, \pi, \rho$  be partitions of  $m, n, p, q > 1$ , respectively. We suppose that  $\ell(\pi) = \ell(\mu)$ . If*

$$s_\nu \circ s_\mu = s_\rho \circ s_\pi$$

then  $\nu = \rho$  and  $\mu = \pi$ .

*Proof.* By the discussion above, we know that we are dealing with a quadruple

$$\mu = q'\alpha, \quad \nu = \rho + (n - q), \quad \pi = n'\alpha, \quad \rho.$$

Comparing the width of the partitions on the left and right of

$$\max_{\succ} (s_{\nu^M} \circ s_{\mu^T}) = \max_{\succ} (s_{\rho^P} \circ s_{\pi^T})$$

we deduce that  $\ell(\mu)n = \ell(\pi)q$ . Thus  $n = q$ ,  $\nu = \rho$ ,  $q' = n'$  and thus  $\mu = \pi$ , as required.  $\square$

We now consider the case where the lengths of the partitions  $\mu$  and  $\pi$  (and hence  $\nu$  and  $\rho$ ) differ. We suppose (without loss of generality) that  $\ell(\mu) < \ell(\pi)$ . We set  $\ell(\mu) = k$  and  $\ell(\pi) = k + c$  for some  $c \geq 1$ . Thus  $\ell(\rho) + c = \ell(\nu) = \ell$ , say. Observe that  $\max_{\succ}(s_{\nu} \circ s_{\mu}) = \max_{\succ}(s_{\rho} \circ s_{\pi})$  if and only if the partitions

$$\begin{array}{cccccccccccc} (n\mu_1 & \dots & n\mu_{k-1} & n\mu_k - n + \nu_1 & \nu_2 & \dots & \nu_c & \nu_{c+1} & \nu_{c+2} & \dots & \nu_{\ell}) \\ (q\pi_1 & \dots & q\pi_{k-1} & q\pi_k & q\pi_{k+1} & \dots & q\pi_{k+c-1} & q\pi_{k+c} - q + \rho_1 & \rho_2 & \dots & \rho_{\ell-c}). \end{array}$$

coincide. We deduce that

$$\mu = q'(\alpha_1, \dots, \alpha_k), \quad \pi = n'(\alpha_1, \dots, \alpha_{k-1}) \sqcup (\pi_k, \dots, \pi_{k+c}) \quad (3.6)$$

for  $\alpha \vdash e$ ,  $(\pi_k, \dots, \pi_{k+c}) \vdash n'\alpha_k$  and

$$\nu = (q\pi_k - n(q'\alpha_k - 1)) \sqcup q(\pi_{k+1}, \dots, \pi_{k+c-1}) \sqcup (q(\pi_{k+c} - 1) + \rho_1) \sqcup (\rho_2, \rho_3, \dots, \rho_{\ell-c}) \quad (3.7)$$

and, in order for  $\nu$  to be a partition, we need

$$q\pi_k - n(q'\alpha_k - 1) \geq q\pi_{k+1}$$

which, rearranging, gives

$$q(\pi_k - \pi_{k+1}) \geq n(\mu_k - 1).$$

We are now ready to complete our proof of Theorem A.

**Theorem 3.5.** *Let  $\mu, \nu, \pi, \rho$  be partitions of  $m, n, p, q > 1$ , respectively. We suppose that both  $\mu$  and  $\pi$  are non-linear and  $\ell(\pi) > \ell(\mu)$ . If*

$$s_{\nu} \circ s_{\mu} = s_{\rho} \circ s_{\pi}$$

then  $\nu = \rho$  and  $\mu = \pi$ .

*Proof.* We set  $\ell(\mu) = k \geq 2$  and  $\ell(\pi) = k + c$  for  $c \geq 1$ . We first see what can be deduced from  $\max_{\succ}(s_{\nu} \circ s_{\mu}) = \max_{\succ}(s_{\rho} \circ s_{\pi})$ . From equation (3.6) we have that

$$\mu = q'(\alpha_1, \dots, \alpha_k) \quad \pi = n'(\alpha_1, \dots, \alpha_{k-1}) \sqcup (\pi_k, \dots, \pi_{k+c}) \quad (3.8)$$

for  $\alpha \vdash e$  and  $(\pi_k, \dots, \pi_{k+c}) \vdash n'\alpha_k$ , and, from equation (3.7), we deduce that  $|\rho| < |\nu|$  and so  $q = q'd < n'd = n$  which implies  $q' < n'$ . From equation (3.6) this implies that  $\mu_1 = q'\alpha_1 < n'\alpha_1 = \pi_1$  in other words  $\ell(\mu^T) < \ell(\pi^T)$ .

We now see what can be deduced from  $\max_{\succ}(s_{\nu^M} \circ s_{\mu^T}) = \max_{\succ}(s_{\rho^P} \circ s_{\pi^T})$ . We have already concluded that  $\ell(\mu^T) < \ell(\pi^T)$ . Therefore applying equation (3.6) (but with the partitions  $\mu^T, \nu^M, \pi^T$  and  $\rho^P$ ) we deduce that

$$\mu^T = q'(\beta_1, \dots, \beta_{\mu_1}) \quad \pi^T = n'(\beta_1, \dots, \beta_{\mu_1-1}) \sqcup (\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T) \quad (3.9)$$

for some  $\beta \vdash e$  and  $(\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T) \vdash n'\beta_{\mu_1}$ .

From equation (3.8) and (3.9) we deduce that  $\mu$  can be built from boxes of size  $q' \times q'$ . In other words,

$$\mu = q'(\underbrace{\gamma_1, \gamma_1, \dots, \gamma_1}_{q'}, \underbrace{\gamma_2, \gamma_2, \dots, \gamma_2}_{q'}, \dots).$$

for some  $\gamma \vdash m/q'^2$ . Since  $\gamma$  might have repeated parts, we write  $\gamma$  in the form

$$\gamma = (a_1^{b_1}, a_2^{b_2}, \dots, a_x^{b_x})$$

where  $a_1 > a_2 > \dots > a_x$ , so

$$\gamma^T = ((b_1 + \dots + b_x)^{a_x}, (b_1 + \dots + b_{x-1})^{a_{x-1} - a_x}, \dots, b_1^{a_1 - a_2}).$$

Now, equation (3.8) reveals that

$$\pi = \underbrace{(n'a_1, \dots, n'a_1)}_{b_1 q'} \underbrace{(n'a_2, \dots, n'a_2)}_{b_2 q'} \dots \underbrace{(n'a_x, \dots, n'a_x)}_{b_x q' - 1}, \pi_k, \dots, \pi_{k+c} \quad (3.10)$$

where  $(\pi_k, \dots, \pi_{k+c}) \vdash n'a_x$  and, from equation (3.9),

$$\pi^T = ((n'(b_1 + \dots + b_x))^{q'a_x}, (n'(b_1 + \dots + b_{x-1}))^{q'(a_x - a_{x-1})}, \dots, (n'b_1)^{q'(a_1 - a_2) - 1}) \sqcup (\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T) \quad (3.11)$$

where  $(\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T) \vdash n'b_1$ . By looking at the first row of  $\pi^T$  we deduce that provided  $x \neq 1$  the last part of  $\pi$  is  $q'a_x$  and that it appears with multiplicity  $n'b_x$ . This implies that

$$(\pi_k, \dots, \pi_{k+c}) = (\dots, \underbrace{q'a_x, \dots, q'a_x}_{n'b_x}) \vdash n'a_x.$$

But the sum over these final  $n'b_x$  rows is  $q'a_x \times n'b_x$  which implies  $q' = 1$  and  $b_x = 1$  and that

$$(\pi_k, \dots, \pi_{k+c}) = (\underbrace{a_x, \dots, a_x}_{n'}) \vdash n'a_x.$$

Now we input this into equation (3.10) to deduce that

$$\ell(\pi) = b_1 + \dots + b_x - 1 + n'.$$

On the other hand by equation (3.11) we know that

$$\ell(\pi) = \pi_1^T = n'(b_1 + \dots + b_x)$$

Therefore

$$n'(b_1 + \dots + b_x - 1) = b_1 + b_2 + \dots + b_x - 1$$

and thus  $n' = 1$  or  $b_1 + b_2 + \dots + b_x = 1$ . If  $n' = 1$  then  $n = q$ , contrary to our earlier observation that  $q < n$ . If  $b_1 + b_2 + \dots + b_x = 1$ , then  $\ell(\gamma) = \ell(\alpha) = \ell(\mu) = 1$ , contrary to our assumption that  $\mu$  is non-linear.

Finally, it remains to consider the  $x = 1$  case. This is the case in which  $\gamma = (a^b)$  is a rectangle. Here we have that  $\mu = q'(a^{bq'})$ ,  $\mu^T = q'(b^{aq'})$  and

$$\pi = ((n'a)^{q'b-1}) \sqcup (\pi_k, \dots, \pi_{k+c}) \quad \text{for } (\pi_k, \dots, \pi_{k+c}) \vdash n'a \quad (3.12)$$

$$\pi = ((q'a - 1)^{n'b}) + (\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T)^T \quad \text{for } (\pi_{\mu_1}^T, \dots, \pi_{\pi_1}^T) \vdash n'b. \quad (3.13)$$

Now, recall that  $q' < n'$ ; and so

$$q'b - 1 < q'b < n'b$$

and so the rectangle in equation (3.12) is at least 2 rows shorter than that in equation (3.13). This implies that  $q' = 1$  and  $a$  or  $b = 1$  and so  $\mu$  is linear, a contradiction.  $\square$

We have now classified all possible equalities between products  $s_\nu \circ s_\mu = s_\rho \circ s_\pi$  where neither, one, or both of  $\pi$  and  $\mu$  are linear partitions. This completes the proof of Theorem A.

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