Cellular Structure of Wreath Product Algebras

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Abstract

We apply the method of iterated inflations to show that the wreath product of a cellular algebra with a symmetric group is cellular, and obtain descriptions of the cell and simple modules together with a semisimplicity condition for such wreath products.

Keywords: wreath products, cellular algebras, iterated inflation

1 Introduction

The wreath product $G \wr S_n$ of a finite group G with a symmetric group S_n is a natural group-theoretic construction with many applications. For example, wreath products $S_m \wr S_n$ of two symmetric groups are of great importance in the representation theory of the symmetric group. It is also natural to consider the wreath product $A \wr S_n$ of an algebra A with a symmetric group S_n, see for example the work of Chuang and Tan in [1]. The notion of a *cellular algebra* was introduced by Graham and Lehrer in [4] and has since found broad application. The question arises as to whether a cellular structure on an algebra A yields a cellular structure on the algebra $A \wr S_n$, and in [3] Geetha and Goodman showed that this is so in the case that A is not only cellular but *cyclic cellular*, meaning that all of the cell modules of A are cyclic [3, Theorem 4.1]. Their proof is quite combinatorial in nature, and draws on the work of Dipper, James, and Mathas in [2] and of Murphy in [11]. However, we shall prove (section 4) that $A \wr S_n$ is cellular for any cellular algebra A, by exhibiting it as an *iterated inflation* of tensor products of group algebras of symmetric groups. Iterated inflations were originally introduced by König and Xi in [8], but we shall use this concept in the form given in [5]. The advantage of taking this approach is a far simpler proof than the one given in [3], and hence much easier access to the powerful machinery of

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cellular algebra theory which allows us to easily prove the nice results on $A \wr S_n$ given in Section 5. The price for this simplicity is that order obtained on the set of cell indices of $A \wr S_n$ is somewhat cruder than the order obtained in [3], and hence contains less representation-theoretic information; see the discussion at the end of Section 4 for more details. Since (as far as the author is aware) all cellular algebras which occur in practice are in fact cyclic cellular, the result presented in this article is in effect a weaker version of the result of Geetha and Goodman. However, we feel that the much simpler proof afforded by the method of iterated inflations is of interest in its own right.

We shall also obtain a convenient graphical description of a well-known method of constructing $A \wr S_n$ modules (section 3), and in section 5 we bring this description together with the cellularity result to deliver results on the representation theory of $A \wr S_n$, in particular a description of the simple modules and a semisimplicity condition. These results require no extra assumptions on the field (e.g. algebraic closedness).

2 Recollections and definitions

We let k be a field of characteristic p (p may be zero or a prime). By a *k-algebra*, we shall mean a finite-dimensional unital associative k-algebra; we shall abbreviate $\otimes_k$ to $\otimes$; all of our modules will be right modules of finite k-dimension. By an *anti-involution* on a k-algebra A, we mean a self-inverse k-linear isomorphism $a \mapsto a^*$ such that $(ab)^* = b^* a^*$ for all $a, b \in A$.

For n a non-negative integer, a *composition of* n is a tuple of non-negative integers whose sum is n, and if $\mu = (\mu_1, \ldots, \mu_t)$ is a tuple of non-negative integers then we call the numbers μ_i the *parts* of μ, and define $|\mu|$ to be the sum $\mu_1 + \cdots + \mu_t$, so that μ is a composition of $|\mu|$. A composition whose entries are positive and appear in non-increasing order is a *partition.* Note that $n = 0$ has exactly one partition, the empty tuple, which we shall write as $()$.

2.1 Cellular algebras

We refer the reader to [4] for basic information and notation on cellular algebras. We shall refer to elements of the poset Λ indexing the cell modules of a cellular algebra as *cell indices*, and we shall write the anti-involution on a cellular algebra A as $a \mapsto a^*$. Recall that to each cell index λ we associate a finite set $M(\lambda)$, and we have a *cellular basis* of A whose elements are indexed by the disjoint union of the sets $M(\lambda) \times M(\lambda)$ for $\lambda \in \Lambda$; we write the cellular basis element indexed by $(S, T) \in M(\lambda) \times M(\lambda)$ as $C^{\lambda}_{S,T}$. We call the tuple (Λ, M, C) the *cellular data* of A with respect to $*$. Since we are using right modules we take the multiplication rule for cellular basis elements to be

$$C^{\lambda}_{S,T}a \equiv \sum_{X \in M(\lambda)} R_a(T,X)C^{\lambda}_{S,X} \tag{1}$$

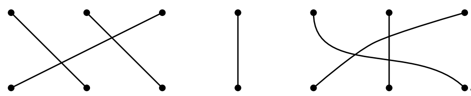
modulo cellular basis elements of lower cell index (where $R_a(T,X) \in k$). Then the right cell module Δ^λ is the vector space with basis $\{C_T : T \in M(\lambda)\}$; our form of the multiplication rule (1) means that the action of A on Δ^λ is

$$C_T a = \sum_{X \in M(\lambda)} R_a(T,X)C_X. \tag{2}$$

Let us recall some basic results on cell modules, see [4, sections 2 and 3]. Indeed, each cell module is equipped with a bilinear form, whose radical is either the whole cell module or else its unique maximal A-submodule; we shall call these bilinear forms the *cell forms* and their radicals the *cell radicals*. We let Λ_0 be the set of $\lambda \in \Lambda$ such that the cell radical of Δ^λ does not equal Δ^λ, and for $\lambda \in \Lambda_0$ we let L^λ be the quotient of Δ^λ by its cell radical; thus L^λ is a simple A-module, and the modules L^λ for $\lambda \in \Lambda_0$ are in fact a complete list of all the simple right A-modules up to isomorphism without redundancy.

2.2 The symmetric group

We let S_n denote the symmetric group on the set $\{1, 2, \ldots, n\}$, and we take S_n to act on the right, so that the product $\sigma\pi$ of permutations is calculated by first applying σ and then applying π; thus we write permutations to the right of their arguments. We shall find it convenient to represent permutations via *permutation diagrams*; for example, we represent $(1,2,3)(5,7) \in S_7$ by the diagram

,

where the i^{th} *node* on the top row is connected by a *string* to the $(i)\sigma^{\text{th}}$ node on the bottom row. To calculate the product $\sigma\pi$ in S_n using permutation diagrams, we connect the diagram for σ above the diagram for π, and then simplify the resulting diagram to yield the permutation diagram of $\sigma\pi$. For μ a composition of n, we write S_μ for the Young subgroup of S_n associated to μ.

Let us denote the dominance order on partitions by $\rhd$. The *reverse dominance order* is the order obtained by reversing all the relations in the dominance order. The group algebra kS_n is known to be a cellular algebra [10, Theorem 3.20], with respect to the anti-involution $*$ defined by setting $\sigma^* = \sigma^{-1}$ for $\sigma \in S_n$, and a tuple of cellular data including the partially ordered set $\mathcal{P}_n$ consisting of all partitions of n endowed with the reverse dominance order. Note that [10, Theorem 3.20] mentions the dominance order rather than the reverse dominance order, but we note that the definition of a cellular algebra used there [10, 2.1], has the opposite convention on the ordering of the elements of the poset of cell indices compared to [4], so that in the sense of our definition of a cellular algebra, we do indeed have the *reverse* dominance order. We shall not need the details of the cellular basis occurring in this structure, but we note that for $\lambda \in \mathcal{P}_n$, the right cell module associated to λ by this structure, which we shall denote by S^λ, is the (contragredient) dual of the right Specht module defined by James in [6][1]. Further, the simple modules are indexed by *p-restricted partitions.* If $p = 0$ then all partitions are considered p-restricted, while if $p > 0$ then a partition is p-restricted if the difference between any two consecutive parts is less than p. Note that () is p-restricted for all $p \geqslant 0$. For λ p-restricted, we denote the associated simple module by D^λ [10, Theorem 3.43].

The following result may easily be proved by directly verifying the axioms for a cellular algebra; in fact, it is merely a special case of the general result that a tensor product of cellular algebras is cellular, see for example section 3.2 of [3].

Proposition 1. *Let $n_1, \ldots, n_r$ be non-negative integers. Then the group algebra $k(S_{n_1} \times \cdots \times S_{n_r})$ is a cellular algebra with respect to the map given by $(\sigma_1, \ldots, \sigma_r) \longmapsto (\sigma_1^{-1}, \ldots, \sigma_r^{-1})$ for $\sigma_i \in S_{n_i}$ and a cellular structure where the poset of cell indices is $\mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_r}$ with the order where $(\lambda_1, \ldots, \lambda_r) \geqslant (\mu_1, \ldots, \mu_r)$ means $\lambda_i \trianglelefteq \mu_i$ for all i; the cell module associated to $(\lambda_1, \ldots, \lambda_r)$ is $S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_r}$ with the action*

$$(x_1 \otimes \cdots \otimes x_r) \cdot (\sigma_1, \ldots, \sigma_r) = (x_1\sigma_1) \otimes \cdots \otimes (x_r\sigma_r)$$

for $x_i \in S^{\lambda_i}$, $\sigma_i \in S_{n_i}$, and the cell form is given on pure tensors by

$$\langle x_1 \otimes \cdots \otimes x_r, y_1 \otimes \cdots \otimes y_r \rangle = \langle x_1, y_1 \rangle \cdots \langle x_r, y_r \rangle$$

where each bilinear form on the right hand side is the appropriate cell form of some S^{λ_i}.

Let $\sigma \in S_n$. Then an *inversion* of σ is a transposition (i, j) in S_n such that $1 \leq i < j \leq n$ but $(i)\sigma > (j)\sigma$, and the *Coxeter length* of σ is defined to be the total number of inversions of σ; we shall simply call this the *length* of σ. It is well-known that if μ is a composition of n, then each right coset $S_\mu\sigma$ of S_μ contains a unique element of minimal length, and further that if $\mu = (\mu_1, \ldots, \mu_r)$, then for any given right S_μ-coset, the element of minimal length is the unique element γ of the coset such that in the sequence

[1] See [10], "Warning" on p.38 and "Note 2" on page 54, but note that the original published text incorrectly states that the cell module obtained is the dual of the right James Specht module associated to the *conjugate* of λ; see the correction to the Warning in the author's errata.

$(1)\gamma^{-1}, \ldots, (n)\gamma^{-1}$, the elements $1, \ldots, \mu_1$ occur in increasing order, as do the elements $\mu_1 + 1, \ldots, \mu_1 + \mu_2$, the elements $\mu_1 + \mu_2 + 1, \ldots, \mu_1 + \mu_2 + \mu_3$, and so on. Equivalently, an element σ of S_n is of minimal length in its coset $S_\mu \sigma$ if and only if, in its permutation diagram, the strings attached to the first μ_1 nodes on the top row do not cross each other, the strings attached to the next μ_2 nodes on the top row do not cross each other, and so on. For example, the permutation whose diagram appears in the diagram (10) below is of minimal length in its S_μ coset for $\mu = (3, 2, 3)$. For any μ a composition of n, we define $\mathcal{R}_\mu$ to be the unique system of minimal-length right S_μ-coset representatives in S_n.

2.3 Iterated inflation of cellular algebras

Iterated inflations of cellular algebras were first introduced by König and Xi in [8], but we shall use them as presented in [5]. We shall now summarise the content of [5]; note however that we give the form using right cell modules, rather than the left cell modules used in [5].

Let A be a k-algebra, with an anti-involution $*$. Suppose that we have, up to isomorphism of k-vector spaces, a decomposition

$$A \cong \bigoplus_{\mu \in I} V_\mu \otimes B_\mu \otimes V_\mu$$

of A, where I is a finite partially ordered set, each V_μ is a k-vector space, and each B_μ is a cellular algebra over k with respect to an anti-involution $*$ and cellular data (Λ_μ, M_μ, C). We shall henceforth consider A to be identified with this direct sum of tensor products, and we shall speak of the subspace $V_\mu \otimes B_\mu \otimes V_\mu$ as the μ-th *layer* of A. Suppose that for each $\mu \in I$, we have a basis $\mathcal{V}_\mu$ for V_μ and a basis $\mathcal{B}_\mu$ for B_μ. Let $\mathcal{A}$ be the basis of A consisting of all elements $u \otimes b \otimes w$ for all $u, w \in \mathcal{V}_\mu$ and all $b \in \mathcal{B}_\mu$, as μ ranges over I. Suppose that for each $\mu \in I$, we have for any $u, w \in \mathcal{V}_\mu$ and any $b \in \mathcal{B}_\mu$ that

$$(u \otimes b \otimes w)^* = w \otimes b^* \otimes u, \tag{3}$$

and suppose further that for any $\mu \in I$ we have maps $\phi_\mu : \mathcal{V}_\mu \times \mathcal{A} \to V_\mu$ and $\theta_\mu : \mathcal{V}_\mu \times \mathcal{A} \to B_\mu$ such that for any $u, w \in \mathcal{V}_\mu$ and any $b \in \mathcal{B}_\mu$, we have for any $a \in \mathcal{A}$ that

$$(u \otimes b \otimes w) \cdot a \equiv u \otimes b\, \theta_\mu(w, a) \otimes \phi_\mu(w, a) \mod J(< \mu), \tag{4}$$

where $J(< \mu) = \bigoplus_{\alpha < \mu} V_\alpha \otimes B_\alpha \otimes V_\alpha$. Then by [5, Theorem 1], A is cellular with respect to $*$ and the cellular data (Λ, M, C), where Λ is the set $\{(\mu, \lambda) : \mu \in I \text{ and } \lambda \in \Lambda_\mu\}$ with the lexicographic order, $M(\mu, \lambda)$ is $\mathcal{V}_\mu \times M_\mu(\lambda)$, and $C^{(\mu,\lambda)}_{(x,X),(y,Y)} = x \otimes C^\lambda_{X,Y} \otimes y$.

Further by [5, Proposition 2], for each $\mu \in I$ there is a unique B_μ-valued k-bilinear form ψ_μ on V_μ such that for any $u, w, x, y \in V_\mu$ and $b, c \in B_\mu$ we have $\psi_\mu(y, u) = \psi_\mu(u, y)^*$ and

$$(x \otimes c \otimes y)(u \otimes b \otimes w) \equiv x \otimes c\, \psi_\mu(y, u) b \otimes w \mod J(< \mu). \tag{5}$$

Finally (see [5, Proposition 3]), let $(\mu, \lambda) \in \Lambda$, and let Δ^λ be the right cell module of B_i corresponding to λ. The right cell module $\Delta^{(\mu,\lambda)}$ of A may be obtained by equipping $\Delta^\lambda \otimes V_\mu$ with the action given, for $a \in \mathcal{A}$, $x \in \mathcal{V}_\mu$ and $z \in \Delta^\lambda$, by $(z \otimes x)a = z\, \theta_\mu(x, a) \otimes \phi_\mu(x, a)$. Moreover, if $\langle \cdot\,, \cdot \rangle$ is the cell form on $\Delta^\lambda \otimes V_\mu$ and $\langle \cdot\,, \cdot \rangle_\lambda$ is the cell form on Δ^λ, then for any $x, y \in V_\mu$ and any $z, v \in \Delta^\lambda$, we have

$$\langle z \otimes x,\, v \otimes y \rangle = \langle z\, \psi_\mu(x, y), v \rangle_\lambda = \langle z, v\, \psi_\mu(y, x) \rangle_\lambda. \tag{6}$$

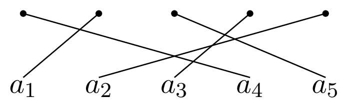
3 Wreath product algebras

We recall the notion of the wreath product of an algebra with a symmetric group from [1]. Indeed, let A be a finite-dimensional unital associative k-algebra. Consider the k-vector space $kS_n \otimes A^{\otimes n}$, and further let us write a pure tensor $x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in this vector space as $(x\,; a_1, a_2, \ldots, a_n)$. Then we have a well-defined multiplication which is given by

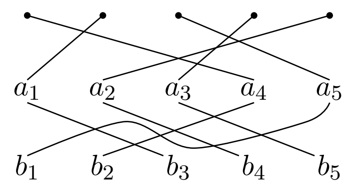
$$(\sigma; a_1, a_2, \ldots, a_n)(\pi; b_1, b_2, \ldots, b_n) = \\ (\sigma\pi; a_{(1)\pi^{-1}} b_1, a_{(2)\pi^{-1}} b_2, \ldots, a_{(n)\pi^{-1}} b_n)$$

for $\sigma, \pi \in S_n$ and $a_i, b_i \in A$; we define the *wreath product* $A \wr S_n$ of A and S_n to be the unital associative k-algebra so obtained.

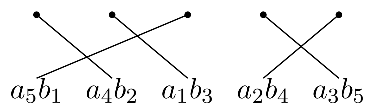
We assume that the reader is familiar with the notion of *diagram algebras*, for example the Brauer or Temperley-Lieb algebras. We can consider $A \wr S_n$ to be a kind of diagram algebra. Indeed, we may represent a pure tensor $(\sigma; a_1, a_2, \ldots, a_n)$ in $A \wr S_n$, where $\sigma \in S_n$ and $a_i \in A$, by a diagram obtained by drawing the permutation diagram associated to σ, with the nodes of the bottom row replaced by the elements a_i. For example, if $n = 5$ and $\sigma = (1, 4, 3, 5, 2)$, then we represent the element $(\sigma; a_1, a_2, a_3, a_4, a_5)$ by



Such diagrams are useful for computing products, as we now show by an example. Indeed, keep $n = 5$ and $\sigma = (1, 4, 3, 5, 2)$, and let $\pi = (1, 3, 5)(2, 4)$. Then to compute the product $(\sigma; a_1, a_2, a_3, a_4, a_5)(\pi; b_1, b_2, b_3, b_4, b_5)$, we draw the diagram corresponding to the first factor above the one corresponding to the second factor, to obtain



and we then slide each a_i down its string to meet some b_j, and then resolve the two connected permutation diagrams into a single diagram, to obtain



which corresponds to the element $\big((1,2,3)(4,5); a_5b_1, a_4b_2, a_1b_3, a_2b_4, a_3b_5\big)$, which is indeed the product of the two elements we started with.

Note that, unlike the usual diagram basis of the Brauer or Temperley-Lieb algebras, the set of all such diagrams is not a basis of $A \wr S_n$. A basis of such diagrams can be formed by fixing a basis $\mathcal{C}$ of A, and then taking the set of all elements $(\sigma; a_1, \ldots, a_n)$ for $\sigma \in S_n$ and $a_i \in \mathcal{C}$; however the product of two such basis elements will not in general be a scalar multiple of another basis element as is the case for the diagram basis of the Brauer or Temperley-Lieb algebras.

It is easy to show that there is a well-defined anti-involution $*$ on $A \wr S_n$ given by

$$(\sigma; a_1, \ldots, a_n)^* = \big(\sigma^{-1}\,;\, a^*_{(1)\sigma}, \ldots, a^*_{(n)\sigma}\big), \tag{7}$$

where $\sigma \in S_n$ and $a_1, \ldots, a_n \in A$. In terms of diagrams, this map corresponds to the operation of taking a diagram, flipping it about the horizontal line half-way between its two rows of nodes (so that the elements a_i lie on the *top* row), replacing each element a_i with its image a_i^* under the anti-involution on A, and then sliding each element a_i^* to the bottom of its string.

Now there is a standard method of constructing modules for $A \wr S_n$ from A-modules and symmetric group modules; see for example Section 3 of [1]. Indeed, let μ be an r-part composition of n, $X_1, \ldots, X_r$ be A-modules, and for each $i = 1, \ldots, r$ let Y_i be a kS_{μ_i} module. We write $A \wr S_\mu$ for the subalgebra of $A \wr S_n$ spanned by all elements $(\sigma; a_1, \ldots, a_n)$ where $a_i \in A$ and $\sigma \in S_\mu$. Then $X_1^{\otimes \mu_1} \otimes \cdots \otimes X_r^{\otimes \mu_r} \otimes Y_1 \otimes \cdots \otimes Y_r$ is naturally a $A \wr S_\mu$-module via the action

$$\begin{aligned}(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r)(\sigma; a_1, \ldots, a_n) = \\ x_{(1)\sigma^{-1}} a_1 \otimes \cdots \otimes x_{(n)\sigma^{-1}} a_n \otimes y_1\sigma_1 \otimes \cdots \otimes y_r\sigma_r,\end{aligned}$$

where the elements $\sigma_i \in S_{\mu_i}$ are such that under the natural identification of S_μ with $S_{\mu_1} \times \cdots \times S_{\mu_r}$, σ is identified with $(\sigma_1, \ldots, \sigma_r)$. Then inducing from $A \wr S_\mu$ to $A \wr S_n$ (that is, applying the functor $- \otimes_{A \wr S_\mu} A \wr S_n$) yields a module which we may easily see is isomorphic as a k-vector space to

$$X_1^{\otimes \mu_1} \otimes \cdots \otimes X_r^{\otimes \mu_r} \otimes Y_1 \otimes \cdots \otimes Y_r \otimes k\mathcal{R}_\mu, \tag{8}$$

where $k\mathcal{R}_\mu$ is the vector space on the basis $\mathcal{R}_\mu$ of minimal-length coset representatives, with the action given by

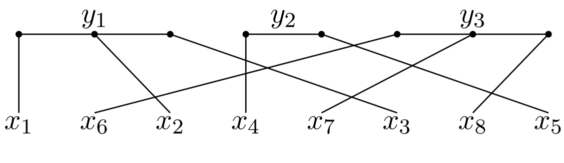
$$\begin{aligned}(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma)(\sigma; a_1, \ldots, a_n) =& \\ x_{(1)\theta^{-1}} a_{(1)\zeta} \otimes \cdots \otimes x_{(n)\theta^{-1}} a_{(n)\zeta} \otimes y_1\theta_1 \otimes \cdots \otimes y_r\theta_r \otimes \zeta,&\end{aligned} \tag{9}$$

where $\gamma \in \mathcal{R}_\mu$, and $\zeta \in \mathcal{R}_\mu$ and $\theta \in S_\mu$ are such that $\gamma\sigma = \theta\zeta$. Letting $\underline{X}$ be the tuple $(X_1, \ldots, X_r)$ and $\underline{Y}$ be the tuple $(Y_1, \ldots, Y_r)$, we denote the module so obtained by $\Theta^\mu(\underline{X}, \underline{Y})$.

We now introduce a diagrammatic representation for certain pure tensors in the module $\Theta^\mu(\underline{X}, \underline{Y})$ which provides a very convenient and intuitive understanding of the action of $A \wr S_n$. Indeed, let us take a pure tensor $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$ in (8), where $\gamma \in \mathcal{R}_\mu$. We represent this element by taking the permutation diagram of γ, labelling the nodes on its lower row from left to right with the elements $x_{(1)\gamma^{-1}}, \ldots, x_{(n)\gamma^{-1}}$, then linking together the first μ_1 nodes on the top row and labelling them with y_1, linking together the next μ_2 nodes on the top row and labelling the linked nodes with y_2, and so on. For example, take $n = 8$, $r = 3$, $\mu = (3, 2, 3)$, and $\gamma = (2, 3, 6)(5, 8, 7)$ (γ may be seen to be an element of $\mathcal{R}_\mu$ from its permutation diagram in (10), since the strings associated to each y_i do not cross each other). We then represent the element

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes x_6 \otimes x_7 \otimes x_8 \otimes y_1 \otimes y_2 \otimes y_3 \otimes \gamma$$

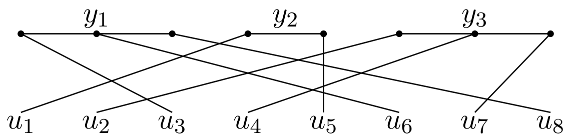
by the diagram



. (10)

Note that each x_i is connected to the i^{th} node on the top row. Note also that for each $i = 1, 2, 3$, the elements of X_i are attached to the strings associated to y_i. We thus identify $\Theta^\mu(\underline{X}, \underline{Y})$ with the k-vector space spanned by diagrams consisting of the permutation diagram of some element of $\mathcal{R}_\mu$ where (as in (10)) for each $i = 1, \ldots, r$, the $(\mu_1 + \cdots + \mu_{i-1} + 1)^{\text{th}}$ to $(\mu_1 + \cdots + \mu_i)^{\text{th}}$ nodes are connected to form a single block which is labelled by an element of Y_i, and where each node on the bottom row is replaced with an element of some X_j such that each top-row node in the i^{th} block is connected to an element of X_i on the bottom row. We note that under this identification, the diagram in $\Theta^{\mu}(\underline{X}, \underline{Y})$ whose top row has labels y_1 to y_r, whose bottom row has labels u_1 to u_n, and whose underlying permutation diagram is that of $\gamma \in \mathcal{R}_{\mu}$ represents the pure tensor $u_{(1)\gamma} \otimes \cdots \otimes u_{(n)\gamma} \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$. Further note that the set of all such diagrams is not linearly independent in $\Theta^{\mu}(\underline{X}, \underline{Y})$, and so they form a spanning set rather than a basis.

This diagram representation of $\Theta^{\mu}(\underline{X}, \underline{Y})$ affords an intuitive realisation of the action of $A \wr S_n$, and we illustrate this by an example. Indeed, keeping $n = 8$, $r = 3$, $\mu = (3, 2, 3)$ as above, let us consider the diagram



(11)

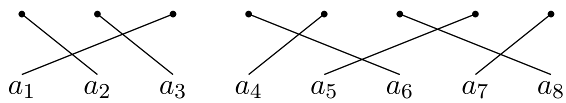
in $\Theta^{\mu}(\underline{X}, \underline{Y})$; note that this diagram represents the pure tensor

$$u_3 \otimes u_6 \otimes u_8 \otimes u_1 \otimes u_5 \otimes u_2 \otimes u_4 \otimes u_7 \otimes \\ y_1 \otimes y_2 \otimes y_3 \otimes (1, 3, 8, 7, 4)(2, 6). \tag{12}$$

Now take the element

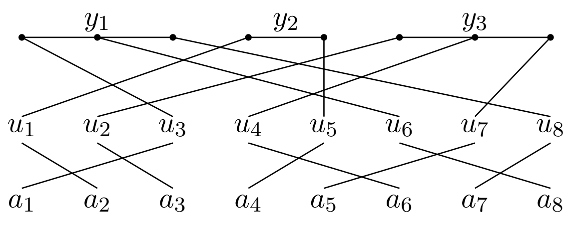
$$\big((1, 2, 3)(4, 6, 8, 7, 5); a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\big) \tag{13}$$

of $A \wr S_8$, which is represented by the diagram



. (14)

The action of the element (14) on (11) is calculated as follows: we connect the diagram (14) below the diagram (11) to get



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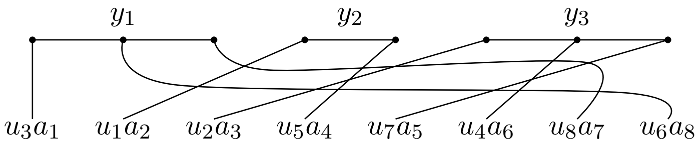
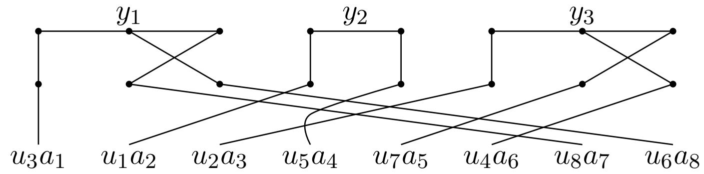
We slide each u_i down its string and simplify the drawing of the resulting partition diagram, to obtain

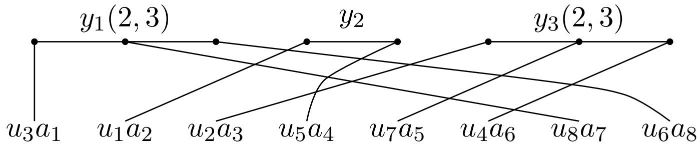


. (15)

The permutation encoded in the strings of this diagram is $(2,8,5,4)(3,7,6)$, which has the factorisation $(2,8,5,4)(3,7,6) = (2,3)(7,8)\cdot(2,7,5,4)(3,8,6)$ where $(2,3)(7,8)\in S_\mu$ and $(2,7,5,4)(3,8,6)\in\mathcal{R}_\mu$; we represent this factorisation by redrawing the diagram (15) as



and we note that in the lower part of this diagram, which represents the permutation $(2,7,5,4)(3,8,6)$, the strings associated to each y_i do not cross each other, which demonstrates that $(2,7,5,4)(3,8,6)$ is in $\mathcal{R}_\mu$. Now in the upper part of the diagram, the arrangement of strings encodes the permutation $(2,3)\in S_3$ below both y_1 and y_3, while the strings below y_2 encode the identity permutation in S_2. We remove the upper part of the diagram and let these permutations act on their respective elements y_i, yielding



.

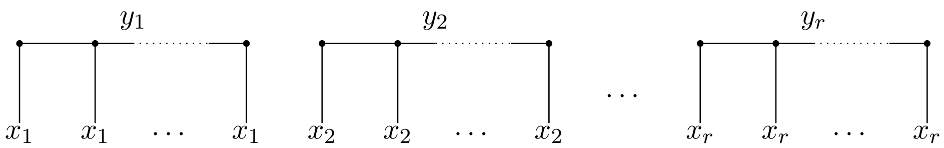
Under our mapping, this corresponds to the pure tensor

$$u_3a_1\otimes u_8a_7\otimes u_6a_8\otimes u_1a_2\otimes u_5a_4\otimes u_2a_3\otimes u_7a_5\otimes u_4a_6\otimes \\ y_1(2,3)\otimes y_2\otimes y_3(2,3)\otimes(2,7,5,4)(3,8,6),$$

and by letting $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = (u_3,u_6,u_8,u_1,u_5,u_2,u_4,u_7)$, $\sigma = (1,2,3)(4,6,8,7,5)$ and $\gamma = (1,3,8,7,4)(2,6)$, and noting as above that then $\gamma\sigma = (2,8,5,4)(3,7,6) = (2,3)(7,8)\cdot(2,7,5,4)(3,8,6)$ where $(2,3)(7,8)\in S_\mu$ and $(2,7,5,4)(3,8,6)\in\mathcal{R}_\mu$, we may verify that this is indeed the image of (12) under the action of (13) as given by (9). In the general case, for the $A\wr S_n$-module $\Theta^\mu(\underline{X},\underline{Y})$, let d be the diagram formed from the permutation diagram of $\gamma\in\mathcal{R}_\mu$ with labels y_1 to y_r on the top row and labels u_1 to u_n on the bottom row, and let a be the element $(\sigma;a_1,\ldots,a_n)$ of $A\wr S_n$. Then we have $\gamma\sigma=\theta\zeta$ where $\theta\in S_\mu$ and $\zeta\in\mathcal{R}_\mu$, and so θ corresponds to some element $(\theta_1,\ldots,\theta_r)$ of $S_{\mu_1}\times\cdots\times S_{\mu_r}$ under the canonical isomorphism. Then the image of d under the action of a is the diagram formed from the permutation diagram of ζ with top row labels $y_1\theta_1$ to $y_r\theta_r$ and bottom row labels $u_{(1)\sigma^{-1}}a_1$ to $u_{(n)\sigma^{-1}}a_n$; we leave it to the reader to convince themselves that in this diagram the nodes of the i^{th} block on the top row are connected to elements of X_i, and moreover that this diagram does indeed represent the action of a on the pure tensor of $\Theta^\mu(\underline{X}, \underline{Y})$ represented by d.

The following result will allow us to prove that the wreath product of a cyclic cellular algebra with S_n is again cyclic cellular, thus obtaining the result of Geetha and Goodman (albeit in a weaker form due to the different ordering on the set of cell indices, as mentioned above).

Proposition 2. *If $X_1, \dots, X_r$ are cyclic A-modules, and for each i, Y_i is a cyclic kS_{μ_i}-module, then $\Theta^\mu(\underline{X}, \underline{Y})$ is a cyclic $A \wr S_n$-module for any r part composition μ of n. Indeed, if x_i is a generator for X_i and y_i is a generator for Y_i, the diagram*



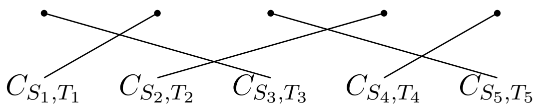
(where each x_i appears μ_i times) generates $\Theta^\mu(\underline{X}, \underline{Y})$.

Proof. Let d_0 be the diagram in the proposition. It is easy to see that we may obtain any diagram in $\Theta^\mu(\underline{X}, \underline{Y})$ by first applying an element $(x; 1, \dots, 1)$ of $A \wr S_n$, where $x \in kS_\mu$, in order to replace each element y_i in d_0 with an arbitrary element of Y_i, then applying $(\gamma; 1, \dots, 1)$ for some $\gamma \in \mathcal{R}_\mu$ to arrange the strings of the diagram, and finally applying an element $(e; a_1, \dots, a_n)$ to replace each element x_i with an arbitrary element of X_i. Since $\Theta^\mu(\underline{X}, \underline{Y})$ is spanned by diagrams, the proof is complete. $\square$

4 The iterated inflation structure of the wreath product algebra

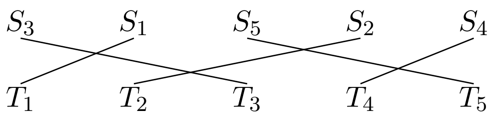
Remark 3. In this section, we work as in the rest of this article with a k-algebra A, where k is a *field*. In doing so, we are conforming to the set-up in the article [5] from which we obtain the crucial result on iterated inflations. However, it is straightforward to check that this result ([5, Theorem 1]) is still valid if we take k to be a commutative ring with 1. Further, the result on the cellularity of kS_n from [10] which we are using is also valid over a commutative ring with 1, and so it follows that Theorem 6 below is valid over a commutative ring with 1. However, for consistency with [5], we shall formally retain the assumption that k is a field.

Now we turn to the case where our interest lies. Let A be a cellular algebra with anti-involution $*$ and cellular data (Λ, M, C). We let $r = |\Lambda|$, and we fix a numbering of the elements of Λ as $\lambda_1, \lambda_2, \dots, \lambda_r$, and moreover we choose this numbering such that $\lambda_i > \lambda_j$ implies $i < j$, so that our numbering is in this sense compatible with the partial ordering on Λ. We write Δ^λ for the right cell module associated to $\lambda \in \Lambda$ as noted above. For convenience we may omit the cell index superscript from elements of the cellular basis, so we write $C_{S,T}$ rather than $C^\lambda_{S,T}$. We have a basis of $A \wr S_n$ consisting of all elements of the form $(\sigma; C_{S_1,T_1}, \dots, C_{S_n,T_n})$ where $\sigma \in S_n$ and each C_{S_i,T_i} is some element of the cellular basis of A; note that we allow the elements C_{S_i,T_i} to be associated to different cell indices. We shall denote this basis by $\mathcal{A}$. Now elements of $\mathcal{A}$ are represented by diagrams like, for example,



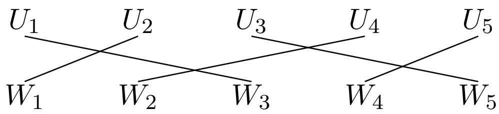
(16)

but we want a slightly different representation. Indeed, in the diagram (16), we replace each C_{S_i,T_i} with the pair S_i, T_i, and then move the S_i up to the top of the associated string, to get



.

We thus obtain a different way of representing elements of $\mathcal{A}$, as diagrams of the form



(17)

consisting of a permutation diagram where the nodes on the top and bottom rows are replaced with elements $U_i, W_i \in \sqcup_{\lambda \in \Lambda} M(\lambda)$, such that if U_i on the top row is connected to W_j on the bottom row, then we must have $U_i, W_j \in M(\lambda)$ for some $\lambda \in \Lambda$ (i.e. U_i and W_j lie in *the same* set $M(\lambda)$). Note that the diagram (17) represents the element

$$\big((1,3,5,4,2);\, C_{U_2,W_1},\, C_{U_4,W_2},\, C_{U_1,W_3},\, C_{U_5,W_4},\, C_{U_3,W_5}\big) \in A \wr S_5.$$

Now given any such diagram, for each $i \in \{1, \dots, r\}$ we let μ_i be the number of elements U_j such that $U_j \in M(\lambda_i)$. We thus obtain a composition $\mu = (\mu_1, \dots, \mu_r)$ of n (note that some of the parts μ_i may be zero in general). We call this the *layer index* of the diagram, and also of the element of $\mathcal{A}$ which it represents. We let $k\mathcal{A}_\mu$ be the k-span of all elements of $\mathcal{A}$ with layer index μ, and we let $I(n,r)$ be the set of all r-part compositions of n with non-negative integer entries. Then $A \wr S_n = \bigoplus_{\mu \in I(n,r)} k\mathcal{A}_\mu$. For a layer index μ, we define a *half diagram of type* μ to be a tuple $(U_1, \dots, U_n)$ of n elements of $\sqcup_{\lambda \in \Lambda} M(\lambda)$, such that there are exactly μ_i elements of $M(\lambda_i)$ for each i. We define $\mathcal{V}_\mu$ to be the set of all half diagrams of type μ. Now if $(U_1,\ldots,U_n)$ is a half diagram of type μ, then we may easily see that there is a unique element ϵ of $\mathcal{R}_\mu$ such that $(U_{(1)\epsilon},\ldots,U_{(n)\epsilon})$ lies in the set $M(\lambda_1)^{\mu_1}\times\cdots\times M(\lambda_r)^{\mu_r}$; we shall call this ϵ the *shape* of the half diagram $(U_1,\ldots,U_n)$.

Let E be the diagram with top row U_1 to U_n, bottom row W_1 to W_n (reading from left to right), and where $\sigma\in S_n$ is the permutation such that U_i is connected to $W_{(i)\sigma}$; then E represents the element

$$\big(\sigma\,;\,C[U_{(1)\sigma^{-1}},W_1],\ldots,C[U_{(n)\sigma^{-1}},W_n]\big)$$

where to ease the notation we allow ourselves to write $C[U,W]$ for $C_{U,W}$. Suppose E has layer index μ. We may decompose E into three pieces of data, namely the half diagrams $(U_1,\ldots,U_n)$, $(W_1,\ldots,W_n)$ of type μ, formed from the top and bottom rows of E respectively, and the element $(\pi_1,\ldots,\pi_r)$ of the group $S_{\mu_i}\times\cdots\times S_{\mu_r}$ where $\pi_i\in S_{\mu_i}$ is such that (counting from left to right) the j^{th} element of $M(\lambda_i)$ on the top row is connected to the $(j)\pi_i{}^{\text{th}}$ element of $M(\lambda_i)$ on the bottom row; thus π_i records how the elements of $M(\lambda_i)$ on the top row are connected to the elements of $M(\lambda_i)$ on the bottom row. For example, suppose that $r=3$ and that the diagram (17) has layer index $(3,0,2)$ with $U_1,U_2,U_4\in M(\lambda_1)$ and $U_3,U_5\in M(\lambda_3)$. Then $(\pi_1,\pi_2,\pi_3)=\big((1,3,2),e,(1,2)\big)$ (note that e here is the unique element of the trivial group $S_{\mu_2}=S_0$). It is easy to see that if ϵ,δ are the shapes of $(U_1,\ldots,U_n)$ and $(W_1,\ldots,W_n)$ respectively, and further if π is the image of $(\pi_1,\ldots,\pi_r)$ under the natural identification of $S_{\mu_i}\times\cdots\times S_{\mu_r}$ with the Young subgroup S_μ of S_n, then $\sigma=\epsilon^{-1}\pi\delta$. If we now let V_μ be the k-vector space with basis $\mathcal{V}_\mu$, then the above decomposition is easily seen to afford a k-linear bijection

$$V_\mu\otimes kS_\mu\otimes V_\mu\longrightarrow k\mathcal{A}_\mu$$

given by mapping

$$(U_1,\ldots,U_n)\otimes\pi\otimes(W_1,\ldots,W_n),$$

to

$$\big(\epsilon^{-1}\pi\delta\,;\;C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}},W_1],\ldots,C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}},W_n]\big)$$

where ϵ is the shape of $(U_1,\ldots,U_n)$ and δ is the shape of $(W_1,\ldots,W_n)$. We thus have a decomposition $A\wr S_n=\bigoplus_{\mu\in I(n,r)}V_\mu\otimes kS_\mu\otimes V_\mu$, and this decomposition will allow us to exhibit the desired iterated inflation structure. For this, we need to equip the set $I(n,r)$ with an ordering. Indeed, if $\mu=(\mu_1,\ldots,\mu_r)$ and $\alpha=(\alpha_1,\ldots,\alpha_r)$ are elements of $I(n,r)$, then we define $\mu\trianglerighteq_\Lambda\alpha$ to mean that for each $q=1,\ldots,r$ we have

$$\sum_{\substack{i\text{ such that}\\ \lambda_i\geq\lambda_q}}\mu_i\;\geq\;\sum_{\substack{i\text{ such that}\\ \lambda_i\geq\lambda_q}}\alpha_i$$

(and of course we define $\rhd_\Lambda$ to match); we call this (partial) order the Λ-*dominance order*.

Now take $\mathcal{V}_\mu$ as above, B_μ to be kS_μ and $\mathcal{B}_\mu$ to be S_μ. We may easily see that our basis $\mathcal{A}$ is indeed the basis of $A \wr S_n$ obtained from the bases $\mathcal{V}_\mu$ and $\mathcal{B}_\mu$ as in section 2.3, and we shall now prove that our decomposition exhibits $A \wr S_n$ as an iterated inflation with respect to the anti-involution given by (7) and the cellular structure on the algebras kS_μ as in Proposition 1. Thus, we must prove that the equations (3) and (4) hold. The fact that equation (3) holds follows easily from the description of the anti-involution on $A \wr S_n$ given after equation (7). To prove that (4) holds, we shall prove the slightly stronger result Proposition 5, below. First, we need a lemma, which will allow us to compare layer indices of elements of $A \wr S_n$.

Lemma 4. *Suppose that we have $s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, \ldots, r\}$ such that $\lambda_{s_j} \geq \lambda_{t_j}$ in the poset Λ for each j. For each $i = 1, \ldots, r$, let μ_i be the number of s_j which are equal to i and α_i be the number of t_j which are equal to i. Let $\mu = (\mu_1, \ldots, \mu_r)$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$ so that $\alpha, \mu \in I(n, r)$. Then $\mu \unrhd_\Lambda \alpha$, and if at least one of the inequalities $\lambda_{s_j} \geq \lambda_{t_j}$ is strict then we have $\mu \rhd_\Lambda \alpha$.*

Proof. This lemma is nothing more than simple combinatorics. We need to show that

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \mu_i \;\geq\; \sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \alpha_i .$$

But we have for each $q = 1, \ldots, r$ that

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \mu_i \;=\; |\{j : \lambda_{s_j} \geq \lambda_q\}|$$

and

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \alpha_i \;=\; |\{j : \lambda_{t_j} \geq \lambda_q\}|$$

and since the set appearing in the right-hand side of the latter equation is a subset of the corresponding set in the first equation, we have the required inequality $\mu \unrhd_\Lambda \alpha$. If there is a strict inequality $\lambda_{s_j} > \lambda_{t_j}$ we clearly have $\mu \neq \alpha$ and hence $\mu \rhd_\Lambda \alpha$. $\square$

Proposition 5. *Let $\mu \in I(n, r)$, and let $u = (U_1, \ldots, U_n), w = (W_1, \ldots, W_n)$ be elements of $\mathcal{V}_\mu$ and $\pi = (\pi_1, \ldots, \pi_r) \in S_\mu$ such that the element of $\mathcal{A}$ corresponding to the pure tensor $u \otimes \pi \otimes w$ has layer index μ. Further, let $a = (\sigma; a_1, \ldots, a_n)$ be a pure tensor in $A \wr S_n$. Then we have $(u \otimes \pi \otimes w) \cdot a \equiv u \otimes \pi\, \theta_\mu(w, a) \otimes \phi_\mu(w, a)$ modulo elements of $\mathcal{A}$ of layer index strictly less (in the Λ-dominance order) than μ, where $\theta_\mu(w, a) \in S_\mu$ and $\phi_\mu(w, a) \in V_\mu$ are independent of u and π.*

Note that in the proposition we allow the a in $\theta_\mu(w,a)$ and $\phi_\mu(w,a)$ to be any pure tensor in $A\wr S_n$ rather than just an element of $\mathcal{A}$ as required in (4).

Proof. Let $\epsilon, \delta \in \mathcal{R}_\mu$ be the shapes of u and w respectively, so that $u \otimes \pi \otimes w$ corresponds to the element

$$\big(\epsilon^{-1}\pi\delta\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}}, W_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}}, W_n]\big).$$

Then

$$\begin{aligned}
&(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n) = \\
&\big(\epsilon^{-1}\pi\delta\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}}, W_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}}, W_n]\big)\big(\sigma; a_1, \ldots, a_n\big) = \\
&\big(\epsilon^{-1}\pi\delta\sigma; C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(1)\sigma^{-1}}]a_1, \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(n)\sigma^{-1}}]a_n\big).
\end{aligned}$$

For each $i = 1, \ldots, n$, let $s_i \in \{1, \ldots, r\}$ be such that $U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(i)\sigma^{-1}} \in M(\lambda_{s_i})$. Then by (1) we have

$$C[U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(i)\sigma^{-1}}]a_i \equiv \sum_{X_i \in M(\lambda_{s_i})} R_{a_i}(W_{(i)\sigma^{-1}}, X_i) C[U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_i]$$

modulo cellular basis elements of lower cell index. Using this, we see that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n)$ is congruent to

$$\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) \big(\epsilon^{-1}\pi\delta\sigma\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_n]\big) \tag{18}$$

modulo elements of $\mathcal{A}$ of the form

$$\big(\epsilon^{-1}\pi\delta\sigma\,; C^{\lambda_{t_1}}[S_1, T_1], \ldots, C^{\lambda_{t_n}}[S_n, T_n]\big) \tag{19}$$

where for each i we have $\lambda_{s_i} \geq \lambda_{t_i}$ and for at least one i this inequality is strict. Now let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be the layer index of (19). By Lemma 4 we have $\mu \rhd_\Lambda \alpha$, so that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n)$ is congruent to (18) modulo elements of lower layer index.

Now X_i lies in the same set $M(\lambda_{s_i})$ as $W_{(i)\sigma^{-1}}$, and from this we may easily see that the shape of $(X_1, \ldots, X_n)$ is the unique element ζ of $\mathcal{R}_\mu$ such that $\delta\sigma = \theta\zeta$ for $\theta \in S_\mu$. Thus in (18) we have

$$\begin{aligned}
&\big(\epsilon^{-1}\pi\delta\sigma\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_n]\big) \\
&\qquad = \big(\epsilon^{-1}\pi\theta\zeta\,;\; C[U_{(1)(\epsilon^{-1}\pi\theta\zeta)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\theta\zeta)^{-1}}, X_n]\big)
\end{aligned}$$

which we now see corresponds to the pure tensor $u \otimes \pi\theta \otimes (X_1, \ldots, X_n)$, and hence (18) is equal to

$$u \otimes \pi\theta \otimes \left(\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) (X_1, \ldots, X_n) \right).$$

Thus, setting $\theta_\mu(w,a)$ to be the unique element θ of S_μ such that $\delta\sigma = \theta\zeta$ for $\zeta \in \mathcal{R}_\mu$ and $\phi_\mu(w,a)$ to be

$$\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) (X_1, \ldots, X_n), \tag{20}$$

we see that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n) \equiv u \otimes \pi\theta_\mu(w,a) \otimes \phi_\mu(w,a)$ modulo lower layers, and furthermore these values depend only on w and a, as required. $\square$

By the results in Section 2.3, we now have that $A \wr S_n$ is a cellular algebra; further, we may use Proposition 1 to see that the set indexing the cell modules of $A \wr S_n$ is the set of all pairs $\big(\mu, (\nu_1, \ldots, \nu_r)\big)$ where μ is an r-component composition $(\mu_1, \ldots, \mu_r)$ of n (recalling that $r = |\Lambda|$), and ν_i is a partition of μ_i. Thus in any such pair we have $\mu = (|\nu_1|, \ldots, |\nu_r|)$, and so we lose no information if we omit the partition μ from these pairs. Hence we may identify the set of cell indices of $A \wr S_n$ with the set of all r-tuples $(\nu_1, \ldots, \nu_r)$ of partitions such that $|\nu_1| + \cdots + |\nu_r| = n$ (with $\nu_i = ()$ allowed); such tuples are called *multipartitions* of n of *length* r. We now give a statement of the cellularity of $A \wr S_n$.

Theorem 6. *Let A be a cellular algebra with anti-involution $*$ and poset Λ of cell indices. Let $\widehat{\mathcal{P}}_n^r$ denote the set of all multipartitions of n of length r. Then $A \wr S_n$ is a cellular algebra with respect to a tuple of cellular data including the anti-involution given for $\sigma \in S_n$ and $a_1, \ldots, a_n \in A$ by*

$$(\sigma; a_1, \ldots, a_n)^* = \big(\sigma^{-1}\,;\, a^*_{(1)\sigma}, \ldots, a^*_{(n)\sigma}\big)$$

and also the poset consisting of $\widehat{\mathcal{P}}_n^r$ with the following partial order: if $(\nu_1, \ldots, \nu_r), (\eta_1, \ldots, \eta_r) \in \widehat{\mathcal{P}}_n^r$ then the relation $(\nu_1, \ldots, \nu_r) \geqslant (\eta_1, \ldots, \eta_r)$ means that either $(|\nu_1|, \ldots, |\nu_r|) \trianglerighteq_\Lambda (|\eta_1|, \ldots, |\eta_r|)$ or that $|\nu_i| = |\eta_i|$ and $\nu_i \trianglelefteq \eta_i$ for each i.

In the next section, we shall consider the cell modules which arise from this structure. In particular we shall follow the work of Geetha and Goodman by proving that if A is cyclic cellular, then so is $A \wr S_n$.

We conclude this section by remarking that the most natural partial order one might hope to have on the poset $\widehat{\mathcal{P}}_n^r$ in Theorem 6 is the Λ-*dominance order on multicompositions* [3, Definition 3.1, (2)] (this is essentially an extension of the Λ-dominance order on compositions to multicompositions).

We also note that, subject to the assumption that A is cyclic cellular, Geetha and Goodman obtained the Λ-dominance order in their cellularity result [3, Theorem 4.1]. The order which we have obtained on $\widehat{\mathcal{P}}_n^r$ is somewhat crude compared to this more subtle dominance order, and thus provides less refined representation-theoretic information (we also note that the Λ-dominance order on multicompositions is not a refinement of the order we have obtained, due to the use of the *reverse* dominance order on partitions in the cellular structure for the group algebra of the symmetric group). This fact is a consequence of the use of the method of iterated inflations, and is due to the structure of the partial orders obtained via this method.

5 The cell and simple modules of the wreath product algebra

Recall that the cell modules Δ^{λ_i} of A are indexed by the cell indices $\lambda_1, \lambda_2, \ldots, \lambda_r$. In the sequel we shall also allow ourselves to write Δ^{λ_i} as $\Delta(\lambda_i)$ when this makes our formulae more readable. We shall now consider the cell modules of $A \wr S_n$. We know that these are indexed by length r multipartitions of n; let $(\nu_1, \ldots, \nu_r)$ be such a multipartition and μ the composition $(|\nu_1|, \ldots, |\nu_r|)$, so that $\mu_i = |\nu_i|$. We shall show that the cell module $\Delta^{(\nu_1,\ldots,\nu_r)}$ is isomorphic to the module $\Theta^\mu\big((\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}), (S^{\nu_1}, \ldots, S^{\nu_r})\big)$ [3, Theorem 4.27].

Now we know from Proposition 1 and the results in section 2.3 that, as a k-vector space, $\Delta^{(\nu_1,\ldots,\nu_r)}$ may naturally be identified with

$$S^{\nu_1} \otimes \cdots \otimes S^{\nu_r} \otimes V_\mu, \tag{21}$$

so let us consider the structure of the vector space V_μ. Indeed, let $\alpha_1, \ldots \alpha_n$ be elements of Λ such that

$$(\alpha_1, \ldots, \alpha_n) = (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{\mu_1 \text{ places}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\mu_2 \text{ places}}, \lambda_3, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{\mu_r \text{ places}}). \tag{22}$$

Let $(X_1, \ldots, X_n)$ be a half diagram in $\mathcal{V}_\mu$. Then the shape of $(X_1, \ldots, X_n)$ is the unique element γ of $\mathcal{R}_\mu$ such that $(X_1, \ldots, X_n)$ lies in $M(\alpha_{(1)\gamma^{-1}}) \times \cdots \times M(\alpha_{(n)\gamma^{-1}})$. We now see that

$$\mathcal{V}_\mu = \bigsqcup_{\gamma \in \mathcal{R}_\mu} M(\alpha_{(1)\gamma^{-1}}) \times \cdots \times M(\alpha_{(n)\gamma^{-1}})$$

and hence if we identify the half diagram $(X_1, \ldots, X_n)$ with the pure tensor $C_{X_1} \otimes \cdots \otimes C_{X_n}$, we obtain a natural identification of k-vector spaces

$$V_\mu = \bigoplus_{\gamma \in \mathcal{R}_\mu} \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}}). \tag{23}$$

We shall henceforth consider these two vector spaces to be thus identified; further, we shall abuse terminology and use the term *pure tensor in* V_μ to mean any pure tensor in any of the summands in the right hand side of (23). For example, using (2), we can show easily using (20) that under the identification (23) we have

$$\phi_\mu\big(C_{W_1}\otimes\cdots\otimes C_{W_n}, (\sigma; a_1, \ldots, a_n)\big) = C_{W_{(1)\sigma^{-1}}}\, a_1 \otimes\cdots\otimes C_{W_{(n)\sigma^{-1}}}\, a_n. \quad (24)$$

In light of (21), we shall further speak of a *pure tensor in* $\Delta^{(\nu_1,\ldots,\nu_r)}$ to mean any pure tensor of the form

$$w_1 \otimes \cdots \otimes w_r \otimes u_1 \otimes \cdots \otimes u_n,$$

where $w_i \in S^{\nu_i}$ and $u_1 \otimes \cdots \otimes u_n$ is a pure tensor in V_μ. Using (24) and the expression for $\theta_\mu(w, a)$ given near the end of the proof of Proposition 5, we may now verify that the map taking the pure tensor

$$x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$$

in $\Theta^\mu\big((\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}), (S^{\nu_1}, \ldots, S^{\nu_r})\big)$ (where $\gamma \in \mathcal{R}_\mu$) to the pure tensor

$$y_1 \otimes \cdots \otimes y_r \otimes x_{(1)\gamma^{-1}} \otimes \cdots \otimes x_{(n)\gamma^{-1}}$$

in $\Delta^{(\nu_1,\ldots,\nu_r)}$ is an isomorphism of $A \wr S_n$-modules (but note that in order to apply the formula given in section 2.3 for the action of an iterated inflation on its cell modules, the arguments w and a in $\theta_\mu(w, a)$ and $\phi_\mu(w, a)$ must be elements of the bases $\mathcal{A}$ and $\mathcal{V}_\mu$, respectively). We may now use Proposition 2 and the fact that all Specht modules are cyclic to obtain the following result. Of course, this is a weaker result than the corresponding result in [3], since (as already mentioned) Geetha and Goodman obtain the Λ-dominance order on their cell indices.

Proposition 7. *(compare [3, Theorem 4.1]) If A is cyclic cellular then so is $A \wr S_n$.*

Now by equation (5), we know that the multiplication within each layer of $A \wr S_n$ is determined by a bilinear form, ψ_μ. Let $(U_1, \ldots, U_n), (W_1, \ldots, W_n)$ be half diagrams in $\mathcal{V}_\mu$, so that $u = C_{U_1}\otimes\cdots\otimes C_{U_n}$ and $w = C_{W_1}\otimes\cdots\otimes C_{W_n}$ are pure tensors in V_μ. Now by equation (5),

$$(u \otimes e \otimes u)(w \otimes e \otimes w) \equiv u \otimes \psi_\mu(u, w) \otimes w \quad (25)$$

modulo lower layers. The element $u \otimes e \otimes u$ of $A \wr S_n$ is represented by the diagram

$$\begin{array}{cccc} U_1 & U_2 & \cdots & U_n \\ | & | & & | \\ U_1 & U_2 & \cdots & U_n \end{array} \quad = \quad \begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{U_1,U_1} & C_{U_2,U_2} & \cdots & C_{U_n,U_n} \end{array}$$

and of course the element $w \otimes e \otimes w$ is represented by a diagram which is the same except that each U is replaced with a W. Thus we find by concatenating and simplifying these diagrams that the product $(u \otimes e \otimes u)(w \otimes e \otimes w)$ corresponds to

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{U_1,U_1}C_{W_1,W_1} & C_{U_2,U_2}C_{W_2,W_2} & \cdots & C_{U_n,U_n}C_{W_n,W_n} \end{array} . \tag{26}$$

We may expand each of the products $C_{U_j,U_j}C_{W_j,W_j}$ in terms of the cellular basis of A and use these expansions to write (26) as a linear combination of diagrams of the form

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{X_1,Y_1} & C_{X_2,Y_2} & \cdots & C_{X_n,Y_n} \end{array} .$$

Now for $j = 1, \ldots, n$, let s_j be such that $U_j \in M(\lambda_{s_j})$. The we know that each product $C_{U_j,U_j}C_{W_j,W_j}$ is a linear combination of cellular basis elements $C^{\lambda_{t_j}}_{X,Y}$ where $\lambda_{t_j} \leq \lambda_{s_j}$. It follows by Lemma 4 that all such diagrams have layer index at most μ (in the Λ-dominance order). Moreover, Lemma 4 also tells us that, if for any j the element W_j do not lie in $M(\lambda_{s_j})$ (so that $C_{U_j,U_j}C_{W_j,W_j}$ is a linear combination of cellular basis elements $C^{\lambda_{t_j}}_{X,Y}$ where $\lambda_{t_j} < \lambda_{s_j}$), then all of the diagrams in the expansion have layer index strictly less than μ, and hence by (25) we see that we must have $\psi_\mu(u, w) = 0$ in this case. Suppose now that $W_j \in M(\lambda_{s_j})$ for each j. By (2.4.1) in [4], we know that $C_{U_j,U_j}C_{W_j,W_j}$ is congruent to $\langle C_{U_j}, C_{W_j}\rangle C_{U_j,W_j}$ modulo cellular basis elements of lower cell index, where $\langle \cdot, \cdot \rangle$ is the appropriate cell form. Using Lemma 4 as above, we see that (26) is congruent modulo lower layers to

$$\langle C_{U_1}, C_{W_1}\rangle\langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle \begin{array}{cccc} U_1 & U_2 & \cdots & U_n \\ | & | & & | \\ W_1 & W_2 & \cdots & W_n \end{array} ,$$

which represents the element $\langle C_{U_1}, C_{W_1}\rangle\langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle\, u \otimes e \otimes w$, and hence we find that in this case

$$\psi_\mu(u, w) = \langle C_{U_1}, C_{W_1}\rangle\langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle.$$

Note in particular that ψ_μ is thus in all cases k-valued. We can now use these values for ψ_μ, together with equation (6) and Proposition 1 to compute the values of the cell form on the cell module $\Delta^{(\nu_1,\ldots,\nu_r)}$; indeed, if $y_1 \otimes \cdots \otimes$

$y_r \otimes u_1 \otimes \cdots \otimes u_n$ and $z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n$ are pure tensors in the cell module $\Delta^{(\nu_1,\ldots,\nu_r)}$, then we see that

$$\langle y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n,\ z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n \rangle = \langle y_1, z_1\rangle \cdots \langle y_r, z_r\rangle \langle u_1, w_1\rangle \cdots \langle u_n, w_n\rangle \quad (27)$$

if u_j and w_j lie in the same $\Delta(\lambda)$ for each $i = 1, \ldots, n$, and

$$\langle y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n,\ z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n \rangle = 0 \quad (28)$$

otherwise.

Next we seek to describe the cell radical of $\Delta^{(\nu_1,\ldots,\nu_r)}$. Using (21) and (23), we have isomorphisms of k-vector spaces

$$\begin{aligned} \Delta^{(\nu_1,\ldots,\nu_r)} &\cong S^{\nu_1} \otimes \cdots \otimes S^{\nu_r} \otimes V_\mu \\ &\cong \bigoplus_{\gamma \in \mathcal{R}_\mu} S^{\nu_1} \otimes \cdots \otimes S^{\nu_r} \otimes \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}}). \end{aligned} \quad (29)$$

For $\gamma \in \mathcal{R}_\mu$, let $\Omega_\gamma = S^{\nu_1} \otimes \cdots \otimes S^{\nu_r} \otimes \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}})$. Now we see from (28) that if γ, β are distinct elements of $\mathcal{R}_\mu$ and $u \in \Omega_\gamma$, $w \in \Omega_\beta$ then $\langle u, w\rangle = 0$. It follows that, if we let R_γ be the radical of the restriction to Ω_γ of $\langle \cdot, \cdot \rangle$, then the cell radical of $\Delta^{(\nu_1,\ldots,\nu_r)}$ is $\bigoplus_{\gamma \in \mathcal{R}_\mu} R_\gamma$.

Let us fix a basis in each Δ^λ and each S^ν; from these bases we obtain a basis of pure tensors in each Ω_γ. Let G_{ν_i} be the Gram matrix of the cell form of S^{ν_i} and G_{α_i} be the Gram matrix of the cell form of Δ^{α_i}, with respect to our chosen bases. If we let B_γ be the Gram matrix of the restriction of the cell form to Ω_γ with respect to our basis, then we see by (27) that B_γ is the matrix Kronecker product $G_{\nu_1} \otimes \cdots \otimes G_{\nu_r} \otimes G_{\alpha_{(1)\gamma^{-1}}} \otimes \cdots \otimes G_{\alpha_{(n)\gamma^{-1}}}$. By fixing some total order on the set R_γ and concatenating our bases of the Ω_γ in this order, we obtain a basis of $\Delta^{(\nu_1,\ldots,\nu_r)}$; using (28), we see that its Gram matrix with respect to this basis is of block diagonal form with diagonal blocks B_γ for $\gamma \in \mathcal{R}_\mu$. From this we see (using the fact that the rank of the Kronecker product of two matrices is the product of their ranks) that the rank of the cell form on $\Delta^{(\nu_1,\ldots,\nu_r)}$ is $|\mathcal{R}_\mu|$ times the product of the ranks of the cell forms of the cell modules $S^{\nu_1}, \ldots, S^{\nu_r}, \Delta^{\alpha_1}, \ldots, \Delta^{\alpha_n}$.

Now in constructing the basis of pure tensors for $\Delta^{(\nu_1,\ldots,\nu_r)}$ as above, we may choose our basis of each cell module of A and kS_n by taking a basis of the cell radical and extending this to a basis of the whole cell module. If we do this, then we see that an element $y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n$ of the basis of pure tensors for $\Delta^{(\nu_1,\ldots,\nu_r)}$ must lie in the cell radical if any y_i or u_i is an element of the cell radical of the cell module in which it lies. By the above calculation of the rank of the cell form on $\Delta^{(\nu_1,\ldots,\nu_r)}$, we see that the number of such elements must be equal to the dimension of the cell radical, and so we have now found a basis of the cell radical inside a basis of the whole cell module.

We can now use the theory of cellular algebras from section 3 of [4] together with our basis of $\Delta^{(\nu_1,\dots,\nu_r)}$ to deduce some results about the simple modules $L^{(\nu_1,\dots,\nu_r)}$ and semisimplicity of $A \wr S_n$. These results are already known for wreath products $A \wr S_n$ with A a general (i.e. not cellular) algebra given extra assumptions on the field (see for example [1, Lemma 3.4]), and in particular for the case $k\big(G \wr S_n\big) \cong (kG) \wr S_n$ where G is a finite group (see for example Chapter 4 of [7] for the case where the field is algebraically closed). However, if A is cellular then our work shows that these results hold *with no restriction on the field at all*. Given the importance of cellular algebras in certain areas of representation theory we are confident that they will prove useful.

Recall that Λ_0 indexes the simple modules of A. Let $\big(\widehat{\mathcal{P}}^r_n\big)_0$ denote the set of elements $(\nu_1,\dots,\nu_r) \in \widehat{\mathcal{P}}^r_n$ such that the cell radical of $\Delta^{(\nu_1,\dots,\nu_r)}$ is a proper submodule of $\Delta^{(\nu_1,\dots,\nu_r)}$, so that $\big(\widehat{\mathcal{P}}^r_n\big)_0$ indexes the simple modules of $A \wr S_n$. Recall that our field k has characteristic p, which may be zero or a prime.

Theorem 8. *The set $\big(\widehat{\mathcal{P}}^r_n\big)_0$ indexing the simple modules of $A \wr S_n$ consists exactly of those $(\nu_1,\dots,\nu_r) \in \widehat{\mathcal{P}}^r_n$ such that $\nu_i = ()$ whenever $\lambda_i \in \Lambda \setminus \Lambda_0$ and all ν_i are p-restricted (recall that $()$ is p-restricted for any p).*

In light of Theorem 8, we see that if we let s be the number of simple modules of A and we let $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_s$ be the subsequence of the sequence $\lambda_1, \lambda_2, \dots, \lambda_r$ consisting of the elements of Λ_0, then the simple $A \wr S_n$-modules may in fact be indexed by the set $\widehat{\mathcal{P}}^s_n(p)$ consisting of all length s multipartitions of n with p-restricted entries. The main idea of the following theorem is well known: see [9, p.204] and also [1, Proposition 3.7] and [3, Theorem 4.25]. As mentioned above, the version presented here is notable for its lack of conditions on the field.

Theorem 9. *Let $(\nu_1,\dots,\nu_r) \in \big(\widehat{\mathcal{P}}^r_n\big)_0$. Then corresponding to the isomorphism* (29)*, we have an isomorphism of k-vector spaces*

$$L^{(\nu_1,\dots,\nu_r)} \cong \bigoplus_{\gamma \in \mathcal{R}_\mu} D^{\nu_1} \otimes \cdots \otimes D^{\nu_r} \otimes L^{\alpha_{(1)\gamma^{-1}}} \otimes \cdots \otimes L^{\alpha_{(n)\gamma^{-1}}}$$

(where $\alpha_1,\dots,\alpha_n$ are as in (22)*). Moreover, $L^{(\nu_1,\dots,\nu_r)}$ has a representation by diagrams of the form* (11) *in exactly the same way as $\Delta^{(\nu_1,\dots,\nu_r)}$, by simply using elements of D^{ν_i} rather than S^{ν_i} and elements of L^{α_i} rather than Δ^{α_i}; the action on such diagrams is exactly the same as described above. We thus see that $L^{(\nu_1,\dots,\nu_r)}$ is isomorphic as an $A \wr S_n$-module to $\Theta^\mu\big((L^{\lambda^1},\dots,L^{\lambda^r}),(D^{\nu_1},\dots,D^{\nu_r})\big)$, where $\mu = (|\nu_1|,\dots,|\nu_r|)$ (a composition of n), and for convenience we let $L^\lambda = 0$ for $\lambda \in \Lambda \setminus \Lambda_0$.*

We thus see that if we index the simple modules by $\widehat{\mathcal{P}}^s_n(p)$ as above, then the simple indexed by $(\hat{\nu}_1,\dots,\hat{\nu}_s)$ (where each $\hat{\nu}_i$ is thus a p-restricted partition) is isomorphic to $\Theta^{\hat{\mu}}\big((L^{\hat{\lambda}^1}, \dots, L^{\hat{\lambda}^s}), (D^{\hat{\nu}_1}, \dots, D^{\hat{\nu}_s})\big)$, where $\hat{\mu} = (|\hat{\nu}_1|, \dots, |\hat{\nu}_s|)$.

Theorem 10. *Let $(\nu_1, \dots, \nu_r) \in \big(\widehat{\mathcal{P}}_n^r\big)_0$. Then we have $L^{(\nu_1,\dots,\nu_r)} \cong \Delta^{(\nu_1,\dots,\nu_r)}$ if and only if $D^{\nu_i} \cong S^{\nu_i}$ for each $i = 1, \dots, r$ and whenever we have $\nu_i \neq ()$ we have $L^{\lambda_i} \cong \Delta^{\lambda_i}$.*

Our final result is a criterion for semisimplicity; compare [1, Lemma 3.5].

Theorem 11. *If A is a cellular algebra, then $A \wr S_n$ is semisimple if and only if both kS_n and A are semisimple.*

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