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Constrained public goods in networks*

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Abstract

This paper analyses the private provision of public goods where agents interact within a fixed network structure and may benefit only from their direct neighbours' provisions. We generalise the public goods in networks model of Bramoullé and Kranton (2007) to allow for constrained provision. In so doing, we characterise Nash equilibria with no intermediate contributors.

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1 Introduction

Voluntary contributions account for the provision of many public goods, ranging from essential infrastructure to education and health care. The seminal contribution of Bergstrom, Blume, and Varian (1986, known as BBV) provides the groundwork for the analysis of the private provision of pure public goods. Recent work on public goods in networks, initiated by the key paper of Bramoullé and Kranton (2007), has many interesting facets and applications. The technology of network analysis allows us to move on from the provision of pure public goods, which benefit all agents, to a more detailed model of local public goods with a heterogeneous benefit structure shaped by a network.

In this paper, we generalise Bramoullé and Kranton's (2007) model to the case where agents' public good provisions may be constrained exogenously so that they may not be able to provide their optimal provisions on their own. In interpretation, there are many economic examples where actions are taken subject to constraints — for example, a budget constraint for consumption decisions, or a time or competence constraint to individual actions or effort. Unlike Tiebout's seminal contribution, and the subsequent vast literature on the local public good model, the public goods in networks model allows for geographic spillovers among nearby communities.[1]

The public goods in networks literature has burgeoned to include more general approaches: Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) incorporate private information; Galeotti and Goyal (2010) investigate issues of network formation; Bramoullé, Kranton, and D'Amours (2014) investigate private provision with linear best replies; Allouch (2015, 2017) extends BBV analysis to networks; and Elliott and Golub (2019) explore decentralised mechanisms for efficient provision. Other important related contributions have been made by Acemoglu, García-Jimeno, and Robinson (2015), Acemoglu, Malekian, and Ozdaglar (2016), López-Pintado (2017), Kinateder and Merlino (2017), and Sun (2017).[2] For more recent contributions on the existence and uniqueness of a Nash equilibrium for network games, including public goods in networks, see Ramachandran and Chaintreau (2015), Naghizadeh and Liu (2017, 2018), Bodwin (2017),

[1] An exception is Bloch and Zenginobuz (2006).

[2] See also Mauleon, Sempere-Monerris, and Vannetelbosch (2016), Mohlmeier, Rusinowska, and Tanimura (2016), and Dev (2018) for related contributions on network formation and stability.

Melo (2018), Parise and Ozdaglar (2018, 2019), and Chen, Zenou, and Zhou (2019).

In our model, the existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem, which then opens the door for an investigation of how, depending on the network structure, agents pool together their equilibrium provisions. To investigate this, we focus on Nash equilibria with no intermediate contributors—that is, Nash equilibria where agents are either full contributors or free-riders. This is because no-intermediate-contributors (NIC hereafter) Nash equilibria are of special interest as they illustrate in an acute form how the network can affect provision patterns. Our analysis shows that NIC Nash equilibria can be characterised by the graph theoretic notion of an *r-insulated set of order t*. An r-insulated set of order t is a set of agents such that each agent in the set is adjacent to at most r other agents in the set and each agent outside the set is adjacent to at least $r + t$ agents in the set.

A key finding in Bramoullé and Kranton (2007) is that specialised Nash equilibria—that is, equilibria with both full contributors and free-riders—correspond to maximal independent sets of the network and therefore always exist, and actually may well be the only Nash equilibria.[3] We obtain their finding as a special case of our result, since specialised equilibria correspond to NIC Nash equilibria with unconstrained provision. In contrast, we find that with constrained provision, NIC Nash equilibria are not always guaranteed to exist. For instance, NIC Nash equilibrium does not exist in the complete network for non-integer optimal provision. Thus, unlike Bramoullé and Kranton (2007), with constrained provision some network structures may not support full specialisation.

This paper is structured as follows. In Section 2, we go over the related literature. Then we describe the model in Section 3 and characterise Nash equilibrium in Section 4. Section 5 concludes the paper.

2 The model

Consider a model of public goods in networks—that is, local public goods with benefits accessible along geographic or social links. There are $n \geq 2$ agents arranged in a connected fixed network $\mathbf{g}$. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} =$

[3] For instance, in the star network.

1 indicates that agent i and agent j are neighbours in the network $\mathbf{g}$ and $g_{ij} = 0$ otherwise. In particular, we assume that $g_{ii} = 0$ for each agent i. We denote by $N = \{1, \ldots, n\}$ the set of agents and by $\mathcal{N}_i = \{j \in N \mid g_{ij} = 1\}$ the set of agent i's neighbours.

Each agent i faces a marginal cost $c > 0$ for providing a public good and his payoffs, for the profile of provisions $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$, are given by:

$$U_i(\mathbf{x}) = b(x_i + \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j) - cx_i,$$

where b is the benefit function, which is differentiable, strictly increasing, and concave.

We assume that for each agent i , public good provision is bounded $x_i \in [0, 1]$. We also assume that $b'(k) = c$, for some positive real number k that denotes optimal provision. Note that the case $k \leq 1$ corresponds to public goods in networks with unconstrained provision as studied in Bramoullé and Kranton (2007). For simplicity of notations, we assume $k \in I = [1, +\infty[$, so that the unconstrained case obtains when $k = 1$. Finally, let $[k]$ denote the integer part of k.

3 Equilibrium provisions

In the following, we characterise Nash equilibria.

Proposition 1. *A profile of provisions* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}_+^N$ *is a Nash equilibrium if and only if for every agent* i *one of the following holds:*

(1) $x_i = 0$ *and* $\sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j \geq k$.

(2) $x_i = 1$ *and* $\sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j \leq k - 1$.

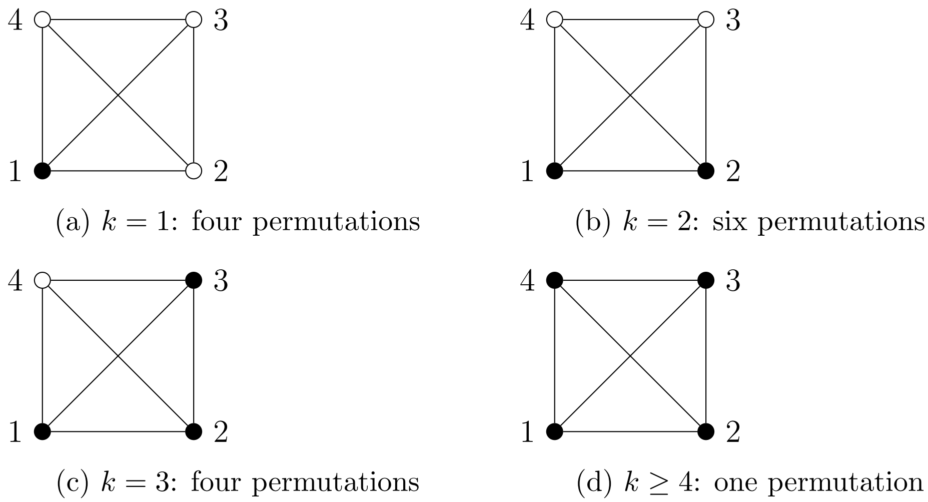
(3) $x_i = k - \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j$ *and* $k - 1 < \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j < k$.

Proof. The best reply of each agent is

$$x_i = f_i(\mathbf{x}_{-i}) = \max\{\min\{k - \sum_{j \in \mathcal{N}_i(\mathbf{g})} x_j, 1\}, 0\},$$

which gives the required results above. ∎

Figure 1: NIC Nash equilibria in the complete network of four agents when $k \in \mathbb{N}^*$.



(a) $k = 1$: four permutations

(b) $k = 2$: six permutations

(c) $k = 3$: four permutations

(d) $k \geq 4$: one permutation

Proposition 1 tells us the public good provision of each agent at a Nash equilibrium. We may distinguish three types of agents: *free-riders*, who contribute 0; *full contributors*, who contribute 1; and the others, *intermediate contributors*—that is, $0 < x_i < 1$. Using a standard fixed-point argument, it is easy to show that there exists a Nash equilibrium for any network. In order to relate the provision profiles to the network structure, we will focus on Nash equilibria with no intermediate contributors, the (NIC) Nash equilibria.

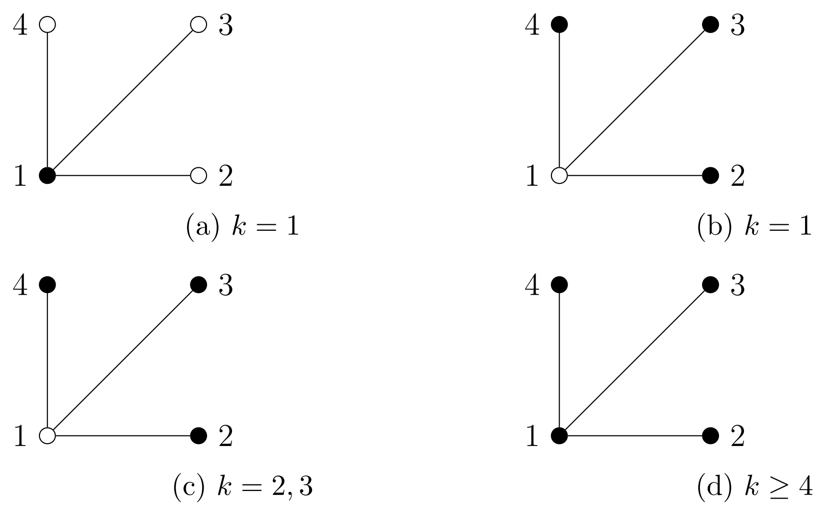
We introduce the following notion from graph theory:

Definition 1. *For positive integers r and $t \geq 1$, an r-insulated set of order t of a network $\mathbf{g}$ is a set of agents $S \subseteq N$ such that each agent in S is adjacent to at most r other agents in S and each agent not in S is adjacent to at least $r + t$ agents in S.*

The notion of an r-insulated set of order t generalises r-insulated set (of order 1) in Jagota, Narasimhan, and Šoltés (2001) and the maximal independent set (or equivalently, 0-insulated set) of order t in Bramoullé and Kranton (2007).

Note that not all networks contain an r-insulated set of order $t \geq 2$. For instance, in the complete graph, there is no r-insulated set of order $t \geq 2$. However, the existence of r-insulated sets of order 1 in every network was established in Jagota, Narasimhan, and Šoltés (2001).

Figure 2: NIC Nash equilibria in the star network of four agents when $k \in \mathbb{N}^*$



3.1 Integer optimal provision

First, we consider the case of integer optimal provision.

Proposition 2. *Suppose* $k \in \mathbb{N}^*$*. A profile of provisions is a NIC Nash equilibrium if and only if its set of full contributors is a* $(k-1)$*-insulated set of order* 1*. Since for every network* $\mathbf{g}$ *there exists a* $(k-1)$*-insulated set of order* 1*, there always exists a NIC Nash equilibrium. Assign* 1 *to agents in a* $(k-1)$*-insulated set and* 0 *to those outside.*

Proof. The proof follows easily from Proposition 1. ■

Proposition 2 holds only for $k \in \mathbb{N}^*$ and shows that in every network structure, there exists a NIC Nash equilibrium. Observe that NIC Nash equilibria with $k = 1$ coincide with specialised equilibria in Bramoullé and Kranton (2007).

Next we illustrate NIC Nash equilibria in a few canonical network structures.

Example 1. Consider the complete network and the star network both with four agents (see Figures 1 and 2). Let vertices in black represent full contributors and vertices in white represent free-riders. For a complete network, observe that at the NIC Nash equilibria, full contributors correspond to subsets of agents of size k if $1 \leq k < 4$ and to the entire set of agents if $4 \leq k$. For a star network, with unconstrained provision if $k = 1$, observe that there are two NIC Nash equilibria consisting of either the core agent or the periphery

Figure 3: NIC Nash equilibria in the star network of four agents when $k \in I \setminus \mathbb{N}^*$

(a) $[k] = 1, 2$ (b) $[k] \geq 4$

agents as full contributors. If $k \geq 2$, then there exists only one NIC Nash equilibrium: all periphery agents are full contributors if $k < 4$ and the entire set of agents are full contributors if $k \geq 4$.

3.2 Non-integer optimal provision

Now, we consider the case of non-integer optimal provision.

Proposition 3. *Suppose $k \in I \setminus \mathbb{N}^*$. A profile of provisions is a NIC Nash equilibrium if and only if its set of full contributors is a $([k]-1)$-insulated set of order 2.*

Proof. The proof follows easily from Proposition 1. ■

Proposition 3 holds only for $k \in I \setminus \mathbb{N}^*$. Since not all networks contain a $([k]-1)$-insulated set of order 2, it follows from Proposition 3 that a NIC Nash equilibrium does not always exist. For example, in the complete graph, there is no NIC Nash equilibrium. However, in the star network, a NIC Nash equilibrium exists if $[k] \neq N - 1$. We illustrate this in the following example.

Example 2. Consider again the star network with four agents (see Figure 3). If $[k] = 1, 2$, then there exists a unique NIC Nash equilibrium consisting of the periphery agents as full contributors. If $[k] \geq 4$, then there exists a unique NIC Nash equilibrium (all agents are full contributors). If $[k] = 3$, then there is no NIC Nash equilibrium.

3.2.1 Stability

While the existence of a NIC Nash equilibrium is not always guaranteed if $k \in I \setminus \mathbb{N}^*$, in the following we will show that, whenever it exists, a NIC Nash equilibrium is stable. To do so, we will consider the following simple notion of stability based on Nash tâtonnement.[4] Let $\mathbf{f} = (f_1, \ldots, f_n)$ denote the collection of individual best-reply functions. A Nash equilibrium $\mathbf{x}$ is stable if and only if there exists $\rho > 0$ such that for any vector $\boldsymbol{\varepsilon}$ satisfying $|\varepsilon_i| < \rho$ and $x_i + \varepsilon_i \geq 0$, for every i, the sequence defined $\mathbf{x}^0 = \mathbf{x} + \boldsymbol{\varepsilon}$ and $\mathbf{x}^{n+1} = \mathbf{f}(\mathbf{x}^n)$ converges to $\mathbf{x}$.

Proposition 4. *Suppose $k \in I \setminus \mathbb{N}^*$. A NIC Nash equilibrium is stable.*

Proof. First, observe that when $k \in I \setminus \mathbb{N}^*$ in a NIC Nash Equilibrium all agents who make the full contribution are connected to at most $[k] - 1$ agents and all agents who make no contributions are connected to at least $[k] + 1$ agents. Consider any small perturbation of agents' provisions at a NIC Nash equilibrium. Then it can be easily checked that, despite the perturbation, free-riders still make no provision and full contributors still make full contribution. Thus the NIC Nash equilibrium is stable. ■

Specialised equilibria are a special case of NIC Nash equilibria and obviously both rule out intermediate contributors, but there is a key difference because NIC Nash equilibria might not have free-riders, unlike specialised equilibria. The absence of free-riders typically happens when the size of each neighbourhood of the network is below the optimal provision level. This key difference can be significant, since whenever the network is connected, NIC Nash equilibria without free-riders are always stable, whereas stable specialised equilibria always have free-riders.

4 Conclusion

We have extended the standard public goods in networks model to allow for constrained provision. We have found that a general definition which relates to both r-insulated sets and maximal independent sets is helpful in establishing the existence of NIC Nash equilibria. We have also found that with constrained provision, the existence of NIC Nash equilibria is not universal. A natural extension of the concept of NIC Nash equilibrium to be investigated in future work is a Nash equilibrium with a minimal number of intermediate contributors. While it can be easily checked that in both complete network and

[4] See Bervoets and Faure (2019) for a recent interesting investigation of stability.

star network a Nash equilibrium with at most one intermediate contributor always exists, it remains to be seen whether similar results can be obtained for more general networks.

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