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Moving frames and Noether's finite difference conservation laws I

E. L. MANSFIELD*, A. ROJO-ECHEBURÚA AND P. E. HYDON
SMSAS, University of Kent, Canterbury, CT2 7FS, UK
*Corresponding author: e.l.mansfield@kent.ac.uk

AND

L. PENG

Waseda Institute for Advanced Study, Waseda University, Tokyo, 169-8050, Japan

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We consider the calculation of Euler–Lagrange systems of ordinary difference equations, including the difference Noether's theorem, in the light of the recently-developed calculus of difference invariants and discrete moving frames. We introduce the difference moving frame, a natural discrete moving frame that is adapted to difference equations by prolongation conditions. For any Lagrangian that is invariant under a Lie group action on the space of dependent variables, we show that the Euler–Lagrange equations can be calculated directly in terms of the invariants of the group action. Furthermore, Noether's conservation laws can be written in terms of a difference moving frame and the invariants. We show that this form of the laws can significantly ease the problem of solving the Euler–Lagrange equations, and we also show how to use a difference frame to integrate Lie group invariant difference equations. In this Part I, we illustrate the theory by applications to Lagrangians invariant under various solvable Lie groups. The theory is also generalized to deal with variational symmetries that do not leave the Lagrangian invariant. Apart from the study of systems that are inherently discrete, one significant application is to obtain geometric (variational) integrators that have finite difference approximations of the continuous conservation laws embedded *a priori*. This is achieved by taking an invariant finite difference Lagrangian in which the discrete invariants have the correct continuum limit to their smooth counterparts. We show the calculations for a discretization of the Lagrangian for Euler's elastica, and compare our discrete solution to that of its smooth continuum limit.

Keywords: Noether's theorem; finite difference; discrete moving frames.

1. Introduction

Conservation laws are among the most fundamental attributes of a given system of partial differential equations. They constrain all solutions and have a topological interpretation as cohomology classes in the restriction of the variational bicomplex to solutions of the given system (Vinogradov, 1984). Their importance has led to a major theme in geometric integration that seeks to construct finite difference approximations that preserve conservation laws in some sense. Approaches include methods that preserve symplectic or multisymplectic structures, energy, and other conservation laws. Variational integrators exploit the structure inherent in variational problems, where conservation laws are associated with symmetries.

Much of physics is governed by variational principles, with symmetries leading to conservation laws and Bianchi identities. Noether's first (and best-known) theorem relates an R -dimensional Lie group of symmetries of a given variational problem to R linearly independent conservation laws of the system

of Euler–Lagrange differential equations (Noether, 1918). A second theorem in her seminal paper deals with variational problems that have gauge symmetries, and there is now a bridging theorem that includes all intermediate cases (Hydon & Mansfield, 2011). For a translation into English of Noether’s paper and an historical survey of the context and impact of Noether’s theorems and subsequent generalizations, see Kosmann-Schwarzbach (2011). Each of these theorems has now been adapted to difference equations (Dorodnitsyn, 2001; Hydon & Mansfield, 2011; Hydon, 2014).

Noether’s (first) theorem may be used to derive conservation laws from a finite-dimensional Lie group of variational point symmetries that act on the space of independent and dependent variables. One can work in terms of the given variables, but for complex problems it is usually more efficient to factor out the Lie group action from the outset and work entirely in terms of invariants and the equivariant frame. Having solved a simplified problem for the invariants, one can then construct the solution to the original problem. This divide-and-conquer approach has recently been achieved, for differential equations, by using moving frame theory. These results were presented in Gonçalves & Mansfield (2012) for all three inequivalent $SL(2)$ actions in the complex plane and in Gonçalves & Mansfield (2013) for the standard $SE(3)$ action. Finally, in Gonçalves & Mansfield (2016) the calculations were extended to cases where the independent variables are not invariant under the group action, which is the case for many physically important models.

The theory and applications of Lie group based moving frames are now well established, and provide an invariant calculus to study differential systems that are either invariant or equivariant under the action of a Lie group. Associated with the name of Élie Cartan (1952), who used *repères mobile* to solve equivalence problems in differential geometry, the ideas go back to earlier works, for example, by Cotton (1905) and Darboux (1887).

From the point of view of symbolic computation, a breakthrough in the understanding of Cartan’s methods for differential systems came in a series of papers by Fels & Olver (1999, 2001), Olver (2001a,b), Hubert (2005, 2007, 2009) and Hubert & Kogan (2007a,b), which provide a coherent, rigorous and constructive moving frame method. The resulting differential invariant calculus is the subject of the textbook by Mansfield (2010). Applications include integration of Lie group invariant differential equations (Mansfield, 2010), the calculus of variations and Noether’s theorem (see e.g., Gonçalves & Mansfield, 2012, 2013; Kogan & Olver, 2003), and integrable systems (e.g., Mansfield & van der Kamp, 2006; Beffa, 2006, 2008, 2010).

Moving frame theory assumes that the Lie group acts on a continuous space. For spaces in which some variables are discrete, the theory must be modified. The first results for the computation of discrete invariants using group-based moving frames were given by Olver (who called them joint invariants) in Olver (2001b); modern applications to date include computer vision (Olver, 2001c) and numerical schemes for systems with a Lie symmetry (Kim & Olver, 2004; Kim, 2007, 2008; Mansfield & Hydon, 2008; Rebelo & Valiquette, 2013). While moving frames for discrete applications as formulated by Olver do give generating sets of discrete invariants, the recursion formulae for differential invariants (which were so successful for the application of moving frames to calculus-based results) do not generalize well to joint invariants. In particular, joint invariants do not seem to have computationally useful recursion formulae under the shift operator (which is defined below). To overcome this problem, Beffa *et al.* (2013) introduced the notion of a *discrete moving frame*, which is essentially a sequence of frames. In that paper discrete recursion formulae were proven for small computable generating sets of invariants, called the *discrete Maurer–Cartan invariants*, and their *syzygies* (i.e., their recursion relations) were investigated.

Difference equations arise as models in their own right, not just as approximations to differential equations. Some have conservation laws; for instance, discrete integrable systems have infinite

hierarchies of conservation laws (Mikhailov *et al.*, 2011). However, the geometry underlying finite difference conservation laws is much less well understood than its counterpart for differential equations. Discrete moving frames are widely applicable to discrete spaces, but they do not incorporate the prolongation structure that is inherent in difference equations. This paper describes the necessary modification to incorporate this structure, difference moving frame theory, and applies it to ordinary difference equations with variational symmetries. This makes it possible to factor out the symmetries and write the Euler–Lagrange equations entirely in terms of invariant variables and the conservation laws in terms of the invariants and the frame.

We consider systems of ordinary difference equations (OΔEs) whose independent variable is $n \in \mathbb{Z}$, with dependent variables $\mathbf{u} = (u^1, \dots, u^q) \in \mathbb{R}^q$. A system of OΔEs is a given system of relations between the quantities $u^\alpha(n+j)$ for a finite set of integers j . The system holds for all n in a given connected domain (interval), which may or may not be finite, so it is helpful to suppress n and use the shorthand u_j^α for $u^\alpha(n+j)$ and \mathbf{u}_j for $\mathbf{u}(n+j)$.

The (forward) shift operator S acts on functions of n as follows:

$$S : n \mapsto n + 1, \quad S : f(n) \mapsto f(n + 1),$$

for all functions f whose domain includes n and $n + 1$. In particular,

$$S : u_j^\alpha \mapsto u_{j+1}^\alpha$$

on any domain where both of these quantities are defined. The forward difference operator is $S - \text{id}$, where id is the identity operator:

$$\text{id} : n \mapsto n, \quad \text{id} : f(n) \mapsto f(n), \quad \text{id} : u_j^\alpha \mapsto u_j^\alpha.$$

A variational system of OΔEs is obtained by extremizing a given functional, $\mathcal{L}[\mathbf{u}] = \sum_n \mathbf{L}(n, \mathbf{u}_0, \dots, \mathbf{u}_J)$, where the sum is taken over all n in a given interval, which need not necessarily be bounded; the Lagrangian \mathbf{L} depends on only a finite number of arguments. The extrema are given by the condition

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \sum_n \mathbf{L}(n, \mathbf{u}_0 + \varepsilon \mathbf{w}_0, \dots, \mathbf{u}_J + \varepsilon \mathbf{w}_J) = 0$$

for all functions $\mathbf{w} : \mathbb{Z} \rightarrow \mathbb{R}^q$. It is well known that the extrema satisfy the following system of Euler–Lagrange (difference) equations (Kupershmidt, 1985; Hydon & Mansfield, 2004):

$$E_{u^\alpha}(\mathbf{L}) := \sum_{j=0}^J S_{-j} \left(\frac{\partial \mathbf{L}}{\partial u_j^\alpha} \right) = 0, \quad \text{where } S_{-j} = (S^{-1})^j. \quad (1.1)$$

Each $E_{u^\alpha}(\mathbf{L})$ depends only on n and $\mathbf{u}_{-J}, \dots, \mathbf{u}_J$, so the Euler–Lagrange equations are of order at most $2J$. In the following, we develop an invariantized version of these equations, together with invariant conservation laws that stem from Noether's theorem.

In Section 2, the natural geometric setting for difference equations is discussed. Just as a given differential equation can be regarded as a subspace of an appropriate jet space, a given difference equation is a subspace of an appropriate difference prolongation space.

Section 3 is a brief review of the difference calculus of variations. The methods that we will develop emulate these calculations as far as possible, but using the invariant difference calculus. In Section 4, we give a short overview of continuous and discrete moving frames, and introduce the difference moving frame, which gives the geometric framework for our results. A running example is used from here on to illustrate how the theory is applied.

In Section 5, we show how a difference moving frame can be used to calculate the difference Euler–Lagrange equations directly in terms of the invariants. This calculation yields boundary terms that can be transformed into the conservation laws, which require both invariants and the frame for their expression. Section 6 introduces the adjoint representation of the frame, enabling us (in Section 7) to state and prove key results on the difference conservation laws that arise via the difference analogue of Noether’s theorem.

Section 8 shows how the difference moving frame may be used to integrate a difference system that is invariant under a Lie group action. Further, we show how the conservation laws and the frame together may be used to ease the integration process, whether or not one can solve for the frame.

In Section 9, we generalize the difference frame version of Noether theory to include variational symmetries that do not leave the Lagrangian invariant. (Their counterparts for differential equations are sometimes called divergence symmetries.) Consequently, difference moving frames may be used to solve or simplify ordinary difference systems with any finite-dimensional Lie group of variational symmetries. This is illustrated in Section 10.

The paper concludes with another use of difference moving frames: to create symmetry-preserving numerical approximations. Section 11 illustrates this for the Euler elastica, which is invariant under the Euclidean group action. Smooth Lagrangians that are invariant under the Euclidean action can be expressed in terms of the Euclidean curvature and arc length, and the Noether laws for these are the conservation of linear and angular momenta. We demonstrate for this example that by taking a difference frame that converges, in some sense, to a smooth frame, then we obtain simultaneous convergence of the Lagrangian, the Euler–Lagrange system and all three conservation laws. The specific difference Lagrangian we consider is a discrete analogue of that for Euler’s elastica, and we show how our results compare with that of the smooth Euler–Lagrange equation, solved using the analogous theory of smooth moving frames. In effect, we show how the design of the approximate Lagrangian can yield a discrete Euler–Lagrange system that is a variational integrator and that respects difference analogues of all three conservation laws.

2. Difference prolongation spaces

A given differential equation can be expressed geometrically as a variety in an appropriate jet space whose coordinates are the independent and dependent variables, together with sufficiently many derivatives of the dependent variables (Olver, 1995). There is an analogous geometric structure for difference equations, but jet spaces and derivatives are replaced by difference prolongation spaces and shifts, respectively. While equations may have singularities, techniques described in this paper are valid only away from these, and hence we do not consider such points here.

The difference prolongation spaces are obtained from the space of independent and dependent variables, $\mathbb{Z} \times \mathbb{R}^q$. Over each base point $n \in \mathbb{Z}$, the dependent variables take values in a continuous fibre $U \subset \mathbb{R}^q$, which has the coordinates $\mathbf{u} = (u^1, \dots, u^q)$. For simplicity, we shall assume that all structures on each fibre are identical; the necessary modifications when this does not hold are obvious but can be messy.

It is useful to regard n as representing a given (arbitrary) base point and to prolong the fibre over n to include the values of \mathbf{u} on other fibres. For all sequences $(\mathbf{u}(m))_{m \in \mathbb{Z}}$, let \mathbf{u}_j denote $\mathbf{u}(n + j)$. Then the fibre over n is $P_n^{(0,0)}(U) \simeq U$, with coordinates \mathbf{u}_0 . The first forward prolongation space over n is $P_n^{(0,1)}(U) \simeq U \times U$ with coordinates $z = (\mathbf{u}_0, \mathbf{u}_1)$. Similarly, the J th forward prolongation space over n is the product space $P_n^{(0,J)}(U) \simeq U \times \cdots \times U$ ($J + 1$ copies) with coordinates $z = (\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_J)$. More generally, one can include both forward and backward shifts, obtaining the prolongation spaces $P_n^{(J_0,J)}(U) \simeq U \times \cdots \times U$ ($J - J_0 + 1$ copies) with coordinates $z = (\mathbf{u}_{J_0}, \dots, \mathbf{u}_J)$, where $J_0 \leq 0$ and $J \geq 0$.

Prolongation spaces are equipped with an ordering that enables sequences to be represented as points in the appropriate prolongation space. Consequently, a given difference equation, $\mathcal{A}(n, \mathbf{u}_{J_0}, \dots, \mathbf{u}_J) = 0$, corresponds to a variety in any prolongation space that contains the submanifold $P_n^{(J_0,J)}(U)$. This makes it possible to apply continuous methods to difference equations (locally, away from singularities).

Every prolongation space $P_n^{(J_0,J)}(U)$ is a submanifold of the *total prolongation space over n* , $P_n^{(-\infty,\infty)}(U)$ with coordinates $z = (\dots, \mathbf{u}_{-2}, \mathbf{u}_{-1}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \dots)$. As n is a free variable, the same structures are repeated over each n . This yields the natural map

$$\pi : P_n^{(-\infty,\infty)}(U) \longrightarrow P_{n+1}^{(-\infty,\infty)}(U), \quad \pi : z \mapsto \hat{z};$$

here the coordinates on $P_{n+1}^{(-\infty,\infty)}(U)$ are distinguished by a caret, so $\hat{\mathbf{u}}_j$ denotes $\mathbf{u}(n + 1 + j)$ for all sequences $(\mathbf{u}(m))_{m \in \mathbb{Z}}$. Consequently, each $\hat{\mathbf{u}}_j$ is represented in $P_n^{(-\infty,\infty)}(U)$ by $\mathbf{u}_{j+1} = S\mathbf{u}_j$, where we regard the shift S as an operator on $P_n^{(-\infty,\infty)}(U)$. Similarly, variables on any prolongation space over a point m may be represented as equivalent variables in $P_n^{(-\infty,\infty)}(U)$ by applying the $(m - n)$ th power of the shift operator S . We denote the j th power of S by S_j , so that $\mathbf{u}_j = S_j\mathbf{u}_0$ for each $j \in \mathbb{Z}$.

For OΔEs, it is enough to use the restriction of S to finite prolongation spaces. To accommodate difference equations on a finite or semi-infinite interval, we add the constraint that $\mathbf{u}_{j+1} = S\mathbf{u}_j$ is defined only if $n + j$ and $n + 1 + j$ are in the interval.

In the remainder of the paper, we treat n as fixed, using powers of the shift operator S to represent structures on prolongation spaces over any base point m as equivalent structures on all sufficiently large prolongation spaces over n . In Section 4, we will use this property to construct difference moving frames.

Throughout, we work formally, without considering convergence of sums or integrals.

3. The difference variational calculus

Consider a functional of the form

$$\mathcal{L}[\mathbf{u}] = \sum \mathcal{L}(n, \mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_J), \quad (3.1)$$

where $\mathbf{u}_j = (u_j^1, \dots, u_j^q) \in \mathbb{R}^q$. Here and henceforth, the unadorned summation symbol denotes summation over n ; the range of this summation is a given interval in \mathbb{Z} , which can be unbounded. For sums over all other variables, we will use the Einstein summation convention as far as possible, to

avoid a proliferation of summation symbols. The *variation* of $\mathcal{L}[\mathbf{u}]$ in the direction \mathbf{w} is

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}[\mathbf{u} + \varepsilon\mathbf{w}] = \sum w_j^\alpha \frac{\partial \mathcal{L}}{\partial u_j^\alpha}. \quad (3.2)$$

Summing by parts, using the identity

$$(S_j f) g = f S_{-j} g + (S_j - \text{id})(f S_{-j} g) \quad (3.3)$$

and pulling out the factor $(S - \text{id})$ from $(S_j - \text{id})$, we obtain

$$w_j^\alpha \frac{\partial \mathcal{L}}{\partial u_j^\alpha} = w_0^\alpha S_{-j} \frac{\partial \mathcal{L}}{\partial u_j^\alpha} + (S - \text{id}) A_{\mathbf{u}}(n, \mathbf{w}), \quad (3.4)$$

where

$$A_{\mathbf{u}}(n, \mathbf{w}) = \sum_{j=1}^J \sum_{l=0}^{j-1} S_l \left\{ w_0^\alpha S_{-j} \frac{\partial \mathcal{L}}{\partial u_j^\alpha} \right\}.$$

The sum over n of the differences $(S - \text{id}) A_{\mathbf{u}}$ telescopes, contributing only boundary terms to the variation. So for all variations to be zero, \mathbf{u} must solve the Euler–Lagrange system of difference equations

$$E_{u^\alpha}(\mathbf{L}) := S_{-j} \frac{\partial \mathcal{L}}{\partial u_j^\alpha} = 0, \quad \alpha = 1, \dots, q. \quad (3.5)$$

Moreover, the boundary terms yield natural boundary conditions that must be satisfied if \mathbf{u} is not fully constrained at the boundary.

EXAMPLE 3.1 As a running example, we will consider a functional with two dependent variables, $\mathbf{u} = (x, u)$, that is of the form

$$\mathcal{L}[x, u] = \sum L(x_0, u_0, x_1, u_1, u_2).$$

Setting $\mathbf{w} = (w^x, w^u)$, the variation is

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \mathcal{L}[x + \varepsilon w^x, u + \varepsilon w^u] = \sum \left\{ w_0^x E_x(\mathbf{L}) + w_0^u E_u(\mathbf{L}) + (S - \text{id}) A_{\mathbf{u}}(n, \mathbf{w}) \right\}.$$

There are two Euler–Lagrange equations, one for each dependent variable:

$$E_x(\mathbf{L}) := \frac{\partial \mathcal{L}}{\partial x_0} + S_{-1} \frac{\partial \mathcal{L}}{\partial x_1} = 0, \quad E_u(\mathbf{L}) := \frac{\partial \mathcal{L}}{\partial u_0} + S_{-1} \frac{\partial \mathcal{L}}{\partial u_1} + S_{-2} \frac{\partial \mathcal{L}}{\partial u_2} = 0.$$

The remaining terms come from $A_{\mathbf{u}}(n, \mathbf{w}) = A^x + A^u$, where

$$A^x = w_0^x S_{-1} \frac{\partial \mathcal{L}}{\partial x_1}, \quad A^u = w_0^u S_{-1} \frac{\partial \mathcal{L}}{\partial u_1} + (S + \text{id}) \left(w_0^u S_{-2} \frac{\partial \mathcal{L}}{\partial u_2} \right).$$

So far, we have considered all variations \mathbf{w} , without reference to the Lagrangian. However, variations $\boldsymbol{\phi}$ that do not change the functional $\mathcal{L}[\mathbf{u}]$ are of special importance. They leave the Lagrangian \mathbf{L} invariant, up to a total difference term.

DEFINITION 3.2 Suppose that a nonzero function $\boldsymbol{\phi} = (\phi^1(n, \mathbf{u}), \dots, \phi^q(n, \mathbf{u}))^T$ satisfies

$$\phi_j^\alpha(n, \mathbf{u}) \frac{\partial \mathbf{L}}{\partial u_j^\alpha} = (S - \text{id})B(n, \mathbf{u}), \quad \text{where } \phi_j^\alpha = S_j \phi_0^\alpha, \quad (3.6)$$

for some $B(n, \mathbf{u})$ (which may be zero). Then the Lagrangian \mathbf{L} is said to have a *variational symmetry*¹ with characteristic $\boldsymbol{\phi}$. The Lagrangian is invariant under this symmetry if $B = 0$.

The relationship between variational symmetries and characteristics will be made clear in Section 6. The next theorem explains why these symmetries are important.

THEOREM 3.3 (Difference Noether's theorem). Suppose that a Lagrangian \mathbf{L} has a variational symmetry with characteristic $\boldsymbol{\phi} \neq \mathbf{0}$. If $\mathbf{u} = \bar{\mathbf{u}}$ is a solution of the Euler–Lagrange system for \mathbf{L} then

$$((S - \text{id})\{A_{\mathbf{u}}(n, \boldsymbol{\phi}) - B(n, \mathbf{u})\})|_{\mathbf{u}=\bar{\mathbf{u}}} = 0. \quad (3.7)$$

Proof. Substituting $\boldsymbol{\phi}$ for \mathbf{w} in (3.4) gives

$$\phi^\alpha(n, \mathbf{u}) E_{u^\alpha}(\mathbf{L}) + (S - \text{id})A_{\mathbf{u}}(n, \boldsymbol{\phi}) = \phi_j^\alpha(n, \mathbf{u}) \frac{\partial \mathbf{L}}{\partial u_j^\alpha} = (S - \text{id})B(n, \mathbf{u}).$$

The result follows immediately. □

The expression in Equation (3.7) is a *conservation law* for the Euler–Lagrange system. As there is only one independent variable, the expression in braces is a first integral, so every solution of the Euler–Lagrange system satisfies

$$\{A_{\mathbf{u}}(n, \boldsymbol{\phi}) - B(n, \mathbf{u})\}|_{\mathbf{u}=\bar{\mathbf{u}}} = c,$$

where c is a constant.

EXAMPLE 3.4 (Example 3.1 cont.) For instance, the Lagrangian

$$\mathbf{L}(x_0, u_0, x_1, u_1, u_2) = \frac{x_1 - x_0}{\{(u_2 - u_1)(u_1 - u_0)\}^{3/2}} \quad (3.8)$$

has three variational symmetries, all with $B = 0$. Table 1 lists the corresponding first integrals for every Lagrangian $\mathbf{L}(x_0, u_0, x_1, u_1, u_2)$ that has these symmetries.

We will see in the sequel that the first symmetry arises from the invariance of the Lagrangian under the scalings $(x, u) \mapsto (\lambda^3 x, \lambda u)$, for $\lambda \in \mathbb{R}^+$, the second arises from invariance under translations in x ,

¹ This is shorthand for a one-parameter local Lie group of variational symmetries, see Olver (1993).

TABLE 1 *Characteristics and first integrals for the Lagrangian (3.8)*

ϕ^x	ϕ^u	First integral: $A_{\mathbf{u}}(n, \phi)$
$3x$	u	$3x_0 S_{-1} \frac{\partial L}{\partial x_1} + u_0 S_{-1} \frac{\partial L}{\partial u_1} + (S + \text{id}) \left(u_0 S_{-2} \frac{\partial L}{\partial u_2} \right)$
1	0	$S_{-1} \frac{\partial L}{\partial x_1}$
0	1	$S_{-1} \frac{\partial L}{\partial u_1} + (S + \text{id}) S_{-2} \frac{\partial L}{\partial u_2} \equiv S_{-1} \frac{\partial L}{\partial u_2} - \frac{\partial L}{\partial u_0}$

that is, $x \mapsto x + a$ for all $a \in \mathbb{R}$ and the third arises from invariance under translations in u , namely $u \mapsto u + b$, $b \in \mathbb{R}$.

Despite having three first integrals for the system of Euler–Lagrange equations, the expressions are unwieldy for the given Lagrangian (3.8) and the system remains difficult to solve. We will show that the necessary insight into the solution set is obtained by using coordinates that are adapted to the three symmetries. So for the rest of the running example, we will consider only those Lagrangians $L(x_0, u_0, x_1, u_1, u_2)$ that have the same three symmetries. A major advantage of using symmetry-adapted coordinates is that one can deal with all such Lagrangians together.

To obtain adapted coordinates for general variational problems, we need the machinery of moving frames and, in particular, difference moving frames.

4. Moving frames

This section outlines the basic theory of the moving frame and its extension to the discrete moving frame on an appropriate prolongation space, as illustrated by the running example. The discrete moving frame is essentially a sequence of moving frames. We introduce the *difference moving frame*, which is equivalent to a discrete moving frame (on a prolongation space) that is subject to further prolongation conditions. It is analogous to the continuous moving frame on a given jet space, adapted to difference equations that are invariant or equivariant under a Lie group action.

4.1 Lie group actions, invariants and moving frames

Given a Lie group G that acts on a manifold M , the group action of G on M is a map

$$G \times M \rightarrow M, \quad (g, z) \mapsto g \cdot z. \quad (4.1)$$

For a left action,

$$g_1 \cdot (g_2 \cdot z) = (g_1 g_2) \cdot z;$$

similarly, for a right action

$$g_1 \cdot (g_2 \cdot z) = (g_2 g_1) \cdot z.$$

Given a left action $(g, z) \mapsto g \cdot z$, it follows that $(g, z) \mapsto g^{-1} \cdot z$ is a right action. In practice both right and left actions occur, and the ease of the calculations can differ considerably, depending on the choice.

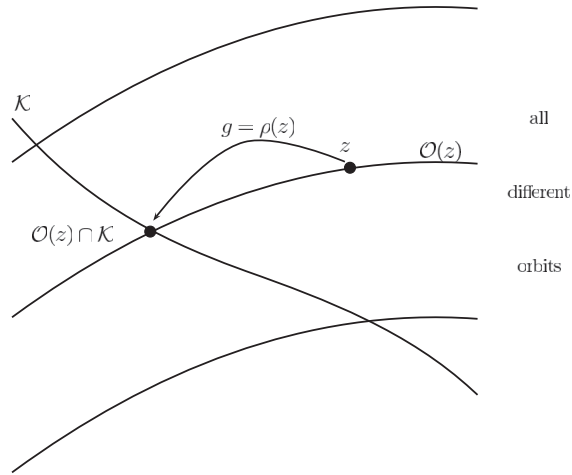


FIG. 1. A moving frame defined by a cross-section.

In a theoretical development, however, only one is needed, so we restrict ourselves in the following to left actions.

Remark on notation. For ease of exposition, a tilde will be used to denote a transformed variable; for instance, $g \cdot z = \tilde{z}$. In this notation, g is implicit.

Important assumption. In the following outline of moving frame theory, the action is assumed to be free and regular on M ; see Mansfield (2010) for details. (If it is not, replace M by a domain \mathcal{M} on which the action is free and regular.) Consequently, there exists a cross section $\mathcal{K} \subset M$ that is transverse to the orbits $\mathcal{O}(z)$ and, for each $z \in M$, the set $\mathcal{K} \cap \mathcal{O}(z)$ has just one element, the projection of z onto \mathcal{K} , as shown in Fig. 1.

Using the cross-section \mathcal{K} , a moving frame for the group action on a neighbourhood $\mathcal{U} \subset M$ of z can be defined as follows.

DEFINITION 4.1 (Moving frame). Given a smooth Lie group action $G \times M \rightarrow M$, a moving frame is an equivariant map $\rho : \mathcal{U} \subset M \rightarrow G$. Here \mathcal{U} is called the domain of the frame.

A left equivariant map satisfies $\rho(g \cdot z) = g\rho(z)$, and a right equivariant map satisfies $\rho(g \cdot z) = \rho(z)g^{-1}$. The frame is called left or right accordingly.

In order to find the frame, let the cross-section \mathcal{K} be given by a system of equations $\psi_r(z) = 0$, for $r = 1, \dots, R$, where R is the dimension of the group G . One then solves the so-called normalization equations,

$$\psi_r(g \cdot z) = 0, \quad r = 1, \dots, R, \quad (4.2)$$

for g as a function of z . The solution is the group element $g = \rho(z)$ that maps z to its projection on \mathcal{K} (see Fig. 1). In other words, the frame ρ satisfies

$$\psi_r(\rho(z) \cdot z) = 0, \quad r = 1, \dots, R.$$

The conditions on the action above are those for the implicit function theorem to hold (Hirsch, 1976), so the solution ρ is unique. A consequence of uniqueness is that

$$\rho(g \cdot z) = \rho(z)g^{-1},$$

that is, the frame is *right equivariant*, as both $\rho(g \cdot z)$ and $\rho(z)g^{-1}$ solve the equation $\psi_r(\rho(g \cdot z) \cdot (g \cdot z)) = 0$. A left frame, which satisfies $\rho(g \cdot z) = g\rho(z)$, is obtained by taking the inverse of a right frame. In practice, the ease of calculation can differ considerably depending on the choice of parity.

EXAMPLE 4.2 (Example 3.1 cont.) It is straightforward to check that the Lagrangian (3.8) is invariant under the scaling and translation group action on \mathbb{R}^2 given by

$$(x, u) \mapsto (\lambda^3 x + a, \lambda u + b), \quad \lambda \in \mathbb{R}^+, a, b \in \mathbb{R};$$

the Lie group is the semi-direct product, $\mathbb{R}^+ \ltimes \mathbb{R}^2$. The action is not free on the space \mathbb{R}^2 over n , which has coordinates (x_0, u_0) . To achieve freeness, one needs to work in a higher-dimensional continuous space. For instance, the action is free on the first forward prolongation space $P_n^{(0,1)}(\mathbb{R}^2)$, which has coordinates (x_0, u_0, x_1, u_1) . On this prolongation space, the action is given by

$$(x_0, u_0, x_1, u_1) \mapsto (\lambda^3 x_0 + a, \lambda u_0 + b, \lambda^3 x_1 + a, \lambda u_1 + b).$$

If we choose the normalization equations to be $g \cdot (x_0, u_0, x_1, u_1) = (0, 0, *, 1)$, where $*$ is unspecified, the values of the parameters for the frame group element are²

$$\lambda = \frac{1}{u_1 - u_0}, \quad a = -\frac{x_0}{(u_1 - u_0)^3}, \quad b = -\frac{u_0}{u_1 - u_0}.$$

A standard matrix representation for the generic group element is

$$g(\lambda, a, b) = \begin{pmatrix} \lambda^3 & 0 & a \\ 0 & \lambda & b \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} g \cdot x \\ g \cdot u \\ 1 \end{pmatrix} = g(\lambda, a, b) \begin{pmatrix} x \\ u \\ 1 \end{pmatrix}.$$

With this representation, which is faithful, the frame is

$$\rho(x_0, u_0, x_1, u_1) = \begin{pmatrix} \frac{1}{(u_1 - u_0)^3} & 0 & -\frac{x_0}{(u_1 - u_0)^3} \\ 0 & \frac{1}{u_1 - u_0} & -\frac{u_0}{u_1 - u_0} \\ 0 & 0 & 1 \end{pmatrix}.$$

² As $\lambda > 0$, this is valid throughout the half-space $\mathcal{U} = \{(x_0, u_0, x_1, u_1) \in P_n^{(0,1)}(\mathbb{R}^2) : u_1 > u_0\}$. For the half-space $u_1 < u_0$, the normalization $g \cdot (x_0, u_0, x_1, u_1) = (0, 0, *, -1)$ would be appropriate.

The equivariance is easily checked in matrix form: for $z = (x_0, u_0, x_1, u_1)$,

$$\begin{aligned} \rho(g \cdot z) &= \begin{pmatrix} 1 & 0 & -\frac{x_0 + a/\lambda^3}{(u_1 - u_0)^3} \\ \frac{1}{\lambda^3(u_1 - u_0)^3} & 0 & -\frac{x_0 + a/\lambda^3}{(u_1 - u_0)^3} \\ 0 & \frac{1}{\lambda(u_1 - u_0)} & -\frac{u_0 + b/\lambda}{u_1 - u_0} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -\frac{x_0}{(u_1 - u_0)^3} \\ \frac{1}{(u_1 - u_0)^3} & 0 & -\frac{x_0}{(u_1 - u_0)^3} \\ 0 & \frac{1}{u_1 - u_0} & -\frac{u_0}{u_1 - u_0} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-3} & 0 & -\frac{a}{\lambda^3} \\ 0 & \lambda^{-1} & -\frac{b}{\lambda} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \rho(z)g(\lambda, a, b)^{-1}. \end{aligned}$$

Returning to the general theory, the requirement that frames are equivariant enables one to obtain invariants of the group action.

LEMMA 4.1 (Normalized invariants). Given a left or right action $G \times M \rightarrow M$ and a right frame ρ , then $\iota(z) = \rho(z) \cdot z$, for z in the domain of the frame ρ , is invariant under the group action.

Proof. First apply the group action to z ; then, by definition,

$$\iota(g \cdot z) = \rho(g \cdot z) \cdot (g \cdot z) = \rho(z) \cdot g^{-1}g \cdot z = \rho(z) \cdot z = \iota(z), \quad (4.3)$$

so $\iota(z)$ is an invariant function. \square

DEFINITION 4.3 The normalized invariants are the components of $\iota(z)$.

DEFINITION 4.4 A set of invariants is said to be a *generating*, or *complete*, set for an algebra of invariants if any invariant in the algebra can be written as a function of elements of the generating set.

We now state the replacement rule (see [Fels & Olver, 1999](#)), from which it follows that the normalized invariants provide a set of generators for the algebra of invariants.

THEOREM 4.5 (Replacement rule). If $F(z)$ is an invariant of the given action $G \times M \rightarrow M$ for a right moving frame ρ on M then $F(z) = F(\iota(z))$.

Proof. As $F(z)$ is invariant, $F(z) = F(g \cdot z)$ for all $g \in G$. Setting $g = \rho(z)$ and using the definition of $\iota(z)$ yields the required result. \square

DEFINITION 4.6 (Invariantization operator). Given a right moving frame ρ , the map $z \mapsto \iota(z) = \rho(z) \cdot z$ is called the *invariantization operator*. This operator extends to functions as $f(z) \mapsto f(\iota(z))$, and $f(\iota(z))$ is called the *invariantization of f* .

If z has components z^α , let $\iota(z^\alpha)$ denote the α th component of $\iota(z)$.

EXAMPLE 4.7 (Example 3.1 cont.) The action of the frame on $z = (x_0, u_0, x_1, u_1) \in P_n^{(0,1)}(\mathbb{R}^2)$ is

$$\rho(z) \cdot z = \left(0, 0, \frac{x_1 - x_0}{(u_1 - u_0)^3}, 1 \right),$$

which is seen to be an invariant of the group action (constants are always invariants). With this same frame, higher forward prolongations of \mathbb{R}^2 yield more invariants. For $z = (x_0, u_0, x_1, u_1, \dots, x_J, u_J) \in P_n^{(0,J)}(\mathbb{R}^2)$,

$$\rho(z) \cdot z = \left(0, 0, \frac{x_1 - x_0}{(u_1 - u_0)^3}, 1, \frac{x_2 - x_0}{(u_1 - u_0)^3}, \frac{u_2 - u_0}{u_1 - u_0}, \dots, \frac{x_J - x_0}{(u_1 - u_0)^3}, \frac{u_J - u_0}{u_1 - u_0} \right).$$

Difference Euler–Lagrange equations typically involve both positive and negative indices j , so in general, one must prolong forwards and backwards. For instance, for elements $z = (x_{-1}, u_{-1}, x_0, u_0, x_1, u_1) \in P_n^{(-1,1)}(\mathbb{R}^2)$,

$$\rho(z) \cdot z = \left(\frac{x_{-1} - x_0}{(u_1 - u_0)^3}, \frac{u_{-1} - u_0}{u_1 - u_0}, 0, 0, \frac{x_1 - x_0}{(u_1 - u_0)^3}, 1 \right),$$

and so on. These results are summarized by

$$\iota(x_j) := \rho \cdot x_j = \frac{x_j - x_0}{(u_1 - u_0)^3}, \quad \iota(u_j) := \rho \cdot u_j = \frac{u_j - u_0}{u_1 - u_0}, \quad j \in \mathbb{Z}.$$

We now calculate recurrence relations for these invariants and show that all of them can be written in terms of two fundamental invariants,

$$\kappa = \iota(u_2) = \rho \cdot u_2, \quad \eta = \iota(x_1) = \rho \cdot x_1, \quad (4.4)$$

and their shifts. For each $j \in \mathbb{Z}$,

$$S\{\iota(u_j)\} = \frac{u_{j+1} - u_1}{u_2 - u_1} = \frac{\frac{u_{j+1} - u_0}{u_1 - u_0} - \frac{u_1 - u_0}{u_1 - u_0}}{\frac{u_2 - u_0}{u_1 - u_0} - \frac{u_1 - u_0}{u_1 - u_0}} = \frac{\iota(u_{j+1}) - 1}{\kappa - 1}.$$

We will later prove that each shift of an invariant is an invariant. So the last equality can be obtained more easily by using the replacement rule (Theorem 4.5):

$$S\{\iota(u_j)\} = \frac{u_{j+1} - u_1}{u_2 - u_1} = \iota \left(\frac{u_{j+1} - u_1}{u_2 - u_1} \right) = \frac{\iota(u_{j+1}) - \iota(u_1)}{\iota(u_2) - \iota(u_1)} = \frac{\iota(u_{j+1}) - 1}{\kappa - 1}.$$

The calculation for $S\{\iota(x_j)\}$ is similar. Together, these identities amount to

$$\iota(u_{j+1}) = (\kappa - 1) S\{\iota(u_j)\} + 1, \quad \iota(x_{j+1}) = (\kappa - 1)^3 S\{\iota(x_j)\} + \eta. \quad (4.5)$$

This shows that the invariants with positive j can be written in terms of κ , η and their forward shifts. For convenience, let $\kappa_k = S_k \kappa$ and $\eta_k = S_k \eta$ for all $k \in \mathbb{Z}$. To find the invariants with negative indices j , invert (4.5):

$$\iota(u_{j-1}) = \frac{S_{-1}\{\iota(u_j)\} - 1}{\kappa_{-1} - 1}, \quad \iota(x_{j-1}) = \frac{S_{-1}\{\iota(x_j)\} - \eta_{-1}}{(\kappa_{-1} - 1)^3}.$$

For instance,

$$\iota(u_{-1}) = \frac{-1}{\kappa_{-1} - 1}, \quad \iota(u_{-2}) = \frac{-\kappa_{-2}}{(\kappa_{-2} - 1)(\kappa_{-1} - 1)},$$

and so on. Note that all $\iota(u_j)$, $\iota(x_j)$ can be written in terms of η and κ and their shifts.

The invariants $\iota(x_j)$, $\iota(u_j)$ do not behave well under the shift map, in the sense that $S\{\iota(x_j)\} \neq \iota(x_{j+1})$ and $S\{\iota(u_j)\} \neq \iota(u_{j+1})$. Even though the shift map takes invariants to invariants, writing shifts of invariantized variables in terms of shifts of η and κ involves complicated expressions. The discrete moving frame, defined next, will lead to the proper geometric setting that explains the origin of these expressions.

4.2 Discrete moving frames

A discrete moving frame is an analogue of a moving frame that is adapted to discrete base points. It amounts to a sequence of frames defined on a product manifold. More detail on discrete moving frames and their applications can be found in [Beffa et al. \(2013\)](#) and [Beffa & Mansfield \(2018\)](#).

In this subsection, the manifold on which G acts will be the Cartesian product manifold $\mathcal{M} = M^N$. We assume that the action on \mathcal{M} is free, taking the number of copies N of the manifold M to be as high as necessary. This happens, for example, when the action is (locally) effective on subsets, see [Boutin \(2002\)](#) for a discussion of this and related issues; further, see [Olver \(2001b\)](#) for a pathological example where the product action is not free for any N . Questions like the regularity and freeness of the action will refer to the diagonal action on the product; given the action $(g, z_j) \mapsto g \cdot z_j$ for $z_j \in M$, the diagonal action of G on $z = (z_1, z_2, \dots, z_N) \in \mathcal{M}$ is

$$g \cdot (z_1, z_2, \dots, z_N) \mapsto (g \cdot z_1, g \cdot z_2, \dots, g \cdot z_N).$$

Important note. Throughout this subsection, no assumptions are made about any relationship between the elements z_1, \dots, z_N .

DEFINITION 4.8 (Discrete moving frames: [Beffa et al., 2013](#); [Beffa & Mansfield, 2018](#)). Let G^N denote the Cartesian product of N copies of the group G . A map

$$\rho : M^N \rightarrow G^N, \quad \rho(z) = (\rho_1(z), \dots, \rho_N(z))$$

is a right discrete moving frame if

$$\rho_k(g \cdot z) = \rho_k(z)g^{-1}, \quad k = 1, \dots, N,$$

and a left discrete moving frame if

$$\rho_k(g \cdot z) = g\rho_k(z), \quad k = 1, \dots, N.$$

Obtaining a frame via the use of normalization equations yields a right frame. As the theory for right and left frames is parallel, we restrict ourselves to studying right frames only. It is advisable when calculating examples, however, to check the parity of actions and frames, see [Mansfield \(2010\)](#) for the subtleties involved.

A discrete moving frame is a sequence of moving frames (ρ_k) with a nontrivial intersection of domains which, locally, are uniquely determined by the cross-section $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_N)$ to the group orbit through z . The right moving frame component ρ_k is the unique element of the group G that takes z to the cross section \mathcal{K}_k . We also define for a right frame, the invariants

$$I_{k,j} := \rho_k(z) \cdot z_j. \quad (4.6)$$

If M is q -dimensional, so that z_j has components z_j^1, \dots, z_j^q , the q components of $I_{k,j}$ are the invariants

$$I_{k,j}^\alpha := \rho_k(z) \cdot z_j^\alpha, \quad \alpha = 1, \dots, q. \quad (4.7)$$

These are invariant by the same reasoning as for Equation (4.3). For later use, let ι_k denote the invariantization operator with respect to the frame $\rho_k(z)$, so that

$$I_{k,j} = \iota_k(z_j), \quad I_{k,j}^\alpha = \iota_k(z_j^\alpha).$$

4.3 Difference moving frames

The discrete moving frame is a powerful construction that can be adapted to any discrete domain. Typically, \mathcal{M} represents the fibres M over a sequence of N discrete points. The geometric context may determine additional structures on \mathcal{M} .

From Section 2, the fact that n is a free variable allows us to replicate the same structures over each base point m , using powers of the natural map π . The shift operator enables these structures to be represented on prolongation spaces over any given n . This suggests that the natural moving frame for a given OΔE has $\mathcal{M} = P_n^{(J_0, J)}(U)$ for some appropriate $J_0 \leq 0$ and $J \geq 0$. Consequently, $N = J - J_0 + 1$; from here on, we replace the indices $1, \dots, N$ by J_0, \dots, J .

We now use \mathcal{K}_k and ρ_k to denote the cross-sections and frames on \mathcal{M} , respectively. The cross-section over n , denoted \mathcal{K}_0 , is replicated for all other base points $n+k$ if and only if the cross-section over $n+k$ is represented on \mathcal{M} by

$$\mathcal{K}_k = S_k \mathcal{K}_0 \quad (4.8)$$

for all k . When this condition holds, $\rho_k = S_k \rho_0$ (by definition) for all k ; consequently, $\mathcal{K}_{k+1} = S \mathcal{K}_k$ and $\rho_{k+1} = S \rho_k$.

DEFINITION 4.9 A *difference moving frame* is a discrete moving frame such that \mathcal{M} is a prolongation space $P_n^{(J_0, J)}(U)$ and (4.8) holds for all $J_0 \leq k \leq J$.

By definition, the invariants $I_{k,j}$ given by a difference moving frame satisfy

$$S(I_{k,j}) = I_{k+1, j+1}, \quad (4.9)$$

so every invariant $I_{k,j}$ can be expressed as a shift of $I_{0, j-k}$.

DEFINITION 4.10 (Discrete Maurer–Cartan invariants). Given a right discrete moving frame ρ , the *right discrete Maurer–Cartan group elements* are

$$K_k = \rho_{k+1}\rho_k^{-1} \quad (4.10)$$

for $J_0 \leq k \leq J - 1$.

As the frame is equivariant, each K_k is invariant under the action of G . We call the components of the Maurer–Cartan elements the *Maurer–Cartan invariants*.

As ρ_k is a frame for each k , the components of $\rho_k(z) \cdot z$ generate the set of all invariants by the replacement rule (Theorem 4.5). Moreover, for the two different frames ρ_{k+1} and ρ_k , and for any invariant $F(z)$, the replacement rule gives

$$F(z) = F(\rho_k(z) \cdot z) = F(\rho_{k+1}(z) \cdot z) = F(\rho_{k+1}(z)\rho_k^{-1}(z) \cdot \rho_k(z) \cdot z).$$

It can be seen that the action of the Maurer–Cartan element $K_k = \rho_{k+1}\rho_k^{-1}$ provides a mechanism for any invariant, written in terms of the components of the invariant $\rho_k(z) \cdot z$, to be expressed in terms of the components of the invariant $\rho_{k+1}(z) \cdot z$. Such a mechanism is an example of a *syzygy*, which we define next.

DEFINITION 4.11 (Syzygy). A syzygy on a set of invariants is a relation between invariants that expresses functional dependency.

Hence, a syzygy on a set of invariants is a function of invariants, which is identically zero when the invariants are expressed in terms of the underlying variables (in this case, $z \in \mathcal{M}$).

The key idea is to use the Maurer–Cartan group elements, which are well-adapted to studying difference equations, to express *all* invariants in terms of a small generating set. Using (4.6) and (4.10) we have

$$K_k \cdot I_{k,j} = \rho_{k+1}\rho_k^{-1} \cdot \rho_k \cdot z_j = \rho_{k+1} \cdot z_j = I_{k+1,j}, \quad (4.11)$$

and iterating this, we have $K_{k+1}K_k \cdot I_{k,j} = I_{k+2,j}$, and so on, which leads to the following result.

THEOREM 4.12 (Beffa *et al.*, 2013, Proposition 3.11). Given a right discrete moving frame ρ , the components of K_k , together with the set of all *diagonal invariants*, $I_{j,j} = \rho_j(z) \cdot z_j$, generate all other invariants.

We refer to the difference identities, or syzygies, (4.11) as *recurrence relations* for the invariants. It is helpful to extend slightly the notion of a generating set from Definition 4.4.

DEFINITION 4.13 A set of invariants is a generating set for an algebra of difference invariants if any difference invariant in the algebra can be written as a function of elements of the generating set and their shifts.

For a right difference moving frame, the identities $I_{j,j} = S_j I_{0,0}$ and $K_k = S_k K_0$ hold, so Theorem 4.12 reduces to the following result.

THEOREM 4.14 Given a right difference moving frame ρ , the set of all invariants is generated by the set of components of $K_0 = \rho_1\rho_0^{-1}$ and $I_{0,0} = \rho_0(z) \cdot z_0$.

Note that as K_0 is invariant, the replacement rule gives the following useful identity:

$$K_0 = \iota_0(\rho_1), \quad (4.12)$$

where ι_0 denotes invariantization with respect to the frame ρ_0 .

EXAMPLE 4.15 (Example 3.1 cont.) As the Lagrangian L in (3.8) is second-order, the Euler–Lagrange equations define a subspace of the prolongation space $\mathcal{M} = P_n^{(-2,2)}(\mathbb{R}^2)$, which we use for the remainder of this example. Early in this section, we found a continuous moving frame ρ for $P_n^{(0,1)}(\mathbb{R}^2)$; this can be used to construct a difference moving frame on \mathcal{M} , setting

$$\rho_0 = \begin{pmatrix} \frac{1}{(u_1 - u_0)^3} & 0 & -\frac{x_0}{(u_1 - u_0)^3} \\ 0 & \frac{1}{u_1 - u_0} & -\frac{u_0}{u_1 - u_0} \\ 0 & 0 & 1 \end{pmatrix} \quad (4.13)$$

and $\rho_k = S_k \rho_0$. It is helpful to review the recurrence relations obtained earlier in the light of Equation (4.11). By definition,

$$\begin{pmatrix} I_{0,j}^x \\ I_{0,j}^u \\ 1 \end{pmatrix} = \iota_0 \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix} = \rho_0 \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix}$$

and therefore

$$\begin{pmatrix} S I_{0,j}^x \\ S I_{0,j}^u \\ 1 \end{pmatrix} = \rho_1 \begin{pmatrix} x_{j+1} \\ u_{j+1} \\ 1 \end{pmatrix} = (\rho_1 \rho_0^{-1}) \rho_0 \begin{pmatrix} x_{j+1} \\ u_{j+1} \\ 1 \end{pmatrix} = K_0 \begin{pmatrix} I_{0,j+1}^x \\ I_{0,j+1}^u \\ 1 \end{pmatrix}. \quad (4.14)$$

Calculating the matrix $K_0 = \rho_1 \rho_0^{-1} = \iota_0(\rho_1)$ yields

$$K_0 = \begin{pmatrix} \frac{1}{(\kappa - 1)^3} & 0 & -\frac{\eta}{(\kappa - 1)^3} \\ 0 & \frac{1}{\kappa - 1} & -\frac{1}{\kappa - 1} \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.15)$$

where η and κ are defined in (4.4). Clearly, equations (4.5) and (4.14) are consistent.

The Maurer–Cartan invariants for this example are the components of K_0 and their shifts. By Theorem 4.14, the algebra of invariants is generated by η , κ and their shifts, because both components of $I_{0,0} = \rho_0 \cdot (x_0, u_0)$ are zero.

A complete discussion of Maurer–Cartan invariants for discrete moving frames, with their recurrence relations and discrete syzygies, is given in [Beffa & Mansfield \(2018\)](#).

4.4 Differential–difference invariants and the differential–difference syzygy

We aim to obtain the Euler–Lagrange equations in terms of the invariants, and also the form of the conservation laws in terms of invariants and a frame. A key ingredient of our method will be a *differential–difference syzygy* between differential and difference invariants, which will feature prominently in our formulae.

Given any smooth path $t \mapsto z(t)$ in the space $\mathcal{M} = M^N$, consider the induced group action on the path and its tangent. We extend the group action to the dummy variable t trivially, so that t is invariant. The action is extended to the first-order jet space of \mathcal{M} as follows:

$$g \cdot \frac{dz(t)}{dt} = \frac{d(g \cdot z(t))}{dt}.$$

If the action is free and regular on \mathcal{M} , it will remain so on the jet space and we may use the same frame to find the first-order differential invariants

$$I_{k,j;t}(t) := \rho_k(z(t)) \cdot \frac{dz_j(t)}{dt}. \quad (4.16)$$

Let $I_{k,j}(t)$ denote the restriction of $I_{k,j}$ to the path $z(t)$. The frame depends on $z(t)$, so, in general,

$$I_{k,j;t}(t) \neq \frac{d}{dt} I_{k,j}(t).$$

The particular differential–difference syzygy that we will need to calculate the invariantized variation of the Euler–Lagrange equations concerns the relationship between the t -derivative of the discrete invariants $I_{k,j}(t)$ and the differential–difference invariants $I_{k,j;t}(t)$. It takes the form

$$\frac{d}{dt} \kappa = \mathcal{H} \sigma, \quad (4.17)$$

where κ is a vector of generating invariants, \mathcal{H} is a linear difference operator with coefficients that are functions of κ and its shifts, and σ is a vector of generating first-order differential invariants of the form (4.16).

If the generating discrete invariants are known, the syzygies can be found by direct differentiation followed by the replacement rule (Theorem 4.5). Recurrence relations for the differential invariants are obtained in a manner analogous to those of the discrete invariants, as illustrated in the running example below. This allows one to write the syzygy in terms of a set of generating differential invariants. Another method is to differentiate the Maurer–Cartan matrix as follows. Given a matrix representation for the right frame ρ_k , restricted throughout the following to the path $z(t)$, apply the product rule to the definition of K_k to obtain

$$\frac{d}{dt} K_k = \frac{d}{dt} (\rho_{k+1} \rho_k^{-1}) = \left(\frac{d}{dt} \rho_{k+1} \right) \rho_{k+1}^{-1} K_k - K_k \left(\frac{d}{dt} \rho_k \right) \rho_k^{-1}. \quad (4.18)$$

This motivates the following definition.

DEFINITION 4.16 (Curvature matrix). The curvature matrix N_k is given by

$$N_k = \left(\frac{d}{dt} \rho_k \right) \rho_k^{-1} \quad (4.19)$$

when ρ_k is in matrix form.

It can be seen that for a right frame, N_k is an invariant matrix that involves the first-order differential invariants. The above derivation applies to all discrete moving frames. For a difference frame, moreover, $N_k = S_k N_0$ and (4.18) simplifies to the set of shifts of a generating syzygy,

$$\frac{d}{dt} K_0 = (S N_0) K_0 - K_0 N_0. \quad (4.20)$$

As N_0 is invariant, the replacement rule yields another useful identity:

$$N_0 = \iota_0 \left(\frac{d}{dt} \rho_0 \right). \quad (4.21)$$

Equating components in (4.20) yields syzygies relating the derivatives of the Maurer–Cartan invariants to the first-order differential invariants and their shifts.

Finally, we need the differential–difference syzygies for the remaining generating invariants, the diagonal invariants (see Theorem 4.14). For a linear (matrix) action,

$$\frac{d}{dt} I_{0,0}(t) = \left(\frac{d}{dt} \rho_0 \right) \rho_0^{-1} \cdot (\rho_0 \cdot z_0(t)) + \rho_0 \cdot \frac{d}{dt} z_0(t) = N_0 I_{0,0}(t) + I_{0,0;t}(t). \quad (4.22)$$

For nonlinear actions, the techniques described in the text Mansfield (2010) may be modified to accommodate difference moving frames.

In all examples in this paper, the diagonal invariants $I_{0,0}^\alpha$ are normalized to be constants; nevertheless, there are examples where this need not hold. In some circumstances, it is necessary to chose a normalization that makes off-diagonal invariants constants, in which case some diagonal invariants may depend on $z(t)$.

EXAMPLE 4.17 (Example 3.1 cont.) We now turn our attention to the differential invariants for our running example. Writing $x_j = x_j(t)$ and $u_j = u_j(t)$, etc., the action on the derivatives $x'_j = dx_j/dt$, $u'_j = du_j/dt$ is induced by the chain rule, as follows:

$$g \cdot x'_j = \frac{d(g \cdot x_j)}{d(g \cdot t)} = \frac{d(g \cdot x_j)}{dt} = \lambda^3 x'_j,$$

and similarly,

$$g \cdot u'_j = \lambda u'_j.$$

Define

$$I_{0,j;t}^x = \rho_0 \cdot x'_j = \frac{x'_j}{(u_1 - u_0)^3}, \quad I_{0,j;t}^u = \rho_0 \cdot u'_j = \frac{u'_j}{u_1 - u_0}. \quad (4.23)$$

We first obtain recurrence relations for the $I_{k,j;t}^x$ and $I_{k,j;t}^u$. As

$$SI_{0,j;t}^x = S(\rho_0 \cdot x'_j) = \rho_1 \cdot x'_{j+1} = (\rho_1 \rho_0^{-1}) \rho_0 \cdot x'_{j+1} = K_0 \cdot I_{0,j+1;t}^x,$$

and similarly for $I_{0,j;t}^u$, it follows that

$$SI_{0,j;t}^x = \frac{I_{0,j+1;t}^x}{(\kappa - 1)^3}, \quad SI_{0,j;t}^u = \frac{I_{0,j+1;t}^u}{\kappa - 1}. \quad (4.24)$$

In the same way, one can use the shift operator and $\rho_k \rho_0^{-1} = K_{k-1} K_{k-2} \cdots K_0$ to obtain all $I_{k,j;t}^x, I_{k,j;t}^u$ in terms of the generating Maurer–Cartan invariants,

$$\begin{aligned} \sigma^x &:= I_{0,0;t}^x = \iota_0(x'_0) = \rho_0 \cdot x'_0 = \frac{x'_0}{(u_1 - u_0)^3}, \\ \sigma^u &:= I_{0,0;t}^u = \iota_0(u'_0) = \rho_0 \cdot u'_0 = \frac{u'_0}{u_1 - u_0}, \end{aligned}$$

and their shifts. We now obtain the differential–difference syzygies (4.20). The simplest way to calculate N_0 is to use the identity (4.21):

$$N_0 = \iota_0 \left(\frac{d}{dt} \rho_0 \right) = \begin{pmatrix} -3(\iota_0(u'_1) - \iota_0(u'_0)) & 0 & -\iota_0(x'_0) \\ 0 & -(\iota_0(u'_1) - \iota_0(u'_0)) & -\iota_0(u'_0) \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.25)$$

Other methods are detailed in Mansfield (2010). Now use the recurrence relations (4.24) to obtain the differential invariants in (4.25) in terms of σ^x, σ^u and their shifts:

$$N_0 = \begin{pmatrix} -3((\kappa - 1)S\sigma^u - \sigma^u) & 0 & -\sigma^x \\ 0 & -((\kappa - 1)S\sigma^u - \sigma^u) & -\sigma^u \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.26)$$

Inserting (4.15) and (4.26) into (4.20) yields, after equating components and simplifying,

$$\begin{aligned} \frac{d\eta}{dt} &= [(\kappa - 1)^3 S - \text{id}] \sigma^x + 3\eta [\text{id} - (\kappa - 1) S] \sigma^u, \\ \frac{d\kappa}{dt} &= (\kappa - 1) [\text{id} - \kappa S + (\kappa_1 - 1) S_2] \sigma^u. \end{aligned} \quad (4.27)$$

Therefore, the differential–difference syzygy between the generating difference invariants, η and κ , and the generating differential invariants, σ^x and σ^u , can be put into the canonical form

$$\frac{d}{dt} \begin{pmatrix} \eta \\ \kappa \end{pmatrix} = \mathcal{H} \begin{pmatrix} \sigma^x \\ \sigma^u \end{pmatrix},$$

where \mathcal{H} is a linear difference operator whose coefficients depend only on the generating difference invariants and their shifts.

5. The Euler–Lagrange equations for a Lie group invariant Lagrangian

We are now ready to present our first main result, the calculation of the Euler–Lagrange equations, in terms of invariants, for a Lie group invariant difference Lagrangian. We emulate the calculation of the Euler–Lagrange equations given in Section 3, but use the computational techniques developed above for difference invariants.

First, recall the summation by parts formula (3.3). This leads to the following similar definition.

DEFINITION 5.1 Given a linear difference operator $\mathcal{H} = c_j S_j$, the adjoint operator \mathcal{H}^* is defined by

$$\mathcal{H}^*(F) = S_{-j}(c_j F)$$

and the associated boundary term $A_{\mathcal{H}}$ is defined by

$$F\mathcal{H}(G) - \mathcal{H}^*(F)G = (S - \text{id})(A_{\mathcal{H}}(F, G)),$$

for all appropriate expressions F and G .

Now suppose we are given a group action $G \times M \rightarrow M$ and that we have found a difference frame for this action. Any group-invariant Lagrangian $L(n, \mathbf{u}_0, \dots, \mathbf{u}_j)$ can be written, in terms of the generating invariants κ and their shifts $\kappa_j = S_j \kappa$, as $L(n, \kappa_0, \dots, \kappa_{J_1})$ for some J_1 ; we adopt this notation from here on. For consistency, we drop the argument from the associated functional, setting

$$\mathcal{L} = \sum L(n, \mathbf{u}_0, \dots, \mathbf{u}_j) = \sum L(n, \kappa_0, \dots, \kappa_{J_1}).$$

THEOREM 5.2 (Invariant Euler–Lagrange equations). Let \mathcal{L} be a Lagrangian functional whose invariant Lagrangian is given in terms of the generating invariants as

$$\mathcal{L} = \sum L(n, \kappa_0, \dots, \kappa_{J_1}),$$

and suppose that the differential–difference syzygies are

$$\frac{d\kappa}{dt} = \mathcal{H} \sigma.$$

Then (with \cdot denoting the sum over all components)

$$E_{\mathbf{u}}(\mathcal{L}) \cdot \mathbf{u}'_0 = (\mathcal{H}^* E_{\kappa}(\mathcal{L})) \cdot \sigma, \tag{5.1}$$

where E_κ is the difference Euler operator with respect to κ . Consequently, the invariantization of the original Euler–Lagrange equations is

$$\iota_0(\overline{E_{\mathbf{u}}(L)}) = \mathcal{H} * E_\kappa(L). \quad (5.2)$$

Proof. In order to effect the variation, set $\mathbf{u} = \mathbf{u}(t)$ and compare

$$\frac{d}{dt} \mathcal{L} = \sum \{E_{\mathbf{u}}(L) \cdot \mathbf{u}'_0 + (S - \text{id})(A_{\mathbf{u}})\} \quad (5.3)$$

with the same calculation in terms of the invariants. This gives $d\mathcal{L}/dt = \sum dL/dt$, where

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial \kappa_j^\alpha} \frac{d\kappa_j^\alpha}{dt} \\ &= \frac{\partial L}{\partial \kappa_j^\alpha} S_j \frac{d\kappa^\alpha}{dt} \\ &= \left(S_{-j} \frac{\partial L}{\partial \kappa_j^\alpha} \right) \frac{d\kappa^\alpha}{dt} + (S - \text{id})(A_\kappa) \\ &= E_\kappa(L) \cdot \frac{d\kappa}{dt} + (S - \text{id})(A_\kappa) \\ &= E_\kappa(L) \cdot \mathcal{H} \sigma + (S - \text{id})(A_\kappa) \\ &= (\mathcal{H} * E_\kappa(L)) \cdot \sigma + (S - \text{id})\{A_\kappa + A_{\mathcal{H}}\}. \end{aligned} \quad (5.4)$$

The divergence terms arising from the first and second summations by parts are $(S - \text{id})A_\kappa$ and $(S - \text{id})A_{\mathcal{H}}$, respectively. (Note that A_κ is linear in the $d\kappa^\alpha/dt$ and their shifts, while $A_{\mathcal{H}}$ is linear in the σ^α and their shifts.) By the fundamental lemma of the calculus of variations, the identity (5.1) holds. To derive (5.2), apply ι_0 to (5.1) and compare components of σ . \square

Consequently, the original Euler–Lagrange equations, in invariant form, are equivalent to

$$\mathcal{H} * E_\kappa(L) = 0.$$

EXAMPLE 5.3 (Example 3.1 cont.) The invariant Lagrangian in our example is of the form

$$\mathcal{L} = \sum L(\eta, \kappa, S\kappa).$$

Using $\eta_j = S_j \eta$ and $\kappa_j = S_j \kappa$ henceforth, we write Equation (4.27) as

$$\begin{aligned} \frac{d\eta}{dt} &= \mathcal{H}_{11} \sigma^x + \mathcal{H}_{12} \sigma^u, \\ \frac{d\kappa}{dt} &= \mathcal{H}_{22} \sigma^u, \end{aligned} \quad (5.5)$$

with

$$\begin{aligned}\mathcal{H}_{11} &= (\kappa - 1)^3 \mathbf{S} - \text{id}, \\ \mathcal{H}_{12} &= 3\eta \{\text{id} - (\kappa - 1) \mathbf{S}\}, \\ \mathcal{H}_{22} &= (\kappa - 1) \{\text{id} - \kappa \mathbf{S} + (\kappa_1 - 1) \mathbf{S}_2\}.\end{aligned}$$

By Theorem 5.2, the invariantized Euler–Lagrange equations are

$$\mathcal{H}_{11}^* \mathbf{E}_\eta(L) = 0, \quad \mathcal{H}_{12}^* \mathbf{E}_\eta(L) + \mathcal{H}_{22}^* \mathbf{E}_\kappa(L) = 0,$$

where

$$\begin{aligned}\mathcal{H}_{11}^* &= (\kappa_{-1} - 1)^3 \mathbf{S}_{-1} - \text{id}, \\ \mathcal{H}_{12}^* &= 3\eta \text{id} - 3\eta_{-1}(\kappa_{-1} - 1) \mathbf{S}_{-1}, \\ \mathcal{H}_{22}^* &= (\kappa - 1) \text{id} - \kappa_{-1}(\kappa_{-1} - 1) \mathbf{S}_{-1} + (\kappa_{-2} - 1)(\kappa_{-1} - 1) \mathbf{S}_{-2}.\end{aligned}$$

The particular Lagrangian (3.8) amounts to $L = \eta(\kappa - 1)^{-3/2}$, so

$$\mathbf{E}_\eta = (\kappa - 1)^{-3/2}, \quad \mathbf{E}_\kappa = -\frac{3}{2} \eta(\kappa - 1)^{-5/2}.$$

Consequently, the invariantized Euler–Lagrange equations are

$$(\kappa_{-1} - 1)^{3/2} - (\kappa - 1)^{-3/2} = 0, \quad (5.6)$$

$$\frac{3}{2} \left\{ \eta(\kappa - 1)^{-3/2} - \eta_{-1}(\kappa_{-1} - 1)^{-1/2} + \eta_{-1}(\kappa_{-1} - 1)^{-3/2} - \eta_{-2}(\kappa_{-1} - 1)(\kappa_{-2} - 1)^{-3/2} \right\} = 0. \quad (5.7)$$

Assuming that L is real-valued ($\kappa > 1$), the general solution of (5.6) is

$$\kappa = 1 + k_1^{2(-1)^n} = 1 + \frac{1}{4} \left[k_1 + k_1^{-1} + (k_1 - k_1^{-1})(-1)^n \right]^2, \quad (5.8)$$

where k_1 is an arbitrary nonzero constant. Therefore (5.7) simplifies to

$$k_1^{3(-1)^{n+1}} \eta + \left(k_1^{3(-1)^n} - k_1^{(-1)^n} \right) \eta_{-1} - k_1^{5(-1)^{n+1}} \eta_{-2} = 0,$$

whose general solution is

$$\eta = k_1^{3(-1)^n} \left\{ k_2 \left((n+1)k_1^{(-1)^{n+1}} - nk_1^{(-1)^n} \right) + k_3(-1)^n \right\}, \quad (5.9)$$

where k_2 and k_3 are arbitrary constants.

6. On infinitesimals and the adjoint action

To state our results concerning the conservation laws, it is necessary to use the infinitesimal generators of a Lie group action on a manifold, together with the adjoint representation of the Lie group.

DEFINITION 6.1 Let $G \times U \rightarrow U$ be a smooth local Lie group action. If $\gamma(t)$ is a path in G with $\gamma(0) = e$, the identity element in G , then

$$\mathbf{v} = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) \cdot \mathbf{u} \quad (6.1)$$

is called the infinitesimal generator of the group action at $\mathbf{u} \in U$, in the direction $\gamma'(0) \in T_e G$, where $T_e G$ is the tangent space to G at e . In coordinates, the components of the infinitesimal generator are $\phi^\alpha = \mathbf{v}(u^\alpha)$, so

$$\mathbf{v} = \phi^\alpha \frac{\partial}{\partial u^\alpha}.$$

The infinitesimal generator is extended to the prolongation space $\mathcal{M} = P_n^{(J_0, J)}(U)$ by the prolongation formula

$$\mathbf{v}(u_j^\alpha) = \left. \frac{d}{dt} \right|_{t=0} \gamma(t) \cdot u_j^\alpha = \phi_j^\alpha = S_j \phi_0^\alpha, \quad J_0 \leq j \leq J,$$

(see [Hydon, 2014](#)). In coordinates, the prolonged infinitesimal generator is

$$\mathbf{v} = \phi_j^\alpha \frac{\partial}{\partial u_j^\alpha}.$$

LEMMA 6.1 If a Lagrangian $\mathbf{L}[\mathbf{u}]$ is invariant under the group action $G \times \mathcal{M} \rightarrow \mathcal{M}$, the components of the infinitesimal generator of the group action given by Definition 6.1 form the characteristic of a variational symmetry of $\mathbf{L}[\mathbf{u}]$, as defined in Definition 3.2.

Proof. The Lagrangian \mathbf{L} is invariant, so

$$\mathbf{L}(\mathbf{u}_0, \mathbf{u}_1, \dots, \mathbf{u}_J) = \mathbf{L}(g \cdot \mathbf{u}_0, g \cdot \mathbf{u}_1, \dots, g \cdot \mathbf{u}_J)$$

for all g . Thus,

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mathbf{L}(\gamma(t) \cdot \mathbf{u}_0, \gamma(t) \cdot \mathbf{u}_1, \dots) = \mathbf{v}(\mathbf{L}) = \phi_j^\alpha \frac{\partial \mathbf{L}}{\partial u_j^\alpha}.$$

By Definition 3.2, the components ϕ^α of the infinitesimal generator are the components of the characteristic of a variational symmetry of \mathbf{L} . \square

Each infinitesimal generator is determined by $\gamma'(0) \in T_e G$; the remainder of the path in G is immaterial. However, $T_e G$ is isomorphic to the Lie algebra \mathfrak{g} , which is the set of right-invariant vector fields on G . Right-invariance yields a Lie algebra homomorphism from \mathfrak{g} to the set \mathcal{X} of infinitesimal generators of symmetries (see [Olver, 1993](#) for details). If the group action is faithful, this is an isomorphism.

The R -dimensional Lie group G can be parametrized by $\mathbf{a} = (a^1, \dots, a^R)$ in a neighbourhood of the identity, e , so that the general group element is $\Gamma(\mathbf{a})$, where $\Gamma(\mathbf{0}) = e$. Given local coordinates

$\mathbf{u} = (u^1, \dots, u^q)$ on U , let $\hat{\mathbf{u}} = \Gamma(\mathbf{a}) \cdot \mathbf{u}$. By varying each independent parameter a^r in turn, the process above yields R infinitesimal generators,

$$\mathbf{v}_r = \xi_r^\alpha(\mathbf{u}) \partial_{u^\alpha}, \quad \text{where} \quad \xi_r^\alpha = \left. \frac{\partial \hat{u}^\alpha}{\partial a^r} \right|_{\mathbf{a}=\mathbf{0}}. \quad (6.2)$$

These form a basis for \mathcal{X} .

As \mathcal{X} is homomorphic to \mathfrak{g} , the adjoint representation of G on \mathfrak{g} gives rise to the adjoint representation of G on \mathcal{X} . Given $g \in G$, the adjoint representation Ad_g is the tangent map on \mathfrak{g} induced by the conjugation $h \mapsto ghg^{-1}$. The corresponding adjoint representation on \mathcal{X} is expressed by a matrix, $Ad(g)$, which is most conveniently obtained as follows³. Having calculated a basis for \mathcal{X} ,

$$\mathbf{v}_r = \xi_r^\alpha(\mathbf{u}) \partial_{u^\alpha}, \quad r = 1, \dots, R,$$

let $\tilde{\mathbf{u}} = g \cdot \mathbf{u}$ and define

$$\tilde{\mathbf{v}}_r = \xi_r^\alpha(\tilde{\mathbf{u}}) \partial_{\tilde{u}^\alpha}, \quad r = 1, \dots, R.$$

Now express each \mathbf{v}_r in terms of $\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_R$ and determine $Ad(g)$ from the identity

$$(\mathbf{v}_1 \cdots \mathbf{v}_R) = (\tilde{\mathbf{v}}_1 \cdots \tilde{\mathbf{v}}_R) Ad(g). \quad (6.3)$$

EXAMPLE 6.2 (Example 3.1 cont.) For our running example, the group parameters are λ, a and b with identity $(\lambda, a, b) = (1, 0, 0)$. Choosing $a^1 = \ln(\lambda), a^2 = a$ and $a^3 = b$, so that the identity corresponds to $\mathbf{a} = \mathbf{0}$, one obtains the following basis for \mathcal{X} :

$$\mathbf{v}_1 = 3x\partial_x + u\partial_u, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = \partial_u.$$

Recall that the action of a fixed group element g , parametrized by (λ, a, b) , gives

$$(\tilde{x}, \tilde{u}) = (\lambda^3 x + a, \lambda u + b).$$

Therefore, by the standard change-of-variables formula,

$$\mathbf{v}_1 = 3(\tilde{x} - a)\partial_{\tilde{x}} + (\tilde{u} - b)\partial_{\tilde{u}}, \quad \mathbf{v}_2 = \lambda^3 \partial_{\tilde{x}}, \quad \mathbf{v}_3 = \lambda \partial_{\tilde{u}}.$$

Consequently,

$$(\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3) = (\tilde{\mathbf{v}}_1 \ \tilde{\mathbf{v}}_2 \ \tilde{\mathbf{v}}_3) Ad(g), \quad \text{where} \quad Ad(g) = \begin{pmatrix} 1 & 0 & 0 \\ -3a & \lambda^3 & 0 \\ -b & 0 & \lambda \end{pmatrix}.$$

³ See the appendix for an alternative construction using Lie algebra structure constants.

Regarding the infinitesimal generators as differential operators and applying the identity (6.3) to each \tilde{u}^α in turn, one obtains

$$(\mathbf{v}_1(\tilde{u}^\alpha) \cdots \mathbf{v}_R(\tilde{u}^\alpha)) = (\xi_1^\alpha(\tilde{\mathbf{u}}) \cdots \xi_R^\alpha(\tilde{\mathbf{u}})) \mathcal{A}d(g). \quad (6.4)$$

This yields a useful matrix identity. Define the matrix of characteristics to be the $q \times R$ matrix

$$\Phi(\mathbf{u}) = (\xi_r^\alpha(\mathbf{u})). \quad (6.5)$$

Then, by the chain rule, (6.4) amounts to

$$\left(\frac{\partial \tilde{\mathbf{u}}}{\partial \mathbf{u}} \right) \Phi(\mathbf{u}) = \Phi(\tilde{\mathbf{u}}) \mathcal{A}d(g), \quad (6.6)$$

where $(\partial \tilde{\mathbf{u}} / \partial \mathbf{u})$ is the Jacobian matrix. This identity is extended to prolongation spaces with coordinates $z = (\mathbf{u}_{J_0}, \dots, \mathbf{u}_J)$, where $J_0 \leq 0$ and $J \geq 0$, as follows. Define the matrix of prolonged infinitesimals to be

$$\Phi(z) = \begin{pmatrix} \Phi(\mathbf{u}_{J_0}) \\ \vdots \\ \Phi(\mathbf{u}_J) \end{pmatrix}.$$

The infinitesimal generators \mathbf{v}_r , prolonged to all variables in z , satisfy (6.3), where the tilde now denotes replacement of z by $g \cdot z$. Applying this identity to $g \cdot z$ gives

$$\left(\frac{\partial (g \cdot z)}{\partial z} \right) \Phi(z) = \Phi(g \cdot z) \mathcal{A}d(g). \quad (6.7)$$

7. Conservation laws

In general, the conservation laws are not invariant. However, as we will show, they are equivariant; indeed, they can be written in terms of invariants and the frame. Our key result is that for difference frames, the R conservation laws can be written in the form

$$(S - \text{id})\{V(I) \mathcal{A}d(\rho_0)\} = 0,$$

where $\mathcal{A}d(\rho_0)$ is the adjoint representation of ρ_0 and $V(I) = (V_1 \cdots V_R)$ is a row vector of invariants.

In the standard (i.e., not invariantized) calculation of the Euler–Lagrange equations and boundary terms, suppose that the dummy variable t effecting the variation is a group parameter for G , under which the Lagrangian is invariant. Then the resulting boundary terms yield conservation laws; this is the difference version of Noether's theorem. So it is useful to identify t with a group parameter by considering the following path in G :

$$t \mapsto \gamma_r(t) = \Gamma \left(a^1(t), \dots, a^R(t) \right), \quad \text{where } a^r(t) = t \quad \text{and} \quad a^l(t) = 0, \quad l \neq r; \quad (7.1)$$

recall from §6 that $\mathbf{a} \mapsto \Gamma(\mathbf{a})$ expresses the general group element in terms of the coordinates \mathbf{a} . On this path, each $(\mathbf{u}_0)'$ at $t = 0$ is an infinitesimal generator, from (6.2).

For the invariantized calculation, we follow essentially the same route to our result, identifying the dummy variable effecting the variation with each group parameter in turn. The proof of Theorem 5.2 uses the identity

$$\frac{d}{dt} L(n, \kappa, \dots, S_{J_1}(\kappa)) = (\mathcal{H}^* E_\kappa(L)) \cdot \sigma + (S - \text{id})\{A_\kappa + A_{\mathcal{H}}\}. \quad (7.2)$$

Recall that A_κ is linear in $d\kappa^\alpha/dt$ and their shifts, while $A_{\mathcal{H}}$ is linear in the σ^α and their shifts. As t is a group parameter and each κ^α is invariant, $d\kappa^\alpha/dt = 0$. Thus, (7.2) reduces to

$$(\mathcal{H}^* E_\kappa(L)) \cdot \sigma + (S - \text{id})A_{\mathcal{H}} = 0, \quad (7.3)$$

so $(S - \text{id})A_{\mathcal{H}} = 0$ on all solutions of the invariantized Euler–Lagrange equations $\mathcal{H}^* E_\kappa(L) = 0$. We now derive the conservation laws from this condition.

THEOREM 7.1 Suppose that the conditions of Theorem 5.2 hold. Write

$$A_{\mathcal{H}} = C_\alpha^j S_j(\sigma^\alpha),$$

where each C_α^j depends only on n, κ and its shifts. Let $\Phi^\alpha(\mathbf{u}_0)$ be the row of the matrix of characteristics corresponding to the dependent variable u_0^α and denote its invariantization by $\Phi_0^\alpha(I) = \Phi^\alpha(\rho_0 \cdot \mathbf{u}_0)$. Then the R conservation laws in row vector form amount to

$$C_\alpha^j S_j\{\Phi_0^\alpha(I) Ad(\rho_0)\} = 0. \quad (7.4)$$

That is, to obtain the conservation laws, it is sufficient to make the replacement

$$\sigma^\alpha \mapsto \{\Phi^\alpha(g \cdot \mathbf{u}_0) Ad(g)\} \Big|_{g=\rho_0} \quad (7.5)$$

in $A_{\mathcal{H}}$.

Proof. Recall that

$$\sigma^\alpha = \rho_0 \cdot (u_0^\alpha)' = \left(\frac{d}{dt} g \cdot u_0^\alpha \right) \Big|_{g=\rho_0}. \quad (7.6)$$

To obtain the conservation laws, conflate t with the group parameter a^r , making the replacement

$$\rho_0 \cdot (u_0^\alpha)' \mapsto \frac{d}{dt} \Big|_{t=0} \rho_0 \cdot \gamma_r(t) \cdot u_0^\alpha \quad (7.7)$$

in $A_{\mathcal{H}}$, where $\gamma_r(t)$ is the path defined in (7.1). For any $g \in G$,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} (g \cdot \gamma_r(t) \cdot u_0^\alpha) &= \left(\frac{\partial (g \cdot \gamma_r(t) \cdot u_0^\alpha)}{\partial (\gamma_r(t) \cdot u_j^\beta)} \right) \Big|_{t=0} \left(\left. \frac{d}{dt} \right|_{t=0} \gamma_r(t) \cdot u_j^\beta \right) \\ &= \frac{\partial (g \cdot u_0^\alpha)}{\partial u_j^\beta} \left(\left. \frac{d}{dt} \right|_{t=0} \gamma_r(t) \cdot u_j^\beta \right). \end{aligned} \quad (7.8)$$

In matrix form, (7.8) amounts to the following (taking (6.7) into account):

$$\left. \frac{d}{dt} \right|_{t=0} (g \cdot \gamma_r(t) \cdot u_0^\alpha) = \left(\frac{\partial (g \cdot z)}{\partial z} \Phi(z) \right)_{(u_0^\alpha, r)} = (\Phi(g \cdot z) \mathcal{A}d(g))_{(u_0^\alpha, r)},$$

where (u_0^α, r) denotes the entry in the row corresponding to u_0^α and the r th column. Setting $g = \rho_0$, the required replacement is

$$\sigma^\alpha \mapsto (\Phi(\rho_0 \cdot z) \mathcal{A}d(\rho_0))_{(u_0^\alpha, r)} = (\Phi(\rho_0 \cdot \mathbf{u}_0) \mathcal{A}d(\rho_0))_r^\alpha.$$

By using each parameter a^r in turn, σ^α is replaced by a row vector,

$$\sigma^\alpha \mapsto \Phi_0^\alpha(I) \mathcal{A}d(\rho_0),$$

as required. □

By the prolongation formula $S_j(\rho_0) = \rho_j$, the conservation laws amount to

$$(S - \text{id}) \left(C_j^\alpha (S_j \Phi_0^\alpha(I)) \mathcal{A}d(\rho_j) \right) = 0, \quad (7.9)$$

As $\mathcal{A}d(\rho_j) \mathcal{A}d(\rho_0)^{-1} = \mathcal{A}d(\rho_j \rho_0^{-1})$ is invariant, this leads to the following corollary.

COROLLARY 7.1 The conservation laws for a difference frame may be written in the form

$$(S - \text{id}) \{V(I) \mathcal{A}d(\rho_0)\} = 0, \quad (7.10)$$

where $V(I) = (V_1 \cdots V_R)$ is an invariant row vector. Specifically,

$$V(I) = C_\alpha^j (S_j \Phi_0^\alpha(I)) \mathcal{A}d(\rho_j \rho_0^{-1}). \quad (7.11)$$

COROLLARY 7.2 On any solution of the invariantized Euler–Lagrange equations,

$$V(I) \mathcal{A}d(\rho_0) = c, \quad (7.12)$$

for some constant row vector $c = (c_1 \cdots c_R)$.

As the conservation laws depend only on the terms arising from $A_{\mathcal{H}}$, the laws can be calculated for all Lagrangians in the relevant invariance class, in terms of the $E_{\kappa}(L)$, independently of the precise form that $L = L(n, \kappa, \dots, S_{j_1} \kappa)$ takes.

EXAMPLE 7.2 (Example 3.1 cont.) Redoing an earlier calculation, but keeping track of the terms in $A_{\mathcal{H}}$,

$$\begin{aligned} \frac{d}{dt} L(\eta, \kappa, S\kappa) &= \iota_0 \{E_x(L)\} \sigma^x + \iota_0 \{E_u(L)\} \sigma^u + (S - \text{id})A_{\kappa} + (S - \text{id}) \left(S_{-1} \{(\kappa - 1)^3 E_{\eta}(L)\} \sigma^x \right) \\ &+ (S - \text{id}) \left(-S_{-1} \{3\eta(\kappa - 1)E_{\eta}(L) + \kappa(\kappa - 1)E_{\kappa}(L)\} \sigma^u \right) \\ &+ (S_2 - \text{id}) \left(S_{-2} \{(\kappa - 1)(\kappa_1 - 1)E_{\kappa}(L)\} \sigma^u \right), \end{aligned}$$

where

$$A_{\kappa} = \frac{d\kappa}{dt} S_{-1} \left(\frac{\partial L}{\partial S\kappa} \right).$$

Hence, the terms that contribute to the conservation laws come from

$$A_{\mathcal{H}} = \mathcal{C}_x^0 \sigma^x + \mathcal{C}_u^0 \sigma^u + \mathcal{C}_u^1 S(\sigma^u), \quad (7.13)$$

where

$$\mathcal{C}_x^0 = S_{-1} \{(\kappa - 1)^3 E_{\eta}(L)\},$$

$$\mathcal{C}_u^0 = -S_{-1} \{3\eta(\kappa - 1)E_{\eta}(L) + \kappa(\kappa - 1)E_{\kappa}(L)\} + S_{-2} \{(\kappa - 1)(\kappa_1 - 1)E_{\kappa}(L)\},$$

$$\mathcal{C}_u^1 = S_{-1} \{(\kappa - 1)(\kappa_1 - 1)E_{\kappa}(L)\}.$$

For this running example,

$$Ad(\rho_0) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3x_0}{(u_1 - u_0)^3} & \frac{1}{(u_1 - u_0)^3} & 0 \\ \frac{u_0}{u_1 - u_0} & 0 & \frac{1}{u_1 - u_0} \end{pmatrix}$$

and the invariantized form of the matrix of characteristics restricted to the variables x_0 and u_0 is

$$\Phi_0(I) = \begin{pmatrix} \Phi_0^x \\ \Phi_0^u \end{pmatrix} = \iota_0 \begin{pmatrix} x_0 & 1 & 0 \\ u_0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, by (7.4), the conservation laws are of the form $(S - \text{id})A = 0$, where

$$\begin{aligned} A &= C_x^0(0 \ 1 \ 0) \mathcal{A}d(\rho_0) + C_u^0(0 \ 0 \ 1) \mathcal{A}d(\rho_0) + C_u^1 S\{(0 \ 0 \ 1) \mathcal{A}d(\rho_0)\} \\ &= \left[C_x^0(0 \ 1 \ 0) + C_u^0(0 \ 0 \ 1) + C_u^1(0 \ 0 \ 1) \mathcal{A}d(\rho_1 \rho_0^{-1}) \right] \mathcal{A}d(\rho_0); \end{aligned}$$

the last equality writes the conservation laws in the form of Equation (7.10). It remains only to write the matrix $\mathcal{A}d(\rho_1 \rho_0^{-1})$ in terms of η and κ :

$$\mathcal{A}d(\rho_1 \rho_0^{-1}) = \mathcal{A}d(K_0) = \iota_0(\mathcal{A}d(\rho_1)) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3\eta}{(\kappa-1)^3} & \frac{1}{(\kappa-1)^3} & 0 \\ \frac{1}{\kappa-1} & 0 & \frac{1}{\kappa-1} \end{pmatrix}.$$

More generally, once one has solved for the frame, each $\mathcal{A}d(\rho_j \rho_0^{-1})$ can be written in terms of κ and its shifts by using invariantization followed by the recurrence formula. Doing the calculation, we find that $A = V(I) \mathcal{A}d(\rho_0)$, where

$$V(I) = \begin{pmatrix} S_{-1}\{(\kappa-1)E_\kappa(L)\} \\ S_{-1}\{(\kappa-1)^3 E_\eta(L)\} \\ -S_{-1}\{3\eta(\kappa-1)E_\eta(L) + (\kappa-1)^2 E_\kappa(L)\} + S_{-2}\{(\kappa-1)(\kappa_1-1)E_\kappa(L)\} \end{pmatrix}^T.$$

For the particular Lagrangian (3.8), the solutions (5.8), (5.9) of the invariantized Euler–Lagrange equations yield

$$\begin{aligned} V_1 &= -\frac{3}{2} \eta_{-1} (\kappa_{-1} - 1)^{-3/2} = -\frac{3}{4} k_2 \left[k_1 + k_1^{-1} + (k_1 - k_1^{-1})(2n-1)(-1)^n \right] + \frac{3}{2} k_3 (-1)^n, \\ V_2 &= (\kappa_{-1} - 1)^{3/2} = k_1^{3(-1)^{n+1}}, \\ V_3 &= -\frac{3}{2} \left[\eta(\kappa-1)^{-3/2} + \eta_{-1} (\kappa_{-1} - 1)^{-3/2} \right] = -3k_2 k_1^{(-1)^{n+1}}. \end{aligned} \tag{7.14}$$

In the coordinates we have used, the first element of $(S - \text{id})A = 0$ is the conservation law due to the scaling invariance, the second is due to invariance under translation of x and the third is due to translation of u .

REMARK 7.1 Calculation of the conservation laws There is another way to calculate the laws for difference frames. By Corollary 7.1, one can use symbolic software to calculate the conservation laws in the original variables, and then use the replacement rule, Theorem 4.5, to obtain the invariantized first integrals $V(I)$ (see Table 2, which is obtained from Table 1 by applying ι_0 , taking the normalization equations into account). This is because the replacement rule sends ρ_0 to the identity matrix. The recurrence formulae can then be used to write $V(I)$ in terms of the generating invariants. This means that the methods to solve for the extremals in the original variables, given in the next section, can still be used without having to perform the more complex, invariantized summation by parts computation.

TABLE 2 *Infinitesimals and invariantized first integrals for the Lagrangian (3.8)*

ϕ^x	ϕ^u	Invariantized first integral: $V(I) = \iota_0\{A_{\mathbf{u}}(n, \boldsymbol{\phi})\}$
3x	u	$\iota_0\left(S_{-1} \frac{\partial L}{\partial u_2}\right)$
1	0	$\iota_0\left(S_{-1} \frac{\partial L}{\partial x_1}\right)$
0	1	$\iota_0\left(S_{-1} \frac{\partial L}{\partial u_2} - \frac{\partial L}{\partial u_0}\right)$

8. Solving for the original dependent variables \mathbf{u}_0 , once the generating invariants are known

In this section we show how to find the solutions \mathbf{u}_0 to the original Euler–Lagrange equations, once the invariant Euler–Lagrange equations have been solved for the generating invariants κ^α . The starting-point is that κ is a known function of n and some arbitrary constants (which are determined if initial data are specified). There are three methods, depending on the information available. We use the running expository example to illustrate each method.

8.1 How to solve for \mathbf{u}_0 from the invariants, knowing only the Maurer–Cartan matrix

This method can be used for any invariant difference system. Indeed, when the adjoint representation of the Lie group is trivial, it is the only available method.

Assume that the Maurer–Cartan matrix $K_0 = \rho_1 \rho_0^{-1}$ is known in terms of the generating invariants, so that it can be written in terms of n (and some arbitrary constants). This yields a system of recurrence relations for ρ_0 , namely

$$\rho_1 = K_0 \rho_0. \quad (8.1)$$

DEFINITION 8.1 The system (8.1) is known as the set of *Maurer–Cartan equations* for the frame ρ .

Once the Maurer–Cartan equations for ρ_0 have been solved, one can obtain \mathbf{u}_0 from

$$u_0^\alpha = \rho_0^{-1} (\rho_0 \cdot u_0^\alpha) = \rho_0^{-1} I_{0,0}^\alpha; \quad (8.2)$$

the invariant $I_{0,0}^\alpha$ is known, either from the normalization equations or from the set of generating invariants already determined.

EXAMPLE 8.2 (Example 3.1 cont.) From Equation (4.15), the Maurer–Cartan matrix is

$$K_0 = \begin{pmatrix} (\kappa - 1)^{-3} & 0 & -\eta(\kappa - 1)^{-3} \\ 0 & (\kappa - 1)^{-1} & -(\kappa - 1)^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, setting λ_k , a_k and b_k to be the parameter values for the group element ρ_k , the set of Maurer–Cartan equations is

$$\begin{pmatrix} \lambda_1^3 & 0 & a_1 \\ 0 & \lambda_1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} = K_0 \begin{pmatrix} \lambda_0^3 & 0 & a_0 \\ 0 & \lambda_0 & b_0 \\ 0 & 0 & 1 \end{pmatrix},$$

which amount to three recurrence relations for the group parameters:

$$\begin{aligned} \lambda_1 &= (\kappa - 1)^{-1} \lambda_0, \\ a_1 &= (\kappa - 1)^{-3} (a_0 - \eta), \\ b_1 &= (\kappa - 1)^{-1} (b_0 - 1). \end{aligned} \tag{8.3}$$

Now suppose that we know the general solution of these recurrence relations. The normalization equations give $\rho_0 \cdot x_0 = 0$ and $\rho_0 \cdot u_0 = 0$, so

$$\begin{aligned} x_0 &= \rho_0^{-1} \cdot (\rho_0 \cdot x_0) = \lambda_0^{-3} (\rho_0 \cdot x_0 - a_0) = -\lambda_0^{-3} a_0, \\ u_0 &= \rho_0^{-1} \cdot (\rho_0 \cdot u_0) = \lambda_0^{-1} (\rho_0 \cdot u_0 - b_0) = -\lambda_0^{-1} b_0. \end{aligned}$$

8.2 Solving for \mathbf{u}_0 from the invariants and conservation laws when the adjoint representation is nontrivial

This method works when the adjoint representation is not the identity representation. The conservation laws give

$$V(I) \mathcal{A}d(\rho_0) = c, \tag{8.4}$$

where c is a constant row vector. The components V_i depend only on κ , and are therefore known functions of n . As $\mathcal{A}d(g)$ is known in terms of the group parameters, Equation (8.4) yields equations for these parameters.

If the adjoint action of the group on its Lie algebra is not transitive, the algebraic system of equations for the parameters may be under-determined. To complete the solution, it is then necessary to supplement this system with the Maurer–Cartan equations (8.1). Even so, the algebraic equations coming from the conservation laws can ease, considerably, the problem of solving the Maurer–Cartan equations alone. Once ρ_0 is known as a function of n , Equation (8.2) yields \mathbf{u}_0 , as before.

EXAMPLE 8.3 (Example 3.1 cont.) For the running example, (8.4) is

$$(V_1 \ V_2 \ V_3) \begin{pmatrix} 1 & 0 & 0 \\ -3a_0 & \lambda_0^3 & 0 \\ -b_0 & 0 & \lambda_0 \end{pmatrix} = (c_1 \ c_2 \ c_3).$$

We obtain immediately $\lambda_0 = c_3/V_3$ and hence, a first integral of the Euler–Lagrange equations:

$$\frac{V_2}{(V_3)^3} = \frac{c_2}{(c_3)^3}. \tag{8.5}$$

The remaining equation is a linear expression for a_0 and b_0 ,

$$3a_0V_2 + b_0V_3 - V_1 + c_1 = 0. \quad (8.6)$$

If one of the second and third equations of (8.3) can be solved, (8.6) yields the remaining parameter.

8.3 Solving for \mathbf{u}_0 from κ from the conservation laws, and with a nontrivial adjoint representation of ρ which is known as a function of \mathbf{u}_0

In this case we consider the conservation laws $V(I)Ad(\rho_0) = c$, taking into account that $\rho_0(\mathbf{u})$ is known as a function of the dependent variables. One can sometimes derive explicit equations for \mathbf{u} that are simple to solve. We illustrate the possibilities in the running example.

EXAMPLE 8.4 (Example 3.1 cont.) The conservation laws amount to

$$(V_1 \ V_2 \ V_3) \begin{pmatrix} 1 & 0 & 0 \\ \frac{3x_0}{(u_1 - u_0)^3} & \frac{1}{(u_1 - u_0)^3} & 0 \\ \frac{u_0}{u_1 - u_0} & 0 & \frac{1}{u_1 - u_0} \end{pmatrix} = (c_1 \ c_2 \ c_3). \quad (8.7)$$

We obtain once more the first integral (8.5) and the simple recurrence relation

$$u_1 - u_0 = V_3/c_3. \quad (8.8)$$

Once this is solved for u_0 , one can obtain x_0 from the first column of (8.7).

For the Lagrangian (3.8), each V_r is given in (7.14) in terms of n and k_i , $i = 1, 2, 3$. The first integral (8.5) yields $c_3 = -3k_2c_2^{1/3}$. Assuming that k_2 is nonzero, it is convenient to define $k_4 = c_2^{-1/3}$; then the general solution of (8.8) is

$$u_0 = \frac{1}{4}k_4 \left[2(k_1 + k_1^{-1})n + (k_1 - k_1^{-1})(-1)^n + k_5 \right],$$

where k_5 is an arbitrary constant. Finally, the first column of (8.7) gives

$$x_0 = k_4^3 \left[k_2nk_1^{(-1)^n} - \frac{1}{2}k_3(-1)^n + k_6 \right],$$

where $k_6 = c_1/3 + k_2(k_1 + k_1^{-1} + k_5)/4$ is the remaining arbitrary constant.

9. Lagrangians invariant up to a divergence

So far, we have considered only Lagrangians L that are invariant under a Lie group G of variational symmetries, that is, $g \cdot L = L$ for all $g \in G$. However, Noether's theorem merely requires the action \mathcal{L} to be invariant. This broader definition of a Lie group of variational symmetries requires that for each

$g \in G$, there exists a function P_g of n and a finite number of shifts of \mathbf{u}_0 such that

$$g \cdot \mathbf{L} = \mathbf{L} + (\mathbf{S} - \text{id})P_g.$$

Without loss of generality, we set $P_e = 0$. This useful generalization can be treated in the invariant framework by introducing a new dependent variable ζ such that

$$g \cdot \zeta_j = S_j(\zeta_0 - P_g), \quad j \in \mathbb{Z}. \quad (9.1)$$

LEMMA 9.1 With the above notation, the group action on $(\mathbf{S} - \text{id})\zeta$ defined by (9.1) is a left action. Furthermore, the modified Lagrangian

$$\bar{\mathbf{L}} = \mathbf{L} + (\mathbf{S} - \text{id})\zeta_0 \quad (9.2)$$

is invariant under all $g \in G$.

Proof. Let $g, h \in G$. By definition, taking the left action of G on \mathbf{u} into account,

$$(\mathbf{S} - \text{id})P_{hg} = (hg) \cdot \mathbf{L} - \mathbf{L} = h \cdot (\mathbf{L} + (\mathbf{S} - \text{id})P_g) - \mathbf{L} = (\mathbf{S} - \text{id})(P_h + h \cdot P_g),$$

where the last equality is a consequence of the prolongation formula. Therefore,

$$(hg) \cdot ((\mathbf{S} - \text{id})\zeta_0) = (\mathbf{S} - \text{id})(\zeta_0 - P_{hg}) = (\mathbf{S} - \text{id})(\zeta_0 - P_h - h \cdot P_g) = h \cdot (g \cdot ((\mathbf{S} - \text{id})\zeta_0)),$$

which extends to a left action on all $(\mathbf{S} - \text{id})\zeta_j$ by the prolongation formula. Invariance of the modified Lagrangian follows

$$g \cdot \bar{\mathbf{L}} = g \cdot \mathbf{L} + g \cdot (\zeta_1 - \zeta_0) = \mathbf{L} + (\mathbf{S} - \text{id})P_g + (\mathbf{S} - \text{id})(\zeta_0 - P_g) = \bar{\mathbf{L}}. \quad \square$$

As $\bar{\mathbf{L}}$ and \mathbf{L} differ by a divergence, they yield the same Euler–Lagrange equations. Therefore, $\bar{\mathbf{L}}$ can be used to obtain the conservation laws by the difference moving frame method. Only the original dependent variables can be used for normalization, because ζ does not appear in the Euler–Lagrange equations. Moreover, ζ does not appear in $A_{\mathcal{H}}$ for the modified Lagrangian, so the expression for the conservation laws in terms of invariants and $\mathcal{A}d(\rho_0)$ is unaffected by the modification.

An example appears in the next section, Section 10.

10. Explicit dependence on n changes nothing

So far, we have considered examples whose Lagrangian and variational symmetries do not depend explicitly on n . However, the difference frame reduction takes place on a prolongation space over a single (arbitrary) base point n . Consequently, n should be regarded in the calculations as a parameter.

Consider the Lagrangian

$$\mathbf{L} = \frac{(v_2 - v_0)(v_3 - v_1)}{(u_0 - v_0)(u_1 - v_1)} + 2 \ln(u_0 - v_0).$$

While this Lagrangian does not depend explicitly on n , there is a three-parameter Lie group of variational symmetries depending explicitly on n , $g \cdot (u, v) = (\tilde{u}, \tilde{v})$, where

$$\begin{aligned}\tilde{u} &= \exp \{(-1)^n a_1\} u + a_2 + a_3(-1)^n = u \cosh a_1 + a_2 + (-1)^n \{u \sinh a_1 + a_3\}, \\ \tilde{v} &= \exp \{(-1)^n a_1\} v + a_2 + a_3(-1)^n = v \cosh a_1 + a_2 + (-1)^n \{v \sinh a_1 + a_3\}.\end{aligned}\quad (10.1)$$

The infinitesimal generators \mathbf{v}_r corresponding to a_r are

$$\mathbf{v}_1 = (-1)^n u \partial_u + (-1)^n v \partial_v, \quad \mathbf{v}_2 = \partial_u + \partial_v, \quad \mathbf{v}_3 = (-1)^n \partial_u + (-1)^n \partial_v.$$

So

$$(\mathbf{v}_1 \cdots \mathbf{v}_R) = (\tilde{\mathbf{v}}_1 \cdots \tilde{\mathbf{v}}_R) \mathcal{A}d(g), \quad \text{where } \mathcal{A}d(g) = \begin{pmatrix} 1 & 0 & 0 \\ -a_3 & \cosh a_1 & \sinh a_1 \\ -a_2 & \sinh a_1 & \cosh a_1 \end{pmatrix}. \quad (10.2)$$

For this group action, the standard representation is not faithful, so we shall use the Adjoint representation, which is faithful.

In this example, \mathbf{L} is not invariant:

$$g \cdot \mathbf{L} = \mathbf{L} + 2a_1(-1)^n = \mathbf{L} + (\mathbf{S} - \text{id})\{a_1(-1)^{n+1}\}.$$

Accordingly, by the result in Section 9, we define a new dependent variable ζ and the (invariant) modified Lagrangian $\bar{\mathbf{L}}(u_0, v_0, \zeta_0, u_1, v_1, \zeta_1, v_2, v_3)$ such that

$$\tilde{\zeta}_0 = \zeta_0 + a_1(-1)^n, \quad \tilde{\zeta}_1 = \zeta_1 + a_1(-1)^{n+1}, \quad \bar{\mathbf{L}} = \mathbf{L} + (\mathbf{S} - \text{id})\zeta_0;$$

here $\tilde{\zeta}_j$ denotes $g \cdot \zeta_j$. The extension of the infinitesimal generators \mathbf{v}_r to include their action on ζ is as follows:

$$\mathbf{v}_1 = (-1)^n u \partial_u + (-1)^n v \partial_v + (-1)^n \partial_\zeta, \quad \mathbf{v}_2 = \partial_u + \partial_v, \quad \mathbf{v}_3 = (-1)^n \partial_u + (-1)^n \partial_v.$$

These generators satisfy the same commutation relations as their unextended counterparts, so (10.2) is unchanged.

We choose the moving frame ρ_0 defined by the normalization

$$\rho_0 \cdot u_0 = 1, \quad \rho_0 \cdot v_0 = 0, \quad \rho_0 \cdot v_1 = 0,$$

which amounts to

$$\begin{aligned}a_1 &= (-1)^{n+1} \ln(u_0 - v_0), \\ a_2 &= -\frac{1}{2} [v_1(u_0 - v_0) + v_0/(u_0 - v_0)], \\ a_3 &= \frac{1}{2}(-1)^n [v_1(u_0 - v_0) - v_0/(u_0 - v_0)].\end{aligned}\quad (10.3)$$

Consequently, the frame (in the adjoint representation) is

$$\rho_0 = \mathcal{A}d(\rho_0) = \begin{pmatrix} 1 & 0 & 0 \\ -a_3 & \cosh a_1 & \sinh a_1 \\ -a_2 & \sinh a_1 & \cosh a_1 \end{pmatrix},$$

where a_1, a_2 and a_3 are given by (10.3); hence,

$$\cosh a_1 = \frac{1}{2} \left[u_0 - v_0 + \frac{1}{u_0 - v_0} \right], \quad \sinh a_1 = \frac{(-1)^{n+1}}{2} \left[u_0 - v_0 - \frac{1}{u_0 - v_0} \right]. \quad (10.4)$$

The fundamental difference invariants are

$$\begin{aligned} \kappa &= \rho_0 \cdot u_1 = (u_0 - v_0)(u_1 - v_1), \\ \mu &= \rho_0 \cdot v_2 = (v_2 - v_0)/(u_0 - v_0), \\ \nu &= \rho_0 \cdot \zeta_0 = \zeta_0 - \ln(u_0 - v_0). \end{aligned}$$

Therefore, in terms of these invariants, \bar{L} amounts to

$$L = \mu\mu_1 + \ln \kappa + \nu_1 - \nu_0, \quad (10.5)$$

and

$$\mathcal{A}d(K_0) = \iota_0(\mathcal{A}d(\rho_1)) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{(-1)^n}{2} \kappa \mu & \frac{1}{2} (\kappa + \kappa^{-1}) & \frac{(-1)^n}{2} (\kappa - \kappa^{-1}) \\ \frac{1}{2} \kappa \mu & \frac{(-1)^n}{2} (\kappa - \kappa^{-1}) & \frac{1}{2} (\kappa + \kappa^{-1}) \end{pmatrix}. \quad (10.6)$$

We now construct the differential–difference syzygies. Noting that

$$\iota_0(u'_j) = (u_0 - v_0)^{(-1)^{j-1}} u'_j, \quad \iota_0(v'_j) = (u_0 - v_0)^{(-1)^{j-1}} v'_j, \quad \iota_0(\zeta'_j) = \zeta'_j,$$

the fundamental differential–difference invariants are

$$\sigma^u = \iota_0(u'_0) = \frac{u'_0}{u_0 - v_0}, \quad \sigma^v = \iota_0(v'_0) = \frac{v'_0}{u_0 - v_0}, \quad \sigma^\zeta = \zeta'_0. \quad (10.7)$$

Consequently,

$$\iota_0(u'_1) = \kappa S \sigma^u, \quad \iota_0(v'_1) = \kappa S \sigma^v, \quad \iota_0(v'_2) = \kappa^{-1} \kappa_1 S_2 \sigma^v, \quad \iota_0(\zeta'_1) = S \sigma^\zeta. \quad (10.8)$$

and thus the differential–difference syzygies needed to calculate the Euler–Lagrange equations are

$$\frac{d}{dt} \begin{pmatrix} \kappa \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & 0 \\ \mathcal{H}_{21} & \mathcal{H}_{22} & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma^u \\ \sigma^v \\ \sigma^\zeta \end{pmatrix}, \quad (10.9)$$

where

$$\mathcal{H}_{11} = \kappa(S + \text{id}), \quad \mathcal{H}_{12} = -\mathcal{H}_{11}, \quad \mathcal{H}_{21} = -\mu \text{id}, \quad \mathcal{H}_{22} = \kappa^{-1}\kappa_1 S_2 + (\mu - 1)\text{id}. \quad (10.10)$$

Therefore,

$$\begin{aligned} \frac{dL}{dt} &= \{\mathcal{H}_{11}^* E_\kappa(L) + \mathcal{H}_{21}^* E_\mu(L) - E_\nu(L)\} \sigma^u + \{\mathcal{H}_{12}^* E_\kappa(L) + \mathcal{H}_{22}^* E_\mu(L) + E_\nu(L)\} \sigma^v \\ &\quad + E_\nu(L) \sigma^\zeta + (S - \text{id})(A_\kappa + A_{\mathcal{H}}), \end{aligned} \quad (10.11)$$

where

$$A_{\mathcal{H}} = [S_{-1}(\kappa E_\kappa(L))] (\sigma^u - \sigma^v) + (S + \text{id}) \left\{ S_{-2} \left[\kappa^{-1} \kappa_1 E_\mu(L) \right] \sigma^v \right\}$$

and, for the particular Lagrangian (10.5), $A_\kappa = \mu_{-1}\mu' + \nu'$.

The invariantized Euler–Lagrange equations are obtained from the coefficients of σ^u , σ^v and σ^ζ in (10.11), ignoring terms in A_κ and $A_{\mathcal{H}}$. By construction, the coefficient of σ^ζ gives $E_\nu(L) = 0$; the remaining Euler–Lagrange equations simplify to

$$0 = (S_{-1} + \text{id}) \{ \kappa E_\kappa(L) \} - \mu E_\mu(L) = 2 - \mu(\mu_{-1} + \mu_1), \quad (10.12)$$

$$0 = S_{-2} \{ \kappa^{-1} \kappa_1 E_\mu(L) \} - E_\mu(L) = \left((\kappa_{-2})^{-1} \kappa_{-1} S_{-2} - \text{id} \right) (\mu_{-1} + \mu_1). \quad (10.13)$$

The general (real-valued) solution of (10.12) is

$$\mu = k_2^{(-1)^n} (1 - (k_1)^2)^{-\lfloor n/2 \rfloor} (1 + k_1^{(-1)^{n+1}})^n, \quad k_1 \neq \pm 1, \quad k_2 \neq 0. \quad (10.14)$$

To complete the solution, the following identities are useful:

$$\mu_2 = \frac{1 + k_1^{(-1)^{n+1}}}{1 + k_1^{(-1)^n}} \mu, \quad \mu\mu_1 = 1 + k_1(-1)^n.$$

The first of these enables (10.13) to be solved:

$$\kappa = c(1 - (k_1)^2)^{-1} (1 + k_1^{(-1)^{n+1}}), \quad (10.15)$$

where the (nonzero) constant c will be determined later.

The conservation laws come from $A_{\mathcal{H}}$, which, for the Lagrangian (10.5), amounts to

$$A_{\mathcal{H}} = \sigma^u - \sigma^v + (S + \text{id}) \left\{ (\kappa_{-2})^{-1} \kappa_{-1} (\mu_{-3} + \mu_{-1}) \sigma^v \right\}. \quad (10.16)$$

Note: by construction, ζ and its invariantization do not contribute to the conservation laws; v' appears in A_{κ} , but not in $A_{\mathcal{H}}$.

One can complete the solution of the problem using (10.16), but it is better to use the Euler–Lagrange equations (10.12) and (10.13) to simplify $A_{\mathcal{H}}$ first; this gives equivalent conservation laws with

$$A_{\mathcal{H}} = \sigma^u - \sigma^v + (S + \text{id})(2\mu^{-1}\sigma^v).$$

To calculate the replacements for σ^u and σ^v , we need the matrix of characteristics, restricted to the original dependent variables:

$$\Phi(u_0, v_0) = \begin{matrix} & a_1 & a_2 & a_3 \\ \begin{matrix} u_0 \\ v_0 \end{matrix} & \begin{pmatrix} (-1)^n u_0 \\ (-1)^n v_0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} (-1)^n \\ (-1)^n \end{pmatrix} \end{matrix}. \quad (10.17)$$

The invariantization of this matrix gives the replacements

$$\sigma^u \mapsto ((-1)^n \ 1 \ (-1)^n) \mathcal{A}d(\rho_0), \quad \sigma^v \mapsto (0 \ 1 \ (-1)^n) \mathcal{A}d(\rho_0). \quad (10.18)$$

The replacement for $S\sigma^v$ is

$$S\sigma^v \mapsto (0 \ 1 \ (-1)^{n+1}) \mathcal{A}d(\rho_1) = (0 \ 1 \ (-1)^{n+1}) \mathcal{A}d(K_0) \mathcal{A}d(\rho_0). \quad (10.19)$$

Collecting terms, the conservation laws are

$$(c_1 \ c_2 \ c_3) = \left((-1)^n \quad \frac{2}{\mu} + \frac{2}{\kappa\mu_1} \quad \frac{2(-1)^n}{\mu} + \frac{2(-1)^{n+1}}{\kappa\mu_1} \right) \mathcal{A}d(\rho_0). \quad (10.20)$$

Bearing in mind that $\kappa = (u_0 - v_0)(u_1 - v_1)$, (10.20) amounts to

$$c_1 = (-1)^n \left[1 - (S - \text{id}) \left\{ \frac{2v_0}{\mu(u_0 - v_0)} \right\} \right], \quad (10.21)$$

$$c_2 = (S + \text{id}) \left\{ \frac{2}{\mu(u_0 - v_0)} \right\}, \quad c_3 = (S + \text{id}) \left\{ \frac{2(-1)^n}{\mu(u_0 - v_0)} \right\}. \quad (10.22)$$

The general solution of (10.22) is

$$u_0 - v_0 = \frac{4}{\mu(c_2 + c_3(-1)^n)}, \quad c_2^2 - c_3^2 \neq 0, \quad (10.23)$$

where μ is given by (10.14). Therefore, the value of the undetermined constant in (10.15) is $c = 16/(c_2^2 - c_3^2)$ and the general solution of (10.21) is

$$v_0 = \frac{2n + c_1(-1)^n + k_3}{\mu(c_2 + c_3(-1)^n)}, \quad (10.24)$$

where k_3 is an arbitrary constant. This yields u_0 from (10.23), completing the solution of the Euler–Lagrange equations in the original variables.

11. Application to the study of Euler’s elastica

As a final example, we study a discrete variational problem analogous to that of the smooth Euler elastica,

$$\mathcal{L} = \int \kappa^2 ds, \quad \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}, \quad ds = \sqrt{1 + u_x^2} dx, \quad (11.1)$$

where κ is the Euclidean curvature and s is the Euclidean arc length. Converting to derivatives with respect to arc length s , the Euler–Lagrange equation is

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0,$$

for which a first integral was found by Euler in his masterpiece [Euler \(1744\)](#) (see also [Levien, 2008](#) for a mathematical history of the problem).

The aim is to design the discrete Lagrangian in such a way that not only the discrete Euler–Lagrange equations, but also all the discrete conservation laws become, in an appropriate continuum limit, the smooth Euler–Lagrange equations and conservation laws of a variational problem. Our method involves taking a difference frame for $SE(2)$ that has for its continuum limit the smooth $SE(2)$ frame, see [Mansfield \(2010\)](#) and [Gonçalves & Mansfield \(2013\)](#); by matching this heart of the two calculations, one for the smooth and one for the difference variational problem, we match not only the Euler–Lagrange equations but all three conservation laws, as all the relevant formulae align allowing convergence to be proven readily.

By contrast, Ge’s famous no-go theorem (see [Ge & Marsden, 1988](#)) states that a symplectic integrator, possibly after reduction so that only the conservation of energy remains, cannot exactly preserve the smooth energy without computing the exact solution. Conservation of energy in the smooth cases arises when a Lagrangian is invariant under translations in the independent variable. When computing a difference analogue, the independent variable must appear as a discrete dependent variable and the difference Lagrangian must be invariant under translation in this, so that conservation of energy in the smooth case becomes a conservation of a linear momentum in the difference analogue. For our example here, it is translation in x in the smooth case which is incorporated.

That our method of works in general is an open conjecture. To evidence this conjecture, we calculate all the relevant quantities in detail.

The Euclidean group of rotations and translations in the plane acts on curves $(x, u(x))$ as

$$\begin{pmatrix} x \\ u \end{pmatrix} \mapsto R_\theta \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Choosing the normalization equations to be

$$\tilde{x} = 0, \quad \tilde{u} = 0, \quad \tilde{u}_x = 0, \quad (11.2)$$

we obtain a smooth frame, denoted by $\widehat{\rho}$, namely

$$\widehat{\rho} = \begin{pmatrix} R_\theta & -R_\theta \begin{pmatrix} x \\ u \end{pmatrix} \\ 0 & 1 \end{pmatrix},$$

where R_θ is the rotation matrix with $\sin \theta = -u_x/\sqrt{1+u_x^2}$, $\cos \theta = 1/\sqrt{1+u_x^2}$. This frame satisfies

$$\widehat{\rho}_s \widehat{\rho}^{-1} = \frac{1}{\sqrt{1+u_x^2}} \widehat{\rho}_x \widehat{\rho}^{-1} = \begin{pmatrix} 0 & \kappa & -1 \\ -\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (11.3)$$

With this frame, the conservation laws for the Lagrangian (11.1) are, in terms of the moving frame $\widehat{\rho}$ (see Mansfield, 2010; Gonçalves & Mansfield, 2013):

$$(-\kappa^2 \quad -2\kappa_s \quad 2\kappa) \underbrace{\begin{pmatrix} x_s & u_s & xu_s - ux_s \\ -u_s & x_s & xx_s + uu_s \\ 0 & 0 & 1 \end{pmatrix}}_{Ad(\widehat{\rho})} = (c_1 \quad c_2 \quad c_3).$$

Using the identity $x_s^2 + u_s^2 = 1$, this amounts to

$$(-\kappa^2 \quad -2\kappa_s \quad 2\kappa) = (c_1 \quad c_2 \quad c_3) \begin{pmatrix} x_s & -u_s & u \\ u_s & x_s & -x \\ 0 & 0 & 1 \end{pmatrix}. \quad (11.4)$$

The same identity gives a first integral for the Euler–Lagrange equation,

$$\kappa^4 + 4\kappa_s^2 = c_1^2 + c_2^2. \quad (11.5)$$

Eliminating x_s from the first two columns of (11.4) gives

$$u_s = \frac{1}{c_1^2 + c_2^2} (2c_1\kappa_s - c_2\kappa^2). \quad (11.6)$$

By solving (11.5), (11.6) and the third column of (11.4) (to determine x), we obtain the smooth solution in Fig. 2, once the constants of integration c_1 and c_2 are determined.

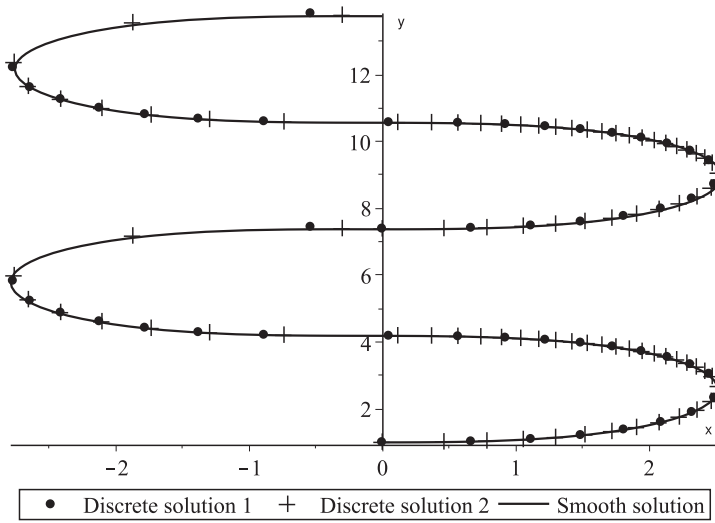


FIG. 2. A plot of an extract of 847 points of the discrete solution for certain initial data and an extract of 507 points of the discrete solution for a variation of the previous initial data. This is compared with an accurate numerical solution of the third column of (11.4), (11.5) and (11.6), using a Fehlberg fourth–fifth order Runge–Kutta method with degree four interpolant, with uniform step 0.1. The conservation laws are used in the solution in order to match the initial data.

The idea is to take a difference frame with matching normalization equations, and to take the discrete analogues of curvature and arc length to be those playing the same role when compared to the smooth Maurer–Cartan invariants given in Equation (11.3). We now explain these remarks. Consider the action of $SE(2)$ in the plane where the points \mathbf{u}_j have coordinates (x_j, u_j) , and take the frame

$$\rho_0 = \begin{pmatrix} R_{\theta_0} & -R_{\theta_0} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \\ 0 & 1 \end{pmatrix},$$

(using the standard representation) such that the normalization equations are

$$\rho_0 \cdot \mathbf{u}_0 = (0, 0), \quad \rho_0 \cdot (\mathbf{u}_1 - \mathbf{u}_0) = (*, 0).$$

In other words, R_{θ_0} is the rotation matrix that sends $\mathbf{u}_1 - \mathbf{u}_0$ to a row vector with a zero second component, so that $\sin \theta_0 = -(u_1 - u_0)/\ell$ and $\cos \theta_0 = (x_1 - x_0)/\ell$, where $\ell = |\mathbf{u}_1 - \mathbf{u}_0|$. These discrete normalization equations match, in some sense, the normalization equations (11.2) for the smooth frame. Then

$$K_0 = \rho_1 \rho_0^{-1} = \begin{pmatrix} R_{h_\theta} & -R_{\theta_1} \begin{pmatrix} x_1 - x_0 \\ u_1 - u_0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{h_\theta} & -R_{h_\theta} \begin{pmatrix} \ell \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix},$$

where $h_\theta = \theta_1 - \theta_0$. Therefore, the generating invariants are h_θ and ℓ .

In order to see the discrete analogues of curvature and arc length, we consider $\widehat{\rho}_x \widehat{\rho}^{-1}$ to be approximated by

$$(\widehat{\rho}(x + h_x) - \widehat{\rho}(x)) \widehat{\rho}(x)^{-1} / h_x = \left(\widehat{\rho}(x + h_x) \widehat{\rho}(x)^{-1} - \text{Id} \right) / h_x,$$

where Id is the identity matrix, and $\widehat{\rho}(x + h_x) \widehat{\rho}(x)^{-1}$ to be approximated by K_0 when $x = x_0$ and $h_x = x_1 - x_0$.

Observing that the component of the first row and second column of the matrix $K_0 - \text{Id}$ is $-\sin h_\theta$ and that, to first-order in h_θ , the component of the first row and third column of the matrix $K_0 - \text{Id}$ is $-\ell$, we take the discrete analogue of ds to be ℓ and the discrete analogue of κ to be

$$\bar{\kappa} = -\ell^{-1} \sin h_\theta.$$

Hence, we consider the variational problem

$$\mathcal{L} = \sum \ell^{-1} \sin^2 h_\theta,$$

with ℓ and h_θ as the fundamental invariants. The differential–difference syzygies are

$$\begin{aligned} \ell' &= \cos h_\theta S \sigma^x + \sin h_\theta S \sigma^u - \sigma^x, \\ h'_\theta &= (S - \text{id}) \left(\ell^{-1} [\sin h_\theta S \sigma^x - \cos h_\theta S \sigma^u + \sigma^u] \right), \end{aligned}$$

where $\sigma^x = \iota_0(x'_0)$ and $\sigma^u = \iota_0(u'_0)$. Applying the theory developed in this paper, the invariantized Euler–Lagrange equations are

$$\begin{aligned} \{S_{-1}(\cos h_\theta) S_{-1} - \text{id}\} E_\ell(L) + \{S_{-1}(\ell^{-1} \sin h_\theta) (S_{-2} - S_{-1})\} E_{h_\theta}(L) &= 0, \\ \{S_{-1}(\sin h_\theta) S_{-1}\} E_\ell(L) + \{\ell^{-1} (S_{-1} - \text{id}) - S_{-1}(\ell^{-1} \cos h_\theta) (S_{-2} - S_{-1})\} E_{h_\theta}(L) &= 0, \end{aligned}$$

where

$$E_{h_\theta}(L) = \frac{\partial L}{\partial h_\theta} = \ell^{-1} \sin(2h_\theta), \quad E_\ell(L) = \frac{\partial L}{\partial \ell} = -\ell^{-2} \sin^2 h_\theta.$$

These equations are then solved for ℓ and h_θ . We note that (the shifts of) these Euler–Lagrange equations can be written in the form,

$$\begin{pmatrix} \cos h_\theta & -\sin h_\theta \\ \sin h_\theta & \cos h_\theta \end{pmatrix} \begin{pmatrix} \ell^{-1} (S_{-1} - \text{id}) E_{h_\theta}(L) \\ E_\ell(L) \end{pmatrix} = S \begin{pmatrix} \ell^{-1} (S_{-1} - \text{id}) E_{h_\theta}(L) \\ E_\ell(L) \end{pmatrix}.$$

The boundary terms can be written in the form

$$A_{\mathcal{H}} = \mathcal{C}_x^0 \iota_0(x'_0) + \mathcal{C}_u^0 \iota_0(u'_0) + \mathcal{C}_x^1 S(\iota_0(x'_0)) + \mathcal{C}_u^1 S(\iota_0(u'_0)),$$

where

$$\begin{aligned} \mathcal{C}_x^0 &= S_{-1} \{ \cos h_\theta E_\ell(L) - \ell^{-1} \sin h_\theta E_{h_\theta}(L) + \ell^{-1} \sin h_\theta S_{-1} (E_{h_\theta}(L)) \}, \\ \mathcal{C}_u^0 &= S_{-1} \{ \sin h_\theta E_\ell(L) + (S(\ell^{-1}) + \ell^{-1} \cos h_\theta - \ell^{-1} \cos h_\theta S_{-1}) E_{h_\theta}(L) \}, \\ \mathcal{C}_x^1 &= \ell^{-1} \sin h_\theta S_{-1} \{ E_{h_\theta}(L) \}, \\ \mathcal{C}_u^1 &= -\ell^{-1} \cos h_\theta S_{-1} \{ E_{h_\theta}(L) \}. \end{aligned}$$

The infinitesimal vector fields are

$$\mathbf{v}_a = \partial_{x_0}, \quad \mathbf{v}_b = \partial_{u_0}, \quad \mathbf{v}_\theta = -u_0 \partial_{x_0} + x_0 \partial_{u_0},$$

so that

$$\Phi(\mathbf{u}_0) = \begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & -x_0 \end{pmatrix}, \quad \Phi(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and

$$\mathcal{A}d(g(\theta, a, b)) = \begin{pmatrix} \cos \theta & -\sin \theta & b \\ \sin \theta & \cos \theta & -a \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying the replacement (7.5) and simplifying and collecting terms, the conservation laws can be written in terms of the row vector of invariants as follows

$$(V_1 \ V_2 \ V_3) \begin{pmatrix} R_{\theta_0} & JR_{\theta_0} \mathbf{u}_0 \\ 0 & 1 \end{pmatrix} = (c_1 \ c_2 \ c_3), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (11.7)$$

where

$$\begin{aligned} V_1 &= S_{-1} (\cos h_\theta E_\ell(L)) + \{ S_{-1} (\ell^{-1} \sin h_\theta) (S_{-2} - S_{-1}) \} E_{h_\theta}(L), \\ V_2 &= S_{-1} (\sin h_\theta E_\ell(L)) - \{ S_{-1} (\ell^{-1} \cos h_\theta) (S_{-2} - S_{-1}) \} E_{h_\theta}(L), \\ V_3 &= -S_{-1} (E_{h_\theta}(L)). \end{aligned}$$

Using Maple, we solve the discrete Euler–Lagrange equations for the invariants as an initial data problem. Note that (11.7) implies that

$$(V_1)^2 + (V_2)^2 = (c_1)^2 + (c_2)^2,$$

which gives a first integral of the discrete Euler–Lagrange equations. Further, there is a linear relation for x and u , in terms of the invariants; using the methods of Section 8.2, the solution in terms of the original variables can be obtained in a straightforward manner. The initial data give the values of the constants c_1, c_2 . We have used these constants and the initial values $(x_0, u_0) = (0, 1)$ to obtain the initial data for the smooth solution. The discrete equations require one more initial datum than the smooth, so that more than one discrete solution will have the same constants and starting point, and hence more than one discrete solution can approximate a given smooth one. In Fig. 2, we compare two discrete solutions with differing initial step sizes, both approximating the single smooth solution. Magnifications of these solutions, verification that conserved quantities are indeed conserved, and relative errors of each discrete solution to the smooth solution are given Figs 3 and 4.

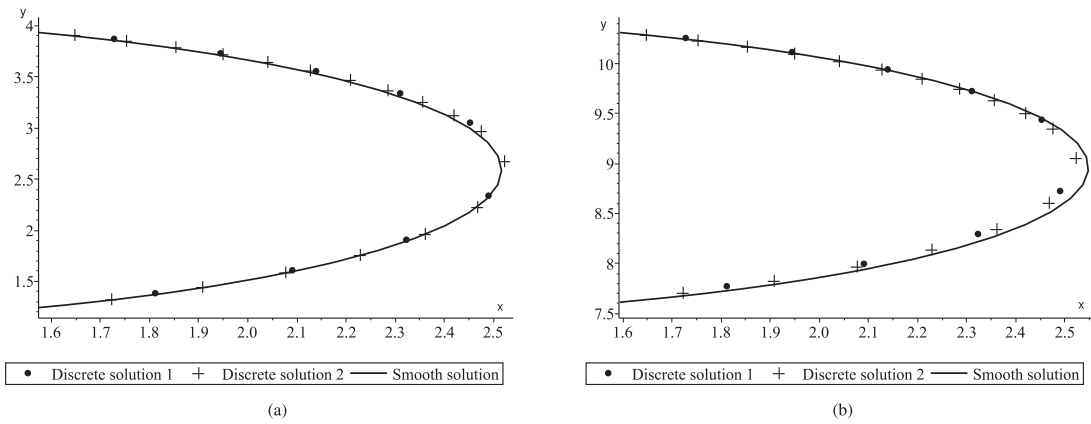


FIG. 3. Plots (a) and (b) magnify two regions of Fig. 2.

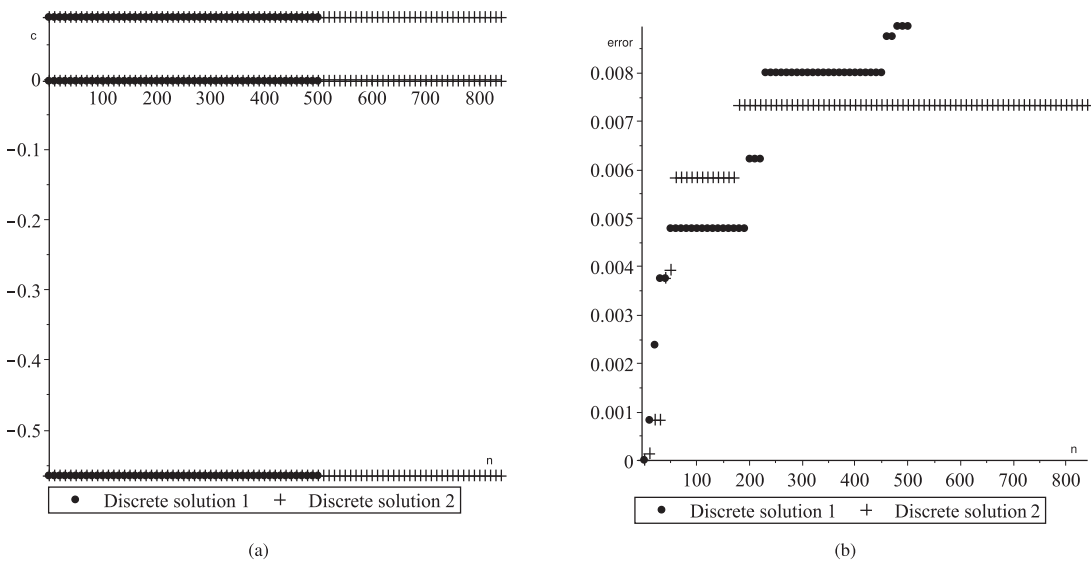


FIG. 4. (a) Graph showing the three conserved quantities are indeed conserved (are constants). The values of the constants in the conservation laws obtained in the discrete case were used to calculate the initial data in the integration of the smooth case. (b) The norm relative error between the solution of the smooth problem and each solution of the discrete problem.

A routine albeit lengthy calculation shows that the Euler–Lagrange equations and the conservation laws converge to the smooth counterparts in the continuum limit. The relative success of this simple example shows that this approach to obtaining symmetry-preserving variational integrators via the difference moving frame merits further research. More sophisticated methods to derive discrete Lagrangians using interpolation are also being explored (Beffa & Mansfield, 2018).

12. Conclusions

In this paper, we have introduced difference moving frames and some applications. In particular, for a discrete Lagrangian with a Lie group of variational symmetries, a difference moving frame expresses the Euler–Lagrange equations in terms of the invariants and Noether’s conservation laws in terms of the frame and a vector of invariants. This makes explicit the equivariance of the conservation laws. The difference frame formulation can allow one to solve for the solutions in terms of the original variables by a divide-and-conquer approach: first the invariantized Euler–Lagrange equations are solved, then the conservation laws and frame are used to construct the complete solution. It is worth noting that one need not use the full Lie group to do this; a Lie subgroup may do the job more simply. For instance, using the translation subgroup in our running example would have been equally effective, with slightly simpler calculations.

We have developed the Noether theory for difference frames to cover all variational symmetries, not just those that leave the given Lagrangian invariant. It also covers symmetries whose generators are n -dependent and Lagrangians that vary with n .

We have shown that, by matching the smooth and difference frames for the smooth and discretized problems respectively, it is possible to create symmetry-preserving variational integrators that can approximate the full set of conservation laws. Optimizing the use of the difference frame in these approximations is a subject for further research.

Part I of this paper has concentrated on relatively straightforward Lie group actions. In Part II, we turn to actions of the semisimple Lie group $SL(2)$.

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Appendix A. From structure constants to $\mathcal{A}d(g)$

There is an alternative way to construct the matrix $\mathcal{A}d(g)$ from the basis

$$\mathbf{v}_r = \xi_r^\alpha(\mathbf{u}) \partial_{t^\alpha}, \quad r = 1, \dots, R,$$

for the Lie algebra of infinitesimal generators. The Lie algebra is closed, giving rise to the commutator relations

$$[\mathbf{v}_i, \mathbf{v}_j] \equiv \mathbf{v}_i \mathbf{v}_j - \mathbf{v}_j \mathbf{v}_i = c_{ij}^k \mathbf{v}_k, \quad i, j = 1, \dots, R,$$

where c_{ij}^k are the structure constants. Define the $R \times R$ matrices C_j by

$$(C_j)_i^k = c_{ij}^k.$$

Every infinitesimal generator $\mathbf{v} = a^r \mathbf{v}_r$ can be exponentiated to produce a one-parameter local Lie group of transformations $\exp(\varepsilon \mathbf{v})$. These act on an arbitrary smooth function $F(\mathbf{u})$ as follows:

$$\exp(\varepsilon \mathbf{v}) F(\mathbf{u}) = F(\widehat{\mathbf{u}}(\varepsilon)), \quad \text{where} \quad \widehat{\mathbf{u}}(\varepsilon) = \exp(\varepsilon \mathbf{v}) \mathbf{u} \equiv \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \mathbf{v}^m(\mathbf{u}).$$

One can obtain $\widehat{\mathbf{u}}(\varepsilon)$ by solving the initial value problem

$$\frac{d\widehat{\mathbf{u}}^\alpha(\varepsilon)}{d\varepsilon} = a^r \xi_r^\alpha(\widehat{\mathbf{u}}(\varepsilon)), \quad \alpha = 1, \dots, q; \quad \widehat{\mathbf{u}}|_{\varepsilon=0} = \mathbf{u}.$$

THEOREM A.1 Suppose that $g \cdot \mathbf{u} = \exp(a^r \mathbf{v}_r) \mathbf{u}$. Then $\mathcal{A}d(g) = \exp(a^r C_r)$.

Proof. Let $\mathbf{v} = a^r \mathbf{v}_r$ and consider the action of the one parameter local Lie group $\exp(\varepsilon \mathbf{v})$. Define

$$g(\varepsilon) \cdot \mathbf{u} = \widehat{\mathbf{u}}(\varepsilon) = \exp(\varepsilon \mathbf{v}) \mathbf{u}$$

and denote the components of $\mathcal{A}d(g(\varepsilon))$ by $A_i^l(\varepsilon)$. The identity (6.3) yields

$$\mathbf{v}_i = A_i^l(\varepsilon) \widehat{\mathbf{v}}_l(\varepsilon),$$

where $\widehat{\mathbf{v}}_l(\varepsilon)$ is \mathbf{v}_l with \mathbf{u} replaced by $\widehat{\mathbf{u}}(\varepsilon)$. If $F(\mathbf{u})$ is an arbitrary smooth function,

$$\exp(-\varepsilon \mathbf{v}) \mathbf{v}_i \exp(\varepsilon \mathbf{v}) F(\mathbf{u}) = \exp(-\varepsilon \mathbf{v}) \{A_i^l(\varepsilon) \widehat{\mathbf{v}}_l(\varepsilon) F(\widehat{\mathbf{u}}(\varepsilon))\} = A_i^l(\varepsilon) \mathbf{v}_l F(\mathbf{u}).$$

Consequently,

$$\frac{d}{d\varepsilon} \{A_i^l(\varepsilon)\} \mathbf{v}_l F(\mathbf{u}) = \exp(-\varepsilon \mathbf{v}) [\mathbf{v}_i, \mathbf{v}] \exp(\varepsilon \mathbf{v}) F(\mathbf{u}) = a^j c_{ij}^k A_k^l(\varepsilon) \mathbf{v}_l F(\mathbf{u}),$$

and so $\mathcal{A}d(g(\varepsilon))$ satisfies the initial value problem

$$\frac{d}{d\varepsilon} \mathcal{A}d(g(\varepsilon)) = a^j \mathcal{A}d(g(\varepsilon)) C_j, \quad \mathcal{A}d(g(0)) = \text{Id}.$$

The $R \times R$ matrix $\exp(\varepsilon a^j C_j)$ satisfies the same initial value problem, so the standard uniqueness theorem for ODEs proves that

$$\mathcal{A}d(g(\varepsilon)) = \exp(\varepsilon a^j C_j).$$

Set $\varepsilon = 1$ to conclude the result. □

COROLLARY A.1 Suppose that

$$g \cdot \mathbf{u} = \exp(a_1^{r_1} \mathbf{v}_{r_1}) \exp(a_2^{r_2} \mathbf{v}_{r_2}) \cdots \exp(a_M^{r_M} \mathbf{v}_{r_M}) \mathbf{u}.$$

Then $\mathcal{A}d(g) = \exp(a_1^{r_1} C_{r_1}) \exp(a_2^{r_2} C_{r_2}) \cdots \exp(a_M^{r_M} C_{r_M})$.

Proof. The adjoint representation preserves left multiplication. □

For every R -dimensional local Lie group of transformations, the action of every g in some neighbourhood of the identity can be written as $\exp(a^r \mathbf{v}_r)$ for some a^r , which are called canonical coordinates of the first kind. So if $\tilde{\mathbf{u}} = g \cdot \mathbf{u}$ is written in this form, Theorem A.1 enables us to write down $\mathcal{A}d(g) = \exp(a^r C_r)$ immediately.

Commonly, $\tilde{\mathbf{u}} = g \cdot \mathbf{u}$ is written as a product of exponentiated generators, in which case Corollary A.1 is applicable. In particular, the action of every g in some neighbourhood of the identity can be written as $\exp(a^R \mathbf{v}_R) \cdots \exp(a^1 \mathbf{v}_1)$; here a^r are canonical coordinates of the second kind.

EXAMPLE A.2 (Example 3.1 cont.) The group action $g \cdot (x, u) = (\tilde{x}, \tilde{u})$ in the running example can be written in terms of canonical coordinates of the second kind as

$$(\tilde{x}, \tilde{u}) = (e^{3a^1} x + a^2, e^{a^1} u + a^3) = \exp(a^3 \mathbf{v}_3) \exp(a^2 \mathbf{v}_2) \exp(a^1 \mathbf{v}_1)(x, u),$$

where $(a^1, a^2, a^3) = (\ln(\lambda), a, b)$ and

$$\mathbf{v}_1 = 3x\partial_x + u\partial_u, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = \partial_u.$$

The only nonzero structure constants are $c_{12}^2 = -c_{21}^2 = -3$ and $c_{13}^3 = -c_{31}^3 = -1$, so Corollary A.1 yields

$$\mathcal{A}d(g) = \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a^3 & 0 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ -3a^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3a^1 & 0 \\ 0 & 0 & a^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3a^2 & e^{3a^1} & 0 \\ -a^3 & 0 & e^{a^1} \end{pmatrix},$$

in agreement with the more straightforward approach of Section 6.