A MULTILINEAR FOURIER EXTENSION IDENTITY ON $\mathbb{R}^n$

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Abstract. We prove an elementary multilinear identity for the Fourier extension operator on $\mathbb{R}^n$, generalising to higher dimensions the classical bilinear extension identity in the plane. In the particular case of the extension operator associated with the paraboloid, this provides a higher dimensional extension of a well-known identity of Ozawa and Tsutsumi for solutions to the free time-dependent Schrödinger equation. We conclude with a similar treatment of more general oscillatory integral operators whose phase functions collectively satisfy a natural multilinear transversality condition. The perspective we present has its origins in work of Drury.

1. Introduction

To a smooth function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ we associate the *Fourier extension operator*

$$Eg(x) = \int_{\mathbb{R}^{n-1}} e^{i(x'\cdot\xi + x_n\phi(\xi))} g(\xi) d\xi;$$

here $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and a-priori $g \in L^1(\mathbb{R}^{n-1})$. The term "extension operator" is used since the adjoint E^*, given by $E^*f(\xi) = \widehat{f}(\xi, \phi(\xi))$, gives a (parametrised) restriction of the Fourier transform of a function f on $\mathbb{R}^n$ to the hypersurface $S = \{(\xi, \phi(\xi)) : \xi \in \mathbb{R}^{n-1}\}$. In practice the function ϕ is often only defined on some compact set $U \subseteq \mathbb{R}^{n-1}$, giving rise to a compact hypersurface S. We gloss over this point in most of what follows since such a feature may be captured by the implicit assertion that the function g is supported in U. In the 1960s Stein observed that if S is compact and has everywhere nonvanishing curvature, then E satisfies estimates of the form

$$\|Eg\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(U)} \tag{1.1}$$

with $q < \infty$; the case $(p, q) = (1, \infty)$ is of course elementary by Minkowski's inequality. The celebrated *Fourier restriction conjecture* asserts that estimates of this type continue to hold for $q > \frac{2n}{n-1}$, with elementary examples preventing an endpoint estimate at $q = \frac{2n}{n-1}$; see for example [16]. Since the 1990s *bilinear*, and more generally *multilinear*, estimates of this type have emerged as particularly natural and useful; see for example [18], [17], [12], [5], [1], [7]. The simplest such example is the well-known and elementary bilinear identity

$$\int_{\mathbb{R}^2} |E_1g_1(x)E_2g_2(x)|^2 dx = (2\pi)^2 \int_{\mathbb{R}^2} \frac{|g_1(\xi_1)|^2|g_2(\xi_2)|^2}{|\phi_1'(\xi_1) - \phi_2'(\xi_2)|} d\xi_1 d\xi_2, \tag{1.2}$$

where E_1, E_2 are extension operators associated with phases ϕ_1, ϕ_2 and curves S_1, S_2 in the plane; see [11] for the origins of this.[1] This particular two-dimensional statement occupies a singular position in Fourier restriction theory in the sense that it is *an identity*. The main purpose of this paper is to establish natural higher-dimensional analogues of this. To this end we consider extension operators $E_1, \ldots, E_n$ associated with the functions $\phi_1, \ldots, \phi_n$, and hypersurfaces $S_1, \ldots, S_n$.

This work was supported by the European Research Council [grant number 307617].

[1] As may be expected, some technical hypotheses relating to the geometry of these curves are needed here, and it will suffice to ask that $\phi_1'(\xi_1) \neq \phi_2'(\xi_2)$ whenever ξ_j belongs to some interval containing the support of g_j, for each $j = 1, 2$.

Theorem 1.1.

$$|E_1g_1|^2 * \cdots * |E_ng_n|^2 \equiv (2\pi)^{n(n-1)} \int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2 \cdots |g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|} d\xi, \tag{1.3}$$

for all functions $g_1, \ldots, g_n$ such that the determinant factor is nonzero whenever ξ_j belongs to the convex hull of the support of g_j, $1 \leq j \leq n$.

It should be remarked that requiring a non-vanishing determinant factor whenever ξ_j belongs to the *support* of g_j ($1 \leq j \leq n$) is necessary in order for the integral on the right hand side of (1.3) to be finite. This is due to a critical lack of local integrability, which is of course also present in (1.2). Our requirement that this continues to hold on the convex hull of the supports is a technical condition used in our proof, and is a product of the generality of the set-up. As we shall see, this is not always necessary, as the particular case where each S_j is the paraboloid reveals. In particular, the following holds.

Theorem 1.2. *Let E be the extension operator on the paraboloid $S = \{(\xi, \phi(\xi)) : \xi \in \mathbb{R}^{n-1}\}$, with $\phi = |\cdot|^2$. Then,*

$$|Eg_1|^2 * \cdots * |Eg_n|^2 \equiv 2^{-(n-1)}(2\pi)^{n(n-1)} \int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2 \cdots |g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \xi_1 & \cdots & \xi_n\end{pmatrix}\right|} d\xi. \tag{1.4}$$

We clarify that while Theorem 1.2 does not impose a support condition on the functions g_j, finiteness in (1.4) requires that the determinant factor on the right hand side does not vanish on their supports. This particular determinant factor is of course just the volume of the parallelepiped in $\mathbb{R}^{n-1}$ with vertices $\xi_1, \ldots, \xi_n$.

Theorem 1.1 tells us that $|E_1g_1|^2 * \cdots * |E_ng_n|^2$ is a constant function. Nevertheless, it is enough to prove (1.3) at the origin, as the right hand side is manifestly modulation-invariant. The case $n = 2$ of Theorem 1.1 immediately reduces to (1.2) on evaluating the convolution at the origin and performing a harmless reflection in either E_1g_1 or E_2g_2. The identity (1.3) may be interpreted as an elementary substitute for the absence of a linear restriction inequality (of the form (1.1)) at the endpoint $q = 2n/(n-1)$. Indeed, notice that the n-fold convolution

$$L^{n/(n-1)}(\mathbb{R}^n) * \cdots * L^{n/(n-1)}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$$

by Young's convolution inequality; therefore, an inequality of the form (1.1) at $q = 2n/(n-1)$ would also imply that $|E_1g_1|^2 * \cdots * |E_ng_n|^2$ is a bounded function. This perspective on the restriction conjecture originates in work of Drury, and the underlying ideas in this paper are closely related to those in [10].

A more geometric interpretation of (1.3) comes from writing

$$E_jg_j = \widehat{f_jd\sigma_j},$$

where $d\sigma_j$ is surface area measure on S_j, and f_j is given by

$$g_j(\xi) = (1 + |\nabla\phi_j(\xi)|^2)^{1/2} f_j(\xi, \phi_j(\xi)).$$

In these terms (1.3) becomes

$$|\widehat{f_1d\sigma_1}|^2 * \cdots * |\widehat{f_nd\sigma_n}|^2 \equiv (2\pi)^{n(n-1)} \int_{S_1\times\cdots\times S_n} \frac{|f_1(y_1)|^2 \cdots |f_n(y_n)|^2}{|v_1(y_1) \wedge \cdots \wedge v_n(y_n)|} d\sigma_1(y_1)\cdots d\sigma_n(y_n), \tag{1.5}$$

where $v_j(y_j)$ denotes a unit normal vector to S_j at the point $y_j \in S_j$. It is instructive to (formally) take the Fourier transform of the identity (1.5), and look to interpret the resulting distribution

$$\prod_{j=1}^{n}(f_j d\sigma_j) * \widetilde{(f_j d\sigma_j)}$$

as a multiple of the delta distribution at the origin; here $\widetilde{\mu}$ denotes the reflection of a measure μ in the origin. The key observation is that each factor $(f_j d\sigma_j) * \widetilde{(f_j d\sigma_j)}$ is supported in the complement of a cone with vertex at 0, and the axes of these cones point in a spanning set of directions. We do not attempt to make these heuristics rigorous here.

We conclude this section with some further contextual remarks and generalisations.

Notice that the vector $(1, -\nabla\phi_j(\xi_j))^T$ is normal to the hypersurface S_j at the point $(\xi_j, \phi_j(\xi_j))$, and so if the surfaces $S_1, \ldots, S_n$ are compact and *transversal*, that is, satisfying[2]

$$|v_1(y_1) \wedge \cdots \wedge v_n(y_n)| \gtrsim 1 \quad \text{for} \quad y_1 \in S_1, \ldots, y_n \in S_n,$$

then (1.3) becomes

$$|E_1 g_1|^2 * \cdots * |E_n g_n|^2 \sim \|g_1\|_2^2 \cdots \|g_n\|_2^2. \tag{1.6}$$

It is interesting to contrast this with the (considerably deeper) endpoint multilinear restriction conjecture

$$\|E_1 g_1 \cdots E_n g_n\|_{L^{\frac{2}{n-1}}(\mathbb{R}^n)} \lesssim \|g_1\|_2 \cdots \|g_n\|_2; \tag{1.7}$$

see [5]. While (1.7) remains open, the weaker

$$\|E_1 g_1 \cdots E_n g_n\|_{L^{\frac{2}{n-1}}(B(0;R))} \lesssim_\varepsilon R^\varepsilon \|g_1\|_2 \cdots \|g_n\|_2, \quad R \gg 1, \tag{1.8}$$

is known; see [5], [1] for a modest improvement, and [3], [20] for generalisations.

Theorem 1.1 is a particular case of a one-parameter family of identities for the multilinear operator $T_\sigma(g_1, \ldots, g_n)(x_1, \ldots, x_n) :=$

$$\int_{(\mathbb{R}^{n-1})^n} \left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|^\sigma \prod_{j=1}^{n} e^{i(x_j'\cdot\xi_j + x_{jn}\phi_j(\xi_j))} g_j(\xi_j)d\xi_j,$$

where $x_1, \ldots, x_n \in \mathbb{R}^n$, $x_j = (x_j', x_{jn}) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and $\sigma \in \mathbb{R}$. Of course $T^0(g_1, \ldots, g_n) = E_1 g_1 \otimes \cdots \otimes E_n g_n$, so that Theorem 1.1 is the $\sigma = 0$ case of the following:

Theorem 1.3. *For each $\sigma \in \mathbb{R}$,*

$$\begin{aligned}\int_{x_1+\cdots+x_n=0} &|T_\sigma(g_1, \ldots, g_n)(x_1, \ldots, x_n)|^2 dx \\ &= (2\pi)^{n(n-1)} \int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2 \cdots |g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|^{1-2\sigma}} d\xi\end{aligned} \tag{1.9}$$

for all functions $g_1, \ldots, g_n$ such that the determinant factor is nonzero whenever ξ_j belongs to the convex hull of the support of g_j, $1 \le j \le n$.

In the case of the extension operator on the paraboloid, the support condition on the functions g_j may be dropped provided $\sigma \ge 0$, as our next theorem clarifies.

[2]Throughout this paper we shall write $A \lesssim B$ if there exists a constant c such that $A \le cB$. The relations $A \gtrsim B$ and $A \sim B$ are defined similarly.

Theorem 1.4. *Suppose $\sigma \geq 0$. In the case of the paraboloid, i.e. for $\phi_1 = \ldots = \phi_n = \phi = |\cdot|^2$,*

$$\begin{aligned}&\int_{x_1+\cdots+x_n=0} |T_\sigma(g_1,\ldots,g_n)(x_1,\ldots,x_n)|^2 dx \\ &\qquad = 2^{-(n-1)}(2\pi)^{n(n-1)} \int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2 \cdots |g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \xi_1 & \cdots & \xi_n\end{pmatrix}\right|^{1-2\sigma}} d\xi.\end{aligned} \tag{1.10}$$

If $\sigma < 0$, (1.10) *continues to hold provided the determinant factor is non-vanishing on the supports of the g_j, $1 \leq j \leq n$.*

Of course when $\sigma = 0$, Theorem 1.4 becomes Theorem 1.2. In contrast with the case $\sigma = 0$, when $\sigma > 0$ finiteness in (1.10) no longer requires that the determinant factor is non-vanishing on the supports of the g_j.

Of course (1.9) ceases to have convolution structure for $\sigma \neq 0$. However, alternative geometric insight may be found in a more elementary Kakeya-type analogue of (1.9), which states that

$$\begin{aligned}&\int_{x_1+\cdots+x_n=0} \left(\sum_{T_1,\ldots,T_n} |e(T_1)\wedge\cdots\wedge e(T_n)|^{2\sigma} c_{T_1}\chi_{T_1}(x_1)\cdots c_{T_n}\chi_{T_n}(x_n)\right) dx \\ &\qquad = c_n \sum_{T_1,\ldots,T_n} \frac{c_{T_1}\cdots c_{T_n}}{|e(T_1)\wedge\cdots\wedge e(T_n)|^{1-2\sigma}};\end{aligned} \tag{1.11}$$

here $T_1,\ldots,T_n$ belong to finite sets $\mathbb{T}_1,\ldots,\mathbb{T}_n$ of doubly infinite 1-tubes (cylinders of cross-sectional volume 1) in $\mathbb{R}^n$, and for such a tube T, $e(T) \in \mathbb{S}^{n-1}$ denotes its direction. Here the coefficients c_{T_j} are nonnegative real numbers, c_n denotes a constant depending only on n, and we make the qualitative transversality assumption that $e(T_1)\wedge\cdots\wedge e(T_n) \neq 0$ whenever $T_j \in \mathbb{T}_j$. When $n = 2$, this is the well-known and elementary bilinear Kakeya theorem in the plane. By multilinearity (1.11) immediately follows, for all σ, from the elementary geometric fact that

$$\chi_{T_1} * \cdots * \chi_{T_n} \equiv \frac{c_n}{|e(T_1)\wedge\cdots\wedge e(T_n)|} \tag{1.12}$$

whenever $e(T_1)\wedge\cdots\wedge e(T_n) \neq 0$. (A simple way to see (1.12) is to begin with its manifest truth for orthogonal axis-parallel *rectangular* tubes $T_1,\ldots,T_n$, and then use multilinearity and scaling to extend it to orthogonal tubes of arbitrary cross section, whereby a change of variables may then be used to establish the claimed dependence on the directions $e(T_1),\ldots,e(T_n)$.) The identity (1.11) with $\sigma = 1/2$ has a similar flavour to the much deeper affine-invariant endpoint multilinear Kakeya inequality

$$\int_{\mathbb{R}^n} \left(\sum_{T_1,\ldots,T_n} |e(T_1)\wedge\cdots\wedge e(T_n)|\, c_{T_1}\chi_{T_1}\cdots c_{T_n}\chi_{T_n}\right)^{\frac{1}{n-1}} \lesssim \left(\sum_{T_1} c_{T_1}\cdots\sum_{T_n} c_{T_n}\right)^{\frac{1}{n-1}}$$

proved in [6] and [8], and the seemingly deeper still (conjectural) variant

$$\begin{aligned}&\int_{\mathbb{R}^n} \left(\sum_{T_1,\ldots,T_n} |e(T_1)\wedge\cdots\wedge e(T_n)|^{2\sigma} c_{T_1}\chi_{T_1}\cdots c_{T_n}\chi_{T_n}\right)^{\frac{1}{n-1}} \\ &\qquad \lesssim \left(\sum_{T_1,\ldots,T_n} \frac{c_{T_1}\cdots c_{T_n}}{|e(T_1)\wedge\cdots\wedge e(T_n)|^{1-2\sigma}}\right)^{\frac{1}{n-1}},\end{aligned} \tag{1.13}$$

for any real number σ. This inequality for $\sigma = 0$, or at least a natural variant of it involving truncated tubes, is easily seen to imply the classical Kakeya maximal conjecture via an application of Drury's inequalities from [10]. The identities in Theorems 1.1 and 1.3 are inspired by the analogous conjectural multilinear extension inequality

$$\|E_1g_1\cdots E_ng_n\|^2_{L^{\frac{2}{n-1}}(\mathbb{R}^n)} \lesssim \int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2\cdots|g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|}d\xi \tag{1.14}$$

and its generalisation

$$\int_{\mathbb{R}^n} |T_\sigma(g_1,\ldots,g_n)(x,\ldots,x)|^{\frac{2}{n-1}}dx \lesssim \left(\int_{(\mathbb{R}^{n-1})^n} \frac{|g_1(\xi_1)|^2\cdots|g_n(\xi_n)|^2}{\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|^{1-2\sigma}}d\xi\right)^{\frac{1}{n-1}}. \tag{1.15}$$

These very strong conjectural inequalities (1.13)–(1.15) arose in discussions with Tony Carbery in 2004, and also recall work of Drury in [10]. Some recent progress in this direction may be found in [15]. Of course (1.3) and (1.9) are much more elementary than (1.14) and (1.15) when $n \geq 3$.

Theorems 1.2 and 1.4 may be formulated in terms of solutions $u_1, \ldots, u_{d+1} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ to the Schrödinger equation $i\partial_t u = \Delta u$ with initial data $f_1, \ldots, f_{d+1}$. Indeed, Theorem 1.2 for $n = d + 1$ becomes

$$\begin{aligned}\int_{\substack{x_1+\cdots+x_{d+1}=0 \\ t_1+\cdots+t_{d+1}=0}} &|u_1(x_1,t_1)|^2\cdots|u_{d+1}(x_{d+1},t_{d+1})|^2dxdt \\ &= \frac{1}{2^d(2\pi)^{d(d+1)}}\int_{(\mathbb{R}^d)^{d+1}} \frac{|\widehat{f}_1(\xi_1)|^2\cdots|\widehat{f}_{d+1}(\xi_{d+1})|^2}{|\rho(\xi)|}d\xi,\end{aligned} \tag{1.16}$$

where

$$\rho(\xi) = \det\begin{pmatrix} 1 & \cdots & 1 \\ \xi_1 & \cdots & \xi_{d+1}\end{pmatrix};$$

here $\xi = (\xi_1, \ldots, \xi_{d+1}) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$. We observe that $\rho(\xi) = 0$ if and only if $\xi_1, \ldots, \xi_{d+1}$ are co-hyperplanar points in $\mathbb{R}^d$, and, in order for the expression in (1.16) to be finite, one needs to stipulate that the determinant factor is non-vanishing for ξ_j in the support of $\widehat{f}_j$, $1 \leq j \leq d+1$. Notice that the tensor product here is a *space-time* tensor product. Thus there are many times in play, and the measure is Lebesgue measure on a linear subspace of space-time. Multilinear expressions of a similar flavour to (1.16) may be found in [2].

A similar reformulation of Theorem 1.4 for $\sigma > 0$ gives an extension of (1.16) that ceases to have local integrability (finiteness) issues, retaining content even if the solutions u_j all coincide. In order to state this, it is natural to define the d-th order differential operator

$$\rho(\nabla_x) := \det\begin{pmatrix} 1 & \cdots & 1 \\ \nabla_{x_1} & \cdots & \nabla_{x_{d+1}}\end{pmatrix},$$

and its fractional power $|\rho(\nabla_x)|^\gamma$ to be the operator with Fourier multiplier $|\rho(\xi)|^\gamma$; here the Fourier variable ξ belongs to $\mathbb{R}^{d(d+1)}$. In this notation, Theorem 1.4 for $\sigma \geq 0$ becomes

Theorem 1.5. *For solutions $u_1, \ldots, u_{d+1}$ of the Schrödinger equation, with initial data $f_1, \ldots, f_{d+1}$ respectively, and for all $\sigma \geq 0$,*

$$\begin{aligned}\int_{\substack{x_1+\cdots+x_{d+1}=0 \\ t_1+\cdots+t_{d+1}=0}} &||\rho(\nabla_x)|^\sigma(u_1(x_1,t_1)\cdots u_{d+1}(x_{d+1},t_{d+1}))|^2dxdt \\ &= \frac{1}{2^d(2\pi)^{d(d+1)}}\int_{(\mathbb{R}^d)^{d+1}} \frac{|\widehat{f}_1(\xi_1)|^2\cdots|\widehat{f}_{d+1}(\xi_{d+1})|^2}{|\rho(\xi)|^{1-2\sigma}}d\xi.\end{aligned}$$

Setting $\sigma = \frac{1}{2}$ is particularly natural, as it reduces to the following:

Corollary 1.6.

$$\int_{\substack{x_1+\cdots+x_{d+1}=0\\ t_1+\cdots+t_{d+1}=0}} ||\rho(\nabla_x)|^{1/2}(u_1(x_1,t_1)\cdots u_{d+1}(x_{d+1},t_{d+1}))|^2 dxdt = \frac{1}{2^d}\|f_1\|_2^2\cdots\|f_{d+1}\|_2^2. \tag{1.17}$$

The case $d=1$ of Corollary 1.6 is due to Ozawa and Tsutsumi [13], and is more usually stated as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |D_x^{1/2}(u_1\overline{u_2})(x,t)|^2 dxdt = \frac{1}{2}\|f_1\|_2^2\|f_2\|_2^2, \tag{1.18}$$

where D_x denotes the scalar derivative operator with Fourier multiplier $|\xi|$. Notice that the complex conjugate and fractional derivative appearing here are encoded in the space-time reflection resulting from the restriction $x_1+x_2=t_1+t_2=0$ in (1.17). Bilinear extensions of (1.18) to higher dimensions are also natural, although these cease to be identities; see [4] for further discussion.

As our proof of Theorem 1.5 reveals, the $\sigma=1$ case may be formulated as

$$\begin{aligned}&\int_{\substack{x_1+\cdots+x_{d+1}=0\\ t_1+\cdots+t_{d+1}=0}} |\rho(\nabla_x)(u_1(x_1,t_1)\cdots u_{d+1}(x_{d+1},t_{d+1}))|^2 dxdt\\ &\qquad = \frac{1}{2^d(2\pi)^{d(d+1)}}\int_{(\mathbb{R}^d)^{d+1}} |\widehat{f}_1(\xi_1)|^2\cdots|\widehat{f}_{d+1}(\xi_{d+1})|^2|\rho(\xi)|d\xi,\end{aligned} \tag{1.19}$$

making it somewhat special since it involves only classical derivatives of the solutions. In [14] (see also [19]), it was shown how to deduce the classical $d=1$ case of (1.19) from certain bilinear virial identities, avoiding explicit reference to the u_j as Fourier extension operators. This convexity-based approach has the noteworthy advantage of applying to certain nonlinear Schrödinger equations, and it may be interesting to extend this approach to (1.19) in higher dimensions. We do not pursue this here.

Organisation of the paper. In Section 2 we give a proof of Theorems 1.3 and 1.4 (thus also proving Theorems 1.1 and 1.2). Finally, in Sections 4 and 5 we establish a version of Theorem 1.1 in the context of more general oscillatory integral operators.

Acknowledgments. We thank Neal Bez, Tony Carbery, Taryn Flock, Susana Gutiérrez and Alessio Martini for many helpful discussions surrounding this work.

2. The proof of Theorem 1.3

The proof we present follows the same lines as the classical case $n=2$: a suitable change of variables that allows the multilinear extension operator to be expressed as a Fourier transform, followed by Plancherel's theorem.

We have

$$T_\sigma(g_1,\ldots,g_n)(x_1,\ldots,x_n) = \int_{(\mathbb{R}^{n-1})^n} e^{i(x_1'\cdot\xi_1+\cdots+x_n'\cdot\xi_n)}e^{i(x_{1n}\phi_1(\xi_1)+\cdots+x_{nn}\phi_n(\xi_n))}G(\xi)d\xi,$$

where $x_j=(x_j',x_{jn})\in\mathbb{R}^{n-1}\times\mathbb{R}$, for each j, and

$$G(\xi) := \left|\det\begin{pmatrix} 1 & \cdots & 1\\ \nabla\phi_1(\xi_1) & \cdots & \nabla\phi_n(\xi_n)\end{pmatrix}\right|^\sigma g_1(\xi_1)\cdots g_n(\xi_n).$$

On the subspace $x_1 + \cdots + x_n = 0$ we therefore have

$$\begin{aligned} E_1 g_1(x_1) \cdots E_n g_n(x_n) &= E_1 g_1(x_1) \cdots E_n g_n(-x_1 - \cdots - x_{n-1}) \\ &= \int_{(\mathbb{R}^{n-1})^n} e^{i(x_1' \cdot (\xi_1 - \xi_n) + \cdots + x_{n-1}' \cdot (\xi_{n-1} - \xi_n))} \\ &\quad \times e^{i(x_{1n}(\phi_1(\xi_1) - \phi_n(\xi_n)) + \cdots + x_{(n-1)n}(\phi_{n-1}(\xi_{n-1}) - \phi_n(\xi_n)))} G(\xi) d\xi. \end{aligned}$$

We now make the change of variables $\eta_j = \xi_j - \xi_n$ for each $1 \leq j \leq n-1$, so that

$$\begin{aligned} &E_1 g_1(x_1) \cdots E_n g_n(-x_1 - \cdots - x_{n-1}) \\ &= \int_{(\mathbb{R}^{n-1})^n} e^{i(x_1' \cdot \eta_1 + \cdots + x_{n-1}' \cdot \eta_{n-1})} \\ &\times e^{i(x_{1n}(\phi_1(\eta_1 + \xi_n) - \phi_n(\xi_n)) + \cdots + x_{(n-1)n}(\phi_{n-1}(\eta_{n-1} + \xi_n) - \phi_n(\xi_n)))} \\ &\times G(\eta_1 + \xi_n, \ldots, \eta_{n-1} + \xi_n, \xi_n) d\eta_1 \cdots d\eta_{n-1} d\xi_n. \end{aligned}$$

Applying Plancherel's theorem in the variables $x_1', \ldots, x_{n-1}'$ gives

$$\begin{aligned} &\int_{x_1 + \cdots + x_n = 0} |E_1 g_1(x_1)|^2 \cdots |E_n g_n(x_n)|^2 dx \\ &= (2\pi)^{(n-1)^2} \int \left| \int e^{i(x_{1n}(\phi_1(\eta_1 + \xi_n) - \phi_n(\xi_n)) + \cdots + x_{(n-1)n}(\phi_{n-1}(\eta_{n-1} + \xi_n) - \phi_n(\xi_n)))} \right. \\ &\quad \left. \times G(\eta_1 + \xi_n, \ldots, \eta_{n-1} + \xi_n, \xi_n) d\xi_n \right|^2 d\eta_1 \cdots d\eta_{n-1} dx_{1n} \cdots dx_{(n-1)n}. \end{aligned} \tag{2.1}$$

For fixed $\eta_1, \ldots, \eta_{n-1}$ we make the change of variables $\xi_n \mapsto t$, where $t_j = \phi_j(\eta_j + \xi_n) - \phi_n(\xi_n)$ for $1 \leq j \leq n-1$. This map is injective on the support of $G_\eta := G(\eta_1 + \cdot, \ldots, \eta_{n-1} + \cdot, \cdot)$. Indeed, if not, then there exist $\xi_1 \neq \xi_2$ in the support of G_η (implying that $\eta_j + \xi_1, \eta_j + \xi_2$ are both in the support of g_j, for all $j = 1, \ldots, n-1$), such that

$$\phi_j(\eta_j + \xi_1) - \phi_n(\xi_1) = \phi_j(\eta_j + \xi_2) - \phi_n(\xi_2) \text{ for all } j = 1, \ldots, n-1,$$

i.e. such that

$$\phi_1(\eta_1 + \xi_1) - \phi_1(\eta_1 + \xi_2) = \ldots = \phi_{n-1}(\eta_{n-1} + \xi_1) - \phi_{n-1}(\eta_{n-1} + \xi_2) = \phi_n(\xi_1) - \phi_n(\xi_2).$$

Note that, for all $j = 1, \ldots, n-1$, the line segment ℓ_j connecting $\eta_j + \xi_1$ with $\eta_j + \xi_2$ is just a parallel translate of the line segment ℓ_n connecting ξ_1 with ξ_2. Of course, for all $j = 1, \ldots, n$, ℓ_j is contained in the convex hull of the support of g_j, and so by our hypotheses, the determinant in the statement of Theorem 1.1 is non-zero whenever $\xi_j \in \ell_j$ for all $1 \leq j \leq n$. By the mean value theorem for each ϕ_j on the line segment ℓ_j, it follows that, for all $j = 1, \ldots, n$, there exists $c_j \in \ell_j$, such that the directional derivative of ϕ_j at c_j, in direction $\xi_1 - \xi_2$, has the same value for all j. In other words,

$$\nabla \phi_j(c_j) \cdot (\xi_1 - \xi_2) = c \text{ for all } j = 1, \ldots, n,$$

for some constant $c \in \mathbb{R}$. Therefore,

$$(1, \nabla \phi_j(c_j)) \cdot (-c, \xi_1 - \xi_2) = 0 \text{ for all } j = 1, \ldots, n.$$

Since $c_j \in \ell_j$ for all j, the vectors $(1, \nabla \phi_j(c_j))$, $j = 1, \ldots, n$, span $\mathbb{R}^n$; thus

$$(-c, \xi_1 - \xi_2) = 0,$$

which is a contradiction, since $\xi_1 \neq \xi_2$. Therefore, our map is injective. Moreover, the Jacobian determinant of the transformation $\xi_n \mapsto t$ is simply

$$\frac{\partial t}{\partial \xi_n} = \left| \det \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \nabla \phi_1(\eta_1 + \xi_n) & \cdots & \nabla \phi_{n-1}(\eta_{n-1} + \xi_n) & \nabla \phi_n(\xi_n) \end{pmatrix} \right|,$$

which does not vanish on the support of G. It follows that

$$\begin{aligned}\int_{x_1+\cdots+x_n=0} &|E_1g_1(x_1)|^2\cdots|E_ng_n(x_n)|^2dx\\ &=(2\pi)^{(n-1)^2}\int\Big|\int e^{i(t_1x_{1n}+\cdots+t_{n-1}x_{(n-1)n})}\\ &\quad\times G(\eta_1+\xi_n,\ldots,\eta_{n-1}+\xi_n,\xi_n)\left(\frac{\partial t}{\partial\xi_n}\right)^{-1}dt\Big|^2dx_{1n}\cdots dx_{(n-1)n}d\eta_1\cdots d\eta_{n-1},\end{aligned}$$

which by Plancherel's theorem again, becomes

$$(2\pi)^{n(n-1)}\int\left|G(\eta_1+\xi_n,\ldots,\eta_{n-1}+\xi_n,\xi_n)\left(\frac{\partial t}{\partial\xi_n}\right)^{-1}\right|^2dt\,d\eta_1\cdots d\eta_{n-1}.$$

Undoing both of the changes of variables above, this expression becomes

$$\begin{aligned}(2\pi)^{n(n-1)}&\int|G(\xi_1,\ldots,\xi_{n-1},\xi_n)|^2\left|\left(\frac{\partial t}{\partial\xi_n}\right)\right|^{-1}d\xi\\ &=(2\pi)^{n(n-1)}\int\frac{|g_1(\xi_1)|^2\cdots|g_n(\xi_n)|^2}{\left|\det\begin{pmatrix}1&\cdots&1\\ \nabla\phi_1(\xi_1)&\cdots&\nabla\phi_n(\xi_n)\end{pmatrix}\right|^{1-2\sigma}}d\xi,\end{aligned}$$

as claimed.

3. The proof of Theorem 1.4

Following the proof of Theorem 1.3, we reach (2.1) and apply the same change of variables $\xi_n\mapsto t$, which, in this case, is explicitly given by

$$t_j=\phi_j(\eta_j+\xi_n)-\phi_n(\xi_n)=|\eta_j|^2+2\eta_j\cdot\xi_n.$$

For every $(\eta_1,\ldots,\eta_{n-1})\in\left(\mathbb{R}^{n-1}\right)^{n-1}$ that span $\mathbb{R}^{n-1}$ (that is, for almost every $(\eta_1,\ldots,\eta_{n-1})$), the above affine transformation is globally injective, with Jacobian determinant

$$2^{n-1}\eta_1\wedge\ldots\wedge\eta_{n-1}=\det\begin{pmatrix}1&\cdots&1&1\\ \nabla\phi_1(\eta_1+\xi_n)&\cdots&\nabla\phi_{n-1}(\eta_{n-1}+\xi_n)&\nabla\phi_n(\xi_n)\end{pmatrix}\neq0.$$

The proof now concludes as in the proof of Theorem 1.3.

4. Variable coefficient generalisations

It is natural to attempt to generalise Theorem 1.1 (at the level of an inequality) to encompass families of more general oscillatory integral operators of the form

$$T_\lambda f(x)=\int_{\mathbb{R}^{n-1}}e^{i\lambda\Phi(x,\xi)}\psi(x,\xi)f(\xi)d\xi,$$

where Φ is a smooth real-valued phase function, ψ is a compactly-supported bump function, and λ is a large real parameter.

To this end, suppose that we have n of these operators, $T_{1,\lambda},\ldots,T_{n,\lambda}$ with phases $\Phi_1,\ldots,\Phi_n$ (and cutoff functions $\psi_1,\ldots,\psi_n$). An appropriate transversality condition is that the kernels of the mappings $d_\xi d_x\Phi_1,\ldots,d_\xi d_x\Phi_n$ span $\mathbb{R}^n$ at every point. In order to be more precise let

$$X(\Phi_j):=\bigwedge_{\ell=1}^{n-1}\frac{\partial}{\partial\xi_\ell}\nabla_x\Phi_j$$

for each $1 \leq j \leq n$; by (Hodge) duality we may interpret each $X(\Phi_j)$ as an $\mathbb{R}^n$-valued function on $\mathbb{R}^n \times \mathbb{R}^{n-1}$. In the extension case where $\Phi_j(x,\xi) = x \cdot \Sigma_j(\xi)$, observe that $X(\Phi_j)(x,\xi)$ is simply a vector normal to the surface S_j at the point $\Sigma_j(\xi)$. A natural transversality condition to impose on the general phases $\Phi_1, \ldots, \Phi_n$ is thus

$$\det\left(X(\Phi_1)(x_1,\xi_1), \ldots, X(\Phi_n)(x_n,\xi_n)\right) \gtrsim 1 \tag{4.1}$$

for all $(x_1,\xi_1) \in \operatorname{supp}(\psi_1), \ldots, (x_n,\xi_n) \in \operatorname{supp}(\psi_n)$. Under this condition it is shown in [5] that

$$\Big\| \prod_{j=1}^{n} T_{j,\lambda} f_j \Big\|_{L^{\frac{2}{n-1}}(\mathbb{R}^n)} \leq C_\varepsilon \lambda^{-\frac{n(n-1)}{2}+\varepsilon} \prod_{j=1}^{n} \|f_j\|_{L^2(\mathbb{R}^{n-1})}, \tag{4.2}$$

generalising (1.8). Here we establish the corresponding generalisation of (1.6).

Theorem 4.1. *Assuming* (4.1)

$$\int_{x_1+\cdots+x_n=0} |T_{1,\lambda} f_1(x_1)|^2 \cdots |T_{n,\lambda} f_n(x_n)|^2 dx \lesssim \lambda^{-n(n-1)} \|f_1\|_2^2 \cdots \|f_n\|_2^2. \tag{4.3}$$

Of course (4.2) with $\varepsilon = 0$ is the same as (4.3) when $n = 2$. Theorem 4.1 is well-known for $n = 2$, and this is a simple exercise using Hörmander's theorem for nondegenerate oscillatory integral operators. More precisely, observe that when $n = 2$,

$$T_{1,\lambda} f_1(x) T_{2,\lambda} f_2(-x) = \int_{(\mathbb{R})^2} e^{i\lambda\Psi(x,\xi)} \psi_1(x,\xi_1)\psi_2(x,\xi_2) f_1(\xi_1) f_2(\xi_2) d\xi_1 d\xi_2,$$

where $\Psi(x,\xi) := \Phi_1(x,\xi_1) + \Phi_2(-x,\xi_2)$, and notice that $\det \operatorname{Hess} \Psi$ coincides with the nonzero quantity in the hypothesis (4.1). Hence (4.3) holds for $n = 2$ by Hörmander's theorem; see [9] and [16] for further context and discussion. As may be expected from Section 2, the higher-dimensional case of Theorem 4.1 will follow by a similar argument, although some additional linear-algebraic ingredients will be required.

5. Proof of Theorem 4.1

We begin by writing

$$\begin{aligned} &T_{1,\lambda} f_1(x_1) \cdots T_{n-1,\lambda} f_{n-1}(x_{n-1}) T_{n,\lambda} f_n(-x_1 - \cdots - x_{n-1}) \\ &\qquad = \int_{(\mathbb{R}^{n-1})^n} e^{i\lambda\Psi(x,\xi)} \psi_1(x,\xi_1) \cdots \psi_n(x,\xi_n) f_1(\xi_1) \cdots f_n(\xi_n) d\xi, \end{aligned}$$

where $\Psi : (\mathbb{R}^n)^{n-1} \times (\mathbb{R}^{n-1})^n \to \mathbb{R}$ is given by

$$\Psi(x,\xi) = \Phi_1(x_1,\xi_1) + \cdots + \Phi_{n-1}(x_{n-1},\xi_{n-1}) + \Phi_n(-x_1 - \cdots - x_{n-1}, \xi_n). \tag{5.1}$$

The difficulty now is that $\operatorname{Hess} \Psi$ is no longer an $n \times n$ matrix, and so some work has to be done to see that its determinant coincides with that in the hypothesis (4.1). Once this is done Theorem 4.1 follows by a direct application of Hörmander's theorem as in the case $n = 2$. Thus matters are reduced to showing the following.

Proposition 5.1.

$$\det \operatorname{Hess} \Psi(x,\xi) = (-1)^{n-1+\frac{(n-1)^2(n-2)}{2}} \det\big(X(\Phi_1)(x_1,\xi_1) \ldots, X(\Phi_n)(-x_1 - \cdots - x_{n-1}, \xi_n)\big);$$

the coefficient above equals 1 *for* $n \equiv 0, 1, 3 \pmod 4$*, and* -1 *for* $n \equiv 2 \pmod 4$.

Note that $\operatorname{Hess}\Psi(x,\xi)$ is of the form

$$\begin{pmatrix} A^{(1)}_{n\times(n-1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(n)}_{n\times(n-1)} \\ \mathbf{0} & A^{(2)}_{n\times(n-1)} & \mathbf{0} & \dots & \mathbf{0} & A^{(n)}_{n\times(n-1)} \\ \mathbf{0} & \mathbf{0} & A^{(3)}_{n\times(n-1)} & \dots & \mathbf{0} & A^{(n)}_{n\times(n-1)} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(n-1)}_{n\times(n-1)} & A^{(n)}_{n\times(n-1)} \end{pmatrix},$$

where

$$A^{(i)}_{n\times(n-1)} = \operatorname{Hess}\Phi_i(x_i,\xi_i) \text{ for } i=1,\dots,n-1$$

and

$$A^{(n)}_{n\times(n-1)} = -\operatorname{Hess}\Phi_n(-x_1-\cdots-x_{n-1},\xi_n).$$

Proposition 5.1 is therefore a special case of Lemma 5.2 that follows, for $k=n-1$.

Lemma 5.2. *For $n\geq 2$ and $1\leq k\leq n-1$, let $A^{(1)}_{n\times(n-1)},\dots,A^{(k)}_{n\times(n-1)}$ be $n\times(n-1)$-block matrices, and $A^{(k+1)}_{n\times k}$ be an $n\times k$-block matrix. Let*

$$M_{n,k} := \begin{pmatrix} A^{(1)}_{n\times(n-1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(k+1)}_{n\times k} \\ \mathbf{0} & A^{(2)}_{n\times(n-1)} & \mathbf{0} & \dots & \mathbf{0} & A^{(k+1)}_{n\times k} \\ \mathbf{0} & \mathbf{0} & A^{(3)}_{n\times(n-1)} & \dots & \mathbf{0} & A^{(k+1)}_{n\times k} \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(k)}_{n\times(n-1)} & A^{(k+1)}_{n\times k} \end{pmatrix}.$$

Then,

$$\det M_{n,k} = (-1)^{(n-1)\frac{k(k-1)}{2}}\Lambda_1^*\wedge\cdots\wedge\Lambda_k^*\wedge\Lambda_{k+1}^*,$$

where Λ_i^* is the (Hodge) dual of the wedge product Λ_i of the columns of $A^{(i)}_{n\times(n-1)}$, for all $i=1,\dots,k$, and $\Lambda_{k+1}^*$is the dual of the wedge product Λ_{k+1} of the columns of $A^{(k+1)}_{n\times k}$.

Proof. For any $1\leq k\leq n-1$, we denote by C_i the i-th column of $A^{(k+1)}_{n\times k}$. By definition,

$\Lambda_1^*\wedge\ldots\wedge\Lambda_k^*\wedge\Lambda_{k+1}^* =$

$$= \det\begin{pmatrix} \langle\Lambda_1^*,C_1\rangle & \langle\Lambda_1^*,C_2\rangle & \dots & \langle\Lambda_1^*,C_k\rangle \\ \langle\Lambda_2^*,C_1\rangle & \langle\Lambda_2^*,C_2\rangle & \dots & \langle\Lambda_2^*,C_k\rangle \\ & & \vdots & \\ \langle\Lambda_k^*,C_1\rangle & \langle\Lambda_k^*,C_2\rangle & \dots & \langle\Lambda_k^*,C_k\rangle \end{pmatrix}$$

$$= \det\begin{pmatrix} \Lambda_1\wedge C_1 & \Lambda_1\wedge C_2 & \dots & \Lambda_1\wedge C_k \\ \Lambda_2\wedge C_1 & \Lambda_2\wedge C_2 & \dots & \Lambda_2\wedge C_k \\ & & \vdots & \\ \Lambda_k\wedge C_1 & \Lambda_k\wedge C_2 & \dots & \Lambda_k\wedge C_k \end{pmatrix}.$$

It thus suffices to show that, for any $n \geq 2$ and $k \leq n$,

$$\det M_{n,k} = (-1)^{(n-1)\frac{k(k-1)}{2}} \det \begin{pmatrix} \Lambda_1 \wedge C_1 & \Lambda_1 \wedge C_2 & \dots & \Lambda_1 \wedge C_k \\ \Lambda_2 \wedge C_1 & \Lambda_2 \wedge C_2 & \dots & \Lambda_2 \wedge C_k \\ & & \vdots & \\ \Lambda_k \wedge C_1 & \Lambda_k \wedge C_2 & \dots & \Lambda_k \wedge C_k \end{pmatrix}. \tag{5.2}$$

We prove (5.2) by induction on k.

Indeed, (5.2) clearly holds for $k = 1$; in that case,

$$\det M_{n,1} = \det \begin{pmatrix} A^{(1)}_{n\times(n-1)} & C_1 \end{pmatrix} = \Lambda_1 \wedge C_1.$$

Let $k \geq 2$, and let us assume that (5.2) holds for $k-1$; we now deduce it for k. We first observe that

$$\det M_{n,k} = \sum_{i=1}^{k} \det B_i,$$

where

$$B_i = \begin{pmatrix} A^{(1)}_{n\times(n-1)} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & 0 & \dots & 0 & C_i & 0 & 0 \\ \mathbf{0} & A^{(2)}_{n\times(n-1)} & \mathbf{0} & \dots & \mathbf{0} & C_1 & \dots & C_{i-1} & 0 & C_{i+1} & C_k \\ \mathbf{0} & \mathbf{0} & A^{(3)}_{n\times(n-1)} & \dots & \mathbf{0} & C_1 & \dots & C_{i-1} & 0 & C_{i+1} & C_k \\ \vdots & \vdots & & \ddots & \vdots & & & \vdots & & & \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(k)}_{n\times(n-1)} & C_1 & \dots & C_{i-1} & 0 & C_{i+1} & C_k \end{pmatrix}.$$

Indeed, let us focus on the last k columns of $M_{n,k}$. By writing the i-th of these columns in the form $(C_i, 0, \dots, 0) + (0, C_i, \dots, C_i)$, for all $i = 1, \dots, k$, multilinearity of the determinant implies that

$$\det M_{n,k} = \sum_{i=1}^{k} \det B_i + \sum_{i\neq j} \det \Gamma_{i,j},$$

where $\Gamma_{i,j}$ is an $nk \times nk$ matrix with $(C_i, 0, \dots, 0)$ and $(C_j, 0, \dots, 0)$ as the i-th and j-th column of its right $nk \times k$ block. These columns, together with the columns of $A^{(1)}_{n\times(n-1)}$, form a set of $n+1$ vectors in $\mathbb{R}^{n-1}$, and are thus linearly dependent, forcing the determinant of $\Gamma_{i,j}$ to be zero.

We now swap the column $(C_i, 0, \dots, 0)$ consecutively with columns on its immediate left until it becomes the n-th column; there are $i - 1 + (n-1)(k-1)$ such swaps involved, therefore

$$\det M_{n,k} = (-1)^{(n-1)(k-1)} \sum_{i=1}^{k} (-1)^{i-1} \det D_i, \tag{5.3}$$

where D_i is the matrix we get from B_i by the above process; in other words,

$$D_i = \begin{pmatrix} A^{(1)}_{n\times(n-1)} & C_i & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & A^{(2)}_{n\times(n-1)} & \mathbf{0} & \dots & \mathbf{0} & \widehat{A}^i_{n,k-1} \\ & \mathbf{0} & \mathbf{0} & A^{(3)}_{n\times(n-1)} & \dots & \mathbf{0} & \widehat{A}^i_{n,k-1} \\ & \vdots & \vdots & & \ddots & \vdots & \vdots \\ & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A^{(k)}_{n\times(n-1)} & \widehat{A}^i_{n,k-1} \end{pmatrix},$$

where $\widehat{A}^i_{n,k-1}$ denotes the $n \times (k-1)$ matrix that we get from $A^{(k+1)}_{n\times k}$ after deleting its i-th column. Since $\begin{pmatrix} A^{(1)}_{n\times(n-1)} & C_i \end{pmatrix}$ is a square matrix, we obtain

$$\det D_i = (\Lambda_1 \wedge C_i) \cdot \det \begin{pmatrix} A^{(2)}_{n\times(n-1)} & \mathbf{0} & \dots & \mathbf{0} & \widehat{A}^i_{n,k-1} \\ \mathbf{0} & A^{(3)}_{n\times(n-1)} & \dots & \mathbf{0} & \widehat{A}^i_{n,k-1} \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & A^{(k)}_{n\times(n-1)} & \widehat{A}^i_{n,k-1} \end{pmatrix}$$

$$= (-1)^{(n-1)\frac{(k-1)(k-2)}{2}} (\Lambda_1 \wedge C_i) \cdot \det \begin{pmatrix} \Lambda_2 \wedge C_1 & \dots & \Lambda_2 \wedge C_{i-1} & \Lambda_2 \wedge C_{i+1} & \dots & \Lambda_2 \wedge C_k \\ \Lambda_3 \wedge C_1 & \dots & \Lambda_3 \wedge C_{i-1} & \Lambda_3 \wedge C_{i+1} & \dots & \Lambda_3 \wedge C_k \\ & & & \vdots & & \\ \Lambda_k \wedge C_1 & \dots & \Lambda_k \wedge C_{i-1} & \Lambda_k \wedge C_{i+1} & \dots & \Lambda_k \wedge C_k \end{pmatrix};$$

the last equality holds by the inductive hypothesis. Plugging this into (5.3), we obtain (5.2) for this k.

□

References

[1] J. Bennett, *Aspects of multilinear harmonic analysis related to transversality*, Harmonic analysis and partial differential equations, 1–28, Contemp. Math., 612, Amer. Math. Soc., Providence, RI, 2014.

[2] J. Bennett, N. Bez, T. C. Flock, S. Gutiérrez and M. Iliopoulou, *A sharp k-plane norm Strichartz inequality for the Schrödinger equation*, arXiv:1611.03692

[3] J. Bennett, N. Bez, T. C. Flock and S. Lee, *Stability of the Brascamp–Lieb constant and applications*, to appear in Amer. J. Math.

[4] J. Bennett, N. Bez, C. Jeavons and N. Pattakos, *On sharp bilinear Strichartz estimates of Ozawa-Tsutsumi type*, J. Math. Soc. Japan **69** (2017), no. 2, 459–476.

[5] J. Bennett, A. Carbery, T. Tao, *On the multilinear restriction and Kakeya conjectures*, Acta Math. **196** (2006), 261–302.

[6] J. Bourgain, L. Guth, *Bounds on oscillatory integral operators based on multilinear estimates*, Geom. Funct. Anal. **21** (2011), 1239–1295.

[7] J. Bourgain, C. Demeter, *The proof of the l^2 Decoupling Conjecture*, Ann. Math. **182** (2015), 351–389.

[8] A. Carbery and S. Valdimarsson, *The multilinear Kakeya theorem via the Borsuk–Ulam theorem*, J. Funct. Anal. **364** (2013), 1643–1663.

[9] L. Carleson, P. Sjölin, *Oscillatory integrals and multiplier problem for the disc*, Studia Mathematica **44** (1972), 287–299.

[10] S. W. Drury, *Estimates for a multilinear form on the sphere*, Mathematical Proceedings of the Cambridge Philosophical Society, **104** (1988), 533–537.

[11] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[12] P. Mattila, *Fourier analysis and Hausdorff dimension*, Cambridge studies in advanced mathematics **150**, 2015.

[13] T. Ozawa, Y. Tsutsumi, *Space-time estimates for null gauge forms and nonlinear Schrödinger equations*, Differential Integral Equations **11** (1998), 201–222.

[14] F. Planchon, L. Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), 261–290.

[15] J. Ramos, *The trilinear restriction estimate with sharp dependence on the transversality*, arXiv:1601.05750.

[16] E. M. Stein, Harmonic Analysis, Princeton University Press, 1993.

[17] T. Tao, *Some recent progress on the restriction conjecture* in Fourier Analysis and Convexity, 217–243, Appl. Numer. Harmon. Anal., Birkhäuser, Boston MA, 2004.

[18] T. Tao, A. Vargas, L. Vega, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc. **11** (1998), 967–1000.

[19] L. Vega, *Bilinear virial identities and oscillatory integrals*, Harmonic analysis and partial differential equations, 219–232, Contemp. Math., 505, Amer. Math. Soc., Providence, RI, 2010.

[20] R. Zhang, *The endpoint perturbed Brascamp–Lieb inequality with examples*, arXiv:1510.09132.

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