Some properties of Specht modules for the wreath product of symmetric groups

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Abstract

We investigate a class of modules for the wreath product $S_m \wr S_n$ of two symmetric groups which are analogous to the Specht modules of the symmetric group, and prove a range of properties for these modules which demonstrate this analogy. In particular, we prove analogues of the Specht module branching rule, we obtain results on homomorphisms and extensions between these modules, and, over an algebraically closed field whose characteristic is neither 2 nor 3, we prove that, if a module for $S_m \wr S_n$ has a filtration by these Specht module analogues, then the multiplicities with which they occur do not depend on the choice of a filtration.

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List of key definitions and notation

k	a field	19
$\otimes$	tensor product over a field, or inner tensor product of group modules	19,29
$\mathrm{char}(k)$	characteristic of the field k	19
$\mathcal{F}, \mathcal{F}_{\langle - \rangle}$	module filtration	20
$\boxtimes$	outer tensor product of group or algebra modules	23,29
kG	group algebra of a group G over a field k	28
e	identity element of a group	28
$\mathbb{1}_G$	trivial group module	28
U^*	contragredient dual of a group module	28
H^g	conjugate subgroup of a subgroup H by a group element g	32
X^g	kH^g-module conjugate to the kH-module X	32
$\gamma \vDash n$	γ is a composition of n	35
$\lambda \vdash n$	λ is a partition of n	35

$\lvert\gamma\rvert$	size of the composition γ	35
Ω_n^t	set of all compositions of n with length t	35
Λ_n	set of partitions of n	35
$[n,i]$	composition of n which is n in the i^{th} place and 0 elsewhere	35
$\underline{\gamma}$	multicomposition or multipartition	37
$\lvert\underline{\gamma}\rvert$	shape of a multicomposition	37
$\lvert\lvert\underline{\gamma}\rvert\rvert$	size of a multicomposition	37
$\underline{\Lambda}_n$	set of multipartitions of n	37
$\underline{\Lambda}_n^t$	set of multipartitions of n with length t	37
$[\gamma,i]$	multicomposition of n with $\gamma \vDash n$ in the i^{th} place and () elsewhere	37
$\underline{\Omega}(n;a_1,\ldots,a_t)$	the set of all multicompositions $\underline{\alpha}=(\alpha^1,\ldots,\alpha^t)$ of n of length t, such that the length of α^i is a_i for $i=1,\ldots,t$	38
$[[n,1],i\,;a_1,\ldots,a_t]$	the element of $\underline{\Omega}(n;a_1,\ldots,a_t)$ with i^{th} component $(n,0,\ldots,0)$ and with a tuple of zeros in every other place	38
$\underline{\underline{\gamma}}$	tuple of multicompositions	38
$\lvert\underline{\underline{\gamma}}\rvert$, $\lvert\lvert\underline{\underline{\gamma}}\rvert\rvert$, $\lvert\lvert\lvert\underline{\underline{\gamma}}\rvert\rvert\rvert$	see text	39
$[[\eta,1],i\,;a_1,\ldots,a_t]$	tuple of multicompositions with $(\eta,(),\ldots,())$ in the i^{th} place and where all other entries are tuples of empty compositions, such that the l^{th} entry has length a_l	40
$\circ$	concatenation of tuples	40
$\triangleright$	dominance order on compositions and multicompositions	36,38

$>$	lexicographic order on compositions	36
$\gtrdot$	reverse lexicographic order on compositions (also the reverse of any order denoted by $>$)	37
λ'	conjugate partition	41
$\underline{\nu}'$	conjugate multipartition $\big((\nu^1)', \ldots, (\nu^t)'\big)$	41
S_α	Young subgroup associated to $\alpha \vDash n$	44
$X{\uparrow}_\alpha^n, Y{\downarrow}_\alpha^n$	induction and restriction between S_n and the Young subgroup S_α	44
$\mathrm{sgn}(\sigma)$	sign function on S_n	44
Sgn_n	sign module for S_n	45
Sgn_H	restriction of Sgn_n to the subgroup H of S_n	45
Sgn_α	restriction of Sgn_n to the Young subgroup S_α of S_n for $\alpha \vDash n$	45
–	inversion of a permutation	45
$\mathrm{len}(\sigma)$	length of a permutation	45
M^α	Young permutation module for kS_n associated to $\alpha \vDash n$	47
$\tau(\alpha)$	basic standard α-tableau	47
S^λ	Specht module for kS_n associated to $\lambda \vdash n$	48
D^λ	simple kS_n-module associated to a p-regular partition $\lambda \vdash n$	49
$K(\nu, \lambda)$	Kostka number, where $\nu, \lambda \vdash n$	50
$c^\lambda_{\alpha,\beta}$	Littlewood-Richardson coefficient, where λ, α, β are partitions	52

$c(\lambda;\underline{\alpha})$	Littlewood-Richardson coefficient, where λ is a partition and $\underline{\alpha}$ a multipartition	54
$S(\underline{\alpha})$	the external tensor product $S^{\alpha^1} \boxtimes \cdots \boxtimes S^{\alpha^t}$, a $kS_{\lvert\underline{\alpha}\rvert}$-module, where $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ is a multipartition	57
$M(\underline{\alpha})$	the external tensor product $M^{\alpha^1} \boxtimes \cdots \boxtimes M^{\alpha^t}$, a $kS_{\lvert\underline{\alpha}\rvert}$-module, where $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ is a multipartition	58
$S_m \wr S_n$	the wreath product of S_n on S_m	68
$(\sigma; \alpha_1, \alpha_2, \ldots, \alpha_n)$	element of the wreath product $S_m \wr S_n$ (or more generally of a wreath product $A \wr S_n$ for A a k-algebra)	68, 96
$G \wr H$	the subgroup of $S_m \wr S_n$ obtained from $G \leqslant S_m$, $H \leqslant S_n$	69
$X\big\uparrow_{m\wr\gamma}^{m\wr n}, Y\big\downarrow_{m\wr\gamma}^{m\wr n}$	induction and restriction for the wreath product	69
$S_{\underline{\gamma}}$	Young subgroup of S_n associated to a multicomposition $\underline{\gamma}$	70
$S_{\underline{\underline{\gamma}}}$	Young subgroup of S_n associated to a tuple $\underline{\underline{\gamma}}$ of multicompositions	71
$\mu^1, \ldots, \mu^r$	the distinct partitions of m	72
$W_{\underline{\gamma}}$	subgroup of $S_m \wr S_n$ associated to a multicomposition $\underline{\gamma}$	72
$W_{\underline{\underline{\gamma}}}$	subgroup of $S_m \wr S_n$ associated to a tuple of multicompositions $\underline{\underline{\gamma}}$	73
$X^{\widetilde{\boxtimes} n}$	module for $k(G \wr S_n)$; see text	74
$\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes} \gamma}$	module for $k(G \wr S_\gamma)$; see text	75
$\mathrm{Inf}_H^{G\wr H} Y$	inflation of a module from kH to $k(G \wr H)$	75
$Z \oslash Y$	module for $k(G \wr H)$; see text	75

$S^{\underline{\nu}}$	Specht module for $k(S_m \wr S_n)$ associated to the multipartition $\underline{\nu}$, $S^{\underline{\nu}} = \left[\left(S^{\mu^1}, \ldots, S^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\nu}|} \oslash \left(S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r}\right)\right] \Big\uparrow_{m\wr\lvert\underline{\nu}\rvert}^{m\wr n}$	82
$M^{\underline{\gamma}}$	Young permutation module for $k(S_m \wr S_n)$ associated to the multicomposition $\underline{\gamma}$, $M^{\underline{\gamma}} = \left[\left(M^{\mu^1}, \ldots, M^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\gamma}|} \oslash \left(M^{\gamma^1} \boxtimes \cdots \boxtimes M^{\gamma^r}\right)\right] \Big\uparrow_{m\wr\lvert\underline{\gamma}\rvert}^{m\wr n}$	82
$a \mapsto a^*$	an anti-involution on a k-algebra, in particular the anti-involution induced on a group algebra by mapping each group element to its inverse	88
$C^{\lambda}_{S,T}, C_{S,T}, C[S,T]$	cellular basis element in a cellular algebra	88,105,106
(Γ, M, C)	tuple of cellular data for a cellular algebra	88
$R_a(T,X)$	coefficient occurring in the decomposition over the cellular basis of a product $C^{\lambda}_{S,T}a$ in a cellular algebra	89
Δ^{λ}	cell module associated to a cell index λ	89
C_T	element of the defining basis of a cell module	89
Γ_0	set of cell indices indexing the simple modules of a cellular algebra	89
L^{λ}	simple module associated to a cell index λ	89
$\mathcal{R}_{\mu}$	for $\mu \vDash n$, the unique system of minimal-length right S_{μ}-coset representatives in S_n	94
$A \wr S_n$	wreath product of the algebra A with S_n	96
$\Theta^{\mu}(\underline{X}, \underline{Y})$	module for $A \wr S_n$; see text	99
$D^{\underline{\lambda}}$	the $k(S_m \wr S_n)$-module $\left[\left(D^{\mu^1}, \ldots, D^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\lambda}|} \oslash \left(D^{\lambda^1} \boxtimes \cdots \boxtimes D^{\lambda^r}\right)\right] \Big\uparrow_{m\wr\lvert\underline{\lambda}\rvert}^{m\wr n}$	121
$[\underline{\epsilon}]$	the multipartition matrix whose $(i,j)^{\text{th}}$ entry is $\underline{\epsilon}^{ij}$	132

$R_i[\underline{\epsilon}]$	the multipartition $\underline{\epsilon}^{i1} \circ \cdots \circ \underline{\epsilon}^{it}$ obtained by concatenating the multipartitions on the i^{th} row of the multipartition matrix $[\underline{\epsilon}]$	132
$C_j[\underline{\epsilon}]$	the multipartition $\underline{\epsilon}^{1j} \circ \cdots \circ \underline{\epsilon}^{sj}$ obtained by concatenating the multipartitions in the j^{th} column of the multipartition matrix $[\underline{\epsilon}]$	132
$R[\underline{\epsilon}]$	the multipartition $R_1[\underline{\epsilon}] \circ \cdots \circ R_s[\underline{\epsilon}]$ obtained from the multipartition matrix $[\underline{\epsilon}]$	132
$C[\underline{\epsilon}]$	the multipartition $C_1[\underline{\epsilon}] \circ \cdots \circ C_t[\underline{\epsilon}]$ obtained from the multipartition matrix $[\underline{\epsilon}]$	132
–	shape of a multipartition matrix	133
$L[\underline{\epsilon}]$	the length matrix of the multipartition matrix $[\underline{\epsilon}]$	133
$\mathrm{Mat}_{\underline{\Lambda}}(L; \alpha\times\beta)$	the set of all multipartition matrices with shape $\alpha\times\beta$ and length matrix L	133
$S(\underline{\epsilon})$	the $kS_{\lvert\underline{\epsilon}\rvert}$-module $S(\underline{\epsilon}^1) \boxtimes \cdots \boxtimes S(\underline{\epsilon}^t)$	135
$c(\eta; \underline{\epsilon})$	the Littlewood-Richardson coefficient $c(\eta; \underline{\epsilon}^1 \circ \cdots \circ \underline{\epsilon}^t)$	135
$S^{\underline{\lambda}}(W_1, \ldots, W_l)$	the $k(S_m \wr S_n)$-module $\Big[\big(W_1, \ldots, W_l\big)^{\widetilde{\boxtimes}\lvert\underline{\lambda}\rvert} \oslash \big(S^{\lambda^1} \boxtimes \cdots \boxtimes S^{\lambda^l}\big)\Big]\Big\uparrow_{m\wr\lvert\underline{\lambda}\rvert}^{m\wr n}$	138
$M(\underline{\eta}; w_1, \ldots, w_s)$	"sparse" multipartition matrix with entries from a multipartition $\underline{\eta}$, in positions given by $w_1, \ldots, w_s$	140
–	tableau of shape α and type γ	155
–	tableau of shape α and type $\underline{\gamma}$	155
τ_{γ}^{α}	standard tableau of shape α and type γ	160
$\tau_{\underline{\gamma}}^{\alpha}$	standard tableau of shape α and type $\underline{\gamma}$	161
–	weakly increasing rows in a tableau	162

–	descent of a permutation	163
$\mathcal{W}^{\alpha}_{\gamma}$	the set of all tableaux of shape α and type γ with weakly increasing rows	166
$\mathcal{W}^{\alpha}_{\underline{\gamma}}$	the set of all tableaux of shape α and type $\underline{\gamma}$ with weakly increasing rows	166
Ω^{α}_{γ}	a complete system of (S_γ, S_α)-double coset representatives in S_n, each of minimal length in its left S_α-coset	166
$\Omega^{\alpha}_{\underline{\gamma}}$	a complete system of $(S_{\underline{\gamma}}, S_\alpha)$-double coset representatives in S_n, each of minimal length in its left S_α-coset	166
$\underline{\underline{\Gamma}}(\tau)$	the tuple of multicompositions associated to a tableau whose entries are pairs of integers	172
$\mathrm{Sgn}_{\beta\wr n}$	the $k(S_\beta \wr S_n)$-module $(\mathrm{Sgn}_\beta)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n$, where β is a composition of m	206
$\widehat{\mathcal{W}}^{\lvert\underline{\nu}\rvert}_{\underline{\gamma}}$	the set of all tableaux τ of shape $\lvert\underline{\nu}\rvert$ and type $\underline{\gamma}$ with weakly increasing rows such that for each $j \in \{1, \ldots, r\}$, no pair $(j, *)$ appears lower than the j ${}^{\text{th}}$ row of τ.	234
$T^{\underline{\lambda}}$	the $k(S_m \wr S_{\lvert\underline{\lambda}\rvert})$-module $\big(S^{\mu^1}, \ldots, S^{\mu^r}\big)^{\widetilde{\boxtimes}\lvert\underline{\lambda}\rvert} \oslash \big(S^{\lambda^1} \boxtimes \cdots \boxtimes S^{\lambda^r}\big)$	180
$\hat{\sigma}$	the element $(\sigma\,;\, e, e, \ldots, e)$ of $S_m \wr S_n$, where $\sigma \in S_n$	225

Chapter 1

Introduction and summary

The representation theory of the symmetric group S_n on the set $\{1, \ldots, n\}$ is of fundamental importance in many branches of mathematics. The study of the symmetric group has an impressive pedigree, with a history stretching back over a century and contributions from many distinguished mathematical figures. In particular, the work of Young, Frobenius and Schur in the early years of the twentieth century helped to lay the foundations of the subject, but it is perhaps best known today in the form introduced by James in the 1970s, in which combinatorial methods and constructions figure prominently.

The main subject of this thesis is a class of modules for the *wreath product* $S_m \wr S_n$ of two symmetric groups (over some field), which are analogous to the *Specht modules* for the symmetric group. Recall that the Specht modules for the symmetric group S_n are a family of combinatorially-defined modules which are indexed by the partitions of n. We shall write the Specht module for S_n which is indexed by the partition λ as S^λ. The Specht modules may be defined in the very general setting of representation theory over a commutative unital ring, but in this thesis we shall be working over a field, and in this setting the Specht modules have many nice properties:

- if the group algebra of the symmetric group is semisimple, then the Specht modules provide a complete list of the isomorphism classes of simple modules without redundancy
- if the group algebra of the symmetric group is not semisimple, then the simple modules arise as the heads of a subset of the Specht modules
- if the characteristic of the field is not 2, then the Specht modules are indecomposable even when they are not simple
- the dimensions of the Specht modules do not depend on the field
- the Specht modules behave well under induction from, and restriction to, important subgroups of the symmetric group, admitting decompositions described by elegant combinatorial *branching rules*.

Because of these and other properties, the Specht modules for S_n have been the subject of intense study for decades, and a large and varied literature has built up around them. Further, the Specht module construction has been generalised in a number of ways, and the properties of the Specht modules have inspired new approaches to various areas of representation theory.

One interesting and recent development in the theory of Specht modules came in [19], in which Hemmer and Nakano demonstrated that, provided that the field of coefficients is algebraically closed and its characteristic is not 2 or 3, then for a module with a filtration by Specht modules, the multiplicities with which the Specht modules occur will be the same for all such filtrations. This result was originally rather surprising, but the "Hemmer-Nakano property" has since been established for other classes of algebra (for example, the Brauer algebra [17]). A highlight of this thesis is a new case of this phenomenon.

Now the wreath product $G \wr S_n$ of a finite group G with a symmetric group S_n is a natural group-theoretic construction with many applications.

In particular, wreath products $S_m \wr S_n$ of two symmetric groups are of great importance in the representation theory of the symmetric group (see for example [5]), and it is the representation theory of $S_m \wr S_n$ which is the principal topic of this thesis. We shall study a class of modules for the wreath product of two symmetric groups which are analogous to the Specht modules of the symmetric group, and we shall justify this analogy by proving that these modules for the wreath product share a range of properties with their symmetric group counterparts. For example, these Specht modules are the *cell modules* of a *cellular algebra* structure on the group algebra of $S_m \wr S_n$, they behave well under induction and restriction, they obey nice combinatorial branching rules, and they exhibit the Hemmer-Nakano property.

In summary, the contents of this thesis are as follows. At the end of each chapter, I have included a brief summary of the original research in that chapter, in order to make clear exactly which material I am claiming as my own work.

Chapter 2 is an introductory chapter in which we recall standard definitions and results on finite-dimensional algebras, group representation theory, and combinatorics.

Chapter 3 recalls basic material on the symmetric group and its Specht modules, before dealing in more depth with some important results about filtrations of symmetric group modules, which will be crucial tools for our later work. Specifically, we shall consider *Young's rule* and the *Littlewood-Richardson filtration rules.* We then recall results on homomorphisms between Specht modules, and further we consider extensions between Specht modules by giving an expanded version of the arguments in [10]. Finally we recall the notion of a *stratifying system* (in the sense given in [11]) and recall the argument given in [10] to demonstrate that the Specht modules for the symmetric group give rise to such a stratifying system. This fact provides an alternative proof of the above-mentioned Hemmer-Nakano property. A major new result of this thesis is that the Specht modules of $S_m \wr S_n$ enjoy the same property, and the proof of this is based on the techniques recalled here.

Chapter 4 recalls the definition and some basic properties of the wreath product $S_m \wr S_n$, and defines some subgroups of the wreath product which will be used later. We then give details of some well-known methods of constructing modules for $S_m \wr S_n$ from modules for S_m and S_n, and use these methods to define our wreath product Specht modules.

Chapter 5 considers the slightly more general situation of the wreath product $A \wr S_n$ of an algebra A (over a field) with a symmetric group, of which the group algebra of $S_m \wr S_n$ is the special case obtained by taking A to be the group algebra of S_m. We show that if A is a cellular algebra, then so is $A \wr S_n$, thus re-proving (with slightly different assumptions) the result of [12]. We consider the cell modules of $A \wr S_n$ in this case, and hence (using the theory of cellular algebras) obtain useful results on the representation theory of $A \wr S_n$. We then apply this work to demonstrate that the group algebra of $S_m \wr S_n$ is a cellular algebra whose cell modules are the Specht modules, and we hence obtain information on the simple modules of $S_m \wr S_n$ and their relationship to the Specht modules. In particular, if the group algebra of $S_m \wr S_n$ is semisimple then the Specht modules provide a complete system of isomorphism classes of simple modules. These properties provide justification for our use of the name "Specht modules" for these wreath product modules, since they mirror the properties of the Specht modules for the symmetric group.

Chapter 6 contains results which explain how the module constructions in Chapter 4 interact with module filtrations, and concludes by applying these methods to prove a wreath-product analogue of Young's rule.

Chapter 7 develops some combinatorial theory of *tableaux*, which will be used in subsequent chapters to understand cosets in $S_m \wr S_n$.

In **Chapter 8** we begin to use the material which we have developed and recalled in previous chapters to prove some more substantial new results. Indeed, an important result in the representation theory of S_n is the *Specht module branching rule*, which describes how the restriction of a Specht module from S_n to S_{n-1} (via the natural embedding of S_{n-1} into S_n) has a filtration by Specht modules. The multiplicity of each Specht module in this filtration is independent of the field of coefficients, and moreover has a simple and elegant combinatorial interpretation. In Chapter 8, we prove two branching rules for Specht modules over the wreath product $S_m \wr S_n$, one for the restriction of a Specht module to $S_{m-1} \wr S_n$, and one for the restriction to $S_m \wr S_{n-1}$. For both rules, we provide a combinatorial interpretation of the multiplicities in the filtration.

Chapter 9 is perhaps the heart of this thesis. It contains novel results on homomorphisms and extensions between our wreath Specht modules and wreath product analogues of the Young permutation modules of the symmetric groups. The proofs of these results are rather complex, and make extensive use of the material from Chapters 6 and 7.

Chapter 10 uses the results from Chapter 9 to prove that the Specht modules for $S_m \wr S_n$ give rise to a stratifying system in the same way as the Specht modules for S_n. Consequently, over an algebraically closed field of characteristic neither 2 nor 3, Specht filtration multiplicities are well-defined, meaning that the Specht modules for the wreath product $S_m \wr S_n$ exhibit the Hemmer-Nakano property, as promised.

Finally, **Appendix A** briefly considers some possible future directions of research which stem from the material in this thesis.

Chapter 2

Background material

In this chapter, we shall collect and review various results from elementary representation theory and from the literature, as well as fixing conventions and notation.

Throughout this thesis, k will denote a field. We shall often have to take the tensor product $\otimes_k$ of k-vector spaces, and we shall thus abbreviate $\otimes_k$ to $\otimes$. Initially, we place no restrictions on the field, so it may have characteristic zero or a prime, and further we shall not assume that k is algebraically closed. However, as we progress through our arguments, we shall find it necessary to require that the characteristic $\mathrm{char}(k)$ of k is not 2 or 3, and in the chapter on stratifying systems we shall also demand that k is algebraically closed. We shall clearly state whenever we are making these assumptions on k.

2.1 Finite-dimensional algebras over fields

This thesis is concerned with the representation theory of the *wreath product* $S_m \wr S_n$ (see Chapter 4), a finite group, over a field k. This is of course none other than the study of the *group algebra* $k(S_m \wr S_n)$. Hence, we shall need some ideas from the representation theory of algebras over a field, and we shall recall these in this section.

By a k**-algebra**, we shall mean a finite-dimensional unital associative algebra over k, such as the group algebras kS_n or $k(S_m \wr S_n)$. We shall work with *right* modules over our algebras, and hence the word "module" will mean "right module" unless stated otherwise. Since we are interested here in group algebras, and the categories of left and right modules over a group algebra are isomorphic, we lose nothing by considering only right modules.

Let A be a k-algebra and B a subalgebra of A. Then we have operations which convert A-modules into B-modules and vice versa. Indeed, if U is an A-module, then we write $U{\downarrow}_B^A$ to denote the B-module obtained by restricting the A-action to a B-action; we call this module the **restriction** of U to B. In the opposite direction, we note that A is itself a *left* B-module under the action defined by multiplication, and so if V is a (right) B-module, we may form the tensor product $V \otimes_B A$, which is then a (right) A-module; we call this module the **induction** of V to A, and write it as $V{\uparrow}_B^A$.

We shall spend much of our time considering filtrations of modules, and so we introduce some notation to help with this. Let A be a k-algebra. Firstly, let us recall that if M is an A-module and $X_1, \dots, X_t$ are also A-modules, then a **filtration of** M **by the modules** $X_1, \dots, X_t$ is a chain of submodules

$$M = M_n \supseteq M_{n-1} \supseteq M_{n-2} \cdots \supseteq M_1 \supseteq M_0 = 0$$

such that each quotient $\frac{M_l}{M_{l-1}}$ is isomorphic to some X_i. We call n the **length** of the filtration. Note in particular that we do not demand that the modules X_i be pairwise non-isomorphic in this definition. Now suppose that for each $i = 1, \dots, t$, α_i is a non-negative integer. We shall write

$$M \sim \mathop{\mathcal{F}}_{i=1}^{t} \alpha_i X_i$$

to mean that there exists a filtration of M

$$M = M_n \supseteq M_{n-1} \supseteq M_{n-2} \cdots \supseteq M_1 \supseteq M_0 = 0$$

and a function $f : \{1, \dots, n\} \longrightarrow \{1, \dots, t\}$ such that for each l we have

$$\frac{M_l}{M_{l-1}} \cong X_{f(l)}$$

and $|f^{-1}(i)| = \alpha_i$ for each i.

We shall routinely make a slight abuse of terminology and say, for example, that "M has a filtration $M \sim \mathcal{F}_{i=1}^{t} \alpha_i X_i$", or talk about "the filtration $M \sim \mathcal{F}_{i=1}^{t} \alpha_i X_i$". In such cases, we are of course referring to some filtration for which a function f as above exists. Further, we shall refer to the integers α_i as **multiplicities**, so we might for example express the situation $M \sim \mathcal{F}_{i=1}^{t} \alpha_i X_i$ by saying that "the module M has a filtration by the modules $X_1, \dots, X_t$ where X_i appears with multiplicity α_i". However, we note that, since we allow there to be isomorphisms between the modules X_i, these multiplicities are not in general uniquely determined by a filtration. Further, even in the case where the modules X_i are pairwise non-isomorphic so that the multiplicities are uniquely determined by a filtration, we note that it is in general perfectly possible to have two filtrations $M \sim \mathcal{F}_{i=1}^{t} \alpha_i X_i$ and $M \sim \mathcal{F}_{i=1}^{t} \beta_i X_i$ with $\alpha_i \neq \beta_i$ for some or all i. In general, it is only for certain special classes of modules $X_1, \dots, X_t$ (for example, the simple A-modules) that any two filtrations of an A-module M by $X_1, \dots, X_t$ will always have the same multiplicities (for the simple A-modules, this is the *Jordan-Hölder Theorem*).

Further, if $r \in \{1, \dots, t\}$ is such that $f(1) = r$, so that $M_1 \cong X_r$, then we will write

$$M \sim \mathop{\mathcal{F}_{\langle r \rangle}}_{i=1}^{t} \alpha_i X_i.$$

Thus, this notation means that M has a filtration by the modules X_i where X_i occurs with multiplicity α_i, and the submodule of M at the very bottom of the filtration is isomorphic to X_r. If all of the integers α_i are equal to 1, then we shall allow ourselves to write

$$M \sim \mathop{\mathcal{F}}_{i=1}^{t} X_i \qquad \text{and} \qquad M \sim \mathop{\mathcal{F}_{\langle r\rangle}}_{i=1}^{t} X_i.$$

The following elementary result will be used repeatedly in the sequel.

Proposition 2.1.1. *Let A be a k-algebra, and let M and N be A-modules. Suppose M has a filtration by modules $X_1,\dots,X_t$, and N has a filtration by modules $Y_1,\dots,Y_s$ (where as above we allow the possibility that there are isomorphisms amongst the modules X_i and Y_j).*

1. *If* $\mathrm{Hom}_A(X_i, Y_j) = 0$ *for all i and j then* $\mathrm{Hom}_A(M, N) = 0$.
2. *If* $\mathrm{Ext}^1_A(X_i, Y_j) = 0$ *for all i and j then* $\mathrm{Ext}^1_A(M, N) = 0$.

Proof. We begin by considering some special cases. Firstly, suppose we have an A-module Y and a short exact sequence of A-modules

$$0 \longrightarrow X_1 \longrightarrow M \longrightarrow X_2 \longrightarrow 0.$$

We apply the (contravariant) functor $\mathrm{Hom}_A(-, Y)$ to obtain a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(X_2, Y) \arrow[r] & \mathrm{Hom}_A(M, Y) \arrow[r] & \mathrm{Hom}_A(X_1, Y) \arrow[dll, bend left] & \\ & \mathrm{Ext}^1_A(X_2, Y) \arrow[r] & \mathrm{Ext}^1_A(M, Y) \arrow[r] & \mathrm{Ext}^1_A(X_1, Y) \arrow[r] & \cdots \end{tikzcd}$$

and hence we see that if $\mathrm{Hom}_A(X_i, Y) = 0$ for $i = 1, 2$ then $\mathrm{Hom}_A(M, Y) = 0$, and that if $\mathrm{Ext}^1_A(X_i, Y) = 0$ for $i = 1, 2$ then $\mathrm{Ext}^1_A(M, Y) = 0$. Similarly, if we have an A-module X and a short exact sequence of A-modules

$$0 \longrightarrow Y_1 \longrightarrow N \longrightarrow Y_2 \longrightarrow 0,$$

then we apply the (covariant) functor $\mathrm{Hom}_A(X,-)$ to obtain a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(X,Y_1) \arrow[r] & \mathrm{Hom}_A(X,N) \arrow[r] & \mathrm{Hom}_A(X,Y_2) \arrow[dll, out=-90, in=90] & \\
& \mathrm{Ext}^1_A(X,Y_1) \arrow[r] & \mathrm{Ext}^1_A(X,N) \arrow[r] & \mathrm{Ext}^1_A(X,Y_2) \arrow[r] & \cdots
\end{tikzcd}$$

and hence we see that if $\mathrm{Hom}_A(X,Y_i)=0$ for $i=1,2$ then $\mathrm{Hom}_A(X,N)=0$, and that if $\mathrm{Ext}^1_A(X,Y_i)=0$ for $i=1,2$ then $\mathrm{Ext}^1_A(X,N)=0$.

We may now easily prove, using induction on the length of the filtration of M by the modules X_i, that if we have some A-module Y such that $\mathrm{Hom}_A(X_i,Y)=0$ for $i=1,\ldots,t$ then we have $\mathrm{Hom}_A(M,Y)=0$. The general result then follows by using induction on the length of the filtration of N by the modules Y_j. $\square$

Now recall that if A and B are k-algebras, then we may form the **tensor product algebra** $A\otimes B$, which is the k-vector space $A\otimes B$ together with the multiplication defined on pure tensors by $(a_1\otimes b_1)(a_2\otimes b_2)=(a_1a_2\otimes b_1b_2)$. Further, if M is an A-module and N is a B-module, then we may form the $A\otimes B$-module whose underlying vector space is $M\otimes N$ and where the action is given on pure tensors by $(x\otimes y)(a\otimes b)=(xa)\otimes(yb)$ for $a\in A$, $b\in B$, $x\in M$ and $y\in N$. We denote this tensor product module by $M\boxtimes N$ and we call it the **outer tensor product** of M and N. Note that we use the symbol $\boxtimes$ rather than $\otimes$ here to clearly distinguish this tensor product from the *inner tensor product* of modules over a group algebra, which we shall introduce below.

The following result is a direct corollary of the proof of Lemma 4.1 in [6], and is in any case easy to prove directly.

Lemma 2.1.2. *Let A and B be k-algebras. Let M be an A-module and N a B-module, with filtrations*

$$M \sim \mathcal{F}_{\langle p \rangle}{}_{i=1}^{t} \, \alpha_i X_i \qquad N \sim \mathcal{F}_{\langle q \rangle}{}_{j=1}^{s} \, \beta_j Y_j.$$

Then the $A \otimes B$-module $M \boxtimes N$ has filtration

$$M \boxtimes N \sim \mathop{\mathcal{F}_{\langle (p,q) \rangle}}_{(i,j) \in \{1,\dots,t\} \times \{1,\dots,s\}} \alpha_i \beta_j \, X_i \boxtimes Y_j.$$

It follows that if $A_1, \dots, A_n$ are k-algebras and for each i, M_i is an A_i-module with a filtration

$$M_i \sim \mathcal{F}_{\langle p_i \rangle}{}_{j=1}^{t_i} \, \alpha^i_j X^i_j,$$

then the $A_1 \otimes \cdots \otimes A_n$-module $M_1 \boxtimes \cdots \boxtimes M_n$ has a filtration

$$M_1 \boxtimes \cdots \boxtimes M_n \sim \mathop{\mathcal{F}_{\langle (p_1,\dots,p_n) \rangle}}_{(j_1,\dots,j_n),\ 1 \leqslant j_i \leqslant t_i} \alpha^1_{j_1} \cdots \alpha^n_{j_n} \; X^1_{j_1} \boxtimes \cdots \boxtimes X^n_{j_n}$$

The following result will be a vital tool for large parts of our work in subsequent chapters.

Proposition 2.1.3. *Let A and B be finite-dimensional algebras over a field k, let M, N be A-modules, and let S, T be B-modules. Then we have isomorphisms of k-vector spaces*

$$\mathrm{Hom}_{A \otimes B}(M \boxtimes S, N \boxtimes T) \cong \mathrm{Hom}_A(M,N) \otimes \mathrm{Hom}_B(S,T) \tag{2.1.1}$$

and

$$\begin{aligned} &\mathrm{Ext}^1_{A \otimes B}(M \boxtimes S, N \boxtimes T) \cong \\ &\quad \left(\mathrm{Ext}^1_A(M,N) \otimes \mathrm{Hom}_B(S,T)\right) \oplus \left(\mathrm{Hom}_A(M,N) \otimes \mathrm{Ext}^1_B(S,T)\right). \end{aligned} \tag{2.1.2}$$

It follows at once that if $A_1, \dots, A_n$ are finite-dimensional k-algebras, and for each $i = 1, \dots, n$, M_i and N_i are A_i-modules, then letting $\underline{A} = \bigotimes_{i=1}^n A_i$ we *have an isomorphism of k-vector spaces*

$$\mathrm{Hom}_{\underline{A}}(M_1 \boxtimes \cdots \boxtimes M_n, N_1 \boxtimes \cdots \boxtimes N_n) \cong \bigotimes_{i=1}^{n} \mathrm{Hom}_{A_i}(M_i, N_i)$$

and an isomorphism of k-vector spaces

$$\mathrm{Ext}^1_{\underline{A}}(M_1 \boxtimes \cdots \boxtimes M_n, N_1 \boxtimes \cdots \boxtimes N_n) \cong \bigoplus_{i=1,\dots,n} \left(\mathrm{Ext}^1_{A_i}(M_i, N_i) \otimes \bigotimes_{\substack{j=1,\dots,n \\ j \neq i}} \mathrm{Hom}_{A_j}(M_j, N_j) \right).$$

Proof. This result is implied by [4, Chapter XI, Theorem 3.1] (see also [35, Lemma 3.2]), but both the statement and the proof of that result take rather a lot of effort to understand (at least, in the author's experience). It is, however, perfectly possible to prove (2.1.1) and (2.1.2) by fairly elementary homological algebra, and we shall now explain how this may be done. As is normal with such arguments, the essential idea of the proof is fairly simple, but there is a substantial volume of details to check. We shall therefore confine ourselves here to sketching the outline of such a proof.

So let us first recall that if $X_\bullet$ and $Y_\bullet$ are chain complexes of k-vector spaces, then the **tensor product** $X_\bullet \otimes Y_\bullet$ of $X_\bullet$ and $Y_\bullet$ is the chain complex with n^{th} term

$$\bigoplus_{i+j=n} X_i \otimes Y_j$$

and boundary map

$$\partial_n = \bigoplus_{i+j=n} (\partial_i \otimes \mathrm{id}) \ + \ (-1)^i \, (\mathrm{id} \otimes \partial_j).$$

Similarly if $X^\bullet$ and $Y^\bullet$ are cochain complexes of k-vector spaces, then their tensor product $X^\bullet \otimes Y^\bullet$ is the cochain complex with n^{th} term

$$\bigoplus_{i+j=n} X^i \otimes Y^j$$

and boundary map

$$\partial^n = \bigoplus_{i+j=n} (\partial^i \otimes \mathrm{id}) \; + \; (-1)^i \, (\mathrm{id} \otimes \partial^j).$$

If $X_\bullet$ is a chain complex of k-vector spaces, let us write $H_i(X_\bullet)$ for the i^{th} homology of $X_\bullet$, and if $X^\bullet$ is a cochain complex of k-vector spaces, let us write $H^i(X^\bullet)$ for the i^{th} cohomology of $X^\bullet$. Recall that if $X_\bullet, Y_\bullet$ are chain complexes of finite dimensional k-vector spaces, then by the well-known *homological Künneth theorem*, we have for each n an isomorphism of k-vector spaces

$$H_n(X_\bullet \otimes Y_\bullet) \cong \bigoplus_{i+j=n} H_i(X_\bullet) \otimes H_j(Y_\bullet). \tag{2.1.3}$$

(See for example [33, Theorem 3.6.3]). If we let $X^\bullet$, $Y^\bullet$ be cochain complexes of finite dimensional k-vector spaces, then by elementary homological algebra, we may obtain from (2.1.3) a "cohomological Künneth theorem", by which we have for each n an isomorphism of k-vector spaces

$$H^n(X^\bullet \otimes Y^\bullet) \cong \bigoplus_{i+j=n} H^i(X^\bullet) \otimes H^j(Y^\bullet). \tag{2.1.4}$$

So with A, B, M, N, S, T as in the proposition, let us take $P_\bullet$ to be a projective resolution of M in the category of A-modules, and $Q_\bullet$ to be a projective resolution of S in the category of B-modules. Viewing $P_\bullet$ and $Q_\bullet$ as complexes of vector spaces, we can form their tensor product $P_\bullet \otimes Q_\bullet$. But we can regard each tensor product space $P_i \otimes Q_j$ as the $A{\otimes}B$-module $P_i \boxtimes Q_j$, so we can regard the complex $P_\bullet \otimes Q_\bullet$ as a complex of $A{\otimes}B$-module, and we shall denote this complex of $A{\otimes}B$-modules by $P_\bullet \boxtimes Q_\bullet$. It is then easy using (2.1.3) (and the easily-proved fact that if P is a projective A-module and Q is a projective B-module then $P \boxtimes Q$ is a projective $A{\otimes}B$-module) to prove that the complex $P_\bullet \boxtimes Q_\bullet$ is a projective resolution of $M \boxtimes S$ in the category of $A{\otimes}B$-modules. We apply the (contravariant) functor $\mathrm{Hom}_{A\otimes B}(-, N \boxtimes T)$

to the complex $P_\bullet \boxtimes Q_\bullet$ to obtain the cochain complex (of k-vector spaces) $\mathrm{Hom}_{A\otimes B}(P_\bullet \boxtimes Q_\bullet, N \boxtimes T)$, whose n^{th} cohomology is $\mathrm{Ext}^n_{A\otimes B}(M\boxtimes S, N\boxtimes T)$. On the other hand, we have the cochain complex of k-vector spaces $\mathrm{Hom}_A(P_\bullet, N)\otimes \mathrm{Hom}_B(Q_\bullet, T)$ (where we have applied (contravariant) Hom-functors to $P_\bullet$ and $Q_\bullet$ to obtain cochain complexes $\mathrm{Hom}_A(P_\bullet, N)$ and $\mathrm{Hom}_B(Q_\bullet, T)$). By (2.1.4), this complex has n^{th} cohomology

$$\bigoplus_{i+j=n} H^i\big(\mathrm{Hom}_A(P_\bullet, N)\big) \otimes H^j\big(\mathrm{Hom}_B(Q_\bullet, T)\big)$$

which is of course the same as

$$\bigoplus_{i+j=n} \mathrm{Ext}^i_A(M, N) \otimes \mathrm{Ext}^j_B(S, T).$$

Thus to establish (2.1.1) and (2.1.2) it suffices to prove that there is a cochain isomorphism between the cochains $\mathrm{Hom}_{A\otimes B}(P_\bullet \boxtimes Q_\bullet, N \boxtimes T)$ and $\mathrm{Hom}_A(P_\bullet, N)\otimes\mathrm{Hom}_B(Q_\bullet, T)$ (note that these are cochains of k-vector spaces) and hence equate their cohomologies in degrees 0 and 1.

Now for any A-modules X, Y and B-modules U, V there is an obvious map of k-vector spaces

$$\mathrm{Hom}_A(X, Y) \otimes \mathrm{Hom}_B(U, V) \longrightarrow \mathrm{Hom}_{A\otimes B}(X \boxtimes U, Y \boxtimes V) \qquad (2.1.5)$$

and thus we can construct a cochain map from $\mathrm{Hom}_A(P_\bullet, N) \otimes \mathrm{Hom}_B(Q_\bullet, T)$ to $\mathrm{Hom}_{A\otimes B}(P_\bullet \boxtimes Q_\bullet, N \boxtimes T)$ (the verification that this is indeed a cochain map is a routine diagram-chasing argument). The only thing remaining is to prove that this is a cochain *isomorphism*, which we do by proving that the map (2.1.5) is an isomorphism when both X and U are projective modules. We do this by reducing it to the case where both X and U are indecomposable projectives, and then using the fact that we then have $X \cong Ax$ and $U \cong Bu$ for idempotents $x \in A$ and $u \in B$. $\square$

The following basic result in the theory of modules over finite-dimensional algebras is well-known, and will be important in our analysis of Ext^1 spaces.

Proposition 2.1.4. *Let A be a k-algebra and M, N be A-modules. Then the k-vector space $\mathrm{Ext}^1_A(M, N)$ is in bijection with the set of equivalence classes of extensions of M by N. In particular, $\mathrm{Ext}^1_A(M, N) = 0$ if and only if any extension of M by N is* split. *Explicitly, this means that $\mathrm{Ext}^1_A(M, N) = 0$ if and only if whenever E is an A-module with a submodule X such that $X \cong N$ and $E/X \cong M$, then E has a direct sum decomposition $E = X \oplus Y$ as an A-module (where we must then have $Y \cong M$).*

2.2 Representation theory of finite groups

As mentioned above, in this thesis we shall be concerned principally with modules for group algebras of finite groups. The theory of such modules has a number of special features, and we recall the necessary results in this section. We shall denote the group algebra of a group G over a field k by kG, and we shall generally denote the identity element of a group by e.

Firstly, recall that for any group G and field k, we have the **trivial** kG-module, which is just a copy of k where all group elements act as the identity map. We write this module as $\mathbb{1}_{kG}$, or just $\mathbb{1}_G$ if k is clear from the context.

Recall that if G is a finite group and U is a (right) kG-module, then the **(contragredient) dual** of U is the (right) kG-module obtained by equipping the k-vector space $\mathrm{Hom}_k(U, k)$ with the action defined by the equation $(\phi g)(u) = \phi(ug^{-1})$ for $\phi \in \mathrm{Hom}_k(U, k)$, $g \in G$, $u \in U$. We write this dual module as U^*. We further recall that $(-)^*$ (when suitably extended to homomorphisms) is a contravariant self-inverse additive isomorphism of categories from the category of right kG-modules to itself. In particular,

$(U^*)^* \cong U$ and $(U \oplus V)^* \cong U^* \oplus V^*$.

There are two common operations of "tensor product" on group modules, and we shall use both of them. Firstly, let G be a finite group with a kG-module X and H be a finite group with a kH-module Y. Then from the previous section, we know that we can form the *external* tensor product of X and Y, which is the $(kG)\otimes(kH)$-module $X \boxtimes Y$. But we have a canonical isomorphism $(kG)\otimes(kH) \cong k(G \times H)$ induced by mapping the pure tensor $g \otimes h$ to (g, h), where $g \in G$ and $h \in H$. Hence, we can regard $X \boxtimes Y$ as a $k(G \times H)$-module, and we will do so from now on without comment. Thus $X \boxtimes Y$ is the $k(G \times H)$-module obtained by equipping the k-vector space $X \otimes Y$ with the action given by $(x \otimes y)(g, h) = (xg) \otimes (yh)$ for $g \in G$, $h \in H$, $x \in X$, and $y \in Y$.

Now let G be a finite group and U, V be kG-modules. Then the **internal tensor product of U and V** is the kG-module obtained by equipping the k-vector space $U \otimes V$ with the G-action given by $(u \otimes v)g = (ug) \otimes (vg)$ for $g \in G$, $u \in U$, and $v \in V$. We shall write this module as $U \otimes V$, but there should not be any chance of confusion with the plain tensor product of vector spaces. It is immediate that $U \otimes V \cong V \otimes U$ as kG-modules.

It turns out that internal and external tensor products behave well under the operation of taking dual modules. Indeed, for modules U, V, X, Y as above, we have module isomorphisms

$$(U \otimes V)^* \cong U^* \otimes V^* \tag{2.2.1}$$

and

$$(X \boxtimes Y)^* \cong X^* \boxtimes Y^*. \tag{2.2.2}$$

These isomorphisms are easily proved by taking k-bases of the modules involved and working through the relevant calculations.

We now recall some important results on the operations of inducing and restricting modules between finite groups and their subgroups. To lighten the notation, if G is a finite group and H is a subgroup of G, then we may abbreviate induction and restriction of modules between kG and kH to just $\uparrow_H^G$ and $\downarrow_H^G$ if the field k is clear from the context. It is a standard and easily-proved fact that if X is a kH-module then

$$\dim_k(X\uparrow_H^G) = [G : H]\dim_k(X) \tag{2.2.3}$$

where $[G : H] = |G|/|H|$ is the index of H in G.

It is immediate from the definition of the inner tensor product of group modules that if G is a finite group with a subgroup H and U, V are kG-modules then we have an isomorphism of kH-modules

$$\big[U \otimes V\big]\big\downarrow_H^G \;\cong\; U\big\downarrow_H^G \otimes V\big\downarrow_H^G. \tag{2.2.4}$$

The outer tensor product behaves well with respect to the operations of induction and restriction. The results in the following lemma are well-known and easily proved.

Lemma 2.2.1. *Let G be a finite group with a subgroup I, and H be a finite group with a subgroup J. Let X be a kG-module, Y a kH-module, U a kI-module, and V a kJ-module. Then we have isomorphisms of group modules*

$$\big(X \boxtimes Y\big)\big\downarrow_{I\times J}^{G\times H} \cong \big(X\big\downarrow_I^G\big) \boxtimes \big(Y\big\downarrow_J^H\big)$$

and

$$\big(U \boxtimes V\big)\big\uparrow_{I\times J}^{G\times H} \cong \big(U\big\uparrow_I^G\big) \boxtimes \big(V\big\uparrow_J^H\big).$$

We shall also make use of the fact that induction and restriction of modules between groups and their subgroups "preserve filtrations" in the sense of the following lemma.

Lemma 2.2.2. *Let G be a finite group and H a subgroup of G. Suppose that we have a kG-module Y with a filtration*

$$Y \sim \mathop{\mathcal{F}_{\langle p \rangle}}_{i=1}^{t} a_i W_i$$

for kG-modules $W_1, \ldots, W_t$. Then we have a filtration of kH-modules

$$Y{\downarrow}_H^G \sim \mathop{\mathcal{F}_{\langle p \rangle}}_{i=1}^{t} a_i \; W_i{\downarrow}_H^G .$$

Conversely, if we have a kH-module X with a filtration

$$X \sim \mathop{\mathcal{F}_{\langle q \rangle}}_{i=1}^{s} b_i V_i$$

for kH-modules $V_1, \ldots, V_s$, then we have a filtration of kG-modules

$$X{\uparrow}_H^G \sim \mathop{\mathcal{F}_{\langle q \rangle}}_{i=1}^{s} b_i \; V_i{\uparrow}_H^G .$$

Proof. Now the operations ${\uparrow}_H^G$ and ${\downarrow}_H^G$ may be extended to homomorphisms in the obvious way, and hence we see that induction and restriction are functors between the relevant module categories. The lemma is then proved by noting the well-known fact that both of these functors are exact. Indeed, we have by [3, Proposition 3.3.1] and [3, Proposition 2.8.1] (which shows that the relevant isomorphisms yield natural isomorphisms of functors) that both $({\uparrow}_H^G, {\downarrow}_H^G)$ and $({\downarrow}_H^G, {\uparrow}_H^G)$ are adjoint pairs, and hence that both functors are both right and left exact, and thus that they are indeed exact. □

Proposition 2.2.3. *Let U, V, W be kG-modules for a finite group G. Then we have an isomorphism of k-vector spaces*

$$\mathrm{Ext}^1_{kG}\left(W, U^* \otimes V\right) \cong \mathrm{Ext}^1_{kG}\left(W \otimes U, V\right).$$

Proof. Now $\mathrm{Hom}_k(U, V)$ has a natural kG-module structure given by setting $(fg)(u) = (f(ug^{-1}))g$ for $g \in G$, $f \in \mathrm{Hom}_k(U, V)$ and $u \in U$ [3, p.50]. From

[3, p.52] we have an isomorphism $\mathrm{Hom}_k(U, V) \cong U^* \otimes V$ of kG-modules, and combining this with [3, Proposition 3.1.8, (ii)] and the fact that $\otimes$ is commutative, we obtain the desired isomorphism. $\square$

From [3], we also have the Eckmann-Shapiro lemma and Mackey's theorem. Note that while [3] uses left modules, we give the right-module versions of these results, which may be obtained by exactly the same arguments as the left module versions.

Theorem 2.2.4. *(Eckmann-Shapiro lemma) Let G be a finite group and H be a subgroup of G. Let X be a right kG-module and Y be a right kH-module. Then we have isomorphisms of k-vector spaces*

$$\begin{aligned}
\mathrm{Hom}_{kG}(Y \uparrow_H^G, X) &\cong \mathrm{Hom}_{kH}(Y, X \downarrow_H^G) \\
\mathrm{Ext}^1_{kG}(Y \uparrow_H^G, X) &\cong \mathrm{Ext}^1_{kH}(Y, X \downarrow_H^G) \\
\mathrm{Hom}_{kG}(X, Y \uparrow_H^G) &\cong \mathrm{Hom}_{kH}(X \downarrow_H^G, Y) \\
\mathrm{Ext}^1_{kG}(X, Y \uparrow_H^G) &\cong \mathrm{Ext}^1_{kH}(X \downarrow_H^G, Y).
\end{aligned}$$

Proof. These can be obtained easily using results from [3]. The first and second isomorphisms are given by [3, Corollary 3.3.2]. The third and fourth can be proved in the same way as [3, Corollary 3.3.2], using [3, Corollary 2.8.4] and the fact given in [3, Section 3.3] that the functors of induction and *co-induction* (see [3, Definition 2.8.1]) coincide for modules over the group algebra of a finite group. $\square$

For the statement of Mackey's Theorem, we need some definitions. Indeed, if G is a finite group and H is a subgroup of G, then for $g \in G$ we define H^g to be the subset $\{g^{-1}hg \mid h \in H\}$ of G. Then H^g is a subgroup of G, the **conjugate subgroup of** H **by** g, and in fact H^g is isomorphic to H. Further, if X is a kH-module, then we define X^g to be the kH^g-module with underlying vector space X and action given by $x(g^{-1}hg) = xh$ for $x \in X$ and $h \in H$. We call this the **conjugate module of X by g.**

Theorem 2.2.5. *(Mackey's Theorem) Let G be a finite group with subgroups H and K, let $\mathcal{U}$ be a complete non-redundant system of (H, K)-double coset representatives in G, and let X be a right kH-module. Then we have a decomposition of right kK-modules*

$$X\uparrow_H^G\downarrow_K^G \cong \bigoplus_{u\in\mathcal{U}} X^u \downarrow^{H^u}_{H^u\cap K}\uparrow^K_{H^u\cap K}.$$

Proof. This is Theorem 3.3.4 of [3]. □

Proposition 2.2.6. *[2, Section 8, Corollary 3] Let G be a finite group with a subgroup H, and let k be a field (note that [2] formally assumes an algebraically closed field, but the proof of this result does not use algebraic closedness). Let U be a kG-module, and suppose that $U\downarrow_H^G$ has a kH-submodule X such that X generates U as a kG-module. If $\dim_k(U) = [G : H]\dim_k(X)$ (where $[G : H] = |G|/|H|$ is as usual the index of H in G), then U is isomorphic as a kG-module to $X\uparrow_H^G$ (note that [2] refers to "relatively free" modules rather than "induced" modules, but as explained in [2] the concepts are equivalent).*

Corollary 2.2.7. *If U is a kG-module with $dim_k(U) = |G|/|H|$ and moreover there is an element $u \in U$ which generates U as a kG-module and which satisfies $uh = u$ for all $h \in H$, then $U \cong \mathbb{1}_H\uparrow_H^G$.*

Proof. Let X be the k-span of u and apply Proposition 2.2.6. □

An elementary but useful result on filtrations of group modules goes as follows.

Lemma 2.2.8. *Let G be a finite group and Z, W kG-modules. Suppose W has a kG-module filtration*

$$W \sim \mathop{\mathcal{F}_{\langle p \rangle}}_{i=1}^{t} a_i U_i.$$

Then the kG-module $Z \otimes W$ has a filtration

$$Z \otimes W \sim \mathop{\mathcal{F}_{\langle p \rangle}}_{i=1}^{t} a_i \; Z \otimes U_i.$$

Symmetrically, if Z has a kG-module filtration

$$Z \sim \mathop{\mathcal{F}_{\langle q \rangle}}_{i=1}^{s} b_i V_i$$

then we have a kG-module filtration

$$Z \otimes W \sim \mathop{\mathcal{F}_{\langle q \rangle}}_{i=1}^{s} b_i \; V_i \otimes W.$$

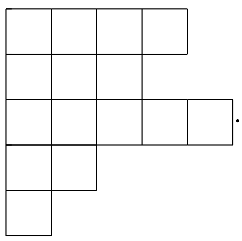
Proof. Now let X, Y be kG-modules such that Y has a submodule U. Let T be the k-subspace of the kG-module $X \otimes Y$ spanned by all pure tensors of the form $x \otimes u$ for $x \in X$ and $u \in U$. Then it is easy to see that T is a kG-submodule of $X \otimes Y$, that $T \cong X \otimes U$ as kG-modules via the obvious map, and moreover that $\frac{X \otimes Y}{T} \cong X \otimes \left(\frac{Y}{U}\right)$ as kG-modules, again via the obvious map. The first part of the lemma now follows via a trivial induction, and the second part via a symmetrical argument. $\square$

2.3 Combinatorial definitions

We now recall some standard combinatorial definitions and notation, and also introduce some more specialised concepts which we shall need for our work. These non-standard definitions and notation will be recalled as they appear in the text, but they are given here together for completeness.

Let n, t be non-negative integers. A **composition of** n **of length** t, or more briefly a t**-composition of** n, is a tuple $\gamma = (\gamma_1, \ldots, \gamma_t)$ of non-negative integers such that $\gamma_1 + \gamma_2 + \cdots + \gamma_t = n$. We define $|\gamma| = n$, the **size** of γ, and $\gamma_1, \ldots, \gamma_t$ are called the **parts** of γ. A **partition** of n is a composition of n with no zero parts and where the parts are weakly decreasing. We shall adopt the standard notation and write $\gamma \vDash n$ to mean that γ is a composition of n, and $\lambda \vdash n$ to mean that λ is a partition of n. Thus, for example, $(0, 2, 3, 1, 0, 1)$ is a composition of 7 which is not a partition, while $(4, 3, 1, 1, 1)$ is a partition of 10. Note that if $n = 0$ then n has exactly one partition, namely the empty tuple $()$, which has length 0. We shall write Ω_n^t for the set of all compositions of n with length t, and Λ_n for the set of all partitions of n. For a given length t, we let $[n, l]$ denote the composition of n of length t whose l^{th} entry is n and which has all other entries 0, so that $[n, l] = (0, 0, \ldots, 0, n, 0, \ldots, 0)$. We shall also adopt the standard convention of allowing ourselves to denote repeated entries in a composition or partition via a superscript, so that for example the partition $(3, 2, 2, 2, 2, 1, 1, 1)$ could be written $(3, 2^4, 1^3)$. We shall use this notation in particular to write the partition of n consisting of n 1's as (1^n).

It is often helpful to think of compositions and partitions in a pictorial way. To this end, if $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a composition of n, the **Young diagram of shape** α is an array of boxes, with α_1 boxes in the top row, α_2 boxes in the next row down, and so on, arranged with the left-most boxes of the rows vertically aligned with each other. For example, if $\alpha = (4, 3, 5, 2, 1) \vDash 15$, then the Young diagram of shape α is



We number the rows of a Young diagram from top to bottom, so that the top row is row 1, the next row down is row 2, and so on. We also speak of "higher" and "lower" rows, where "higher" means nearer the top of the diagram and "lower" means nearer the bottom of the diagram. Thus we note that the i^{th} row is *higher* than the j^{th} row if i is *lower* than j.

If $\alpha = (\alpha_1, \ldots, \alpha_t)$ and $\beta = (\beta_1, \ldots, \beta_s)$ are compositions of n, then we say that α **dominates** β, and we write $\alpha \trianglerighteq \beta$, if we have $\sum_{i=1}^{r} \alpha_i \geq \sum_{i=1}^{r} \beta_i$ for each $r = 1, \ldots, \max(t, s)$ (where if one composition is shorter than the other, we pad the shorter composition on the right with zeros to give them the same length). This relation induces a partial ordering on compositions of n, called the **dominance order**, and we make Λ_n a poset by equipping it with this partial order. For example with $n = 8$, we have $(3, 3, 2) \trianglerighteq (3, 2, 2, 1)$, but $(3, 2, 2, 1)$ and $(4, 1, 1, 1, 1)$ are not comparable in the dominance order. The dominance order is a non-strict order, and we shall of course use the symbol $\triangleright$ to denote the associated strict order. Informally, we see that in the dominance order, compositions whose Young diagrams are "shorter and wider" rank higher than those whose Young diagrams are "taller and thinner".

The dominance order is in general a partial order, but sometimes we shall need a total order on compositions and partitions of n. The total order we shall use is the **lexicographic order**, which we shall denote by $>$. If $\alpha = (\alpha_1, \ldots, \alpha_t)$ and $\beta = (\beta_1, \ldots, \beta_s)$ are compositions of n, then we define $(\alpha_1, \ldots, \alpha_t) > (\beta_1, \ldots, \beta_s)$ to mean that there exists an i with $0 < i \leq \min(t, s)$ such that $\alpha_j = \beta_j$ for all $j < i$ and $\alpha_i > \beta_i$. We note that the lexicographic order is an extension of the dominance order, meaning that if $\alpha \triangleright \beta$ then $\alpha > \beta$. Now if we have any partial order on a set, then the **reverse** of this order is the order obtained by simply reversing all the relations of the order. We shall make use of the **reverse lexicographic order**, which we shall denote by

$\gtrdot$. We shall also use the symbol $\gtrdot$ to denote the reverse order of any order denoted by the symbol $>$.

Keep n, t as non-negative integers. A **multicomposition of** n **of length** t, or more briefly a t**-multicomposition of** n, is a tuple $\underline{\gamma} = (\gamma^1, \ldots, \gamma^t)$ of compositions (which will in general have different sizes, and where $\gamma^i = ()$ is allowed), such that $|\gamma^1| + |\gamma^2| + \cdots + |\gamma^t| = n$. The compositions $\gamma^1, \ldots, \gamma^t$ are called the **components** of $\underline{\gamma}$. We shall write the j^{th} part of the composition γ^i as γ^i_j. A multicomposition $\underline{\nu} = (\nu^1, \ldots, \nu^t)$ is a **multipartition** if each component ν^i is a partition. Note that a multipartition can have empty components (i.e. $\nu^i = ()$ is allowed). For example, $\big((3,2,2,1),(4,1),(),(3),(3,2),()\big)$ is a 6-multipartition of 21. For $\underline{\gamma} = (\gamma^1, \ldots, \gamma^t)$ a multicomposition of n, we define $||\underline{\gamma}|| = n$, the **size** of $\underline{\gamma}$, and $|\underline{\gamma}| = (|\gamma^1|, |\gamma^2|, \ldots, |\gamma^t|)$, a composition of n, which we shall call the **shape** of $\underline{\gamma}$. Thus for example we have

$$| \big((3,2,2,1),(4,1),(),(3),(3,2),()\big) | = (8,5,0,3,5,0)$$
$$|| \big((3,2,2,1),(4,1),(),(3),(3,2),()\big) || = 21.$$

Note that even if $\underline{\gamma}$ is a multi*partition*, $|\underline{\gamma}|$ will in general be a *composition*. We shall write $\underline{\Lambda}_n$ for the set of all multipartitions of n, and $\underline{\Lambda}^t_n$ for the set of all t-multipartitions of n. Note that if $n = 0$ then n has exactly one multipartition with t components for each $t \geq 0$. Indeed, for $t = 0$ it is $()$ (the unique multicomposition with zero components), and for $t \geq 1$ it is the t-tuple $\big((), \ldots, ()\big)$. We shall be particularly interested in t-multicompositions $\underline{\gamma}$ of n where all components are $()$ except for one, which must then be a composition of n. Indeed, for a given length t, let us denote the t-multicomposition of n which has $\gamma \vDash n$ in the i^{th} place and empty compositions $()$ everywhere else as $[\gamma, i]$. Thus we have

$$[\gamma, i] = \big((), (), \ldots, (), \gamma, (), \ldots, ()\big)$$

where the γ on the right-hand side occurs in the i^{th} place.

In Corollary 6.2.2 below, and its applications, we shall have a tuple $(a_1, \ldots, a_t)$ of non-negative integers, and we shall wish to work with the set of all t-multicompositions $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ of n where α^i has length a_i. We shall write $\underline{\Omega}(n; a_1, \ldots, a_t)$ for this set. Further, we let $[[n, 1], i; a_1, \ldots, a_t]$ be the element of $\underline{\Omega}(n; a_1, \ldots, a_t)$ whose i^{th} component is $(n, 0, \ldots, 0)$ and with an a_l-tuple of zeros in the l^{th} place for each $l \neq i$. Thus

$$[[n, 1], i; a_1, \ldots, a_t] = \big((0, \ldots, 0), \ldots, (n, 0, \ldots, 0), \ldots, (0, \ldots, 0)\big).$$

There is a natural analogue of the dominance order on compositions for the set of multicompositions of n of length t (which of course then restricts to a partial ordering on the set of multi*partitions* of n of length t), see for example in [8, Definition 3.11]. For multicompositions $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ and $\underline{\beta} = (\beta^1, \ldots, \beta^t)$ of n, we define $\underline{\alpha} \trianglerighteq \underline{\beta}$ to mean that for any $p \in \{1, \ldots, t\}$ and $q \geq 0$, we have (taking any parts α_i^p or β_i^p which would otherwise be undefined to be zero as necessary)

$$\sum_{i=1}^{p-1} |\alpha^i| + \sum_{i=1}^{q} \alpha_i^p \;\geq\; \sum_{i=1}^{p-1} |\beta^i| + \sum_{i=1}^{q} \beta_i^p.$$

Note that $\underline{\alpha} \trianglerighteq \underline{\beta}$ implies $|\underline{\alpha}| \trianglerighteq |\underline{\beta}|$ (taking $q = 0$ for each p), and if $|\underline{\alpha}| = |\underline{\beta}|$ then $\underline{\alpha} \trianglerighteq \underline{\beta}$ if and only if $\alpha^i \trianglerighteq \beta^i$ for all i. When we refer to $\underline{\Lambda}_n$ and $\underline{\Lambda}_n^t$ as posets, it is to this order that we will be referring. For example, in the poset $\underline{\Lambda}_{15}^6$, we have

$$\big((2, 2, 1), (2, 1), (2, 1, 1), (1), (), (2)\big) \trianglerighteq \big((2, 2), (3, 1), (2, 1), (1), (1), (1, 1)\big).$$

We shall also need to make use of tuples of multicompositions. We shall typically denote such a tuple with a double underlined symbol such as $\underline{\underline{\gamma}}$. We shall write the i^{th} component of $\underline{\underline{\gamma}}$ as $\underline{\gamma}^i$ (a multicomposition), the j^{th} component of $\underline{\gamma}^i$ as γ^{ij} (a composition), and the l^{th} part of γ^{ij} as γ^{ij}_l (a non-negative integer). If $\underline{\underline{\gamma}} = (\underline{\gamma}^1, \ldots, \underline{\gamma}^t)$, then we write $|\underline{\underline{\gamma}}|$ for the multicomposition $(|\underline{\gamma}^1|, \ldots, |\underline{\gamma}^t|)$, $||\underline{\underline{\gamma}}||$ for the composition $(||\underline{\gamma}^1||, \ldots, ||\underline{\gamma}^t||)$, and $|||\underline{\underline{\gamma}}|||$ for the integer $||\underline{\gamma}^1|| + \cdots + ||\underline{\gamma}^t||$.

We now recapitulate some of the above, in order to draw the reader's attention to the notational conventions which we have established, and which we shall maintain throughout this thesis. Indeed, compositions and partitions are denoted by Greek letters (generally lowercase), and their parts are indexed with *subscript* numerals. For example, the i^{th} part of the composition α is α_i. Multicompositions and multipartitions are denoted by underlined Greek letters, and when we have lists or tuples of compositions, the index of an element of this list is written as a *superscript*. For example, the i^{th} component of the multicomposition $\underline{\alpha}$ is α^i, where we note that the symbol α is *not* underlined because α^i is a composition, not a multicomposition. Similarly, tuples of multicompositions are denoted by double-underlined Greek letters, and when we have lists or tuples of multicompositions, the index of an element of this list is written as a *superscript*. For example, the i^{th} component of the tuple $\underline{\underline{\gamma}}$ of multicompositions is $\underline{\gamma}^i$ (where the symbol γ is underlined once, since this object is a multicomposition), and the j^{th} component of $\underline{\gamma}^i$ is γ^{ij} (no underline, since this object is a composition). We therefore emphasise the following points of notation to the reader.

- Symbols based around a double-underlined Greek letter denote tuples of multicompositions.
- Symbols based around a single-underlined Greek letter (perhaps with one or more superscripts) denote multicompositions (or multipartitions).
- Symbols based around a Greek letter with neither underlining nor a sub-

script (but perhaps with one or more superscripts) denote compositions (or partitions).

- Symbols based around a non-underlined Greek letter with a subscript (perhaps with one or more superscripts) denote non-negative integers occurring as parts of compositions or partitions.

It is the author's experience that the use of these conventions is of great assistance when performing or reading the kinds of calculations that will feature prominently throughout this thesis, which involve the use of multicompositions and multipartitions or tuples thereof. The author hopes that readers will also find these conventions useful. Note, however, the interaction of these conventions with the notations we have established above involving the vertical bar symbol $|$. For example, if $\underline{\gamma}$ is a tuple of multicompositions, then $|\underline{\gamma}|$ is a *composition.* The symbol γ is underlined here because the expression $|\underline{\gamma}|$ is the result of applying the operation $|\cdot|$ to the multicomposition $\underline{\gamma}$.

In (6.4.8) below, we shall need notation for a specific kind of tuple of multicompositions. Indeed, let $(a_1, \dots, a_t)$ be a tuple of non-negative integers such that we have some $i \in \{1, \dots, t\}$ with $a_i \neq 0$. Given a composition η, we define $[[\eta, 1], i; a_1, \dots, a_t]$ to be the t-tuple of multicompositions whose i^{th} entry is the multicomposition $\big(\eta, (), \dots, ()\big)$ of length a_i, and where for $l \neq i$, the l^{th} entry is a tuple of empty compositions of length a_l. Thus

$$[[\eta, 1], i; a_1, \dots, a_t] = \big(((), \dots, ()), \dots, (\eta, (), \dots, ()), \dots, ((), \dots, ())\big).$$

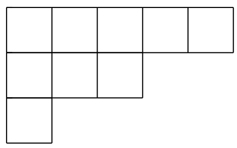
We shall use the symbol $\circ$ to denote the concatenation of tuples. Thus if $\nu^1, \dots, \nu^t$ are compositions, then $\nu^1 \circ \cdots \circ \nu^t$ denotes their concatenation, so that for example we have

$$(3, 2, 1) \circ (2, 2) \circ (4, 1, 1) \circ () \circ (2) = (3, 2, 1, 2, 2, 4, 1, 1, 2).$$

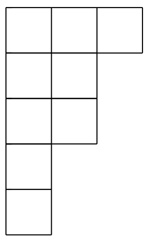
Similarly, if $\underline{\nu}^1, \dots, \underline{\nu}^t$ are tuples of compositions, then $\underline{\nu}^1 \circ \dots \circ \underline{\nu}^t$ denotes their concatenation, so that for example

$$\big((3,2,1),(2,2),(3,1)\big) \circ \big((4,1,1),(),(2)\big) = \\ \big((3,2,1),(2,2),(3,1),(4,1,1),(),(2)\big).$$

Now if λ is a partition of n, we associate to λ the **conjugate partition** λ' of λ, which is the partition whose Young diagram is obtained by reflecting the Young diagram of λ about its leading diagonal axis. For example, if $\lambda = (5,3,1)$ then we take the Young diagram



and reflect it as described to obtain



and hence we see that $\lambda' = (3,2,2,1,1)$. We see that the map $\lambda \mapsto \lambda'$ is a self-inverse bijection on Λ_n.

If $\underline{\nu} = (\nu^1, \dots, \nu^t)$ is a multipartition, then we define the **conjugate** of $\underline{\nu}$ to be the multipartition $\big((\nu^1)', \dots, (\nu^t)'\big)$, and we denote this by $\underline{\nu}'$.

When we come to discuss the simple modules of the group algebra of the symmetric group, we shall need the concepts of p**-regular** and p**-singular** partitions, where p is either 0 or a prime (indeed, p will be the characteristic of our field k). Let n be a non-negative integer. If $p = 0$, then all partitions of n are p-regular, and none are p-singular. If $p > 0$ is a prime number, then a partition is p-singular if it contains a constant subsequence of length p, and is p-regular otherwise. For example, the partition $(4,2,2,1,1,1,1)$ of 12 is 3-singular but 5-regular.

Original research in Chapter 2: There is no original research in Chapter 2.

Chapter 3

Representation theory of symmetric groups

In this chapter, we recall the material which we shall require on the representation theory of the symmetric group. Our main source is the classic monograph [20] of James, but we shall also recall some results from other sources.

3.1 The symmetric group, Young permutation modules, Specht modules, and simple modules

For n a non-negative integer, we take S_n to be the **symmetric group** of all permutations of the set $\{1, 2, \ldots, n\}$ under composition. Thus if $n = 0$ we have the set of permutations of the empty set, which is the trivial group. We take S_n to act on $\{1, 2, \ldots, n\}$ from the *right*, meaning that the product $\sigma\pi$ of permutations $\sigma, \pi \in S_n$ is the permutation obtained by first applying σ and then applying π. Thus we write permutations on the right of their arguments, for example $(i)\sigma$, so that we have the expected formula

$$(i)(\sigma\pi) = ((i)\sigma)\pi.$$

We shall typically write permutations in cycle notation, so that for example the element of S_6 which maps $1, 2, 3, 4, 5, 6$ to $4, 6, 1, 3, 5, 2$ respectively will be written as $(1, 4, 3)(2, 6)$.

For $\alpha = (\alpha_1, \ldots, \alpha_t)$ a composition of $n \geq 0$, we define as usual the **Young subgroup** S_α of S_n as follows. For each $i = 0, \ldots, t$, let $\hat{\alpha}_i = \alpha_1 + \cdots + \alpha_i$ (so note that $\hat{\alpha}_0 = 0$). Then S_α is the subgroup of S_n consisting of all permutations which for each $i = 1, \ldots, t$ map the set $\{\hat{\alpha}_{i-1} + 1, \ldots, \hat{\alpha}_i\}$ to itself. We shall be making frequent use of the operations of induction and restriction between group algebras of symmetric groups and of their Young subgroups, for example

$$X{\uparrow}_{kS_\alpha}^{kS_n} \text{ and } Y{\downarrow}_{kS_\alpha}^{kS_n}.$$

To de-clutter such expressions, we shall abbreviate the notation by replacing the full symbols for the group algebras with the subscripts used to identify the various subgroups of S_n involved, so for example the above would be abbreviated to

$$X{\uparrow}_{\alpha}^{n} \text{ and } Y{\downarrow}_{\alpha}^{n}.$$

Similarly, we abbreviate, for example, Hom_{kS_n} to Hom_n and $\mathrm{Ext}^1_{kS_\alpha}$ to Ext^1_α, and also $\mathbb{1}_{kS_n}$ and $\mathbb{1}_{kS_\alpha}$ to $\mathbb{1}_n$ and $\mathbb{1}_\alpha$.

Now it is a standard fact that any permutation may be expressed as the product of **transpositions**, which are elements of S_n of the form (a, b) (expressed in cycle notation). Further, we may show for $\sigma \in S_n$ that if $\sigma = t_1 t_2 \cdots t_p = t'_1 t'_2 \cdots t'_{p'}$ where t_i and t'_i are transpositions, then p and p' are congruent modulo 2, and thus we may define $\mathrm{sgn}(\sigma) \in k$ to be $(-1)^p$, where p is the length of any factorisation of σ into transpositions. We say that σ is **even** if $\operatorname{sgn}(\sigma) = 1$ and **odd** if $\operatorname{sgn}(\sigma) = -1$. The map $\operatorname{sgn} : S_n \to k$ yields a one-dimensional kS_n-module consisting of a copy of k upon which $\sigma \in S_n$ acts as multiplication by $\operatorname{sgn}(\sigma)$. This module is called the **sign module** for kS_n, which we write (in keeping with our abbreviated notation $\mathbb{1}_n$) as Sgn_n (with the field k being understood). Further, the sign module can of course be restricted to any subgroup H of S_n, and we write the resulting kH-module as Sgn_H, with our convention as above that if, for example, $H = S_\alpha$, we will write Sgn_{S_α} as Sgn_α. If $\alpha = (\alpha_1, \ldots, \alpha_t)$ is a composition of n then we may easily see that

$$\mathrm{Sgn}_\alpha = \mathrm{Sgn}_n{\downarrow}^n_\alpha \cong \mathrm{Sgn}_{\alpha_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\alpha_t} \tag{3.1.1}$$

where we have identified S_α with $S_{\alpha_1} \times \cdots \times S_{\alpha_t}$ in the usual way.

Now if $\sigma \in S_n$, then an **inversion** of σ is a pair (i, j) such that $1 \leq i < j \leq n$ and $(i)\sigma > (j)\sigma$, and the **length** of σ, $\operatorname{len}(\sigma)$, is the total number of inversions of σ.

Remark 3.1.1. Recall that a **basic transposition** in S_n is a transposition of the form $(i, i+1)$ for some i such that $1 \leqslant i < n$. It is a standard fact that S_n is generated by its basic transpositions, and it is also well-known that for $\sigma \in S_n$, $\operatorname{len}(\sigma)$ is equal to the minimal length of an expression of σ as a product of basic transpositions (see for example [34, Lemma 2.1]). Hence, this concept of the length of a permutation agrees with the *Coxeter length* of a permutation when S_n is regarded as a *Coxeter group* generated by the basic transpositions. See for example [30, Chapter 1, Section 1] for details. It also follows that $\operatorname{sgn}(\sigma) = (-1)^{\operatorname{len}(\sigma)}$.

In the previous chapter, we recalled the notion of the *Young diagram* of a composition. We now recall briefly the related definitions of *Young tableaux* and *Young tabloids* in order to sketch the definition of the *Young permutation* *modules* and *Specht modules* for the symmetric group. Note that we shall not need the definitions of these modules in our work, but we include them here for completeness.

If α is a composition of n, then a **Young tableau** of shape α, or more briefly an α**-tableau**, is a Young diagram of shape α, with the numbers 1 to n inserted in the boxes, with one number per box and each number appearing once. For example, if we take $\alpha = (4, 3, 5, 2, 1) \vDash 15$, then one α-tableau is

$$\begin{array}{ccccc}
\cline{1-4}
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{12} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{9} & \\
\cline{1-4}
\multicolumn{1}{|c|}{15} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{10} & & \\
\cline{1-5}
\multicolumn{1}{|c|}{14} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{13} & \multicolumn{1}{c|}{1} \\
\cline{1-5}
\multicolumn{1}{|c|}{11} & \multicolumn{1}{c|}{2} & & & \\
\cline{1-2}
\multicolumn{1}{|c|}{6} & & & & \\
\cline{1-1}
\end{array}.$$

Given some composition $\alpha \vDash n$, we define an equivalence relation on the set of all α-tableaux by making two tableaux equivalent if each number appears on the same row in both tableaux. We call the resulting equivalence classes α**-tabloids**. We think of a tabloid as "a tableau with unordered rows".

We also define a more general kind of tableau, where repeated entries are allowed. Indeed, given two compositions α and β of n, an α**-tableau of type** β is a Young diagram of shape α with a positive integer in each box, such that 1 appears β_1 times, 2 appears β_2 times, and so on. Thus a Young tableau as defined above is more fully an α-tableau of type $(1, 1, \ldots, 1)$. So for example keeping $\alpha = (4, 3, 5, 2, 1)$ and taking $\beta = (3, 3, 2, 4, 1, 0, 2)$ then one α-tableau of type β is

$$\begin{array}{ccccc}
\cline{1-4}
\multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \\
\cline{1-4}
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{7} & & \\
\cline{1-5}
\multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{2} \\
\cline{1-5}
\multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{1} & & & \\
\cline{1-2}
\multicolumn{1}{|c|}{2} & & & & \\
\cline{1-1}
\end{array}.$$

We now use tabloids to define modules for kS_n, see Chapter 4 of [20] for full details. For $\alpha \vDash n$, $\sigma \in S_n$ acts on the set of α-tableaux by replacing each entry i in a tableau with $(i)\sigma$, and this action induces an action on the set of α-tabloids. We let M^α be the k-vector space with the set of all α-tableaux as a basis. Then the S_n-action on the set of tabloids induces a right kS_n-module structure on M^α. We call this module M^α the **Young permutation module** associated to α. It is clear that M^α is a cyclic module, generated by any tabloid. Let us take $\tau(\alpha)$ to be the tabloid obtained from the tableau of shape α where the first row contains the numbers $1, \ldots, \alpha_1$, the second row contains the numbers $\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2$, and so on, so that for example if $\alpha = (4, 3, 3, 1)$ then $\tau(\alpha)$ is the tabloid obtained from the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & 9 & 10 & \multicolumn{1}{c}{} \\ \cline{1-3} 11 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} .$$

The tableau $\tau(\alpha)$ is sometimes called the **basic standard α-tableau**. It is clear that $\tau(\alpha)$ generates M^α as a kS_n-module, and that $\tau(\alpha)\sigma = \tau(\alpha)$ for all $\sigma \in S_\alpha$. Further, by counting tabloids it is easily seen that

$$\dim_k(M^\alpha) = \frac{n!}{\prod_i \alpha_i!} \tag{3.1.2}$$

(note that this is independent of k), and this of course is equal to $[S_n : S_\alpha]$, so that by Corollary 2.2.7 we have

$$M^\alpha \cong \mathbb{1}_\alpha{\uparrow}_\alpha^n. \tag{3.1.3}$$

Proposition 3.1.2. *If α and β are compositions of n such that one may be obtained by reordering the parts of the other, then we have $M^\alpha \cong M^\beta$ as kS_n-modules.*

Proof. We know by the isomorphism (3.1.3) that $M^\alpha \cong \mathbb{1}_\alpha{\uparrow}_\alpha^n$ and that $M^\beta \cong \mathbb{1}_\beta{\uparrow}_\beta^n$, so we need only prove that $\mathbb{1}_\alpha{\uparrow}_\alpha^n \cong \mathbb{1}_\beta{\uparrow}_\beta^n$. But it is a well-known and easily proved result that if H is a subgroup of a finite group G, then the kG-module $\mathbb{1}_H{\uparrow}_H^G$ is the permutation module for kG obtained by taking the k-linearisation of the right H-coset space of G (recall that this is the right G-set formed by equipping the set of right H-cosets Hg in G with the action $(Hg_1)g_2 = H(g_1g_2)$ for $g_1, g_2 \in G$). Further, it is also a well-known and easily proved result that if H and K are subgroups of a finite group G, then the space of right H-cosets in G and the space of right K-cosets in G are isomorphic as G-sets if H and K are conjugate in G, meaning that $H = g^{-1}Kg$ for some $g \in G$. Thus it suffices to prove that the Young subgroups S_α and S_β are conjugate in S_n. This is a well-known fact, which may easily be proved using the standard result that if $\delta, \epsilon \in S_n$, then the conjugate $\epsilon^{-1}\delta\epsilon$ of δ by ϵ is the permutation whose cycle notation is obtained by replacing each number i in the cycle notation of δ with $(i)\epsilon$. $\square$

Now if λ is a partition of n, then we identify a certain submodule S^λ of M^λ, called the **Specht module** associated to λ. For the full definition we refer the reader to Chapter 4 of [20], but in summary the Specht module S^λ is defined as the k-span of all *polytabloids* in M^λ, where a polytabloid is a certain element of M^λ, with one polytabloid associated to each λ-tableau. If char(k) is 0 or greater than n, then the collection of all Specht modules S^λ for $\lambda \vdash n$ forms a complete system of isomorphism classes of simple kS_n-modules without redundancy [20, Chapter 11]. If char(k) is greater than 0 but less than or equal to n, then the Specht modules are not in general simple. We define $\mathcal{F}_S$ to be the category of finite-dimensional kS_n-modules with a Specht filtration. If kS_n is semisimple, then $\mathcal{F}_S$ is the category of all finite-dimensional kS_n-modules, but $\mathcal{F}_S$ is still of interest when kS_n is not semisimple. We shall review some theory of the category $\mathcal{F}_S$ for a general field k, although some of our results will require that k have characteristic different from 2 and 3 or that k be algebraically closed.

Amongst the Specht modules, we mention two which are of particular interest. Firstly, the Specht module $S^{(n)}$ turns out to be isomorphic to the trivial kS_n-module $\mathbb{1}_n$, and secondly the Specht module $S^{(1,1,\ldots,1)}$ is isomorphic to the sign module Sgn_n (see [20, page 14] for these facts). We shall thus freely interchange the notations $S^{(n)}$ and $\mathbb{1}_n$ and also $S^{(1,1,\ldots,1)}$ and Sgn_n.

The following fact relating dual Specht modules to conjugation of partitions will be useful in our work.

Proposition 3.1.3. *([20, Theorem 8.15]) Let ν be a partition of n. Then we have an isomorphism of kS_n-modules*

$$(S^{\nu})^* \cong \mathrm{Sgn}_n \otimes S^{\nu'}.$$

As mentioned above, if kS_n is not semisimple, the Specht modules are no longer simple in general. However, even in this situation the Specht modules may be used to obtain a complete list of the isomorphism classes of simple kS_n-modules. Recall that, with p the characteristic of our field k, so that p is either zero or a prime, a partition is *p-singular* if $p > 0$ and the partition contains a constant subsequence of length p, and a partition is *p-regular* otherwise. Recall from [20, Definition 11.2] that to each p-regular partition λ of n, we associate a kS_n-module D^{λ}. We summarise the relevant properties of these modules in the following theorem.

Theorem 3.1.4. *[20, Theorem 11.5, Corollary 12.2] Let k be a field of characteristic p (so p is zero or a prime), and let n be a non-negative integer. If λ is a p-regular partition of n, then D^{λ} is a simple kS_n-module. Further, as λ varies over all p-regular partitions of n, so D^{λ} varies over all isomorphism classes of simple kS_n-modules without repetition.*

Now let λ be any partition of n. Then the composition factors of S^λ are all of the form D^μ for $\mu \trianglerighteq \lambda$. Thus if λ is p-singular, the composition factors of S^λ are all of the form D^μ for $\mu \triangleright \lambda$. Further, if λ is p-regular, then the multiplicity of D^λ in S^λ is exactly one, and moreover in any composition series of S^λ, the top factor is D^λ and all the other factors are D^μ for $\mu \triangleright \lambda$.

3.2 The Littlewood-Richardson rule and Young's rule

In this section, we recall some important results which give filtrations of certain modules for kS_n, and which moreover give information about the multiplicities occurring in those filtrations.

Firstly, let $\lambda \vdash n$. Then **Young's rule** [20, 14.1 and 17.14] tells us that we have coefficients $K(\nu, \lambda)$ such that

$$M^\lambda \sim \mathop{\mathcal{F}_{\langle\lambda\rangle}}_{\nu \vdash n} K(\nu, \lambda) S^\nu. \tag{3.2.1}$$

The coefficients $K(\nu, \lambda)$ are called **Kostka numbers**, and they have a pleasing combinatorial interpretation, for which we need another definition. Indeed, a tableau of shape ν and type β, where ν is a partition and β is a composition, is **semistandard** if the entries are non-decreasing from left to right in each row, and the entries are strictly increasing down each column. For example, with $\nu = (5, 4, 4, 1, 1)$ and $\beta = (3, 3, 2, 4, 1, 0, 2)$ as above, then one semistandard ν-tableau of type β is

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 7 \\ \hline 2 & 2 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 4 & 4 & 7 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & \multicolumn{4}{c}{} \\ \cline{1-1} 5 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \; .$$

Then the Kostka number $K(\nu, \lambda)$ is equal to the number of semistandard ν-tableaux of type λ [20, 14.1]. From this, it follows easily that the Kostka numbers satisfy

$$K(\nu, \lambda) = \begin{cases} 1 & \text{if } \nu = \lambda \\ 0 & \text{if } \nu \not\trianglerighteq \lambda. \end{cases} \tag{3.2.2}$$

We shall often need to consider the restriction of a Specht module to a Young subgroup, or the module obtained by inducing an outer tensor product of Specht modules up from a Young subgroup to the full symmetric group. In particular, we shall be interested in obtaining useful filtrations of such modules. The tools for this task are the **Littlewood-Richardson filtration rules**. In order to present these results, we must first consider the combinatorial **Littlewood-Richardson rule** and the associated **Littlewood-Richardson coefficients**, and to do this, we must recall some material on *symmetric functions*. Our source is [32, Chapter 7], but since we shall not make any further use of this material, our presentation of it here will be a rough sketch only, so we refer the reader to [32] for more details if they are desired.

Let us denote by $\mathcal{S}_{\mathbb{C}}$ the **ring of symmetric functions** over $\mathbb{C}$ in the variables $x_1, x_2, \ldots$, and further for $n \geq 0$ let $\mathcal{S}_{\mathbb{C}}^n$ denote the $\mathbb{C}$-vector space consisting of all homogeneous symmetric functions of degree n together with the zero element. The definition of a symmetric function need not concern us here; it suffices to know that $\mathcal{S}_{\mathbb{C}}$ is an infinite-dimensional commutative unital associative $\mathbb{C}$-algebra, with a grading

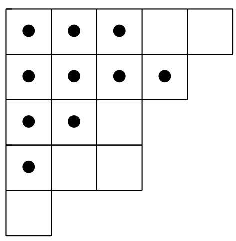
$$\mathcal{S}_{\mathbb{C}} = \bigoplus_{n \geq 0} \mathcal{S}_{\mathbb{C}}^n.$$

To each partition λ of each $n \geq 0$, we associate the **Schur function** $s_\lambda \in \mathcal{S}_{\mathbb{C}}^n$. Again, we do not need the definition of s_λ here; we need only know that for each n, the set of all Schur functions s_λ for $\lambda \vdash n$ forms a $\mathbb{C}$-basis of $\mathcal{S}_{\mathbb{C}}^n$ [32, Corollary 7.10.6], and that $s_{()} = 1$ [32, Definition 7.10.1]. Now if α, β are partitions (not necessarily of the same integer), then $s_\alpha s_\beta$ is a homogeneous symmetric function of degree $|\alpha|+|\beta|$, and hence we have for each $\lambda \vdash |\alpha|+|\beta|$ a uniquely-defined coefficient $c^\lambda_{\alpha,\beta} \in \mathbb{C}$ such that

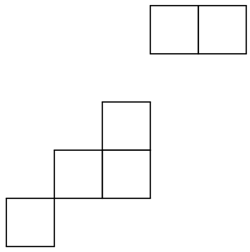
$$s_\alpha s_\beta = \sum_{\lambda \vdash |\alpha|+|\beta|} c^\lambda_{\alpha,\beta} s_\lambda.$$

The coefficients $c^\lambda_{\alpha,\beta}$ are called **Littlewood-Richardson coefficients**. We extend the definition by defining $c^\lambda_{\alpha,\beta} = 0$ for any three partitions α, β, λ where $|\lambda| \neq |\alpha| + |\beta|$.

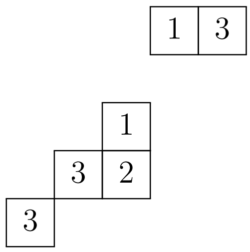
Like the Kostka numbers, the Littlewood-Richardson coefficients have a nice combinatorial interpretation, and as for the Kostka numbers we need to recall some more combinatorics to state this. Indeed, if α and β are compositions and we have $\alpha_i \leq \beta_i$ for all i (with any parts that would otherwise be undefined taken to be zero as usual), then we say that α **lies inside** β and write $\alpha \subseteq \beta$; this necessarily implies $|\alpha| \leq |\beta|$. This terminology makes sense if one notes that α lies inside β if and only if the Young diagram of α is a subdiagram of the Young diagram of β. For example, we have $(3, 4, 2, 1) \subseteq (5, 4, 3, 3, 1)$, which we see by drawing the Young diagram of $(5, 4, 3, 3, 1)$ and picking out the Young diagram of $(3, 4, 2, 1)$ inside it,


.

If $\alpha \subseteq \beta$, then the **skew Young diagram** of shape $\beta \setminus \alpha$ is the diagram obtained by starting with the Young diagram of β and removing the boxes which form the copy of the Young diagram of α inside it. For example, the skew Young diagram of shape $(5, 4, 3, 3, 1) \setminus (3, 4, 2, 1)$ is



(note that this diagram is disconnected, as skew Young diagrams may in general be). We extend the definition of a Young tableau to allow its underlying Young diagram to be a skew Young diagram, and we call such an object a **skew tableau**. Thus for example



is a skew tableau of shape $(5, 4, 3, 3, 1) \setminus (3, 4, 2, 1)$ and type $(2, 1, 3)$. We extend the definition of semistandardness to skew tableaux, noting that there may be gaps in the columns of a skew tableau.

The **Littlewood-Richardson rule** [32, Theorem A1.3.3] states that $c^{\lambda}_{\alpha,\beta}$ is equal to the number of skew semistandard tableaux of shape $\lambda \setminus \alpha$ and type β where the sequence obtained by concatenating its reversed rows is a **lattice word**. A lattice word is a finite sequence of integers, allowing repetitions, such that if for any $r \geq 0$ and any $i \geq 0$ we let $\#^i_r$ be the number of times i appears in the first r places of the sequence, then for each r we have $\#_r^1 \geqslant \#_r^2 \geqslant \#_r^3 \geqslant \cdots$. For example

$$\begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & & & \\ & & \boxed{2} & & \\ & \boxed{1} & \boxed{3} & & \\ \boxed{2} & & & & \end{array}$$

is such a tableau of shape $(5,4,3,3,1) \setminus (3,4,2,1)$ and type $(3,2,1)$. In particular, every Littlewood-Richardson coefficient is in fact a non-negative integer.

From the Littlewood-Richardson rule we may easily deduce the following result.

Proposition 3.2.1. *(see [29, page 142]) Suppose that we have $c^{\lambda}_{\alpha,\beta} \neq 0$ for partitions α, β, λ (so that by definition we have $|\alpha| + |\beta| = |\lambda|$). Then we must have $\alpha, \beta \subseteq \lambda$, and further we must have*

$$\sum_{i=1}^{q} \lambda_i \leqslant \sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{q} \beta_i \qquad \textit{for } q = 1, 2, 3, \ldots$$

(i.e. $\alpha + \beta \trianglerighteq \lambda$, where addition of partitions is defined pointwise) taking any parts of partitions which would otherwise be undefined to be 0 as usual.

We generalise the above definition of Littlewood-Richardson coefficients as follows: for any partition λ and any multipartition $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ (for $t \geq 0$), we define the Littlewood-Richardson coefficient $c(\lambda; \underline{\alpha})$ to be the coefficient of s_λ in the product $s_{\alpha^1} s_{\alpha^2} \cdots s_{\alpha^t}$. Thus in particular $c(\lambda; \underline{\alpha}) = 0$ unless $|\lambda| = |\alpha^1| + \cdots + |\alpha^t| = ||\underline{\alpha}||$. Further, if either λ or $\underline{\alpha}$ is $()$ then $c(\lambda; \underline{\alpha}) = 0$ unless both λ and $\underline{\alpha}$ are $()$, in which case we have $c\big((); ()\big) = 1$; this last fact follows from the fact that $s_{()} = 1$. Another consequence of the fact that $s_{()} = 1$ is that we have

$$c(\lambda; \underline{\alpha}) = c(\lambda; \widehat{\underline{\alpha}}) \tag{3.2.3}$$

where $\widehat{\underline{\alpha}}$ is the multipartition obtained from $\underline{\alpha}$ by removing any empty partitions, so that for example if $\underline{\alpha} = \big((2,1),(),(1),(3,1,1),()\big)$ then we have $\widehat{\underline{\alpha}} = \big((2,1),(1),(3,1,1)\big)$. Keeping our partition λ and multipartition $\underline{\alpha} = (\alpha^1,\dots,\alpha^t)$ of n, if we assume that $t \geq 2$ then we have

$$s_{\alpha^2}\cdots s_{\alpha^t} = \sum_{\beta \vdash n-|\alpha^1|} c\big(\beta;(\alpha^2,\dots,\alpha^t)\big)s_\beta$$

and hence

$$\begin{aligned} s_{\alpha^1}s_{\alpha^2}\cdots s_{\alpha^t} &= \sum_{\beta \vdash n-|\alpha^1|} c\big(\beta;(\alpha^2,\dots,\alpha^t)\big)s_{\alpha^1}s_\beta \\ &= \sum_{\beta \vdash n-|\alpha^1|} c\big(\beta;(\alpha^2,\dots,\alpha^t)\big)\Big(\sum_{\lambda\vdash n} c^\lambda_{\alpha^1,\beta}\, s_\lambda\Big) \\ &= \sum_{\lambda\vdash n}\Bigg(\sum_{\beta \vdash n-|\alpha^1|} c^\lambda_{\alpha^1,\beta}\, c\big(\beta;(\alpha^2,\dots,\alpha^t)\big)\Bigg)s_\lambda. \end{aligned}$$

Thus we see that

$$c\big(\lambda;\underline{\alpha}\big) = \sum_{\beta \vdash n-|\alpha^1|} c^\lambda_{\alpha^1,\beta}\, c\big(\beta;(\alpha^2,\dots,\alpha^t)\big). \tag{3.2.4}$$

We also note that, for the case $t=1$, we have

$$c(\lambda;\underline{\alpha}) = c\big(\lambda;(\alpha^1)\big) = \begin{cases} 1 & \text{if } \underline{\alpha} = (\lambda) \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.5}$$

Further, for the case $t=2$, we have

$$c(\lambda;\underline{\alpha}) = c\big(\lambda;(\alpha^1,\alpha^2)\big) = c^\lambda_{\alpha^1,\alpha^2}. \tag{3.2.6}$$

The following lemma is a trivial consequence of the commutativity of $\mathcal{S}_\mathbb{C}$.

Lemma 3.2.2. *Let λ be a partition and $\underline{\alpha} = (\alpha^1,\dots,\alpha^t)$ a multipartition (with $t \geq 0$). Then $c(\lambda;\underline{\alpha})$ is invariant under permutation of the components of $\underline{\alpha}$.*

In particular, we see from Lemma 3.2.2 that $c^{\lambda}_{\alpha,\beta} = c^{\lambda}_{\beta,\alpha}$.

Lemma 3.2.3. *Suppose we have a partition λ and a multipartition $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ such that $c(\lambda; \underline{\alpha}) \neq 0$. Then*

1. $|\lambda| = |\alpha^1| + \cdots + |\alpha^t|$.

2. *For each $i = 1, \ldots, t$ we have $\alpha^i \subseteq \lambda$, from which it follows that*

$$\sum_{j=1}^{q} \lambda_j \geqslant \sum_{j=1}^{q} \alpha^i_j \qquad \text{for } q = 1, 2, 3, \ldots,$$

   taking any parts of partitions which would otherwise be undefined to be 0 as usual.

3. *For each $i = 1, \ldots, t$ we have a partition β^i of $|\lambda| - |\alpha^i|$ such that*

$$\sum_{j=1}^{q} \lambda_j \leqslant \sum_{j=1}^{q} \alpha^i_j + \sum_{j=1}^{q} \beta^i_j \qquad \text{for } q = 1, 2, 3, \ldots$$

   (i.e. $\alpha^i + \beta^i \trianglerighteq \lambda$, where addition of compositions is defined pointwise) taking any parts of partitions which would otherwise be undefined to be 0 as usual.

Proof. Part (1) is immediate from the definition of $c(\lambda; \underline{\alpha})$. Parts (2) and (3) may be obtained as follows. First, we use Lemma 3.2.2 to see that, for each $i = 1, \ldots, t$, the coefficient $c(\lambda; \underline{\alpha})$ is equal to $c(\lambda; \hat{\underline{\alpha}}^i)$, where $\hat{\underline{\alpha}}^i$ represents the multipartition obtained from $\underline{\alpha}$ by moving α^i to the first place. Then we apply (3.2.4) to $c(\lambda; \hat{\underline{\alpha}}^i) \neq 0$ to see that we must have some partition β such that $c^{\lambda}_{\alpha^i,\beta} \neq 0$ (by looking at the first factor in the summand on the right-hand side of the resulting equation). Applying Proposition 3.2.1 to the coefficient $c^{\lambda}_{\alpha^i,\beta} \neq 0$ and taking $\beta^i = \beta$ and, we obtain parts (2) and (3) of the lemma. $\square$

As mentioned above, our interest in Littlewood-Richardson coefficients lies in their relation to the representation theory of the symmetric group. The following results are well-known over the complex numbers (where they describe direct sum decompositions), but they hold over any field, and in this generality the proof is due to James and Peel in [24] (Theorems 3.1 and 5.5). See also the remark on page 70 of [20], or [27, Theorem 2.4]. Let λ and μ be partitions (not necessarily of the same size) and let $n = |\lambda| + |\mu|$. Then we have a kS_n-module $(S^\lambda \boxtimes S^\mu)\big\uparrow^n_{(|\lambda|,|\mu|)}$, and this module has a filtration by Specht modules S^ν for $\nu \vdash n$, and the multiplicity with which S^ν occurs is the Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$. That is,

$$(S^\lambda \boxtimes S^\mu)\big\uparrow^n_{(|\lambda|,|\mu|)} \sim \mathop{\mathcal{F}}_{\nu \vdash n} c^\nu_{\lambda,\mu}\, S^\nu. \tag{3.2.7}$$

There is a "dual" version of this result, which states that if $\nu \vdash n$ and a, b are integers such that $a + b = n$, then the $k(S_a \times S_b)$-module $S^\nu\big\downarrow^n_{(a,b)}$ has a filtration by modules $S^\lambda \boxtimes S^\mu$ for $\lambda \vdash a$, $\mu \vdash b$, where $S^\lambda \boxtimes S^\mu$ occurs with multiplicity $c^\nu_{\lambda,\mu}$, so that

$$S^\nu\big\downarrow^n_{(a,b)} \sim \mathop{\mathcal{F}}_{\lambda \vdash a,\, \mu \vdash b} c^\nu_{\lambda,\mu}\, S^\lambda \boxtimes S^\mu. \tag{3.2.8}$$

The name "Littlewood-Richardson rule" is sometimes applied to (3.2.7) and (3.2.8), but we shall reserve that name for the combinatorial characterisation of the Littlewood-Richardson coefficients, and call (3.2.7) and (3.2.8) the **Littlewood-Richardson filtration rules**.

We shall require versions of the Littlewood-Richardson filtration rules (3.2.7) and (3.2.8) where the Young subgroup involved corresponds to a general composition rather that just a two-part composition, and we introduce some notation for this. Indeed, if $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ is a multipartition of n, then we define a $kS_{|\underline{\alpha}|}$-module $S(\underline{\alpha})$ by setting

$$S(\underline{\alpha}) = S^{\alpha^1} \boxtimes S^{\alpha^2} \boxtimes \cdots \boxtimes S^{\alpha^t}, \tag{3.2.9}$$

recalling that $|\underline{\alpha}|$ is the composition $(|\alpha^1|, \ldots, |\alpha^t|)$ of n. We also define the $kS_{|\underline{\alpha}|}$-module $M(\underline{\alpha})$ by setting

$$M(\underline{\alpha}) = M^{\alpha^1} \boxtimes M^{\alpha^2} \boxtimes \cdots \boxtimes M^{\alpha^t}. \tag{3.2.10}$$

Now let $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ be a multipartition of n. Then, defining $a = |\alpha^2| + \cdots + |\alpha^t|$ we have

$$\begin{aligned}
S(\underline{\alpha})\big\uparrow_{|\underline{\alpha}|}^{n} &= \left(S^{\alpha^1} \boxtimes \cdots \boxtimes S^{\alpha^t}\right)\big\uparrow_{(|\alpha^1|,\ldots,|\alpha^t|)}^{n} \\
&= \left(S^{\alpha^1} \boxtimes \cdots \boxtimes S^{\alpha^t}\right)\big\uparrow_{(|\alpha^1|,\ldots,|\alpha^t|)}^{(|\alpha^1|,a)}\big\uparrow_{(|\alpha^1|,a)}^{n} \\
&\qquad \text{(by transitivity of induction)} \\
&= \left[S^{\alpha^1} \boxtimes \left(S^{\alpha^2} \boxtimes \cdots \boxtimes S^{\alpha^t}\right)\big\uparrow_{(|\alpha^2|,\ldots,|\alpha^t|)}^{a}\right]\Big\uparrow_{(|\alpha^1|,a)}^{n} \\
&\qquad \text{(by Lemma 2.2.1).}
\end{aligned}$$

By using the Littlewood-Richardson filtration rule (3.2.7) and (3.2.4), we may now easily prove by induction on t that

$$S(\underline{\alpha})\big\uparrow_{|\underline{\alpha}|}^{n} \sim \mathop{\mathcal{F}}_{\nu \vdash n} c(\nu; \underline{\alpha})\, S^{\nu}. \tag{3.2.11}$$

By a very similar argument involving transitivity of restriction and the Littlewood-Richardson filtration rule (3.2.8), we may prove that if $\nu \vdash n$ and $\gamma = (\gamma_1, \ldots, \gamma_t)$ is a composition of n, then

$$S^{\nu}\big\downarrow_{\gamma}^{n} \sim \mathop{\mathcal{F}}_{|\underline{\alpha}|=\gamma} c(\nu; \underline{\alpha})\, S(\underline{\alpha}) \tag{3.2.12}$$

where $\underline{\alpha}$ ranges over all multipartitions $\underline{\alpha}$ of n such that $|\underline{\alpha}| = \gamma$.

3.3 Homomorphisms and extensions between Specht modules

By corollary 13.17 in [20], we know that if $\mathrm{char}(k) \neq 2$ and $\lambda, \nu \vdash n$, then the k-vector space $\mathrm{Hom}_n(S^{\nu}, M^{\lambda})$ is zero if $\nu \ntrianglerighteq \lambda$, and is one-dimensional if $\nu = \lambda$. Now if γ is a composition of n such that λ is the partition of n obtained by rearranging the parts of γ into non-increasing order, then we clearly have $\lambda \trianglerighteq \gamma$. It follows by Proposition 3.1.2 that if $\nu \vdash n$ and $\gamma \vDash n$, then we have

$$\mathrm{Hom}_n(S^\nu, M^\gamma) \cong \begin{cases} 0 & \text{if } \nu \ntrianglerighteq \gamma \\ k & \text{if } \nu = \gamma. \end{cases} \tag{3.3.1}$$

Since S^λ is a submodule of M^λ for any partition λ of n, it follows that $\mathrm{Hom}_n(S^\nu, S^\lambda)$ is a subspace of $\mathrm{Hom}_n(S^\nu, M^\lambda)$, from which we see that for $\lambda, \nu \vdash n$ we have

$$\mathrm{Hom}_n(S^\nu, S^\lambda) \cong \begin{cases} 0 & \text{if } \nu \ntrianglerighteq \lambda \\ k & \text{if } \nu = \lambda. \end{cases} \tag{3.3.2}$$

The following proposition is immediate from (3.3.2).

Proposition 3.3.1. *([20, 13.18]) If* $\mathrm{char}(k) \neq 2$*, then* S^λ *is indecomposable for any* $\lambda \vdash n$.

We shall next consider extensions between Specht modules. In [10], Erdmann proved the following theorem.

Theorem 3.3.2. *Let k be a field of characteristic not 2 or 3, and let μ, λ be partitions of n such that $\mu \ntriangleright \lambda$. Then*

$$\mathrm{Ext}^1_n\left(S^\mu, S^\lambda\right) = 0.$$

We shall now give a full and detailed proof of this theorem. In later chapters, we shall use a method inspired by these arguments to prove a corresponding result for the wreath product $S_m \wr S_n$. We begin with a lemma.

Lemma 3.3.3. *Let k be a field of characteristic not 2, and let δ, ϵ be partitions of n such that $\delta \ntriangleright \epsilon$. Then we have an injective map of k-vector spaces*

$$\mathrm{Ext}^1_n\left(S^\delta, S^\epsilon\right) \hookrightarrow \mathrm{Ext}^1_n\left(S^\delta, M^\epsilon\right).$$

Proof. We have a short exact sequence of kS_n-modules

$$0 \longrightarrow S^{\epsilon} \longrightarrow M^{\epsilon} \longrightarrow \frac{M^{\epsilon}}{S^{\epsilon}} \longrightarrow 0$$

and hence we may apply the functor $\mathrm{Hom}_n\left(S^{\delta}, -\right)$ to obtain a long exact sequence of k-vector spaces

$$\begin{tikzcd}
0 \to \mathrm{Hom}_n\left(S^{\delta}, S^{\epsilon}\right) \arrow[r] & \mathrm{Hom}_n\left(S^{\delta}, M^{\epsilon}\right) \arrow[r] & \mathrm{Hom}_n\left(S^{\delta}, \frac{M^{\epsilon}}{S^{\epsilon}}\right) \arrow[dll, bend left] \\
\mathrm{Ext}^1_n\left(S^{\delta}, S^{\epsilon}\right) \arrow[r] & \mathrm{Ext}^1_n\left(S^{\delta}, M^{\epsilon}\right) \arrow[r] & \cdots .
\end{tikzcd}$$

Now by Young's rule (3.2.1) and the properties of the Kostka numbers (3.2.2), $\frac{M^{\epsilon}}{S^{\epsilon}}$ is filtered by Specht modules S^{θ} for $\theta \rhd \epsilon$. Now if $\theta \rhd \epsilon$, then $\delta \not\trianglerighteq \theta$ (otherwise we have $\delta \trianglerighteq \theta \rhd \epsilon$, and hence $\delta \rhd \epsilon$, a contradiction). By (3.3.2), $\delta \not\trianglerighteq \theta$ implies

$$\mathrm{Hom}_n(S^{\delta}, S^{\theta}) = 0.$$

It now follows by Proposition 2.1.1 that

$$\mathrm{Hom}_n\left(S^{\delta}, \frac{M^{\epsilon}}{S^{\epsilon}}\right) = 0,$$

and hence by exactness of our long exact sequence we obtain the desired injection. $\square$

By Lemma 3.3.3, we see that proving the following proposition will establish Theorem 3.3.2.

Proposition 3.3.4. *If k is a field of characteristic not 2 or 3, then for any partition μ of n and any composition α of n, we have*

$$\mathrm{Ext}^1_n\left(S^{\mu}, M^{\alpha}\right) = 0.$$

Note that Proposition 3.3.4 does *not* require any ordering condition on μ and α: they may be *any* partition and composition of n.

The following reduction will provide our path to proving Proposition 3.3.4 and thus Theorem 3.3.2.

Reduction 3.3.5. *To prove Proposition 3.3.4, it is enough to prove that if k is a field of characteristic not 2 or 3, then for any partition $\gamma \vdash n$ we have*

$$\mathrm{Ext}^1_n\left(S^\gamma, \mathbb{1}_n\right) = 0.$$

Proof. If μ is any partition of n and α is any composition of n, then

$$\begin{aligned}\mathrm{Ext}^1_n\left(S^\mu, M^\alpha\right) &\cong \mathrm{Ext}^1_n\left(S^\mu, \mathbb{1}_\alpha\uparrow^n_\alpha\right)\\ &\cong \mathrm{Ext}^1_\alpha\left(S^\mu\downarrow^n_\alpha, \mathbb{1}_\alpha\right) \qquad \text{(by Theorem 2.2.4)}.\end{aligned}$$

Thus it suffices to show that $\mathrm{Ext}^1_\alpha\left(S^\mu\downarrow^n_\alpha, \mathbb{1}_\alpha\right) = 0$. Let $\alpha = (\alpha_1, \dots, \alpha_r)$, so that S_α is canonically isomorphic to $S_{\alpha_1} \times \cdots \times S_{\alpha_r}$. Then by (3.2.12), $S^\mu\downarrow^n_\alpha$ has a filtration whose factors are modules of the form

$$S^{\gamma^1} \boxtimes S^{\gamma^2} \boxtimes \cdots \boxtimes S^{\gamma^r}$$

where each γ^i is a partition of α_i. Thus by Proposition 2.1.1 it suffices to prove that if $\gamma^i \vdash \alpha_i$ for each $i = 1, \dots, r$, then

$$\mathrm{Ext}^1_\alpha\left(S^{\gamma^1} \boxtimes S^{\gamma^2} \boxtimes \cdots \boxtimes S^{\gamma^r}, \mathbb{1}_\alpha\right) = 0.$$

But we have

$$\mathrm{Ext}^1_\alpha\left(S^{\gamma^1}\boxtimes S^{\gamma^2}\boxtimes\cdots\boxtimes S^{\gamma^r}, \mathbb{1}_\alpha\right) \cong \mathrm{Ext}^1_\alpha\left(S^{\gamma^1}\boxtimes S^{\gamma^2}\boxtimes\cdots\boxtimes S^{\gamma^r}, \mathbb{1}_{\alpha_1}\boxtimes\cdots\boxtimes\mathbb{1}_{\alpha_r}\right)$$

and hence by Proposition 2.1.3 it suffices to prove that $\mathrm{Ext}^1_n\left(S^\gamma, \mathbb{1}_n\right) = 0$. □

Proposition 3.3.6. *Let k be a field whose characteristic is not 2. Let $\gamma \vdash n$. Then*

$$\mathrm{Ext}^1_n(S^\gamma, \mathbb{1}_n) \cong \mathrm{Ext}^1_n(\mathrm{Sgn}_n, S^{\gamma'}).$$

Proof. We have

$$\begin{aligned}\mathrm{Ext}^1_n\left(\mathrm{Sgn}_n, S^{\gamma'}\right) &\cong \mathrm{Ext}^1_n\left(\mathrm{Sgn}_n, \left((S^{\gamma'})^*\right)^* \otimes \mathbb{1}_n\right)\\ &\cong \mathrm{Ext}^1_n\left(\mathrm{Sgn}_n \otimes (S^{\gamma'})^*, \mathbb{1}_n\right) \quad \text{(by Proposition 2.2.3)}\\ &\cong \mathrm{Ext}^1_n\left(S^\gamma, \mathbb{1}_n\right) \quad \text{(by Proposition 3.1.3)}.\end{aligned}$$

□

We now prove Theorem 3.3.2 by proving that if k is a field of characteristic not 2 or 3, then for any partition $\gamma \vdash n$ we have $\mathrm{Ext}^1_n(\mathrm{Sgn}_n, S^\gamma) = 0$. The special case where $\gamma = (n)$, so that $S^\gamma \cong \mathbb{1}_n$, provides the necessary stepping-stone to the general result.

Lemma 3.3.7. *Let k be a field with* $\mathrm{char}(k) \notin \{2, 3\}$. *Then*

$$\mathrm{Ext}^1_n(\mathrm{Sgn}_n, \mathbb{1}_n) = 0.$$

Proof. Suppose we have a kS_n-module E with $x \in E$ such that kx (the k-span of x in E) is a kS_n-submodule of E with $kx \cong \mathbb{1}_n$ and $\frac{E}{kx} \cong \mathrm{Sgn}_n$. By Proposition 2.1.4, it is enough to show that E has a direct sum decomposition $E = kx \oplus Z$ as a kS_n-module.

Choose $y \in E$ such that x and y form a basis of E. Thus $x\sigma = x$ for all $\sigma \in S_n$ and

$$(y + kx)\sigma = \mathrm{sgn}(\sigma)(y + kx)$$

so that

$$y\sigma + kx = \mathrm{sgn}(\sigma)y + kx$$

and hence for each $\sigma \in S_n$ we have $u_\sigma \in k$ such that

$$y\sigma = \mathrm{sgn}(\sigma)y + u_\sigma x.$$

Now for each $i \in \{1, \ldots, n-1\}$ define σ_i to be the basic transposition $(i,\, i+1)$, so that we have

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

(this is one of the *braid relations* for the symmetric group) and further define $u_i = u_{\sigma_i}$ so that

$$y\sigma_i = u_i x - y.$$

Then for each $i \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned}
y(\sigma_i\sigma_{i+1}\sigma_i) &= (u_i x - y)(\sigma_{i+1}\sigma_i) \\
&= (u_i x - (u_{i+1}x - y))\sigma_i \\
&= (u_i x - u_{i+1}x + y)\sigma_i \\
&= u_i x - u_{i+1}x + u_i x - y \\
&= (2u_i - u_{i+1})x - y.
\end{aligned}$$

Similarly,

$$y(\sigma_{i+1}\sigma_i\sigma_{i+1}) = (2u_{i+1} - u_i)x - y$$

so that

$$\begin{aligned}
&2u_{i+1} - u_i = 2u_i - u_{i+1} \\
&\Rightarrow\ 3(u_i - u_{i+1}) = 0 \\
&\Rightarrow\ u_i - u_{i+1} = 0 \qquad \text{(because char}(k) \neq 3) \\
&\Rightarrow\ u_i = u_{i+1}.
\end{aligned}$$

Thus all of the scalars u_i have a common value. Let us denote this by u. Now define

$$z = y - \frac{u}{2}x$$

(recalling that $\mathrm{char}(k) \neq 2$), so that z and x form a basis of E. To establish the claim, we need only prove that kz is a kS_n-submodule of E, and since S_n is generated by the basic transpositions, it is enough to prove that $z\sigma_i \in kz$

for each $i \in \{1, \dots, n-1\}$. Indeed,

$$\begin{aligned}
z\sigma_i &= (y - \frac{u}{2}x)\sigma_i \\
&= y\sigma_i - \frac{u}{2}x\sigma_i \\
&= (ux - y) - \frac{u}{2}x \\
&= \frac{u}{2}x - y \\
&= -z.
\end{aligned}$$

□

Proposition 3.3.8. *Let k be a field of characteristic not 2 or 3, and let $\gamma \vdash n$. We have*

$$\mathrm{Ext}_n^1\left(\mathrm{Sgn}_n, S^\gamma\right) = 0.$$

By Reduction 3.3.5 and Proposition 3.3.6, establishing Proposition 3.3.8 will prove Proposition 3.3.4 and hence prove Theorem 3.3.2.

Proof. We have

$$\begin{aligned}
\mathrm{Ext}_n^1(\mathrm{Sgn}_n, M^\gamma) &\cong \mathrm{Ext}_n^1(\mathrm{Sgn}_n, \mathbb{1}_\gamma\!\uparrow_\gamma^n) \\
&\cong \mathrm{Ext}_\gamma^1(\mathrm{Sgn}_n\!\downarrow_\gamma^n, \mathbb{1}_\gamma) \qquad \text{(by Theorem 2.2.4)}.
\end{aligned}$$

Now if we let $\gamma = (\gamma_1, \dots, \gamma_t)$, then by (3.1.1) we have $\mathrm{Sgn}_n\!\downarrow_\gamma^n \cong \mathrm{Sgn}_{\gamma_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\gamma_t}$. Thus, using Proposition 2.1.3 and Lemma 3.3.7, we get

$$\mathrm{Ext}_n^1(\mathrm{Sgn}_n, M^\gamma) = 0.$$

We know from above that $\mathrm{Sgn}_n \cong S^{(1,1,\dots,1)}$, and since $(1, 1, \dots, 1) \ntrianglerighteq \beta$ for all $\beta \vdash n$, we see by Lemma 3.3.3 that

$$\mathrm{Ext}_n^1\left(\mathrm{Sgn}_n, S^\gamma\right) = 0.$$

□

3.4 Stratifying systems

The original definition of a *stratifying system* for an algebra was given by Cline, Parshall, and Scott in [7] as part of their work on *standardly stratified algebras*. However, we shall use the definition of a *stratifying system* given by Erdmann and Sáenz in [11], which is based on the work of Xi in [36], of Dlab and Ringel in [9], and of Ágoston et al in [1]. Indeed, if A is a finite-dimensional algebra over an algebraically closed field k, then a **stratifying system** for A consists of A-modules $\Theta_1, \dots, \Theta_r$ and indecomposable A-modules $Y_1, \dots, Y_r$ such that

1. $\mathrm{Hom}_A(\Theta_i, \Theta_j) = 0$ if $i > j$
2. for each i, there is a short exact sequence $0 \to \Theta_i \to Y_i \to Z_i \to 0$ where Z_i has a filtration by Θ_j with $j < i$
3. if X is an A-module with a filtration by the modules $\Theta_1, \dots, \Theta_r$, then $\mathrm{Ext}^1_A(X, Y) = 0$ where $Y = \bigoplus_{i=1}^r Y_i$.

Further, it was proved in [11] that if we have a collection $\Theta_1, \dots, \Theta_r$ of indecomposable A-modules satisfying the conditions

1. $\mathrm{Hom}_A(\Theta_i, \Theta_j) = 0$ if $i > j$
2. $\mathrm{Ext}^1_A(\Theta_i, \Theta_j) = 0$ if $i \geq j$

then there exist A-modules $Y_1, \dots, Y_r$ which together with the A-modules $\Theta_1, \dots, \Theta_r$ form a stratifying system.

Our interest in stratifying systems comes from the following result, and more particularly its corollary.

Proposition 3.4.1 ([11], Lemma 1.4)**.** *Let A be a finite-dimensional algebra over an algebraically closed field k with a stratifying system given by modules $\Theta_1, \dots, \Theta_r, Y_1, \dots, Y_r$ as above. Suppose that some A-module M has a* *filtration by $\Theta_1, \ldots, \Theta_r$. Then for any two filtrations of M by $\Theta_1, \ldots, \Theta_r$, the multiplicity with which each Θ_i occurs is the same in both filtrations.*

If the conclusion of Proposition 3.4.1 holds, we say that "Θ-filtration multiplicities are well-defined".

Corollary 3.4.2. *Let A be a finite-dimensional algebra over an algebraically closed field. If $\Theta_1, \ldots, \Theta_r$ are indecomposable A-modules such that*

1. $\mathrm{Hom}_A(\Theta_i, \Theta_j) = 0$ *if* $i > j$

2. $\mathrm{Ext}^1_A(\Theta_i, \Theta_j) = 0$ *if* $i \geq j$

then Θ-filtration multiplicities are well-defined.

In order to apply this result to the Specht modules, we only need to put a suitable total order on them. This is equivalent to putting a total order on the set of partitions of n. The order we use is the *reverse lexicographic order*, which is the order obtained by reversing every relation in the lexicographic order on partitions. We denote this order by $\gtrdot$. Thus for $n = 9$ we have

$$(2, 2, 2, 2, 1) \gtrdot (3, 2, 2, 2) \gtrdot (3, 3, 1, 1, 1) \gtrdot (3, 3, 2, 1) \gtrdot (4, 3, 2).$$

Note in particular the the partition $(1, 1, \ldots, 1)$ is always greatest in the reverse lexicographic order, and (n) is always least. We have the following easily obtained relations between the dominance and reverse lexicographic orders: for $\nu, \lambda \vdash n$,

- $\nu \gtrdot \lambda \Rightarrow \nu \not\trianglerighteq \lambda$
- $\nu \geqslant \lambda \Rightarrow \nu \not\triangleright \lambda$.

We thus obtain the following theorem. The argument we have followed is that of Erdmann in [10], but this result was originally given (in the more general Hecke algebra setting, and by different methods) by Hemmer and Nakano in [19].

Theorem 3.4.3. *[[10], Corollary 3.3; [19], Theorem 3.7.1] Over an algebraically closed field of characteristic not 2 or 3, Specht filtration multiplicities are well-defined.*

Proof. Combine Corollary 3.4.2 with Theorem 3.3.2, Proposition 3.3.1, and the results (3.3.2). $\square$

The assumption that the characteristic of the field is not 2 or 3 is necessary by some well-known examples. Indeed, if $\mathrm{char}(k) = 2$ then $S^{(n)} \cong \mathbb{1}_n \cong \mathrm{Sgn}_n \cong S^{(1,1,\ldots,1)}$ and this isomorphism between Specht modules means that Specht filtration multiplicities cannot possibly be well-defined. If $\mathrm{char}(k) = 3$, we find for example that if $n = 3$ then the Specht module $S^{(2,1)}$ has an obvious filtration by Specht modules by virtue of *being* a Specht module, but also a submodule isomorphic to the trivial module $S^{(3)}$ such that the quotient of $S^{(2,1)}$ by this submodule is isomorphic to the sign module $S^{(1,1,1)}$.

Theorem 3.4.3 was originally a rather surprising result. Before it was known, it seems that it was assumed that the failure of Specht filtration multiplicities to be well-defined in characteristic 2 and 3 was indicative of the situation in general prime characteristic. A major new result of this thesis is a generalisation of Theorem 3.4.3 to the wreath product $S_m \wr S_n$.

Original research in Chapter 3: There is no original research in Chapter 3. I have filled in some of the details of the proof of Theorem 3.3.2 myself, but the overall argument is clearly given in [10].

Chapter 4

The wreath product $S_m \wr S_n$

In many areas of Mathematics, one finds that *wreath products* of groups arise naturally. The main topic of this thesis is the representation theory of one specific kind of wreath product, the wreath product of two symmetric groups. We begin by recalling the definition of this wreath product. Most of the material in this chapter is drawn from [6] and [21].

Let n and m be non-negative integers. We denote by $S_m \wr S_n$ the **wreath product** of S_n on S_m. This is the group whose underlying set is the Cartesian product of S_n with n copies of S_m. We shall write elements of $S_m \wr S_n$ as

$$(\sigma; \alpha_1, \alpha_2, \ldots, \alpha_n)$$

for $\alpha_1, \alpha_2, \ldots, \alpha_n \in S_m$ and $\sigma \in S_n$. Multiplication is given by the formula

$$(\sigma; \alpha_1, \alpha_2, \ldots, \alpha_n)(\pi; \beta_1, \beta_2, \ldots, \beta_n) = \Big(\sigma\pi;\ (\alpha_{(1)\pi^{-1}}\beta_1),\ (\alpha_{(2)\pi^{-1}}\beta_2),\ \ldots\ ,(\alpha_{(n)\pi^{-1}}\beta_n)\Big).$$

It is easy to show that inversion in $S_m \wr S_n$ is given by the formula

$$(\sigma; \alpha_1, \alpha_2, \ldots, \alpha_n)^{-1} = \left(\sigma^{-1};\ \left(\alpha_{(1)\sigma}\right)^{-1}, \left(\alpha_{(2)\sigma}\right)^{-1}, \ldots, \left(\alpha_{(n)\sigma}\right)^{-1}\right).$$

Now let G be a subgroup of S_m and H a subgroup of S_n. Then we shall write $G \wr H$ for the subgroup of $S_m \wr S_n$ consisting of all elements

$$(\sigma; \alpha_1, \alpha_2, \ldots, \alpha_n)$$

for $\alpha_1, \alpha_2, \ldots, \alpha_n \in G$ and $\sigma \in H$. The special case $S_m \wr H$ where H is a Young subgroup of S_n will be of particular importance below, but we shall also make use of the case where G and H are each either the full symmetric group or a Young subgroup thereof, and we shall make frequent use of the operations of induction and restriction between such subgroups, for example

$$X\uparrow_{k(S_m \wr S_\gamma)}^{k(S_m \wr S_n)} \text{ and } Y\downarrow_{k(S_m \wr S_\gamma)}^{k(S_m \wr S_n)}$$

where γ is some composition of n. As with the symmetric group, we shall de-clutter such expressions by replacing the full symbols for the group algebras with the subscripts used to identify the various subgroups of S_n and S_m and also suppressing the field k, so for example the above would be abbreviated to

$$X\uparrow_{m \wr \gamma}^{m \wr n} \text{ and } Y\downarrow_{m \wr \gamma}^{m \wr n}.$$

4.1 Subgroups of the symmetric group associated to multicompositions and tuples of multicompositions

In the representation theory of the symmetric groups, an important role is played by the Young subgroups associated to compositions. In this section, we extend the notion of a Young subgroup to encompass multicompositions and tuples of multicompositions.

Let $\underline{\gamma} = (\gamma^1, \ldots, \gamma^t)$ be a t-multicomposition of n (t some non-negative integer) and let $\hat{\gamma}$ be the composition $\gamma^1 \circ \cdots \circ \gamma^t$ of n (recall that $\circ$ denotes concatenation). We define the **Young subgroup** of S_n associated to $\underline{\gamma}$ to be the Young subgroup $S_{\hat{\gamma}}$ associated to $\hat{\gamma}$, and we write $S_{\underline{\gamma}}$ for this subgroup. Thus we have a canonical isomorphism

$$S_{\underline{\gamma}} \cong S_{\gamma^1} \times S_{\gamma^2} \times \cdots \times S_{\gamma^t}.$$

Further, we note that $S_{\underline{\gamma}}$ is a subgroup of $S_{|\underline{\gamma}|}$. For example, if $n = 20$ and $\underline{\gamma}$ is $\big((1,2,1,0,4),(),(4,2,1),(0,1,1,3)\big)$, then $\hat{\gamma}$ is $(1,2,1,0,4,4,2,1,0,1,1,3)$ and

$$S_{\underline{\gamma}} = S_{\{1\}} \times S_{\{2,3\}} \times S_{\{4\}} \times S_{\{5,6,7,8\}} \times S_{\{9,10,11,12\}} \times \\ S_{\{13,14\}} \times S_{\{15\}} \times S_{\{16\}} \times S_{\{17\}} \times S_{\{18,19,20\}},$$

where for a subset Ω of $\{1,\ldots,n\}$, we are writing S_Ω for the subgroup of S_n consisting of all permutations which fix every element of $\{1,\ldots,n\} \setminus \Omega$.

Now let $\underline{\underline{\gamma}} = (\underline{\gamma}^1, \ldots, \underline{\gamma}^t)$ be a t-tuple of r-multicompositions $\underline{\gamma}^i$ (r some positive integer) such that $|||\underline{\underline{\gamma}}||| = n$. The definition we are about to make works just as well if we allow the multicompositions to have different lengths, but we shall not need this. Thus each $\underline{\gamma}^i$ is a multicomposition

$$\underline{\gamma}^i = \big(\gamma^{i,1}, \gamma^{i,2}, \ldots, \gamma^{i,r}\big),$$

where each $\gamma^{i,j}$ is thus a composition (with $\gamma^{i,j} = ()$ allowed)

$$\gamma^{i,j} = \big(\gamma_1^{i,j}, \gamma_2^{i,j}, \ldots, \gamma_{l_{ij}}^{i,j}\big),$$

(where l_{ij} is the length of $\gamma^{i,j}$) where the integers $\gamma_s^{i,j}$ are the parts of the composition $\gamma^{i,j}$, and further the sum of the integers $\gamma_s^{i,j}$ over all i, j and s is n. For each i, let $\hat{\gamma}^i$ be the composition $\gamma^{i,1} \circ \gamma^{i,2} \circ \cdots \circ \gamma^{i,r}$ of $||\underline{\gamma}^i||$, so that

$$\hat{\gamma}^i = \big(\gamma_1^{i,1}, \gamma_2^{i,1}, \ldots, \gamma_{l_{i1}}^{i,1}, \gamma_1^{i,2}, \ldots, \gamma_{l_{i2}}^{i,2}, \gamma_1^{i,3}, \ldots\ldots, \gamma_{l_{ir}}^{i,r}\big),$$

and let $\hat{\gamma}$ be the composition $\hat{\gamma}^1 \circ \hat{\gamma}^2 \circ \cdots \circ \hat{\gamma}^t$ of n, so that

$$\begin{aligned}\hat{\gamma} = \big(&\gamma_1^{1,1}, \gamma_2^{1,1}, \dots, \gamma_{l_{1,1}}^{1,1}, \gamma_1^{1,2}, \dots, \gamma_{l_{1,2}}^{1,2}, \gamma_1^{1,3}, \dots\dots, \gamma_{l_{1,r}}^{1,r}, \\ &\gamma_1^{2,1}, \gamma_2^{2,1}, \dots, \gamma_{l_{2,1}}^{2,1}, \gamma_1^{2,2}, \dots\dots, \gamma_{l_{2,r}}^{2,r}, \\ &\gamma_1^{3,1}, \dots\dots \\ &\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad \dots\dots, \gamma_{l_{t,r}}^{t,r}\big).\end{aligned}$$

Then we define $S_{\underline{\underline{\gamma}}}$ to be the Young subgroup $S_{\hat{\gamma}}$ of S_n associated to $\hat{\gamma}$, and we call this the **Young subgroup** associated to $\underline{\underline{\gamma}}$. Thus we have canonical isomorphisms

$$\begin{aligned} S_{\underline{\underline{\gamma}}} &\cong S_{\underline{\gamma}^1} \times \cdots \times S_{\underline{\gamma}^t} \\ &\cong S_{\gamma^{1,1}} \times \cdots \times S_{\gamma^{1,r}} \\ &\qquad \times S_{\gamma^{2,1}} \times \cdots \times S_{\gamma^{2,r}} \\ &\qquad\qquad \vdots \\ &\qquad \times S_{\gamma^{t,1}} \times \cdots \times S_{\gamma^{t,r}}. \end{aligned}$$

Further, recalling that $|\underline{\underline{\gamma}}| = \big(|\underline{\gamma}^1|, \dots, |\underline{\gamma}^t|\big)$ (a multicomposition of n) and $||\underline{\underline{\gamma}}|| = \big(||\underline{\gamma}^1||, \dots, ||\underline{\gamma}^t||\big)$ (a composition of n), we note that we have subgroup inclusions

$$S_{\underline{\underline{\gamma}}} \;\leq\; S_{|\underline{\underline{\gamma}}|} \;\leq\; S_{||\underline{\underline{\gamma}}||} \;\leq\; S_n. \tag{4.1.1}$$

For example, if we take $n = 37$, $t = 4$, $r = 3$, and we let

$$\begin{aligned}\underline{\underline{\gamma}} = \Big(&\big((3,1,2,0),(1,0,2),()\big), \big((0,0),(1,1,2,5),(0,1,0,2,1)\big), \\ &\big((0,1,3),(),(3,2,0,1)\big), \big((),(2,0,1),(2)\big)\Big)\end{aligned}$$

then

$$\hat{\gamma} = (3,1,2,0,1,0,2,0,0,1,1,2,5,0,1,0,2,1,0,1,3,3,2,0,1,2,0,1,2),$$

and $S_{\underline{\underline{\gamma}}}$ is the Young subgroup $S_{\hat{\gamma}}$. Further, we see that

$$|\underline{\underline{\gamma}}| = \big((6,3,0),(0,9,4),(4,0,6),(0,3,2)\big)$$

and $||\underline{\underline{\gamma}}|| = \big(9,13,10,5\big)$ and thus we see that we have the subgroup inclusions (4.1.1).

4.2 Subgroups of the wreath product associated to multicompositions and tuples of multicompositions

We now define certain subgroups of the wreath product $S_m \wr S_n$ which will be fundamental to our work below. To do so, we must first fix a total order on the partitions of m. We choose the lexicographic order. We take r to be the number of distinct partitions of m, and we enumerate them in the lexicographic order as follows

$$(m) = \mu^1 > \mu^2 > \ \dots\ > \mu^r = (1^m).$$

So for example if $m = 4$, then $r = 5$ and we have

$$(4) = \mu^1 > (3,1) = \mu^2 > (2,2) = \mu^3 > (2,1,1) = \mu^4 > (1,1,1,1) = \mu^5.$$

Now let $\underline{\gamma} = (\gamma^1, \dots, \gamma^r)$ be an r-multicomposition of n (the fact that $\underline{\gamma}$ has length r is crucial for the following construction). We associate to $\underline{\gamma}$ a subgroup of $S_m \wr S_n$, which we shall think of as an analogue of the Young subgroup of a symmetric group associated to a composition. Indeed, we define $W_{\underline{\gamma}}$ to be the subgroup of $S_m \wr S_n$ consisting of all elements of the form

$$\big(\ \sigma\ ;\ \underbrace{\alpha^1_1\,,\ \alpha^1_2\,,\ \dots\ ,\ \alpha^1_{|\gamma^1|}}_{\in S_{\mu^1} \leq S_m}\,,\ \underbrace{\alpha^2_1\,,\ \dots\ ,\ \alpha^2_{|\gamma^2|}}_{\in S_{\mu^2} \leq S_m}\,,\ \alpha^3_1\,,\ \dots\dots\ ,\ \alpha^r_{|\gamma^r|}\big)$$

where $\sigma \in S_{\underline{\gamma}}$ and, as indicated, each α^i_j lies in the Young subgroup S_{μ^i} of S_m associated to the partition $\mu^i \vdash m$. Thus we have a canonical isomorphism

$$W_{\underline{\gamma}} \cong \big(S_{\mu^1} \wr S_{\gamma^1}\big) \times \big(S_{\mu^2} \wr S_{\gamma^2}\big) \times \cdots \times \big(S_{\mu^r} \wr S_{\gamma^r}\big), \tag{4.2.1}$$

and further we note that $W_{\underline{\gamma}}$ is a subgroup of $S_m \wr S_{|\underline{\gamma}|}$.

Now as in the previous section we let $\underline{\underline{\gamma}} = (\underline{\gamma}^1, \ldots, \underline{\gamma}^t)$ be a t-tuple of r-multicompositions $\underline{\gamma}^i$ such that $|||\underline{\underline{\gamma}}||| = n$ where we allow $||\underline{\gamma}^i|| = 0$ (in which case we have $\underline{\gamma}^i = \big((), (), \ldots, ()\big)$, an r-tuple of empty compositions). We take

$$\underline{\gamma}^i = \big(\gamma^{i,1}, \gamma^{i,2}, \ldots, \gamma^{i,r}\big)$$

for $i = 1, \ldots, t$, so that each $\gamma^{i,j}$ is a composition (with $\gamma^{i,j} = ()$ allowed), and further we take

$$\gamma^{i,j} = \big(\gamma^{i,j}_1, \gamma^{i,j}_2, \ldots, \gamma^{i,j}_{l_{ij}}\big)$$

for each i and j, so that each $\gamma^{i,j}_s$ is a non-negative integer. We associate to $\underline{\underline{\gamma}}$ a subgroup of $S_m \wr S_n$, which we shall write as $W_{\underline{\underline{\gamma}}}$ and which we define to be the subgroup consisting of all elements of $S_m \wr S_n$ of the form

$$\begin{array}{l}
\Big(\sigma; \\[1em]
\underbrace{\alpha^{1,1}_1, \alpha^{1,1}_2, \ldots, \alpha^{1,1}_{|\gamma^{1,1}|}}_{\in S_{\mu^1} \le S_m}, \underbrace{\alpha^{1,2}_1, \ldots, \alpha^{1,2}_{|\gamma^{1,2}|}}_{\in S_{\mu^2} \le S_m}, \alpha^{1,3}_1, \quad \ldots\ldots, \underbrace{\alpha^{1,r}_1, \ldots, \alpha^{1,r}_{|\gamma^{1,r}|}}_{\in S_{\mu^r} \le S_m}, \\[1em]
\underbrace{\alpha^{2,1}_1, \alpha^{2,1}_2, \ldots, \alpha^{2,1}_{|\gamma^{2,1}|}}_{\in S_{\mu^1} \le S_m}, \alpha^{2,2}_1, \ldots\ldots \qquad\qquad \ldots\ldots, \underbrace{\alpha^{2,r}_1, \ldots, \alpha^{2,r}_{|\gamma^{2,r}|}}_{\in S_{\mu^r} \le S_m}, \\[1em]
\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\[1em]
\underbrace{\alpha^{t,1}_1, \alpha^{t,1}_2, \ldots, \alpha^{t,1}_{|\gamma^{t,1}|}}_{\in S_{\mu^1} \le S_m}, \underbrace{\alpha^{t,2}_1, \ldots, \alpha^{t,2}_{|\gamma^{t,2}|}}_{\in S_{\mu^2} \le S_m}, \alpha^{t,3}_1, \quad \ldots\ldots, \underbrace{\alpha^{t,r}_1, \ldots, \alpha^{t,r}_{|\gamma^{t,r}|}}_{\in S_{\mu^r} \le S_m} \Big)
\end{array}$$

where $\sigma \in S_{\underline{\underline{\gamma}}}$ and, as indicated, for each i, j, s we have $\alpha_s^{i,j} \in S_{\mu^j}$. Thus note that we have canonical isomorphisms

$$\begin{aligned} W_{\underline{\underline{\gamma}}} &\cong W_{\underline{\gamma}^1} \times \cdots \times W_{\underline{\gamma}^t} \qquad (4.2.2) \\ &\qquad\qquad (\text{where } W_{\underline{\gamma}^i} \text{ is a subgroup of } S_m \wr S_{||\underline{\gamma}^i||}) \\ &\cong \left(S_{\mu^1} \wr S_{\gamma^{1,1}}\right) \times \left(S_{\mu^2} \wr S_{\gamma^{1,2}}\right) \times \cdots \times \left(S_{\mu^r} \wr S_{\gamma^{1,r}}\right) \\ &\quad \times \left(S_{\mu^1} \wr S_{\gamma^{2,1}}\right) \times \cdots \times \left(S_{\mu^r} \wr S_{\gamma^{2,r}}\right) \\ &\qquad\qquad \vdots \\ &\quad \times \left(S_{\mu^1} \wr S_{\gamma^{t,1}}\right) \times \cdots \times \left(S_{\mu^r} \wr S_{\gamma^{t,r}}\right), \end{aligned}$$

where we recall that for each i and j we have that $S_{\gamma^{i,j}}$ is a subgroup of $S_{|\gamma^{i,j}|}$.

4.3 Construction of wreath product modules

We now recall several standard methods for constructing modules for wreath products, as described in [21, Section 4.3] and [6, Section 3]. Recall that we are using *right* modules.

Firstly, let G be a subgroup of S_m, and let X be a kG-module. We define $X^{\widetilde{\boxtimes} n}$ to be the $k(G \wr S_n)$-module obtained by equipping the k-vector space $X^{\otimes n}$ (that is, the tensor product over k of n copies of X) with the action given by the formula

$$(x_1 \otimes \cdots \otimes x_n)(\sigma; \alpha_1, \ldots, \alpha_n) = (x_{(1)\sigma^{-1}}\alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}}\alpha_n)$$

for $x_1, \ldots, x_n \in X$, $\alpha_1, \ldots, \alpha_n \in G$, $\sigma \in S_n$.

More generally, let $X_1, \ldots, X_t$ be kG-modules, and $\gamma = (\gamma_1, \ldots, \gamma_t)$ a composition of n of length t. We form a $k(G \wr S_\gamma)$-module by equipping the k-vector space $\left(X_1^{\otimes \gamma_1}\right) \otimes \left(X_2^{\otimes \gamma_2}\right) \otimes \cdots \otimes \left(X_t^{\otimes \gamma_t}\right)$ with the action given by the formula

$$(x_1 \otimes \cdots \otimes x_n)(\sigma; \alpha_1, \ldots, \alpha_n) = (x_{(1)\sigma^{-1}}\alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}}\alpha_n)$$

where each x_i lies in the appropriate X_j, $\alpha_1, \ldots, \alpha_n \in G$, and $\sigma \in S_\gamma$. We denote this module by $\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma}$, and we note that $X^{\widetilde{\boxtimes}n}$ is the special case of this construction where γ has an n in one place and all the other parts are 0.

Now let G be a subgroup of S_m, H be a subgroup of S_n, and Y a kH-module. It is easy to check that we may make Y into a $k(G \wr H)$-module via the formula

$$y(\sigma; \alpha_1, \ldots, \alpha_n) = y\sigma \tag{4.3.1}$$

for $y \in Y$, $\alpha_1, \ldots, \alpha_n \in G$, and $\sigma \in H$. This module may be understood by noting that $G \wr H$ is the semidirect product of the normal subgroup consisting of all elements $(e; \alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in G$ with the subgroup consisting of all elements $(\sigma; e, \ldots, e)$ for $\sigma \in H$. This latter subgroup is canonically isomorphic to H, and hence we see that the module obtained from Y via (4.3.1) is the *inflation* of Y from H to $G \wr H$ with respect to the semidirect product structure. Hence we shall denote this module by $\mathrm{Inf}_H^{G \wr H} Y$. We shall be particularly interested in the case where H is S_n or a Young subgroup S_γ of S_n and G is S_m or a Young subgroup S_λ of S_m, and in accordance with our notational conventions, we shall write these modules as, for example, $\mathrm{Inf}_n^{m \wr n} Y$ or $\mathrm{Inf}_\gamma^{\lambda \wr \gamma} Y$.

Now let H be a subgroup of S_n, G be a subgroup of S_m, Y be a kH-module, and further let Z be a $k(G \wr H)$-module. Then we define a $k(G \wr H)$-module $Z \oslash Y$ as follows: the underlying k-vector space is $Z \otimes Y$, and the action is given by the formula

$$(z \otimes y)(\sigma; \alpha_1, \ldots, \alpha_n) = (z(\sigma; \alpha_1, \ldots, \alpha_n)) \otimes (y\sigma)$$

for $z \in Z$, $y \in Y$, $\alpha_1, \ldots, \alpha_n \in G$, $\sigma \in H$. Thus we see that we have an equality of $k(G \wr H)$-modules

$$Z \oslash Y = Z \otimes \mathrm{Inf}_H^{G \wr H} Y \qquad (4.3.2)$$

where the module on the right-hand side is the internal tensor product of the $k(G \wr H)$-modules Z and $\mathrm{Inf}_H^{G \wr H} Y$.

We can combine the above constructions as follows: if G is a subgroup of S_m, $X_1, \ldots, X_t$ are kG-modules and Y is a kS_γ-module for γ a composition of n, then we obtain a $k(G \wr S_\gamma)$-module

$$\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma} \oslash Y$$

with underlying vector space $\left(X_1^{\otimes \gamma_1}\right) \otimes \left(X_2^{\otimes \gamma_2}\right) \otimes \cdots \otimes \left(X_t^{\otimes \gamma_t}\right) \otimes Y$ and action given by the formula

$$(x_1 \otimes \cdots \otimes x_n \otimes y)(\sigma; \alpha_1, \ldots, \alpha_n) = \\ (x_{(1)\sigma^{-1}} \alpha_1) \otimes \cdots \otimes (x_{(n)\sigma^{-1}} \alpha_n) \otimes (y\sigma) \qquad (4.3.3)$$

for $x_i \in X$, $\alpha_i \in G$, $y \in Y$, $\sigma \in S_\gamma$. Further, the $k(G \wr S_\gamma)$-module

$$\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma} \oslash Y$$

is exactly the inner tensor product

$$\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma} \otimes \mathrm{Inf}_{S_\gamma}^{G \wr S_\gamma} Y$$

of $k(G \wr S_\gamma)$-modules, as explained above. We shall often be interested (for the case $G = S_m$) in inducing such modules from $S_m \wr S_\gamma$ to the full wreath product $S_m \wr S_n$, that is, in modules

$$\left[\left(X_1, \ldots, X_t\right)^{\widetilde{\boxtimes}\gamma} \oslash Y\right] \Big\uparrow_{m \wr \gamma}^{m \wr n}.$$

We now recall an elementary construction for producing kS_γ-modules Y for use in the above constructions. Indeed, for each $i \in \{1, \ldots, t\}$, let Y_i be a right kS_{γ_i}-module. Now recall that we have a canonical identification of the group S_γ with the direct product $S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t}$ of groups. Thus any module for $k(S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t})$ may be regarded as a kS_γ-module in a canonical way, and vice versa. In particular, if Y_i is a kS_{γ_i}-module for each i, then the external tensor product $Y_1 \boxtimes Y_2 \boxtimes \cdots \boxtimes Y_t$, which is a $k\,(S_{\gamma_1} \times S_{\gamma_2} \times \cdots \times S_{\gamma_t})$-module, may be regarded as a kS_γ-module.

Now recall further that we have a canonical isomorphism between $G \wr S_\gamma$ and $(G \wr S_{\gamma_1}) \times (G \wr S_{\gamma_2}) \times \cdots \times (G \wr S_{\gamma_t})$, and hence we have a canonical identification of algebras

$$k(G \wr S_\gamma) = k\,(G \wr S_{\gamma_1}) \otimes k\,(G \wr S_{\gamma_2}) \otimes \cdots \otimes k\,(G \wr S_{\gamma_t})\,. \tag{4.3.4}$$

Suppose that we have for each $i = 1, \ldots, t$ a $k(G \wr S_{\gamma_i})$-module Z_i. Thus with modules Y_i as above, we see that for each i, $Z_i \oslash Y_i$ is a $k(G \wr S_{\gamma_i})$-module. Hence via the identification (4.3.4), we see that both $Z_1 \boxtimes \cdots \boxtimes Z_t$ and $(Z_1 \oslash Y_1) \boxtimes \cdots \boxtimes (Z_t \oslash Y_t)$ may be considered to be $k(G \wr S_\gamma)$-modules. It is now easy to see that we have an isomorphism of $k(G \wr S_\gamma)$-modules

$$(Z_1 \boxtimes \cdots \boxtimes Z_t) \oslash (Y_1 \boxtimes \cdots \boxtimes Y_t) \cong (Z_1 \oslash Y_1) \boxtimes \cdots \boxtimes (Z_t \oslash Y_t). \tag{4.3.5}$$

There is an important special case of the isomorphism (4.3.5). Indeed, if we have kG-modules $X_1, \ldots, X_t$, then we may form for each i the $k(G \wr S_{\gamma_i})$-module $X_i^{\widetilde{\boxtimes}\gamma_i}$. We have

$$\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\gamma} = X_1^{\widetilde{\boxtimes}\gamma_1} \boxtimes \cdots \boxtimes X_t^{\widetilde{\boxtimes}\gamma_t}$$

by the definition of the left-hand side. We now see via (4.3.5) that we have an isomorphism

$$\big(X_1,\,\dots\,,X_t\big)^{\widetilde{\boxtimes}\gamma}\oslash\big(Y_1\boxtimes Y_2\boxtimes\cdots\boxtimes Y_t\big)\cong \\ \big(X_1^{\widetilde{\boxtimes}\gamma_1}\oslash Y_1\big)\boxtimes\big(X_2^{\widetilde{\boxtimes}\gamma_2}\oslash Y_2\big)\boxtimes\cdots\boxtimes\big(X_t^{\widetilde{\boxtimes}\gamma_t}\oslash Y_t\big) \quad (4.3.6)$$

(this isomorphism was given in [6, Lemma 3.2 (1)]). In subsequent chapters we shall be particularly interested in modules of the form

$$\Big[\big(X_1,\,\dots\,,X_t\big)^{\widetilde{\boxtimes}\gamma}\oslash\big(Y_1\boxtimes Y_2\boxtimes\cdots\boxtimes Y_t\big)\Big]\Big\uparrow_{m\wr\gamma}^{m\wr n}. \quad (4.3.7)$$

In the next chapter (Section 5.2), we shall generalise the construction (4.3.7) to the wreath product of a k-algebra with a symmetric group, and moreover we shall develop a graphical representation of pure tensors in such modules using modified permutation diagrams, which gives a more intuitive way of understanding their structure.

We now give some basic properties of the above constructions.

Proposition 4.3.1. *Let G be a subgroup of S_m and H a subgroup of S_n. Let W and Y be kH-modules. Then we have an isomorphism of $k(G\wr H)$-modules*

$$\mathrm{Inf}_H^{G\wr H}(W\otimes Y)\cong\mathrm{Inf}_H^{G\wr H}(W)\otimes\mathrm{Inf}_H^{G\wr H}(Y).$$

Proof. Both modules have underlying vector space $W\otimes Y$, and it is easy to verify that the identity map on this space yields the required isomorphism of group modules. $\square$

Proposition 4.3.2. *Let G be a subgroup of S_m and H a subgroup of S_n. Let Z be a $k(G\wr H)$-module and Y a kH-module, so that $Z\oslash Y$ is a $k(G\wr H)$-module. Then we have an isomorphism of $k(G\wr H)$-modules*

$$(Z\oslash Y)^*\cong Z^*\oslash Y^*.$$

Proof. We have

$$\begin{aligned}(Z \oslash Y)^* &= \left(Z \otimes \operatorname{Inf}_H^{G \wr H} Y\right)^* \\ &\cong Z^* \otimes \left(\operatorname{Inf}_H^{G \wr H} Y\right)^* \qquad \text{(by (2.2.1))}\end{aligned}$$

And then by using the easily-verified fact that $\left(\operatorname{Inf}_H^{G \wr H} Y\right)^* \cong \operatorname{Inf}_H^{G \wr H}(Y^*)$ and a second application of (2.2.1), the claim is established. □

Proposition 4.3.3. *Let G be a subgroup of S_m and let U, V be kG-modules. Then we have an isomorphism of $k(G \wr S_n)$-modules*

$$U^{\widetilde{\boxtimes} n} \otimes V^{\widetilde{\boxtimes} n} \cong (U \otimes V)^{\widetilde{\boxtimes} n}.$$

Proof. Recall that, as k-vector spaces, we have $U^{\widetilde{\boxtimes} n} \otimes V^{\widetilde{\boxtimes} n} = U^{\otimes n} \otimes V^{\otimes n}$ and $(U \otimes V)^{\widetilde{\boxtimes} n} = (U \otimes V)^{\otimes n}$. It is a routine calculation to verify that the formula

$$(u_1 \otimes \cdots \otimes u_n) \otimes (v_1 \otimes \cdots \otimes v_n) \longmapsto (u_1 \otimes v_1) \otimes \cdots \otimes (u_n \otimes v_n)$$

where $u_i \in U$ and $v_i \in V$ yields a well-defined k-linear map $U^{\otimes n} \otimes V^{\otimes n} \longmapsto (U \otimes V)^{\otimes n}$ which is a homomorphism of $k(G \wr S_n)$-modules. This map is then immediately seen to be onto, and hence must be an isomorphism because the two spaces in question clearly have the same dimension. □

Proposition 4.3.4. *Let G be a subgroup of S_m and let U be a kG-module. Then we have an isomorphism of $k(G \wr S_n)$-modules*

$$\left(U^{\widetilde{\boxtimes} n}\right)^* \cong (U^*)^{\widetilde{\boxtimes} n}.$$

Proof. Firstly, recall that $\left(U^{\widetilde{\boxtimes} n}\right)^*$ is equal as a k-vector space to $\operatorname{Hom}_k(U^{\otimes n}, k)$ while $(U^*)^{\widetilde{\boxtimes} n}$ is $\left(\operatorname{Hom}_k(U, k)\right)^{\otimes n}$.

Now if we take $g_1, \ldots, g_n \in U^* = \operatorname{Hom}_k(U, k)$, then it is a routine calculation to check that we have a well-defined k-linear map from $U^{\otimes n}$ to k given on pure tensors by the formula $u_1 \otimes \cdots \otimes u_n \longmapsto g_1(u_1)\cdots g_n(u_n)$. Another routine calculation then shows us that we have a well-defined k-linear map from $\big(\mathrm{Hom}_k(U,k)\big)^{\otimes n}$ to $\mathrm{Hom}_k(U^{\otimes n},k)$ which is defined by mapping the pure tensor $g_1 \otimes \ldots \otimes g_n$ in $\big(\mathrm{Hom}_k(U,k)\big)^{\otimes n}$ to the map defined above. We shall denote this map by Φ, and it is a routine calculation to show that Φ is then a $k(G \wr S_n)$-module homomorphism from $(U^*)^{\widetilde{\boxtimes} n}$ to $\big(U^{\widetilde{\boxtimes} n}\big)^*$. Now it is clear that these two modules have the same k-dimension, and so to prove that Φ is an isomorphism, it suffices to prove that it is onto.

So let us fix a k-basis $u_1, \ldots, u_d$ for U. Then U^* has a k-basis $f_1, \ldots, f_d$, where $f_i : U \longrightarrow k$ is defined by $f_i(u_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker delta). Now for any n-tuple $\tau = (t_1, \ldots, t_n)$ over the set $\{1, \ldots, d\}$, we define u_τ to be the pure tensor $u_{t_1} \otimes \cdots \otimes u_{t_n} \in U^{\widetilde{\boxtimes} n}$. We now see that $U^{\widetilde{\boxtimes} n}$ has k-basis

$$\{u_\tau : \tau \in \{1, \ldots, d\}^n\}.$$

Thus $U^{\widetilde{\boxtimes} n}$ has k-basis

$$\{f_\tau : \tau \in \{1, \ldots, d\}^n\}$$

where f_τ is the k-linear map defined by $f_\tau(u_\theta) = \delta_{\tau\theta}$. But it is now clear that if $\tau = (t_1, \ldots, t_n)$, then $\Phi(f_{t_1} \otimes \cdots \otimes f_{t_n}) = f_\tau$ so that Φ is onto as required. $\square$

Proposition 4.3.5. *Let $G_1 \subseteq G_2$ be subgroups of S_m and X a kG_2-module. Then we have an isomorphism of $k(G_1 \wr S_n)$-modules*

$$\left[X^{\widetilde{\boxtimes} n}\right]\Big\downarrow^{G_2 \wr S_n}_{G_1 \wr S_n} \cong \left[X\big\downarrow^{G_2}_{G_1}\right]^{\widetilde{\boxtimes} n}$$

Proof. This is immediate from the definition of $(-)^{\widetilde{\boxtimes} n}$. $\square$

Proposition 4.3.6. *[6, Lemma 3.1] Let G be a subgroup of S_m. Let $\alpha = (\alpha_1, \ldots, \alpha_t)$ be a composition of n, and for $i = 1, \ldots, t$ let X_i be a $k(G \wr S_{\alpha_i})$-module, so that $X_1 \boxtimes \cdots \boxtimes X_t$ is naturally a $k(G \wr S_\alpha)$-module. Let $\pi \in S_t$.*

Then

$$\left[X_1 \boxtimes \cdots \boxtimes X_t\right]\big\uparrow_{G \wr \alpha}^{G \wr n} \;\cong\; \left[X_{(1)\pi} \boxtimes \cdots \boxtimes X_{(t)\pi}\right]\big\uparrow_{G \wr ({}^{\pi}\alpha)}^{G \wr n}$$

where ${}^{\pi}\alpha$ *represents the composition* $(\alpha_{(1)\pi}, \ldots, \alpha_{(t)\pi})$*, and where as usual the symbols* n*,* α*, and* ${}^{\pi}\alpha$ *represent the subgroups* S_n*,* S_α*, and* $S_{{}^{\pi}\alpha}$ *of* S_n*, respectively.*

Proposition 4.3.7. *[6, Lemma 3.2] Let* G *be a subgroup of* S_m*. Let* $\alpha = (\alpha_1, \ldots, \alpha_t)$ *be a composition of* n *and let* V *be a* $k(G \wr S_n)$*-module,* W *be a* $k(G \wr S_\alpha)$*-module,* X *be a* kS_n*-module and* Y *be a* kS_α*-module. Then we have module isomorphisms*

1. $\left[V \oslash X\right]\big\downarrow_{G \wr \alpha}^{G \wr n} \cong \left(V\downarrow_{G \wr \alpha}^{G \wr n}\right) \oslash \left(X\downarrow_{\alpha}^{n}\right)$
2. $V \oslash \left(Y\uparrow_{\alpha}^{n}\right) \cong \left[\left(V\downarrow_{G \wr \alpha}^{G \wr n}\right) \oslash Y\right]\Big\uparrow_{G \wr \alpha}^{G \wr n}$
3. $\left(W\uparrow_{G \wr \alpha}^{G \wr n}\right) \oslash X \cong \left[W \oslash \left(X\downarrow_{\alpha}^{n}\right)\right]\big\uparrow_{G \wr \alpha}^{G \wr n}$

where as usual the symbols n *and* α *represent the subgroups* S_n *and* S_α *of* S_n*, respectively.*

4.4 Analogues of Specht and Young permutation modules for $k(S_m \wr S_n)$

We now define analogues for the wreath product $S_m \wr S_n$ of the Specht and Young permutation modules of the symmetric group. Our justification for calling these modules analogues of the Specht and Young permutation modules will come in subsequent chapters, where we shall show that they have a range of properties analogous to the corresponding symmetric group modules. In particular, in the next chapter we shall use the theory of cellular algebras to prove results which will justify our use of the name "Specht module" here.

As above, let us fix the distinct partitions of m, in the lexicographic order, to be

$$(m) = \mu^1 > \mu^2 > \ \dots \ > \mu^r = (1^m).$$

Then our Specht module analogues for $S_m \wr S_n$ are indexed by the set $\underline{\Lambda}_n^r$ of r-multipartitions of n. Indeed, for such an r-multipartition $\underline{\nu} = (\nu^1, \dots, \nu^r)$, we define a $k(S_m \wr S_n)$-module

$$S^{\underline{\nu}} \ = \ \Big[\big(S^{\mu^1}, \dots, S^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\nu}|} \oslash \big(S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r}\big)\Big]\Big\uparrow_{m\wr|\underline{\nu}|}^{m\wr n},$$

and we call $S^{\underline{\nu}}$ the the **Specht module** for $S_m \wr S_n$ associated to $\underline{\nu}$.

A special case of the above which will be of particular interest is the case of $S^{\underline{\nu}}$ for $\underline{\nu} = [\nu, i]$ (recall that $[\nu, i]$ is the r-multipartition with $\nu \vdash n$ in the i^{th} place, and () in all the other places). In this case, we have

$$\big(S^{\mu^1}, \dots, S^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\nu}|} = \big(S^{\mu^i}\big)^{\widetilde{\boxtimes} n}$$

and $S_{|\underline{\nu}|} = S_n$, from which it follows that

$$S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r} \cong S^{\nu}$$

as modules for $kS_{|\underline{\nu}|} = kS_n$. Thus we have

$$S^{[\nu,i]} = \big(S^{\mu^i}\big)^{\widetilde{\boxtimes} n} \oslash S^{\nu}.$$

Now let $\underline{\gamma} = (\gamma^1, \gamma^2, \dots, \gamma^r)$ be an r-multicomposition of n (recalling that we allow $\gamma^i = () \vDash 0$, and we allow the compositions γ^i to have zero parts). We define the **Young permutation module** for $S_m \wr S_n$ associated to $\underline{\gamma}$ to be

$$M^{\underline{\gamma}} = \Big[\big(M^{\mu^1}, \dots, M^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\gamma}|} \oslash \big(M^{\gamma^1} \boxtimes \cdots \boxtimes M^{\gamma^r}\big)\Big]\Big\uparrow_{m\wr|\underline{\gamma}|}^{m\wr n}.$$

We know from (3.1.3) that in the representation theory of kS_n, we have for any composition γ of n that $\mathbb{1}_\gamma\big\uparrow_\gamma^n \cong M^\gamma$. We shall need the analogous result for $k(S_m \wr S_n)$, and we shall establish it by the same method as (3.1.3).

Proposition 4.4.1. *For $\underline{\gamma}$ an r-multicomposition of n, we have an isomorphism*

$$\mathbb{1}\uparrow_{W_{\underline{\gamma}}}^{m\wr n} \cong M^{\underline{\gamma}}$$

of $k(S_m \wr S_n)$-modules.

Proof. We shall apply Corollary 2.2.7. By the definition of $M^{\underline{\gamma}}$ and (2.2.3), we have

$$\dim_k(M^{\underline{\gamma}}) = \left(\prod_{i=1}^{r} \dim_k(M^{\mu^i})^{|\gamma^i|}\right)\left(\prod_{i=1}^{r} \dim_k(M^{\gamma^i})\right)[S_n : S_{|\underline{\gamma}|}]$$

so that using (3.1.2) we have

$$\begin{aligned}
\dim_k(M^{\underline{\gamma}}) &= \left(\prod_{i=1}^{r} \dim_k(M^{\mu^i})^{|\gamma^i|}\right)\left(\prod_{i=1}^{r} \dim_k(M^{\gamma^i})\right)[S_n : S_{|\underline{\gamma}|}] \\
&= \left(\prod_{i=1}^{r}\left(\frac{m!}{\prod_j \mu^i_j!}\right)^{|\gamma^i|}\right)\left(\prod_{i=1}^{r}\frac{|\gamma^i|!}{\prod_j \gamma^i_j!}\right)\frac{n!}{\prod_{i=1}^{r}|\gamma^i|!} \\
&= \frac{n!\left(m!^{|\gamma^1|+\cdots+|\gamma^r|}\right)}{\prod_{i=1}^{r}\left((\prod_j \mu^i_j!)^{|\gamma^i|}\cdot\prod_j \gamma^i_j!\right)} \\
&= \frac{n!(m!)^n}{\prod_{i=1}^{r}|S_{\mu^i}\wr S_{\gamma^i}|} \\
&= \frac{|S_m\wr S_n|}{|W_{\underline{\gamma}}|} \qquad \text{(by the isomorphism (4.2.1)).}
\end{aligned}$$

Recall that for any composition α, we have an element $\tau(\alpha) \in M^\alpha$ upon which S_α acts trivially but which generates M^α as a $kS_{|\alpha|}$-module. Now from the definition of $M^{\underline{\gamma}}$, we see that as a k-vector space, $M^{\underline{\gamma}}$ is

$$\left(M^{\mu^1}\right)^{\otimes|\gamma^1|}\otimes\cdots\otimes\left(M^{\mu^r}\right)^{\otimes|\gamma^r|}\otimes M^{\gamma^1}\otimes\cdots\otimes M^{\gamma^r}\underset{m\wr|\underline{\gamma}|}{\otimes} k(S_m\wr S_n)$$

(where, note, all the tensor products are taken over k, except the one over $k(S_m\wr S_{|\underline{\gamma}|})$). Using the definition of the action of $k(S_m\wr S_{|\underline{\gamma}|})$ on the module

$\big(M^{\mu^1},\dots,M^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\gamma}|}\oslash\big(M^{\gamma^1}\boxtimes\cdots\boxtimes M^{\gamma^r}\big)$ given by (4.3.3), it is now easy to prove that the element

$$\tau(\mu^1)^{\otimes|\gamma^1|}\otimes\cdots\otimes\tau(\mu^r)^{\otimes|\gamma^r|}\otimes\tau(\gamma^1)\otimes\cdots\otimes\tau(\gamma^r)\otimes e$$

(where e is the group identity element of $S_m \wr S_n$) generates $M^{\underline{\gamma}}$ as a $k(S_m \wr S_n)$-module but is acted upon trivially by $W_{\underline{\gamma}}$. The proposition now follows by Corollary 2.2.7. □

Original research in Chapter 4: Most of the material in this chapter is taken more-or-less directly from the literature, although Propositions 4.3.1, 4.3.2, 4.3.3, 4.3.4 and 4.3.5 (all of which are fairly routine properties of the module constructions given in Section 4.3) are my own work, as is Proposition 4.4.1 (again, this is a fairly routine result).

Chapter 5

Cellular structure of wreath product algebras

In this chapter, we shall offer some justification for our use of the name "Specht module" for the modules $S^{\underline{\nu}}$ in the previous chapter. We shall do this by proving that the group algebra $k(S_m \wr S_n)$ is a *cellular algebra* with the modules $S^{\underline{\nu}}$ as its *cell modules*, and further that if $k(S_m \wr S_n)$ is semisimple, then the modules $S^{\underline{\nu}}$ form a complete system of isomorphism classes of simple $k(S_m \wr S_n)$-modules without redundancy (as $\underline{\nu}$ ranges over all r-multipartitions of n). This is exactly the situation which holds for the Specht modules of the group algebra of the symmetric group. We shall also give a description of these modules in terms of a certain class of diagram, which affords a more intuitive understanding of their structure. In fact, this description is valid for any $k(S_m \wr S_n)$-modules of the form

$$\Big[\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\gamma} \oslash \big(Y_1 \boxtimes Y_2 \boxtimes \cdots \boxtimes Y_t\big)\Big]\Big\uparrow_{m\wr\gamma}^{m\wr n}$$

(see (4.3.7)).

In contrast to the rest of this thesis, we shall in this chapter work in the more general situation of the wreath product $A \wr S_n$ of a finite-dimensional k-algebra A with S_n (see Section 5.2 below for the definition of $A \wr S_n$), noting that this setting includes $k(S_m \wr S_n)$ since we have $(kS_m) \wr S_n \cong k(S_m \wr S_n)$. In particular, we shall be concerned with the case where A is a *cellular algebra*. Cellular algebras were introduced by Graham and Lehrer in [13] and the concept has since found broad application.

In [12], Geetha and Goodman showed that the algebra $A \wr S_n$ is cellular in the case that A is not only cellular but *cyclic cellular*, meaning that all of the cell modules of A are cyclic [12, Theorem 4.1]. Their proof is quite combinatorial in nature, and draws on the work of Dipper, James, and Mathas in [8] and of Murphy in [31]. However, we shall prove (section 5.3) that $A \wr S_n$ is cellular for any cellular algebra A, by exhibiting it as an *iterated inflation* of tensor products of group algebras of symmetric groups. Iterated inflations were originally introduced by König and Xi in [23], but we shall use this concept in the form given in [16]. The advantage of taking this approach is a far simpler proof than the one given in [12], and hence much easier access to the powerful machinery of cellular algebra theory which allows us to easily prove the nice results on $A \wr S_n$ given in Section 5.4. The price for this simplicity is that order obtained on the set of cell indices of $A \wr S_n$ contains more relations than the order obtained in [12], and hence contains less representation-theoretic information; see the discussion at the end of Section 5.3 for more details. Since (as far as the author is aware) all cellular algebras which occur in practice are in fact cyclic cellular, the result presented here is in effect a weaker version of the result of Geetha and Goodman. However, the much simpler proof afforded by the method of iterated inflations is of interest in its own right.

In Section 5.2, we shall generalise the construction of modules of the form

(4.3.7) to $A \wr S_n$, and as mentioned above we shall also obtain a convenient graphical description of such modules. In Section 5.4 we bring this description together with the cellularity result to deliver results on the representation theory of $A \wr S_n$, in particular a description of the simple modules and a semisimplicity condition. These results require no extra assumptions on the field (e.g. algebraic closedness).

We shall conclude by applying (Section 5.5) this work to the case where $A = kS_m$, in which case $A \wr S_n$ is in fact the group algebra $k(S_m \wr S_n)$ which is the main topic of this thesis.

Note that we shall not use the contents of this chapter again in this thesis. Indeed, this material is intended to provide motivation for our study of the Specht modules for $k(S_m \wr S_n)$, and to place the study of the representation theory of $k(S_m \wr S_n)$ in the broader context of the study of wreath product algebras. In particular, our arguments in subsequent chapters do *not* make use of the cellular structure on $k(S_m \wr S_n)$.

This chapter is an adapted version of an article preprint [15] which the author has submitted for publication to the *Journal of Pure and Applied Algebra* (Elsevier) under the title *Cellular Structure of Wreath Product Algebras*, and which has subsequently been resubmitted to this journal in revised form following review. This chapter is based on the revised version.

Some of the material in this chapter is based on material from [14], a thesis for which the author was awarded an M.Sc. at the University of Kent in 2016. In particular, the whole of Section 5.1.3 appeared in essentially the same form in [14]. The rest of the material in question comprises whole of Section 5.3 and the first part of Section 5.4, from the start of the section up to and including Proposition 5.4.1. All of this material appeared in [14], but the version presented here is an improvement on the version given in

[14] because it makes use of a more sophisticated order on the layers of the iterated inflation structure (the Γ-*dominance order*; see below). Other than the relatively minor modifications to the arguments necessary to make use of this improved order, the material is in essentially the same form as in [14] (the version in [14] used a slightly different cellular structure on the group algebra of the symmetric group, with the *duals* of the Specht modules appearing as the cell modules, but this makes no difference to the arguments).

5.1 Recollections and definitions

An **anti-involution** on a k-algebra A is a self-inverse k-linear isomorphism $a \mapsto a^*$ such that $(ab)^* = b^*a^*$ for all $a, b \in A$.

5.1.1 Cellular algebras

We refer the reader to [13] for basic information and notation on cellular algebras. However, in order to avoid confusion with our established notations for various sets of (multi)partitions based on the symbol Λ, we shall use the symbol Γ to denote the poset indexing the cell modules of a cellular algebra. We shall refer to elements of the poset Γ as *cell indices*, and we shall write the anti-involution on a cellular algebra A as $a \mapsto a^*$. Recall that to each cell index λ we associate a finite set $M(\lambda)$, and we have a *cellular basis* of A whose elements are indexed by the disjoint union of the sets $M(\lambda) \times M(\lambda)$ for $\lambda \in \Gamma$. We write the cellular basis element indexed by $(S, T) \in M(\lambda) \times M(\lambda)$ as $C^{\lambda}_{S,T}$. We call the tuple (Γ, M, C) the *cellular data* of A with respect to $*$. Since we are using right modules we take the multiplication rule for cellular basis elements to be

$$C^{\lambda}_{S,T}a \equiv \sum_{X\in M(\lambda)} R_a(T,X)C^{\lambda}_{S,X} \tag{5.1.1}$$

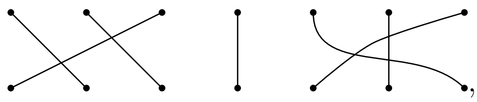
modulo cellular basis elements of lower cell index, where the coefficients $R_a(T,X) \in k$ are independent of S. Then the right cell module Δ^{λ} is the vector space with basis $\{C_T : T \in M(\lambda)\}$. Our form of the multiplication rule (5.1.1) means that the action of A on Δ^{λ} is

$$C_T a = \sum_{X\in M(\lambda)} R_a(T,X)C_X. \tag{5.1.2}$$

Let us recall some basic results on cell modules, see [13, Sections 2 and 3]. Indeed, each cell module is equipped with a bilinear form, whose radical is either the whole cell module or else its unique maximal A-submodule. We shall call these bilinear forms the *cell forms* and their radicals the *cell radicals*. We let Γ_0 be the set of $\lambda \in \Gamma$ such that the cell radical of Δ^{λ} does not equal Δ^{λ}, and for $\lambda \in \Gamma_0$ we let L^{λ} be the quotient of Δ^{λ} by its cell radical. Thus L^{λ} is a simple A-module, and the modules L^{λ} for $\lambda \in \Gamma_0$ are in fact a complete list of all the simple right A-modules up to isomorphism, without redundancy.

5.1.2 Permutation diagrams and cellularity of kS_n

We shall find it convenient to represent permutations in the symmetric group S_n via *permutation diagrams*. For example, we represent $(1,2,3)(5,7) \in S_7$ by the diagram



where the i^{th} *node* on the top row is connected by a *string* to the $(i)\sigma^{\text{th}}$ node on the bottom row. To calculate the product $\sigma\pi$ in S_n using permutation diagrams, we connect the diagram for σ above the diagram for π, and then simplify the resulting diagram to yield the permutation diagram of $\sigma\pi$.

Now from [30], kS_n is known to be cellular with respect to our map $*$ and a tuple of cellular data including the set Λ_n of all partitions of n with the *reverse* dominance order. Further, the cell module associated to a partition $\lambda \in \Lambda_n$ by this cellular structure is the dual Specht module $\left(S^\lambda\right)^*$, where S^λ is as in previous chapters the (right) Specht module of James in [20]. Our source for these facts is [30], in particular Theorem 3.20, the "Warning" on page 38, and "Note 2" on page 54. Note however that the original published text of [30] incorrectly states that the cell module obtained is the dual of the right James Specht module associated to the *conjugate* of λ; see the correction to the "Warning" on page 38 in the author's errata to [30]. Note further that [30, Theorem 3.20] mentions the dominance order on Λ_n rather than the reverse dominance order. However, looking at the definition of a cellular algebra used there [30, 2.1], we see that [30] uses the opposite convention on ordering when defining a cellular algebra compared to our definition, so in the sense of our definition of a cellular algebra the order is indeed the *reverse* dominance order.

For our work in this chapter, we would like a cellular structure on kS_n with the Specht modules S^λ themselves as cell modules. To obtain such a structure we use the work on *dual bases* of *Frobenius cellular algebras* of Li and Xiao in [26] and of Li in [25] (see also [30, Chapter 2, exercise 11]). For this, we must recall that if A is a k-algebra and $\langle -, - \rangle$ is a k-valued bilinear form on A, then $\langle -, - \rangle$ is *associative* if we have $\langle ab, c\rangle = \langle a, bc\rangle$ for all $a, b, c \in A$. From Section 3 of [26] we know that if a cellular algebra A is endowed with a symmetric, non-degenerate, associative bilinear form (and hence the algebra is a *symmetric Frobenius algebra*), then we may take the *dual cellular basis* of our cellular basis, and that this basis is indeed a cellular basis of A. Further, the cellular structure on A associated to this dual cellular basis has the same set of cell indices as the original cellular structure, but with the reverse order. Moreover, by [26, Proposition 3.3], if we take Δ^λ to be the cell module associated to a cell index λ by the original cellular structure on A, then the cell module associated to λ by the new cellular structure is the A-module obtained by equipping $\mathrm{Hom}_k(\Delta^\lambda, k)$ with the A-action given by the formula $(fa)(x) = f(xa^*)$ for $f \in \mathrm{Hom}_k(\Delta^\lambda, k)$, $a \in A$, $x \in \Delta^\lambda$.

Turning to the algebra kS_n, we may easily show that the form defined on kS_n by letting $\langle a, b\rangle$ be the coefficient of the identity element of S_n in the expansion of ab over the basis S_n is symmetric, non-degenerate, associative and bilinear. It follows that we may obtain a cellular structure on kS_n involving the anti-involution $*$ and the set Λ_n with the dominance order, where the cell module associated to a partition λ is $\left(\left(S^\lambda\right)^*\right)^*$ (the dual of the dual of S^λ), which is trivially isomorphic to S^λ. It is this cellular structure on kS_n which we shall use in this chapter. Note, however, that we shall not require any details about the definition of the associated cellular basis.

Now our cellular structure on kS_n yields an indexing of the simple kS_n-modules as L^λ for $\lambda \in \left(\Lambda_n\right)_0$, where $\left(\Lambda_n\right)_0$ is a subset of Λ_n. It turns out that this set $\left(\Lambda_n\right)_0$ is the set of all p-regular partitions of n, and moreover that the simple module L^λ associated to a p-regular partition λ is isomorphic to the simple module D^λ as in Theorem 3.1.4. We shall now justify these assertions.

Lemma 5.1.1. *([13, Section 3]) Let A be a cellular algebra with poset of cell indices Λ, cell modules Δ^λ and simple modules L^λ for $\lambda \in \Lambda_0$. For $\lambda \in \Lambda_0$, the composition factors of the cell module Δ^λ include one copy of L^λ, and all other composition factors are L^μ for $\mu \in \Lambda_0$ with $\mu > \lambda$. For $\lambda \in \Lambda \setminus \Lambda_0$, the composition factors of the cell module Δ^λ are all of the form L^μ for $\mu \in \Lambda_0$*

with $\mu > \lambda$.

Proposition 5.1.2. *In our cellular structure on kS_n with the Specht modules as cell modules, the set $\left(\Lambda_n\right)_0$ of cell indices indexing the simple modules is exactly the set of p-regular partitions. Further, if λ is p-regular then the simple module L^λ obtained from this cellular structure is isomorphic to D^λ.*

Proof. We shall prove by (strong) induction on $\lambda \in \Lambda_n$ (where Λ_n is equipped with the dominance order) that for each $\lambda \vdash n$, we have $\lambda \in \left(\Lambda_n\right)_0$ if and only if λ is p-regular, and that if λ is p-regular then $D^\lambda \cong L^\lambda$. Indeed, assume that for some $\lambda \vdash n$ the desired statement holds for all partitions $\mu \vdash n$ with $\mu \rhd \lambda$. It follows that

$$\left\{\mu \vdash n \mid \mu \text{ is } p\text{-regular and } \mu \rhd \lambda\right\} = \left\{\mu \in \left(\Lambda_n\right)_0 \mid \mu \rhd \lambda\right\},$$

and that we have $D^\mu \cong L^\mu$ for all μ in this set.

Suppose that λ is p-singular. By Theorem 3.1.4, S^λ has a filtration by simple modules D^μ for p-regular partitions $\mu \vdash n$ such that $\mu \rhd \lambda$. Hence all composition factors of S^λ are of the form L^μ for $\mu \in \left(\Lambda_n\right)_0$ such that $\mu \rhd \lambda$. Suppose for a contradiction that $\lambda \in \left(\Lambda_n\right)_0$. Then by Lemma 5.1.1, the composition factors of S^λ must include a factor of the simple module L^λ satisfying $L^\lambda \ncong L^\mu$ for all $\mu \in \left(\Lambda_n\right)_0$ such that $\mu \rhd \lambda$, a contradiction.

Now suppose that λ is p-regular. Then by Theorem 3.1.4, S^λ has a filtration by simple modules D^μ where D^λ occurs exactly once and all other factors are of the form D^μ for p-regular partitions $\mu \vdash n$ such that $\mu \rhd \lambda$. Suppose for a contradiction that $\lambda \in \Lambda_n \setminus \left(\Lambda_n\right)_0$. Then by Lemma 5.1.1, we must have $D^\lambda \cong L^\mu$ for some $\mu \in \left(\Lambda_n\right)_0$ with $\mu \rhd \lambda$, which yields a contradiction since we than have $L^\mu \cong D^\mu \ncong D^\lambda$. Thus $\lambda \in \left(\Lambda_n\right)_0$. Hence by Lemma 5.1.1, L^λ must be amongst the composition factors of S^λ, which forces $D^\lambda \cong L^\lambda$ since $L^\lambda \ncong L^\mu \cong D^\mu$ for all $\mu \in \left(\Lambda_n\right)_0$ with $\mu \rhd \lambda$. $\square$

Thus we see that the existing literature on the symmetric group and cellular algebras yields the following theorem.

Theorem 5.1.3. *The group algebra kS_n is cellular with respect to the anti-involution $*$ defined by setting $\sigma^* = \sigma^{-1}$ for $\sigma \in S_n$, and a tuple of cellular data including the partially ordered set Λ_n consisting of all partitions of n endowed with the dominance order $\trianglerighteq$. The cell module associated to $\lambda \in \Lambda_n$ by this structure is the Specht module S^λ as defined above. Further, the set $\left(\Lambda_n\right)_0$ of cell indices indexing the simple modules is exactly the set of p-regular partitions (recall that for $p = 0$ all partitions are p-regular), and moreover for $\lambda \in \left(\Lambda_n\right)_0$, the simple module L^λ obtained from the cell module S^λ is isomorphic to the simple module D^λ.*

The following result may easily be proved by directly verifying the axioms for a cellular algebra. In fact, it is merely a special case of the general result that a tensor product of cellular algebras is cellular, see for example Section 3.2 of [12].

Proposition 5.1.4. *Let $n_1, \ldots, n_t$ be non-negative integers. Then the group algebra $k(S_{n_1} \times \cdots \times S_{n_t})$ is a cellular algebra with respect to the map given by $(\sigma_1, \ldots, \sigma_t) \longmapsto (\sigma_1^{-1}, \ldots, \sigma_t^{-1})$ for $\sigma_i \in S_{n_i}$ and a cellular structure where the poset of cell indices is $\Lambda_{n_1} \times \cdots \times \Lambda_{n_t}$ with the order where $(\lambda^1, \ldots, \lambda^t) \geqslant (\nu^1, \ldots, \nu^t)$ means $\lambda^i \trianglerighteq \nu^i$ for all i. The cell module associated to $(\lambda^1, \ldots, \lambda^t)$ is $S^{\lambda^1} \boxtimes \cdots \boxtimes S^{\lambda^t}$ (where we identify the algebra $k(S_{n_1} \times \cdots \times S_{n_t})$ with the algebra $kS_{n_1} \otimes \cdots \otimes kS_{n_t}$ in the canonical way) with the action*

$$(x_1 \otimes \cdots \otimes x_t) \cdot (\sigma_1, \ldots, \sigma_t) = (x_1\sigma_1) \otimes \cdots \otimes (x_t\sigma_t)$$

for $x_i \in S^{\lambda^i}$, $\sigma_i \in S_{n_i}$. The cell form on this cell module is given on pure tensors by

$$\langle x_1 \otimes \cdots \otimes x_t, y_1 \otimes \cdots \otimes y_t \rangle = \langle x_1, y_1 \rangle \cdots \langle x_t, y_t \rangle$$

where each bilinear form on the right hand side is the appropriate cell form of some S^{λ^i}.

Recall from page 45 the notion of the length of a permutation, which is defined to be the total number of inversions of the permutation, where an inversion of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $(i)\sigma > (j)\sigma$. From these definitions, it is easy to see that if μ is a composition of n, then each right coset $S_\mu\sigma$ of S_μ contains a unique element of minimal length, and further that if $\mu = (\mu_1, \ldots, \mu_r)$, then for any given right S_μ-coset, the element of minimal length is the unique element γ of the coset such that in the sequence $(1)\gamma^{-1}, \ldots, (n)\gamma^{-1}$, the elements $1, \ldots, \mu_1$ occur in increasing order, as do the elements $\mu_1 + 1, \ldots, \mu_1 + \mu_2$, the elements $\mu_1 + \mu_2 + 1, \ldots, \mu_1 + \mu_2 + \mu_3$, and so on. Equivalently, an element σ of S_n is of minimal length in its coset $S_\mu\sigma$ if and only if, in its permutation diagram, the strings attached to the first μ_1 nodes on the top row do not cross each other, the strings attached to the next μ_2 nodes on the top row do not cross each other, and so on. For example, the permutation whose diagram appears in the diagram (5.2.5) below is of minimal length in its S_μ-coset for $\mu = (3, 2, 3)$. For any μ a composition of n, we define $\mathcal{R}_\mu$ to be the unique system of minimal-length right S_μ-coset representatives in S_n.

5.1.3 Iterated inflation of cellular algebras

Iterated inflations of cellular algebras were first introduced by König and Xi in [23], but we shall use them as presented in [16], and this section is a summary of the contents of that article. However, we give the form of these results using right cell modules, rather than the left cell modules used in [16]. Note that all of the material in this section formed part of the author's M.Sc. thesis [14], and thus is not to be considered as new research in this present thesis.

Let A be a k-algebra, with an anti-involution $*$. Suppose that we have, up to isomorphism of k-vector spaces, a decomposition

$$A \cong \bigoplus_{\mu \in I} V_\mu \otimes B_\mu \otimes V_\mu$$

of A, where I is a finite partially ordered set, each V_μ is a k-vector space, and each B_μ is a cellular algebra over k with respect to an anti-involution $*$ and cellular data (Γ_μ, M_μ, C). We shall henceforth consider A to be identified with this direct sum of tensor products, and we shall speak of the subspace $V_\mu \otimes B_\mu \otimes V_\mu$ as the μ-th *layer* of A. Suppose that for each $\mu \in I$, we have a basis $\mathcal{V}_\mu$ for V_μ and a basis $\mathcal{B}_\mu$ for B_μ. Let $\mathcal{A}$ be the basis of A consisting of all elements $u \otimes b \otimes w$ for all $u, w \in \mathcal{V}_\mu$ and all $b \in \mathcal{B}_\mu$, as μ ranges over I. Suppose that for each $\mu \in I$, we have for any $u, w \in \mathcal{V}_\mu$ and any $b \in \mathcal{B}_\mu$ that

$$(u \otimes b \otimes w)^* = w \otimes b^* \otimes u, \tag{5.1.3}$$

and suppose further that for any $\mu \in I$ we have maps $\phi_\mu : \mathcal{V}_\mu \times \mathcal{A} \to V_\mu$ and $\theta_\mu : \mathcal{V}_\mu \times \mathcal{A} \to B_\mu$ such that for any $u, w \in \mathcal{V}_\mu$ and any $b \in \mathcal{B}_\mu$, we have for any $a \in \mathcal{A}$ that

$$(u \otimes b \otimes w) \cdot a \equiv u \otimes b\, \theta_\mu(w, a) \otimes \phi_\mu(w, a) \mod J(< \mu), \tag{5.1.4}$$

where $J(< \mu) = \bigoplus_{\alpha < \mu} V_\alpha \otimes B_\alpha \otimes V_\alpha$. Then by [16, Theorem 1], A is cellular with respect to $*$ and the cellular data (Γ, M, C), where Γ is the set $\{(\mu, \lambda) : \mu \in I \text{ and } \lambda \in \Gamma_\mu\}$ with the lexicographic order, $M(\mu, \lambda)$ is $\mathcal{V}_\mu \times M_\mu(\lambda)$, and $C^{(\mu,\lambda)}_{(x,X),(y,Y)} = x \otimes C^\lambda_{X,Y} \otimes y$.

Further by [16, Proposition 2], for each $\mu \in I$ there is a unique B_μ-valued k-bilinear form ψ_μ on V_μ such that for any $u, w, x, y \in V_\mu$ and $b, c \in B_\mu$ we have $\psi_\mu(y, u) = \psi_\mu(u, y)^*$ and

$$(x \otimes c \otimes y)(u \otimes b \otimes w) \equiv x \otimes c\, \psi_\mu(y, u) b \otimes w \mod J(< \mu). \tag{5.1.5}$$

Finally (see [16, Proposition 3]), let $(\mu, \lambda) \in \Gamma$, and let Δ^λ be the right cell module of B_i corresponding to λ. The right cell module $\Delta^{(\mu,\lambda)}$ of A may be obtained by equipping $\Delta^\lambda \otimes V_\mu$ with the action given, for $a \in \mathcal{A}$, $x \in \mathcal{V}_\mu$ and $z \in \Delta^\lambda$, by $(z \otimes x)a = z\,\theta_\mu(x,a) \otimes \phi_\mu(x,a)$. Moreover, if $\langle \cdot\,,\cdot \rangle$ is the cell form on $\Delta^\lambda \otimes V_\mu$ and $\langle \cdot\,,\cdot \rangle_\lambda$ is the cell form on Δ^λ, then for any $x, y \in V_\mu$ and any $z, v \in \Delta^\lambda$, we have

$$\langle z \otimes x,\, v \otimes y \rangle = \langle z\,\psi_\mu(x,y), v \rangle_\lambda = \langle z, v\,\psi_\mu(y,x) \rangle_\lambda. \tag{5.1.6}$$

5.2 Wreath product algebras

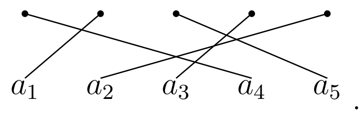
We recall the notion of the wreath product of an algebra with a symmetric group from [6]. Indeed, let A be a finite-dimensional unital associative k-algebra. Consider the k-vector space $kS_n \otimes A^{\otimes n}$, and further let us write a pure tensor $x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in this vector space as $(x\,; a_1, a_2, \ldots, a_n)$. Then we have a well-defined multiplication which is given by

$$(\sigma; a_1, a_2, \ldots, a_n)(\pi; b_1, b_2, \ldots, b_n) = \\ (\sigma\pi; a_{(1)\pi^{-1}}b_1, a_{(2)\pi^{-1}}b_2, \ldots, a_{(n)\pi^{-1}}b_n)$$

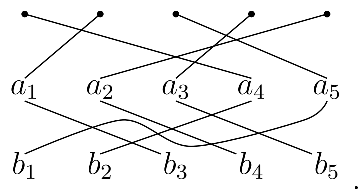
for $\sigma, \pi \in S_n$ and $a_i, b_i \in A$. We define the *wreath product* $A \wr S_n$ of A and S_n to be the unital associative k-algebra so obtained. We note in particular the case where $A = kS_m$, where we see that the algebra $(kS_m) \wr S_n$ is isomorphic to the algebra $k(S_m \wr S_n)$ via the obvious isomorphism. We shall return to this special case in Section 5.5 below, where we shall relate our work in this chapter on $A \wr S_n$ to our work on $k(S_m \wr S_n)$ in the rest of the thesis.

We assume that the reader is familiar with the notion of *diagram algebras*, for example the Brauer or Temperley-Lieb algebras. We can consider $A \wr S_n$ to be a kind of diagram algebra. Indeed, we may represent a pure tensor

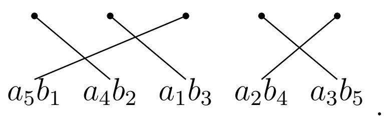
$(\sigma; a_1, a_2, \ldots, a_n)$ in $A \wr S_n$, where $\sigma \in S_n$ and $a_i \in A$, by a diagram obtained by drawing the permutation diagram associated to σ, with the nodes of the bottom row replaced by the elements a_i. For example, if $n = 5$ and $\sigma = (1, 4, 3, 5, 2)$, then we represent the element $(\sigma; a_1, a_2, a_3, a_4, a_5)$ by



Such diagrams are useful for computing products, as we now show by an example. Indeed, keep $n = 5$ and $\sigma = (1, 4, 3, 5, 2)$, and let $\pi = (1, 3, 5)(2, 4)$. Then to compute the product $(\sigma; a_1, a_2, a_3, a_4, a_5)(\pi; b_1, b_2, b_3, b_4, b_5)$, we draw the diagram corresponding to the first factor above the one corresponding to the second factor, to obtain



We then slide each a_i down its string to meet some b_j, and then resolve the two connected permutation diagrams into a single diagram, to obtain



This diagram corresponds to the element

$$\big((1, 2, 3)(4, 5); a_5b_1, a_4b_2, a_1b_3, a_2b_4, a_3b_5\big),$$

which is indeed the product of the two elements we started with.

Note that, unlike the usual diagram basis of the Brauer or Temperley-Lieb algebras, the set of all such diagrams is not a basis of $A \wr S_n$. A basis of such diagrams can be formed by fixing a basis $\mathcal{C}$ of A, and then taking the set of all elements $(\sigma; a_1, \ldots, a_n)$ for $\sigma \in S_n$ and $a_i \in \mathcal{C}$. However, the product of two such basis elements will not in general be a scalar multiple of another basis element as is the case for the diagram basis of the Brauer or Temperley-Lieb algebras.

It is easy to show that there is a well-defined anti-involution $*$ on $A \wr S_n$ given by

$$(\sigma; a_1, \ldots, a_n)^* = \big(\sigma^{-1}\,;\, a^*_{(1)\sigma}, \ldots, a^*_{(n)\sigma}\big), \tag{5.2.1}$$

where $\sigma \in S_n$ and $a_1, \ldots, a_n \in A$. In terms of diagrams, this map corresponds to the operation of taking a diagram, flipping it about the horizontal line half-way between its two rows of nodes (so that the elements a_i lie on the *top* row), replacing each element a_i with its image a_i^* under the anti-involution on A, and then sliding each element a_i^* to the bottom of its string. For the case $A = kS_m$ where we have $A \wr S_n \cong k(S_m \wr S_n)$, we may easily see that if we take the anti-involution induced on kS_m by mapping an element of S_m to its inverse, then we obtain the anti-involution on $k(S_m \wr S_n)$ which is induced by mapping each element of $S_m \wr S_n$ to its inverse.

We now give the well-known generalisation of the construction of the $k(S_m \wr S_n)$-module (4.3.7) to the algebra $A \wr S_n$ (see for example Section 3 of [6]). The construction is essentially unchanged from the $k(S_m \wr S_n)$ case. Indeed, let μ be an r-part composition of n (where r is some integer), $X_1, \ldots, X_r$ be A-modules, and for each $i = 1, \ldots, r$ let Y_i be a kS_{μ_i} module. We write $A \wr S_\mu$ for the subalgebra of $A \wr S_n$ spanned by all elements $(\sigma; a_1, \ldots, a_n)$ where $a_i \in A$ and $\sigma \in S_\mu$. Then $X_1^{\otimes \mu_1} \otimes \cdots \otimes X_r^{\otimes \mu_r} \otimes Y_1 \otimes \cdots \otimes Y_r$ is naturally a

$A \wr S_\mu$-module via the action

$$(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r)(\sigma; a_1, \ldots, a_n) = \\ x_{(1)\sigma^{-1}} a_1 \otimes \cdots \otimes x_{(n)\sigma^{-1}} a_n \otimes y_1\sigma_1 \otimes \cdots \otimes y_r\sigma_r,$$

where the elements $\sigma_i \in S_{\mu_i}$ are such that under the natural identification of S_μ with $S_{\mu_1} \times \cdots \times S_{\mu_r}$, σ is identified with $(\sigma_1, \ldots, \sigma_r)$. Then inducing from $A \wr S_\mu$ to $A \wr S_n$ (that is, applying the functor $- \otimes_{A\wr S_\mu} A \wr S_n$) yields a module which we may easily see is isomorphic as a k-vector space to

$$X_1^{\otimes\mu_1} \otimes \cdots \otimes X_r^{\otimes\mu_r} \otimes Y_1 \otimes \cdots \otimes Y_r \otimes k\mathcal{R}_\mu, \tag{5.2.2}$$

where $k\mathcal{R}_\mu$ is the vector space on the basis $\mathcal{R}_\mu$ of minimal-length coset representatives, with the action given by

$$(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma)(\sigma; a_1, \ldots, a_n) = \\ x_{(1)\theta^{-1}} a_{(1)\zeta} \otimes \cdots \otimes x_{(n)\theta^{-1}} a_{(n)\zeta} \otimes y_1\theta_1 \otimes \cdots \otimes y_r\theta_r \otimes \zeta, \tag{5.2.3}$$

where $\gamma \in \mathcal{R}_\mu$, and $\zeta \in \mathcal{R}_\mu$ and $\theta \in S_\mu$ are such that $\gamma\sigma = \theta\zeta$. Letting $\underline{X}$ be the tuple $(X_1, \ldots, X_r)$ and $\underline{Y}$ be the tuple $(Y_1, \ldots, Y_r)$, we denote the module so obtained by $\Theta^\mu(\underline{X}, \underline{Y})$. Comparing this construction to our work in Section 4.3, and in particular the $k(S_m \wr S_n)$-module (4.3.7), we see that in the case $A = kS_m$ where $A \wr S_n \cong k(S_m \wr S_n)$ we have an equality of $k(S_m \wr S_n)$-modules

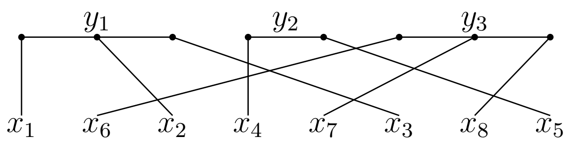
$$\Theta^\mu(\underline{X}, \underline{Y}) = \Big[\big(X_1, \ldots, X_r\big)^{\widetilde{\boxtimes}\mu} \oslash \big(Y_1 \boxtimes \cdots \boxtimes Y_r\big)\Big]\Big\uparrow_{m\wr\mu}^{m\wr n} \tag{5.2.4}$$

where now each X_i is a kS_m-module.

We now introduce a diagrammatic representation for certain pure tensors in the module $\Theta^\mu(\underline{X}, \underline{Y})$ which provides a very convenient and intuitive understanding of the action of $A \wr S_n$. Indeed, let us take a pure tensor $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$ in (5.2.2), where $\gamma \in \mathcal{R}_\mu$. We represent this element by taking the permutation diagram of γ, labelling the nodes on its lower row from left to right with the elements $x_{(1)\gamma^{-1}}, \ldots, x_{(n)\gamma^{-1}}$, then linking together the first μ_1 nodes on the top row and labelling them with y_1, linking together the next μ_2 nodes on the top row and labelling the linked nodes with y_2, and so on. For example, take $n = 8$, $r = 3$, $\mu = (3, 2, 3)$, and $\gamma = (2, 3, 6)(5, 8, 7)$ (γ may be seen to be an element of $\mathcal{R}_\mu$ from its permutation diagram in (5.2.5), since the strings associated to each y_i do not cross each other). We then represent the element

$$x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5 \otimes x_6 \otimes x_7 \otimes x_8 \otimes y_1 \otimes y_2 \otimes y_3 \otimes \gamma$$

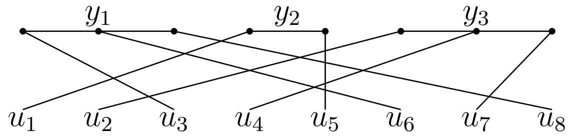
by the diagram



. (5.2.5)

Note that each x_i is connected to the i^{th} node on the top row. Note also that for each $i = 1, 2, 3$, the elements of X_i are attached to the strings associated to y_i. We thus identify $\Theta^\mu(\underline{X}, \underline{Y})$ with the k-vector space spanned by diagrams consisting of the permutation diagram of some element of $\mathcal{R}_\mu$ where (as in (5.2.5)) for each $i = 1, \ldots, r$, the $(\mu_1 + \cdots + \mu_{i-1} + 1)^{\text{th}}$ to $(\mu_1 + \cdots + \mu_i)^{\text{th}}$ nodes are connected to form a single block which is labelled by an element of Y_i, and where each node on the bottom row is replaced with an element of some X_j such that each top-row node in the i^{th} block is connected to an element of X_i on the bottom row. We note that under this identification, the diagram in $\Theta^\mu(\underline{X}, \underline{Y})$ whose top row has labels y_1 to y_r, whose bottom row has labels u_1 to u_n, and whose underlying permutation diagram is that of $\gamma \in \mathcal{R}_\mu$ represents the pure tensor $u_{(1)\gamma} \otimes \cdots \otimes u_{(n)\gamma} \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$. Further note that the set of all such diagrams is not linearly independent in $\Theta^\mu(\underline{X}, \underline{Y})$, and so they form a spanning set rather than a basis.

This diagram representation of $\Theta^\mu(\underline{X}, \underline{Y})$ affords an intuitive realisation of the action of $A \wr S_n$, and we illustrate this by an example. Indeed, keeping $n = 8$, $r = 3$, $\mu = (3, 2, 3)$ as above, let us consider the diagram



(5.2.6)

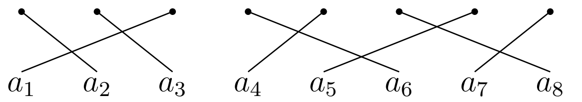
in $\Theta^\mu(\underline{X}, \underline{Y})$; note that this diagram represents the pure tensor

$$u_3 \otimes u_6 \otimes u_8 \otimes u_1 \otimes u_5 \otimes u_2 \otimes u_4 \otimes u_7 \otimes \\ y_1 \otimes y_2 \otimes y_3 \otimes (1, 3, 8, 7, 4)(2, 6). \tag{5.2.7}$$

Now take the element

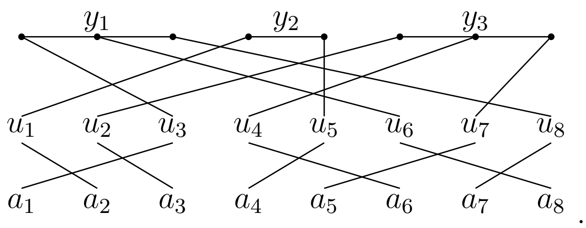
$$\big((1, 2, 3)(4, 6, 8, 7, 5); a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\big) \tag{5.2.8}$$

of $A \wr S_8$, which is represented by the diagram

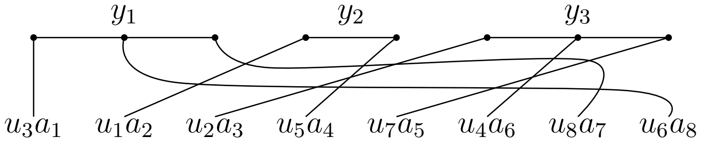


(5.2.9)

The action of the element (5.2.9) on (5.2.6) is calculated as follows: we connect the diagram (5.2.9) below the diagram (5.2.6) to get

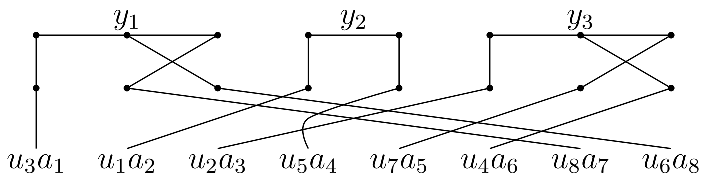


We slide each u_i down its string and simplify the drawing of the resulting partition diagram, to obtain

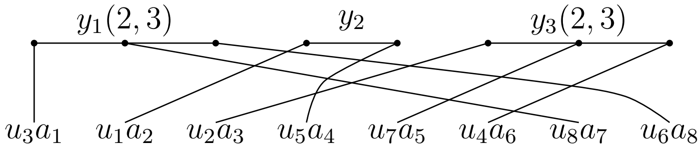


(5.2.10)

The permutation encoded in the strings of this diagram is $(2,8,5,4)(3,7,6)$, which has the factorisation $(2,8,5,4)(3,7,6) = (2,3)(7,8) \cdot (2,7,5,4)(3,8,6)$ where $(2,3)(7,8) \in S_\mu$ and $(2,7,5,4)(3,8,6) \in \mathcal{R}_\mu$; we represent this factorisation by redrawing the diagram (5.2.10) as



and we note that in the lower part of this diagram, which represents the permutation $(2,7,5,4)(3,8,6)$, the strings associated to each y_i do not cross each other, which demonstrates that $(2,7,5,4)(3,8,6)$ is in $\mathcal{R}_\mu$. Now in the upper part of the diagram, the arrangement of strings encodes the permutation $(2,3) \in S_3$ below both y_1 and y_3, while the strings below y_2 encode the identity permutation in S_2. We remove the upper part of the diagram and let these permutations act on their respective elements y_i, yielding



.

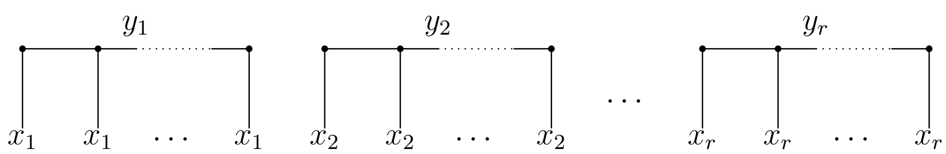
Under our mapping, this corresponds to the pure tensor

$$u_3a_1 \otimes u_8a_7 \otimes u_6a_8 \otimes u_1a_2 \otimes u_5a_4 \otimes u_2a_3 \otimes u_7a_5 \otimes u_4a_6 \otimes \\ y_1(2,3) \otimes y_2 \otimes y_3(2,3) \otimes (2,7,5,4)(3,8,6).$$

By letting $(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) = (u_3,u_6,u_8,u_1,u_5,u_2,u_4,u_7)$, $\sigma = (1,2,3)(4,6,8,7,5)$ and $\gamma = (1,3,8,7,4)(2,6)$, and noting as above that then $\gamma\sigma = (2,8,5,4)(3,7,6) = (2,3)(7,8) \cdot (2,7,5,4)(3,8,6)$ where $(2,3)(7,8) \in S_\mu$ and $(2,7,5,4)(3,8,6) \in \mathcal{R}_\mu$, we may verify that this is indeed the image of (5.2.7) under the action of (5.2.8) as given by (5.2.3). In the general case, for the $A \wr S_n$-module $\Theta^\mu(\underline{X}, \underline{Y})$, let d be the diagram formed from the permutation diagram of $\gamma \in \mathcal{R}_\mu$ with labels y_1 to y_r on the top row and labels u_1 to u_n on the bottom row, and let a be the element $(\sigma; a_1, \ldots, a_n)$ of $A \wr S_n$. Then we have $\gamma\sigma = \theta\zeta$ where $\theta \in S_\mu$ and $\zeta \in \mathcal{R}_\mu$, and so θ corresponds to some element $(\theta_1, \ldots, \theta_r)$ of $S_{\mu_1} \times \cdots \times S_{\mu_r}$ under the canonical isomorphism. Then the image of d under the action of a is the diagram formed from the permutation diagram of ζ with top row labels $y_1\theta_1$ to $y_r\theta_r$ and bottom row labels $u_{(1)\sigma^{-1}}a_1$ to $u_{(n)\sigma^{-1}}a_n$. We leave it to the reader to convince themselves that in this diagram the nodes of the i^{th} block on the top row are connected to elements of X_i, and moreover that this diagram does indeed represent the action of a on the pure tensor of $\Theta^\mu(\underline{X}, \underline{Y})$ represented by d.

The following result will allow us to prove that the wreath product of a cyclic cellular algebra with S_n is again cyclic cellular, thus obtaining the result of Geetha and Goodman (albeit in a weaker form due to the different ordering on the set of cell indices, as mentioned above).

Proposition 5.2.1. *If $X_1, \ldots, X_r$ are cyclic A-modules, and for each i, Y_i is a cyclic kS_{μ_i}-module, then $\Theta^\mu(\underline{X}, \underline{Y})$ is a cyclic $A \wr S_n$-module for any r part composition μ of n. Indeed, if x_i is a generator for X_i and y_i is a generator for Y_i, the diagram*



(where each x_i appears μ_i times) generates $\Theta^\mu(\underline{X}, \underline{Y})$.

Proof. Let d_0 be the diagram in the proposition. It is easy to see that we may obtain any diagram in $\Theta^\mu(\underline{X}, \underline{Y})$ by first applying an element $(\theta; 1, \ldots, 1)$

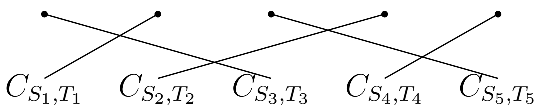
of $A \wr S_n$, where $\theta \in S_\mu$, in order to replace each element y_i in d_0 with an arbitrary element of Y_i, then applying $(\gamma; 1, \ldots, 1)$ for some $\gamma \in \mathcal{R}_\mu$ to arrange the strings of the diagram, and finally applying an element $(e; a_1, \ldots, a_n)$ to replace each element x_i with an arbitrary element of X_i. Since $\Theta^\mu(\underline{X}, \underline{Y})$ is spanned by diagrams, the proof is complete. $\square$

5.3 The iterated inflation structure of the wreath product algebra

We now turn to the case where our interest lies, which is the case where A is a cellular algebra. We shall exhibit the wreath product $A \wr S_n$ as an iterated inflation of cellular algebras, and hence show that it is in turn a cellular algebra.

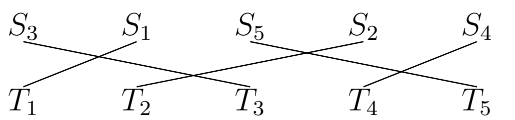
Note that a version of the material in this section formed part of the author's M.Sc. thesis [14] as mentioned at the start of the chapter. The version presented here is an improvement on the version given in [14] because it makes use of a more sophisticated order on the layers of the iterated inflation structure. Other than the relatively minor modifications to the arguments necessary to make use of this improved order, the material is in essentially the same form as in [14] (the version in [14] used a slightly different cellular structure on the group algebra of the symmetric group, with the *duals* of the Specht modules appearing as the cell modules, but this makes no difference to the arguments).

Let A be a cellular algebra with anti-involution $*$ and cellular data (Γ, M, C). We let $r = |\Gamma|$, and we fix a numbering of the elements of Γ as $\lambda_1, \lambda_2, \ldots, \lambda_r$, and moreover we choose this numbering such that $\lambda_i > \lambda_j$ implies $i < j$, so that our numbering is in this sense compatible with the partial ordering on Γ. We write Δ^λ for the right cell module associated to $\lambda \in \Gamma$ as noted above. For convenience we may omit the cell index superscript from elements of the cellular basis, so we write $C_{S,T}$ rather than $C^\lambda_{S,T}$. We have a basis of $A \wr S_n$ consisting of all elements of the form $(\sigma; C_{S_1,T_1}, \dots, C_{S_n,T_n})$ where $\sigma \in S_n$ and each C_{S_i,T_i} is some element of the cellular basis of A; note that we allow the elements C_{S_i,T_i} to be associated to different cell indices. We shall denote this basis by $\mathcal{A}$. Now elements of $\mathcal{A}$ are represented by diagrams like, for example,

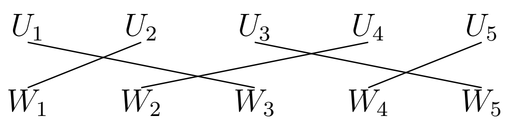


(5.3.1)

but we want a slightly different representation. Indeed, in the diagram (5.3.1), we replace each C_{S_i,T_i} with the pair S_i, T_i, and then move the S_i up to the top of the associated string, to get



We thus obtain a different way of representing elements of $\mathcal{A}$, as diagrams of the form



(5.3.2)

consisting of a permutation diagram where the nodes on the top and bottom rows are replaced with elements $U_i, W_i \in \sqcup_{\lambda\in\Gamma} M(\lambda)$, such that if U_i on the top row is connected to W_j on the bottom row, then we must have $U_i, W_j \in M(\lambda)$ for some $\lambda \in \Gamma$ (i.e. U_i and W_j lie in *the same* set $M(\lambda)$). Note that the diagram (5.3.2) represents the element

$$\big((1,3,5,4,2);\, C_{U_2,W_1},\, C_{U_4,W_2},\, C_{U_1,W_3},\, C_{U_5,W_4},\, C_{U_3,W_5}\big) \in A \wr S_5.$$

Now given any such diagram, for each $i \in \{1, \dots, r\}$ we let μ_i be the number of elements U_j such that $U_j \in M(\lambda_i)$. We thus obtain a composition $\mu = (\mu_1, \dots, \mu_r)$ of n (note that some of the parts μ_i may be zero in general). We call this the *layer index* of the diagram, and also of the element of $\mathcal{A}$ which it represents. We let $k\mathcal{A}_\mu$ be the k-span of all elements of $\mathcal{A}$ with layer index μ, and we recall that Ω_n^r is the set of all r-part compositions of n with non-negative integer entries. Then $A \wr S_n = \bigoplus_{\mu \in \Omega_n^r} k\mathcal{A}_\mu$. For a layer index μ, we define a *half diagram of type* μ to be a tuple $(U_1, \dots, U_n)$ of n elements of $\sqcup_{\lambda \in \Gamma} M(\lambda)$, such that there are exactly μ_i elements of $M(\lambda_i)$ for each i. We define $\mathcal{V}_\mu$ to be the set of all half diagrams of type μ. Now if $(U_1, \dots, U_n)$ is a half diagram of type μ, then we may easily see that there is a unique element ϵ of $\mathcal{R}_\mu$ such that $(U_{(1)\epsilon}, \dots, U_{(n)\epsilon})$ lies in the set $M(\lambda_1)^{\mu_1} \times \cdots \times M(\lambda_r)^{\mu_r}$. We shall call this ϵ the *shape* of the half diagram $(U_1, \dots, U_n)$.

Let E be the diagram with top row U_1 to U_n, bottom row W_1 to W_n (reading from left to right), and where $\sigma \in S_n$ is the permutation such that U_i is connected to $W_{(i)\sigma}$. Thus E represents the element

$$\big(\sigma\,;\, C[U_{(1)\sigma^{-1}}, W_1], \dots, C[U_{(n)\sigma^{-1}}, W_n]\big)$$

where to ease the notation we allow ourselves to write $C[U, W]$ for $C_{U,W}$. Suppose E has layer index μ. We may decompose E into three pieces of data, namely the half diagrams $(U_1, \dots, U_n)$, $(W_1, \dots, W_n)$ of type μ, formed from the top and bottom rows of E respectively, and the element $(\pi_1, \dots, \pi_r)$ of the group $S_{\mu_i} \times \cdots \times S_{\mu_r}$ where $\pi_i \in S_{\mu_i}$ is such that (counting from left to right) the j^{th} element of $M(\lambda_i)$ on the top row is connected to the $(j)\pi_i{}^{\text{th}}$ element of $M(\lambda_i)$ on the bottom row. Thus π_i records how the elements of $M(\lambda_i)$ on the top row are connected to the elements of $M(\lambda_i)$ on the bottom row. For example, suppose that $r = 3$ and that the diagram (5.3.2) has layer index $(3, 0, 2)$ with $U_1, U_2, U_4 \in M(\lambda_1)$ and $U_3, U_5 \in M(\lambda_3)$. Then

$(\pi_1, \pi_2, \pi_3) = \big((1,3,2), e, (1,2)\big)$ (note that e here is the unique element of the trivial group $S_{\mu_2} = S_0$). It is easy to see that if ϵ, δ are the shapes of $(U_1, \ldots, U_n)$ and $(W_1, \ldots, W_n)$ respectively, and further if π is the image of $(\pi_1, \ldots, \pi_r)$ under the natural identification of $S_{\mu_i} \times \cdots \times S_{\mu_r}$ with the Young subgroup S_μ of S_n, then $\sigma = \epsilon^{-1}\pi\delta$. If we now let V_μ be the k-vector space with basis $\mathcal{V}_\mu$, then the above decomposition is easily seen to afford a k-linear bijection

$$V_\mu \otimes kS_\mu \otimes V_\mu \longrightarrow k\mathcal{A}_\mu$$

given by mapping

$$(U_1, \ldots, U_n) \otimes \pi \otimes (W_1, \ldots, W_n),$$

to

$$\big(\epsilon^{-1}\pi\delta\,;\ C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}}, W_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}}, W_n]\big)$$

where ϵ is the shape of $(U_1, \ldots, U_n)$ and δ is the shape of $(W_1, \ldots, W_n)$. We thus have a decomposition $A \wr S_n = \bigoplus_{\mu \in \Omega_n^r} V_\mu \otimes kS_\mu \otimes V_\mu$, and this decomposition will allow us to exhibit the desired iterated inflation structure. For this, we need to equip the set Ω_n^r with an ordering. The ordering which we shall use is neither the lexicographic order nor the dominance order, but rather a variation on the dominance order which takes account of the partial order on the set Γ. Indeed, if $\mu = (\mu_1, \ldots, \mu_r)$ and $\alpha = (\alpha_1, \ldots, \alpha_r)$ are elements of Ω_n^r, then we define $\mu \trianglerighteq_\Gamma \alpha$ to mean that for each $q = 1, \ldots, r$ we have

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \mu_i \ \geq \sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \alpha_i$$

(and of course we define $\triangleright_\Gamma$ to match). We call this (partial) order the Γ-*dominance order*[1].

[1] The use of the Γ-dominance order on Ω_n^r was suggested by the anonymous reviewer

Now take $\mathcal{V}_\mu$ as above, B_μ to be kS_μ and $\mathcal{B}_\mu$ to be S_μ. We may easily see that our basis $\mathcal{A}$ is indeed the basis of $A \wr S_n$ obtained from the bases $\mathcal{V}_\mu$ and $\mathcal{B}_\mu$ as in section 5.1.3, and we shall now prove that our decomposition exhibits $A \wr S_n$ as an iterated inflation with respect to the anti-involution given by (5.2.1) and the cellular structure on the algebras kS_μ as in Proposition 5.1.4. Thus, we must prove that the equations (5.1.3) and (5.1.4) hold. The fact that equation (5.1.3) holds follows easily from the description of the anti-involution on $A \wr S_n$ given after equation (5.2.1). To prove that (5.1.4) holds, we shall prove the slightly stronger result Proposition 5.3.2, below. First, we need a lemma, which will allow us to compare layer indices of elements of $A \wr S_n$.

Lemma 5.3.1. *Suppose that we have $s_1, \dots, s_n, t_1, \dots, t_n \in \{1, \dots, r\}$ such that $\lambda_{s_j} \geq \lambda_{t_j}$ in the poset Γ for each j. For each $i = 1, \dots, r$, let μ_i be the number of s_j which are equal to i and α_i be the number of t_j which are equal to i. Let $\mu = (\mu_1, \dots, \mu_r)$ and $\alpha = (\alpha_1, \dots, \alpha_r)$ so that $\alpha, \mu \in \Omega_n^r$. Then $\mu \trianglerighteq_\Gamma \alpha$, and if at least one of the inequalities $\lambda_{s_j} \geq \lambda_{t_j}$ is strict then we have $\mu \triangleright_\Gamma \alpha$.*

Proof. This lemma is nothing more than simple combinatorics. We need to show that

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \mu_i \;\geq\; \sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \alpha_i.$$

But we have for each $q = 1, \dots, r$ that

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \mu_i \;=\; |\{j : \lambda_{s_j} \geq \lambda_q\}|$$

and

$$\sum_{\substack{i \text{ such that} \\ \lambda_i \geq \lambda_q}} \alpha_i \;=\; |\{j : \lambda_{t_j} \geq \lambda_q\}|$$

who reviewed my article *Cellular Structure of Wreath Product Algebras* for the *Journal of Pure and Applied Algebra.*

and since the set appearing in the right-hand side of the latter equation is a subset of the corresponding set in the first equation, we have the required inequality $\mu \trianglerighteq_\Gamma \alpha$. If there is a strict inequality $\lambda_{s_j} > \lambda_{t_j}$ we clearly have $\mu \neq \alpha$ and hence $\mu \triangleright_\Gamma \alpha$. $\square$

Proposition 5.3.2. *Let $\mu \in \Omega_n^r$, let $u = (U_1, \ldots, U_n)$, $w = (W_1, \ldots, W_n)$ be elements of $\mathcal{V}_\mu$ and $\pi = (\pi_1, \ldots, \pi_r) \in S_\mu$ such that the element of $\mathcal{A}$ corresponding to the pure tensor $u \otimes \pi \otimes w$ has layer index μ. Further, let $a = (\sigma; a_1, \ldots, a_n)$ be a pure tensor in $A \wr S_n$. Then we have $(u \otimes \pi \otimes w) \cdot a \equiv u \otimes \pi\, \theta_\mu(w, a) \otimes \phi_\mu(w, a)$ modulo elements of $\mathcal{A}$ of layer index strictly less (in the Γ-dominance order) than μ, where $\theta_\mu(w, a) \in S_\mu$ and $\phi_\mu(w, a) \in V_\mu$ are independent of u and π.*

Note that in the proposition we allow the a in $\theta_\mu(w, a)$ and $\phi_\mu(w, a)$ to be any pure tensor in $A \wr S_n$ rather than just an element of $\mathcal{A}$ as required in (5.1.4).

Proof. Let $\epsilon, \delta \in \mathcal{R}_\mu$ be the shapes of u and w respectively, so that $u \otimes \pi \otimes w$ corresponds to the element

$$\big(\epsilon^{-1}\pi\delta\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}}, W_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}}, W_n]\big).$$

Then

$$\begin{aligned}
&(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n) = \\
&\quad \big(\epsilon^{-1}\pi\delta\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta)^{-1}}, W_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta)^{-1}}, W_n]\big)\big(\sigma; a_1, \ldots, a_n\big) = \\
&\qquad \big(\epsilon^{-1}\pi\delta\sigma; C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(1)\sigma^{-1}}]a_1, \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(n)\sigma^{-1}}]a_n\big).
\end{aligned}$$

For each $i = 1, \ldots, n$, let $s_i \in \{1, \ldots, r\}$ be such that $U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(i)\sigma^{-1}} \in M(\lambda_{s_i})$. Then by (5.1.1) we have

$$C[U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, W_{(i)\sigma^{-1}}]a_i \equiv \sum_{X_i \in M(\lambda_{s_i})} R_{a_i}(W_{(i)\sigma^{-1}}, X_i) C[U_{(i)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_i]$$

modulo cellular basis elements of lower cell index. Using this, we see that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n)$ is congruent to

$$\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) \big(\epsilon^{-1}\pi\delta\sigma\,;\; C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_n]\big) \quad (5.3.3)$$

modulo elements of $\mathcal{A}$ of the form

$$\big(\epsilon^{-1}\pi\delta\sigma\,; C^{\lambda_{t_1}}[S_1, T_1], \ldots, C^{\lambda_{t_n}}[S_n, T_n]\big) \quad (5.3.4)$$

where for each i we have $\lambda_{s_i} \geq \lambda_{t_i}$ and for at least one i this inequality is strict. Now let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be the layer index of (5.3.4). By Lemma 5.3.1 we have $\mu \rhd_\Gamma \alpha$, so that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n)$ is congruent to (5.3.3) modulo elements of lower layer index.

Now X_i lies in the same set $M(\lambda_{s_i})$ as $W_{(i)\sigma^{-1}}$, and from this we may easily see that the shape of $(X_1, \ldots, X_n)$ is the unique element ζ of $\mathcal{R}_\mu$ such that $\delta\sigma = \theta\zeta$ for $\theta \in S_\mu$. Thus in (5.3.3) we have

$$\begin{aligned}\big(\epsilon^{-1}\pi\delta\sigma\,;\; & C[U_{(1)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\delta\sigma)^{-1}}, X_n]\big) \\ &= \big(\epsilon^{-1}\pi\theta\zeta\,;\; C[U_{(1)(\epsilon^{-1}\pi\theta\zeta)^{-1}}, X_1], \ldots, C[U_{(n)(\epsilon^{-1}\pi\theta\zeta)^{-1}}, X_n]\big)\end{aligned}$$

which we now see corresponds to the pure tensor $u \otimes \pi\theta \otimes (X_1, \ldots, X_n)$, and hence (5.3.3) is equal to

$$u \otimes \pi\theta \otimes \left(\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) (X_1, \ldots, X_n) \right).$$

Thus, setting $\theta_\mu(w, a)$ to be the unique element θ of S_μ such that $\delta\sigma = \theta\zeta$ for $\zeta \in \mathcal{R}_\mu$ and $\phi_\mu(w, a)$ to be

$$\sum_{X_1} \cdots \sum_{X_n} \left(\prod_{i=1}^{n} R_{a_i}\big(W_{(i)\sigma^{-1}}, X_i\big) \right) (X_1, \ldots, X_n), \quad (5.3.5)$$

we see that $(u \otimes \pi \otimes w)(\sigma; a_1, \ldots, a_n) \equiv u \otimes \pi\theta_\mu(w,a) \otimes \phi_\mu(w,a)$ modulo lower layers, and furthermore these values depend only on w and a, as required. $\square$

By the results in Section 5.1.3, we now have that $A \wr S_n$ is a cellular algebra. Further, we may use Proposition 5.1.4 to see that the set indexing the cell modules of $A \wr S_n$ is the set of all pairs $\big(\mu, (\nu^1, \ldots, \nu^r)\big)$ where μ is an r-component composition $(\mu_1, \ldots, \mu_r)$ of n (recalling that $r = |\Gamma|$), and ν^i is a partition of μ_i. Thus in any such pair we have $\mu = (|\nu^1|, \ldots, |\nu^r|)$, and so we lose no information if we omit the partition μ from these pairs. Hence we may identify the set of cell indices of $A \wr S_n$ with the set of all r-multipartitions $(\nu^1, \ldots, \nu^r)$ of n. We now give a statement of the cellularity of $A \wr S_n$.

Theorem 5.3.3. *Let A be a cellular algebra with anti-involution $*$ and poset Γ of cell indices. Recall that $\underline{\Lambda}_n^r$ denotes the set of all multipartitions of n of length r. Then $A \wr S_n$ is a cellular algebra with respect to a tuple of cellular data including the anti-involution given for $\sigma \in S_n$ and $a_1, \ldots, a_n \in A$ by*

$$(\sigma; a_1, \ldots, a_n)^* = \big(\sigma^{-1}\,;\, a^*_{(1)\sigma}, \ldots, a^*_{(n)\sigma}\big)$$

and also the poset consisting of $\underline{\Lambda}_n^r$ with the following partial order: if $(\nu^1, \ldots, \nu^r), (\eta^1, \ldots, \eta^r) \in \underline{\Lambda}_n^r$ then $(\nu^1, \ldots, \nu^r) \geqslant (\eta^1, \ldots, \eta^r)$ means either that $(|\nu^1|, \ldots, |\nu^r|) \trianglerighteq_\Gamma (|\eta^1|, \ldots, |\eta^r|)$ or that $|\nu^i| = |\eta^i|$ and $\nu^i \trianglerighteq \eta^i$ for each i.

In the next section, we shall consider the cell modules which arise from this structure. In particular we shall follow the work of Geetha and Goodman by proving that if A is cyclic cellular, then so is $A \wr S_n$.

Note that the partial order we have obtained on $\underline{\Lambda}_n^r$ is not the dominance order. Indeed, the dominance order is a strictly smaller order than our order (meaning that any relation between multipartitions which holds in the dominance order also holds in the order in Theorem 5.3.3). In their cellularity result [12, Theorem 4.1], Geetha and Goodman obtained (subject to the assumption that A is cyclic cellular) the Γ-*dominance order* on $\underline{\Lambda}_n^r$ (see [12, Definition 3.1, (2)]), an order which is in general strictly smaller than the dominance order on $\underline{\Lambda}_n^r$ and moreover preserves the representation-theoretic information present in the ordering on Γ. The fact that we have ended up with our larger ordering on the set $\underline{\Lambda}_n^r$ is fundamentally due to our use of the method of iterated inflations, which will always yield an order with the kind of "layered" structure which our order exhibits. This is important to note, since a smaller ordering on the set of cell indices of a cellular algebra (i.e. an ordering with fewer relations) provides better representation-theoretic information about the algebra. Thus we note that in order to use the method of iterated inflations rather than the intricate arguments of Geetha and Goodman (and hence obtain a much simpler proof of the cellularity of $A \wr S_n$ than the proof given in [12]), we must be content with a slightly weaker result. However, our result is still sufficient to obtain the desirable results in the next section.

5.4 The cell and simple modules of the wreath product algebra

In this section we shall use the theory of cellular algebras to prove nice results about the simple modules of $A \wr S_n$, and to establish a condition for the semisimplicity of $A \wr S_n$ (where A is a cellular algebra as above).

Note that a version of the material in the first part of this section, up to and including Proposition 5.4.1, formed part of the author's M.Sc. thesis [14] as mentioned at the start of the chapter. The version presented here is an improvement on the version given in [14] because it makes use of the more sophisticated Γ-dominance order on the layers of the iterated inflation structure. Other than the relatively minor modifications to the arguments necessary to make use of this improved order, the material is in essentially the same form as in [14] (the version in [14] used a slightly different cellular structure on the group algebra of the symmetric group, with the *duals* of the Specht modules appearing as the cell modules, but this makes no difference to the arguments).

Recall that the cell modules Δ^{λ_i} of A are indexed by the cell indices $\lambda_1, \lambda_2, \ldots, \lambda_r$. In the sequel we shall also allow ourselves to write Δ^{λ_i} as $\Delta(\lambda_i)$ when this makes our formulae more readable. We shall now consider the cell modules of $A \wr S_n$. We know that these are indexed by length r multipartitions of n. Let $\underline{\nu} = (\nu^1, \ldots, \nu^r)$ be such a multipartition. We shall show that the cell module $\Delta^{\underline{\nu}}$ is isomorphic to the module $\Theta^{|\underline{\nu}|}\big((\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}), (S^{\nu^1}, \ldots, S^{\nu^r})\big)$ [12, Theorem 4.27].

Now we know from Proposition 5.1.4 and the results in section 5.1.3 that, as a k-vector space, $\Delta^{\underline{\nu}}$ may naturally be identified with

$$S^{\nu^1} \otimes \cdots \otimes S^{\nu^r} \otimes V_{|\underline{\nu}|}, \tag{5.4.1}$$

so let us consider the structure of the vector space $V_{|\underline{\nu}|}$. Indeed, let $\alpha_1, \ldots \alpha_n$ be elements of Γ such that

$$(\alpha_1, \ldots, \alpha_n) = (\underbrace{\lambda_1, \lambda_1, \ldots, \lambda_1}_{|\nu^1| \text{ places}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{|\nu^2| \text{ places}}, \lambda_3, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{|\nu^r| \text{ places}}). \tag{5.4.2}$$

Let $(X_1, \ldots, X_n)$ be a half diagram in $\mathcal{V}_{|\underline{\nu}|}$. Then the shape of $(X_1, \ldots, X_n)$ is the unique element γ of $\mathcal{R}_{|\underline{\nu}|}$ such that $(X_1, \ldots, X_n)$ lies in $M(\alpha_{(1)\gamma^{-1}}) \times \cdots \times M(\alpha_{(n)\gamma^{-1}})$. We now see that

$$\mathcal{V}_{|\underline{\nu}|} = \bigsqcup_{\gamma \in \mathcal{R}_{|\underline{\nu}|}} M(\alpha_{(1)\gamma^{-1}}) \times \cdots \times M(\alpha_{(n)\gamma^{-1}})$$

and hence if we identify the half diagram $(X_1, \dots, X_n)$ with the pure tensor $C_{X_1} \otimes \cdots \otimes C_{X_n}$, we obtain a natural identification of k-vector spaces

$$V_{|\underline{\nu}|} = \bigoplus_{\gamma \in \mathcal{R}_{|\underline{\nu}|}} \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}}). \tag{5.4.3}$$

We shall henceforth consider these two vector spaces to be thus identified. Further, we shall abuse terminology and use the term *pure tensor in* $V_{|\underline{\nu}|}$ to mean any pure tensor in any of the summands in the right hand side of (5.4.3). For example, we can easily show using (5.1.2) and (5.3.5) that under the identification (5.4.3) we have

$$\phi_{|\underline{\nu}|}\big(C_{W_1} \otimes \cdots \otimes C_{W_n}, (\sigma; a_1, \dots, a_n)\big) = C_{W_{(1)\sigma^{-1}}}\, a_1 \otimes \cdots \otimes C_{W_{(n)\sigma^{-1}}}\, a_n, \tag{5.4.4}$$

where $\phi_{|\underline{\nu}|}$ is of course the function $\mathcal{V}_{|\underline{\nu}|} \times \mathcal{A} \longrightarrow V_{|\underline{\nu}|}$ which forms part of the iterated inflation structure on $A \wr S_n$. In light of (5.4.1), we shall further speak of a *pure tensor in* $\Delta^{\underline{\nu}}$ to mean any pure tensor of the form

$$w_1 \otimes \cdots \otimes w_r \otimes u_1 \otimes \cdots \otimes u_n,$$

where $w_i \in S^{\nu^i}$ and $u_1 \otimes \cdots \otimes u_n$ is a pure tensor in $V_{|\underline{\nu}|}$. Using (5.4.4) and the expression for $\theta_{|\underline{\nu}|}(w, a)$ given near the end of the proof of Proposition 5.3.2, we may now verify that the map taking the pure tensor

$$x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_r \otimes \gamma$$

in $\Theta^{|\underline{\nu}|}\big((\Delta^{\lambda_1}, \dots, \Delta^{\lambda_r}), (S^{\nu^1}, \dots, S^{\nu^r})\big)$ (where $\gamma \in \mathcal{R}_{|\underline{\nu}|}$) to the pure tensor

$$y_1 \otimes \cdots \otimes y_r \otimes x_{(1)\gamma^{-1}} \otimes \cdots \otimes x_{(n)\gamma^{-1}}$$

in $\Delta^{\underline{\nu}}$ is an isomorphism of $A \wr S_n$-modules (but note that in order to apply the formula given in section 5.1.3 for the action of an iterated inflation on its cell modules, the arguments w and a in $\theta_{|\underline{\nu}|}(w, a)$ and $\phi_{|\underline{\nu}|}(w, a)$ must be elements of the bases $\mathcal{A}$ and $\mathcal{V}_{|\underline{\nu}|}$, respectively). We have thus proved the following result.

Proposition 5.4.1. *[12, Theorem 4.27] We have for any r-multipartition $\underline{\nu} = (\nu^1, \dots, \nu^r)$ of n an isomorphism of $A \wr S_n$-modules*

$$\Theta^{|\underline{\nu}|}\big((\Delta^{\lambda_1}, \dots, \Delta^{\lambda_r}), (S^{\nu^1}, \dots, S^{\nu^r})\big) \cong \Delta^{\underline{\nu}}.$$

We may now use Proposition 5.2.1 and the fact that all Specht modules are cyclic to obtain the following result. Of course, this is a weaker result than the corresponding result in [12], since (as already mentioned) Geetha and Goodman obtain the Γ-dominance order on their cell indices.

Proposition 5.4.2. *(compare [12, Theorem 4.1]) If A is cyclic cellular then so is $A \wr S_n$.*

Let $\mu \in \Omega_n^r$. Then by (5.1.5), we know that the multiplication within the layer of $A \wr S_n$ indexed by μ is determined by a bilinear form, ψ_μ. Let $(U_1, \dots, U_n), (W_1, \dots, W_n)$ be half diagrams in $\mathcal{V}_\mu$, so that $u = C_{U_1} \otimes \cdots \otimes C_{U_n}$ and $w = C_{W_1} \otimes \cdots \otimes C_{W_n}$ are pure tensors in V_μ. Now by equation (5.1.5),

$$(u \otimes e \otimes u)(w \otimes e \otimes w) \equiv u \otimes \psi_\mu(u, w) \otimes w \tag{5.4.5}$$

modulo lower layers. The element $u \otimes e \otimes u$ of $A \wr S_n$ is represented by the diagram

$$\begin{array}{cccc} U_1 & U_2 & \cdots & U_n \\ | & | & & | \\ U_1 & U_2 & \cdots & U_n \end{array} \quad = \quad \begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{U_1,U_1} & C_{U_2,U_2} & \cdots & C_{U_n,U_n} \end{array}$$

and of course the element $w \otimes e \otimes w$ is represented by a diagram which is the same except that each U is replaced with a W. Thus we find by concatenating and simplifying these diagrams that the product $(u \otimes e \otimes u)(w \otimes e \otimes w)$ corresponds to

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{U_1,U_1}C_{W_1,W_1} & C_{U_2,U_2}C_{W_2,W_2} & \cdots & C_{U_n,U_n}C_{W_n,W_n} \end{array}. \tag{5.4.6}$$

We may expand each of the products $C_{U_j,U_j}C_{W_j,W_j}$ in terms of the cellular basis of A and use these expansions to write (5.4.6) as a linear combination of diagrams of the form

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ | & | & & | \\ C_{X_1,Y_1} & C_{X_2,Y_2} & \cdots & C_{X_n,Y_n} \end{array}.$$

Now for $j = 1, \ldots, n$, let s_j be such that $U_j \in M(\lambda_{s_j})$. The we know that each product $C_{U_j,U_j}C_{W_j,W_j}$ is a linear combination of cellular basis elements $C^{\lambda_{t_j}}_{X,Y}$ where $\lambda_{t_j} \leq \lambda_{s_j}$. It follows by Lemma 5.3.1 that all such diagrams have layer index at most μ (in the Γ-dominance order). Moreover, Lemma 5.3.1 also tells us that, if for any j the element W_j do not lie in $M(\lambda_{s_j})$ (so that $C_{U_j,U_j}C_{W_j,W_j}$ is a linear combination of cellular basis elements $C^{\lambda_{t_j}}_{X,Y}$ where $\lambda_{t_j} < \lambda_{s_j}$), then all of the diagrams in the expansion have layer index strictly less than μ, and hence by (5.4.5) we see that we must have $\psi_\mu(u, w) = 0$ in this case. Suppose now that $W_j \in M(\lambda_{s_j})$ for each j. By (2.4.1) in [13], we know that $C_{U_j,U_j}C_{W_j,W_j}$ is congruent to $\langle C_{U_j}, C_{W_j}\rangle C_{U_j,W_j}$ modulo cellular basis elements of lower cell index, where $\langle \cdot, \cdot \rangle$ is the appropriate cell form. Using Lemma 5.3.1 as above, we see that (5.4.6) is congruent modulo lower layers to

$$\langle C_{U_1}, C_{W_1}\rangle \langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle \begin{array}{cccc} U_1 & U_2 & \cdots & U_n \\ | & | & & | \\ W_1 & W_2 & \cdots & W_n \end{array},$$

which represents the element $\langle C_{U_1}, C_{W_1}\rangle \langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle\, u \otimes e \otimes w$, and hence we find that in this case

$$\psi_\mu(u, w) = \langle C_{U_1}, C_{W_1}\rangle \langle C_{U_2}, C_{W_2}\rangle \cdots \langle C_{U_n}, C_{W_n}\rangle.$$

Note in particular that ψ_μ is thus in all cases k-valued. We can now use these values for ψ_μ in the case where $\mu = |\underline{\nu}|$, together with equation (5.1.6) and

Proposition 5.1.4 to compute the values of the cell form on the cell module $\Delta^{\underline{\nu}}$. Indeed, if $y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n$ and $z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n$ are pure tensors in the cell module $\Delta^{\underline{\nu}}$, then we see that

$$\begin{aligned}\langle y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n,\ z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n\rangle = \\ \langle y_1, z_1\rangle \cdots \langle y_r, z_r\rangle\langle u_1, w_1\rangle \cdots \langle u_n, w_n\rangle \end{aligned} \tag{5.4.7}$$

if u_j and w_j lie in the same $\Delta(\lambda)$ for each $i = 1, \ldots, n$, and

$$\langle y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n,\ z_1 \otimes \cdots \otimes z_r \otimes w_1 \otimes \cdots \otimes w_n\rangle = 0 \tag{5.4.8}$$

otherwise.

Next we seek to describe the cell radical of $\Delta^{\underline{\nu}}$. Using (5.4.1) and (5.4.3), we have isomorphisms of k-vector spaces

$$\begin{aligned}\Delta^{\underline{\nu}} &\cong S^{\nu^1} \otimes \cdots \otimes S^{\nu^r} \otimes V_{|\underline{\nu}|} \\ &\cong \bigoplus_{\gamma \in \mathcal{R}_{|\underline{\nu}|}} S^{\nu^1} \otimes \cdots \otimes S^{\nu^r} \otimes \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}}).\end{aligned} \tag{5.4.9}$$

For $\gamma \in \mathcal{R}_{|\underline{\nu}|}$, let $\Upsilon_\gamma = S^{\nu^1} \otimes \cdots \otimes S^{\nu^r} \otimes \Delta(\alpha_{(1)\gamma^{-1}}) \otimes \cdots \otimes \Delta(\alpha_{(n)\gamma^{-1}})$. Now we see from (5.4.8) that if γ, β are distinct elements of $\mathcal{R}_{|\underline{\nu}|}$ and $u \in \Upsilon_\gamma$, $w \in \Upsilon_\beta$, then $\langle u, w\rangle = 0$. It follows that, if we let R_γ be the radical of the restriction to Υ_γ of $\langle \cdot, \cdot\rangle$, then the cell radical of $\Delta^{\underline{\nu}}$ is $\bigoplus_{\gamma \in \mathcal{R}_{|\underline{\nu}|}} R_\gamma$.

Let us fix a basis in each Δ^λ and each S^ν. From these bases we obtain a basis of pure tensors in each Υ_γ. Let G_{ν^i} be the Gram matrix of the cell form of S^{ν^i} and G_{α_i} be the Gram matrix of the cell form of Δ^{α_i}, with respect to our chosen bases. If we let B_γ be the Gram matrix of the restriction of the cell form to Υ_γ with respect to our basis, then we see by (5.4.7) that B_γ is the matrix Kronecker product $G_{\nu^1} \otimes \cdots \otimes G_{\nu^r} \otimes G_{\alpha_{(1)\gamma^{-1}}} \otimes \cdots \otimes G_{\alpha_{(n)\gamma^{-1}}}$. By fixing some total order on the set R_γ and concatenating our bases of the Υ_γ in this order, we obtain a basis of $\Delta^{\underline{\nu}}$. Using (5.4.8), we see that the Gram matrix of the cell form with respect to this basis is of block diagonal form with diagonal blocks B_γ for $\gamma \in \mathcal{R}_{|\underline{\nu}|}$. From this we see (using the fact that the rank of the Kronecker product of two matrices is the product of their ranks) that the rank of the cell form on $\Delta^{\underline{\nu}}$ is $|\mathcal{R}_{|\underline{\nu}|}|$ times the product of the ranks of the cell forms of the cell modules $S^{\nu^1}, \ldots, S^{\nu^r}, \Delta^{\alpha_1}, \ldots, \Delta^{\alpha_n}$.

Now in constructing the basis of pure tensors for $\Delta^{\underline{\nu}}$ as above, we may choose our basis of each cell module of A and kS_n by taking a basis of the cell radical and extending this to a basis of the whole cell module. If we do this, then we see that an element $y_1 \otimes \cdots \otimes y_r \otimes u_1 \otimes \cdots \otimes u_n$ of the basis of pure tensors for $\Delta^{\underline{\nu}}$ must lie in the cell radical if any y_i or u_i is an element of the cell radical of the cell module in which it lies. By the above calculation of the rank of the cell form on $\Delta^{\underline{\nu}}$, we see that the number of such elements must be equal to the dimension of the cell radical, and so we have now found a basis of the cell radical inside a basis of the whole cell module.

We can now use the theory of cellular algebras from section 3 of [13] together with our basis of $\Delta^{\underline{\nu}}$ to deduce some results about the simple modules $L^{\underline{\nu}}$ and semisimplicity of $A \wr S_n$. These results are already known for wreath products $A \wr S_n$ with A a general (i.e. not cellular) algebra given extra assumptions on the field (see for example [6, Lemma 3.4]), and in particular for the case $k\big(G \wr S_n\big) \cong (kG) \wr S_n$ where G is a finite group (see for example Chapter 4 of [21] for the case where the field is algebraically closed). However, if A is cellular then our work shows that these results hold *with no restriction on the field at all.* Given the importance of cellular algebras in certain areas of representation theory we are confident that they will prove useful.

Recall that Γ_0 indexes the simple modules of A. Let $\big(\underline{\Lambda}_n^r\big)_0$ denote the set of elements $\underline{\nu} \in \underline{\Lambda}_n^r$ such that the cell radical of $\Delta^{\underline{\nu}}$ is a proper submodule of $\Delta^{\underline{\nu}}$, so that $\big(\underline{\Lambda}_n^r\big)_0$ indexes the simple modules of $A \wr S_n$. Recall that our field

k has characteristic p, which may be zero or a prime.

Theorem 5.4.3. *The set $\left(\underline{\Lambda}_n^r\right)_0$ indexing the simple modules of $A \wr S_n$ consists exactly of those $(\nu^1, \ldots, \nu^r) \in \underline{\Lambda}_n^r$ such that $\nu^i = ()$ whenever $\lambda_i \in \Gamma \setminus \Gamma_0$ and all ν^i are p-regular (recall that $()$ is p-regular for any p).*

In light of Theorem 5.4.3, we see that if we let s be the number of simple modules of A and we let $\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_s$ be the subsequence of the sequence $\lambda_1, \lambda_2, \ldots, \lambda_r$ consisting of the elements of Γ_0, then the simple $A \wr S_n$-modules may in fact be indexed by the set $\underline{\Lambda}_n^s(p)$ consisting of all length s multipartitions of n with p-regular entries. The main idea of the following theorem is well known: see [28, p.204] and also [6, Proposition 3.7] and [12, Theorem 4.25]. As mentioned above, the version presented here is notable for its lack of conditions on the field.

Theorem 5.4.4. *Let $\underline{\nu} = (\nu^1, \ldots, \nu^r) \in \left(\underline{\Lambda}_n^r\right)_0$. Then corresponding to the isomorphism* (5.4.9)*, we have an isomorphism of k-vector spaces*

$$L^{\underline{\nu}} \cong \bigoplus_{\gamma \in \mathcal{R}_{|\underline{\nu}|}} D^{\nu^1} \otimes \cdots \otimes D^{\nu^r} \otimes L^{\alpha_{(1)\gamma^{-1}}} \otimes \cdots \otimes L^{\alpha_{(n)\gamma^{-1}}}$$

(where $\alpha_1, \ldots, \alpha_n$ are as in (5.4.2)*). Moreover, $L^{\underline{\nu}}$ has a representation by diagrams of the form* (5.2.6) *in exactly the same way as $\Delta^{\underline{\nu}}$, by simply using elements of D^{ν^i} rather than S^{ν^i} and elements of L^{α_i} rather than Δ^{α_i}. The action on such diagrams is exactly the same as described above. We thus see that $L^{\underline{\nu}}$ is isomorphic as an $A \wr S_n$-module to $\Theta^{|\underline{\nu}|}\big((L^{\lambda_1}, \ldots, L^{\lambda_r}), (D^{\nu^1}, \ldots, D^{\nu^r})\big)$, where for notational convenience we let $L^{\lambda} = 0$ for $\lambda \in \Gamma \setminus \Gamma_0$.*

We thus see that if we index the simple modules by $\underline{\Lambda}_n^s(p)$ as above, then the simple indexed by $\underline{\hat{\nu}} = (\hat{\nu}^1, \ldots, \hat{\nu}^s)$ (where each $\hat{\nu}^i$ is thus a p-regular partition) is isomorphic to $\Theta^{|\underline{\hat{\nu}}|}\big((L^{\hat{\lambda}_1}, \ldots, L^{\hat{\lambda}_s}), (D^{\hat{\nu}^1}, \ldots, D^{\hat{\nu}^s})\big)$.

Theorem 5.4.5. *Let $\underline{\nu} = (\nu^1, \ldots, \nu^r) \in \left(\underline{\Lambda}_n^r\right)_0$. Then we have $L^{\underline{\nu}} \cong \Delta^{\underline{\nu}}$ if and only if $D^{\nu^i} \cong S^{\nu^i}$ for each $i = 1, \ldots, r$ and whenever we have $\nu^i \neq ()$ we have $L^{\lambda_i} \cong \Delta^{\lambda_i}$.*

Our final result is a criterion for semisimplicity; compare [6, Lemma 3.5].

Theorem 5.4.6. *If A is a cellular algebra, then $A \wr S_n$ is semisimple if and only if both kS_n and A are semisimple.*

5.5 Cellularity results for $k\big(S_m \wr S_n\big)$

Let us conclude this chapter by considering the case $A = kS_m$ so that $A \wr S_n$ is the group algebra $k(S_m \wr S_n)$, and thus applying the work which has been done in this chapter to the situation which is considered in the rest of the thesis. Recall that k is a field of characteristic p, where p may be zero or a prime.

Indeed, we know from Theorem 5.1.3 that kS_m is cellular with respect to the anti-involution induced by mapping each element of S_m to its inverse, and a tuple of cellular data including the poset Λ_m of all partitions of m endowed with the dominance order. Moreover, the cell module associated to $\mu \in \Lambda_m$ by this structure is the Specht module S^μ, the set $\left(\Lambda_m\right)_0$ of cell indices indexing the simple modules in this structure is the set of all p-regular partitions of m, and the simple module associated to a p-regular partition μ by this structure is D^μ.

Taking $A = kS_m$ and thus considering the wreath product $(kS_m) \wr S_n \cong k(S_m \wr S_n)$, we recall from (5.2.4) that we have for any $\alpha \vDash n$ an equality of $k(S_m \wr S_n)$-modules

$$\Theta^\alpha(\underline{X}, \underline{Y}) = \Big[\big(X_1, \ldots, X_r\big)^{\widetilde{\boxtimes}\alpha} \oslash \big(Y_1 \boxtimes \cdots \boxtimes Y_r\big)\Big]\Big\uparrow_{m\wr\alpha}^{m\wr n}$$

where each X_i is a kS_m-module and Y_i is a kS_{α_i}-module. We thus have for any r-multipartition $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of n an equality of $k(S_m \wr S_n)$-modules

$$S^{\underline{\lambda}} \;=\; \Theta^{|\underline{\lambda}|}\big((S^{\mu^1}, \ldots, S^{\mu^r}), (S^{\lambda^1}, \ldots, S^{\lambda^r})\big).$$

We can now apply the results of this chapter to obtain a cellular structure on $k(S_m \wr S_n)$. Before we do so, we require one further definition. Indeed, let $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ be an r-multipartition of n, and further suppose that each λ^i is p-regular, and moreover that for each i such that μ^i is p-singular we have $\lambda^i = ()$. Then we define a $k(S_m \wr S_n)$-module

$$D^{\underline{\lambda}} = \Big[\big(D^{\mu^1}, \ldots, D^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\lambda}|} \oslash \big(D^{\lambda^1} \boxtimes \cdots \boxtimes D^{\lambda^r}\big)\Big]\Big\uparrow_{m \wr |\underline{\lambda}|}^{m \wr n}$$

where for notational convenience we take $D^{\mu} = 0$ for any p-singular partition μ of m.

Using the foregoing information together with Theorem 5.3.3, Proposition 5.4.1 and Theorems 5.4.3 and 5.4.4, we have the following result.

Theorem 5.5.1. *The algebra $k(S_m \wr S_n)$ is cellular with respect to the anti-involution induced by mapping each element of $S_m \wr S_n$ to its inverse, and a tuple of cellular data including a poset whose underlying set is the set $\underline{\Lambda}_n^r$ of all r-multipartitions of n. The cell module associated to $\underline{\lambda} \in \underline{\Lambda}_n^r$ by this cellular structure is $S^{\underline{\lambda}}$. The subset $\big(\underline{\Lambda}_n^r\big)_0$ of $\underline{\Lambda}_n^r$ which indexes the simple modules under this cellular structure is the set of all r-multipartitions $\underline{\lambda}$ of n such that each λ^i is p-regular and for each i such that μ^i is p-singular we have $\lambda^i = ()$. If $\underline{\lambda}$ is a multipartition in $\big(\underline{\Lambda}_n^r\big)_0$, then the simple module associated to $\underline{\lambda}$ by this cellular structure is $D^{\underline{\lambda}}$.*

The conclusion of Theorem 5.4.5 is trivial in this case (it is easily obtained directly from the definition of $D^{\underline{\lambda}}$). Theorem 5.4.6 says simply that $k(S_m \wr S_n)$ is semisimple when both kS_m and kS_n are, which by Maschke's theorem occurs if and only if either $p = 0$ or p is greater than both m and n. But we also know by using Maschke's theorem directly that $k(S_m \wr S_n)$ is semisimple if and only if p does not divide $|S_m \wr S_n| = (m!)^n n!$, and this occurs exactly when $p = 0$ or p is greater than both m and n. Thus we see that the conclusion of Theorem 5.4.6 is in agreement with the conclusion of Maschke's theorem applied to the group $S_m \wr S_n$.

Finally we recall from [13, Theorem 3.8] that if a cellular algebra is semisimple, then its cell modules are all simple and form a complete system of pairwise non-isomorphic simple modules. We thus have the following result.

Theorem 5.5.2. *If $k(S_m \wr S_n)$ is semisimple, then we have $\left(\underline{\Lambda}^r_n\right)_0 = \underline{\Lambda}^r_n$ and $D^{\underline{\lambda}} = S^{\underline{\lambda}}$ for each $\underline{\lambda} \in \underline{\Lambda}^r_n$. Furthermore, the Specht modules $S^{\underline{\lambda}}$ for $\underline{\lambda} \in \underline{\Lambda}^r_n$ form a complete system of pairwise non-isomorphic simple $k(S_m \wr S_n)$-modules.*

The results in this section all support our assertion that the modules $S^{\underline{\lambda}}$ should indeed be considered as the wreath product analogues of the Specht modules for the symmetric groups, and hence do indeed deserve the name *Specht module.*

Original research in Chapter 5: The diagrammatic representation of the module $\Theta^\mu(\underline{X}, \underline{Y})$ in Section 5.2 is original research.

The use of the method of iterated inflation to establish cellularity of $A \wr S_n$ is my own work, but as already stated a version of some of this work formed part of my M.Sc. thesis [14]. As indicated above, the material which was included in [14] was: the whole of Section 5.1.3; the whole of Section 5.3; the first part of Section 5.4, from the start of the section up to and including Proposition 5.4.1. The use of the Γ-dominance order on the set Ω^r_n of layer indexes here does, however, represent an improvement over the work in [14]. Thus the use of the Γ-dominance order is the only aspect of this work which constitutes original research for the purposes of this present thesis.

The material about the simple modules of $A \wr S_n$ and in particular $k(S_m \wr S_n)$ appearing in Section 5.4 after Proposition 5.4.1 and in Section 5.5 might perhaps not be regarded as entirely new, since versions of these results, albeit with additional assumptions, are well-known, as indicated in the text. However, I am not aware of published versions of these results which place no restrictions on the field of coefficients like the results given here, and further the method of obtaining these results using the theory of cellular algebras is my own work, and hence aspects of this material are certainly original research.

Finally, Proposition 5.1.2 is my own work, although it is fairly trivial.

Chapter 6

Filtration of modules for wreath products

Let m, n be non-negative integers. In this chapter, we shall consider how the constructions of $k(S_m \wr S_n)$-modules from Chapter 4 interact with module filtrations, and as a first application of the results we shall obtain a $k(S_m \wr S_n)$ analogue of Young's rule (3.2.1). Much of this material is derived from [6], but the use of multipartition matrices is novel, and we believe that this is a useful and efficient way of presenting the results.

6.1 Filtrations and the operation $\oslash$

Our first result is the following elementary lemma, which shows how the operation $\oslash$ from Chapter 4 preserves module filtrations.

Lemma 6.1.1. *Let G be a subgroup of S_m and H a subgroup of S_n. Let Z be a $k(G \wr H)$-module, and Y a kH-module. Suppose Y has a kH-module filtration*

$$Y \sim \mathop{\mathcal{F}_{\langle l \rangle}}_{i=1}^{s} a_i Q_i.$$

Then $Z \oslash Y$ has a $k(G\wr H)$-module filtration

$$Z \oslash Y \sim \mathop{\mathcal{F}_{\langle l\rangle}}_{i=1}^{s} a_i\, Z \oslash Q_i.$$

On the other hand, if Z has a $k(G\wr H)$-module filtration

$$Z \sim \mathop{\mathcal{F}_{\langle l\rangle}}_{i=1}^{s} b_i V_i,$$

then $Z \oslash Y$ has a filtration

$$Z \oslash Y \sim \mathop{\mathcal{F}_{\langle l\rangle}}_{i=1}^{s} b_i\, V_i \oslash Y.$$

Proof. Recall from above that we have

$$Z \oslash Y \cong Z \otimes \mathrm{Inf}_H^{G\wr H} Y,$$

where the right-hand side is an internal tensor product of $k(G \wr H)$-modules. Now trivially the given filtration of Y by the modules Q_i yields a filtration of $\mathrm{Inf}_H^{G\wr H} Y$ by modules $\mathrm{Inf}_H^{G\wr H} Q_i$, and both parts of the claim now follow by Lemma 2.2.8. □

6.2 Filtrations and the operation $(-)^{\widetilde{\boxtimes} n}$

Let G be a subgroup of S_m. We now investigate how kG-module filtrations yield $k(G\wr S_n)$-module filtrations under the operation $(-)^{\widetilde{\boxtimes} n}$. From [6], we have the following result.

Proposition 6.2.1. *[6, Lemma 4.2] Let G be a subgroup of S_m. Let M be a kG-module with a filtration*

$$M \sim \mathop{\mathcal{F}_{\langle w\rangle}}_{i=1}^{t} X_i.$$

where $1 \leqslant w \leqslant t$ and each X_i is a kG-module. Then the $k(G \wr S_n)$-module $M^{\widetilde{\boxtimes} n}$ has a filtration

$$M^{\widetilde{\boxtimes} n} \sim \mathop{\mathcal{F}_{\langle [n,w] \rangle}}_{\alpha \in \Omega_n^t} \left[(X_1, \dots, X_t)^{\widetilde{\boxtimes} \alpha} \right] \Big\uparrow_{G \wr S_\alpha}^{G \wr S_n}$$

where Ω_n^t is the set of all compositions of n with exactly t parts, and $[n, w]$ represents the composition $(0, 0, \dots, 0, n, 0, \dots, 0)$ of length t where the n occurs in the w^{th} place. Thus the module at the bottom of this filtration is isomorphic to $X_w^{\widetilde{\boxtimes} n}$.

Our proof of Proposition 6.2.1 is the same as the proof given for Lemma 4.2 in [6]. However, [6] formally has slightly different assumptions to us (mainly that $n!$ is invertible in k). Further, the statement of [6, Lemma 4.2] does not explicitly identify the bottom-most factor in the filtration. Due to these slight differences, we present the proof here in full.

Proof. By renumbering the X_i if necessary, we have without loss of generality a chain of kG-modules

$$M = M_t \supseteq M_{t-1} \supseteq \cdots \supseteq M_1 \supseteq M_0 = 0$$

where $\frac{M_i}{M_{i-1}} \cong X_i$ for $i = 1, \dots, t$. Note in particular that the module X_w occurring at the bottom of the filtration in the statement of the proposition has been renumbered to X_1 here. Let us choose a k-basis $b_1^1, \dots, b_{d_1}^1$ for M_1, (where d_1 is thus the k-dimension of X_1), which we may then extend by adding elements $b_1^2, \dots, b_{d_2}^2$ to a k-basis for M_2, and so on. We thus obtain a k-basis

$$b_1^1, \dots, b_{d_1}^1, b_1^2, \dots, \dots, b_1^t, \dots, b_{d_t}^t$$

for M, where for each i the elements up to and including $b_{d_i}^i$ are a k-basis of M_i (in particular, $d_i = \dim_k(X_i)$). It follows that the set of all pure tensors

$$b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}$$

forms a k-basis of $M^{\widetilde{\boxtimes} n}$. Let us call this basis $\mathcal{B}$.

For an element $b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}$ of $\mathcal{B}$ (where thus $\delta_i \in \{1, \ldots, t\}$ for $i = 1, \ldots, n$), we define $\mathcal{R}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big) = \delta_1 + \cdots + \delta_n$, and we call this the *rank* of the element. We note that the rank satisfies

$$n \leqslant \mathcal{R}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big) \leqslant nt.$$

For each $N = n, \ldots, nt$, we define Z_N to be the k-span in $M^{\widetilde{\boxtimes} n}$ of all elements of $\mathcal{B}$ of rank equal to or less than N. We see that each Z_N is a $k(G \wr S_n)$-submodule of $M^{\widetilde{\boxtimes} n}$. Thus, defining Z_{n-1} to be 0, we have a filtration of the $k(G \wr S_n)$-module $M^{\widetilde{\boxtimes} n}$

$$M^{\widetilde{\boxtimes} n} = Z_{nt} \supseteq Z_{nt-1} \supseteq \cdots \supseteq Z_n \supseteq Z_{n-1} = 0. \tag{6.2.1}$$

Let $\overline{Z}_N$ be the quotient module $\frac{Z_N}{Z_{N-1}}$ for each $N = n, \ldots, nt$, and for $z \in Z_N$ let $\overline{z}$ represent $z + Z_{N-1}$. Then $\overline{Z}_N$ has a basis $\overline{\mathcal{B}}_N$, where

$$\overline{\mathcal{B}}_N = \left\{\overline{b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}} \,:\, \mathcal{R}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big) = N\right\}.$$

For an element $b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}$ of $\mathcal{B}$, we define $\mathcal{W}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big)$ to be the t-composition $(\alpha_1, \ldots, \alpha_t)$ of n where α_i is the number of $j \in \{1, \ldots, n\}$ for which $\delta_j = i$. We call this composition the *weight* of $b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}$. Thus $\mathcal{W}$ maps $\mathcal{B}$ to Ω_n^t, the set of compositions of n of length t. Now for an element $b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}$ of $\mathcal{B}$ with weight $(\alpha_1, \ldots, \alpha_t)$, we see that $\mathcal{R}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big) = \alpha_1 + 2\alpha_2 + \cdots + t\alpha_t$, and thus the rank of an element of $\mathcal{B}$ depends only on its weight. Thus we may regard the rank function as being defined on Ω_n^t by the formula

$$\mathcal{R}\big((\alpha_1, \ldots, \alpha_t)\big) = \alpha_1 + 2\alpha_2 + \cdots + t\alpha_t$$

and we may hence speak of the *rank* of an element of Ω_n^t.

Let $\alpha = (\alpha_1, \ldots, \alpha_t) \in \Omega_n^t$ and $N = \mathcal{R}(\alpha)$. We define $\overline{\mathcal{B}}_\alpha$ to be the set of all elements $\overline{b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}}$ of $\overline{\mathcal{B}}_N$ such that $\mathcal{W}\big(b_{\epsilon_1}^{\delta_1} \otimes \cdots \otimes b_{\epsilon_n}^{\delta_n}\big) = \alpha$. Further, we define V_α to be the k-span of $\overline{\mathcal{B}}_\alpha$ in $\overline{Z}_N$. Thus $\overline{\mathcal{B}}_\alpha$ is a k-basis of V_α. We see that V_α is then a $k(G \wr S_n)$-submodule of $\overline{Z}_N$, and moreover that we have a $k(G \wr S_n)$-module decomposition

$$\overline{Z}_N = \bigoplus_{\substack{\alpha \in \Omega_n^t \\ \mathcal{R}(\alpha) = N}} V_\alpha. \tag{6.2.2}$$

In combination with the filtration (6.2.1), we now see that, in order to establish the filtration as claimed in the proposition, it is enough to prove that we have a $k(G \wr S_n)$-module isomorphism

$$V_\alpha \cong \Big[(X_1, \ldots, X_t)^{\widetilde{\boxtimes}\alpha}\Big]\Big\uparrow_{G \wr S_\alpha}^{G \wr S_n}. \tag{6.2.3}$$

In particular, note that the only $\alpha \in \Omega_n^t$ with $\mathcal{R}(\alpha) = n$ is $(n, 0, \ldots, 0)$, which implies by (6.2.2) that $Z_n = V_{(n,0,\ldots,0)}$. Thus we see that $V_{(n,0,\ldots,0)}$ is the bottom-most module in our filtration of $M^{\widetilde{\boxtimes}n}$. Recalling that the module denoted by X_1 here is the module denoted by X_w in the statement of the proposition due to our renumbering of the modules X_i at the start of this proof, we see that proving (6.2.3) will establish that the bottom-most factor in our filtration of $M^{\widetilde{\boxtimes}n}$ is as in the claimed filtration in the proposition.

We shall use Proposition 2.2.6 to establish (6.2.3). We thus see that we need to prove that

$$\dim_k(V_\alpha) = \dim_k\Big(\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\alpha}\Big)\big[G \wr S_n : G \wr S_\alpha\big] \tag{6.2.4}$$

and to find a $k(G \wr S_\alpha)$-submodule Y of $V_\alpha\big\downarrow_{G \wr S_\alpha}^{G \wr S_n}$ which is isomorphic to the $k(G \wr S_\alpha)$-module $\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\alpha}$, and which generates V_α as a $k(G \wr S_n)$-module. The dimension condition (6.2.4) is straightforward. Indeed, we can calculate $\dim_k(V_\alpha)$ by counting the elements of the basis $\overline{\mathcal{B}}_\alpha$, which is the same as counting the number of elements $b^{\delta_1}_{\epsilon_1} \otimes \cdots \otimes b^{\delta_n}_{\epsilon_n}$ of $\mathcal{B}$ which satisfy $\mathcal{W}\big(b^{\delta_1}_{\epsilon_1} \otimes \cdots \otimes b^{\delta_n}_{\epsilon_n}\big) = \alpha$. Now $\mathcal{W}\big(b^{\delta_1}_{\epsilon_1} \otimes \cdots \otimes b^{\delta_n}_{\epsilon_n}\big) = \alpha$ if and only if, for each i, exactly α_i of the elements b^{δ}_{ϵ} are elements of the list $b^i_1, \ldots, b^i_{d_i}$, where $d_i = \dim_k(X_i)$. Now there are

$$\dim_k(X_1)^{\alpha_1} \cdots \dim_k(X_t)^{\alpha_t}$$

ways of choosing α_1 elements b^1_*, α_2 elements b^2_*, up to α_t elements b^t_*. Then for each such choice, there are

$$\frac{n!}{\alpha_1! \cdots \alpha_t!}$$

ways of arranging these elements to form an element of $\overline{\mathcal{B}}_\alpha$. It follows that V_α has k-dimension

$$\frac{n! \dim_k(X_1)^{\alpha_1} \cdots \dim_k(X_t)^{\alpha_t}}{\alpha_1! \cdots \alpha_t!}.$$

On the other hand, by the construction of $\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\alpha}$ we see that it has k-dimension $\dim_k(X_1)^{\alpha_1} \cdots \dim_k(X_t)^{\alpha_t}$, and we have

$$\begin{aligned}\big[G \wr S_n : G \wr S_\alpha\big] &= \frac{|G \wr S_n|}{|G \wr S_\alpha|}\\ &= \frac{|G|^n \cdot n!}{|G|^n \cdot \alpha_1! \cdots \alpha_t!}\\ &= \frac{n!}{\alpha_1! \cdots \alpha_t!}\end{aligned}$$

and thus we see that (6.2.4) holds. To find the required $k(G \wr S_\alpha)$-submodule Y, we let $\overline{\mathcal{B}}^*_\alpha$ be the subset of $\overline{\mathcal{B}}_\alpha$ consisting of all elements

$$\overline{b^1_{\epsilon(1)} \otimes \cdots \otimes b^1_{\epsilon(\alpha_1)} \otimes b^2_{\epsilon(\alpha_1+1)} \otimes \cdots \otimes b^2_{\epsilon(\alpha_1+\alpha_2)} \otimes b^3_{\epsilon(\alpha_1+\alpha_2+1)} \cdots\cdots \otimes b^t_n}$$

where for the sake of readability we write $\epsilon(i)$ rather than ϵ_i. We define Y to be the k-span of $\overline{\mathcal{B}}^*_\alpha$ in V_α. It is now easy to see that Y is a $k(G \wr S_\alpha)$-submodule of V_α which is isomorphic to the $k(G \wr S_\alpha)$-module $\big(X_1, \ldots, X_t\big)^{\widetilde{\boxtimes}\alpha}$ and which generates V_α as a $k(G \wr S_n)$-module, as required. $\square$

Proposition 6.2.1 does not allow us to pass information about the multiplicities with which isomorphism classes of modules appear in the filtration of M to the filtration of $M^{\widetilde{\boxtimes} n}$. For this, we need the following more sophisticated result.

Corollary 6.2.2. *Let G be a subgroup of S_m. Let M be a kG-module with a filtration*

$$M \sim \mathop{\mathcal{F}_{\langle w \rangle}}_{i=1}^{t} a_i X_i$$

where $1 \leqslant w \leqslant t$, each X_i is a kG-module, and each a_i is a non-negative integer (so note that $a_w \geqslant 1$, since X_w occurs at the bottom of the filtration). Then the $kG \wr S_n$-module $M^{\widetilde{\boxtimes} n}$ has a filtration

$$M^{\widetilde{\boxtimes} n} \sim \mathop{\mathcal{F}_{\langle [[n,1],w;a_1,\ldots,a_t] \rangle}}_{\underline{\alpha} \in \underline{\Omega}(n;a_1,\ldots,a_t)} \left[(X_1, \ldots, X_t)^{\widetilde{\boxtimes} |\underline{\alpha}|} \right] \Big\downarrow^{G\wr S_{|\underline{\alpha}|}}_{G\wr S_{\underline{\alpha}}} \Big\uparrow^{G\wr S_n}_{G\wr S_{\underline{\alpha}}}$$

where $\underline{\Omega}(n; a_1, \ldots, a_t)$ is the set of all t-multicompositions $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ of n such that the length of α^i is a_i for $i = 1, \ldots, t$, and $[[n,1], w; a_1, \ldots, a_t]$ represents the element

$$\big((0,\ldots,0),\ldots,(0,\ldots,0),(n,0,\ldots,0),(0,\ldots,0),\ldots,(0,\ldots,0)\big)$$

of $\underline{\Omega}(n; a_1, \ldots, a_t)$, where the $(n, 0, \ldots, 0)$ occurs in the w^{th} place. Thus the module at the bottom of this filtration is isomorphic to $X_w^{\widetilde{\boxtimes} n}$.

Proof. Let

$$(Y_1, \ldots, Y_s) = (\underbrace{X_1, \ldots, X_1}_{a_1 \text{ places}}, \underbrace{X_2, \ldots, X_2}_{a_2 \text{ places}}, \ldots, \underbrace{X_t, \ldots, X_t}_{a_t \text{ places}})$$

so that $s = a_1 + \cdots + a_t$, and let $l = a_1 + \cdots + a_{w-1} + 1$. So

$$M \sim \mathop{\mathcal{F}_{\langle l \rangle}}_{i=1}^{s} Y_i.$$

So by Proposition 6.2.1,

$$M^{\widetilde{\boxtimes} n} \sim \mathcal{F}_{\substack{\langle [n,l] \rangle \\ \alpha \in \Omega_n^s}} \left[(Y_1, \ldots, Y_s)^{\widetilde{\boxtimes}\alpha} \right] \Big\uparrow_{G \wr S_\alpha}^{G \wr S_n}.$$

Now let $\alpha \in \Omega_n^s$, and take $\alpha = (\alpha(1), \ldots, \alpha(s))$, where we use function notation rather than subscript notation for the indices of the parts of α in order to make the formulae below more readable. Now define

$$\begin{aligned}
\alpha^1 &= \big(\alpha(1), \ldots, \alpha(a_1)\big) \\
\alpha^2 &= \big(\alpha(a_1+1), \ldots, \alpha(a_1+a_2)\big) \\
&\;\;\vdots \\
\alpha^t &= \big(\alpha(a_1 + \cdots + a_{t-1} + 1) \ldots, \alpha(s)\big)
\end{aligned}$$

so that α^i is a composition of length a_i and $\alpha = \alpha^1 \circ \cdots \circ \alpha^t$. We define $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$, a t-multicomposition of n, and we note that $\underline{\alpha} \in \underline{\Omega}(n; a_1, \ldots, a_t)$. Moreover, we note by the definition of $S_{\underline{\alpha}}$ that $S_{\underline{\alpha}} = S_\alpha$. Hence $G \wr S_\alpha = G \wr S_{\underline{\alpha}}$, so that in particular $(Y_1, \ldots, Y_s)^{\widetilde{\boxtimes}\alpha}$ is thus a $k(G \wr S_{\underline{\alpha}})$-module. Then we have equalities of $k(G \wr S_{\underline{\alpha}})$-modules

$$\begin{aligned}
(Y_1, \ldots, Y_s)^{\widetilde{\boxtimes}\alpha} &= Y_1^{\widetilde{\boxtimes}\alpha(1)} \boxtimes \cdots \boxtimes Y_s^{\widetilde{\boxtimes}\alpha(s)} \\
&= (X_1)^{\widetilde{\boxtimes}\alpha(1)} \boxtimes \cdots \boxtimes (X_1)^{\widetilde{\boxtimes}\alpha(a_1)} \boxtimes (X_2)^{\widetilde{\boxtimes}\alpha(a_1+1)} \boxtimes \cdots \\
&\qquad\qquad \cdots \boxtimes (X_t)^{\widetilde{\boxtimes}\alpha(s)} \\
&= \left(X_1^{\widetilde{\boxtimes}|\alpha^1|} \Big\downarrow_{G \wr S_{\alpha^1}}^{G \wr S_{|\alpha^1|}} \right) \boxtimes \cdots \boxtimes \left(X_t^{\widetilde{\boxtimes}|\alpha^t|} \Big\downarrow_{G \wr S_{\alpha^t}}^{G \wr S_{|\alpha^t|}} \right) \\
&= (X_1, \ldots, X_t)^{\widetilde{\boxtimes}|\underline{\alpha}|} \Big\downarrow_{G \wr S_{\underline{\alpha}}}^{G \wr S_{|\underline{\alpha}|}}.
\end{aligned}$$

And if $\alpha = [n, l] = (0, \ldots, 0, n, 0, \ldots, 0)$, then we see that

$$\begin{aligned}
\underline{\alpha} &= \big((0, \ldots, 0), \ldots, (0, \ldots, 0), (n, 0, \ldots, 0), (0, \ldots, 0), \ldots, (0, \ldots, 0)\big) \\
&= [[n, 1], w; a_1, \ldots, a_t].
\end{aligned}$$

The result now follows. $\square$

6.3 Multipartition matrices

In order to give further results on filtrations in modules for the wreath product, we need to introduce some new concepts. Recall that a multipartition is simply a tuple of partitions, where we allow the empty partition () to occur as an entry. We also allow the length of the tuple to be zero, yielding the empty multipartition, which we also denote by (). We define a **multipartition matrix** to be a matrix whose entries are multipartitions. We shall typically denote the multipartition matrix whose $(i,j)^{\mathrm{th}}$ entry is the multipartition $\underline{\epsilon}^{ij}$ as $[\underline{\epsilon}]$. Let us fix an $s \times t$ multipartition matrix $[\underline{\epsilon}]$. Then we define multipartitions

$$R_i[\underline{\epsilon}] = \underline{\epsilon}^{i1} \circ \cdots \circ \underline{\epsilon}^{it} \qquad \text{for each } i = 1, \ldots, s$$
$$C_j[\underline{\epsilon}] = \underline{\epsilon}^{1j} \circ \cdots \circ \underline{\epsilon}^{sj} \qquad \text{for each } j = 1, \ldots, t$$

where, recall, $\circ$ denotes the concatenation of compositions. Note that thus $R_i[\underline{\epsilon}]$ is the concatenation of all of the multipartitions from the i^{th} row of $[\underline{\epsilon}]$, while $C_j[\underline{\epsilon}]$ is the concatenation of all the multipartitions from the j^{th} column of $[\underline{\epsilon}]$. We also define multipartitions

$$\begin{aligned} R[\underline{\epsilon}] &= R_1[\underline{\epsilon}] \circ \cdots \circ R_s[\underline{\epsilon}] = \underline{\epsilon}^{11} \circ \underline{\epsilon}^{12} \circ \cdots \circ \underline{\epsilon}^{1t} \circ \underline{\epsilon}^{21} \circ \cdots \cdots \circ \underline{\epsilon}^{st} \\ C[\underline{\epsilon}] &= C_1[\underline{\epsilon}] \circ \cdots \circ C_t[\underline{\epsilon}] = \underline{\epsilon}^{11} \circ \underline{\epsilon}^{21} \circ \cdots \circ \underline{\epsilon}^{s1} \circ \underline{\epsilon}^{12} \circ \cdots \cdots \circ \underline{\epsilon}^{st} \end{aligned}$$

so that $R[\underline{\epsilon}]$ is the concatenation of all the entries of $[\underline{\epsilon}]$ taken "row-wise", while $C[\underline{\epsilon}]$ is the concatenation of all the entries of $[\underline{\epsilon}]$ taken "column-wise". Recall that if $\underline{\gamma} = (\gamma^1, \ldots, \gamma^w)$ is a multicomposition, then we have defined $||\underline{\gamma}||$ to be the integer $|\gamma^1| + \cdots + |\gamma^w|$. We have

$$||R_i[\underline{\epsilon}]|| = \sum_{j=1}^{t} ||\underline{\epsilon}^{ij}||, \quad ||C_j[\underline{\epsilon}]|| = \sum_{i=1}^{s} ||\underline{\epsilon}^{ij}||.$$

Further, if we let $\alpha = \big(||R_1[\underline{\epsilon}]||, \ldots, ||R_s[\underline{\epsilon}]||\big)$ and $\beta = \big(||C_1[\underline{\epsilon}]||, \ldots, ||C_t[\underline{\epsilon}]||\big)$, so that α and β are both compositions of $\sum_{i,j} ||\underline{\epsilon}^{ij}||$, then we say that $[\underline{\epsilon}]$ has

shape $\alpha \times \beta$.

For example, if $[\underline{\epsilon}]$ is the 2×2 multipartition matrix

$$\begin{bmatrix} \big((1,1),(),(2,1)\big) & \big((),(3,2,1)\big) \\ \big((),(),(1)\big) & () \end{bmatrix}$$

then

$$\begin{aligned} R_1[\underline{\epsilon}] &= \big((1,1),(),(2,1),(),(3,2,1)\big) \\ R_2[\underline{\epsilon}] &= \big((),(),(1)\big) \\ C_1[\underline{\epsilon}] &= \big((1,1),(),(2,1),(),(),(1)\big) \\ C_2[\underline{\epsilon}] &= \big((),(3,2,1)\big). \end{aligned}$$

and

$$\begin{aligned} R[\underline{\epsilon}] &= \big((1,1),(),(2,1),(),(3,2,1),(),(),(1)\big) \\ C[\underline{\epsilon}] &= \big((1,1),(),(2,1),(),(),(1),(),(3,2,1)\big) \end{aligned}$$

and further we have $\big(||R_1[\underline{\epsilon}]||, ||R_2[\underline{\epsilon}]||\big) = (11,1)$, $\big(||C_1[\underline{\epsilon}]||, ||C_2[\underline{\epsilon}]||\big) = (6,6)$, so that $[\underline{\epsilon}]$ has shape $(11,1)\times(6,6)$.

We also define $L[\underline{\epsilon}]$ to be the $s\times t$ matrix with $(i,j)^{\text{th}}$ entry the length of $\underline{\epsilon}^{ij}$. We call $L[\underline{\epsilon}]$ the **length matrix** of $[\underline{\epsilon}]$. In the example given above, we have

$$L[\underline{\epsilon}] \quad = \quad \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}.$$

Finally, if α, β are compositions of the same integer n and with lengths s and t respectively, and L is an $s\times t$ matrix with non-negative integer entries, then we define $\mathrm{Mat}_{\underline{\Lambda}}(L; \alpha\times\beta)$ to be the set of all $s\times t$ multipartition matrices $[\underline{\epsilon}]$ of shape $\alpha\times\beta$ such that $L[\underline{\epsilon}] = L$.

6.4 Filtrations and the operation $\left[(-,\ldots,-)^{\widetilde{\boxtimes}|\underline{\eta}|} \oslash \left(S^{\eta^1} \boxtimes \cdots \boxtimes S^{\eta^s}\right)\right]\Big\uparrow_{m\wr|\underline{\eta}|}^{m\wr n}$

In this section we shall develop a result for obtaining filtrations for $k(S_m \wr S_n)$-modules of the form

$$\left[(Y_1,\ldots,Y_s)^{\widetilde{\boxtimes}|\underline{\eta}|} \oslash (S^{\eta^1} \boxtimes \cdots \boxtimes S^{\eta^s})\right]\Big\uparrow_{m\wr|\underline{\eta}|}^{m\wr n} \tag{6.4.1}$$

(where $\underline{\eta}$ is an s-multipartition of n) if we have filtrations of the modules Y_i. The material in this section is contained in Section 4 of [6], but our presentation of it is somewhat different. In particular the use of multipartition matrices is an innovation which we believe helps the application of these results in the cases we are interested in.

We shall begin by showing how a filtration of a kS_m-module Y yields a filtration of the $k(S_m \wr S_n)$-module $Y^{\widetilde{\boxtimes}n} \oslash S^\eta$, where η is a partition of n. We will then use the filtration obtained, (6.4.9) below, to tackle the general case (6.4.1).

So indeed let the kS_m-module Y have the filtration

$$Y \sim \mathop{\mathcal{F}_{\langle w \rangle}}_{j=1}^{t} a_j X_j.$$

Let η be a partition of n. By Corollary 6.2.2, we have a filtration

$$Y^{\widetilde{\boxtimes}n} \sim \mathop{\mathcal{F}_{\langle [[n,1],w;a_1,\ldots,a_t] \rangle}}_{\underline{\alpha}\in\underline{\Omega}(n;a_1,\ldots,a_t)} (X_1,\ldots,X_t)^{\widetilde{\boxtimes}|\underline{\alpha}|}\Big\downarrow_{m\wr\underline{\alpha}}^{m\wr|\underline{\alpha}|}\Big\uparrow_{m\wr\underline{\alpha}}^{m\wr n}$$

so by Lemma 6.1.1, we have

$$Y^{\widetilde{\boxtimes}n} \oslash S^\eta \sim \mathop{\mathcal{F}_{\langle [[n,1],w;a_1,\ldots,a_t] \rangle}}_{\underline{\alpha}\in\underline{\Omega}(n;a_1,\ldots,a_t)} \left((X_1,\ldots,X_t)^{\widetilde{\boxtimes}|\underline{\alpha}|}\Big\downarrow_{m\wr\underline{\alpha}}^{m\wr|\underline{\alpha}|}\Big\uparrow_{m\wr\underline{\alpha}}^{m\wr n}\right) \oslash S^\eta. \tag{6.4.2}$$

Now if $\underline{\alpha} = (\alpha^1,\ldots,\alpha^t) \in \underline{\Omega}(n;a_1,\ldots,a_t)$, then letting α be the composition

$\alpha^1 \circ \cdots \circ \alpha^t$ of n, we have $S_{\underline{\alpha}} = S_\alpha$. Hence by Proposition 4.3.7 we have

$$\left((X_1, \ldots, X_t)^{\tilde{\boxtimes} |\underline{\alpha}|} \Big\downarrow^{m\wr|\underline{\alpha}|}_{m\wr\underline{\alpha}} \Big\uparrow^{m\wr n}_{m\wr\underline{\alpha}} \right) \oslash S^\eta \cong \left[(X_1, \ldots, X_t)^{\tilde{\boxtimes} |\underline{\alpha}|} \Big\downarrow^{m\wr|\underline{\alpha}|}_{m\wr\underline{\alpha}} \oslash S^\eta \big\downarrow^n_{\underline{\alpha}} \right] \Big\uparrow^{m\wr n}_{m\wr\underline{\alpha}}. \quad (6.4.3)$$

We shall now obtain a filtration of $S^\eta\downarrow^n_{\underline{\alpha}}$, and hence by Lemma 6.1.1 and Lemma 2.2.2 a filtration of the module (6.4.3). Combining the filtration (6.4.2) with these filtrations of the modules (6.4.3) (for all $\underline{\alpha}$), we shall obtain our desired filtration of $Y^{\tilde{\boxtimes} n} \oslash S^\eta$.

Let us keep $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t) \in \underline{\Omega}(n; a_1, \ldots, a_t)$ and $\alpha = \alpha^1 \circ \cdots \circ \alpha^t \vDash n$ as above. We have $S_{\underline{\alpha}} = S_\alpha$, and by (3.2.12) we have that

$$S^\eta \big\downarrow^n_\alpha \sim \mathop{\mathcal{F}}_{\substack{\underline{\epsilon} \text{ is a multipartition} \\ |\underline{\epsilon}| = \alpha}} c(\eta; \underline{\epsilon}) S(\underline{\epsilon}). \quad (6.4.4)$$

We wish to reformulate (6.4.4) slightly to obtain a statement where our tuple $\underline{\alpha} = (\alpha^1, \ldots, \alpha^t)$ appears in place of the composition α. The indexing set for this filtration will be the set of all t-tuples of multipartitions $\underline{\underline{\epsilon}} = (\underline{\epsilon}^1, \ldots, \underline{\epsilon}^t)$ such that $|\underline{\epsilon}^j| = \alpha^j$, and we note that this set is in bijection with the indexing set in (6.4.4) (namely the set of multipartitions $\underline{\epsilon}$ such that $|\underline{\epsilon}| = \alpha$) by mapping the tuple $\underline{\underline{\epsilon}} = (\underline{\epsilon}^1, \ldots, \underline{\epsilon}^t)$ to the multipartition $\underline{\epsilon}^1 \circ \cdots \circ \underline{\epsilon}^t$. Recall that the Young subgroup $S_{|\underline{\underline{\epsilon}}|}$ associated to the multicomposition $|\underline{\underline{\epsilon}}| = (|\underline{\epsilon}^1|, \ldots, |\underline{\epsilon}^t|)$ is canonically isomorphic to $S_{|\underline{\epsilon}^1|} \times \cdots \times S_{|\underline{\epsilon}^t|}$. Recall from (3.2.9) that for a multipartition $\underline{\epsilon} = (\epsilon^1, \ldots, \epsilon^s)$, we have defined $S(\underline{\epsilon})$ to be the $kS_{|\underline{\epsilon}|}$-module $S^{\epsilon^1} \boxtimes \cdots \boxtimes S^{\epsilon^s}$. We can thus define for a tuple of multipartitions $\underline{\underline{\epsilon}} = (\underline{\epsilon}^1, \ldots, \underline{\epsilon}^t)$ a $kS_{|\underline{\underline{\epsilon}}|}$-module

$$S(\underline{\underline{\epsilon}}) = S(\underline{\epsilon}^1) \boxtimes \cdots \boxtimes S(\underline{\epsilon}^t).$$

Further, given a partition η, we define a Littlewood-Richardson coefficient

$$c(\eta; \underline{\underline{\epsilon}}) = c(\eta; \underline{\epsilon}^1 \circ \cdots \circ \underline{\epsilon}^t). \quad (6.4.5)$$

We note that if $\underline{\underline{\epsilon}}$ is a t-tuple of multipartitions and we let $\underline{\epsilon} = \underline{\epsilon}^1 \circ \cdots \circ \underline{\epsilon}^t$, then $c(\eta; \underline{\underline{\epsilon}}) = c(\eta; \underline{\epsilon})$, and further by the definition of $S(\underline{\epsilon})$ we have $S(\underline{\underline{\epsilon}}) = S(\underline{\epsilon})$. Noting that $S_{\underline{\alpha}} = S_\alpha$, we now see easily that (6.4.4) is equivalent to

$$S^\eta{\downarrow}^n_{\underline{\alpha}} \sim \mathop{\mathcal{F}}_{\substack{\underline{\underline{\epsilon}} \text{ is a } t\text{-tuple of multipartitions} \\ |\underline{\underline{\epsilon}}| = \underline{\alpha}}} c(\eta; \underline{\underline{\epsilon}}) S(\underline{\underline{\epsilon}}). \tag{6.4.6}$$

By applying Lemmas 6.1.1 and 2.2.2 and the filtration (6.4.6) to the module (6.4.3), and furthermore noting that if $\underline{\underline{\epsilon}}$ is a t-tuple of multipartitions such that $|\underline{\underline{\epsilon}}| = \underline{\alpha}$ then we have $||\underline{\underline{\epsilon}}|| = |\underline{\alpha}|$, we obtain a filtration

$$\begin{aligned}
&\left[(X_1, \ldots, X_t)^{\widetilde{\boxtimes}|\underline{\alpha}|} \Big\downarrow^{m\wr|\underline{\alpha}|}_{m\wr\underline{\alpha}} \oslash S^\eta \Big\downarrow^n_{\underline{\alpha}} \right] \Big\uparrow^{m\wr n}_{m\wr\underline{\alpha}} \sim \\
&\qquad \mathop{\mathcal{F}}_{\substack{\underline{\underline{\epsilon}} \text{ is a } t\text{-tuple of multipartitions} \\ |\underline{\underline{\epsilon}}| = \underline{\alpha}}} c(\eta; \underline{\underline{\epsilon}}) \ \left[(X_1, \ldots, X_t)^{\widetilde{\boxtimes}||\underline{\underline{\epsilon}}||} \Big\downarrow^{m\wr||\underline{\underline{\epsilon}}||}_{m\wr|\underline{\underline{\epsilon}}|} \oslash S(\underline{\underline{\epsilon}}) \right] \Big\uparrow^{m\wr n}_{m\wr|\underline{\underline{\epsilon}}|}.
\end{aligned} \tag{6.4.7}$$

It remains only to consider the bottom-most factor in the filtration (6.4.2). Indeed, if $\underline{\alpha} = [[n, 1], w; a_1, \ldots, a_t]$ and $\underline{\underline{\epsilon}} = (\underline{\epsilon}^1, \ldots, \underline{\epsilon}^t)$ is a t-tuple of multipartitions such that $|\underline{\underline{\epsilon}}| = \underline{\alpha}$, then we have

$$|\underline{\epsilon}^j| = \begin{cases} (n, 0, \ldots, 0) & \text{if } j = w \\ (0, \ldots, 0) & \text{if } j \neq w \end{cases}$$

so that we must have $\underline{\epsilon}^j = \big((), \ldots, ()\big)$ for $j \neq w$ and $\underline{\epsilon}^w = \big(\nu, (), \ldots, ()\big)$ for some $\nu \vdash n$. Thus we have

$$\begin{aligned}
c(\eta; \underline{\underline{\epsilon}}) &= c(\eta; \underline{\epsilon}^1 \circ \cdots \circ \underline{\epsilon}^t) \\
&= c\big(\eta; \big((), (), \ldots, (), \nu, (), \ldots, ()\big)\big) \\
&= c\big(\eta; (\nu)\big) \qquad \text{(by (3.2.3))}
\end{aligned}$$

We thus have by (3.2.5) that

$$c(\eta; \underline{\underline{\epsilon}}) = \begin{cases} 1 & \text{if } \nu = \eta \\ 0 & \text{otherwise.} \end{cases}$$

So the only tuple of multipartitions $\underline{\underline{\epsilon}}$ appearing with non-zero multiplicity in the filtration (6.4.7) for $\underline{\alpha} = [[n,1], w; a_1, \ldots, a_t]$ is the t-tuple of multipartitions whose w^{th} component is the multicomposition $(\eta, (), \ldots, ())$ of length a_w, and whose i^{th} component for $i \neq w$ is the multicomposition of length a_i with all components equal to $()$. We denote this tuple by $[[\eta, 1], w; a_1, \ldots, a_t]$, and so we have

$$[[\eta, 1], w; a_1, \ldots, a_t] = \big(((), \ldots, ()), \ldots, (\eta, (), \ldots, ()), \ldots, ((), \ldots, ())\big) \tag{6.4.8}$$

where the $(\eta, (), \ldots, ())$ occurs in the w-th place and all the other entries are tuples of empty partitions.

Thus, if we write $\operatorname{len}(\underline{\epsilon})$ for the length of a multipartition $\underline{\epsilon}$, we obtain the following filtration by combining the filtration given for each $\underline{\alpha}$ by (6.4.7) with the filtration (6.4.2) and the equation (6.4.3).

$$Y^{\widetilde{\boxtimes} n} \oslash S^{\eta} \sim \mathcal{F}_{\substack{\langle [[\eta,1],w;a_1,\ldots,a_t]\rangle \\ \underline{\underline{\epsilon}} \text{ is a } t\text{-tuple of multipartitions} \\ |||\underline{\underline{\epsilon}}|||=n,\ \operatorname{len}(\underline{\epsilon}^j)=a_j}} c(\eta; \underline{\underline{\epsilon}}) \left[(X_1, \ldots, X_t)^{\widetilde{\boxtimes} ||\underline{\underline{\epsilon}}||} \Big\downarrow^{m\wr ||\underline{\underline{\epsilon}}||}_{m\wr |\underline{\underline{\epsilon}}|} \oslash S(\underline{\underline{\epsilon}}) \right] \Big\uparrow^{m\wr n}_{m\wr |\underline{\underline{\epsilon}}|},$$

where, recall, $|||\underline{\underline{\epsilon}}|||$ is the sum of all of the parts of all of the partitions occurring as components in all of the multipartitions $\underline{\epsilon}^i$ which are the components of $\underline{\underline{\epsilon}}$. We want to reformulate this result slightly. Firstly, it is easy to see that we have an isomorphism of $k(S_m \wr S_{|\underline{\epsilon}|})$-modules

$$
\begin{aligned}
&(X_1,\dots,X_t)^{\widetilde{\boxtimes}||\underline{\epsilon}||}\Big\downarrow^{m\wr||\underline{\epsilon}||}_{m\wr|\underline{\epsilon}|} \oslash S(\underline{\epsilon})\\
&\quad\cong \left(\left(X_1^{\widetilde{\boxtimes}||\epsilon^1||}\Big\downarrow^{m\wr||\underline{\epsilon}^1||}_{m\wr|\underline{\epsilon}^1|}\right)\boxtimes\cdots\boxtimes\left(X_t^{\widetilde{\boxtimes}||\epsilon^t||}\Big\downarrow^{m\wr||\underline{\epsilon}^t||}_{m\wr|\underline{\epsilon}^t|}\right)\right)\oslash\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\big(S(\underline{\epsilon}^1)\boxtimes\cdots\boxtimes S(\underline{\epsilon}^t)\big)\,,
\end{aligned}
$$

and from (4.3.5) this is isomorphic to

$$
\mathop{\boxtimes}_{j=1}^{t}\left(X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^j||}\Big\downarrow^{m\wr||\underline{\epsilon}^j||}_{m\wr|\underline{\epsilon}^j|}\oslash S(\underline{\epsilon}^j)\right).
$$

Hence we obtain our desired filtration

$$
\begin{aligned}
&Y^{\widetilde{\boxtimes}n}\oslash S^\eta \sim\\
&\qquad \mathcal{F}_{\substack{\langle[[\eta,1],w;a_1,\dots,a_t]\rangle\\ \underline{\epsilon}\text{ is a }t\text{-tuple of multipartitions}\\ |||\underline{\epsilon}|||=n,\ \mathrm{len}(\underline{\epsilon}^j)=a_j}} c(\eta;\underline{\epsilon})\left[\mathop{\boxtimes}_{j=1}^{t}\left(X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^j||}\Big\downarrow^{m\wr||\underline{\epsilon}^j||}_{m\wr|\underline{\epsilon}^j|}\oslash S(\underline{\epsilon}^j)\right)\right]\Bigg\uparrow^{m\wr n}_{m\wr|\underline{\epsilon}|}. \qquad (6.4.9)
\end{aligned}
$$

This filtration, though rather complicated and unpleasant-looking at first sight, is none-the-less the key to obtaining our filtration of (6.4.1).

So let us now derive the desired filtration for the module (6.4.1). In order to do this, we introduce some further notation. If $W_1,\dots,W_l$ are kS_m-modules and $\underline{\lambda}=(\lambda^1,\dots,\lambda^l)$ is an l-component multipartition of n, then we define the $k(S_m\wr S_n)$-module $S^{\underline{\lambda}}(W_1,\dots,W_l)$ by setting

$$
S^{\underline{\lambda}}(W_1,\dots,W_l) \;=\; \left[\big(W_1,\dots,W_l\big)^{\widetilde{\boxtimes}|\underline{\lambda}|}\oslash\big(S^{\lambda^1}\boxtimes\cdots\boxtimes S^{\lambda^l}\big)\right]\Big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}. \qquad (6.4.10)
$$

Note that taking $l=r$ and $W_i=S^{\mu^i}$ in this construction yields the Specht module $S^{\underline{\lambda}}$. Now suppose we have kS_m-modules $Y_1,\dots,Y_s$ and an s-component multipartition $\underline{\eta}$ of n. We consider the module (6.4.1), which in our new notation is denoted by $S^{\underline{\eta}}(Y_1,\dots,Y_s)$. Moreover suppose we have kS_m-modules $X_1, \dots, X_t$ such that for each $i = 1, \dots, s$ there exists a filtration

$$Y_i \sim \mathcal{F}_{\langle w_i \rangle}{}_{j=1}^{t} \, a_j^i X_j.$$

We shall next obtain a filtration of the module $S^{\underline{\eta}}(Y_1, \dots, Y_s)$ by modules $S^{\underline{\nu}}(X_1, \dots, X_t)$ for t-component multipartitions $\underline{\nu}$ of n. Indeed, we have from (4.3.6) an isomorphism of $k(S_m \wr S_n)$-modules

$$S^{\underline{\eta}}(Y_1, \dots, Y_s) \cong \left[\left(Y_1^{\widetilde{\boxtimes}|\eta^1|} \oslash S^{\eta^1} \right) \boxtimes \cdots \boxtimes \left(Y_s^{\widetilde{\boxtimes}|\eta^s|} \oslash S^{\eta^s} \right) \right] \Big\uparrow_{m \wr |\underline{\eta}|}^{m \wr n} . \quad (6.4.11)$$

Now by (6.4.9), we have for each i a filtration of $k(S_m \wr S_{|\eta^i|})$-modules

$$\left(Y_i\right)^{\widetilde{\boxtimes}|\eta^i|} \oslash S^{\eta^i} \sim \mathcal{F}_{\substack{\langle [[\eta^i,1], w_i; a_1^i, \dots, a_t^i] \rangle \\ \underline{\underline{\epsilon}}^i \text{ is a } t\text{-tuple of multipartitions} \\ |||\underline{\underline{\epsilon}}^i||| = |\eta^i|, \ \mathrm{len}(\underline{\epsilon}^{ij}) = a_j^i}} c(\eta^i; \underline{\underline{\epsilon}}^i) \left[\mathop{\boxtimes}_{j=1}^{t} \left(X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^{ij}||} \Big\downarrow_{m \wr |\underline{\epsilon}^{ij}|}^{m \wr ||\underline{\epsilon}^{ij}||} \oslash S(\underline{\epsilon}^{ij}) \right) \right] \Big\uparrow_{m \wr |\underline{\underline{\epsilon}}^i|}^{m \wr |\eta^i|} . \quad (6.4.12)$$

Hence by Lemma 2.1.2, we may obtain a filtration of the $k(S_m \wr S_{|\underline{\eta}|})$-module

$$\left(Y_1^{\widetilde{\boxtimes}|\eta^1|} \oslash S^{\eta^1} \right) \boxtimes \cdots \boxtimes \left(Y_s^{\widetilde{\boxtimes}|\eta^s|} \oslash S^{\eta^s} \right)$$

and hence by (6.4.11) and Lemma 2.2.2 we may obtain a filtration of $S^{\underline{\eta}}(Y_1, \dots, Y_s)$. The indexing set of the filtration so obtained is rather complicated, so we shall simplify it before we state the filtration. Indeed, the indexing set is the set of all s-tuples $(\underline{\underline{\epsilon}}^1, \dots, \underline{\underline{\epsilon}}^s)$ where $\underline{\underline{\epsilon}}^i$ is a tuple of multipartitions as in (6.4.12). Thus each $\underline{\underline{\epsilon}}^i$ is a t-tuple of multipartitions such that $|||\underline{\underline{\epsilon}}^i||| = |\eta^i|$ and $\mathrm{len}(\underline{\epsilon}^{ij}) = a_j^i$ for each j. By identifying the tuple $(\underline{\underline{\epsilon}}^1, \dots, \underline{\underline{\epsilon}}^s)$ with the $s \times t$ multipartition matrix $[\underline{\epsilon}]$ whose i^{th} row is the tuple $\underline{\underline{\epsilon}}^i$ of multipartitions, we may instead index the filtration with the set of $s \times t$ multipartition matrices $[\underline{\epsilon}]$ satisfying $||R_i[\underline{\epsilon}]|| = |\eta^i|$ for $i = 1, \dots, s$ and $L[\underline{\epsilon}] = A$ where A is the $s \times t$

integer matrix whose (i,j)-th entry is a^i_j. Note that under this identification, the j^{th} entry $\underline{\underline{\epsilon}}^{ij}$ of the tuple $\underline{\underline{\epsilon}}^i = (\underline{\epsilon}^{i1}, \ldots, \underline{\epsilon}^{it})$ becomes the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$, and hence we have an equality $R_i[\underline{\epsilon}] = \underline{\epsilon}^{i1} \circ \cdots \circ \underline{\epsilon}^{it}$ of multipartitions. It follows that under this identification we have $S_{|\underline{\underline{\epsilon}}^i|} = S_{|R_i[\underline{\epsilon}]|}$, so that the operations $\big\uparrow^{m\wr|\eta^i|}_{|\underline{\underline{\epsilon}}^i|}$ and $\big\uparrow^{m\wr|\eta^i|}_{|R_i[\underline{\epsilon}]|}$ coincide, and further that $c(\eta^i; \underline{\underline{\epsilon}}^i) = c(\eta^i; R_i[\underline{\epsilon}])$ (by (6.4.5)). Turning to the bottom-most factor in our filtration, we see that index of this factor before applying our identification with multipartition matrices is the tuple $(\underline{\underline{\epsilon}}^1, \ldots, \underline{\underline{\epsilon}}^s)$ where $\underline{\underline{\epsilon}}^i = [[\eta^i, 1], w_i; a^i_1, \ldots, a^i_t]$. Under our identification of indices $(\underline{\underline{\epsilon}}^1, \ldots, \underline{\underline{\epsilon}}^s)$ with multipartitions, we may now easily see that the module at the bottom of the filtration is indexed by the multipartition matrix with length matrix A and whose (i,j)-th entry is $[\eta^i, 1]$ if $j = w_i$ and a tuple of empty partitions otherwise. We shall denote this multipartition matrix by $M(\underline{\eta}; w_1, \ldots, w_s)$. Thus as explained above we have by (6.4.12), Lemma 2.1.2, (6.4.11), and Lemma 2.2.2 a filtration of $k(S_m \wr S_n)$-modules

$$S^{\underline{\eta}}(Y_1, \ldots, Y_s) \sim \mathop{\mathcal{F}_{\langle M(\underline{\eta}; w_1, \ldots, w_s)\rangle}}_{\substack{[\underline{\epsilon}]\ s\times t \text{ multipartition matrix} \\ ||R_i[\underline{\epsilon}]|| = |\eta^i|,\ L[\underline{\epsilon}] = A}} \left(\prod_{i=1}^{s} c(\eta^i; R_i[\underline{\epsilon}])\right) \cdot$$

$$\left[\mathop{\boxtimes}_{i=1}^{s} \left[\mathop{\boxtimes}_{j=1}^{t} X_j^{\tilde{\boxtimes}||\underline{\epsilon}^{ij}||} \Big\downarrow^{m\wr||\underline{\epsilon}^{ij}||}_{m\wr|\underline{\epsilon}^{ij}|} \oslash S(\underline{\epsilon}^{ij})\right] \Bigg\uparrow^{m\wr|\eta^i|}_{m\wr|R_i[\underline{\epsilon}]|}\right] \Bigg\uparrow^{m\wr n}_{m\wr|\underline{\eta}|}. \tag{6.4.13}$$

We now wish to reformulate the filtration (6.4.13). For any $[\underline{\epsilon}]$ an $s \times t$ multipartition matrix as in the filtration (6.4.13) and any i, j, we define a $k(S_m \wr S_{|\underline{\epsilon}^{ij}|})$-module (recalling that, since $\underline{\epsilon}^{ij}$ is a multipartition, $|\underline{\epsilon}^{ij}|$ is a composition of the integer $||\underline{\epsilon}^{ij}||$) by setting

$$Z^{ij}_{[\underline{\epsilon}]} = X_j^{\tilde{\boxtimes}||\underline{\epsilon}^{ij}||} \Big\downarrow^{m\wr||\underline{\epsilon}^{ij}||}_{m\wr|\underline{\epsilon}^{ij}|} \oslash S(\underline{\epsilon}^{ij}).$$

Thus we write the modules occurring in the filtration (6.4.13) as

$$\left[\bigboxtimes_{i=1}^{s}\left[\bigboxtimes_{j=1}^{t} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|R_i[\underline{\epsilon}]|}^{m\wr|\eta^i|}\right]\Bigg\uparrow_{m\wr|\underline{\eta}|}^{m\wr n}.$$

Recalling that $R[\underline{\epsilon}]$ is the multicomposition $R_1[\underline{\epsilon}] \circ \cdots \circ R_s[\underline{\epsilon}]$, we see that $S_{|R_1[\underline{\epsilon}]|} \times \cdots \times S_{|R_s[\underline{\epsilon}]|} \cong S_{(|R_1[\underline{\epsilon}]|,\ldots,|R_s[\underline{\epsilon}]|)} = S_{|R[\underline{\epsilon}]|}$, and so we have an equality of $k(S_m\wr S_n)$-modules

$$\left[\bigboxtimes_{i=1}^{s}\left[\bigboxtimes_{j=1}^{t} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|R_i[\underline{\epsilon}]|}^{m\wr|\eta^i|}\right]\Bigg\uparrow_{m\wr|\underline{\eta}|}^{m\wr n} = \left[\bigboxtimes_{i=1}^{s}\bigboxtimes_{j=1}^{t} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|R[\underline{\epsilon}]|}^{m\wr|\underline{\eta}|}\Bigg\uparrow_{m\wr|\underline{\eta}|}^{m\wr n}$$

and by transitivity of induction, this is

$$\left[\bigboxtimes_{i=1}^{s}\bigboxtimes_{j=1}^{t} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|R[\underline{\epsilon}]|}^{m\wr n}.$$

Now $|R[\underline{\epsilon}]|$ is the composition $(|\epsilon^{11}|,|\epsilon^{12}|,\ldots,|\epsilon^{1t}|,|\epsilon^{21}|,\ldots,|\epsilon^{st}|)$ of n. On the other hand, we recall that $C[\underline{\epsilon}]$ is the multicomposition $C_1[\underline{\epsilon}] \circ \cdots \circ C_t[\underline{\epsilon}]$, and so $|C[\underline{\epsilon}]|$ is the composition $(|\epsilon^{11}|,|\epsilon^{21}|,\ldots,|\epsilon^{s1}|,|\epsilon^{12}|,\ldots,|\epsilon^{st}|)$ of n, and hence by Proposition 4.3.6 we have

$$\left[\bigboxtimes_{i=1}^{s}\bigboxtimes_{j=1}^{t} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|R[\underline{\epsilon}]|}^{m\wr n} \cong \left[\bigboxtimes_{j=1}^{t}\bigboxtimes_{i=1}^{s} Z^{ij}_{[\underline{\epsilon}]}\right]\Bigg\uparrow_{m\wr|C[\underline{\epsilon}]|}^{m\wr n}.$$

Now we have $S_{|C[\underline{\epsilon}]|} = S_{(|C_1[\underline{\epsilon}]|,\ldots,|C_t[\underline{\epsilon}]|)} \cong S_{|C_1[\underline{\epsilon}]|} \times \cdots \times S_{|C_t[\underline{\epsilon}]|}$, and hence we note that $\bigboxtimes_{i=1}^{s} Z^{ij}_{[\underline{\epsilon}]}$ is a $k(S_m\wr S_{|C_j[\underline{\epsilon}]|})$-module for each $j = 1,\ldots,t$. We shall now obtain a filtration of this module. Indeed, we have

$$\begin{aligned}
\bigboxtimes_{i=1}^{s} Z_{[\underline{\epsilon}]}^{ij} &= \bigboxtimes_{i=1}^{s} X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^{ij}||}\Big\downarrow_{m\wr|\underline{\epsilon}^{ij}|}^{m\wr||\underline{\epsilon}^{ij}||} \oslash S(\underline{\epsilon}^{ij}) \\
&= \left(X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^{1j}||}\Big\downarrow_{m\wr|\underline{\epsilon}^{1j}|}^{m\wr||\underline{\epsilon}^{1j}||} \oslash S(\underline{\epsilon}^{1j}) \right) \boxtimes \cdots \boxtimes \left(X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^{sj}||}\Big\downarrow_{m\wr|\underline{\epsilon}^{sj}|}^{m\wr||\underline{\epsilon}^{sj}||} \oslash S(\underline{\epsilon}^{sj}) \right) \\
&= X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||}\Big\downarrow_{m\wr|C_j[\underline{\epsilon}]|}^{m\wr||C_j[\underline{\epsilon}]||} \oslash \big(S(\underline{\epsilon}^{1j}) \boxtimes \cdots \boxtimes S(\underline{\epsilon}^{sj}) \big) \\
&\qquad \big(\text{by (4.3.5) and the fact that } ||C_j[\underline{\epsilon}]|| = ||\underline{\epsilon}^{1j}|| + \cdots + ||\underline{\epsilon}^{sj}||\big) \\
&= X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||}\Big\downarrow_{m\wr|C_j[\underline{\epsilon}]|}^{m\wr||C_j[\underline{\epsilon}]||} \oslash S(C_j[\underline{\epsilon}])
\end{aligned}$$

where the final equality follows from the definition of the module $S(\underline{\alpha})$ for $\underline{\alpha}$ a multipartition and the fact that $C_j[\underline{\epsilon}] = \underline{\epsilon}^{1j} \circ \cdots \circ \underline{\epsilon}^{sj}$. We thus have a $k(S_m \wr S_n)$-module isomorphism

$$\left[\bigboxtimes_{j=1}^{t} \bigboxtimes_{i=1}^{s} Z_{[\underline{\epsilon}]}^{ij} \right] \Bigg\uparrow_{m\wr|C[\underline{\epsilon}]|}^{m\wr n} \cong \left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||}\Big\downarrow_{m\wr|C_j[\underline{\epsilon}]|}^{m\wr||C_j[\underline{\epsilon}]||} \oslash S(C_j[\underline{\epsilon}]) \right] \Bigg\uparrow_{m\wr|C[\underline{\epsilon}]|}^{m\wr n} . \tag{6.4.14}$$

Now for each $j = 1, \dots, t$, we have that $|C_j[\underline{\epsilon}]|$ is a composition of the integer $||C_j[\underline{\epsilon}]||$. It follows that the Young subgroup $S_{(|C_1[\underline{\epsilon}]|,\dots,|C_t[\underline{\epsilon}]|)} \cong S_{|C_1[\underline{\epsilon}]|} \times \cdots \times S_{|C_t[\underline{\epsilon}]|}$ of S_n is a subgroup of $S_{(||C_1[\underline{\epsilon}]||,\dots,||C_t[\underline{\epsilon}]||)} \cong S_{||C_1[\underline{\epsilon}]||} \times \cdots \times S_{||C_t[\underline{\epsilon}]||}$. By transitivity of induction, we now see that the right-hand side of (6.4.14) is

$$\left[\bigboxtimes_{j=1}^{t} \left[X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||}\Big\downarrow_{m\wr|C_j[\underline{\epsilon}]|}^{m\wr||C_j[\underline{\epsilon}]||} \oslash S(C_j[\underline{\epsilon}]) \right] \Bigg\uparrow_{m\wr|C_j[\underline{\epsilon}]|}^{m\wr||C_j[\underline{\epsilon}]||} \right] \Bigg\uparrow_{m\wr(||C_1[\underline{\epsilon}]||,\dots,||C_t[\underline{\epsilon}]||)}^{m\wr n}$$

and by Proposition 4.3.7, this is

$$\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash \left(S(C_j[\underline{\epsilon}])\Big\uparrow_{|C_j[\underline{\epsilon}]|}^{||C_j[\underline{\epsilon}]||} \right) \right] \Bigg\uparrow_{m\wr(||C_1[\underline{\epsilon}]||,\dots,||C_t[\underline{\epsilon}]||)}^{m\wr n} .$$

Hence, recalling (6.4.13), we have now shown that

$$\left[\bigboxtimes_{i=1}^{s}\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||\underline{\epsilon}^{ij}||}\Big\downarrow^{m\wr||\underline{\epsilon}^{ij}||}_{m\wr|\underline{\epsilon}^{ij}|} \oslash S(\underline{\epsilon}^{ij})\right]\Bigg\uparrow^{m\wr|\eta^i|}_{m\wr|R_i[\underline{\epsilon}]|}\right]\Bigg\uparrow^{m\wr n}_{m\wr|\underline{\eta}|}$$

$$\cong \left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash \left(S(C_j[\underline{\epsilon}])\Big\uparrow^{||C_j[\underline{\epsilon}]||}_{|C_j[\underline{\epsilon}]|}\right)\right]\Bigg\uparrow^{m\wr n}_{m\wr(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)}. \quad (6.4.15)$$

Now by (3.2.11), we have (recalling that $C_j[\underline{\epsilon}]$ is a multipartition)

$$S(C_j[\underline{\epsilon}])\Big\uparrow^{||C_j[\underline{\epsilon}]||}_{|C_j[\underline{\epsilon}]|} \sim \mathop{\mathcal{F}}_{\nu^j \vdash ||C_j[\underline{\epsilon}]||} c(\nu^j; C_j[\underline{\epsilon}]) S^{\nu^j}$$

and so (using Lemmas 2.2.2, 2.1.2 and 6.1.1), the right-hand side of (6.4.15) has a filtration

$$\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash \left(S(C_j[\underline{\epsilon}])\Big\uparrow^{||C_j[\underline{\epsilon}]||}_{|C_j[\underline{\epsilon}]|}\right)\right]\Bigg\uparrow^{m\wr n}_{m\wr(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)} \sim$$

$$\mathop{\mathcal{F}}_{\substack{\underline{\nu} \text{ is a } t\text{-multipartition} \\ \text{of } n \\ |\underline{\nu}|=(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)}} \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])\right)\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash S^{\nu^j}\right]\Bigg\uparrow^{m\wr n}_{m\wr(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)}.$$

Using the fact that for any multipartition $\underline{\nu}$ as in the filtration we have

$$\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash S^{\nu^j}\right]\Bigg\uparrow^{m\wr n}_{m\wr(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)} = S^{\underline{\nu}}(X_1, \ldots, X_t),$$

this filtration becomes

$$\left[\bigboxtimes_{j=1}^{t} X_j^{\widetilde{\boxtimes}||C_j[\underline{\epsilon}]||} \oslash \left(S(C_j[\underline{\epsilon}])\Big\uparrow^{||C_j[\underline{\epsilon}]||}_{|C_j[\underline{\epsilon}]|}\right)\right]\Bigg\uparrow^{m\wr n}_{m\wr(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)} \sim$$

$$\mathop{\mathcal{F}}_{\substack{\underline{\nu} \text{ is a } t\text{-multipartition of } n \\ |\underline{\nu}|=(||C_1[\underline{\epsilon}]||,\ldots,||C_t[\underline{\epsilon}]||)}} \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])\right) S^{\underline{\nu}}(X_1, \ldots, X_t). \quad (6.4.16)$$

In light of the isomorphism (6.4.15), we see that we may use the filtration (6.4.16) to refine the filtration (6.4.13) of $S^{\underline{\eta}}(Y_1, \ldots, Y_s)$, to obtain a filtration

$$S^{\underline{\eta}}(Y_1, \ldots, Y_s) \sim \mathop{\mathcal{F}}_{([\underline{\epsilon}], \underline{\nu})} \left(\prod_{i=1}^{s} c(\eta^i; R_i[\underline{\epsilon}])\right)\left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])\right) S^{\underline{\nu}}(X_1, \ldots, X_t) \quad (6.4.17)$$

where $([\underline{\epsilon}], \underline{\nu})$ ranges over all pairs where $[\underline{\epsilon}]$ is an $s \times t$ multipartition matrix such that $||R_i[\underline{\epsilon}]|| = |\eta^i|$ for each $i = 1, \ldots, s$ and $L[\underline{\epsilon}] = A$, and $\underline{\nu}$ is a t-multipartition of n such that $|\underline{\nu}| = (||C_1[\underline{\epsilon}]||, \ldots, ||C_t[\underline{\epsilon}]||)$. This is our desired filtration of $S^{\underline{\eta}}(Y_1, \ldots, Y_s)$, but before stating this result in its final form as a proposition, we shall pause to consider a special case which will be important in our work below.

Suppose that we have $s = t$ and moreover that we have $w_i = i$ for each $i = 1, \ldots, t$. Then the multipartition matrix $M(\underline{\eta}; w_1, \ldots, w_t) = M(\underline{\eta}; 1, \ldots, t)$ which indexes the bottom-most factor in the filtration (6.4.13) has $(i, i)^{\text{th}}$ entry $(\eta^i, (), \ldots, ())$, and all entries off the main diagonal are tuples of empty partitions. Thus $M(\underline{\eta}; 1, \ldots, t)$ has the form

$$\begin{bmatrix} (\eta^1, (), \ldots, ()) & ((), \ldots, ()) & \cdots & \cdots & ((), \ldots, ()) \\ ((), \ldots, ()) & (\eta^2, (), \ldots, ()) & ((), \ldots, ()) & \cdots & ((), \ldots, ()) \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & ((), \ldots, ()) \\ ((), \ldots, ()) & ((), \ldots, ()) & \cdots & ((), \ldots, ()) & (\eta^t, (), \ldots, ()) \end{bmatrix}.$$

Hence, if we take $[\underline{\epsilon}] = M(\underline{\eta}; 1, \ldots, t)$ in (6.4.16), then we find that

$$C_j[\underline{\epsilon}] = \big((), \ldots, (), \eta^j, (), \ldots, ()\big)$$

for each j. Now if we have a partition $\nu \vdash n$ and a multipartition $\underline{\alpha} = \big((), \ldots, (), \eta, (), \ldots, ()\big)$, where η is some partition, such that the Littlewood-Richardson coefficient $c(\nu; \underline{\alpha})$ is non-zero, then we see by (3.2.3) and (3.2.5) that we must have $\eta = \nu$, and that $c(\nu; \underline{\alpha}) = 1$ in this case. It now follows that for any t-multipartition $\underline{\nu}$ of n, we have that $\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}])$ is equal to 1 if $\underline{\nu} = \underline{\eta}$, and is zero if $\underline{\nu} \neq \underline{\eta}$. Thus we see that for $[\underline{\epsilon}] = M(\underline{\eta}; 1, \ldots, t)$, the only module occurring with non-zero multiplicity in the filtration (6.4.16) is $S^{\underline{\eta}}(X_1, \ldots, X_t)$. Since the filtration (6.4.17) was obtained by refining the filtration (6.4.13) using the filtration (6.4.16), it follows that in this special case, the module occurring at the very bottom of the filtration (6.4.17) is $S^{\underline{\eta}}(X_1, \dots, X_t)$.

From (6.4.17), we thus see that we have obtained the following result, which is essentially a reformulated version of [6, Lemma 4.4, (1)].

Proposition 6.4.1. *[6, Lemma 4.4, (1)] Let $Y_1, \dots, Y_s$ and $X_1, \dots, X_t$ be kS_m-modules such that for each $i = 1, \dots, s$ we have a filtration*

$$Y_i \sim \mathop{\mathcal{F}_{\langle w_i \rangle}}_{j=1}^{t} a^i_j X_j,$$

and let $\underline{\eta}$ be an s-component multipartition of n. Then we have a filtration

$$S^{\underline{\eta}}(Y_1, \dots, Y_s) \sim \mathop{\mathcal{F}}_{\underline{\nu}} \left[\sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\eta}| \times |\underline{\nu}|)} \left(\prod_{i=1}^{s} c(\eta^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]) \right) \right] S^{\underline{\nu}}(X_1, \dots, X_t)$$

where $\underline{\nu}$ runs over all t-multipartitions of n and where A is the $s \times t$ integer matrix whose $(i,j)^{th}$ entry is a^i_j. Further, suppose that we have $s = t$ and moreover that we have $w_i = i$ for each $i = 1, \dots, t$. Then the module occurring at the bottom of this filtration is $S^{\underline{\eta}}(X_1, \dots, X_t)$.

6.5 Unitriangular systems and Young's rule for the wreath product

If we assume certain extra conditions on the system of filtrations for the modules Y_i in Proposition 6.4.1, we can obtain more precise results on the filtration obtained, as the following proposition (which is essentially a reformulation of [6, Proposition 4.7]) shows.

Proposition 6.5.1. *[6, Lemma 4.7] With the hypotheses of Proposition 6.4.1, let us further assume that $s = t$ and moreover that*

1. *for all i and j we have that $j > i$ implies $a^i_j = 0$*

2. *we have $a^i_i = 1$ for each $i = 1, \dots, t$*

so that the matrix A is in fact square and lower unitriangular. Then the multiplicity of $S^{\underline{\nu}}(X_1, \dots, X_t)$ in the filtration of $S^{\underline{\eta}}(Y_1, \dots, Y_s)$ in Proposition 6.4.1 is 1 if $\underline{\nu} = \underline{\eta}$ and 0 if $\underline{\nu} \not\trianglerighteq \underline{\eta}$.

Proof. Firstly, note that the multiplicity of $S^{\underline{\nu}}(X_1, \dots, X_t)$ in the filtration of $S^{\underline{\eta}}(Y_1, \dots, Y_t)$ in proposition 6.4.1 (with $s = t$) is

$$\sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\eta}| \times |\underline{\nu}|)} \left(\prod_{i=1}^{t} c(\eta^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]) \right). \tag{6.5.1}$$

Assume that for some $\underline{\nu}$ and some $\underline{\eta}$, (6.5.1) is non-zero. So we must have some $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\eta}| \times |\underline{\nu}|)$ such that

$$\left(\prod_{i=1}^{t} c(\eta^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]) \right) \neq 0.$$

This implies that

$$c(\eta^i; R_i[\underline{\epsilon}]) \neq 0 \quad \text{for } i = 1, \dots, t \tag{6.5.2}$$

and

$$c(\nu^j; C_j[\underline{\epsilon}]) \neq 0 \quad \text{for } j = 1, \dots, t. \tag{6.5.3}$$

Further, we have of course that

$$||R_i[\underline{\epsilon}]|| = |\eta^i| \quad \text{for } i = 1, \dots, t \tag{6.5.4}$$

and

$$||C_j[\underline{\epsilon}]|| = |\nu^j| \quad \text{for } j = 1, \dots, t. \tag{6.5.5}$$

We claim that we must have $\underline{\nu} \trianglerighteq \underline{\eta}$, and so by the definition of the dominance order on multipartitions, we need to show that for any $p = 1, \ldots, t$ and $q = 1, 2, 3, \ldots$, we have

$$\sum_{i=1}^{p-1} |\nu^i| + \sum_{j=1}^{q} \nu_j^p \geqslant \sum_{i=1}^{p-1} |\eta^i| + \sum_{j=1}^{q} \eta_j^p.$$

Indeed, let us fix such p and q. Then by (6.5.2) and (6.5.3) we have

$$0 \neq c(\eta^p; R_p[\underline{\epsilon}]) = c\left(\eta^p; \underline{\epsilon}^{p,1} \circ \cdots \circ \underline{\epsilon}^{p,t}\right) \tag{6.5.6}$$

and

$$0 \neq c(\nu^p; C_p[\underline{\epsilon}]) = c\left(\nu^p; \underline{\epsilon}^{1,p} \circ \cdots \circ \underline{\epsilon}^{t,p}\right). \tag{6.5.7}$$

By our assumptions about the coefficients a_j^i, we know that the matrix A is lower unitriangular. Since we know that $L[\underline{\epsilon}] = A$ (where, recall, $L[\underline{\epsilon}]$ is the matrix whose $(i,j)^{\text{th}}$ entry is the length of the multipartition $\underline{\epsilon}^{ij}$ which is the $(i,j)^{\text{th}}$ entry of the multipartition matrix $[\underline{\epsilon}]$), it follows that $\underline{\epsilon}^{ii} = (\epsilon^i)$ for partitions $\epsilon^1, \ldots, \epsilon^t$, and that $\underline{\epsilon}^{ij} = ()$ if $j > i$. Thus $[\underline{\epsilon}]$ has the form

$$[\underline{\epsilon}] \quad = \quad \begin{bmatrix} (\epsilon^1) & () & \cdots & \cdots & () \\ * & (\epsilon^2) & () & \cdots & () \\ \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & \ddots & () \\ * & * & \cdots & * & (\epsilon^t) \end{bmatrix}. \tag{6.5.8}$$

We thus see that

$$R_p[\underline{\epsilon}] = \underline{\epsilon}^{p,1} \circ \cdots \circ \underline{\epsilon}^{p,t} = (*, \ldots, *, \epsilon^p) \tag{6.5.9}$$

and

$$C_p[\underline{\epsilon}] = \underline{\epsilon}^{1,p} \circ \cdots \circ \underline{\epsilon}^{t,p} = (\epsilon^p, *, \ldots, *). \tag{6.5.10}$$

Since $c(\nu^p; C_p[\underline{\epsilon}]) \neq 0$ by (6.5.7), we thus have by (6.5.10) and Lemma 3.2.3 (2) that

$$\sum_{j=1}^{q} \nu_j^p \geqslant \sum_{j=1}^{q} \epsilon_j^p. \tag{6.5.11}$$

Further, since $c(\eta^p; R_p[\underline{\epsilon}]) \neq 0$ by (6.5.6), we see by (6.5.9) and Lemma 3.2.3 (3) that we have a partition ζ of $|\eta^p| - |\epsilon^p|$ such that

$$\sum_{j=1}^{q} \eta_j^p \leqslant \sum_{j=1}^{q} \epsilon_j^p + \sum_{j=1}^{q} \zeta_j. \tag{6.5.12}$$

Further we have by (6.5.5) that $\sum_{i=1}^{p-1} |\nu^i|$ is equal to the sum of all of the sizes of the partitions occurring as components of the multipartitions in the first $p-1$ columns of $[\underline{\epsilon}]$. By (6.5.4) we have that $\sum_{i=1}^{p} |\eta^i|$ is the sum of all of the sizes of the partitions occurring as components of the multipartitions in the first p rows of $[\underline{\epsilon}]$. Further, from (6.5.8) we see that all of the partitions which occur as components of the multipartitions in the first p rows of $[\underline{\epsilon}]$, except for the ϵ^p occurring in the $(p,p)^{\text{th}}$ entry, lie in the first $p-1$ columns of $[\underline{\epsilon}]$. Hence

$$\begin{aligned}
\sum_{i=1}^{p-1} |\nu^i| &\geqslant \left(\sum_{i=1}^{p} |\eta^i|\right) - |\epsilon^p| \\
&= \left(\sum_{i=1}^{p-1} |\eta^i|\right) + |\eta^p| - |\epsilon^p| \\
&= \left(\sum_{i=1}^{p-1} |\eta^i|\right) + |\zeta| \\
&\geqslant \sum_{i=1}^{p-1} |\eta^i| + \sum_{j=1}^{q} \zeta_j.
\end{aligned} \tag{6.5.13}$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^{p-1}|\nu^i| + \sum_{j=1}^{q}\nu_j^p &\geqslant \sum_{i=1}^{p-1}|\nu^i| + \sum_{j=1}^{q}\epsilon_j^p \qquad \text{(by (6.5.11))} \\
&\geqslant \sum_{i=1}^{p-1}|\eta^i| + \sum_{j=1}^{q}\zeta_j + \sum_{j=1}^{q}\epsilon_j^p \qquad \text{(by (6.5.13))} \\
&\geqslant \sum_{i=1}^{p-1}|\eta^i| + \sum_{j=1}^{q}\eta_j^p \qquad \text{(by (6.5.12))}
\end{aligned}$$

as required, so indeed $\underline{\nu} \trianglerighteq \underline{\eta}$.

Finally, we consider the case $\underline{\nu} = \underline{\eta}$. We seek to show that in this case (6.5.1) is equal to 1. Indeed, if $\underline{\nu} = \underline{\eta}$ then (6.5.1) becomes

$$\sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A;|\underline{\eta}|\times|\underline{\eta}|)} \left(\prod_{i=1}^{t} c(\eta^i; R_i[\underline{\epsilon}])\right)\left(\prod_{j=1}^{t} c(\eta^j; C_j[\underline{\epsilon}])\right). \tag{6.5.14}$$

Now we know that any multipartition matrix $[\underline{\epsilon}]$ in $\mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\eta}| \times |\underline{\eta}|)$ is of the form (6.5.8), and moreover that the sum of the sizes of all partitions occurring as components of the multipartitions in the i^{th} column is $|\eta^i|$, and that the sum of the sizes of all partitions occurring as components of the multipartitions in the i^{th} row is also $|\eta^i|$. Considering the first row of (6.5.8), we see that we must have $||\underline{\epsilon}^{1,1}|| = |\epsilon^1| = |\eta^1|$. But then considering the first column of (6.5.8), we see that all of the multipartitions $\underline{\epsilon}^{i,1}$ in the first column of $[\underline{\epsilon}]$ where $i \neq 1$ must satisfy $||\underline{\epsilon}^{i,1}|| = 0$. Moving on to the second row and second column of (6.5.8), we see by the same logic that all multipartitions $\underline{\epsilon}^{ij}$ appearing on the second row or second column of $[\underline{\epsilon}]$ satisfy $||\underline{\epsilon}^{ij}|| = 0$, except for $\underline{\epsilon}^{2,2}$, which satisfies $||\underline{\epsilon}^{2,2}|| = |\epsilon^2| = |\eta^2|$. Continuing in this manner down the rows and columns, we see that the multipartitions $\underline{\epsilon}^{ij}$ which are the entries of $[\underline{\epsilon}]$ must satisfy

$$||\underline{\epsilon}^{ij}|| = \begin{cases} |\eta^i| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and hence $[\underline{\epsilon}]$ is a $t{\times}t$ multipartition matrix whose (i,i)-th entry is (ϵ^i) for some $\epsilon^i \vdash |\eta^i|$, and where all the other entries are either empty multipartitions or tuples of empty partitions. We thus see that for each $i = 1,\ldots,t$, both $R_i[\underline{\epsilon}]$ and $C_i[\underline{\epsilon}]$ are multipartitions where one component is ϵ^i and all other components are $()$. It follows by (3.2.3) and (3.2.5) that for all i and j we have

$$c(\eta^i; R_i[\underline{\epsilon}]) = \begin{cases} 1 & \text{if } \eta^i = \epsilon^i \\ 0 & \text{otherwise} \end{cases}$$

and

$$c(\eta^j; C_j[\underline{\epsilon}]) = \begin{cases} 1 & \text{if } \eta^j = \epsilon^j \\ 0 & \text{otherwise} \end{cases}$$

by which we see that the only $\underline{\epsilon}$ for which the summand in (6.5.14) is non-zero is the $\underline{\epsilon}$ where we have $\epsilon^i = \eta^i$ for $i = 1,\ldots,t$ and where all the other entries are either empty multipartitions or tuples of empty partitions. Further, we see that for this $\underline{\epsilon}$ the summand is 1. Thus (6.5.14) equals 1 as claimed. □

We conclude by applying Proposition 6.5.1 to obtain a filtration of the module $M^{\underline{\lambda}}$ by modules $S^{\underline{\nu}}$, thus proving an analogue of Young's rule for $k(S_m \wr S_n)$.

Recall that we have fixed the distinct partitions of m, in the lexicographic order, to be

$$(m) = \mu^1 > \mu^2 > \ \ldots\ > \mu^r = (1^m).$$

Let us define K^i_j to be the Kostka number $K(\mu^j, \mu^i)$, which recall is the multiplicity of S^{μ^j} in M^{μ^i} in the filtration (3.2.1) given by Young's rule.

$$M^{\mu^i} \sim \mathop{\mathcal{F}_{\langle i \rangle}}_{j=1}^{r} K^i_j S^{\mu^j}. \tag{6.5.15}$$

Thus we have by (3.2.2) that $K^i_i = 1$ and $K^i_j = 0$ if $i < j$. Let us define K to be the $r \times r$ matrix whose $(i,j)^{\text{th}}$ entry is K^i_j, so that K is lower unitriangular.

Proposition 6.5.2. *Let $\underline{\lambda}$ be an r-multipartition of n. Then we have a filtration of $k(S_m \wr S_n)$-modules*

$$M^{\underline{\lambda}} \sim$$
$$\mathcal{F}_{\substack{\langle\underline{\lambda}\rangle \\ \underline{\nu}\in\underline{\Lambda}_n^r}} \left(\sum_{([\underline{\epsilon}],\underline{\eta})} \left(\prod_{l=1}^{r} K(\eta^l,\lambda^l) \right) \left(\prod_{i=1}^{r} c(\eta^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{r} c(\nu^j; C_j[\underline{\epsilon}]) \right) \right) S^{\underline{\nu}}$$

where for each multipartition $\underline{\nu} \in \underline{\Lambda}_n^r$ (recalling that $\underline{\Lambda}_n^r$ is the set of all multipartitions of n with length r), the pair $([\underline{\epsilon}],\underline{\eta})$ ranges over all pairs consisting of a multipartition matrix $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(K; |\underline{\lambda}| \times |\underline{\nu}|)$ and a multipartition $\underline{\eta} \in \underline{\Lambda}_n^r$ such that $|\underline{\eta}| = |\underline{\lambda}|$.

The multiplicity of $S^{\underline{\nu}}$ in this filtration is 0 if $\underline{\nu} \ntrianglerighteq \underline{\lambda}$ and 1 if $\underline{\nu} = \underline{\lambda}$. Thus $M^{\underline{\lambda}}$ has a filtration by modules $S^{\underline{\nu}}$ for $\underline{\nu} \trianglerighteq \underline{\lambda}$, in which $S^{\underline{\lambda}}$ occurs exactly once at the very bottom of the filtration.

We may regard this result as a wreath product analogue of Young's rule, and the coefficients with which the modules $S^{\underline{\nu}}$ appear as wreath product analogues of the Kostka numbers.

Proof. We have $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$, and

$$M^{\underline{\lambda}} = \left[\left(M^{\mu^1}, \ldots, M^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\lambda}|} \oslash \left(M^{\lambda^1} \boxtimes \cdots \boxtimes M^{\lambda^r} \right) \right] \Big\uparrow_{m\wr|\underline{\lambda}|}^{m\wr n}.$$

We may use Young's rule (3.2.1) and Lemma 2.1.2 to see that the $k(S_m \wr S_{|\underline{\lambda}|})$-module $M^{\lambda^1} \boxtimes \cdots \boxtimes M^{\lambda^r}$ has a filtration

$$M^{\lambda^1} \boxtimes \cdots \boxtimes M^{\lambda^r} \sim \mathcal{F}_{\substack{\langle\underline{\lambda}\rangle \\ \underline{\eta}\in\underline{\Lambda}_n^r \\ |\underline{\eta}|=|\underline{\lambda}|}} \left(\prod_{l=1}^{r} K(\eta^l,\lambda^l) \right) S^{\eta^1} \boxtimes \cdots \boxtimes S^{\eta^r}.$$

Thus by Lemmas 6.1.1 and 2.2.2, we have a filtration

$$M^{\underline{\lambda}} \sim \mathcal{F}_{\substack{\langle\underline{\lambda}\rangle \\ \underline{\eta}\in\underline{\Lambda}_n^r \\ |\underline{\eta}|=|\underline{\lambda}|}} \left(\prod_{l=1}^{r} K(\eta^l,\lambda^l) \right) S^{\underline{\eta}}\left(M^{\mu^1}, \ldots, M^{\mu^r}\right). \tag{6.5.16}$$

Now suppose that $\underline{\eta}$ is a multipartition of n with $|\underline{\eta}| = |\underline{\lambda}|$ such that

$$\prod_{l=1}^{r} K\left(\eta^l, \lambda^l\right) \neq 0.$$

Then by (3.2.2), we must have $\eta^l \trianglerighteq \lambda^l$ for $l = 1, \ldots, r$, which in turn implies $\underline{\eta} \trianglerighteq \underline{\lambda}$ (since $|\underline{\eta}| = |\underline{\lambda}|$). Hence the multiplicity of $S^{\underline{\eta}}\big(M^{\mu^1}, \ldots, M^{\mu^r}\big)$ in (6.5.16) is zero unless $\underline{\eta} \trianglerighteq \underline{\lambda}$. Further, if $\underline{\eta} = \underline{\lambda}$, then we have by (3.2.2) that

$$\prod_{l=1}^{r} K\left(\eta^l, \lambda^l\right) = \prod_{l=1}^{r} K\left(\lambda^l, \lambda^l\right) = 1$$

so the multiplicity of $S^{\underline{\lambda}}\big(M^{\mu^1}, \ldots, M^{\mu^r}\big)$ in the filtration (6.5.16) is 1.

Now the filtrations (6.5.15) of the modules M^{μ^i} by the modules S^{μ^j} satisfy the additional condition in Proposition 6.4.1 (i.e. that "$s = t$ and moreover that we have $w_i = i$ for each $i = 1, \ldots, t$"), and also all of the conditions in Proposition 6.5.1. Hence by applying Propositions 6.4.1 and 6.5.1 to the module $S^{\underline{\eta}}(M^{\mu^1}, \ldots, M^{\mu^r})$, and using the fact that $S^{\underline{\nu}}(S^{\mu^1}, \ldots, S^{\mu^r}) = S^{\underline{\nu}}$, we have for any r-multipartition $\underline{\eta}$ of n a filtration

$$S^{\underline{\eta}}(M^{\mu^1}, \ldots, M^{\mu^r}) \sim$$
$$\mathcal{F}_{\substack{\langle \underline{\eta} \rangle \\ \underline{\nu} \in \underline{\Lambda}_n^r}} \left[\sum_{[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(K; |\underline{\eta}| \times |\underline{\nu}|)} \left(\prod_{i=1}^{r} c(\eta^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{r} c(\nu^j; C_j[\underline{\epsilon}]) \right) \right] S^{\underline{\nu}} \quad (6.5.17)$$

where the multiplicity of $S^{\underline{\eta}}$ equals 1 and the multiplicity of $S^{\underline{\nu}}$ equals zero if $\underline{\nu} \not\trianglerighteq \underline{\eta}$. The filtration of $M^{\underline{\lambda}}$ in the statement of the proposition now follows by combining the filtration (6.5.16) with the filtration (6.5.17), noting in particular that $S^{\underline{\lambda}}\big(M^{\mu^1}, \ldots, M^{\mu^r}\big)$ occurs at the bottom of the filtration (6.5.16) and that $S^{\underline{\lambda}}$ occurs at the bottom of the filtration of $S^{\underline{\lambda}}\big(M^{\mu^1}, \ldots, M^{\mu^r}\big)$ given by (6.5.17), so that $S^{\underline{\lambda}}$ is the bottom-most factor in the filtration of $M^{\underline{\lambda}}$ obtained by using the filtration (6.5.17) to refine the filtration (6.5.16).

The claims made in the proposition about multiplicities follow easily from the facts about multiplicities in the filtrations (6.5.16) and (6.5.17), namely that in the filtration (6.5.16) the multiplicity of $S^{\underline{\eta}}\big(M^{\mu^1}, \dots, M^{\mu^r}\big)$ is zero unless $\underline{\eta} \trianglerighteq \underline{\lambda}$ and the multiplicity of $S^{\underline{\lambda}}\big(M^{\mu^1}, \dots, M^{\mu^r}\big)$ is 1, and that in the filtration (6.5.17) the multiplicity of $S^{\underline{\eta}}$ equals 1 and the multiplicity of $S^{\underline{\nu}}$ equals zero if $\underline{\nu} \ntrianglerighteq \underline{\eta}$. $\square$

Original research in Chapter 6: The work in this chapter is almost all a reformulation of results from [6] and thus is not really original research. However, the use of multipartition matrices to state and prove these results is a novel idea which I believe improves the clarity and usability of the results, and furthermore allows for more transparent proofs. Proposition 6.5.2, while technically new, is really just an application of the results of [6] and thus is not really original. Corollary 6.2.2 is my own work, although this is a fairly straightforward result.

Chapter 7

Tableau combinatorics

In the coming chapters, we shall make crucial use of Mackey's theorem (Theorem 2.2.5). In our applications of Mackey's theorem, we shall be confronted with the problem of finding a system of (H, K)-double coset representatives for certain subgroups H, K of S_n, and moreover of understanding the subgroups $H^x \cap K$ where x is one of our chosen coset representatives (recalling that H^x denotes the conjugate subgroup $x^{-1}Hx$ of H by x). In particular, we shall be interested in the case where K is the Young subgroup S_α for some $\alpha \vDash n$, and where H is either S_γ for γ a composition of n, or else H is $S_{\underline{\gamma}}$ for some multicomposition $\underline{\gamma}$ of n. In this chapter, we shall develop the theory of certain kinds of Young tableau which provide a natural and convenient way of dealing with these questions. We draw on the account given by Wildon in his unpublished note [34]: the material for the case where $H = S_\gamma$ is taken more-or-less directly from this note, while the corresponding material for the case $H = S_{\underline{\gamma}}$ is of course closely based on the $H = S_\gamma$ case. However, the material in section 7.4 is original.

Throughout this chapter we fix $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ to be an l part composition of n, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_t)$ to be a t part composition of n, and

$\underline{\gamma} = (\gamma^1, \gamma^2, \ldots, \gamma^t)$ to be a t component multicomposition of n, where thus $\gamma^i = (\gamma^i_1, \gamma^i_2, \ldots, \gamma^i_{l_i}) \vDash |\gamma^i|$ for each i, with $|\gamma^1| + \cdots + |\gamma^t| = n$.

7.1 Tableaux and the action of S_n

A **tableau of shape** α **and type** γ is a Young diagram of shape α where each box contains a positive integer i such that for each $i \in \{1, \ldots, t\}$, i occurs exactly γ_i times, while a **tableau of shape** α **and type** $\underline{\gamma}$ is a Young diagram of shape α where each box contains a pair (i, j) of positive integers, such that for each i and j the pair (i, j) occurs exactly γ^i_j times.

Example 7.1.1. Take $n = 31$, $\alpha = (7, 5, 6, 4, 7, 2)$ and $\gamma = (5,\ 9,\ 3,\ 8,\ 6)$. Then one possible tableau of shape α and type γ is

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & 4 & 2 & 4 & 5 & 3 & 5 \\
\hline
4 & 1 & 2 & 4 & 1 \\
\cline{1-6}
2 & 5 & 2 & 5 & 5 & 2 \\
\hline
5 & 1 & 3 & 2 \\
\hline
1 & 4 & 2 & 3 & 4 & 4 & 1 \\
\hline
2 & 4 \\
\cline{1-2}
\end{array} .$$

Further, if

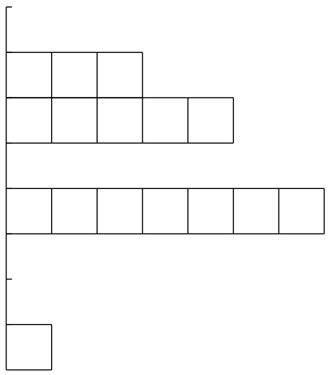
$$\underline{\gamma} = \big((2,3),\ (3,4,2),\ (3),\ (4,4),\ (2,3,1)\big),$$

then one possible tableau of shape α and type $\underline{\gamma}$ is

(2,1)	(4,2)	(2,3)	(4,2)	(5,2)	(3,1)	(5,1)
(4,1)	(1,1)	(2,2)	(4,2)	(1,2)		
(2,2)	(5,3)	(2,1)	(5,2)	(5,2)	(2,3)	
(5,1)	(1,1)	(3,1)	(2,2)			
(1,2)	(4,1)	(2,1)	(3,1)	(4,1)	(4,2)	(1,2)
(2,2)	(4,1)					

.

Since we allow α to have zero parts, our Young diagrams can have rows with no boxes in them. For example if $\alpha = (0, 3, 5, 0, 7, 0, 0, 1)$, then the Young diagram of shape α might be drawn as



Example 7.1.2. Take $n = 16$, $\alpha = (0, 3, 5, 0, 7, 0, 0, 1)$ and $\gamma = (0,\ 3,\ 8,\ 0,\ 5)$.

Then one possible tableau of shape α and type γ is

$$\begin{array}{|ccccccc}
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} & & & & \\
\hline
\multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & & \\
\hline
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{3} \\
\hline
 & & & & & & \\
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{2} & & & & & & \\
\hline
\end{array}\ .$$

Further, if

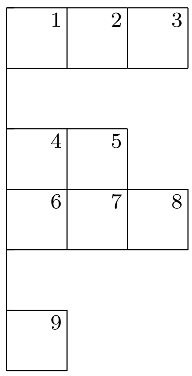
$$\underline{\gamma} = \big((),\ (1,0,2),\ (3,5,0),\ (),\ (0,0,2,3)\big),$$

then one possible tableau of shape α and type $\underline{\gamma}$ is

$$\begin{array}{|ccccccc}
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{(3,2)} & \multicolumn{1}{c|}{(5,3)} & \multicolumn{1}{c|}{(3,1)} & & & & \\
\hline
\multicolumn{1}{|c|}{(5,3)} & \multicolumn{1}{c|}{(2,3)} & \multicolumn{1}{c|}{(3,2)} & \multicolumn{1}{c|}{(3,2)} & \multicolumn{1}{c|}{(5,4)} & & \\
\hline
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{(3,2)} & \multicolumn{1}{c|}{(3,1)} & \multicolumn{1}{c|}{(5,4)} & \multicolumn{1}{c|}{(2,1)} & \multicolumn{1}{c|}{(3,2)} & \multicolumn{1}{c|}{(5,4)} & \multicolumn{1}{c|}{(3,1)} \\
\hline
 & & & & & & \\
 & & & & & & \\
\hline
\multicolumn{1}{|c|}{(2,3)} & & & & & & \\
\hline
\end{array}\ .$$

Tableaux are of little interest as static objects: in order to make use of them, we must introduce an action of S_n which transforms one tableau into another. For our fixed composition α of n, we let S_n act (from the right) on both the set of tableaux of shape α and type γ, and the set of tableaux of shape α and type $\underline{\gamma}$, by permuting the entries of a tableau in the manner which we shall now describe.

Firstly, we introduce a numbering of the boxes in a Young diagram of shape α. Indeed, we number the boxes of the tableau from 1 to n going from left to right across each row in turn, starting with the top row and working down. Thus if $n = 9$ and $\alpha = (3, 0, 2, 3, 0, 1)$, the numbering of the boxes is



.

Now let τ be a tableau of shape α and type γ or $\underline{\gamma}$, and let $\sigma \in S_n$. Then $\tau\sigma$ is defined to be the tableau obtained from τ by moving the number or pair of numbers in box number i to box number $(i)\sigma$, for each $i = 1, \ldots, n$.

Example 7.1.3. For example, let us take $n = 13$, $\alpha = (5, 3, 4, 1)$, $\underline{\gamma} =$

$\big((2,1,0,1),\ (3,2),\ (1,3)\big)$, $\sigma = (1,12,3,6)(5,7,13)(8,10) \in S_{13}$, and

$$\tau \quad = \quad \begin{array}{|c|c|c|c|c|}
\hline
(1,1) & (2,1) & (1,2) & (3,1) & (2,1) \\
\hline
(2,2) & (3,2) & (2,2) \\
\cline{1-4}
(2,1) & (3,2) & (1,1) & (3,2) \\
\cline{1-4}
(1,4) \\
\cline{1-1}
\end{array} .$$

We write the box numbers into τ to obtain

$$\begin{array}{|c|c|c|c|c|}
\hline
(1,1)^{1} & (2,1)^{2} & (1,2)^{3} & (3,1)^{4} & (2,1)^{5} \\
\hline
(2,2)^{6} & (3,2)^{7} & (2,2)^{8} \\
\cline{1-4}
(2,1)^{9} & (3,2)^{10} & (1,1)^{11} & (3,2)^{12} \\
\cline{1-4}
(1,4)^{13} \\
\cline{1-1}
\end{array} ,$$

and then performing the above operation yields

$$\begin{array}{|c|c|c|c|c|}
\hline
(2,2)^{1} & (2,1)^{2} & (3,2)^{3} & (3,1)^{4} & (1,4)^{5} \\
\hline
(1,2)^{6} & (2,1)^{7} & (3,2)^{8} \\
\cline{1-4}
(2,1)^{9} & (2,2)^{10} & (1,1)^{11} & (1,1)^{12} \\
\cline{1-4}
(3,2)^{13} \\
\cline{1-1}
\end{array} .$$

Thus we have

$$
\tau\sigma \quad = \quad
\begin{array}{lllll}
\boxed{(2,2)} & \boxed{(2,1)} & \boxed{(3,2)} & \boxed{(3,1)} & \boxed{(1,4)} \\
\boxed{(1,2)} & \boxed{(2,1)} & \boxed{(3,2)} & & \\
\boxed{(2,1)} & \boxed{(2,2)} & \boxed{(1,1)} & \boxed{(1,1)} & \\
\boxed{(3,2)} & & & &
\end{array}
\quad .
$$

It is easy to see that this definition does indeed yield S_n actions as claimed, and it is obvious that these S_n actions are transitive. It is natural to ask what the stabilizer of a given tableau is under this action, and in order to answer this we now consider certain special tableaux of shape α and type γ or $\underline{\gamma}$. Indeed, for our compositions α and γ, we construct the **standard tableau of shape α and type γ** as follows: we begin with a Young diagram of shape α with the boxes numbered as described above, and then working from box 1 to box n we enter first γ_1 1's, then γ_2 2's, and so on. We denote this tableau by τ^α_γ. For example, if we take $n = 13$, $\alpha = (2, 0, 3, 1, 3, 4)$ and $\gamma = (3, 5, 0, 4, 1)$, then we have

$$
\tau^\alpha_\gamma \quad = \quad
\begin{array}{llll}
\boxed{1} & \boxed{1} & & \\
 & & & \\
\boxed{1} & \boxed{2} & \boxed{2} & \\
\boxed{2} & & & \\
\boxed{2} & \boxed{2} & \boxed{4} & \\
\boxed{4} & \boxed{4} & \boxed{4} & \boxed{5}
\end{array}
\quad .
$$

Similarly, for our multicomposition $\underline{\gamma}$, we begin with a Young diagram of shape α with the boxes numbered as described above, and then working from box 1 to box n we enter symbols (i, j) one per box, starting with γ^1_1 pairs $(1, 1)$, then

γ^1_2 pairs $(1,2)$, and so on until we run out of parts in γ^1. We then enter γ^2_1 pairs $(2,1)$, then γ^2_2 pairs $(2,2)$, and so on, and so on. We thus obtain a tableau of shape α and type $\underline{\gamma}$, and we define $\tau^\alpha_{\underline{\gamma}}$ to be this tableau. For example, take $n = 13$, $\alpha = (2,0,3,1,3,4)$ and $\underline{\gamma} = \big((1,3),(2,2),(),(1,0,3),(1)\big)$. Then from $\underline{\gamma}$ we see that, to form a tableau of shape α and type $\underline{\gamma}$, we need 1 pair $(1,1)$, 3 pairs $(1,2)$, 2 pairs $(2,1)$, 2 pairs $(2,2)$, 1 pair $(4,1)$, 3 pairs $(4,3)$, and 1 pair $(5,1)$. Entering these into a Young diagram of shape α in the manner described above gives

$$\tau^\alpha_{\underline{\gamma}} \quad = \quad \begin{array}{llll}
\boxed{(1,1)} & \boxed{(1,2)} & & \\
 & & & \\
\boxed{(1,2)} & \boxed{(1,2)} & \boxed{(2,1)} & \\
\boxed{(2,1)} & & & \\
\boxed{(2,2)} & \boxed{(2,2)} & \boxed{(4,1)} & \\
\boxed{(4,3)} & \boxed{(4,3)} & \boxed{(4,3)} & \boxed{(5,1)}
\end{array} \quad .$$

Now it is clear from the definition of τ^α_γ and $\tau^\alpha_{\underline{\gamma}}$ that their stabilizers under the action of S_n are the Young subgroups S_γ and $S_{\underline{\gamma}}$, respectively. Now let $\sigma \in S_n$. Then for any $\theta \in S_n$ we have (writing $\mathrm{Stab}(-)$ to denote a stabilizer)

$$\begin{aligned}
& \theta \in \mathrm{Stab}\left(\tau^\alpha_\gamma \sigma\right) \\
\iff \quad & \tau^\alpha_\gamma \sigma = \tau^\alpha_\gamma \sigma\theta \\
\iff \quad & \tau^\alpha_\gamma = \tau^\alpha_\gamma \left(\sigma\theta\sigma^{-1}\right) \\
\iff \quad & \sigma\theta\sigma^{-1} \in \mathrm{Stab}\left(\tau^\alpha_\gamma\right) = S_\gamma \\
\iff \quad & \theta \in \sigma^{-1} S_\gamma \sigma.
\end{aligned}$$

We may apply the same argument to prove that

$$\theta \in \operatorname{Stab}\bigl(\tau^{\alpha}_{\underline{\gamma}}\sigma\bigr) \quad \Longleftrightarrow \quad \theta \in \sigma^{-1} S_{\underline{\gamma}} \sigma$$

and hence we have the following.

Proposition 7.1.4. *(See for example [34], proof of Proposition 5.2) For any* $\sigma \in S_n$*, we have* $\operatorname{Stab}\bigl(\tau^{\alpha}_{\gamma}\sigma\bigr) = \bigl(S_{\gamma}\bigr)^{\sigma}$ *and* $\operatorname{Stab}\bigl(\tau^{\alpha}_{\underline{\gamma}}\sigma\bigr) = \bigl(S_{\underline{\gamma}}\bigr)^{\sigma}$.

7.2 Weakly increasing rows and double cosets

Our purpose in studying tableaux is to gain an understanding of certain kinds of double cosets in S_n, and in this section we shall show how we may use a particular subset of tableaux to index these double cosets in a natural way.

We say that a tableau of shape α and type γ has **weakly increasing rows** if the entries in its rows are weakly increasing from left to right. We say that a tableau of shape α and type $\underline{\gamma}$ has **weakly increasing rows** if the entries in its rows are weakly increasing from left to right when we equip the pairs (i, j) with the lexicographic order

$$(i, j) < (p, q) \quad \Longleftrightarrow \quad (i < p) \text{ or } (i = p \text{ and } j < q).$$

Example 7.2.1. None of the tableaux in Examples 7.1.1 or 7.1.2 have weakly increasing rows, but if we keep α and γ as in Example 7.1.1, then

1	1	1	3	4	4	5
2	2	3	4	5		
1	2	2	2	5	5	
3	4	4	5			
1	2	2	2	4	4	5
2	4					

is a tableau of shape α and type γ with weakly increasing rows. Further, if we keep $\underline{\gamma}$ as in Example 7.1.1, then

(1,1)	(1,2)	(1,2)	(3,1)	(4,1)	(4,2)	(5,2)
(2,2)	(2,3)	(3,1)	(4,1)	(5,1)		
(1,1)	(2,1)	(2,2)	(2,2)	(5,1)	(5,3)	
(3,1)	(4,1)	(4,2)	(5,2)			
(1,2)	(2,1)	(2,2)	(2,3)	(4,2)	(4,2)	(5,2)
(2,1)	(4,1)					

is a tableau of shape α and type $\underline{\gamma}$ with weakly increasing rows.

We now seek a condition on $\sigma \in S_n$ which ensures that the tableaux $\tau_{\gamma}^{\alpha}\sigma$ and $\tau_{\underline{\gamma}}^{\alpha}\sigma$ have weakly increasing rows. To do this, we recall from page 45 the notion of the *length* of a permutation, which is defined to be the total number of inversions of the permutation, where an *inversion* of a permutation $\sigma \in S_n$ is a pair (i, j) such that $1 \leq i < j \leq n$ and $(i)\sigma > (j)\sigma$.

We shall prove that if $\sigma \in S_n$ is of minimal length in its S_α-coset σS_α, then the tableaux $\tau_{\gamma}^{\alpha}\sigma$ and $\tau_{\underline{\gamma}}^{\alpha}\sigma$ have weakly increasing rows. For this, we shall need a well-known combinatorial fact. We define a **descent** of σ to be an inversion $(j, j+1)$ of σ for some $1 \leq j < n$.

Lemma 7.2.2. *Let $\sigma \in S_n$, and suppose that $(j, j+1)$ is a descent of σ^{-1}. Then* $\operatorname{len}\big(\sigma(j, j+1)\big) = \operatorname{len}(\sigma) - 1$.

Proof. ([34], Lemma 2.1) We establish the claim by proving the following two properties for any $\theta \in S_n$

1. $\mathrm{len}(\theta) = \mathrm{len}(\theta^{-1})$

2. if $(j, j+1)$ is a descent of θ, then $\mathrm{len}\big((j, j+1)\theta\big) = \mathrm{len}(\theta) - 1$.

For the first property, we have that

$$\begin{aligned}(x, y) \text{ is an inversion of } \theta \quad &\Longleftrightarrow \quad x < y \text{ and } (x)\theta > (y)\theta \\ &\Longleftrightarrow \quad (x\theta)\theta^{-1} < (y\theta)\theta^{-1} \text{ and } (x)\theta > (y)\theta \\ &\Longleftrightarrow \quad ((y)\theta, (x)\theta) \text{ is an inversion of } \theta^{-1}\end{aligned}$$

and clearly the map $(x, y) \longmapsto ((y)\theta, (x)\theta)$ is a bijection from $\{1, \ldots, n\} \times \{1, \ldots, n\}$ to itself. Hence the inversions of θ and θ^{-1} are in bijection, so that $\mathrm{len}(\theta) = \mathrm{len}(\theta^{-1})$.

For the second property, we have trivially for any $x, y \in \{1, \ldots, n\}$ that

$$\begin{aligned}(x)\theta > (y)\theta &\Longleftrightarrow (x)\Big((j, j+1)(j, j+1)\theta\Big) > (y)\Big((j, j+1)(j, j+1)\theta\Big) \\ &\Longleftrightarrow \big((x)(j, j+1)\big)\big((j, j+1)\theta\big) > \big((y)(j, j+1)\big)\big((j, j+1)\theta\big).\end{aligned}$$

From this we may easily see that if $x < y$ and the pair (x, y) does not equal the pair $(j, j+1)$, then $(x)(j, j+1) < (y)(j, j+1)$. Moreover, (x, y) is then an inversion of θ if and only if $\Big((x)(j, j+1),\ (y)(j, j+1)\Big)$ is an inversion of $(j, j+1)\theta$. Further, the pair $(j, j+1)$ is by assumption a descent of θ but is not a descent of $(j, j+1)\theta$, and the second property is now established.

The proof of the claim is now trivial: we have

$$\begin{aligned}\mathrm{len}\big(\sigma(j, j+1)\big) &= \mathrm{len}\big(((j, j+1)\sigma^{-1})^{-1}\big) \\ &= \mathrm{len}\big((j, j+1)\sigma^{-1}\big) \\ &= \mathrm{len}\big(\sigma^{-1}\big) - 1 \\ &= \mathrm{len}(\sigma) - 1.\end{aligned}$$

$\square$

Proposition 7.2.3. *(Compare [34], Proposition 5.2 and Theorem 4.1) If $\sigma \in S_n$ is of minimal length in its left S_α-coset σS_α, then both $\tau^\alpha_\gamma \sigma$ and $\tau^\alpha_{\underline{\gamma}} \sigma$ have weakly increasing rows.*

Proof. The proofs for $\tau^\alpha_\gamma \sigma$ and $\tau^\alpha_{\underline{\gamma}} \sigma$ are identical (indeed, γ and $\underline{\gamma}$ play no role in the argument), and hence we give both in parallel by writing τ^α to represent either τ^α_γ or $\tau^\alpha_{\underline{\gamma}}$.

Suppose that $\tau^\alpha \sigma$ does not have weakly increasing rows. Indeed, suppose that the i^{th} row of $\tau^\alpha \sigma$ is not weakly increasing, and let us define

$$a = 1 + \sum_{j=1}^{i-1} \alpha_i$$

and

$$b = \sum_{j=1}^{i} \alpha_i$$

so that (with our numbering of the boxes of a Young diagram as above) the boxes on the i^{th} row of $\tau^\alpha \sigma$ are numbered from a to b. Thus the numbering of the boxes on the i^{th} row of $\tau^\alpha \sigma$ looks like

$$\begin{array}{|c|c|}\hline a & a+1 \\ & \\ \hline \end{array} \cdots\cdots \begin{array}{|c|c|}\hline b-1 & b \\ & \\ \hline \end{array}\,.$$

The fact that the i^{th} row of $\tau^\alpha \sigma$ is not weakly increasing means that we have some (p, q) with $a \leq p < q \leq b$ such that the entry in the box of $\tau^\alpha \sigma$ with number p is greater (in the appropriate ordering) than the entry in the box of $\tau^\alpha \sigma$ with number q. Now by the definition of the action of S_n on tableaux, we have for any j that the entry which is in box number j in $\tau^\alpha \sigma$ is the entry from box number $(j)\sigma^{-1}$ in τ^α. By the definition of τ^α, if $i < j$ then the entry in the box of τ^α with number i is less (in the appropriate ordering) than the entry in the box of τ^α with number j. Hence we must have $(p)\sigma^{-1} > (q)\sigma^{-1}$, and so (p, q) is an inversion of σ^{-1}. This implies that there must be a *descent* $(j, j+1)$ of σ^{-1} such that $a \leq j < b$, for if not then we must have

$$(a)\sigma^{-1} < (a+1)\sigma^{-1} < \cdots < (b-1)\sigma^{-1} < (b)\sigma^{-1},$$

a contradiction. But then $\sigma(j, j+1) \in \sigma S_\alpha$ since $(j, j+1) \in S_\alpha$, and by Lemma 7.2.2, $\sigma(j, j+1)$ has length one less than σ, contradicting the minimality of the length of σ in σS_α. $\square$

We now demonstrate how tableaux with weakly increasing rows can be used to index double cosets. Let us define $\mathcal{W}^\alpha_\gamma$ to be the set of all tableaux of shape α and type γ with weakly increasing rows, and $\mathcal{W}^\alpha_{\underline{\gamma}}$ to be the set of all tableaux of shape α and type $\underline{\gamma}$ with weakly increasing rows. Further, let us take Ω^α_γ to be a complete system of (S_γ, S_α)-double coset representatives in S_n and $\Omega^\alpha_{\underline{\gamma}}$ to be a complete system of $(S_{\underline{\gamma}}, S_\alpha)$-double coset representatives in S_n, where each element σ of Ω^α_γ or $\Omega^\alpha_{\underline{\gamma}}$ is of minimal length in its left coset σS_α.

Proposition 7.2.4. *([34], Corollary 5.1) The maps*

$$\begin{aligned} f_\gamma : \Omega^\alpha_\gamma &\longrightarrow \mathcal{W}^\alpha_\gamma \\ \sigma &\longmapsto \tau^\alpha_\gamma \sigma \end{aligned}$$

and

$$\begin{aligned} f_{\underline{\gamma}} : \Omega^\alpha_{\underline{\gamma}} &\longrightarrow \mathcal{W}^\alpha_{\underline{\gamma}} \\ \sigma &\longmapsto \tau^\alpha_{\underline{\gamma}} \sigma \end{aligned}$$

are bijections.

Proof. To prove that f_γ is onto, let τ be an element of $\mathcal{W}^\alpha_\gamma$. Then certainly $\tau = \tau^\alpha_\gamma \theta$ for some $\theta \in S_n$, since our action of S_n on tableaux is transitive. But $\theta = u\sigma v$ for some $\sigma \in \Omega^\alpha_\gamma$, $u \in S_\gamma$, $v \in S_\alpha$, so that $\tau = \tau^\alpha_\gamma u\sigma v$. Now by

Proposition 7.1.4, the stabilizer of τ_γ^α under the action of S_n is S_γ, and so $\tau = \tau_\gamma^\alpha \sigma \upsilon$. Hence $\tau \upsilon^{-1} = \tau_\gamma^\alpha \sigma$. But σ is certainly of minimal length in its left S_α-coset, and hence by Proposition 7.2.3 $\tau_\gamma^\alpha \sigma$ has weakly increasing rows, so $\tau \upsilon^{-1}$ has weakly increasing rows. But $\upsilon^{-1} \in S_\alpha$, and so the action of υ^{-1} on τ just permutes the elements within each row of τ. The fact that $\tau \upsilon^{-1}$ and τ both have weakly increasing rows now implies that $\tau = \tau \upsilon^{-1}$ and thus that $\tau = \tau_\gamma^\alpha \sigma$. Hence f_γ is onto.

To see that f_γ is one-to-one, suppose that $\tau_\gamma^\alpha \sigma_1 = \tau_\gamma^\alpha \sigma_2$ for $\sigma_1, \sigma_2 \in \Omega_\gamma^\alpha$. Thus $\tau_\gamma^\alpha \sigma_1 \sigma_2^{-1} = \tau_\gamma^\alpha$ and hence by Proposition 7.1.4 $\sigma_1 \sigma_2^{-1} \in S_\gamma$. It now follows at once that $S_\gamma \sigma_1 S_\alpha = S_\gamma \sigma_2 S_\alpha$ and hence that $\sigma_1 = \sigma_2$. Thus f_γ is one-to-one.

The proof for $f_{\underline{\gamma}}$ works in exactly the same way, using the fact that by Proposition 7.1.4 $S_{\underline{\gamma}}$ is the stabilizer of $\tau_{\underline{\gamma}}^\alpha$ under the action of S_n. $\square$

Corollary 7.2.5. *Suppose that we have $\sigma_1, \ldots, \sigma_N \in S_n$ such that if $i \neq j$ then $\tau_\gamma^\alpha \sigma_i \neq \tau_\gamma^\alpha \sigma_j$ and further $\{\tau_\gamma^\alpha \sigma_i \mid 1 \leqslant i \leqslant N\} = \mathcal{W}_\gamma^\alpha$. Then $\sigma_1, \ldots, \sigma_N$ is a complete system of (S_γ, S_α)-double coset representatives in S_n without redundancy. Further, this corollary remains true if one replaces γ with $\underline{\gamma}$ throughout.*

Proof. With our system of (S_γ, S_α)-double coset representatives Ω_γ^α as above, we may by Proposition 7.2.4 list the distinct elements of Ω_γ^α as $\omega_1, \ldots, \omega_N$ such that $\tau_\gamma^\alpha \sigma_i = \tau_\gamma^\alpha \omega_i$. This implies that $\tau_\gamma^\alpha = \tau_\gamma^\alpha \omega_i \sigma_i^{-1}$, and hence that $\omega_i \sigma_i^{-1} \in \mathrm{Stab}(\tau_\gamma^\alpha)$, so that by Proposition 7.1.4 we have $\omega_i \sigma_i^{-1} \in S_\gamma$. Hence $S_\gamma \sigma_i S_\alpha = S_\gamma (\omega_i \sigma_i^{-1}) \sigma_i S_\alpha = S_\gamma \omega_i S_\alpha$, and so $\sigma_1, \ldots, \sigma_N$ is a complete system of (S_γ, S_α)-double coset representatives in S_n without redundancy.

The above argument remains valid if we simply replace γ with $\underline{\gamma}$ throughout.

$\square$

7.3 Tableaux and subgroups of S_n

As mentioned at the start of the chapter, now that we have identified our sets Ω^{α}_{γ} and $\Omega^{\alpha}_{\underline{\gamma}}$ of double coset representatives, we want to understand the subgroups $S_\alpha \cap \big(S_\gamma\big)^\sigma$ and $S_\alpha \cap \big(S_{\underline{\gamma}}\big)^\sigma$, where σ lies in Ω^{α}_{γ} or $\Omega^{\alpha}_{\underline{\gamma}}$, respectively.

Now by Proposition 7.1.4, $\mathrm{Stab}\big(\tau^{\alpha}_{\gamma}\sigma\big) = \big(S_\gamma\big)^\sigma$. Further, it is clear that $\mathrm{Stab}\big(\tau^{\alpha}_{\underline{\gamma}}\sigma\big)$ consists exactly of those elements of S_n which permute the equal entries in the tableau $\tau^{\alpha}_{\underline{\gamma}}\sigma$. Further, for *any* tableau of shape α, it is immediate that the Young subgroup S_α of S_n is exactly the set of elements of S_n which permute the entries within the rows of the tableau (i.e. that do not move any entries between rows). We may apply the same arguments to $\mathrm{Stab}\big(\tau^{\alpha}_{\underline{\gamma}}\sigma\big) = \big(S_{\underline{\gamma}}\big)^\sigma$. We thus obtain the following result.

Lemma 7.3.1. *([34], proof of Proposition 5.2) For any $\sigma \in S_n$, $S_\alpha \cap \big(S_\gamma\big)^\sigma$ is the set of all elements of S_n which permute the equal entries within each row of the tableau $\tau^{\alpha}_{\gamma}\sigma$, while $S_\alpha \cap \big(S_{\underline{\gamma}}\big)^\sigma$ is the set of all elements of S_n which permute the equal entries within each row of the tableau $\tau^{\alpha}_{\underline{\gamma}}\sigma$.*

Example 7.3.2. Let us take $n = 13$, $\underline{\gamma} = \big((2,1),(3),(2,2),(2),(1)\big)$, $\alpha = (4,3,4,2)$ and $\sigma = (1,11,13,12,5,2,8,3)(9,10)$. Note that

$$S_\alpha = S_{\{1,2,3,4\}} \times S_{\{5,6,7\}} \times S_{\{8,9,10,11\}} \times S_{\{12,13\}}$$

and

$$S_{\underline{\gamma}} = S_{\{1,2\}} \times S_{\{3\}} \times S_{\{4,5,6\}} \times S_{\{7,8\}} \times S_{\{9,10\}} \times S_{\{11,12\}} \times S_{\{13\}}.$$

We have

$$\tau^{\alpha}_{\underline{\gamma}} \quad = \quad \begin{array}{|c|c|c|c|}
\hline
(1,1) & (1,1) & (1,2) & (2,1) \\
\hline
(2,1) & (2,1) & (3,1) \\
\cline{1-4}
(3,1) & (3,2) & (3,2) & (4,1) \\
\hline
(4,1) & (5,1) \\
\cline{1-2}
\end{array}$$

and

$$\tau^{\alpha}_{\underline{\gamma}}\sigma \quad = \quad \begin{array}{|c|c|c|c|}
\hline
(1,2) & (2,1) & (3,1) & (2,1) \\
\hline
(4,1) & (2,1) & (3,1) \\
\cline{1-4}
(1,1) & (3,2) & (3,2) & (1,1) \\
\hline
(5,1) & (4,1) \\
\cline{1-2}
\end{array}$$

so that

$$\left(S_{\underline{\gamma}}\right)^{\sigma} = S_{\{1\}} \times S_{\{2,4,6\}} \times S_{\{3,7\}} \times S_{\{5,13\}} \times S_{\{8,11\}} \times S_{\{9,10\}} \times S_{\{12\}}$$

and

$$S_{\alpha} \cap \left(S_{\underline{\gamma}}\right)^{\sigma} = S_{\{2,4\}} \times S_{\{8,11\}} \times S_{\{9,10\}}.$$

Proposition 7.3.3. *([34], Proposition 5.2) Suppose that $\sigma \in S_n$ is of minimal length in its left S_α-coset σS_α. Then $S_\alpha \cap \left(S_\gamma\right)^{\sigma}$ and $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^{\sigma}$ are Young subgroups of S_n.*

Proof. By Proposition 7.2.3, $\tau^{\alpha}_{\gamma}\sigma$ has weakly increasing rows, and so within each row the equal entries occur in contiguous blocks. Since by Lemma 7.3.1 $S_\alpha \cap \left(S_\gamma\right)^{\sigma}$ is the set of all elements of S_n which permute the equal entries in each row of $\tau^{\alpha}_{\gamma}\sigma$, $S_\alpha \cap \left(S_\gamma\right)^{\sigma}$ is indeed a Young subgroup. An identical argument works for $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^{\sigma}$. $\square$

So note that the σ in Example 7.3.2 is not of minimal length in its left S_α-coset, since the subgroup $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma$ is not a Young subgroup of S_n.

In fact if $\sigma \in S_n$ is of minimal length in σS_α, we may use the tableaux τ_γ^α and $\tau_{\underline{\gamma}}^\alpha$ to read off compositions ϵ, δ of n such that $S_\alpha \cap \left(S_\gamma\right)^\sigma = S_\delta$ and $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma = S_\epsilon$, as the following example demonstrates.

Example 7.3.4. Take $n = 16$ and

$$\begin{aligned}
\alpha &= (5,4,5,2)\\
\underline{\gamma} &= \big((3,4),(4,1),(1),(3)\big)\\
\sigma &= (1,6,2,15,5,10,16,14,13,9,8,11,12,4)(3,7).
\end{aligned}$$

Then σ is of minimal length in σS_α (see below for a justification of this) and we have by direct calculation that

$$\tau_{\underline{\gamma}}^\alpha \sigma = \begin{array}{|c|c|c|c|c|}
\hline
(1,2) & (1,2) & (1,2) & (2,2) & (4,1)\\
\hline
(1,1) & (1,1) & (2,1) & (3,1) & \\
\hline
(1,2) & (2,1) & (2,1) & (4,1) & (4,1)\\
\hline
(1,1) & (2,1) & & & \\
\hline
\end{array}$$

so that $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma$ is the Young subgroup

$$S_{\{1,2,3\}} \times S_{\{6,7\}} \times S_{\{11,12\}} \times S_{\{13,14\}}$$

associated to the composition $\Gamma = (3,1,1,2,1,1,1,2,2,1,1) \vDash 16$ and so $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma = S_\Gamma$.

Now it is not immediately clear that the σ in this example is indeed of minimal length in its coset σS_α. However, there is an easy way of seeing this.

Indeed, we find by direct calculation that $\tau^{\alpha}_{(1^{16})}\sigma$ is

$$\begin{array}{|c|c|c|c|c|}\hline 4 & 6 & 7 & 12 & 15 \\ \hline 1 & 3 & 9 & 13 & \multicolumn{1}{c}{} \\ \hline 5 & 8 & 11 & 14 & 16 \\ \hline 2 & 10 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

and we see that this tableau has weakly (indeed, strictly) increasing rows. Now choose $\theta \in S_\alpha$ such that $\sigma\theta$ is of minimal length in σS_α. By Proposition 7.2.3, $\tau^{\alpha}_{(1^{16})}\sigma\theta$ has weakly increasing rows. But $\theta \in S_\alpha$, so the action of θ on any tableau of shape α is to permute the entries within each row, and since $\tau^{\alpha}_{(1^{16})}\sigma\theta$ and $\tau^{\alpha}_{(1^{16})}\sigma$ both have weakly increasing rows, it follows that they are equal. But tableaux of type (1^{16}) have distinct entries, and so we immediately see that $\tau^{\alpha}_{(1^{16})}\epsilon = \tau^{\alpha}_{(1^{16})}\delta$ implies $\epsilon = \delta$ for any $\epsilon, \delta \in S_n$. Thus $\sigma = \sigma\theta$ and so σ is of minimal length as claimed.

7.4 The tuple of multicompositions associated to a tableau

In Example 7.3.4, we have seen how $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma$ may be characterised as the Young subgroup associated to a composition of n, provided that σ is of minimal length in σS_α. We now show how we can characterise $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma$ in a slightly different way, namely as the Young subgroup associated to an l-tuple of length t multicompositions with total size n. This characterisation will prove critical to our work on the structure of the spaces $\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ in Chapter 9.

Recall that we have fixed a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ of n and a t component multicomposition of n, $\underline{\gamma} = (\gamma^1, \gamma^2, \ldots, \gamma^t)$.

First, we explain how obtain an l-tuple of multicompositions of length t from a tableau τ of shape α and type $\underline{\gamma}$. We shall illustrate this process with an example once we have given the definition. Indeed, for each i, j, s let $\Gamma^{i,j}_s(\tau)$ be the number of times the pair (j, s) occurs on the i^{th} row of τ (so $\Gamma^{i,j}_s(\tau)$ is an integer). We then define for each $i \in \{1, \dots, l\}$ and each $j \in \{1, \dots, t\}$ a composition

$$\Gamma^{i,j}(\tau) = \left(\Gamma^{i,j}_1(\tau), \Gamma^{i,j}_2(\tau), \dots, \Gamma^{i,j}_p(\tau)\right)$$

where p is the highest integer such that a pair (j, p) occurs on the i^{th} row of τ. If there are no pairs (j, s) for any s on the i^{th} row, then $\Gamma^{i,j}(\tau) = ()$. Thus $\Gamma^{i,j}(\tau)$ records how many of each pair (j, s) occur on the i^{th} row of τ for different integers s. We then define for each $i \in \{1, \dots, l\}$ a length t multicomposition

$$\underline{\Gamma}^i(\tau) = \left(\Gamma^{i,1}(\tau), \Gamma^{i,2}(\tau), \dots, \Gamma^{i,t}(\tau)\right).$$

Thus $\underline{\Gamma}^i(\tau)$ records how many of each pair (j, s) occur on the i^{th} row of τ for different integers j and s. Finally, we define an l-tuple of t-multicompositions

$$\underline{\underline{\Gamma}}(\tau) = \left(\underline{\Gamma}^1(\tau), \underline{\Gamma}^2(\tau), \dots, \underline{\Gamma}^l(\tau)\right).$$

We call $\underline{\underline{\Gamma}}(\tau)$ the **tuple of multicompositions associated to** τ.

Example 7.4.1. Let us take

$$\begin{aligned}
n &= 20 \\
\alpha &= (5, 4, 5, 2, 4) \\
\underline{\gamma} &= \left((3, 4, 1), (4, 3), (1, 1), (3)\right)
\end{aligned}$$

and thus in this case $l = 5$ and $t = 4$. Then we let τ be the tableau of shape α and type $\underline{\gamma}$ given by

$$\tau = \begin{array}{|c|c|c|c|c|} \hline (2,2) & (4,1) & (1,2) & (1,2) & (2,2) \\ \hline (1,1) & (3,2) & (2,1) & (1,1) \\ \cline{1-5} (1,2) & (4,1) & (2,1) & (4,1) & (2,1) \\ \hline (3,1) & (2,1) \\ \cline{1-4} (1,3) & (2,2) & (1,1) & (1,2) \\ \cline{1-4} \end{array} \quad .$$

We can now read off $\underline{\underline{\Gamma}}(\tau)$. Indeed, from the first row we have

$$\underline{\Gamma}^1(\tau) = \big((0,2),(0,2),(),(1)\big)$$

and similarly from the second, third, fourth, and fifth rows, we have

$$\begin{aligned} \underline{\Gamma}^2(\tau) &= \big((2),(1),(0,1),()\big) \\ \underline{\Gamma}^3(\tau) &= \big((0,1),(2),(),(2)\big) \\ \underline{\Gamma}^4(\tau) &= \big((),(1),(1),()\big) \\ \underline{\Gamma}^5(\tau) &= \big((1,1,1),(0,1),(),()\big) \end{aligned}$$

and so

$$\begin{aligned} \underline{\underline{\Gamma}}(\tau) =& (\underline{\Gamma}^1, \underline{\Gamma}^2, \underline{\Gamma}^3, \underline{\Gamma}^4, \underline{\Gamma}^5) \\ =& \Big(\big((0,2),(0,2),(),(1)\big), \\ & \big((2),(1),(0,1),()\big), \\ & \big((0,1),(2),(),(2)\big), \\ & \big((),(1),(1),()\big), \\ & \big((1,1,1),(0,1),(),()\big)\Big). \end{aligned}$$

Proposition 7.4.2. *If $\sigma \in S_n$ is of minimal length in its left S_α-coset σS_α, then we have*

$$S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma = S_{\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^\alpha \sigma)}.$$

Proof. By Lemma 7.3.1, $S_\alpha \cap \left(S_{\underline{\gamma}}\right)^\sigma$ is the subgroup of S_n consisting of all permutations which permute the equal elements in each row of the tableau $\tau_{\underline{\gamma}}^\alpha \sigma$. We see by the definition of $\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^\alpha \sigma)$ that if $\tau_{\underline{\gamma}}^\alpha \sigma$ has weakly increasing rows, then $S_{\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^\alpha \sigma)}$ is also the set of all permutations which permute the equal elements in each row of the tableau $\tau_{\underline{\gamma}}^\alpha \sigma$. But Proposition 7.3.3 tells us that $\tau_{\underline{\gamma}}^\alpha \sigma$ does indeed have weakly increasing rows, and so the proposition follows. $\square$

Example 7.4.3. Let us return to the α, $\underline{\gamma}$, σ of Example 7.3.4, so that

$$\begin{aligned}
n &= 16 \\
\alpha &= (5,4,5,2) \\
\underline{\gamma} &= \big((3,4),(4,1),(1),(3)\big) \\
\sigma &= (1,6,2,15,5,10,16,14,13,9,8,11,12,4)(3,7)
\end{aligned}$$

and thus in this case $l = t = 4$. Recall from Example 7.3.4 that this σ is of minimal length in its left S_α-coset σS_α. Recall also that

$$\tau_{\underline{\gamma}}^\alpha \sigma = \begin{array}{|c|c|c|c|c|}
\hline
(1,2) & (1,2) & (1,2) & (2,2) & (4,1) \\
\hline
(1,1) & (1,1) & (2,1) & (3,1) & \multicolumn{1}{c}{} \\
\hline
(1,2) & (2,1) & (2,1) & (4,1) & (4,1) \\
\hline
(1,1) & (2,1) & \multicolumn{3}{c}{} \\
\cline{1-2}
\end{array}$$

We can now read off $\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^\alpha \sigma)$ from this tableau. Indeed, from the first row we have

$$\underline{\Gamma}^1 = \big((0,3),(0,1),(),(1)\big)$$

and similarly from the second, third, and fourth rows, we have

$$\begin{aligned}
\underline{\Gamma}^2 &= \big((2),(1),(1),()\big)\\
\underline{\Gamma}^3 &= \big((0,1),(2),(),(2)\big)\\
\underline{\Gamma}^4 &= \big((1),(1),(),()\big).
\end{aligned}$$

Thus

$$\begin{aligned}
\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{\alpha}\sigma) =& (\underline{\Gamma}^1,\underline{\Gamma}^2,\underline{\Gamma}^3,\underline{\Gamma}^4)\\
=& \Big(\big((0,3),(0,1),(),(1)\big),\\
& \big((2),(1),(1),()\big),\\
& \big((0,1),(2),(),(2)\big),\\
& \big((1),(1),(),()\big)\Big).
\end{aligned}$$

We conclude this chapter with a combinatorial result which will be vital to our work in Chapter 9 on the structure of the spaces $\mathrm{Hom}_{m\wr n}(S^{\underline{\nu}}, M^{\underline{\gamma}})$ and $\mathrm{Ext}^1_{m\wr n}(S^{\underline{\nu}}, M^{\underline{\gamma}})$.

Proposition 7.4.4. *Let $\underline{\gamma}$ and $\underline{\nu}$ be multicompositions of n of length t, and let τ be a tableau of shape $|\underline{\nu}|$ and type $\underline{\gamma}$.*

1. *Let $i, j \in \{1, \ldots, t\}$. Then $\Gamma^{i,j}(\tau) \neq ()$ if and only if some pair $(j, *)$ appears on the i^{th} row of τ.*

2. *Suppose that $\underline{\nu} \not\trianglerighteq \underline{\gamma}$. Suppose further that for each $j \in \{1, \ldots, t\}$, no pair $(j, *)$ occurs lower than the j^{th} row of τ, where by "lower" we mean further down the tableau, i.e. in the l^{th} row for some $l > j$. Then we*

have some i and some j such that the first $i-1$ entries of $\underline{\Gamma}^i$ are $()$, so that we have

$$\underline{\Gamma}^i(\tau) = \big((), (), \ldots, (), \Gamma^{i,i}(\tau), \Gamma^{i,i+1}(\tau), \ldots, \Gamma^{i,t}(\tau)\big)\,,$$

and such that we also have

$$\sum_{q=1}^{j} \Gamma^{i,i}_q(\tau) > \sum_{q=1}^{j} \nu^i_q.$$

Proof. The first claim of the proposition is simply a restatement of part of the definition of the composition $\Gamma^{i,j}(\tau)$, so all we need to do is prove the second claim.

Since $\underline{\nu} \not\trianglerighteq \underline{\gamma}$, we have i and j such that

$$\sum_{p=1}^{i-1} |\gamma^p| + \sum_{q=1}^{j} \gamma^i_q > \sum_{p=1}^{i-1} |\nu^p| + \sum_{q=1}^{j} \nu^i_q. \tag{7.4.1}$$

The fact that the first $i-1$ entries of $\underline{\Gamma}^i$ are $()$ follows from the first part of the proposition and our assumptions, and so all that remains is to prove the final inequality.

Now there are $\sum_{p=1}^{i-1} |\nu^p|$ boxes on the first $i-1$ rows of τ (since τ has shape $|\underline{\nu}|$). Also, τ is of type $\underline{\gamma}$, and hence the number of pairs $(p,*)$ for $p<i$ occurring in τ is $\sum_{p=1}^{i-1} |\gamma^p|$. By our assumption that no pair $(j,*)$ occurs lower than the j^{th} row of τ, these pairs must all occur on the first $i-1$ rows. Thus we see that there can be at most

$$\sum_{p=1}^{i-1} |\nu^p| - \sum_{p=1}^{i-1} |\gamma^p|$$

pairs $(i,*)$ on the first $i-1$ rows of τ, and hence in particular at most this many pairs (i,q) for $q \le j$ on the first $i-1$ rows of τ. But τ is of type $\underline{\gamma}$, and this means in particular that the total number of pairs (i,q) for $q \le j$

occurring in the tableau τ is $\sum_{q=1}^{j} \gamma_q^i$. By our assumption that no pair $(j,*)$ occurs lower than the j^{th} row of τ, all of these pairs must occur within the first i rows of τ. Thus at least

$$\left(\sum_{q=1}^{j} \gamma_q^i\right) - \left(\sum_{p=1}^{i-1} |\nu^p| \ - \ \sum_{p=1}^{i-1} |\gamma^p|\right)$$

pairs (i,q) with $q \leq j$ occur on the i^{th} row of τ. By definition of $\underline{\underline{\Gamma}}(\tau)$, we see that the number of pairs (i,q) for $q \leq j$ occurring on the i^{th} row of τ is $\sum_{q=1}^{j} \Gamma_q^{i,i}(\tau)$, and thus we have

$$\begin{aligned}
\sum_{q=1}^{j} \Gamma_q^{i,i}(\tau) &\geq \left(\sum_{q=1}^{j} \gamma_q^i\right) - \left(\sum_{p=1}^{i-1} |\nu^p| \ - \ \sum_{p=1}^{i-1} |\gamma^p|\right) \\
&= \sum_{p=1}^{i-1} |\gamma^p| + \sum_{q=1}^{j} \gamma_q^i - \sum_{p=1}^{i-1} |\nu^p| \\
&> \sum_{q=1}^{j} \nu_q^i. \qquad \text{(by (7.4.1))}
\end{aligned}$$

as required. $\square$

Original research in Chapter 7: Most of the material in this chapter is taken from [34] with adaptations for use in subsequent chapters of this thesis, although the use of tableaux containing *pairs* of numbers and the associated wreath product action are original. The contents of Section 7.4, and in particular Proposition 7.4.4, are original research.

Chapter 8

Specht branching rules for the wreath product

A famous and fundamental result in the representation theory of the symmetric group is the result which Kleschchev in [22] calls the "Classical Branching Theorem" [22, Theorem 3.1], but which we shall call the "Specht branching rule". This gives a Specht filtration for the restriction of a Specht module from kS_n to kS_{n-1} with an elegant combinatorial description of the set of Specht modules occurring in this filtration. Indeed, recall that we have for each $n > 0$ a natural embedding of the symmetric group S_{n-1} into S_n by letting $\sigma \in S_{n-1}$ act on $1, \ldots, n$ by fixing n and permuting the other elements as it does in S_{n-1}. Thus we can regard S_{n-1} as a subgroup of S_n and hence we may induce a module X from kS_{n-1} to kS_n, or restrict a module Y from kS_n to kS_{n-1}. In keeping with notations we have already introduced, we shall write these operations as

$$X{\uparrow}_{n-1}^{n} \text{ and } Y{\downarrow}_{n-1}^{n}.$$

Theorem 8.0.1. *(Specht branching rule) ([20], Theorem 9.3) Let $\lambda \vdash n$ where*

$n > 0$, and let k be a field. Then the $kS_{(n-1)}$-module $S^{\lambda}\big\downarrow^{n}_{n-1}$ has a Specht filtration where for $\nu \vdash (n-1)$, S^{ν} occurs exactly once if the Young diagram of ν can be obtained from the Young diagram of λ by removing a single box, and S^{ν} does not occur otherwise.

Recall that if kS_n is semisimple (which occurs if and only if $\mathrm{char}(k) = 0$ or $\mathrm{char}(k) > n$), the Specht modules (indexed by all partitions of n) form a complete system of simple kS_n-modules without redundancy. Hence in this semisimple case, Theorem 8.0.1 describes the composition series of the restriction of a simple module from S_n to S_{n-1} in the chain of nested groups $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$, and thus is a "branching rule" in the more usual sense of this term.

In this chapter, we shall produce analogues of the Specht branching rule for the wreath product $S_m \wr S_n$ of two symmetric groups. Now since our wreath product has *two* parameters m and n, we have two branching rules to investigate. Firstly, we can embed $S_{m-1} \wr S_n$ into $S_m \wr S_n$ using the canonical embedding of S_{m-1} into S_m, thus identifying $S_{m-1} \wr S_n$ with the subgroup of $S_m \wr S_n$ consisting of all elements $(\sigma; \alpha_1, \dots, \alpha_n)$ where $\sigma \in S_n$ and each α_i is an element of the subgroup S_{m-1} of S_m. Hence we can consider the restriction

$$S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1) \wr n}$$

of a Specht module $S^{\underline{\lambda}}$ from $k(S_m \wr S_n)$ to $k(S_{m-1} \wr S_n)$. Secondly, we can embed $S_m \wr S_{n-1}$ into $S_m \wr S_n$ via the mapping

$$(\sigma; \alpha_1, \dots, \alpha_{n-1}) \longmapsto (\sigma; \alpha_1, \dots, \alpha_{n-1}, e)$$

where $\sigma \in S_{n-1}$, $\alpha_i \in S_m$, and e represents the identity element of S_m (note that we are making use of the canonical embedding of S_{n-1} into S_n). We thus have the restriction

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{m\wr(n-1)}$$

of a Specht module $S^{\underline{\lambda}}$ from $k(S_m \wr S_n)$ to this copy of $k(S_m \wr S_{n-1})$ inside $k(S_m \wr S_n)$. We shall see below that it is the Specht branching rule for the latter restriction which is most closely analogous to the Specht branching rule for the symmetric group.

As in previous chapters, we let the distinct partitions of m, in the lexicographic order, be

$$(m) = \mu^1 > \mu^2 > \ldots > \mu^r = (1^m).$$

8.1 Specht branching rule for $S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n}$

Let us fix some r-multipartition $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of n and consider the $k(S_{m-1} \wr S_n)$-module

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n}.$$

Let us define a $k(S_m \wr S_{|\underline{\lambda}|})$-module

$$T^{\underline{\lambda}} = \left(S^{\mu^1}, \ldots, S^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\lambda}|} \oslash \left(S^{\lambda^1} \boxtimes \cdots \boxtimes S^{\lambda^r}\right)$$

so that

$$S^{\underline{\lambda}} = T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}.$$

We then have

$$\begin{aligned} S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n} &\cong T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{(m-1)\wr n} \\ &\cong \bigoplus_{u\in\mathcal{U}} \left(T^{\underline{\lambda}}\right)^u \big\downarrow^{(m\wr|\underline{\lambda}|)^u}_{(m\wr|\underline{\lambda}|)^u \cap (m-1)\wr n} \big\uparrow^{(m-1)\wr n}_{(m\wr|\underline{\lambda}|)^u \cap (m-1)\wr n} \end{aligned} \tag{8.1.1}$$

where the last isomorphism follows by Mackey's Theorem (Theorem 2.2.5), with $\mathcal{U}$ representing a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_{(m-1)} \wr S_n)$-double coset representatives in $S_m \wr S_n$, and where we allow ourselves a slight abuse of notation by writing $(m \wr |\underline{\lambda}|)^u$ to represent the subgroup $(S_m \wr S_{|\underline{\lambda}|})^u$ conjugate to $S_m \wr S_{|\underline{\lambda}|}$ by u, and $(m \wr |\underline{\lambda}|)^u \cap (m-1) \wr n$ for the intersection of this subgroup with $S_{(m-1)} \wr S_n$. But it turns out that in fact the group $S_m \wr S_n$ is a single $(S_m \wr S_{|\underline{\lambda}|}, S_{(m-1)} \wr S_n)$-double coset. Indeed, choosing $(\sigma; \alpha_1, \ldots, \alpha_n) \in S_m \wr S_n$, we have equalities of double cosets

$$\begin{aligned}
&S_m \wr S_{|\underline{\lambda}|}\, (\sigma; \alpha_1, \ldots, \alpha_n)\, S_{(m-1)} \wr S_n \\
&\qquad = S_m \wr S_{|\underline{\lambda}|}\, (e; \alpha_{(1)\sigma}, \ldots, \alpha_{(n)\sigma})(e; e, \ldots, e)(\sigma; e, \ldots, e)\, S_{(m-1)} \wr S_n \\
&\qquad = S_m \wr S_{|\underline{\lambda}|}\, (e; e, \ldots, e)\, S_{(m-1)} \wr S_n
\end{aligned}$$

and so we may take $\mathcal{U} = \{(e; e, \ldots, e)\}$. We then have by (8.1.1) that

$$S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1) \wr n} \cong T^{\underline{\lambda}}\big\downarrow^{m \wr |\underline{\lambda}|}_{m \wr |\underline{\lambda}| \cap (m-1) \wr n}\big\uparrow^{(m-1) \wr n}_{m \wr |\underline{\lambda}| \cap (m-1) \wr n}$$

and clearly $\big(S_m \wr S_{|\underline{\lambda}|}\big) \cap \big(S_{(m-1)} \wr S_n\big) = S_{(m-1)} \wr S_{|\underline{\lambda}|}$ (note that formally these are subgroups of $S_m \wr S_n$, so that $S_{(m-1)} \wr S_{|\underline{\lambda}|}$ is the subgroup of $S_m \wr S_n$ consisting of all elements $(\sigma; \alpha_1, \ldots, \alpha_n)$ for $\sigma \in S_{|\underline{\lambda}|}$ and $\alpha_i \in S_{(m-1)} \leqslant S_m$). Thus we have

$$S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1) \wr n} \cong T^{\underline{\lambda}}\big\downarrow^{m \wr |\underline{\lambda}|}_{(m-1) \wr |\underline{\lambda}|}\big\uparrow^{(m-1) \wr n}_{(m-1) \wr |\underline{\lambda}|}.$$

We then have

$$
\begin{aligned}
T^{\underline{\lambda}}\Big\downarrow^{m\wr|\underline{\lambda}|}_{(m-1)\wr|\underline{\lambda}|}\Big\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} &\cong \left[\boxtimes_{i=1}^{r}\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|}\oslash S^{\lambda^i}\right]\Bigg\downarrow^{m\wr|\underline{\lambda}|}_{(m-1)\wr|\underline{\lambda}|}\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\cong \left[\boxtimes_{i=1}^{r}\left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|}\oslash S^{\lambda^i}\right]\Bigg\downarrow^{m\wr|\lambda^i|}_{(m-1)\wr|\lambda^i|}\right]\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\cong \left[\boxtimes_{i=1}^{r}\left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|}\Bigg\downarrow^{m\wr|\lambda^i|}_{(m-1)\wr|\lambda^i|}\right]\oslash S^{\lambda^i}\right]\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\qquad \text{(easy to see directly)} \\
&\cong \left[\boxtimes_{i=1}^{r}\left(S^{\mu^i}\Big\downarrow^{m}_{m-1}\right)^{\widetilde{\boxtimes}|\lambda^i|}\oslash S^{\lambda^i}\right]\Bigg\uparrow^{(m-1)\wr n}_{(m-1)\wr|\underline{\lambda}|} \\
&\qquad \text{(by Proposition 4.3.5)} \\
&\cong S^{\underline{\lambda}}\left(S^{\mu^1}\big\downarrow^{m}_{m-1},\dots,S^{\mu^r}\big\downarrow^{m}_{m-1}\right) \\
&\qquad \text{(using the isomorphism (4.3.6); see} \\
&\qquad\qquad \text{(6.4.10) for the definition of this notation).}
\end{aligned}
$$

We thus see that

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n} \cong S^{\underline{\lambda}}\left(S^{\mu^1}\big\downarrow^{m}_{m-1},\dots,S^{\mu^r}\big\downarrow^{m}_{m-1}\right).$$

Now let us fix the partitions of $m-1$ just as we have done for m. Indeed, let t be the number of distinct partitions of $m-1$, and let

$$(m-1)=\theta^1>\theta^2>\ \dots\ >\theta^t=(1^{m-1})$$

be the partitions of $m-1$ in lexicographic order. Then by Theorem 8.0.1, we have for any $i\in\{1,\dots,r\}$ that

$$S^{\mu^i}\Big\downarrow^{m}_{m-1}\sim\mathcal{F}_{j=1}^{t}a^i_jS^{\theta^j}$$

where

$$a^i_j=\begin{cases}1 & \text{if } \theta^j \text{ can be obtained by removing a box from } \mu^i\\ 0 & \text{otherwise.}\end{cases}$$

It now follows by Proposition 6.4.1 that we have a filtration

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{(m-1)\wr n} \sim \mathop{\mathcal{F}}_{\substack{\underline{\nu} \text{ is a } t\text{-multipartition} \\ \text{of } n}} \left[\sum_{[\underline{\epsilon}] \,\in\, \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}|\times|\underline{\nu}|)} \left(\prod_{i=1}^{r} c(\lambda^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]) \right) \right] S^{\underline{\nu}} \tag{8.1.2}$$

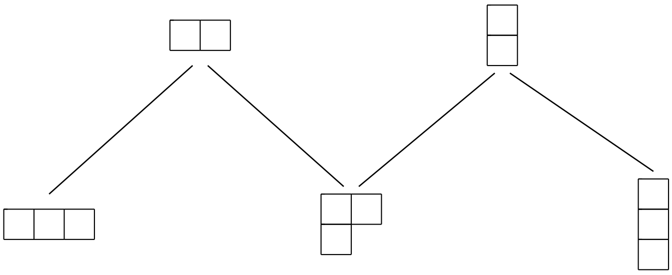
where A is the $r \times t$ integer matrix whose $(i,j)^{\text{th}}$ entry is a^i_j. This filtration is the basis of our desired Specht branching rule, but we would like some kind of combinatorial interpretation of the coefficient with which $S^{\underline{\nu}}$ occurs. Our task is now to find such an interpretation.

So with $\underline{\lambda}$ as above and $\underline{\nu}$ as in (8.1.2), consider, for a given multipartition matrix $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ the coefficient

$$\left(\prod_{i=1}^{r} c(\lambda^i; R_i[\underline{\epsilon}]) \right) \left(\prod_{j=1}^{t} c(\nu^j; C_j[\underline{\epsilon}]) \right) \tag{8.1.3}$$

occurring in (8.1.2). Recall that the length matrix A of $[\underline{\epsilon}]$ is defined to be the integer matrix whose $(i,j)^{\text{th}}$ entry is the length of the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$. Thus A is the integer matrix whose $(i,j)^{\text{th}}$ entry is 1 if θ^j can be obtained by removing a box from μ^i, and 0 otherwise. Thus the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$ is a multipartition of length 1, say (ϵ^{ij}), if θ^j can be obtained by removing a box from μ^i, and $()$ otherwise. This gives us an alternative way to think of such multipartition matrices and calculate the associated coefficient (8.1.3), as we shall now explain.

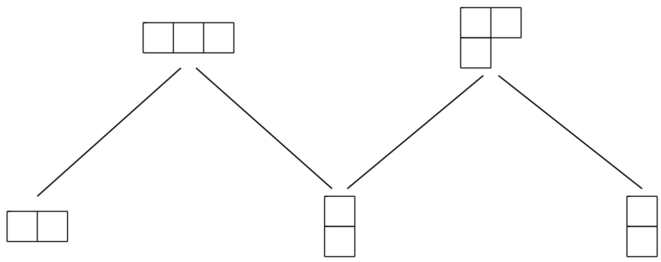
Recall that we can arrange the set of all partitions of all non-negative integers in a graphical structure called the **Young graph**, by arranging the partitions in layers, with the partitions of size s forming the s^{th} layer, and then for each partition $\lambda \vdash s$ in the s^{th} layer, drawing an edge from λ to each partition of $s-1$ in the $(s-1)^{\text{th}}$ layer which can be obtained from λ by removing a single box. For example, the second and third rows of the Young graph, together with the edges connecting them, look like this



. (8.1.4)

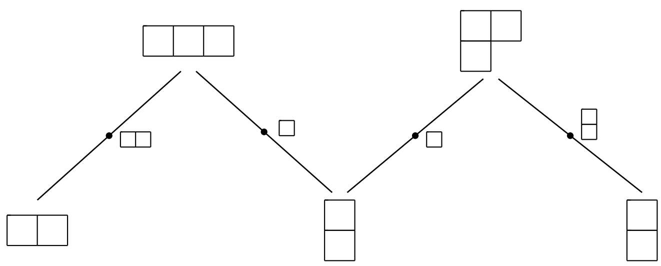
For our purposes, we are interested in the subgraph of the Young graph consisting of the m^{th} and $(m-1)^{\text{th}}$ layers together with the edges connecting them. Let us call this subgraph $\mathcal{Y}_m$. So for example if $m = 3$, $\mathcal{Y}_3$ is the graph (8.1.4). We see that there is a natural one-to-one correspondence between the 1's in the matrix A and the edges in $\mathcal{Y}_m$. Indeed, a 1 in the $(i,j)^{\text{th}}$ place of A corresponds to an edge linking $\theta^j \vdash m-1$ and $\mu^i \vdash m$ in $\mathcal{Y}_m$. We now see that a multipartition matrix $[\underline{\epsilon}] \in \mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ may be identified with a labelling of the edges in $\mathcal{Y}_m$ by partitions. Indeed, to obtain such a labelling from such a matrix $[\underline{\epsilon}]$, we label the edge linking θ^j and μ^i in $\mathcal{Y}_m$, if it exists, with the partition ϵ^{ij} which is the unique entry of the length 1 multipartiton which is the $(i,j)^{\text{th}}$ entry of $[\underline{\epsilon}]$. We may easily see that we have now established a one-to-one correspondence between on the one hand the set $\mathrm{Mat}_{\underline{\Lambda}}(A; |\underline{\lambda}| \times |\underline{\nu}|)$ and on the other hand labellings of the edges of $\mathcal{Y}_m$ by integer partitions, such that for each $i = 1, \ldots, r$ the sizes of the partitions labelling the edges touching the node $\mu^i \vdash m$ of $\mathcal{Y}_m$ add up to $|\lambda^i|$, and similarly for each $j = 1, \ldots, t$ the sizes of the partitions labelling the edges touching the node $\theta^j \vdash m-1$ of $\mathcal{Y}_m$ add up to $|\nu^i|$. We shall henceforth call such a labelling of $\mathcal{Y}_m$ a **labelling of shape** $|\underline{\lambda}| \times |\underline{\nu}|$. The diagram (8.1.6) below is an example of such a labelling.

We now explain how to calculate the coefficient (8.1.3) associated to a labelling of $\mathcal{Y}_m$ of shape $|\underline{\lambda}| \times |\underline{\nu}|$. In order to do this, we need to introduce a graph which is a modified version of $\mathcal{Y}_m$. Indeed, recall that we have multipartitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ and $\underline{\nu} = (\nu^1, \ldots, \nu^t)$ of n. We define $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ to be the graph obtained by replacing each partition $\mu^i \vdash m$ with λ^i, and each partition $\theta^j \vdash m-1$ with ν^j. Thus for example if $m = 3$ (so that $r = 3$ and $t = 2$) and $n = 6$, and we take $\underline{\lambda} = \big((2), (1,1), (1,1)\big)$ and $\underline{\nu} = \big((3), (2,1)\big)$, then $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$ is the graph



. (8.1.5)

We now see that a labelling of $\mathcal{Y}_m$ of shape $|\underline{\lambda}| \times |\underline{\nu}|$ corresponds to a labelling of the edges $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ by partitions in such a way that, for each partition γ lying at a node of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$, the sizes of the partitions labelling all the edges touching γ add up to $|\gamma|$. We call such a labelling of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ a **good labelling** of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$. To continue our example, one good labelling of the graph $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$ depicted in (8.1.5) is



. (8.1.6)

Looking back through our arguments, we see that this labelling corresponds to the multipartition matrix

$$\begin{array}{c c} & \begin{array}{cc} (3) & (2,1) \end{array} \\ \begin{array}{c} (2) \\ (1,1) \\ (1,1) \end{array} & \begin{bmatrix} \big((2)\big) & () \\ \big((1)\big) & \big((1)\big) \\ () & \big((1,1)\big) \end{bmatrix} \end{array}$$

(where we have labelled the rows and columns with the entries of $\underline{\lambda}$ and $\underline{\nu}$ respectively) and further we see that the coefficient (8.1.3) associated to this multipartition matrix is

$$c\Big((2); \big((2)\big)\Big) \cdot c\Big((1,1); \big((1),(1)\big)\Big) \cdot c\Big((1,1); \big((1,1)\big)\Big) \cdot \\ c\Big((3); \big((2),(1)\big)\Big) \cdot c\Big((2,1); \big((1),(1,1)\big)\Big).$$

By using (3.2.5) and (3.2.6), and by counting the appropriate kinds of skew tableaux as per the Littlewood-Richardson rule, we may see that each of these Littlewood-Richardson coefficients is 1, and hence the coefficient associated to the graph (8.1.6) is 1.

In the general case, we see that the coefficient associated to a good labelling of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ is formed by taking the product, over all partitions γ which are nodes of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ (that is, over all partitions of m and of $m-1$), of the Littlewood-Richardson coefficients $c\big(\gamma; (\delta^1, \ldots, \delta^s)\big)$, where $\delta^1, \ldots, \delta^s$ are the partitions labelling all of the edges which touch γ in $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$. If $\mathcal{L}$ is a good labelling of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$, we denote this coefficient by $\mathcal{M}\big(\mathcal{L}\big)$.

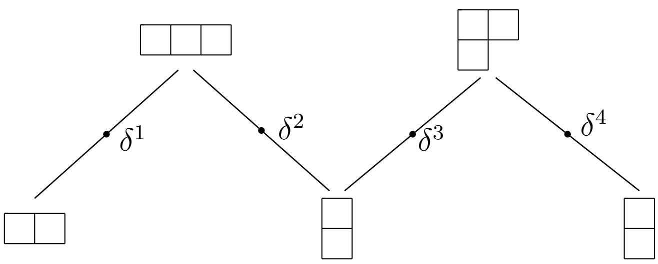
We have now proved the following Specht branching rule.

Theorem 8.1.1. *Let $m > 0$, and as above let r be the number of distinct partitions of m and t the number of distinct partitions of $m-1$. Let $\underline{\lambda}$*

be an r-multipartition of n. Then we have a filtration of the $k(S_{m-1} \wr S_n)$-module $S^{\underline{\lambda}}\big\downarrow^{m \wr n}_{(m-1)\wr n}$ by Specht modules $S^{\underline{\nu}}$ for t-multipartitions $\underline{\nu}$ of n, where the multiplicity of $S^{\underline{\nu}}$ is the sum over all good labellings $\mathcal{L}$ of $\mathcal{Y}_m(\underline{\lambda}, \underline{\nu})$ of the coefficients $\mathcal{M}\big(\mathcal{L}\big)$.

We note that the multiplicities in this theorem are independent of the field k.

Let us now extend our example to calculate the multiplicity with which $S^{((3),(2,1))}$ occurs in our filtration of $S^{((2),(1,1),(1,1))}\big\downarrow^{3\wr 6}_{2\wr 6}$. We have already calculated that the coefficient $\mathcal{M}\big(\mathcal{L}\big)$ is equal to 1 when $\mathcal{L}$ is the labelling (8.1.6). We shall show that if $\underline{\lambda} = \big((2),(1,1),(1,1)\big)$ and $\underline{\nu} = \big((3),(2,1)\big)$, then for any good labelling $\mathcal{L}$ of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$ other than (8.1.6), we have $\mathcal{M}\big(\mathcal{L}\big) = 0$. Thus the multiplicity which we seek is in fact 1. Indeed, suppose that we have some good labelling $\mathcal{L}$ of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$. Then $\mathcal{L}$ is equal to



for some integer partitions $\delta^1, \delta^2, \delta^3, \delta^4$. Now by the definition of a good labelling of $\mathcal{Y}_3(\underline{\lambda}, \underline{\nu})$, we see that we must have $|\delta^1| = 2, |\delta^2| = 1, |\delta^3| = 1, |\delta^4| = 2$, so that $\delta^2 = \delta^3 = (1)$. We now see that

$$\mathcal{M}\big(\mathcal{L}\big) = c\Big((2);\big(\delta^1\big)\Big) \cdot c\Big((1,1);\big((1),(1)\big)\Big) \cdot c\Big((1,1);\big(\delta^4\big)\Big) \cdot \\ c\Big((3);\big(\delta^1,(1)\big)\Big) \cdot c\Big((2,1);\big((1),\delta^4\big)\Big).$$

By (3.2.5), the only case where this is nonzero is the case where $\delta^1 = (2)$ and $\delta^4 = (1,1)$, as in (8.1.6).

8.2 Specht branching rule for $S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{m\wr(n-1)}$

Now let us take $n > 0$, and fix some r-multipartition $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$ of n and consider the $k(S_m \wr S_{n-1})$-module

$$S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{m\wr(n-1)}$$

where we recall that we are regarding S_{n-1} as the subgroup of S_n consisting of all permutations of $\{1, \ldots, n\}$ which fix n, and hence regarding $S_m \wr S_{n-1}$ as the subgroup of all elements of the form $(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)$ where $\sigma \in S_{n-1}$, $\alpha_i \in S_m$, and e represents the identity element of S_m. So as in the previous section we have

$$\begin{aligned} S^{\underline{\lambda}}\big\downarrow^{m\wr n}_{m\wr(n-1)} &\cong T^{\underline{\lambda}}\big\uparrow^{m\wr n}_{m\wr|\underline{\lambda}|}\big\downarrow^{m\wr n}_{m\wr(n-1)} \\ &\cong \bigoplus_{u\in\mathcal{U}} \left(T^{\underline{\lambda}}\right)^u \big\downarrow^{(m\wr|\underline{\lambda}|)^u}_{(m\wr|\underline{\lambda}|)^u \cap m\wr(n-1)} \big\uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^u \cap m\wr(n-1)} \end{aligned} \tag{8.2.1}$$

where again we have used Mackey's Theorem (Theorem 2.2.5) with minor notational abuses as above, and where $\mathcal{U}$ now represents a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$. We thus want to find such a set of double coset representatives. Indeed, recall that for $\sigma \in S_n$, we write $\hat{\sigma}$ for the element $(\sigma; e, \ldots, e)$ of $S_m \wr S_n$. Let $\sigma_1, \ldots, \sigma_N$ be a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n. We claim that $\hat{\sigma}_1, \ldots, \hat{\sigma}_N$ is then a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$. Indeed, if $(\theta; \alpha_1, \ldots, \alpha_n) \in S_m \wr S_n$, then we have $\theta = \epsilon\sigma_i\delta$ for some $i \in \{1, \ldots, N\}$, $\epsilon \in S_{|\underline{\lambda}|}$ and $\delta \in S_{n-1}$, and it follows that

$$(\theta; \alpha_1, \ldots, \alpha_n) = \underbrace{(\epsilon; \alpha_{(1)\sigma_i}, \ldots, \alpha_{(n)\sigma_i})}_{\in S_m\wr S_{|\underline{\lambda}|}} \underbrace{(\sigma_i; e, \ldots, e)}_{=\hat{\sigma}_i} \underbrace{(\delta; e, \ldots, e)}_{\in S_m\wr S_{n-1}}$$

which establishes completeness. For non-redundancy, suppose that we have some i, j such that

$$\left(S_m \wr S_{|\underline{\lambda}|}\right) \hat{\sigma}_i \left(S_m \wr S_{n-1}\right) = \left(S_m \wr S_{|\underline{\lambda}|}\right) \hat{\sigma}_j \left(S_m \wr S_{n-1}\right).$$

Hence $\hat{\sigma}_i \in \left(S_m \wr S_{|\underline{\lambda}|}\right) \hat{\sigma}_j \left(S_m \wr S_{n-1}\right)$, so that we have $\epsilon \in S_{|\underline{\lambda}|}$, $\delta \in S_{n-1}$ and elements α_i, β_i of S_m such that

$$(\sigma_i; e, \ldots, e) = (\epsilon; \alpha_1, \ldots, \alpha_n)(\sigma_j; e, \ldots, e)(\delta; \beta_1, \ldots, \beta_{n-1}, e)$$

from which it follows that $\sigma_i = \epsilon \sigma_j \delta$ and hence that $i = j$. Thus we now seek such $\sigma_1, \ldots, \sigma_N$, and to do this we shall make use of our work on tableaux.

Now recall that if α, γ are compositions of n, then we have defined the tableau τ_γ^α to be the tableau of shape α whose entries, read from left to right across each row in turn starting with the top row, consist of γ_1 1's, then γ_2 2's, then γ_3 3's, and so on. So for example if $n = 9$, $\alpha = (8, 1)$ and $\gamma = (3, 1, 0, 2, 3)$, then

$$\tau_\gamma^\alpha = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 4 & 4 & 5 & 5 \\ \hline 5 \\ \cline{1-1} \end{array}.$$

Further, we know by Corollary 7.2.5 that if we have $\sigma_1, \ldots, \sigma_N \in S_n$ such that $\tau_\gamma^\alpha \sigma_1, \ldots, \tau_\gamma^\alpha \sigma_N$ is a complete list, with no repetition, of the tableaux of shape α and type γ with weakly increasing rows, then $\sigma_1, \ldots, \sigma_N$ is in fact a complete system of (S_γ, S_α)-double coset representatives without redundancy. We now apply this in the case where $\alpha = (n-1, 1)$ and $\gamma = |\underline{\lambda}|$ to obtain our desired system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n, noting that the subgroup S_{n-1} of S_n is exactly the Young subgroup $S_{(n-1,1)}$. The following example should serve to illustrate the general argument which we shall give below.

Keep $n = 9$, and suppose that $|\underline{\lambda}| = (3, 1, 0, 2, 3)$ as above. Then the possible tableaux of shape $(n-1, 1)$ and type $|\underline{\lambda}|$ with weakly increasing rows are

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 4 & 5 & 5 \\ \hline 5 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 4 & 5 & 5 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 4 & 4 & 5 & 5 & 5 \\ \hline 2 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 4 & 4 & 5 & 5 & 5 \\ \hline 1 \\ \cline{1-1} \end{array}.$$

Thus, a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{(n-1,1)})$-double coset representatives is $e, (6, 9, 8, 7), (4, 9, 8, 7, 6, 5), (3, 9, 8, 7, 6, 5, 4)$, recalling that in our action of S_n on tableaux, $\sigma \in S_n$ acts by moving the contents of the i^{th} box to the $(i)\sigma^{\text{th}}$ box, where the boxes of a tableau are numbered with the numbers $1, \ldots, n$ from left to right across each row, working from the top row to the bottom row.

The general case works in exactly the same way as the example. Indeed, recall that $\underline{\lambda} = (\lambda^1, \ldots, \lambda^r)$. For $i = 1, \ldots, r$ we let $b_i = |\lambda^1| + \cdots + |\lambda^i|$, so that we have a sequence $0 \le b_1 \le b_2 \le \cdots \le b_r = n$. Then for each $i = 1, \ldots, r$ such that $b_i \neq 0$ we define an element ρ_i of S_n by letting

$$\rho_i = \begin{cases} (b_i, n, n-1, \ldots, b_i + 1) & \text{if } b_i < n \\ e \quad \text{if } b_i = n \end{cases}$$

(where e is the identity element). By letting i run through all $1, \ldots, r$ such that $|\lambda^i| > 0$, we obtain a complete list of all the distinct ρ_i without repetition.

As in the above example, we see that the set of all tableaux $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ for i such that $|\lambda^i| > 0$ forms a complete list of all of the tableaux of shape $(n-1,1)$ and type $|\underline{\lambda}|$ with weakly increasing rows. Hence by Corollary 7.2.5 we see that the collection of all ρ_i for i such that $|\lambda^i| > 0$ forms a complete non-redundant system of $(S_{|\underline{\lambda}|}, S_{n-1})$-double coset representatives in S_n, and hence the collection of all $\hat{\rho}_i$ for i such that $|\lambda^i| > 0$ forms a complete non-redundant system of $(S_m \wr S_{|\underline{\lambda}|}, S_m \wr S_{n-1})$-double coset representatives in $S_m \wr S_n$.

Looking back to (8.2.1), we see that we want to understand the module

$$\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)} \uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)}$$

for i such that $|\lambda^i| > 0$. Our first step in doing so will be to understand the subgroup $\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i} \cap \left(S_m \wr S_{n-1}\right)$ of $S_m \wr S_n$ and its action on the module $\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}$.

So choose i such that $|\lambda^i| > 0$. It is easy to show directly that $\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i}$ is equal to $S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i}$. Thus we have

$$\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i} \cap \left(S_m \wr S_{n-1}\right) = S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap \left(S_m \wr S_{n-1}\right)$$

and it is easy to show directly that $S_m \wr \left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap \left(S_m \wr S_{n-1}\right)$ is equal to the subgroup of $S_m \wr S_n$ consisting of all elements of the form

$$(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e) \tag{8.2.2}$$

where σ is an element of the subgroup $\left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap S_{n-1}$ of S_n and $\alpha_i \in S_m$. We thus wish to understand the subgroup $\left(S_{|\underline{\lambda}|}\right)^{\rho_i} \cap S_{n-1}$ of S_n. By Proposition 7.1.4, $\left(S_{|\underline{\lambda}|}\right)^{\rho_i}$ is the stabilizer (under the action of S_n) of the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$. It is easy to see that the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ is the unique tableau of shape $(n-1,1)$ and type $|\underline{\lambda}|$ with weakly increasing rows which has an i in the box on the second row; such tableaux are illustrated in the above example.

For any subset Ω of $\{1,\dots,n\}$, let us write $S(\Omega)$ to denote the subgroup of S_n consisting of all permutations which fix any number not lying in Ω. We easily see that the stabilizer of the tableau $\tau_{|\underline{\lambda}|}^{(n-1,1)}\rho_i$ is the subgroup $X^i_{|\underline{\lambda}|}$ of S_n, where we define (recalling that $|\lambda^i|>0$ and hence $b_i>b_{i-1}$, where b_0 is taken to be 0)

$$\begin{aligned}X^i_{|\underline{\lambda}|} &= S\big(\{1,\dots,b_1\}\big)\times S\big(\{b_1+1,\dots,b_2\}\big)\times\cdots\\ &\times S\big(\{b_{i-1}+1,\dots,b_i-1,n\}\big)\times S\big(\{b_i,\dots,b_{i+1}-1\}\big)\times S\big(\{b_{i+1},\dots,b_{i+2}-1\}\big)\times\\ &\cdots\times S\big(\{b_{r-1},\dots,b_r-1\,=\,n-1\}\big)\end{aligned}$$

(note that here we are using the $\times$ symbol to denote an *internal* direct product of subgroups, and that if $b_i=b_{i+1}$ then $\{b_i,\dots,b_{i+1}-1\}$ represents the empty set, and that if $b_i=b_{i-1}+1$ then $\{b_{i-1}+1,\dots,b_i-1,n\}=\{n\}$), and hence $\big(S_{|\underline{\lambda}|}\big)^{\rho_i} = X^i_{|\underline{\lambda}|}$. We now introduce a small piece of notation. Indeed, if $\gamma=(\gamma_1,\dots,\gamma_r)$ is a composition of n, and $i\in\{1,\dots,r\}$ such that $\gamma_i>0$, then we write $[\gamma]_i$ for the composition $(\gamma_1,\dots,\gamma_{i-1},\gamma_i-1,\gamma_{i+1},\gamma_r)$ of $n-1$. We see that $X^i_{|\underline{\lambda}|}\cap S_{n-1}$ is the subgroup

$$\begin{aligned}&S\big(\{1,\dots,b_1\}\big)\times S\big(\{b_1+1,\dots,b_2\}\big)\times\cdots\\ &\times S\big(\{b_{i-1}+1,\dots,b_i-1\}\big)\times S\big(\{b_i,\dots,b_{i+1}-1\}\big)\times S\big(\{b_{i+1},\dots,b_{i+2}-1\}\big)\times\\ &\cdots\times S\big(\{b_{r-1},\dots,b_r-1\,=\,n-1\}\big)\end{aligned}$$

of S_n, and under our embedding of S_{n-1} into S_n this is exactly the subgroup $S_{[|\underline{\lambda}|]_i}$ of S_{n-1}. Hence, recalling that we are viewing $S_m\wr S_{n-1}$ as a subgroup of $S_m\wr S_n$ via the embedding $(\sigma;\alpha_1,\dots,\alpha_{n-1})\longmapsto(\sigma;\alpha_1,\dots,\alpha_{n-1},e)$, we see that the subgroup $\big(S_m\wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i}\cap\big(S_m\wr S_{n-1}\big)$ of $S_m\wr S_n$ is equal to the subgroup $S_m\wr S_{[|\underline{\lambda}|]_i}$ of the subgroup $S_m\wr S_{n-1}$ of $S_m\wr S_n$.

We now turn our attention to the action of $\big(S_m\wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i}\cap\big(S_m\wr S_{n-1}\big)$ on the $k\big(S_m\wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i}$-module $\big(T^{\underline{\lambda}}\big)^{\hat{\rho}_i}$. We know by the definition of conjugate modules (see page 32) that $\big(T^{\underline{\lambda}}\big)^{\hat{\rho}_i}$ is the module formed by equipping $T^{\underline{\lambda}}$ with the $k\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i}$-action $*$ given for $x \in T^{\underline{\lambda}}$ and $y \in \big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i}$ by $x * y = x(\hat{\rho}_i\, y\, \hat{\rho}_i^{-1})$ (where the action on the right-hand side is the action of $S_m \wr S_{|\underline{\lambda}|}$ on $T^{\underline{\lambda}}$, noting that $\hat{\rho}_i\, y\, \hat{\rho}_i^{-1}$ does indeed lie in $S_m \wr S_{|\underline{\lambda}|}$). Thus to calculate the action of an element

$$(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e) \in \big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i} \cap \big(S_m \wr S_{n-1}\big)$$

on the module $\big(T^{\underline{\lambda}}\big)^{\hat{\rho}_i}$, we need to calculate $\hat{\rho}_i(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)\hat{\rho}_i^{-1}$. We have

$$\begin{aligned}
\hat{\rho}_i(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)\hat{\rho}_i^{-1} &= (\rho_i; e, \ldots, e)(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)(\rho_i^{-1}; e, \ldots, e) \\
&= (\rho_i \sigma \rho_i^{-1}; \alpha_{(1)\rho_i}, \ldots, \alpha_{(n)\rho_i}) \qquad \text{(taking } \alpha_n = e\text{)} \\
&= (\rho_i \sigma \rho_i^{-1}; \alpha_1, \alpha_2, \ldots, \alpha_{b_i - 1}, e, \alpha_{b_i}, \alpha_{b_i + 1}, \\
&\qquad\qquad\qquad\qquad \ldots, \alpha_{n-2}, \alpha_{n-1}).
\end{aligned}$$

But by our description (8.2.2) of the elements of $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i} \cap \big(S_m \wr S_{n-1}\big)$, we see that $\sigma \in \big(S_{|\underline{\lambda}|}\big)^{\rho_i} \cap S_{n-1}$, which implies that $\rho_i \sigma \rho_i^{-1} \in S_{|\underline{\lambda}|} \cap \big(S_{n-1}\big)^{\rho_i^{-1}}$. By direct calculation, any element of $\big(S_{n-1}\big)^{\rho_i^{-1}}$ fixes b_i, and hence we see that $\rho_i \sigma \rho_i^{-1}$ is an element of $S_{|\underline{\lambda}|}$ which fixes b_i. Now we know that the subgroup $S_{|\underline{\lambda}|}$ of S_n has an internal direct product factorisation

$$\begin{aligned}
&S\big(\{1, \ldots, b_1\}\big) \times S\big(\{b_1 + 1, \ldots, b_2\}\big) \times \cdots \\
&\qquad \cdots \times S\big(\{b_{i-1} + 1, \ldots, b_i\}\big) \times S\big(\{b_i + 1, \ldots, b_{i+1}\}\big) \times \cdots \\
&\qquad\qquad \cdots \times S\big(\{b_{r-1} + 1, \ldots, b_r = n\}\big).
\end{aligned}$$

Thus any element π of $S_{|\underline{\lambda}|}$ has a unique factorisation $\pi = \theta_1 \cdots \theta_r$ where $\theta_j \in S\big(\{b_{j-1} + 1, \ldots, b_j\}\big)$ (with b_0 taken to be 0). We thus see that $\rho_i \sigma \rho_i^{-1}$ has such a factorisation $\rho_i \sigma \rho_i^{-1} = \theta_1 \cdots \theta_r$, where θ_i fixes b_i. Thus we see that our element $(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)$ of $\big(S_m \wr S_{|\underline{\lambda}|}\big)^{\hat{\rho}_i} \cap \big(S_m \wr S_{n-1}\big)$ acts on the module $\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}$ as the element

$$\left(\theta_1 \cdots \theta_r; \alpha_1, \alpha_2, \ldots, \alpha_{b_i-1}, e, \alpha_{b_i}, \alpha_{b_i+1}, \ldots, \alpha_{n-2}, \alpha_{n-1}\right)$$

of $S_m \wr S_{|\underline{\lambda}|}$ acts on $T^{\underline{\lambda}}$ (recalling that $\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}$ and $T^{\underline{\lambda}}$ are equal as k-vector spaces). But we know that $\left(S_m \wr S_{|\underline{\lambda}|}\right)^{\hat{\rho}_i} \cap \left(S_m \wr S_{n-1}\right)$ is equal to the subgroup $S_m \wr S_{[|\underline{\lambda}|]_i}$ of the subgroup $S_m \wr S_{n-1}$ of $S_m \wr S_n$, and we now see that if we identify $S_m \wr S_{[|\underline{\lambda}|]_i}$ with

$$\begin{aligned}&(S_m \wr S_{|\lambda^1|}) \times (S_m \wr S_{|\lambda^2|}) \times \cdots \\ &\qquad \cdots \times (S_m \wr S_{|\lambda^{i-1}|}) \times (S_m \wr S_{|\lambda^i|-1}) \times (S_m \wr S_{|\lambda^{i+1}|}) \times \cdots \times (S_m \wr S_{|\lambda^r|})\end{aligned}$$

in the canonical way, then by the definition of the $k(S_m \wr S_{|\underline{\lambda}|})$-module $T^{\underline{\lambda}}$, the $k(S_m \wr S_{[|\underline{\lambda}|]_i})$-module

$$\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i} \cap m\wr(n-1)} = \left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i} \big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i}$$

is isomorphic to

$$\begin{aligned}&\left(\left(S^{\mu^1}\right)^{\widetilde{\boxtimes}|\lambda^1|} \oslash S^{\lambda^1}\right) \boxtimes \cdots \boxtimes \left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right) \big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)} \boxtimes \cdots \\ &\qquad\qquad \cdots \boxtimes \left(\left(S^{\mu^r}\right)^{\widetilde{\boxtimes}|\lambda^r|} \oslash S^{\lambda^r}\right). \qquad (8.2.3)\end{aligned}$$

Thus, we want to investigate the $k(S_m \wr S_{|\lambda^i|-1})$-module

$$\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right) \big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)}.$$

Now the restriction operation $\big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)}$ may be expressed as

$$\big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} \big\downarrow^{m\wr(|\lambda^i|-1,1)}_{m\wr(|\lambda^i|-1)},$$

where, we recall, $m \wr (|\lambda^i| - 1, 1)$ represents the subgroup $S_m \wr S_{(|\lambda^i|-1,1)}$ of $S_m \wr S_{|\lambda^i|}$ consisting of all elements of the form $(\sigma; \alpha_1, \ldots, \alpha_n)$ for $\alpha_i \in S_m$

and $\sigma \in S_{(|\lambda^i|-1,1)}$, while $m \wr (|\lambda^i|-1)$ represents the subgroup $S_m \wr S_{(|\lambda^i|-1)}$ of $S_m \wr S_{|\lambda^i|}$ consisting of all elements of the form $(\sigma; \alpha_1, \ldots, \alpha_{n-1}, e)$ for $\alpha_i \in S_m$ and $\sigma \in S_{(|\lambda^i|-1,1)}$. Now we have by Proposition 4.3.7 that

$$\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right)\Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} = \left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|}\Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1,1)} \oslash S^{\lambda^i}\Big\downarrow^{|\lambda^i|}_{(|\lambda^i|-1,1)}.$$

Upon further restriction to $S_m \wr S_{(|\lambda^i|-1)}$, we see that this is isomorphic to the direct sum of $\dim_k(S^{\mu^i})$ copies of

$$\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|-1} \oslash S^{\lambda^i}\Big\downarrow^{|\lambda^i|}_{|\lambda^i|-1}.$$

It now follows by Theorem 8.0.1 and Lemma 6.1.1 that we have a filtration

$$\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|} \oslash S^{\lambda^i}\right)\Big\downarrow^{m\wr|\lambda^i|}_{m\wr(|\lambda^i|-1)} \sim \mathop{\mathcal{F}}_{\delta\in \mathrm{R}(\lambda^i)} \dim_k(S^{\mu^i})\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}|\lambda^i|-1} \oslash S^{\delta}$$

where if ϵ is any partition, we define $\mathrm{R}(\epsilon)$ to be the set of all partitions of $|\epsilon|-1$ which may be obtained from ϵ by removing a box. Using (8.2.3), it now follows that we have a filtration of $k(S_m \wr S_{[|\underline{\lambda}|]_i})$-modules

$$\begin{aligned}\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}\Big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}\cap m\wr(n-1)} &= \left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}\Big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i} \\ &\sim \mathop{\mathcal{F}}_{\substack{\underline{\delta}\text{ is an }r\text{-multipartition of }n-1\\ \delta^j=\lambda^j\text{ for }j\neq i\\ \delta^i\in\mathrm{R}(\lambda^i)}} \dim_k(S^{\mu^i})\, T^{\underline{\delta}}.\end{aligned}$$

By Lemma 2.2.2, it now follows that we have a filtration of $k(S_m \wr S_{n-1})$-modules

$$\begin{aligned}&\left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}\Big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}\cap m\wr(n-1)}\Big\uparrow^{m\wr(n-1)}_{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}\cap m\wr(n-1)} \\ &\qquad= \left(T^{\underline{\lambda}}\right)^{\hat{\rho}_i}\Big\downarrow^{(m\wr|\underline{\lambda}|)^{\hat{\rho}_i}}_{m\wr[|\underline{\lambda}|]_i}\Big\uparrow^{m\wr(n-1)}_{m\wr[|\underline{\lambda}|]_i} \\ &\qquad\sim \mathop{\mathcal{F}}_{\substack{\underline{\delta}\text{ is an }r\text{-multipartition of }n-1\\ \delta^j=\lambda^j\text{ for }j\neq i\\ \delta^i\in\mathrm{R}(\lambda^i)}} \dim_k(S^{\mu^i})\, S^{\underline{\delta}}.\end{aligned}$$

Refering back to the decomposition (8.2.1), we now see that we have proved the following result, which is our desired Specht branching rule.

Theorem 8.2.1. *Let $n > 0$, and let $\underline{\lambda}$ be an r-multipartition of n. Then we have a filtration of the $k(S_m \wr S_{n-1})$-module $S^{\underline{\lambda}}\downarrow^{m\wr n}_{m\wr(n-1)}$ by Specht modules $S^{\underline{\delta}}$ for r-multipartitions $\underline{\delta}$ of $n-1$. For a multipartition $\underline{\delta}$ of $n-1$, if $\underline{\delta}$ may be obtained from $\underline{\lambda}$ by removing a single box from the partition λ^i for some i (while leaving all other partitions λ^j unchanged), then $S^{\underline{\delta}}$ occurs with multiplicity $\dim_k(S^{\mu^i})$ in the filtration, and otherwise $S^{\underline{\delta}}$ does not occur in the filtration.*

We note that the multiplicities $\dim_k(S^{\mu^i})$ occuring in this filtration have a simple and elegant combinatorial interpretation via the *hook length formula* (see for example [20, Chapter 20]), from which we see that they are in fact independent of the field k. We also note the similarity of this result to Theorem 8.0.1.

Original research in Chapter 8: All of the material in Sections 8.1 and 8.2 is original research.

Chapter 9

Structure of $\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$

This chapter contains the most substantial new results of this thesis. Let m and n be non-negative integers and k a field as in previous chapters, and let $\underline{\nu}$ be an r component multipartition of n and $\underline{\gamma}$ an r component multicomposition of n, where r is the number of distinct partitions of m as before. We shall prove that if the characteristic of k is not 2, then

$$\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right) \cong \begin{cases} k & \text{if } \underline{\nu} = \underline{\gamma}, \\ 0 & \text{if } \underline{\nu} \ntrianglerighteq \underline{\gamma}, \end{cases} \tag{9.0.1}$$

and further if the characteristic of k is neither 2 nor 3, then

$$\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right) = 0.$$

These results are wreath product analogues of the symmetric group results (3.3.1) and Proposition 3.3.4. In the next chapter we shall use these results to prove wreath product versions of (3.3.2) and Theorem 3.3.2, and hence by Corollary 3.4.2 show (as was done in Section 3.4 for the symmetric group) that Specht filtration multiplicities are well-defined in the wreath product case over an algebraically closed field whose characteristic is neither 2 nor 3.

Recall that, for our fixed non-negative integer m, we have fixed the distinct partitions of m, in the lexicographic order, to be

$$(m) = \mu^1 > \mu^2 > \ \ldots \ > \mu^r = (1^m).$$

Recall also that if α is some composition of n and $i \in \{1, \ldots, r\}$, then we write $[\alpha, i]$ to denote the r-multicomposition of n which has α in the i^{th} place and () in all other places.

We shall begin by considering the structure of the spaces $\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ for $\underline{\nu} = [\nu, i]$ and $\underline{\gamma} = [\gamma, j]$. From this we move to the case where $\underline{\nu} = [\nu, i]$ but $\underline{\gamma}$ is any multicomposition. Finally, we shall use this special case to derive our desired results in the general case.

9.1 Structure of $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right)$

We begin by proving (9.0.1) in the case where $\underline{\nu} = [\nu, i]$ and $\underline{\gamma} = [\gamma, j]$.

Proposition 9.1.1. *Let k be a field such that* $\mathrm{char}(k) \neq 2$. *Let $\nu \vdash n$, and $\gamma \vDash n$, and let $i, j \in \{1, \ldots, r\}$. Then*

$$\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right) \cong \begin{cases} 0 & \text{if } i > j, \\ 0 & \text{if } i = j \text{ and } \nu \ntrianglerighteq \gamma, \\ k & \text{if } i = j \text{ and } \nu = \gamma. \end{cases}$$

Proof. Recall that

$$\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right) = \mathrm{Hom}_{m\wr n}\left(\left(S^{\mu^i}\right)^{\tilde{\boxtimes} n} \oslash S^{\nu}, \left(M^{\mu^j}\right)^{\tilde{\boxtimes} n} \oslash M^{\gamma}\right).$$

For the first case where $i > j$, note that $i > j$ implies that $\mu^j > \mu^i$ in the lexicographic order. Hence $\mu^j \not\leq \mu^i$, and so $\mu^i \not\trianglerighteq \mu^j$. Thus by (3.3.1) we have

$$\mathrm{Hom}_m\big(S^{\mu^i}, M^{\mu^j}\big) = 0. \tag{9.1.1}$$

Now the subgroup of $S_m \wr S_n$ consisting of all elements of the form $(e; \sigma, e, \ldots, e)$ for $\sigma \in S_m$ may be identified with S_m in the obvious way, and hence any $k(S_m \wr S_n)$-module X becomes a kS_m-module under the action of this subgroup. Let us denote the resulting kS_m-module by $\overline{X}$. We now see that if U is a kS_m-module and V is a kS_n-module, then the kS_m-module $\overline{U^{\tilde{\boxtimes} n} \oslash V}$ is the k-vector space $U^{\otimes n} \otimes V$ with action given on pure tensors by $(u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v)\sigma = u_1\sigma \otimes u_2 \otimes \cdots \otimes u_n \otimes v$, and hence this module is isomorphic to a direct sum of copies of U. Hence the kS_m-modules

$$\overline{\big(S^{\mu^i}\big)^{\tilde{\boxtimes} n} \oslash S^{\nu}} \quad \text{and} \quad \overline{\big(M^{\mu^j}\big)^{\tilde{\boxtimes} n} \oslash M^{\gamma}}$$

are isomorphic to direct sums of copies of S^{μ^i} and M^{μ^j}, respectively. It follows by (9.1.1) that any kS_m-module homomorphism from the former module to the latter must be zero. But if X and Y are $k(S_m \wr S_n)$-modules, then any $k(S_m \wr S_n)$-module homomorphism $f : X \to Y$ must be a kS_m-module homomorphism $f : \overline{X} \to \overline{Y}$, and it follows that

$$\mathrm{Hom}_{m \wr n}\Big(\big(S^{\mu^i}\big)^{\tilde{\boxtimes} n} \oslash S^{\nu}, \big(M^{\mu^j}\big)^{\tilde{\boxtimes} n} \oslash M^{\gamma}\Big) = 0.$$

The second and third cases will follow from (3.3.1) by proving that for any $\mu \vdash m$, we have

$$\mathrm{Hom}_{m \wr n}\Big(\big(S^{\mu}\big)^{\tilde{\boxtimes} n} \oslash S^{\nu}, \big(M^{\mu}\big)^{\tilde{\boxtimes} n} \oslash M^{\gamma}\Big) \cong \mathrm{Hom}_n\big(S^{\nu}, M^{\gamma}\big) \tag{9.1.2}$$

as k-vector spaces. Indeed, let B be the subgroup of $S_m \wr S_n$ consisting of all elements of the form $(e; \sigma_1, \sigma_2, \ldots, \sigma_n)$ for $\sigma_l \in S_m$ (where e is as usual the identity element of S_n), so that B is canonically isomorphic to $\big(S_m\big)^n$ (the direct product of n copies of S_m). Now let $x_1, \ldots, x_p$ be a k-basis of S^ν. Then

$$\Big[\big(S^\mu\big)^{\widetilde{\boxtimes} n} \oslash S^\nu\Big]\Big\downarrow_{kB}^{k(S_m \wr S_n)} = \bigoplus_{i=1}^{p} W_i$$

where W_i is the k-subspace spanned by pure tensors $u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes x_i$ for $u_t \in S^\mu$. Thus W_i is a kB-submodule which is isomorphic to $\big(S^\mu\big)^{\boxtimes n}$ (the external tensor product of n copies of S^μ) when B is identified with $\big(S_m\big)^n$ in the canonical way; the isomorphism is given by $u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes x_i \mapsto u_1 \otimes u_2 \otimes \cdots \otimes u_n$. Similarly, if $y_1, \ldots, y_q$ is a k-basis of M^γ, then

$$\Big[\big(M^\mu\big)^{\widetilde{\boxtimes} n} \oslash M^\gamma\Big]\Big\downarrow_{kB}^{k(S_m \wr S_n)} = \bigoplus_{j=1}^{q} Z_j$$

where Z_j is the k-subspace spanned by pure tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes y_j$ for $v_t \in M^\mu$. Thus Z_j is a kB-submodule which is isomorphic to $\big(M^\mu\big)^{\boxtimes n}$ when B is identified with $\big(S_m\big)^n$ in the canonical way; the isomorphism is given by $v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes y_j \mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_n$. By using Proposition 2.1.3 and the fact that (by (3.3.1)) $\mathrm{Hom}_m\big(S^\mu, M^\mu\big) \cong k$, we may now easily see that for each i and j, the k-vector space $\mathrm{Hom}_{kB}\left(W_i, Z_j\right)$ has dimension 1, and hence we have

$$\mathrm{Hom}_{kB}\left(W_i, Z_j\right) = \{\alpha\Phi_{i,j} \mid \alpha \in k\}$$

where $\Phi_{i,j}$ is the map given on pure tensors by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes x_i \;\longmapsto\; u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes y_j$$

(recalling that $S^\mu \subseteq M^\mu$). Now

$$\mathrm{Hom}_{m \wr n}\left(\big(S^\mu\big)^{\widetilde{\boxtimes} n} \oslash S^\nu, \big(M^\mu\big)^{\widetilde{\boxtimes} n} \oslash M^\gamma\right) \tag{9.1.3}$$

is a k-subspace of

$$\mathrm{Hom}_{kB}\left(\bigoplus_{i=1}^{p} W_i, \bigoplus_{j=1}^{q} Z_j\right)$$

which we may identify in the obvious way with

$$\bigoplus_{i=1}^{p} \bigoplus_{j=1}^{q} \mathrm{Hom}_{kB}\left(W_i, Z_j\right).$$

Under this identification, for any element f of (9.1.3) we have

$$f = \sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_{i,j} \Phi_{i,j} \tag{9.1.4}$$

for unique scalars $\alpha_{i,j} \in k$. For each such f, let us define a k-linear map $\hat{f} : S^\nu \to M^\gamma$ by letting

$$\hat{f}(x_i) = \sum_{j=1}^{q} \alpha_{i,j} y_j.$$

By a simple direct calculation, it is easy to show that for any $x \in S^\nu$ and any $u_1, \ldots, u_n \in S^\mu$ we have

$$f(u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes x) = u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes \hat{f}(x).$$

Then if $x \in S^\nu$ and $\sigma \in S_n$, we may fix some non-zero $u \in S^\mu$ and note that

$$\begin{aligned} u \otimes \cdots \otimes u \otimes (\hat{f}(x)\sigma) &= \Big(u \otimes \cdots \otimes u \otimes \hat{f}(x)\Big)\,(\sigma; e, \ldots, e) \\ &= f(u \otimes \cdots \otimes u \otimes x)(\sigma; e, \ldots, e) \\ &= f\left((u \otimes \cdots \otimes u \otimes x)(\sigma; e, \ldots, e)\right) \\ &= f(u \otimes \cdots \otimes u \otimes x\sigma) \\ &= u \otimes \cdots \otimes u \otimes \hat{f}(x\sigma) \end{aligned}$$

which implies that $\hat{f}(x)\sigma = \hat{f}(x\sigma)$ and hence that $\hat{f}$ is a kS_n-module homomorphism. We claim that the map $f \longmapsto \hat{f}$ is the required isomorphism (9.1.2). To see that this map is onto, let $g \in \mathrm{Hom}_n\big(S^\nu, M^\gamma\big)$ and define f to be the k-linear map from $\left(S^\mu\right)^{\widetilde{\boxtimes} n} \oslash S^\nu$ to $\left(M^\mu\right)^{\widetilde{\boxtimes} n} \oslash M^\gamma$ given on pure tensors by

$$f(u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes x) = u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes g(x).$$

We may easily show by direct calculation that f is a $k(S_m \wr S_n)$-module homomorphism and moreover it is clear that $\hat{f} = g$, and so the map $f \longmapsto \hat{f}$ is onto. To see that the map $f \longmapsto \hat{f}$ is injective, let $f_1, f_2 \in \mathrm{Hom}_{m\wr n}\left(\left(S^\mu\right)^{\tilde{\boxtimes} n} \oslash S^\nu, \left(M^\mu\right)^{\tilde{\boxtimes} n} \oslash M^\gamma\right)$ and suppose that $\hat{f}_1 = \hat{f}_2$. Now if we decompose f_1 as in (9.1.4) with coefficients $\alpha^1_{i,j}$ and similarly decompose f_2 with coefficients $\alpha^2_{i,j}$, then by the definition of $\hat{f}_1$ and $\hat{f}_2$, we see that we must have $\alpha^1_{i,j} = \alpha^2_{i,j}$ for all i and j, which implies that $f_1 = f_2$. Thus $f \longmapsto \hat{f}$ is indeed an isomorphism as required. $\square$

9.2 Structure of $\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right)$

In this section, we shall prove the following result.

Proposition 9.2.1. *Suppose that k has characteristic not 2 or 3, and let $\nu \vdash n$ and $\gamma \vDash n$ and $i, j \in \{1, \ldots, r\}$. Then*

$$\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right) = 0.$$

The arguments in this section are based on the arguments given in Section 3.3 to prove Proposition 3.3.4 about extensions between modules S^μ and M^α over the symmetric group (and those arguments are themselves based on the work of Erdmann in [10]). This section is rather long and the arguments it contains are somewhat complicated, but once we have completed them the greater part of the work in this chapter will be behind us.

For the rest of this section, we take k to be a field whose characteristic is neither 2 nor 3.

Recall that if $\underline{\delta} = (\delta^1, \ldots, \delta^t)$ is a multipartition of m, then $|\underline{\delta}|$ is the composition $(|\delta^1|, \ldots, |\delta^t|)$ of m, so that we have a Young subgroup $S_{|\underline{\delta}|}$ of S_m.

Recall further from page 57 that we define a $kS_{|\underline{\delta}|}$-module $S(\underline{\delta})$ by setting

$$S(\underline{\delta}) = S^{\delta^1} \boxtimes S^{\delta^2} \boxtimes \cdots \boxtimes S^{\delta^t}.$$

Reduction 9.2.2. *To prove Proposition 9.2.1, it is enough to prove that for any $\mu \vdash m$, any $\underline{\delta}$ a multipartition of m with $|\underline{\delta}| = \mu$, and any $\epsilon \vdash n$, we have*

$$\mathrm{Ext}^1_{\mu \wr n}\left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}, \mathbb{1}_{\mu \wr n}\right) = 0.$$

Proof. With the hypotheses of Proposition 9.2.1, we have by Proposition 4.4.1 that $M^{[\gamma,j]} = \mathbb{1}\big\uparrow^{m\wr n}_{W_{[\gamma,j]}}$. But of course $[\gamma, j] = \big((), \ldots, (), \gamma, (), \ldots, ()\big)$ with γ appearing in the j^{th} place, and hence we see easily that $W_{[\gamma,j]}$ (see page 72 for the definition of $W_{\underline{\gamma}}$ for a multipartition $\underline{\gamma}$) is the subgroup $S_{\mu^j} \wr S_\gamma$ of $S_m \wr S_n$. Thus we have

$$\begin{aligned}
\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right) &\cong \mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, \mathbb{1}_{\mu^j\wr\gamma}\big\uparrow^{m\wr n}_{\mu^j\wr\gamma}\right) \\
&\cong \mathrm{Ext}^1_{\mu^j\wr\gamma}\left(S^{[\nu,i]}\big\downarrow^{m\wr n}_{\mu^j\wr\gamma}, \mathbb{1}_{\mu^j\wr\gamma}\right) \qquad (9.2.1)
\end{aligned}$$

(by Theorem 2.2.4).

Now we have

$$\begin{aligned}
S^{[\nu,i]}\big\downarrow^{m\wr n}_{\mu^j\wr\gamma} &= \left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \oslash S^{\nu}\right]\Big\downarrow^{m\wr n}_{\mu^j\wr\gamma} \\
&= \left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \oslash S^{\nu}\right]\Big\downarrow^{m\wr n}_{\mu^j\wr n}\Big\downarrow^{\mu^j\wr n}_{\mu^j\wr\gamma}
\end{aligned}$$

(by transitivity of induction).

Now recall from (4.3.2) that $\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \oslash S^{\nu} = \left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \otimes \mathrm{Inf}^{m\wr n}_{n} S^{\nu}$ (recalling from page 75 that if G is a subgroup of S_m, H is a subgroup of S_n, and Y is a kH-module, then $\mathrm{Inf}^{G\wr H}_{H} Y$ is the $k(G \wr H)$-module obtained by equipping Y with the action defined by $y(\sigma; \alpha_1, \ldots, \alpha_n) = y\sigma$ for $y \in Y$, $\alpha_1, \ldots, \alpha_n \in G$, and $\sigma \in H$). Thus by (2.2.4), we have

$$\left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \oslash S^{\nu}\right]\Big\downarrow^{m\wr n}_{\mu^j\wr n} \cong \left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n}\Big\downarrow^{m\wr n}_{\mu^j\wr n} \otimes \mathrm{Inf}^{m\wr n}_{n} S^{\nu}\big\downarrow^{m\wr n}_{\mu^j\wr n}.$$

But it is clear that $\mathrm{Inf}_n^{m\wr n} S^\nu \big\downarrow_{\mu^j\wr n}^{m\wr n}$ is the $k(S_{\mu^j}\wr S_n)$-module $\mathrm{Inf}_n^{\mu^j\wr n} S^\nu$, while $\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n}\Big\downarrow_{\mu^j\wr n}^{m\wr n}$ is $\left(S^{\mu^i}\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes} n}$ by Proposition 4.3.5. Thus we have

$$\begin{aligned} S^{[\nu,i]}\big\downarrow_{\mu^j\wr\gamma}^{m\wr n} &\cong \left[\left(S^{\mu^i}\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes} n} \oslash S^\nu\right]\Big\downarrow_{\mu^j\wr\gamma}^{\mu^j\wr n} \\ &\cong \left(S^{\mu^i}\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes} n}\big\downarrow_{\mu^j\wr\gamma}^{\mu^j\wr n} \oslash S^\nu\big\downarrow_\gamma^n \\ &\qquad\qquad \text{(by Proposition 4.3.7).} \end{aligned}$$

Now let t be the length of γ, so that $\gamma=(\gamma_1,\ldots,\gamma_t)$. Then by (3.2.12) $S^\nu\downarrow_\gamma^n$ is filtered by modules of the form $S^{\lambda^1}\boxtimes\cdots\boxtimes S^{\lambda^t}$ where $\lambda^l\vdash\gamma_l$. Hence by Lemma 6.1.1, the $k(S_{\mu^j}\wr S_\gamma)$-module $S^{[\nu,i]}\big\downarrow_{\mu^j\wr\gamma}^{m\wr n}$ has a filtration by modules of the form

$$\begin{aligned} &\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes} n}\big\downarrow_{\mu^j\wr\gamma}^{\mu^j\wr n} \oslash \left(S^{\lambda^1}\boxtimes\cdots\boxtimes S^{\lambda^t}\right) \cong \\ &\qquad\qquad \left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_1}\boxtimes\cdots\boxtimes\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_t}\right) \oslash \left(S^{\lambda^1}\boxtimes\cdots\boxtimes S^{\lambda^t}\right) \end{aligned}$$

and under the natural isomorphism of $S_{\mu^j}\wr S_\gamma$ with $S_{\mu^j}\wr S_{\gamma_1}\times\cdots\times S_{\mu^j}\wr S_{\gamma_t}$, this means by the isomorphism (4.3.5) that $S^{[\nu,i]}\big\downarrow_{\mu^j\wr\gamma}^{m\wr n}$ has a filtration by modules

$$\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_1}\oslash S^{\lambda^1}\right)\boxtimes\cdots\boxtimes\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_t}\oslash S^{\lambda^t}\right).$$

Hence by (9.2.1) and Proposition 2.1.1, we see that to prove that the space $\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{[\gamma,j]}\right)$ is zero, it suffices to prove that if $\lambda^l\vdash\gamma_l$ for $l=1,\ldots,t$, then the module

$$\mathrm{Ext}^1_{\mu^j\wr\gamma}\left(\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_1}\oslash S^{\lambda^1}\right)\boxtimes\cdots\boxtimes\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_t}\oslash S^{\lambda^t}\right), \mathbb{1}_{\mu^j\wr\gamma}\right) \qquad (9.2.2)$$

is zero. By writing the $k(S_{\mu^j}\wr S_\gamma)$-module $\mathbb{1}_{\mu^j\wr\gamma}$ as $\mathbb{1}_{\mu^j\wr\gamma_1}\boxtimes\cdots\boxtimes\mathbb{1}_{\mu^j\wr\gamma_t}$ and applying Proposition 2.1.3, we see that the module (9.2.2) is isomorphic as a k-vector space to a direct sum of the terms

$$\mathrm{Ext}^1_{\mu^j\wr\gamma_l}\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_l}\oslash S^{\lambda^l}, \mathbb{1}_{\mu^j\wr\gamma_l}\right) \otimes \bigotimes_{\substack{p=1,\ldots,t\\ p\neq l}} \mathrm{Hom}_{\mu^j\wr\gamma_p}\left(\left(S^{\mu^i}\big\downarrow_{\mu^j}^m\right)^{\widetilde{\boxtimes}\gamma_p}\oslash S^{\lambda^p}, \mathbb{1}_{\mu^j\wr\gamma_p}\right)$$

for $l = 1, \ldots, t$. Hence we see that it suffices to prove that

$$\operatorname{Ext}^1_{\mu^j \wr n}\Big(\big(S^{\mu^i}\downarrow^m_{\mu^j}\big)^{\widetilde{\boxtimes} n} \oslash S^\lambda, \mathbb{1}_{\mu^j \wr n}\Big) = 0 \tag{9.2.3}$$

for any partition λ of a non-negative integer n and any $i, j \in \{1, \ldots, r\}$. So let us fix some such λ and i, j. Now by (3.2.12), the kS_{μ^j}-module $S^{\mu^i}\downarrow^m_{\mu^j}$ is filtered by modules $S(\underline{\delta}) = S^{\delta^1} \boxtimes \cdots \boxtimes S^{\delta^{L_j}}$ where $\underline{\delta}$ is a multipartition of m with $|\underline{\delta}| = \mu^j$, and L_j represents the length of μ^j. Explicitly, we have a filtration

$$S^{\mu^i}\downarrow^m_{\mu^j} \sim \mathop{\mathcal{F}}_{l=1}^{p} S(\underline{\delta}^l)$$

for some (not necessarily pairwise unequal) multipartitions $\underline{\delta}^1, \ldots, \underline{\delta}^p$ of m (p some integer), where $|\underline{\delta}^l| = \mu^j$ for each l. Thus by Lemma 6.1.1 and Proposition 6.2.1 we find that the $k(S_{\mu^j} \wr S_n)$-module

$$\Big(S^{\mu^i}\downarrow^m_{\mu^j}\Big)^{\widetilde{\boxtimes} n} \oslash S^\lambda$$

has a filtration by modules of the form

$$\begin{aligned}\Big[\big(S(\underline{\delta}^1), \ldots, S(\underline{\delta}^p)\big)^{\widetilde{\boxtimes}\alpha}\Big]\Big\uparrow^{\mu^j \wr n}_{\mu^j \wr \alpha} \oslash S^\lambda = \\ \Big[S(\underline{\delta}^1)^{\widetilde{\boxtimes}\alpha_1} \boxtimes \cdots \boxtimes S(\underline{\delta}^p)^{\widetilde{\boxtimes}\alpha_p}\Big]\Big\uparrow^{\mu^j \wr n}_{\mu^j \wr \alpha} \oslash S^\lambda\end{aligned} \tag{9.2.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)$ is some composition of n. Then by Proposition 4.3.7, (9.2.4) is isomorphic to

$$\Big[\Big(S(\underline{\delta}^1)^{\widetilde{\boxtimes}\alpha_1} \boxtimes \cdots \boxtimes S(\underline{\delta}^p)^{\widetilde{\boxtimes}\alpha_p}\Big) \oslash S^\lambda\downarrow^n_\alpha\Big]\Big\uparrow^{\mu^j \wr n}_{\mu^j \wr \alpha} \tag{9.2.5}$$

and by (3.2.12) $S^\lambda\downarrow^n_\alpha$ has a filtration by modules $S^{\epsilon^1} \boxtimes \cdots \boxtimes S^{\epsilon^p}$ where $\epsilon^l \vdash \alpha_l$, so (using the isomorphism (4.3.5)) the module (9.2.5) is filtered by modules

$$\Big[\Big(S(\underline{\delta}^1)^{\widetilde{\boxtimes}\alpha_1} \oslash S^{\epsilon^1}\Big) \boxtimes \cdots \boxtimes \Big(S(\underline{\delta}^p)^{\widetilde{\boxtimes}\alpha_p} \oslash S^{\epsilon^p}\Big)\Big]\Big\uparrow^{\mu^j \wr n}_{\mu^j \wr \alpha}.$$

Hence, by Proposition 2.1.1, to prove (9.2.3) it is enough to prove that

$$\operatorname{Ext}^1_{\mu^j \wr n}\left(\left[\bigboxtimes_{l=1}^{p} S(\underline{\delta}^l)^{\widetilde{\boxtimes}\alpha_l} \oslash S^{\epsilon^l}\right]\Bigg\uparrow_{\mu^j\wr\alpha}^{\mu^j\wr n}, \mathbb{1}_{\mu^j\wr n}\right) = 0.$$

But by Theorem 2.2.4 (the Eckmann-Shapiro lemma),

$$\begin{aligned}\operatorname{Ext}^1_{\mu^j \wr n}&\left(\left[\bigboxtimes_{l=1}^{p} S(\underline{\delta}^l)^{\widetilde{\boxtimes}\alpha_l} \oslash S^{\epsilon^l}\right]\Bigg\uparrow_{\mu^j\wr\alpha}^{\mu^j\wr n}, \mathbb{1}_{\mu^j\wr n}\right)\\ &= \operatorname{Ext}^1_{\mu^j\wr\alpha}\left(\bigboxtimes_{l=1}^{p} S(\underline{\delta}^l)^{\widetilde{\boxtimes}\alpha_l} \oslash S^{\epsilon^l}, \mathbb{1}_{\mu^j\wr n}\Big\downarrow_{\mu^j\wr\alpha}^{\mu^j\wr n}\right)\\ &= \operatorname{Ext}^1_{\mu^j\wr\alpha}\left(\bigboxtimes_{l=1}^{p} S(\underline{\delta}^l)^{\widetilde{\boxtimes}\alpha_l} \oslash S^{\epsilon^l}, \bigboxtimes_{l=1}^{p} \mathbb{1}_{\mu^j\wr\alpha_l}\right).\end{aligned}$$

By Proposition 2.1.3 this is isomorphic to the direct sum of the terms

$$\operatorname{Ext}^1_{\mu^j\wr\alpha_l}\left(S(\underline{\delta}^l)^{\widetilde{\boxtimes}\alpha_l} \oslash S^{\epsilon^l}, \mathbb{1}_{\mu^j\wr\alpha_l}\right) \otimes \bigotimes_{\substack{s=1,\dots,p\\ s\neq l}} \operatorname{Hom}_{\mu^j\wr\alpha_s}\left(S(\underline{\delta}^s)^{\widetilde{\boxtimes}\alpha_s} \oslash S^{\epsilon^s}, \mathbb{1}_{\mu^j\wr\alpha_s}\right)$$

for $l = 1, \dots, p$, and hence proving that $\operatorname{Ext}^1_{\mu\wr n}\left(S(\underline{\delta})^{\widetilde{\boxtimes}n} \oslash S^{\epsilon}, \mathbb{1}_{\mu\wr n}\right) = 0$ for any $\mu \vdash m$, any $\underline{\delta}$ a multipartition of m with $|\underline{\delta}| = \mu$, and any $\epsilon \vdash n$ will indeed prove Proposition 9.2.1. □

We shall now reformulate the condition in Reduction 9.2.2 to obtain the condition which we shall prove below. For this reformulation, we shall need a lemma and some new notation. For $\beta = (\beta_1, \dots, \beta_t) \vDash m$, recall from page 45 that we have defined a sign module $\operatorname{Sgn}_\beta = \operatorname{Sgn}_n\downarrow^n_\beta \cong \operatorname{Sgn}_{\beta_1} \boxtimes \cdots \boxtimes \operatorname{Sgn}_{\beta_t}$. We define a sign module $\operatorname{Sgn}_{\beta\wr n}$ for $k(S_\beta \wr S_n)$ by letting

$$\operatorname{Sgn}_{\beta\wr n} = (\operatorname{Sgn}_\beta)^{\widetilde{\boxtimes}n} \oslash \operatorname{Sgn}_n.$$

We note that

$$\operatorname{Sgn}_{\beta\wr n} \cong \left(\operatorname{Sgn}_{\beta_1} \boxtimes \operatorname{Sgn}_{\beta_2} \boxtimes \cdots \boxtimes \operatorname{Sgn}_{\beta_t}\right)^{\widetilde{\boxtimes}n} \oslash \operatorname{Sgn}_n.$$

We may easily verify that we have an isomorphism of $k(S_\beta \wr S_n)$-modules

$$\mathrm{Sgn}_{\beta\wr n} \otimes \mathrm{Sgn}_{\beta\wr n} \cong \mathbb{1}_{\beta\wr n}. \tag{9.2.6}$$

Further, recall that we are writing ν' for the conjugate partition of a partition ν and that if $\underline{\nu} = (\nu^1, \ldots, \nu^t)$ is a multipartition, then we have defined $\underline{\nu}'$ to be the multipartition $\big((\nu^1)', \ldots, (\nu^t)'\big)$.

Lemma 9.2.3. *With μ, $\underline{\delta}$, and ϵ as in Reduction 9.2.2,*

$$\left(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right)^* \cong \mathrm{Sgn}_{\mu\wr n} \otimes \left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right)$$

as $k(S_\mu \wr S_n)$-modules (recall that U^* denotes the dual of a module U).

Proof. Since we have $\epsilon'' = \epsilon$ and $\underline{\delta}'' = \underline{\delta}$, the claim in the lemma is equivalent to

$$\left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right)^* \cong \mathrm{Sgn}_{\mu\wr n} \otimes \left(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right),$$

and it is this result which we shall prove. By Proposition 4.3.2 have

$$\left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right)^* \cong \left(S(\underline{\delta})^{\widetilde{\boxtimes} n}\right)^* \oslash (S^{\epsilon})^*.$$

Now by Proposition 3.1.3 we have $(S^\epsilon)^* \cong \mathrm{Sgn}_n \otimes S^{\epsilon'}$, and by Proposition 4.3.4 we have $\big(S(\underline{\delta})^{\widetilde{\boxtimes} n}\big)^* \cong \big(S(\underline{\delta})^*\big)^{\widetilde{\boxtimes} n}$. Further, if we let t be the length of $\underline{\delta}$ (and hence also the length of μ, since $|\underline{\delta}| = \mu$), then

$$\begin{aligned}
S(\underline{\delta})^* &\cong \left(S^{\delta^1} \boxtimes \cdots \boxtimes S^{\delta^t}\right)^* \\
&\cong \left(S^{\delta^1}\right)^* \boxtimes \cdots \boxtimes \left(S^{\delta^t}\right)^* \qquad \text{(by (2.2.2))} \\
&\cong \left(\mathrm{Sgn}_{\mu_1} \otimes S^{(\delta^1)'}\right) \boxtimes \cdots \boxtimes \left(\mathrm{Sgn}_{\mu_t} \otimes S^{(\delta^t)'}\right) \qquad \text{(by Proposition 3.1.3)} \\
&\cong \left(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\right) \otimes \left(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\right),
\end{aligned}$$

where the last isomorphism is due to the easily proved fact that if $G_1, \ldots, G_t$ are finite groups and for each $i = 1, \ldots, t$, X_i and Y_i are kG_i-modules, then the two $k(G_1 \times \cdots \times G_t)$-modules $\big(X_1 \otimes Y_1\big) \boxtimes \cdots \boxtimes \big(X_t \otimes Y_t\big)$ and $\big(X_1 \boxtimes \cdots \boxtimes X_t\big) \otimes \big(Y_1 \boxtimes \cdots \boxtimes Y_t\big)$ are isomorphic in the obvious way. Thus we have

$$\begin{aligned}(S(\underline{\delta})^*)^{\widetilde{\boxtimes} n} &\cong \Big(\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big) \otimes \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)\Big)^{\widetilde{\boxtimes} n} \\ &\cong \big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \otimes \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n} \\ &\qquad \text{(by Proposition 4.3.3).}\end{aligned}$$

Thus $\Big(S(\underline{\delta})^{\widetilde{\boxtimes} n}\Big)^* \oslash (S^{\epsilon})^*$ is isomorphic as a $k(S_\mu \wr S_n)$-module to

$$\Big(\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \otimes \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n}\Big) \oslash \Big(\mathrm{Sgn}_n \otimes S^{\epsilon'}\Big)$$

which by (4.3.2) is isomorphic to

$$\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \otimes \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n} \otimes \mathrm{Inf}_n^{m\wr n} \Big(\mathrm{Sgn}_n \otimes S^{\epsilon'}\Big)$$

which by Proposition 4.3.1 is isomorphic to

$$\begin{aligned}&\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \otimes \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n} \otimes \\ &\qquad\qquad \mathrm{Inf}_n^{m\wr n}(\mathrm{Sgn}_n) \otimes \mathrm{Inf}_n^{m\wr n}(S^{\epsilon'}).\end{aligned}$$

By commutativity of the inner tensor product of group modules, this is isomorphic to

$$\begin{aligned}&\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \otimes \mathrm{Inf}_n^{m\wr n}(\mathrm{Sgn}_n) \otimes \\ &\qquad\qquad \big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n} \otimes \mathrm{Inf}_n^{m\wr n}(S^{\epsilon'})\end{aligned}$$

which by (4.3.2) is isomorphic to

$$\Big(\big(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\big)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n\Big) \otimes \Big(\big(S^{(\delta^1)'} \boxtimes \cdots \boxtimes S^{(\delta^t)'}\big)^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\Big)$$

which is

$$\mathrm{Sgn}_{\mu\wr n} \otimes \Big(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\Big)$$

as required. □

Proposition 9.2.4. *With μ, $\underline{\delta}$ and ϵ as in Reduction 9.2.2, we have a k-vector space isomorphism*

$$\operatorname{Ext}^1_{\mu\wr n}\left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}, \mathbb{1}_{\mu\wr n}\right) \cong \operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right).$$

Proof. We have

$$\begin{aligned}
&\operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right)\\
&\quad\cong \operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, \left(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right)^{**} \otimes \mathbb{1}_{\mu\wr n}\right)\\
&\qquad\qquad \text{(since both } (-)^{**} \text{ and } - \otimes \mathbb{1}_{\mu\wr n} \text{ are operations which do not}\\
&\qquad\qquad\quad \text{change the isomorphism class of a module)}\\
&\quad\cong \operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n} \otimes \left(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right)^{*}, \mathbb{1}_{\mu\wr n}\right) \quad \text{(by Proposition 2.2.3)}
\end{aligned}$$

and by Lemma 9.2.3,

$$\operatorname{Sgn}_{\mu\wr n} \otimes \left(S(\underline{\delta}')^{\widetilde{\boxtimes} n} \oslash S^{\epsilon'}\right)^{*}$$

is isomorphic to

$$\operatorname{Sgn}_{\mu\wr n} \otimes \operatorname{Sgn}_{\mu\wr n} \otimes \left(S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right)$$

which by (9.2.6) is isomorphic to $S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}$. □

In light of Proposition 9.2.4 and Reduction 9.2.2, we see that in order to establish Proposition 9.2.1, it suffices to prove that for μ, $\underline{\delta}$ and ϵ as in Reduction 9.2.2, we have

$$\operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right) = 0 \tag{9.2.7}$$

(by taking $\underline{\delta}'$ in place of $\underline{\delta}$ and ϵ' in place of ϵ, noting that of course we have $\underline{\delta}'' = \underline{\delta}$ and $\epsilon'' = \epsilon$ and $|\underline{\delta}'| = |\underline{\delta}| = \mu$). In order to do this, we shall require a number of lemmas. Recall from (3.2.10) that for $\underline{\delta}$ a multipartition of m with $|\underline{\delta}| = \mu$ where μ is of length t, we have defined a kS_μ-module

$$M(\underline{\delta}) = M^{\delta^1} \boxtimes \cdots \boxtimes M^{\delta^t}.$$

Lemma 9.2.5. *With μ, $\underline{\delta}$ and ϵ as in Reduction 9.2.2, the $k(S_\mu \wr S_n)$-module $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon$ has a filtration where $S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$ occurs at the bottom and all other factors Q satisfy*

$$\mathrm{Hom}_{\mu \wr n}\left(\mathrm{Sgn}_{\mu \wr n}, Q\right) = 0.$$

Proof. By Young's rule (3.2.1) and properties of Kostka numbers (3.2.2), together with Lemma 6.1.1, the module $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon$ has a filtration by modules $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\nu$ for $\nu \vdash n$, where the bottom-most factor is $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$ and all the other factors satisfy $\nu \rhd \epsilon$. Let $\nu \vdash n$ with $\nu \rhd \epsilon$, and let us consider the k-vector space

$$\begin{aligned} &\mathrm{Hom}_{\mu \wr n}(\mathrm{Sgn}_{\mu \wr n}, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\nu) \cong \\ &\qquad\qquad \mathrm{Hom}_{\mu \wr n}\big((\mathrm{Sgn}_\mu)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\nu\big). \end{aligned} \tag{9.2.8}$$

Now the subgroup of $S_\mu \wr S_n$ consisting of all elements $(\sigma; e, \ldots, e)$ for $\sigma \in S_n$ (where e is the identity element of S_μ) is isomorphic to S_n, and hence a $k(S_\mu \wr S_n)$-module may be considered to be a kS_n-module by virtue of restriction to this subgroup. If we thus view $(\mathrm{Sgn}_\mu)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n$ and $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\nu$ as kS_n-modules, then we see easily that the former is isomorphic to the kS_n-module Sgn_n while the latter is a direct sum of copies of S^ν. It now follows that the k-vector space (9.2.8) may be exhibited as a subspace of a direct sum of copies of the space $\mathrm{Hom}_n(\mathrm{Sgn}_n, S^\nu) = \mathrm{Hom}_n(S^{(1^n)}, S^\nu)$. But $\nu \trianglerighteq \epsilon \rhd (1^n)$ and so by (3.3.1) we have $\mathrm{Hom}_n(S^{(1^n)}, S^\nu) = 0$, so that in turn the space (9.2.8) is zero.

To prove the lemma, it now suffices to show that $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$ has a filtration where the bottom-most factor is $S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$ and all other factors Q satisfy

$$\mathrm{Hom}_{\mu \wr n}\left(\mathrm{Sgn}_{\mu \wr n}, Q\right) = 0.$$

Now if we let t be the length of μ, then $\underline{\delta}$ also has length t since $\mu = |\underline{\delta}|$. We have that $M(\underline{\delta}) = M^{\delta^1} \boxtimes \cdots \boxtimes M^{\delta^t}$ with $\delta^i \vdash \mu_i$ for each i. By Young's rule (3.2.1) and the properties of the Kostka numbers (3.2.2), M^{δ^i} has a filtration by modules S^γ for $\gamma \vdash \mu_i$, where in the bottom-most factor we have $\gamma = \delta^i$ and in all other factors we have $\gamma \rhd \delta^i$. It now follows by Lemma 2.1.2 that $M(\underline{\delta})$ has a filtration by modules $S(\underline{\gamma}) = S^{\gamma^1} \boxtimes \cdots \boxtimes S^{\gamma^t}$ with $|\underline{\gamma}| = \mu$ where in the bottom-most factor we have $\underline{\gamma} = \underline{\delta}$ and in all the other factors we have $\gamma^l \rhd \delta^l$ for at least one $l \in \{1, \dots, t\}$. Explicitly, we have

$$M(\underline{\delta}) \sim \mathop{\mathcal{F}_{\langle 1 \rangle}}_{i=1}^{p} S(\underline{\gamma}^i)$$

for some (not necessarily pairwise unequal) multipartitions $\underline{\gamma}^1, \dots, \underline{\gamma}^p$ of m (p some integer), where $|\underline{\gamma}^i| = \mu$ and $\underline{\gamma}^1 = \underline{\delta}$ but for each $i > 1$ we have some l such that $\gamma^{i,l} \rhd \delta^l$ (where $\underline{\gamma}^i = (\gamma^{i,1}, \dots, \gamma^{i,t})$). Hence by Lemma 6.2.1 and Lemma 6.1.1, $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$ has a filtration

$$M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon \sim \mathop{\mathcal{F}_{\langle [n,1] \rangle}}_{\alpha \in \Omega_n^p} \left[\left(S(\underline{\gamma}^1), \dots, S(\underline{\gamma}^p) \right)^{\widetilde{\boxtimes} \alpha} \right] \Big\uparrow_{\mu \wr \alpha}^{\mu \wr n} \oslash S^\epsilon.$$

The bottom-most factor in this filtration (indexed by $\alpha = [n, 1]$) is $S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon$, and so to prove our claim it now suffices to show that if $\alpha \neq [n, 1]$ then

$$\mathrm{Hom}_{\mu \wr n} \left(\mathrm{Sgn}_{\mu \wr n}, \left[\left(S(\underline{\gamma}^1), \dots, S(\underline{\gamma}^p) \right)^{\widetilde{\boxtimes} \alpha} \right] \Big\uparrow_{\mu \wr \alpha}^{\mu \wr n} \oslash S^\epsilon \right) = 0. \tag{9.2.9}$$

But for $\alpha = (\alpha_1, \dots, \alpha_p) \in \Omega_n^p$, we have by Proposition 4.3.7 that

$$\left[\left(S(\underline{\gamma}^1), \dots, S(\underline{\gamma}^p) \right)^{\widetilde{\boxtimes} \alpha} \right] \Big\uparrow_{\mu \wr \alpha}^{\mu \wr n} \oslash S^\epsilon \cong \left[\left(S(\underline{\gamma}^1)^{\widetilde{\boxtimes} \alpha_1} \boxtimes \cdots \boxtimes S(\underline{\gamma}^p)^{\widetilde{\boxtimes} \alpha_p} \right) \oslash S^\epsilon \big\downarrow_\alpha^n \right] \Big\uparrow_{\mu \wr \alpha}^{\mu \wr n}$$

and so by Theorem 2.2.4 we see that the left-hand side of (9.2.9) is isomorphic to

$$\mathrm{Hom}_{\mu \wr \alpha} \left(\left[(\mathrm{Sgn}_\mu)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n \right] \Big\downarrow_{\mu \wr \alpha}^{\mu \wr n}, \left(S(\underline{\gamma}^1)^{\widetilde{\boxtimes} \alpha_1} \boxtimes \cdots \boxtimes S(\underline{\gamma}^p)^{\widetilde{\boxtimes} \alpha_p} \right) \oslash S^\epsilon \big\downarrow_\alpha^n \right). \tag{9.2.10}$$

Now let $\alpha = (\alpha_1, \ldots, \alpha_p) \in \Omega_n^p$ with $\alpha \neq [n, 1]$. So we have some $i \in \{2, \ldots, p\}$ such that $\alpha_i > 0$, and from above we then have some l such that $\gamma^{i,l} \rhd \delta^l$. Now let us consider the subgroup of $S_\mu \wr S_\alpha$ consisting of all elements of the form $(e; e, \ldots, e, \sigma, e, \ldots, e)$ where the element $\sigma \in S_\mu$ occurs in place $\alpha_1 + \cdots + \alpha_{i-1} + 1$. This subgroup is isomorphic to S_μ, and hence a $k(S_\mu \wr S_\alpha)$-module may be considered to be a kS_μ-module by virtue of restriction to this subgroup. If we thus consider the module

$$\left[(\mathrm{Sgn}_\mu)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n\right]\Big\downarrow^{\mu\wr n}_{\mu\wr\alpha} = \left[\left(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\right)^{\widetilde{\boxtimes} n} \oslash \mathrm{Sgn}_n\right]\Big\downarrow^{\mu\wr n}_{\mu\wr\alpha}$$

to be a kS_μ-module, then we easily see that it is isomorphic to the module $\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}$. Further, if we view the module

$$\left(S(\underline{\gamma}^1)^{\widetilde{\boxtimes}\alpha_1} \boxtimes \cdots \boxtimes S(\underline{\gamma}^p)^{\widetilde{\boxtimes}\alpha_p}\right) \oslash S^\epsilon\big\downarrow^n_\alpha$$

as a kS_μ-module in this way, then we see that it is isomorphic to a direct sum of copies of $S(\underline{\gamma}^i)$. It now follows that, as a k-vector space, (9.2.10) is contained in a direct sum of copies of

$$\mathrm{Hom}_\mu(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}, S(\underline{\gamma}^i)), \tag{9.2.11}$$

and hence to prove our claim, it now suffices to prove that the module (9.2.11) is zero. We have $S(\underline{\gamma}^i) = S^{\gamma^{i,1}} \boxtimes \cdots \boxtimes S^{\gamma^{i,t}}$ as a kS_μ-module, and hence by Proposition 2.1.3, (9.2.11) is isomorphic as a k-vector space to

$$\bigboxtimes_{s=1}^{t} \mathrm{Hom}_{\mu_s}(\mathrm{Sgn}_{\mu_s}, S^{\gamma^{i,s}}).$$

Recall that we have some l such that $\gamma^{i,l} \rhd \delta^l$. But then Sgn_{μ_l} is the Specht module indexed by the partition (1^{μ_l}), and we have $\gamma^{i,l} \rhd \delta_l \trianglerighteq (1^{\mu_l})$, so by (3.3.2) we see that $\mathrm{Hom}_{\mu_l}(\mathrm{Sgn}_{\mu_l}, S^{\gamma^{i,l}}) = 0$. It follows that (9.2.11) is zero, and the proof is complete. $\square$

Lemma 9.2.6. *For m, n non-negative integers and any composition μ of m, we have*

$$\mathrm{Ext}^1_{\mu\wr n}(\mathrm{Sgn}_{\mu\wr n}, \mathbb{1}_{\mu\wr n}) = 0.$$

Proof. Now suppose we have some $k(S_\mu \wr S_n)$-module E with $x \in E$ such that kx is a $k(S_\mu \wr S_n)$-submodule of E with $kx \cong \mathbb{1}_{\mu\wr n}$ (where kx denotes the k-span of x in E), and $\frac{E}{kx} \cong \mathrm{Sgn}_{\mu\wr n}$. By Proposition 2.1.4, it is enough to show that E has a direct sum decomposition

$$E = kx \oplus Z \tag{9.2.12}$$

as a $k(S_\mu \wr S_n)$-module. Before we do this, we must do a little preliminary work.

Firstly, we let

$$B = \{(e; \alpha_1, \ldots, \alpha_n) \mid \alpha_i \in S_\mu\}$$

and

$$T = \{(\sigma; e, \ldots, e) \mid \sigma \in S_n\},$$

so that B and T are subgroups of $S_\mu \wr S_n$, with B isomorphic to the direct product of n copies of S_μ and T isomorphic to S_n.

Let t be the length of μ. Recall that we have defined a sign module $\mathrm{Sgn}_\mu = \mathrm{Sgn}_n{\downarrow}^n_\mu$ for S_μ, and under the canonical isomorphism $S_\mu \cong S_{\mu_1} \times \cdots \times S_{\mu_t}$ we have from (3.1.1) that Sgn_μ corresponds to the module

$$\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}.$$

We therefore have that

$$\mathrm{Ext}^1_\mu(\mathrm{Sgn}_\mu, \mathbb{1}_\mu) \cong \mathrm{Ext}^1_\mu(\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}, \mathbb{1}_{\mu_1} \boxtimes \cdots \boxtimes \mathbb{1}_{\mu_t})$$

and then by applying Proposition 2.1.3 and Lemma 3.3.7 we see that

$$\mathrm{Ext}^1_\mu(\mathrm{Sgn}_\mu, \mathbb{1}_\mu) = 0. \tag{9.2.13}$$

Now under our identification of T with S_n, we see that kx is a kS_n-submodule of $E{\downarrow}_T^{\mu\wr n}$, and moreover that $kx \cong \mathbb{1}_n$, while the quotient of $E{\downarrow}_T^{\mu\wr n}$ by kx is isomorphic to Sgn_n. It follows by Proposition 2.1.4 and Lemma 3.3.7 that we have some $y \in E$ such that, as a kT-module,

$$E{\downarrow}_T^{\mu\wr n} \cong kx \oplus ky$$

and moreover that an element $(\sigma; e, \ldots, e)$ of T acts on y as multiplication by $\mathrm{sgn}(\sigma)$. We can perform a similar analysis on the module $E{\downarrow}_B^{\mu\wr n}$. For this, we recall that B is isomorphic to $(S_\mu)^n$, and under this identification we may easily see that kx is a $k(S_\mu)^n$-submodule of the $k(S_\mu)^n$-module $E{\downarrow}_B^{\mu\wr n}$, and moreover that $kx \cong \mathbb{1}_{(S_\mu)^n}$, while the quotient of $E{\downarrow}_B^{\mu\wr n}$ by kx is isomorphic to the $k(S_\mu)^n$-module $(\mathrm{Sgn}_\mu)^{\boxtimes n}$ (the external tensor product of n copies of Sgn_μ). But we have

$$\mathrm{Ext}^1_{k(S_\mu)^n}\left((\mathrm{Sgn}_\mu)^{\boxtimes n}, \mathbb{1}_{(S_\mu)^n}\right) \cong \mathrm{Ext}^1_{k(S_\mu)^n}\left((\mathrm{Sgn}_\mu)^{\boxtimes n}, \left(\mathbb{1}_\mu\right)^{\boxtimes n}\right)$$

and by applying Proposition 2.1.3 and (9.2.13) we see that this space is zero. It follows by Proposition 2.1.4 that we have some $z \in E$ such that, as a kB-module

$$E{\downarrow}_{kB}^{\mu\wr n} \cong kx \oplus kz$$

and moreover an element $(e; \alpha_1, \ldots, \alpha_n)$ of B acts on z as multiplication by $\mathrm{sgn}(\alpha_1) \cdots \mathrm{sgn}(\alpha_n)$.

So we have $y = ax + bz$ for some $a, b \in k$. Since B and T generate the group $S_\mu \wr S_n$, in order to establish the desired decomposition (9.2.12), it suffices to prove that $a = 0$. Now if n or m is 1 then the proposition reduces to Lemma 3.3.7, so let us assume that both n and m are at least 2. Let us fix some elements of T and B. Indeed, let us define

$$\begin{aligned}
\sigma &= \big((1,2); e, \dots, e\big) \in T \\
\tau_1 &= \big(e; (1,2), e \dots, e\big) \in B \\
\tau_2 &= \big(e; e, (1,2), e \dots, e\big) \in B.
\end{aligned}$$

We have

$$\begin{aligned}
& x\sigma = x\tau_1 = x\tau_2 = x \\
& y\sigma = -y \\
& z\tau_1 = z\tau_2 = -z.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
y\tau_1\sigma &= (ax+bz)\tau_1\sigma \\
&= (ax-bz)\sigma \\
&= (2ax-(ax+bz))\sigma \\
&= (2ax-y)\sigma \\
&= 2ax+y.
\end{aligned}$$

We also have

$$\begin{aligned}
y\sigma\tau_2 &= -y\tau_2 \\
&= -(ax+bz)\tau_2 \\
&= -(ax-bz) \\
&= -ax+bz.
\end{aligned}$$

But we also have $\tau_1\sigma = \sigma\tau_2$, and so we have

$$\begin{aligned} & 2ax + y = -ax + bz \\ \implies & 2ax + ax + bz = -ax + bz \\ \implies & 4ax = 0 \\ \implies & a = 0 \end{aligned}$$

where the last implication uses the facts that $x \neq 0$ and that $\mathrm{char}(k)$ is not 2. $\square$

Lemma 9.2.7. *With μ, $\underline{\delta}$, and ϵ as in Reduction 9.2.2,*

$$\mathrm{Ext}^1_{\mu \wr n}\left(\mathrm{Sgn}_{\mu\wr n},\, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon\right) = 0.$$

Proof. Firstly let us prove that

$$M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon \cong \mathbb{1}\!\uparrow_{\underline{\delta}\wr\epsilon}^{\mu\wr n} \tag{9.2.14}$$

for which we shall use Corollary 2.2.7. Recall from (3.1.2) that if α is some composition then $\dim_k(M^\alpha) = \frac{|S_{|\alpha|}|}{|S_\alpha|}$. Let t be the length of μ, so that t is also the length of $\underline{\delta}$ since $|\underline{\delta}| = \mu$. Recalling that $M(\underline{\delta}) = M^{\delta^1} \boxtimes \cdots \boxtimes M^{\delta^t}$, we see that

$$\begin{aligned} \dim_k\big(M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon\big) &= \dim_k\big(M(\underline{\delta})\big)^n \dim_k\big(M^\epsilon\big) \\ &= \left(\prod_{i=1}^t \frac{|S_{|\delta^i|}|}{|S_{\delta^i}|}\right)^n \left(\frac{|S_{|\epsilon|}|}{|S_\epsilon|}\right) \\ &= \left(\prod_{i=1}^t \frac{|S_{\mu_i}|}{|S_{\delta^i}|}\right)^n \left(\frac{|S_n|}{|S_\epsilon|}\right) \\ &\qquad \text{(because } |\underline{\delta}| = \mu \text{ and } \epsilon \vdash n\text{)} \\ &= \left(\frac{|S_\mu|}{|S_{\underline{\delta}}|}\right)^n \left(\frac{|S_n|}{|S_\epsilon|}\right) \\ &= \frac{|S_\mu \wr S_n|}{|S_{\underline{\delta}} \wr S_\epsilon|} \end{aligned}$$

as required. Now recall from page 47 that for any composition α, we have an element $\tau(\alpha) \in M^\alpha$ upon which S_α acts trivially, but which generates M^α as a $kS_{|\alpha|}$-module. It is easy to check that the pure tensor

$$\big(\tau(\delta^1) \otimes \cdots \otimes \tau(\delta^t)\big)^{\otimes n} \otimes \tau(\epsilon) \ \in\ M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon$$

(where $(x)^{\otimes n}$ denotes the tensor product of n copies of an element x) generates $M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon$ as a $k(S_\mu \wr S_n)$-module, but is acted upon trivially by $S_{\underline{\delta}} \wr S_\epsilon$. Hence all of the conditions of Corollary 2.2.7 are satisfied. Thus (9.2.14) holds. We now have, using Theorem 2.2.4 (the Eckmann-Shapiro lemma), that

$$\begin{aligned}\operatorname{Ext}^1_{\mu\wr n}\big(\operatorname{Sgn}_{\mu\wr n},\, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon\big) &\cong \operatorname{Ext}^1_{\mu\wr n}\Big(\operatorname{Sgn}_{\mu\wr n},\, \mathbb{1}_{\underline{\delta}\wr\epsilon}\big\uparrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon}\Big)\\ &\cong \operatorname{Ext}^1_{\underline{\delta}\wr\epsilon}\big(\operatorname{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon},\, \mathbb{1}_{\underline{\delta}\wr\epsilon}\big).\end{aligned}$$

Now we have by transitivity of induction that $\operatorname{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} = \operatorname{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon}\big\downarrow^{\mu\wr\epsilon}_{\underline{\delta}\wr\epsilon}$. But using Proposition 4.3.7, we have

$$\operatorname{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon} = \Big[\big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n} \oslash \operatorname{Sgn}_n\Big]\Big\downarrow^{\mu\wr n}_{\mu\wr\epsilon} = \big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon} \oslash \operatorname{Sgn}_n\big\downarrow^n_\epsilon.$$

By the isomorphism (4.3.2), this is in fact the internal tensor product $\big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon} \otimes \operatorname{Inf}^{\mu\wr\epsilon}_\epsilon\big(\operatorname{Sgn}_n\big\downarrow^n_\epsilon\big)$ of $k(S_\mu \wr S_\epsilon)$-modules. Hence, we see that

$$\operatorname{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} \cong \Big[\big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon} \otimes \operatorname{Inf}^{\mu\wr\epsilon}_\epsilon\big(\operatorname{Sgn}_n\big\downarrow^n_\epsilon\big)\Big]\Big\downarrow^{\mu\wr\epsilon}_{\underline{\delta}\wr\epsilon}$$

as $k(S_{\underline{\delta}} \wr S_\epsilon)$-modules. By (2.2.4), the right-hand side is isomorphic to

$$\big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\mu\wr\epsilon}\big\downarrow^{\mu\wr\epsilon}_{\underline{\delta}\wr\epsilon} \otimes \operatorname{Inf}^{\mu\wr\epsilon}_\epsilon\big(\operatorname{Sgn}_n\big\downarrow^n_\epsilon\big)\big\downarrow^{\mu\wr\epsilon}_{\underline{\delta}\wr\epsilon}$$

which is clearly equal to

$$\big(\operatorname{Sgn}_\mu\big)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} \otimes \operatorname{Inf}^{\underline{\delta}\wr\epsilon}_\epsilon\big(\operatorname{Sgn}_n\big\downarrow^n_\epsilon\big)$$

and by (4.3.2), this is the $k(S_{\underline{\delta}} \wr S_\epsilon)$-module $\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes} n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} \oslash \mathrm{Sgn}_n\big\downarrow^{n}_{\epsilon}$. Using (3.1.1), we see that if $\epsilon = (\epsilon_1, \ldots, \epsilon_s)$, then this module is isomorphic to

$$\left(\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_1}\big\downarrow^{\mu\wr\epsilon_1}_{\underline{\delta}\wr\epsilon_1} \boxtimes \cdots \boxtimes \left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_s}\big\downarrow^{\mu\wr\epsilon_s}_{\underline{\delta}\wr\epsilon_s}\right) \oslash \left(\mathrm{Sgn}_{\epsilon_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\epsilon_s}\right)$$

and so by (4.3.5) we have

$$\mathrm{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} \cong \left(\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_1}\big\downarrow^{\mu\wr\epsilon_1}_{\underline{\delta}\wr\epsilon_1} \oslash \mathrm{Sgn}_{\epsilon_1}\right) \boxtimes \cdots \boxtimes \left(\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_s}\big\downarrow^{\mu\wr\epsilon_s}_{\underline{\delta}\wr\epsilon_s} \oslash \mathrm{Sgn}_{\epsilon_s}\right). \tag{9.2.15}$$

But we have by Proposition 4.3.5 that

$$\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_i}\big\downarrow^{\mu\wr\epsilon_i}_{\underline{\delta}\wr\epsilon_i} = \left(\mathrm{Sgn}_\mu\big\downarrow^{\mu}_{\underline{\delta}}\right)^{\widetilde{\boxtimes}\epsilon_i}$$

and using (3.1.1) we have

$$\begin{aligned}
\mathrm{Sgn}_\mu\big\downarrow^{\mu}_{\underline{\delta}} &= \left[\mathrm{Sgn}_{\mu_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\mu_t}\right]\big\downarrow^{\mu}_{\underline{\delta}} \\
&= \left(\mathrm{Sgn}_{\mu_1}\big\downarrow^{\mu_1}_{\delta^1}\right) \boxtimes \cdots \boxtimes \left(\mathrm{Sgn}_{\mu_t}\big\downarrow^{\mu_t}_{\delta^t}\right) \\
&= \mathrm{Sgn}_{\delta^1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\delta^t}
\end{aligned}$$

and this is just Sgn_δ, where δ is the composition of m defined by $\delta = \delta^1 \circ \cdots \circ \delta^t$. Hence, we have for each $i = 1, \ldots, s$ that

$$\left(\mathrm{Sgn}_\mu\right)^{\widetilde{\boxtimes}\epsilon_i}\big\downarrow^{\mu\wr\epsilon_i}_{\underline{\delta}\wr\epsilon_i} \oslash \mathrm{Sgn}_{\epsilon_i} \cong \left(\mathrm{Sgn}_\delta\right)^{\widetilde{\boxtimes}\epsilon_i} \oslash \mathrm{Sgn}_{\epsilon_i} = \mathrm{Sgn}_{\delta\wr\epsilon_i}.$$

It now follows from (9.2.15) that $\mathrm{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\underline{\delta}\wr\epsilon} \cong \mathrm{Sgn}_{\delta\wr\epsilon_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\delta\wr\epsilon_s}$. Thus, we have

$$\begin{aligned}
&\mathrm{Ext}^1_{\underline{\delta}\wr\epsilon}\left(\mathrm{Sgn}_{\mu\wr n}\big\downarrow^{\mu\wr n}_{\delta\wr\epsilon},\ \mathbb{1}_{\delta\wr\epsilon}\right) \\
&\qquad \cong \mathrm{Ext}^1_{\delta\wr\epsilon}\left(\mathrm{Sgn}_{\delta\wr\epsilon_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\delta\wr\epsilon_s},\ \mathbb{1}_{\delta\wr\epsilon}\right) \\
&\qquad \cong \mathrm{Ext}^1_{\delta\wr\epsilon}\left(\mathrm{Sgn}_{\delta\wr\epsilon_1} \boxtimes \cdots \boxtimes \mathrm{Sgn}_{\delta\wr\epsilon_s},\ \mathbb{1}_{\delta\wr\epsilon_1} \boxtimes \cdots \boxtimes \mathbb{1}_{\delta\wr\epsilon_s}\right).
\end{aligned}$$

By Proposition 2.1.3, we see that this k-vector space is the direct sum of the terms

$$\operatorname{Ext}^1_{\delta\wr\epsilon_i}\left(\operatorname{Sgn}_{\delta\wr\epsilon_i}, \mathbb{1}_{\delta\wr\epsilon_i}\right) \otimes \bigotimes_{\substack{p=1,\ldots,s \\ p\neq i}} \operatorname{Hom}_{\delta\wr\epsilon_p}\left(\operatorname{Sgn}_{\delta\wr\epsilon_p}, \mathbb{1}_{\delta\wr\epsilon_p}\right)$$

for $i = 1, \ldots, s$, and by Lemma 9.2.6 these terms are all zero, thus establishing the claim. $\square$

We now complete our proof of Proposition 9.2.1 by proving that (9.2.7) holds.

Proposition 9.2.8. *For $\mu \vdash m$, $\underline{\delta}$ a multipartition of m with $|\underline{\delta}| = \mu$, and $\epsilon \vdash n$, we have*

$$\operatorname{Ext}^1_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon\right) = 0.$$

Proof. Using the filtration from Lemma 9.2.5, we have a short exact sequence

$$0 \longrightarrow S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon \longrightarrow M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon \longrightarrow \frac{M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon}{S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon} \longrightarrow 0 \tag{9.2.16}$$

where the module

$$\frac{M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon}{S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon}$$

has a filtration by modules Q such that $\operatorname{Hom}_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, Q\right) = 0$, whence we have by Proposition 2.1.1 that

$$\operatorname{Hom}_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, \frac{M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^\epsilon}{S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^\epsilon}\right) = 0. \tag{9.2.17}$$

Now let us apply the functor $\operatorname{Hom}_{\mu\wr n}\left(\operatorname{Sgn}_{\mu\wr n}, -\right)$ to (9.2.16) to obtain a long exact sequence

$$
\begin{array}{l}
0 \to \mathrm{Hom}_{\mu\wr n}\left(\mathrm{Sgn}_{\mu\wr n}, S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right) \longrightarrow \mathrm{Hom}_{\mu\wr n}\left(\mathrm{Sgn}_{\mu\wr n}, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^{\epsilon}\right) \\
\longrightarrow \mathrm{Hom}_{\mu\wr n}\left(\mathrm{Sgn}_{\mu\wr n}, \frac{M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^{\epsilon}}{S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}}\right) \longrightarrow \mathrm{Ext}^1_{\mu\wr n}\left(\mathrm{Sgn}_{\mu\wr n}, S(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash S^{\epsilon}\right) \\
\longrightarrow \mathrm{Ext}^1_{\mu\wr n}\left(\mathrm{Sgn}_{\mu\wr n}, M(\underline{\delta})^{\widetilde{\boxtimes} n} \oslash M^{\epsilon}\right) \longrightarrow \cdots
\end{array}
$$

and the result now follows by (9.2.17) and Lemma 9.2.7. $\square$

9.3 Structure of $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$

We now consider the spaces $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$, where $\underline{\gamma}$ may be any multicomposition. We begin by proving the following lemma, which is essentially a consequence of [6, Lemma 3.3 (2)].

Lemma 9.3.1. *Let $\nu \vdash n$ and $\alpha = (\alpha_1, \ldots, \alpha_t) \vDash n$. Let $i \in \{1, \ldots, r\}$. Then the $k(S_m \wr S_n)$-module*

$$S^{[\nu,i]}\big\downarrow^{m\wr n}_{m\wr\alpha}$$

has (identifying $S_m \wr S_\alpha$ with $(S_m \wr S_{\alpha_1}) \times \cdots \times (S_m \wr S_{\alpha_t})$ via the canonical isomorphism) a filtration by modules of the form

$$S^{[\epsilon^1,i]} \boxtimes S^{[\epsilon^2,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^t,i]},$$

where for each $j \in \{1, \ldots, t\}$, we have $\epsilon^j \vdash \alpha_j$ and $\epsilon^j \subseteq \nu$.

Proof. We have using Proposition 4.3.7 that

$$
\begin{aligned}
S^{[\nu,i]}\big\downarrow^{m\wr n}_{m\wr\alpha} &\cong \left[\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n} \oslash S^{\nu}\right]\Big\downarrow^{m\wr n}_{m\wr\alpha} \\
&\cong \left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n}\Big\downarrow^{m\wr n}_{m\wr\alpha} \oslash \left(S^{\nu}\big\downarrow^{n}_{\alpha}\right) \qquad (9.3.1)
\end{aligned}
$$

as $k(S_m \wr S_\alpha)$-modules. Now by Lemma 3.2.3 and (3.2.12), we may see that $S^{\nu}\big\downarrow^{n}_{\alpha}$ has a filtration by modules $S^{\epsilon^1} \boxtimes \cdots \boxtimes S^{\epsilon^t}$ where $\epsilon^j \vdash \alpha_j$ and $\epsilon^j \subseteq \nu$ for $j = 1, \ldots, t$. Thus using Lemma 6.1.1, we see that the module (9.3.1) has a filtration by modules of the form

$$
\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n}\Big\downarrow^{m\wr n}_{m\wr\alpha} \oslash \left(S^{\epsilon^1} \boxtimes \cdots \boxtimes S^{\epsilon^t}\right) \qquad (9.3.2)
$$

where $\epsilon^j \vdash \alpha_j$ and $\epsilon^j \subseteq \nu$ for $j = 1, \ldots, t$. Using the canonical identification of $S_m \wr S_\alpha$ with $(S_m \wr S_{\alpha_1}) \times \cdots \times (S_m \wr S_{\alpha_t})$ and the fact that under this isomorphism the module $\left(S^{\mu^i}\right)^{\widetilde{\boxtimes} n}\Big\downarrow^{m\wr n}_{m\wr\alpha}$ corresponds to $\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}\alpha_1} \boxtimes \cdots \boxtimes \left(S^{\mu^i}\right)^{\widetilde{\boxtimes}\alpha_t}$, we see by the isomorphism (4.3.5) that the module (9.3.2) is isomorphic to

$$
\left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}\alpha_1} \oslash S^{\epsilon^1}\right) \boxtimes \cdots \boxtimes \left(\left(S^{\mu^i}\right)^{\widetilde{\boxtimes}\alpha_t} \oslash S^{\epsilon^t}\right)
$$

which is precisely

$$
S^{[\epsilon^1,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^t,i]}
$$

and the claim now follows. $\square$

Proposition 9.3.2. *Let k be a field whose characteristic is not 2. Let $\nu \vdash n$ and $i \in \{1, \ldots, r\}$, and further let $\underline{\gamma} = (\gamma^1, \ldots, \gamma^r)$ be an r component multicomposition of n. We have*

$$
\operatorname{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) \cong \begin{cases} 0 & \text{if } [\nu,i] \ntrianglerighteq \underline{\gamma} \\ k & \text{if } [\nu,i] = \underline{\gamma}. \end{cases}
$$

Further, if the characteristic of k is neither 2 nor 3, then for any such $[\nu, i]$ and $\underline{\gamma}$ we have

$$
\operatorname{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) = 0.
$$

Proof. Firstly, if we have $[\nu, i] = \underline{\gamma}$, then we have $M^{\underline{\gamma}} = M^{[\nu,i]}$ and thus $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$ is isomorphic to k by Proposition 9.1.1.

Now it is clear by the definition of $M^{\underline{\gamma}}$ and the isomorphism (4.3.6) that we have

$$M^{\underline{\gamma}} = \left[M^{[\gamma^1,1]} \boxtimes M^{[\gamma^2,2]} \boxtimes \cdots \boxtimes M^{[\gamma^r,r]}\right] \Big\uparrow_{m\wr|\underline{\gamma}|}^{m\wr n},$$

and hence by the Eckmann-Shapiro lemma (Theorem 2.2.4) we have an isomorphism of k-vector spaces

$$\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) \cong \mathrm{Hom}_{m\wr|\underline{\gamma}|}\left(S^{[\nu,i]}\Big\downarrow_{m\wr|\underline{\gamma}|}^{m\wr n}, M^{[\gamma^1,1]} \boxtimes M^{[\gamma^2,2]} \boxtimes \cdots \boxtimes M^{[\gamma^r,r]}\right).$$

Now by Lemma 9.3.1, we know that the module $S^{[\nu,i]}\big\downarrow_{m\wr|\underline{\gamma}|}^{m\wr n}$ is filtered by modules of the form $S^{[\epsilon^1,i]} \boxtimes S^{[\epsilon^2,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^r,i]}$ where $\epsilon^j \vdash |\gamma^j|$ and $\epsilon^j \subseteq \nu$ for each j. Thus in order to prove that $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) = 0$, it suffices by Proposition 2.1.1 to prove that for any such $\epsilon^1, \ldots, \epsilon^r$, we have

$$\mathrm{Hom}_{m\wr|\underline{\gamma}|}\left(S^{[\epsilon^1,i]} \boxtimes S^{[\epsilon^2,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^r,i]}, M^{[\gamma^1,1]} \boxtimes M^{[\gamma^2,2]} \boxtimes \cdots \boxtimes M^{[\gamma^r,r]}\right) = 0.$$

But by Proposition 2.1.3,

$$\mathrm{Hom}_{m\wr|\underline{\gamma}|}\left(S^{[\epsilon^1,i]} \boxtimes S^{[\epsilon^2,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^r,i]}, M^{[\gamma^1,1]} \boxtimes M^{[\gamma^2,2]} \boxtimes \cdots \boxtimes M^{[\gamma^r,r]}\right)$$

is isomorphic as a k-vector space to

$$\mathrm{Hom}_{m\wr|\gamma^1|}\left(S^{[\epsilon^1,i]}, M^{[\gamma^1,1]}\right) \otimes \cdots \otimes \mathrm{Hom}_{m\wr|\gamma^r|}\left(S^{[\epsilon^r,i]}, M^{[\gamma^r,r]}\right)$$

and thus it suffices to prove that whenever we have $[\nu, i] \ntrianglerighteq \underline{\gamma}$ and $\epsilon^1, \ldots, \epsilon^r$ such that $\epsilon^j \vdash |\gamma^j|$ and $\epsilon^j \subseteq \nu$ for each j, then we must have some $l \in \{1, \ldots, r\}$ such that the space $\mathrm{Hom}_{m\wr|\gamma^l|}\left(S^{[\epsilon^l,i]}, M^{[\gamma^l,l]}\right)$ is zero. Now it is easy to see that the condition $[\nu, i] \ntrianglerighteq \underline{\gamma}$ is equivalent to having either $\gamma^j \neq ()$ for some $j < i$ or else having some s such that $\sum_{j=1}^{s} \nu_j < \sum_{j=1}^{s} \gamma_j^i$. Now suppose

$[\nu, i] \not\trianglerighteq \underline{\gamma}$. If we have some $j < i$ such that $\gamma^j \neq ()$, then for any partition $\epsilon \vdash |\gamma^j|$ we have by Proposition 9.1.1 that

$$\mathrm{Hom}_{m \wr |\gamma^j|}\left(S^{[\epsilon,i]}, M^{[\gamma^j,j]}\right) = 0,$$

and hence $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) = 0$. On the other hand, if we have some s such that $\sum_{j=1}^{s} \nu_j < \sum_{j=1}^{s} \gamma^i_j$, then for any partition $\epsilon \vdash |\gamma^i|$ such that $\epsilon \subseteq \nu$, we certainly have $\epsilon \not\trianglerighteq \gamma^i$, and so we have by Proposition 9.1.1 that

$$\mathrm{Hom}_{m \wr |\gamma^i|}\left(S^{[\epsilon,i]}, M^{[\gamma^i,i]}\right) = 0,$$

and hence $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right) = 0$.

Finally, we assume that the characteristic of k is neither 2 nor 3, and we consider the space $\mathrm{Ext}^1_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$. By the same argument that was used above for the space $\mathrm{Hom}_{m\wr n}\left(S^{[\nu,i]}, M^{\underline{\gamma}}\right)$ (using the Eckmann-Shapiro lemma, Lemma 9.3.1, and Proposition 2.1.1), we find that it is enough to show that for any $\epsilon^1, \ldots, \epsilon^r$ where $\epsilon^j \vdash |\gamma^j|$, we have

$$\mathrm{Ext}^1_{m\wr|\underline{\gamma}|}\left(S^{[\epsilon^1,i]} \boxtimes S^{[\epsilon^2,i]} \boxtimes \cdots \boxtimes S^{[\epsilon^r,i]}, M^{[\gamma^1,1]} \boxtimes M^{[\gamma^2,2]} \boxtimes \cdots \boxtimes M^{[\gamma^r,r]}\right) = 0,$$

and this follows at once from Proposition 2.1.3 and Proposition 9.2.1. □

9.4 Structure of $\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$

We now consider the spaces $\mathrm{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$ and $\mathrm{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\gamma}}\right)$, where $\underline{\nu}$ is a multipartition of n of length r and $\underline{\gamma}$ is a multicomposition of n of length r (n, m, and r as above). Recall from page 180 that we have defined a $k(S_m \wr S_{|\underline{\nu}|})$-module

$$T^{\underline{\nu}} = \left(S^{\mu^1}, \ldots, S^{\mu^r}\right)^{\widetilde{\boxtimes}|\underline{\nu}|} \oslash \left(S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r}\right)$$

so that we have $S^{\underline{\nu}} = T^{\underline{\nu}}\uparrow^{m\wr n}_{m\wr|\underline{\nu}|}$. Thus we have (see page 72 for the definition of the subgroup $W_{\underline{\gamma}}$ of $S_m \wr S_n$)

$$\begin{aligned}
\operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}},\, M^{\underline{\gamma}}\right) &\cong \operatorname{Hom}_{m\wr n}\left(T^{\underline{\nu}}\uparrow^{m\wr n}_{m\wr|\underline{\nu}|},\, \mathbb{1}\uparrow^{m\wr n}_{W_{\underline{\gamma}}}\right) \\
&\qquad \text{(by Proposition 4.4.1)} \\
&\cong \operatorname{Hom}_{m\wr|\underline{\nu}|}\left(T^{\underline{\nu}},\, \mathbb{1}\uparrow^{m\wr n}_{W_{\underline{\gamma}}}\downarrow^{m\wr n}_{m\wr|\underline{\nu}|}\right) \\
&\qquad \text{(by Theorem 2.2.4)}
\end{aligned} \tag{9.4.1}$$

and by exactly the same reasoning we have

$$\operatorname{Ext}^1_{m\wr n}\left(S^{\underline{\nu}},\, M^{\underline{\gamma}}\right) \cong \operatorname{Ext}^1_{m\wr|\underline{\nu}|}\left(T^{\underline{\nu}},\, \mathbb{1}\uparrow^{m\wr n}_{W_{\underline{\gamma}}}\downarrow^{m\wr n}_{m\wr|\underline{\nu}|}\right). \tag{9.4.2}$$

Our main work in this section will be to use our work on tableaux to obtain a direct sum decomposition of the module

$$\mathbb{1}\uparrow^{m\wr n}_{W_{\underline{\gamma}}}\downarrow^{m\wr n}_{m\wr|\underline{\nu}|} \tag{9.4.3}$$

where the summands are indexed by tableaux of shape $|\underline{\nu}|$ and type $\underline{\gamma}$ with weakly increasing rows, and hence obtain corresponding direct sum decompositions of the above Hom and Ext^1 spaces. These decompositions will be the key to proving our desired results.

So let us consider the module (9.4.3). The natural tool to apply to this module is Mackey's theorem (Theorem 2.2.5), and so we want to obtain a complete non-redundant system $\mathcal{U}$ of $\left(W_{\underline{\gamma}},\, S_m\wr S_{|\underline{\nu}|}\right)$-double coset representatives in $S_m\wr S_n$, since then by Mackey's theorem we shall have

$$\mathbb{1}\uparrow^{m\wr n}_{W_{\underline{\gamma}}}\downarrow^{m\wr n}_{m\wr|\underline{\nu}|} \quad\cong\quad \bigoplus_{u\in\mathcal{U}} \mathbb{1}^u\downarrow^{(W_{\underline{\gamma}})^u}_{(W_{\underline{\gamma}})^u\cap(S_m\wr S_{|\underline{\nu}|})}\uparrow^{S_m\wr S_{|\underline{\nu}|}}_{(W_{\underline{\gamma}})^u\cap(S_m\wr S_{|\underline{\nu}|})}$$

where superscript u denotes conjugation of subgroups and modules (see page 32). But clearly

$$\mathbb{1}^u\downarrow^{(W_{\underline{\gamma}})^u}_{(W_{\underline{\gamma}})^u\cap(S_m\wr S_{|\underline{\nu}|})} \;=\; \mathbb{1}$$

as modules for $(W_{\underline{\gamma}})^u \cap (S_m \wr S_{|\underline{\nu}|})$, so

$$\mathbb{1}\uparrow_{W_{\underline{\gamma}}}^{m\wr n}\downarrow_{m\wr|\underline{\nu}|} \quad\cong\quad \bigoplus_{u\in\mathcal{U}} \mathbb{1}\uparrow_{(W_{\underline{\gamma}})^u\cap(S_m\wr S_{|\underline{\nu}|})}^{S_m\wr S_{|\underline{\nu}|}}. \tag{9.4.4}$$

Thus we wish to understand the modules $\mathbb{1}\uparrow_{(W_{\underline{\gamma}})^u\cap(S_m\wr S_{|\underline{\nu}|})}^{S_m\wr S_{|\underline{\nu}|}}$. For this, a good choice of the system of coset representatives $\mathcal{U}$ is key. The following lemma allows us to obtain such a set $\mathcal{U}$ from a system of $(S_{\underline{\gamma}},\, S_{|\underline{\nu}|})$-double coset representatives in S_n. For $\sigma \in S_n$, let us write $\hat{\sigma}$ for the element $(\sigma\,;\, e, e, \ldots, e)$ of $S_m \wr S_n$. Thus the map $\sigma \longmapsto \hat{\sigma}$ is an isomorphic embedding of S_n into $S_m \wr S_n$.

Lemma 9.4.1. *Let $\sigma_1, \sigma_2, \ldots, \sigma_N$ be a complete non-redundant system of $(S_{\underline{\gamma}},\, S_{|\underline{\nu}|})$-double coset representatives in S_n. Then $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_N$ is a complete non-redundant system of $\big(W_{\underline{\gamma}}\,,\, S_m\wr S_{|\underline{\nu}|}\big)$-double coset representatives in $S_m \wr S_n$.*

Proof. The proof is essentially by direct calculation.

To prove that the system is complete, let $(\sigma; \alpha_1, \ldots, \alpha_n) \in S_m \wr S_n$. Then we have some $i \in \{1, \ldots, N\}$ such that

$$S_{\underline{\gamma}}\sigma S_{|\underline{\nu}|} = S_{\underline{\gamma}}\sigma_i S_{|\underline{\nu}|}.$$

Thus $\sigma = \epsilon\sigma_i\delta$ for some $\epsilon \in S_{\underline{\gamma}}$ and some $\delta \in S_{|\underline{\nu}|}$. Then

$$\begin{aligned}
W_{\underline{\gamma}}(\sigma; \alpha_1, \ldots, \alpha_n) S_m\wr S_{|\underline{\nu}|} &= W_{\underline{\gamma}}(\sigma; e, \ldots, e)\underbrace{(e; \alpha_1, \ldots, \alpha_n)}_{\in\, S_m\wr S_{|\underline{\nu}|}} S_m\wr S_{|\underline{\nu}|} \\
&= W_{\underline{\gamma}}(\sigma; e, \ldots, e) S_m\wr S_{|\underline{\nu}|} \\
&= W_{\underline{\gamma}}(\epsilon\sigma_i\delta; e, \ldots, e) S_m\wr S_{|\underline{\nu}|} \\
&= W_{\underline{\gamma}}\underbrace{(\epsilon; e, \ldots, e)}_{\in\, W_{\underline{\gamma}}}(\sigma_i; e, \ldots, e)\underbrace{(\delta; e, \ldots, e)}_{\in\, S_m\wr S_{|\underline{\nu}|}} S_m\wr S_{|\underline{\nu}|} \\
&= W_{\underline{\gamma}}(\sigma_i; e, \ldots, e) S_m\wr S_{|\underline{\nu}|} \\
&= W_{\underline{\gamma}}\ \hat{\sigma}_i\ S_m\wr S_{|\underline{\nu}|}.
\end{aligned}$$

Thus the elements $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_N$ are indeed a complete set of $\big(W_{\underline{\gamma}}\,,\ S_m \wr S_{|\underline{\nu}|}\big)$-double coset representatives in $S_m \wr S_n$.

For non-redundancy, suppose that

$$W_{\underline{\gamma}}\ \hat{\sigma}_i\ S_m \wr S_{|\underline{\nu}|}\ \cong\ W_{\underline{\gamma}}\ \hat{\sigma}_j\ S_m \wr S_{|\underline{\nu}|}$$

for some $i, j \in \{1, \ldots, N\}$. So we have an element $(x\,;\,\alpha_1, \ldots, \alpha_n)$ of $W_{\underline{\gamma}}$ and an element $(y\,;\,\beta_1, \ldots, \beta_n)$ of $S_m \wr S_{|\underline{\nu}|}$ such that

$$\begin{aligned}\hat{\sigma}_i &= (x\,;\,\alpha_1, \ldots, \alpha_n)\hat{\sigma}_j(y\,;\,\beta_1, \ldots, \beta_n)\\ &= (x\,;\,\alpha_1, \ldots, \alpha_n)(\sigma_j\,;\,e, \ldots, e)(y\,;\,\beta_1, \ldots, \beta_n)\\ &= (x\sigma_j y\,;\,\alpha_{(1)\sigma_j^{-1}}\beta_1, \ldots, \alpha_{(n)\sigma_j^{-1}}\beta_n)\end{aligned}$$

and so $\sigma_i = x\sigma_j y$, where $x \in S_{\underline{\gamma}}$ and $y \in S_{|\underline{\nu}|}$, so indeed $i = j$. □

We now fix a complete non-redundant system $\sigma_1, \sigma_2, \ldots, \sigma_N$ of $(S_{\underline{\gamma}}, S_{|\underline{\nu}|})$-double coset representatives in S_n, where moreover *each σ_i is of minimal length in its left $S_{|\underline{\nu}|}$-coset $\sigma_i S_{|\underline{\nu}|}$*. This extra assumption will allow us to apply our work on tableaux from Chapter 7 to the situation at hand. We have by Lemma 9.4.1 that $\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_N$ is a complete non-redundant system of $\big(W_{\underline{\gamma}}\,,\ S_m \wr S_{|\underline{\nu}|}\big)$-double coset representatives in $S_m \wr S_n$. Thus we have by (9.4.4) that

$$\mathbb{1}\big\uparrow_{W_{\underline{\gamma}}}^{m\wr n}\big\downarrow_{m\wr|\underline{\nu}|}^{m\wr n} \quad\cong\quad \bigoplus_{i=1}^{N} \mathbb{1}\big\uparrow_{(W_{\underline{\gamma}})^{\hat{\sigma}_i} \cap (S_m \wr S_{|\underline{\nu}|})}^{S_m \wr S_{|\underline{\nu}|}}. \tag{9.4.5}$$

We therefore need to understand the subgroups $(W_{\underline{\gamma}})^{\hat{\sigma}_i} \cap (S_m \wr S_{|\underline{\nu}|})$, and it is here that we shall use our work on tableaux, by means of the following lemma. Recall that if α is a composition of n and $\underline{\gamma}$ is a multicomposition of n, then we have defined (see page 161) a tableau $\tau_{\underline{\gamma}}^{\alpha}$ of shape α and type $\underline{\gamma}$. Further, if we let l be the length of α and t be the length of $\underline{\gamma}$, then to any tableau τ of shape α and type $\underline{\gamma}$, we have associated (see page 172) an l-tuple $\underline{\underline{\Gamma}}(\tau) = \big(\underline{\Gamma}^1(\tau), \underline{\Gamma}^2(\tau), \ldots, \underline{\Gamma}^l(\tau)\big)$ of t-multicompositions such that $||\underline{\Gamma}^1(\tau)|| + ||\underline{\Gamma}^2(\tau)|| + \cdots + ||\underline{\Gamma}^l(\tau)|| = n$. So in particular for our multicomposition $\underline{\gamma}$ and multipartition $\underline{\nu}$ which both have length r, we have for any $\sigma \in S_n$ an r-tuple $\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ of r-multicompositions. Recall further that if $\underline{\underline{\gamma}}$ is a tuple of r-multicompositions such that $|||\underline{\underline{\gamma}}||| = n$, then we have associated (see page 73) to $\underline{\underline{\gamma}}$ a subgroup $W_{\underline{\underline{\gamma}}}$ of $S_m \wr S_n$.

Lemma 9.4.2. *Let $\sigma \in S_n$ be of minimal length in its left $S_{|\underline{\nu}|}$-coset $\sigma S_{|\underline{\nu}|}$. Then we have*

$$(W_{\underline{\gamma}})^{\hat{\sigma}} \cap (S_m \wr S_{|\underline{\nu}|}) \;=\; W_{\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)}.$$

Proof. Now by definition, $W_{\underline{\gamma}}$ consists exactly of the elements of $S_m \wr S_n$ of the form

$$\big(\,\pi\,;\,\underbrace{\alpha^1_1\,,\,\alpha^1_2\,,\,\ldots\,,\,\alpha^1_{|\gamma^1|}}_{\in S_{\mu^1} \leq S_m}\,,\,\underbrace{\alpha^2_1\,,\,\ldots\,,\,\alpha^2_{|\gamma^2|}}_{\in S_{\mu^2} \leq S_m}\,,\,\alpha^3_1\,,\,\ldots\ldots\,,\,\alpha^r_{|\gamma^r|}\big)$$

where $\pi \in S_{\underline{\gamma}}$ and, as indicated, each α^i_* lies in S_{μ^i}. Further, $(W_{\underline{\gamma}})^{\hat{\sigma}}$ consists exactly of all elements $\hat{\sigma}^{-1}(\pi\,;\,\alpha_1, \ldots, \alpha_n)\hat{\sigma}$ for $(\pi\,;\,\alpha_1, \ldots, \alpha_n) \in W_{\underline{\gamma}}$. Now

$$\begin{aligned} \hat{\sigma}^{-1}(\pi\,;\,\alpha_1, \ldots, \alpha_n)\hat{\sigma} &= (\sigma^{-1}\,;\,e, \ldots, e)(\pi\,;\,\alpha_1, \ldots, \alpha_n)(\sigma\,;\,e, \ldots, e) \\ &= (\sigma^{-1}\pi\sigma\,;\,\alpha_{(1)\sigma^{-1}}, \alpha_{(2)\sigma^{-1}}, \ldots, \alpha_{(n)\sigma^{-1}}). \end{aligned} \tag{9.4.6}$$

Now define $X_{\underline{\gamma}}$ to be the n-tuple

$$X_{\underline{\gamma}} = (\underbrace{1, 1, \ldots, 1}_{|\gamma^1| \text{ places}}, \underbrace{2, \ldots, 2}_{|\gamma^2| \text{ places}}, \ldots, \underbrace{r, \ldots, r}_{|\gamma^r| \text{ places}})$$

and define x_i to be the i^{th} entry of $X_{\underline{\gamma}}$ for $i = 1, \ldots, n$. Now define $X^\sigma_{\underline{\gamma}}$ to be the n-tuple

$$X^\sigma_{\underline{\gamma}} = (x_{(1)\sigma^{-1}},\, x_{(2)\sigma^{-1}},\, \ldots,\, x_{(n)\sigma^{-1}})$$

and define x^σ_i to be the i^{th} entry of $X^\sigma_{\underline{\gamma}}$ for $i = 1, \ldots, n$. Then by (9.4.6) we see that $(W_{\underline{\gamma}})^{\hat{\sigma}}$ is exactly the set of elements of $S_m \wr S_n$ of the form

$(\theta\,;\,\beta_1,\beta_2,\ldots,\beta_n)$ for $\theta\in(S_{\underline{\gamma}})^{\sigma}$ and $\beta_i\in S_{\hat{\mu}^i}$ where $\hat{\mu}^i=\mu^{x_i^{\sigma}}$. Further, since $S_m\wr S_{|\underline{\nu}|}$ is exactly the set of elements $(\delta\,;\,\epsilon_1,\epsilon_2,\ldots,\epsilon_n)$ for $\delta\in S_{|\underline{\nu}|}$ and $\epsilon_i\in S_m$, we see that $(W_{\underline{\gamma}})^{\hat{\sigma}}\cap(S_m\wr S_{|\underline{\nu}|})$ consists exactly of those elements of the form $(\theta\,;\,\beta_1,\beta_2,\ldots,\beta_n)$ for $\theta\in(S_{\underline{\gamma}})^{\sigma}\cap S_{|\underline{\nu}|}$ and $\beta_i\in S_{\hat{\mu}^i}$ where $\hat{\mu}^i=\mu^{x_i^{\sigma}}$.

Now to ease the notation let us define $\underline{\underline{\Gamma}}=\underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$, and as usual denote the i^{th} component of $\underline{\underline{\Gamma}}$ as $\underline{\Gamma}^i$ (a multicomposition), the j^{th} component of $\underline{\Gamma}^i$ as $\Gamma^{i,j}$ (a composition), and the s^{th} part of $\Gamma^{i,j}$ as $\Gamma^{i,j}_s$ (an integer). We consider $W_{\underline{\underline{\Gamma}}}$. By definition, $W_{\underline{\underline{\Gamma}}}$ is the subgroup of $S_m\wr S_n$ consisting of all elements of the form

$$
\begin{aligned}
\Big(\sigma;\\
&\underbrace{\alpha_1^{1,1},\alpha_2^{1,1},\ldots,\alpha_{|\Gamma^{1,1}|}^{1,1}}_{\in S_{\mu^1}\le S_m},\underbrace{\alpha_1^{1,2},\ldots,\alpha_{|\Gamma^{1,2}|}^{1,2}}_{\in S_{\mu^2}\le S_m},\alpha_1^{1,3},\quad\ldots\ldots,\underbrace{\alpha_1^{1,r},\ldots,\alpha_{|\Gamma^{1,r}|}^{1,r}}_{\in S_{\mu^r}\le S_m},\\
&\underbrace{\alpha_1^{2,1},\alpha_2^{2,1},\ldots,\alpha_{|\Gamma^{2,1}|}^{2,1}}_{\in S_{\mu^1}\le S_m},\alpha_1^{2,2},\ldots\ldots\qquad\qquad\ldots\ldots,\underbrace{\alpha_1^{2,r},\ldots,\alpha_{|\Gamma^{2,r}|}^{2,r}}_{\in S_{\mu^r}\le S_m},\\
&\qquad\qquad\vdots\qquad\qquad\qquad\vdots\qquad\qquad\qquad\vdots\\
&\underbrace{\alpha_1^{r,1},\alpha_2^{r,1},\ldots,\alpha_{|\Gamma^{r,1}|}^{r,1}}_{\in S_{\mu^1}\le S_m},\underbrace{\alpha_1^{r,2},\ldots,\alpha_{|\Gamma^{r,2}|}^{r,2}}_{\in S_{\mu^2}\le S_m},\alpha_1^{r,3},\quad\ldots\ldots,\underbrace{\alpha_1^{r,r},\ldots,\alpha_{|\Gamma^{r,r}|}^{r,r}}_{\in S_{\mu^r}\le S_m}\Big)
\end{aligned}
$$

where $\sigma \in S_{\underline{\underline{\Gamma}}}$. Let us define $Y_{\underline{\underline{\Gamma}}}$ to be the n-tuple

$$\begin{aligned} Y_{\underline{\underline{\Gamma}}} \;=\; \Big(& \underbrace{1,\,1,\,\ldots,\,1}_{|\Gamma^{1,1}|\text{ places}},\, \underbrace{2,\,2,\,\ldots,\,2}_{|\Gamma^{1,2}|\text{ places}},\, 3,\,\ldots\ldots,\, \underbrace{r,\,r,\,\ldots,\,r}_{|\Gamma^{1,r}|\text{ places}}, \\ & \underbrace{1,\,1,\,\ldots,\,1}_{|\Gamma^{2,1}|\text{ places}},\, 2,\,\ldots\ldots,\,\ldots\ldots,\, r, \\ & \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ & \underbrace{1,\,1,\,\ldots,\,1}_{|\Gamma^{r,1}|\text{ places}},\, \underbrace{2,\,2,\,\ldots,\,2}_{|\Gamma^{r,2}|\text{ places}},\, 3,\,\ldots\ldots,\, \underbrace{r,\,r,\,\ldots,\,r}_{|\Gamma^{r,r}|\text{ places}} \Big), \end{aligned}$$

and let us define y_i to be the i^{th} entry of $Y_{\underline{\underline{\Gamma}}}$. Thus $W_{\underline{\underline{\Gamma}}}$ consists exactly of the elements of $S_m \wr S_n$ of the form $(\theta\,;\,\beta_1, \beta_2, \ldots, \beta_n)$ for $\theta \in S_{\underline{\underline{\Gamma}}}$ an $\beta_i \in S_{\tilde{\mu}^i}$ where $\tilde{\mu}^i = \mu^{y_i}$. But by Proposition 7.4.2 and the minimality of the length of σ, we have $S_{\underline{\underline{\Gamma}}} = (S_{\underline{\gamma}})^{\sigma} \cap S_{|\underline{\nu}|}$. Hence, to prove that $(W_{\underline{\gamma}})^{\hat{\sigma}} \cap (S_m \wr S_{|\underline{\nu}|}) \;=\; W_{\underline{\underline{\Gamma}}}$ it is now sufficient to prove that $X_{\underline{\gamma}}^{\sigma} = Y_{\underline{\underline{\Gamma}}}$.

Let us define $Z_{\underline{\gamma}}$ to be the n-tuple

$$Z_{\underline{\gamma}} = \Big(\underbrace{(1,1),\,(1,1),\,\ldots,(1,1)}_{\gamma_1^1\text{ places}},\, \underbrace{(1,2),\,\ldots,(1,2)}_{\gamma_2^1\text{ places}},\,(1,3),\,\ldots\ldots\Big)$$

where, recall, $\underline{\gamma} = (\gamma^1, \ldots, \gamma^r)$ and $\gamma^i = (\gamma_1^i, \ldots, \gamma_r^i)$. Further, let z_i be the i^{th} entry of $Z_{\underline{\gamma}}$, and let us define $Z_{\underline{\gamma}}^{\sigma}$ to be the n-tuple

$$Z_{\underline{\gamma}}^{\sigma} = \big(z_{(1)\sigma^{-1}}, z_{(2)\sigma^{-1}}, \ldots, z_{(n)\sigma^{-1}}\big).$$

Recall the numbering of the boxes of a Young diagram from Section 7.1, and recall also that the tableau $\tau_{\underline{\gamma}}^{|\underline{\nu}|}$ is defined by entering the l^{th} entry of $Z_{\underline{\gamma}}$ into box number l of a Young diagram of shape $|\underline{\nu}|$, for each $l = 1, \ldots, n$. Then $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$ is obtained by moving the pair in box number l of $\tau_{\underline{\gamma}}^{|\underline{\nu}|}$ into box number $(l)\sigma$, for each $l = 1, \ldots, n$. Thus $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$ is the tableau of shape $|\underline{\nu}|$ where for each $l = 1, \ldots, n$, the box numbered l contains the $\big((l)\sigma^{-1}\big)^{\text{th}}$ entry of $Z_{\underline{\gamma}}$.

Thus in fact $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$ is the tableau obtained by entering the l^{th} entry of $Z_{\underline{\gamma}}^{\sigma}$ into box number l of a Young diagram of shape $|\underline{\nu}|$, for each $l = 1, \ldots, n$. Thus, if we form an n-tuple $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ of numbers from the set $\{1, \ldots, r\}$ by taking the l^{th} entry of $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ to be the first element of the pair in box number l of $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$ (i.e. if the pair in the box with number l is (i, j) then the l^{th} entry of $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ is i), then it is immediate that $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ is equal to the n-tuple $\widehat{Z}_{\underline{\gamma}}^{\sigma}$ whose l^{th} entry is defined to be the first element of the pair which appears in the l^{th} place of the tuple $Z_{\underline{\gamma}}^{\sigma}$. But it is clear that this tuple $\widehat{Z}_{\underline{\gamma}}^{\sigma}$ is also the n-tuple obtained by first forming the n-tuple whose l^{th} entry is the first element of the pair which appears in the l^{th} place of $Z_{\underline{\gamma}}$, and then for each $l = 1, \ldots, n$ moving the entry from the l^{th} place of this tuple to the $(l)\sigma^{\text{th}}$ place. But this tuple is $X_{\underline{\gamma}}^{\sigma}$ (by the definition of $X_{\underline{\gamma}}^{\sigma}$). Thus $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma) = X_{\underline{\gamma}}^{\sigma}$.

On the other hand, recall from the definition of $\underline{\underline{\Gamma}} = \underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ (see page 172) that $|\Gamma^{i,j}|$ is the number of pairs $(j, *)$ on the i^{th} row of $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$. Further, σ is of minimal length in $\sigma S_{|\underline{\nu}|}$, and so by Proposition 7.2.3 $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$ has weakly increasing rows. Hence, on the i^{th} row of $\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma$, all of the $|\Gamma^{i,1}|$ pairs of the form $(1, *)$ come first (reading left-to-right), followed by all of the $|\Gamma^{i,2}|$ pairs of the form $(2, *)$, and so on. It now follows at once that $U(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ is equal to $Y_{\underline{\underline{\Gamma}}}$ (by the definition of $Y_{\underline{\underline{\Gamma}}}$). Hence $X_{\underline{\gamma}}^{\sigma} = Y_{\underline{\underline{\Gamma}}}$ as required. $\square$

So looking back to (9.4.5), we see that, with $\underline{\underline{\Gamma}} = \underline{\underline{\Gamma}}(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma)$ as in the foregoing proof, we have

$$\mathbb{1}\Big\uparrow_{(W_{\underline{\gamma}})^{\hat{\sigma}_i} \cap (S_m \wr S_{|\underline{\nu}|})}^{S_m \wr S_{|\underline{\nu}|}} \cong \mathbb{1}\Big\uparrow_{W_{\underline{\underline{\Gamma}}}}^{S_m \wr S_{|\underline{\nu}|}}$$

and to understand this module, we can use the following result.

Lemma 9.4.3. *Let n, m, and r be as above. Let $\underline{\underline{\gamma}}$ be an r-tuple of r-multicompositions such that*

$$||\underline{\gamma}^1|| + \cdots + ||\underline{\gamma}^r|| = n.$$

Let $\alpha = (||\underline{\gamma}^1||, \ldots, ||\underline{\gamma}^r||) \vDash n$. Then the subgroup $W_{\underline{\underline{\gamma}}}$ of $S_m \wr S_n$ lies in the subgroup $S_m \wr S_\alpha$, and further we have a module isomorphism

$$\mathbb{1}\uparrow_{W_{\underline{\underline{\gamma}}}}^{m\wr\alpha} \;\cong\; M^{\underline{\gamma}^1} \boxtimes \cdots \boxtimes M^{\underline{\gamma}^r}$$

where the $k\big(S_m \wr S_{||\underline{\gamma}^1||} \times \cdots \times S_m \wr S_{||\underline{\gamma}^r||}\big)$ module on the right-hand side is viewed as a $k(S_m \wr S_\alpha)$ via the canonical isomorphism

$$S_m \wr S_\alpha \;\cong\; S_m \wr S_{||\underline{\gamma}^1||} \times \cdots \times S_m \wr S_{||\underline{\gamma}^r||}.$$

Proof. The fact that $W_{\underline{\underline{\gamma}}}$ lies in $S_m \wr S_\alpha$ is immediate from the definition of $W_{\underline{\underline{\gamma}}}$ (see 73). Recall that for each $i = 1, \ldots, r$, we have a $k(S_m \wr S_{||\underline{\gamma}^i||})$-module $M^{\underline{\gamma}^i}$. Indeed, recalling that $|\underline{\gamma}^i|$ is a composition of $||\underline{\gamma}^i||$ (of length r), we have by Proposition 4.4.1 that

$$M^{\underline{\gamma}^i} \cong \mathbb{1}\uparrow_{W_{\underline{\gamma}^i}}^{m\wr||\underline{\gamma}^i||}.$$

Now let us identify $S_m \wr S_\alpha$ with $\big(S_m \wr S_{||\underline{\gamma}^1||}\big) \times \cdots \times \big(S_m \wr S_{||\underline{\gamma}^r||}\big)$ via the canonical isomorphism. Under this identification $W_{\underline{\underline{\gamma}}}$ corresponds to the subgroup

$$W_{\underline{\gamma}^1} \times \cdots \times W_{\underline{\gamma}^r}$$

(see (4.2.2)). Thus we have

$$\begin{aligned}
\mathbb{1}\uparrow_{W_{\underline{\underline{\gamma}}}}^{m\wr\alpha} \;&=\; \mathbb{1}\uparrow_{W_{\underline{\gamma}^1} \times \cdots \times W_{\underline{\gamma}^r}}^{S_m \wr S_{||\underline{\gamma}^1||} \times \cdots \times S_m \wr S_{||\underline{\gamma}^r||}} \\
&\cong\; \mathbb{1}\uparrow_{W_{\underline{\gamma}^1}}^{m\wr||\underline{\gamma}^1||} \boxtimes \cdots \boxtimes \mathbb{1}\uparrow_{W_{\underline{\gamma}^r}}^{m\wr||\underline{\gamma}^r||} \\
&\cong\; M^{\underline{\gamma}^1} \boxtimes \cdots \boxtimes M^{\underline{\gamma}^r}
\end{aligned}$$

where the last isomorphism is again by Proposition 4.4.1. □

We can now combine Lemmas 9.4.3 and 9.4.2 with (9.4.5), to obtain

$$\mathbb{1}\uparrow_{W_{\underline{\gamma}}}^{m\wr n}\downarrow_{m\wr|\underline{\nu}|}^{m\wr n} \cong \bigoplus_{i=1}^{N} M^{\underline{\Gamma}^1(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma_i)} \boxtimes \cdots \boxtimes M^{\underline{\Gamma}^r(\tau_{\underline{\gamma}}^{|\underline{\nu}|}\sigma_i)}.$$

Further, by Proposition 7.2.4, we know that our complete non-redundant system of $(S_{\underline{\gamma}}, S_{|\underline{\nu}|})$-double coset representatives $\sigma_1, \ldots, \sigma_N$ (each of minimal length in its left $S_{|\underline{\nu}|}$-coset $\sigma_i S_{|\underline{\nu}|}$) is in bijective correspondence with the set $\mathcal{W}_{\underline{\gamma}}^{|\underline{\nu}|}$ of tableaux of shape $|\underline{\nu}|$ and type $\underline{\gamma}$ with weakly increasing rows, via the map

$$\sigma_i \longmapsto \tau_{\underline{\gamma}}^{|\underline{\nu}|} \sigma_i.$$

Hence, we can use $\mathcal{W}_{\underline{\gamma}}^{|\underline{\nu}|}$ to index the summation. We thus obtain

$$\mathbb{1}\uparrow_{W_{\underline{\gamma}}}^{m\wr n}\downarrow_{m\wr|\underline{\nu}|}^{m\wr n} \cong \bigoplus_{\tau\in\mathcal{W}_{\underline{\gamma}}^{|\underline{\nu}|}} M^{\underline{\Gamma}^1(\tau)} \boxtimes \cdots \boxtimes M^{\underline{\Gamma}^r(\tau)} \tag{9.4.7}$$

and this form makes it clear that this decomposition is independent of a choice of double coset representatives. Note that we have not made any assumptions about the characteristic of k in obtaining (9.4.7).

We now apply (9.4.7) to (9.4.1), and we thus have

$$\operatorname{Hom}_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) \;\cong\; \bigoplus_{\tau\,\in\,\mathcal{W}_{\underline{\gamma}}^{|\underline{\nu}|}} \operatorname{Hom}_{m\wr|\underline{\nu}|}\big(T^{\underline{\nu}},\, M^{\underline{\Gamma}^1(\tau)} \boxtimes \cdots \boxtimes M^{\underline{\Gamma}^r(\tau)}\big). \tag{9.4.8}$$

Now recall that

$$T^{\underline{\nu}} = \big(S^{\mu^1}, \ldots, S^{\mu^r}\big)^{\widetilde{\boxtimes}|\underline{\nu}|} \oslash \big(S^{\nu^1} \boxtimes \cdots \boxtimes S^{\nu^r}\big),$$

so that, using the isomorphism (4.3.6), $T^{\underline{\nu}}$ is the module

$$\Big((S^{\mu^1})^{\widetilde{\boxtimes}|\nu^1|} \oslash S^{\nu^1}\Big) \boxtimes \cdots \boxtimes \Big((S^{\mu^r})^{\widetilde{\boxtimes}|\nu^r|} \oslash S^{\nu^r}\Big) = S^{[\nu^1,1]} \boxtimes \cdots \boxtimes S^{[\nu^r,r]}.$$

Hence, by (9.4.8), we have a direct sum decomposition of k-vector spaces

$$\operatorname{Hom}_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) \;\cong$$
$$\bigoplus_{\tau\,\in\,\mathcal{W}_{\underline{\gamma}}^{|\underline{\nu}|}} \operatorname{Hom}_{m\wr|\underline{\nu}|}\big(S^{[\nu^1,1]} \boxtimes \cdots \boxtimes S^{[\nu^r,r]},\, M^{\underline{\Gamma}^1(\tau)} \boxtimes \cdots \boxtimes M^{\underline{\Gamma}^r(\tau)}\big), \tag{9.4.9}$$

and we shall return to this decomposition after dealing with the Ext^1 case. Indeed, by applying (9.4.7) to (9.4.2) as we have done for the Hom-space, we also obtain a decomposition

$$\begin{aligned}&\mathrm{Ext}^1_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) \;\cong\\ &\qquad\bigoplus_{\tau\in\mathcal{W}^{|\underline{\nu}|}_{\underline{\gamma}}} \mathrm{Ext}^1_{m\wr|\underline{\nu}|}\big(S^{[\nu^1,1]}\boxtimes\cdots\boxtimes S^{[\nu^r,r]},\, M^{\underline{\Gamma}^1(\tau)}\boxtimes\cdots\boxtimes M^{\underline{\Gamma}^r(\tau)}\big).\end{aligned}\tag{9.4.10}$$

If we take k to be a field whose characteristic is neither 2 nor 3, then all of the summands on the right-hand side of the decomposition (9.4.10) are easily seen to be zero via Proposition 2.1.3 and Proposition 9.3.2. We have thus proved the following result, which is our desired result on Ext^1-spaces.

Theorem 9.4.4. *Let k be a field whose characteristic is neither 2 nor 3. Let $\underline{\nu}$ be a multipartition of n with length r, and $\underline{\gamma}$ a multicomposition of n with length r. Then*

$$\mathrm{Ext}^1_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) = 0.$$

Now let us take k to be a field whose characteristic is not 2. Returning to the Hom-space decomposition (9.4.9), we find that by Proposition 2.1.3 we have an isomorphism of k-vector spaces

$$\begin{aligned}&\mathrm{Hom}_{m\wr|\underline{\nu}|}\big(S^{[\nu^1,1]}\boxtimes S^{[\nu^2,2]}\boxtimes\cdots\boxtimes S^{[\nu^r,r]},\, M^{\underline{\Gamma}^1(\tau)}\boxtimes\cdots\boxtimes M^{\underline{\Gamma}^r(\tau)}\big) \;\cong\\ &\qquad\qquad \mathrm{Hom}_{m\wr|\nu^1|}\big(S^{[\nu^1,1]}, M^{\underline{\Gamma}^1(\tau)}\big)\otimes\cdots\otimes \mathrm{Hom}_{m\wr|\nu^r|}\big(S^{[\nu^r,r]}, M^{\underline{\Gamma}^r(\tau)}\big).\end{aligned}$$

Thus we have obtained a decomposition of Hom-spaces

$$\begin{aligned}&\mathrm{Hom}_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) \;\cong\\ &\bigoplus_{\tau\in\mathcal{W}^{|\underline{\nu}|}_{\underline{\gamma}}} \mathrm{Hom}_{m\wr|\nu^1|}\big(S^{[\nu^1,1]}, M^{\underline{\Gamma}^1(\tau)}\big)\otimes\cdots\otimes \mathrm{Hom}_{m\wr|\nu^r|}\big(S^{[\nu^r,r]}, M^{\underline{\Gamma}^r(\tau)}\big).\end{aligned}\tag{9.4.11}$$

We can in fact refine the indexing set in (9.4.11) somewhat. Indeed, let $\tau \in \mathcal{W}^{|\underline{\nu}|}_{\underline{\gamma}}$ and suppose that for some $j \in \{1, \ldots, r\}$, a pair $(j, *)$ appears in some row of τ which lies lower than the j^{th} row, say on the i^{th} row (so we have $i > j$). Then by the first part of Proposition 7.4.4, we have $\Gamma^{i,j}(\tau) \neq ()$, which implies that $[\nu^i, i] \not\trianglerighteq \underline{\Gamma}^i(\tau)$, and so by Proposition 9.3.2 we have

$$\mathrm{Hom}_{m\wr|\nu^i|}\big(S^{[\nu^i,i]}, M^{\underline{\Gamma}^i(\tau)}\big) \;=\; 0.$$

Thus the τ^{th} summand of (9.4.11) is zero unless for each $j \in \{1, \ldots, r\}$, no pair $(j, *)$ appears lower than the j^{th} row of τ. We have thus proved the following result.

Theorem 9.4.5. *Let k be a field whose characteristic is not 2. Let $\underline{\nu}$ be a multipartition of n with length r, and $\underline{\gamma}$ a multicomposition of n with length r. Let $\widehat{\mathcal{W}}^{|\underline{\nu}|}_{\underline{\gamma}}$ be the set of all tableaux τ of shape $|\underline{\nu}|$ and type $\underline{\gamma}$ with weakly increasing rows such that for each $j \in \{1, \ldots, r\}$, no pair $(j, *)$ appears lower than the j^{th} row of τ. Then we have an isomorphism of k-vector spaces*

$$\begin{aligned}\mathrm{Hom}_{m\wr n}\big(S^{\underline{\nu}},\, M^{\underline{\gamma}}\big) &\cong \\ &\bigoplus_{\tau \in \widehat{\mathcal{W}}^{|\underline{\nu}|}_{\underline{\gamma}}} \mathrm{Hom}_{m\wr|\nu^1|}\big(S^{[\nu^1,1]}, M^{\underline{\Gamma}^1(\tau)}\big) \otimes \cdots \otimes \mathrm{Hom}_{m\wr|\nu^r|}\big(S^{[\nu^r,r]}, M^{\underline{\Gamma}^r(\tau)}\big).\end{aligned}$$

We can now prove our desired result on the structure of the Hom-space $\mathrm{Hom}_{m\wr n}\,(S^{\underline{\nu}}, M^{\underline{\gamma}})$.

Theorem 9.4.6. *Let k be a field whose characteristic is not 2. Let $\underline{\nu}$ be a multipartition of n with length r, and $\underline{\gamma}$ a multicomposition of n with length r. Then*

$$\mathrm{Hom}_{m\wr n}\,(S^{\underline{\nu}}, M^{\underline{\gamma}}) \cong \begin{cases} k & \text{if } \underline{\nu} = \underline{\gamma}, \\ 0 & \text{if } \underline{\nu} \not\trianglerighteq \underline{\gamma}. \end{cases}$$

Proof. If $\underline{\nu} = \underline{\gamma}$, then it is easy to see that $\widehat{\mathcal{W}}_{\underline{\gamma}}^{|\nu|} = \{\tau_{\underline{\nu}}^{|\nu|}\}$. It is also easy to see that

$$\underline{\underline{\Gamma}}(\tau_{\underline{\nu}}^{|\nu|}) = \big([\nu^1, 1], [\nu^2, 2], \ldots, [\nu^r, r]\big)\,,$$

so that by Theorem 9.4.5, we find that $\mathrm{Hom}_{m\wr n}\,(S^{\underline{\nu}}, M^{\underline{\nu}})$ is isomorphic to

$$\mathrm{Hom}_{m\wr|\nu^1|}\big(S^{[\nu^1,1]}, M^{[\nu^1,1]}\big) \otimes \cdots \otimes \mathrm{Hom}_{m\wr|\nu^r|}\big(S^{[\nu^r,r]}, M^{[\nu^r,r]}\big).$$

By Proposition 9.3.2, this is indeed just k.

If $\underline{\nu} \not\trianglerighteq \underline{\gamma}$, then by Proposition 7.4.4 we have for each $\tau \in \widehat{\mathcal{W}}_{\underline{\gamma}}^{|\nu|}$ an $i \in \{1, \ldots, r\}$ and some j such that

$$\underline{\Gamma}^i(\tau) = \big((), (), \ldots, (), \Gamma^{i,i}(\tau), \Gamma^{i,i+1}(\tau), \ldots, \Gamma^{i,r}(\tau)\big)$$

and such that

$$\sum_{q=1}^{j} \Gamma_q^{i,i}(\tau) > \sum_{q=1}^{j} \nu_q^i,$$

and the existence of such a j implies that $[\nu, i] \not\trianglerighteq \underline{\Gamma}^i(\tau)$. Hence by Proposition 9.3.2 we have

$$\mathrm{Hom}_{m\wr|\nu^i|}\big(S^{[\nu^i,i]}, M^{\underline{\Gamma}^i(\tau)}\big) = 0$$

and hence the τ^{th} summand of the summation in Theorem 9.4.5 is zero. Thus indeed

$$\mathrm{Hom}_{m\wr n}\big(S^{\underline{\nu}}, M^{\underline{\gamma}}\big) = 0$$

as required. $\square$

Original research in Chapter 9: Everything in Chapter 9 is original research.

Chapter 10

Homomorphisms and extensions between wreath Specht modules, and a stratifying system for $k\,(S_m \wr S_n)$

In this short final chapter, we shall prove wreath product analogues of (3.3.2) and Theorem 3.3.2, and using these we shall deliver the promised proof that, if k is algebraically closed and has characteristic neither 2 nor 3, then the Specht modules for $k(S_m \wr S_n)$ yield a stratifying system as defined in Section 3.4, and hence that Specht filtration multiplicities are well-defined for the wreath product algebra $k(S_m \wr S_n)$ as for the symmetric group algebra kS_n. We shall use an argument closely based on the corresponding work for kS_n in Section 3.4. Since the symmetric group S_n is a special case of the wreath product $S_m \wr S_n$, the same counter examples which prove that Specht filtration multiplicities for kS_n are not well-defined in characteristic 2 or 3 (see page 67) also prove that Specht filtration multiplicities for $k(S_m \wr S_n)$ are not well-defined in characteristic 2 or 3. Thus the result which we shall obtain is the best we could hope for.

As in previous chapters, we let m and n be non-negative integers and we let r be the number of distinct partitions of m.

10.1 Homomorphisms and extensions between wreath Specht modules

Now if k is a field whose characteristic is not 2, and $\underline{\nu}, \underline{\lambda}$ are multipartitions of n with length r, then we have by Theorem 9.4.6 that

$$\operatorname{Hom}_{m \wr n}\left(S^{\underline{\nu}}, M^{\underline{\lambda}}\right) \cong \begin{cases} k & \text{if } \underline{\nu} = \underline{\lambda}, \\ 0 & \text{if } \underline{\nu} \not\trianglerighteq \underline{\lambda}, \end{cases}$$

and by Proposition 6.5.2 we know that $S^{\underline{\lambda}}$ is a submodule of $M^{\underline{\lambda}}$. We have thus established the following theorem.

Theorem 10.1.1. *Let k be a field whose characteristic is not 2. Let $\underline{\nu}$ and $\underline{\lambda}$ be multipartitions of n with length r. Then we have*

$$\operatorname{Hom}_{m \wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) \cong \begin{cases} k & \textit{if } \underline{\nu} = \underline{\lambda}, \\ 0 & \textit{if } \underline{\nu} \not\trianglerighteq \underline{\lambda}. \end{cases}$$

Corollary 10.1.2. *Let k be a field whose characteristic is not 2 and let $\underline{\nu}$ be a multipartition of n with length r. Then the $k(S_m \wr S_n)$-module $S^{\underline{\nu}}$ is indecomposable.*

Proof. If $S^{\underline{\nu}}$ were not indecomposable, we could project to any non-zero proper summand and thus obtain an endomorphism of $S^{\underline{\nu}}$ which is not a scalar multiple of the identity, contradicting Theorem 10.1.1. $\square$

Theorem 10.1.3. *Let k be a field whose characteristic is neither 2 nor 3. Let $\underline{\nu}$ and $\underline{\lambda}$ be multipartitions of n with length r such that $\underline{\nu} \not\triangleright \underline{\lambda}$. Then we have*

$$\operatorname{Ext}^1_{m\wr n}\big(S^{\underline{\nu}},\, S^{\underline{\lambda}}\big) = 0.$$

Proof. By Proposition 6.5.2, we know that $M^{\underline{\lambda}}$ has a filtration by modules $S^{\underline{\alpha}}$, where $S^{\underline{\lambda}}$ occurs exactly once at the bottom of the filtration, and all the other modules $S^{\underline{\alpha}}$ which appear satisfy $\underline{\alpha} \triangleright \underline{\lambda}$. Thus we have a short exact sequence

$$0 \longrightarrow S^{\underline{\lambda}} \longrightarrow M^{\underline{\lambda}} \longrightarrow \frac{M^{\underline{\lambda}}}{S^{\underline{\lambda}}} \longrightarrow 0$$

where $\frac{M^{\underline{\lambda}}}{S^{\underline{\lambda}}}$ has a filtration by modules $S^{\underline{\alpha}}$ for multipartitions $\underline{\alpha} \triangleright \underline{\lambda}$. But $\underline{\alpha} \triangleright \underline{\lambda}$ implies $\underline{\nu} \not\trianglerighteq \underline{\alpha}$ (for if $\underline{\nu} \trianglerighteq \underline{\alpha}$ then we have $\underline{\nu} \trianglerighteq \underline{\alpha} \triangleright \underline{\lambda}$, contradicting $\underline{\nu} \not\triangleright \underline{\lambda}$), and so if $\underline{\alpha} \triangleright \underline{\lambda}$ then we have by Theorem 10.1.1 that

$$\operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, S^{\underline{\alpha}}\right) = 0.$$

It follows by Proposition 2.1.1 that

$$\operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, \frac{M^{\underline{\lambda}}}{S^{\underline{\lambda}}}\right) = 0. \tag{10.1.1}$$

We apply the functor $\operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, -\right)$ to our short exact sequence to obtain a long exact sequence

$$\begin{array}{l}
0 \to \operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) \longrightarrow \operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\lambda}}\right) \longrightarrow \operatorname{Hom}_{m\wr n}\left(S^{\underline{\nu}}, \frac{M^{\underline{\lambda}}}{S^{\underline{\lambda}}}\right) \\
\longrightarrow \operatorname{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) \longrightarrow \operatorname{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, M^{\underline{\lambda}}\right) \longrightarrow \operatorname{Ext}^1_{m\wr n}\left(S^{\underline{\nu}}, \frac{M^{\underline{\lambda}}}{S^{\underline{\lambda}}}\right) \\
\longrightarrow \operatorname{Ext}^2_{m\wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) \longrightarrow \cdots
\end{array}$$

and so by (10.1.1) and Theorem 9.4.4 we have

$$\operatorname{Ext}^1_{m\wr n}\big(S^{\underline{\nu}},\, S^{\underline{\lambda}}\big) = 0.$$

$\square$

10.2 A stratifying system for $k\left(S_m \wr S_n\right)$

From now on, we take k to be an algebraically closed field whose characteristic is neither 2 nor 3.

We now require a total ordering of the set $\underline{\Lambda}_n^r$ of r-multipartitions of n, where r is as above the number of distinct partitions of m. Let $>$ be any (strict) total order on $\underline{\Lambda}_n^r$ such that $>$ extends the dominance order $\triangleright$ (that is, such that for any multipartitions $\underline{\alpha}, \underline{\beta} \in \underline{\Lambda}_n^r$, we have that $\underline{\alpha} \triangleright \underline{\beta}$ implies $\underline{\alpha} > \underline{\beta}$). Beyond this requirement, the exact choice of the order $>$ does not matter. Let us write $\gtrdot$ for the (strict) total order on $\underline{\Lambda}_n^r$ obtained by reversing $>$ (that is, by defining $\underline{\alpha} \gtrdot \underline{\beta}$ to mean $\underline{\alpha} < \underline{\beta}$). It is now easy to prove that

$$\underline{\alpha} \gtrdot \underline{\beta} \Rightarrow \underline{\alpha} \ntrianglerighteq \underline{\beta} \qquad \text{and} \qquad \underline{\alpha} \mathrel{\dot{\geqslant}} \underline{\beta} \Rightarrow \underline{\alpha} \ntriangleright \underline{\beta}. \tag{10.2.1}$$

Recall from Corollary 3.4.2 that in order to show that Specht filtration multiplicities are well-defined for $k(S_m \wr S_n)$, it suffices to prove that

- for any $\underline{\nu} \in \underline{\Lambda}_n^r$, $S^{\underline{\nu}}$ is indecomposable
- $\mathrm{Hom}_{m \wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) = 0$ if $\underline{\nu} \gtrdot \underline{\lambda}$
- $\mathrm{Ext}^1_{m \wr n}\left(S^{\underline{\nu}}, S^{\underline{\lambda}}\right) = 0$ if $\underline{\nu} \mathrel{\dot{\geqslant}} \underline{\lambda}$.

The first result is just Corollary 10.1.2, and the other two conditions are immediate from Theorems 10.1.1 and 10.1.3 by (10.2.1). Thus we have proved the following theorem, which we might say establishes the "Hemmer-Nakano property" for the group algebra $k(S_m \wr S_n)$.

Theorem 10.2.1. *Over an algebraically closed field k whose characteristic is neither 2 nor 3, if a $k(S_m \wr S_n)$-module has a filtration by Specht modules then the multiplicities with which the Specht modules appear are independent of the choice of a filtration.*

Original research in Chapter 10: Everything in Chapter 10 is original research, based on the argument used in [10] to establish the corresponding result for the symmetric group.

Appendix A

Future directions

The new results presented in this thesis offer a number of interesting possibilities for future work, and in this short appendix we shall briefly consider a selection of these.

Firstly, we note that our definition of the modules $S^{\underline{\lambda}}$ and $M^{\underline{\lambda}}$ is by means of a general method of constructing modules for the wreath product. However, these modules are clearly analogous to the modules S^{λ} and M^{λ} for the symmetric group, and so we might expect them to have a combinatorial construction parallel to the construction of S^{λ} and M^{λ} given in [20]. Indeed, by Proposition 4.4.1, we know that $M^{\underline{\lambda}}$ is the permutation module for $S_m \wr S_n$ on the cosets by the subgroup $W_{\underline{\lambda}}$, and it is easy to imagine that such cosets would have some tableau representation, perhaps involving tableaux whose entries are pairs of numbers like those in Chapter 7. Such a combinatorial construction could allow us to apply methods analogous to those in [20].

Continuing this theme, we note that Proposition 6.5.2 may be regard as a wreath-product analogue of Young's rule (3.2.1), since it gives a filtration of $M^{\underline{\lambda}}$ by modules $S^{\underline{\nu}}$. However, Proposition 6.5.2 lacks a combinatorial interpretation of the multiplicities which occur in the filtration, and one might hope to find an interpretation analogous to the combinatorial characterisation of the Kostka numbers. Going further in this direction, we note that the Specht branching rule and Young's rule, as presented in [20], give not only the multiplicities with which factors occur, but also some information about the order in which those factors occur in the filtrations, and it seems likely that by taking more care in the arguments one might be able to get similar information in the wreath product results.

Now the Specht branching rule and Young's rule for the symmetric group may be regarded as special cases of the general results (3.2.12) and (3.2.11) for induction and restriction of Specht modules and tensor products thereof, which feature Littlewood-Richardson coefficients as multiplicities. A rather more ambitious aspiration than those mentioned above would be to formulate and prove an appropriate generalisation of these results to the wreath product case, including a combinatorial interpretation of the multiplicities occurring therein, analogous to the Littlewood-Richardson rule. A much more ambitious goal would be to use these coefficients as a starting point to forge a connection between the representation theory of wreath products, the theory of symmetric functions, and the representation theory of general linear groups (or some extension or generalisation of these), paralleling the deep and fruitful connections enjoyed by the symmetric group.

Returning to rather more humble and concrete possibilities, we note that the treatment of the spaces $\mathrm{Hom}_n(S^\lambda, M^\gamma)$ in [20] makes use of the notion of *semistandard homomorphisms*, and indeed constructs a basis of this space using them. It seems very probable that Theorem 9.4.5, which provides a decomposition of the Hom-space $\mathrm{Hom}_{m \wr n}\big(S^{\underline{\nu}},\ M^{\underline{\gamma}}\big)$, could be a starting-point for an analogous result in the wreath product case.

Another possible direction would be to explore the consequences of the existence of the stratifying system from Chapter 10. Indeed, our sole use for this fact in Chapter 10 was to allow us to establish the Hemmer-Nakano property for $k(S_m \wr S_n)$, but there are other interesting corollaries to this fact, as mentioned in [10]. For example (see [10, Lemma 2.2]), the stratifying system allows us to associate a certain algebra A to $k(S_m \wr S_n)$ (A is in fact the endomorphism algebra of a certain $k(S_m \wr S_n)$-module) which can be viewed as analogous to the *classical Schur algebra*. The classical Schur algebra and its relatives appear prominently in, and are intimately connected to, the representation theory of the symmetric group and its generalisations. Moreover, the classical Schur algebra features in the famous and profound *Schur-Weyl duality* which connects the representation theory of the symmetric group with the polynomial representation theory of the general linear group. The algebra A defined above would enjoy the same close relationship to $k(S_m \wr S_n)$, and moreover by Theorem 10.1.1 we know (again, see [10, Lemma 2.2]) that A is a *quasi-hereditary algebra*, which would provide a good starting point for studying its representation theory.

Let us consider now the setting of Chapter 5, in which we study the wreath product $A \wr S_n$ where A is a cellular algebra. One possible extension of this work would be to attempt to augment the cellular structure on $A \wr S_n$ with cohomological information via the identification of suitable idempotents within the layers of the iterated inflation structure, thus exhibiting a *cohomological stratification*, a concept introduced in [18] (such a structure would certainly require some extra assumptions on the algebra A). Doing so would be one possible route by which we could seek to generalise the results of Chapters 9 and 10 to algebras of the form $A \wr S_n$, given suitable assumptions on the algebra A. Of course, it might be possible to directly generalise the arguments of Chapters 9 and 10 to algebras $A \wr S_n$, since those arguments make use of only a relatively limited set of properties of the Specht modules of kS_m, and any algebra A with a suitable set of modules might be amenable to those methods. The main impediment to some initial attempts to carry out this latter generalisation seems to be the fact that, for general algebras A, the operations of induction and coinduction (see [3, Definition 2.8.1]) do not coincide, and so we do not have the same very nice form of the Eckmann-Shapiro lemma (Theorem 2.2.4).

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