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# Quantum unique factorisation domains

S Launois, T H Lenagan and L Rigal \*

## Abstract

We prove a general theorem showing that iterated skew polynomial extensions of the type which fit the conditions needed by Cauchon's deleting derivations theory and by the Goodearl-Letzter stratification theory are unique factorisation rings in the sense of Chatters and Jordan. This general result applies to many quantum algebras; in particular, generic quantum matrices and quantized enveloping algebras of the nilpotent part of a semisimple Lie algebra are unique factorisation domains in the sense of Chatters. By using noncommutative dehomogenisation, the result also extends to generic quantum grassmannians.

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*Key words:* Unique factorisation domain, quantum algebra, quantum matrices, quantum grassmannian

## Introduction

In [4], Chatters introduced the notion of a noncommutative unique factorisation domain in the following way. An element  $p$  of a noetherian domain  $R$  is said to be *prime* if (i)  $pR = Rp$ , (ii)  $pR$  is a height one prime ideal of  $R$ , and (iii)  $R/pR$  is an integral domain. A noetherian domain  $R$  is then said to be a *unique factorisation domain*, noetherian UFD for short, if  $R$  has at least one height one prime ideal, and every height one prime ideal is generated by a prime element. As well as the usual commutative noetherian UFDs, examples include universal enveloping algebras of finite dimensional solvable Lie algebras over  $\mathbb{C}$ . However, one of the deficiencies of this definition is that the class of noetherian UFDs is not closed

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under polynomial extensions, as [4, Example 2.11] shows. The problem is that the condition of height one prime factors being domains does not pass up to polynomial extensions.

In order to remedy this deficiency, in a later paper, the notion of a *noetherian unique factorisation ring*, noetherian UFR for short, was introduced by Chatters and Jordan, [5]. For a large class of rings (namely, the noetherian prime rings satisfying the descending chain condition on prime ideals), being a noetherian unique factorisation ring amounts having height one primes principal (that is generated by a normal element). This condition is closed under polynomial extensions, and, indeed, they then are able to prove theorems about skew polynomial extensions of the type  $R[x; \sigma]$  and  $R[x; \delta]$ . However, they do not prove any results about general skew polynomial extensions of type  $R[x; \sigma, \delta]$ .

In many quantum algebras, in the generic case where the deformation parameter  $q$  is not a root of unity, it is known that all prime ideals are completely prime, and then the distinction between a noetherian domain being a noetherian UFD and a noetherian UFR disappears and so the results of [5] on noetherian UFRs also apply to noetherian UFDs in this setting.

The purpose of this paper is to obtain a theorem on unique factorisation for certain extensions of the type  $R[x; \sigma, \delta]$  that arise naturally in the study of quantum algebras. Once this theorem is proved, an iterated version is obtained which is sufficient to show that many quantum algebras are noetherian UFDs. In particular, we show that the algebra of generic quantum matrices,  $\mathcal{O}_q(M_{m,n})$  is a noetherian UFD, as is the quantized enveloping algebra  $U_q^+(g)$ .

Roughly speaking, an iterated skew polynomial extension will be a noetherian UFD provided that the Cauchon theory of deleting derivations can be applied, and that there is a torus action  $\mathcal{H}$  for which the Goodearl-Letzer stratification theory applies. Exact requirements will be given as they become necessary.

In the case of quantum matrices, we can go further, since we can identify the height one prime ideals that are  $\mathcal{H}$ -primes for the natural torus that acts.

In the final section, we use the idea of noncommutative dehomogenisation, developed in [13] to deduce that generic quantum grassmannians are noetherian UFDs.

For general results concerning noetherian rings and localisation, we refer the reader to [12] or [17].

Throughout the paper,  $k$  denotes a field.

# 1 Non commutative unique factorisation rings

This section investigates the behaviour of the notion of a *noetherian unique factorisation ring*, as defined in [5] by Chatters and Jordan, under localisation by normal elements.

To start with, we recall the definition of noetherian unique factorisation ring; further details concerning this notion can be found in [5].

An ideal  $I$  in a ring  $A$  is called *principal* if there exists a normal element  $x$  in  $A$  such that  $I = \langle x \rangle (= xA = Ax)$ .

**Definition 1.1** *A ring  $A$  is called a noetherian unique factorisation ring (noetherian UFR for short) if: (i)  $A$  is a prime noetherian ring, and (ii) any nonzero prime ideal in  $A$  contains a nonzero principal prime ideal.*

**Definition 1.2** *A noetherian UFR  $A$  is said to be a unique factorisation domain (noetherian UFD for short) if  $A$  is a domain and each height one prime ideal  $P$  of  $A$  is completely prime; that is,  $A/P$  is a domain for each height one prime ideal  $P$  of  $A$ .*

**Remark 1.3** If  $A$  is a prime noetherian ring that satisfies the descending chain condition for prime ideals, then  $A$  is a noetherian UFR if and only if height one primes are principal (see [5]). Hence, the notions of noetherian UFR and noetherian UFD are good generalisations of the usual notion of unique factorisation domain for commutative rings (see in particular Corollaries 10.3 and 10.6 in [7]).

Note that the algebras we are dealing with are all noetherian and have finite Gelfand-Kirillov dimension; so, they satisfy the descending chain condition for prime ideals, see for example, [14, Corollary 3.16].

We start by proving a noncommutative analogue of Nagata's Lemma (in the commutative case, see [7] 19.20 p. 487). The following result is taken from [6], where it appears without proof. We include a proof here, for the convenience of the reader, since it is crucial to a part of our argument.

If  $A$  is a prime noetherian ring and  $x$  a nonzero normal element of  $A$ , we denote by  $A_x$  the right localisation of  $A$  with respect to the powers of  $x$ .

**Lemma 1.4** *Let  $A$  be a prime noetherian ring and  $x$  a nonzero, nonunit, normal element of  $A$  such that  $\langle x \rangle$  is a completely prime ideal of  $A$ .*

*(i) If  $P$  is a prime ideal of  $A$  not containing  $x$  and such that the prime ideal  $PA_x$  of  $A_x$  is*

principal, then  $P$  is principal.

(ii) If  $A_x$  is a noetherian unique factorisation ring, then so is  $A$ .

(iii) If  $A_x$  is a noetherian unique factorisation domain, then so is  $A$ .

*Proof.* (i) The result is trivial if  $P = 0$ , so we assume that  $P \neq 0$ . Since  $x$  is a nonzero normal element of the prime ring  $A$  one may localise  $A$  with respect to the multiplicative set of powers of  $x$  and there is canonical embedding  $A \hookrightarrow A_x$ . Moreover,  $Q := PA_x$  is a prime ideal of  $A_x$  whose contraction to  $A$  is  $P$ , since  $P$  is a prime ideal of  $A$  not containing  $x$ . Let us suppose that  $Q$  is a principal ideal. Then, clearly, there exists  $q \in A$ , normal in  $A_x$ , such that  $Q = qA_x$ . Moreover, one may assume the right ideal  $qA$  maximal for this property, since  $A$  is right noetherian. Suppose that  $q \in Ax$ . Then there exists  $p$  in  $A$  such that  $q = px$  (in particular  $qA \subseteq pA$ ). But then,  $Q = pA_x$  and  $p$  is normal in  $A_x$ . The maximality of  $qA$  leads to  $qA = pA$  from which follows the existence of  $r \in A$  such that  $p = qr$  and hence  $q = qrx$ . Since  $q$  is a non-zero normal element in the prime ring  $A_x$ , the above equality gives  $1 = rx$  (with  $r \in A$ ), a contradiction, since  $x$  is not a unit. Thus,  $q \notin Ax$ . Now, let  $p \in P \subseteq Q$ ; so that there exist  $r \in A$  and  $t \in \mathbb{N}$  with  $p = qrx^{-t}$ , and we may choose  $t$  minimal for this property. If  $t > 0$  then  $r \notin Ax$ , by the minimality of  $t$ . The above equation then leads to  $px^t = qr$ ; and so either  $q$  or  $r$  must be in  $Ax$  which is a contradiction. Thus,  $t = 0$  and so  $p \in qA$ . Hence,  $P \subseteq qA$ . Also,  $qA \subseteq qA_x \cap A = Q \cap A = P$ ; so that  $P = qA$ . A similar argument gives  $P = Aq$ . Hence  $P = Aq = qA$  which proves the first claim.

(ii) Let us now assume that  $A_x$  is a noetherian UFR. If  $Q_0$  is a non-zero prime ideal of  $A$  not containing  $x$ , then  $Q_0A_x$  is a non-zero prime ideal of  $A_x$ . Since  $A_x$  is a noetherian UFR,  $Q_0A_x$  contains a nonzero principal prime ideal  $P$  which is the extension to  $A_x$  of its contraction  $P_0$  in  $A$ . By part (i), the ideal  $P_0$  is principal, since  $P$  is principal. Thus,  $P_0$  is a nonzero principal prime ideal contained in  $Q_0$ . Moreover, if  $Q_0$  is a prime ideal of  $A$  containing  $x$ , then it contains the nonzero principal prime ideal  $\langle x \rangle$ . We have proved that each nonzero prime ideal of  $A$  contains a nonzero principal prime ideal, which means that  $A$  is a noetherian UFR.

(iii) Suppose that  $A_x$  is a noetherian UFD. Then part (ii) shows that  $A$  is a noetherian UFR. Let  $P$  be a prime ideal of height one in  $A$ . If  $x \in P$  then  $P = \langle x \rangle$  and so  $P$  is completely prime, by assumption. Otherwise, standard localisation theory shows that  $PA_x$  is a prime ideal of height one in  $A_x$  and that  $P = PA_x \cap A$ . Thus,  $A/P$  embeds in  $A_x/PA_x$ , which is a domain; and so  $A/P$  is a domain, as required.  $\square$

Proposition 1.6 below will be of central use later. It gives a way to pull back the unique

factorisation property from a certain type of localisation to the initial ring. The following lemma is needed in the proof of the proposition.

**Lemma 1.5** *Let  $R$  be a prime noetherian ring and suppose that  $d, s$  are normal elements of  $R$  such that  $dR$  is prime and  $s \notin dR$ . Then, there exist units  $u, v \in R$  such that  $ds = sdu$  and  $sd = vds$ .*

*Proof.* If either  $d$  or  $s$  is zero, then the result is trivial; so we assume that  $d, s \neq 0$ . Since  $s$  is normal in a prime ring,  $s$  is regular and we can associate to it an automorphism  $\sigma : R \rightarrow R$  such that  $xs = s\sigma(x)$ , for all  $x \in R$ . Set  $P := dR = Rd$ . Then  $s\sigma(P) = Ps \subseteq P$ ; and so  $\sigma(P) \subseteq P$ , since  $s$  is normal and not in  $P$ . Hence,  $P \subseteq \sigma^{-1}(P)$ , and it follows that there is an ascending chain  $P \subseteq \sigma^{-1}(P) \subseteq \sigma^{-2}(P) \subseteq \dots$  of ideals of  $R$ . The noetherian hypothesis then ensures that there exists  $n \in \mathbb{N}$  such that  $\sigma^{-n}(P) = \sigma^{-(n+1)}(P)$ , and so  $\sigma(P) = P$ ; that is,  $\sigma(d)R = dR$ . From this it follows that  $dsR = sdR$ , which gives the existence of  $u, u' \in R$  such that  $ds = sdu$  and  $sd = dsu'$ . But then,  $ds = sdu = dsu'u$ ; and so  $u'u = 1$  which shows  $u$  is a unit in  $R$ . We also have  $Rds = Rsd$ , since  $d$  and  $s$  are normal, and it follows in a similar manner that there exists a unit  $v$  in  $R$  such that  $sd = vds$ .  $\square$

**Proposition 1.6** *Let  $R$  be a prime noetherian ring and suppose that  $d_1, \dots, d_t$  are nonzero normal elements of  $R$  such that the ideals  $d_1R, \dots, d_tR$  are completely prime and pairwise distinct. Denote by  $T$  the right quotient ring of  $R$  with respect to the right denominator set generated by  $d_1, \dots, d_t$ . If  $T$  is a noetherian UFR then so is  $R$ . Also, if  $T$  is a noetherian UFD then so is  $R$ .*

*Proof.* We proceed by induction on  $t$ , the result being true for  $t = 1$  by Lemma 1.4 (ii). Assume that the result is true up to order  $t \in \mathbb{N}^*$ . We will work in the right quotient ring of fractions of  $R$  in which all the localisations of  $R$  are naturally embedded. Denote by  $\mathcal{S}_{t+1}$  the multiplicative subset of  $R$  generated by  $d_1, \dots, d_{t+1}$  and by  $\mathcal{S}_t$  the multiplicative subset of  $R$  generated by  $d_1, \dots, d_t$ . Hence  $T = R\mathcal{S}_{t+1}^{-1}$ . We first show, using the above lemma, that  $d_{t+1}$  is a nonzero normal element of  $R\mathcal{S}_t^{-1}$ . Let  $(a, s) \in R \times \mathcal{S}_t$ ; hence  $s$  is normal in  $R$  and, due to the hypothesis that the ideals  $d_iR$  are completely prime and pairwise distinct,  $s \notin d_{t+1}R$  (by the principal ideal theorem). So, by the lemma above, there exist elements  $u, v \in R$  such that  $d_{t+1}s = sd_{t+1}u$  and  $sd_{t+1} = vd_{t+1}s$ . In addition, since  $d_{t+1}$  is normal in  $R$ , there exist  $b, c \in R$  such that  $ad_{t+1} = d_{t+1}b$  and  $d_{t+1}a = cd_{t+1}$ . Hence, we have  $as^{-1}d_{t+1} = ad_{t+1}us^{-1} = d_{t+1}bus^{-1}$  and  $d_{t+1}as^{-1} = cd_{t+1}s^{-1} = cs^{-1}vd_{t+1}$ . It follows that  $d_{t+1}$  is indeed a nonzero normal element of  $R\mathcal{S}_t^{-1}$ .

Let  $\mathcal{S}$  be the multiplicative subset of  $R\mathcal{S}_t^{-1}$  generated by  $d_{t+1}$ . Notice that,  $R\mathcal{S}_{t+1}^{-1} = (R\mathcal{S}_t^{-1})\mathcal{S}^{-1}$ , as is easily verified. Of course,  $R\mathcal{S}_t^{-1}$  is prime noetherian and the ideal  $d_{t+1}R\mathcal{S}_t^{-1}$  is completely prime since  $d_{t+1}R$  is completely prime and does not intersect  $\mathcal{S}_t$ .

Now assume that  $T = (R\mathcal{S}_t^{-1})\mathcal{S}^{-1}$  is a noetherian UFR. By the comments above, Lemma 1.4 (ii) may be applied and we get that  $R\mathcal{S}_t^{-1}$  is a noetherian UFR. Now, the induction hypothesis gives that  $R$  is a noetherian UFR, as required.

Finally, suppose that  $T$  is a noetherian UFD. Then  $T$  is certainly a noetherian UFR; and so  $R$  is a noetherian UFR, by the first part of this result. That  $R$  is a noetherian UFD then follows by standard localisation theory (cf. the proof of Lemma 1.4 (iii)).  $\square$

## 2 Height one $\mathcal{H}$ -primes in Cauchon extensions

Most of the algebras that we are considering in this paper have groups acting on them in natural ways. The study of the prime spectra of such algebras is often facilitated by first studying ideals invariant under the natural group action. We begin this section by recalling some standard terminology concerning ideals invariant under group actions. A convenient reference is [1, II.1.8, II.1.9]. Let  $\mathcal{H}$  be a group acting by automorphisms on a ring  $R$ . An ideal  $I$  of  $R$  is an  $\mathcal{H}$ -ideal provided that  $h(I) = I$  for all  $h \in \mathcal{H}$ . A proper  $\mathcal{H}$ -ideal is an  $\mathcal{H}$ -prime ideal provided that whenever  $IJ \subseteq P$  for  $\mathcal{H}$ -ideals  $I, J$  of  $R$  then either  $I \subseteq P$  or  $J \subseteq P$ . The set of  $\mathcal{H}$ -prime ideals of  $R$  is denoted by  $\mathcal{H}\text{-Spec}(R)$ . It is obvious that a prime ideal  $P$  that is an  $\mathcal{H}$ -ideal is an  $\mathcal{H}$ -prime ideal. The converse is not true in general; however, it will usually be true for the algebras that interest us in this paper (see comments after Definition 3.1).

**Hypothesis 2.1** *Let  $A$  be a domain that is a noetherian  $k$ -algebra and suppose that  $\sigma$  is a  $k$ -automorphism of  $A$ . Suppose that there is a group  $\mathcal{H}$  acting as automorphisms on the skew Laurent extension  $A[X^{\pm 1}; \sigma]$  in such a way that  $X$  is an  $\mathcal{H}$ -eigenvector and  $A$  is stable under  $\mathcal{H}$ . Further, suppose that the action of  $\sigma$  on  $A$  coincides with the action of an element  $h_0 \in \mathcal{H}$ . Finally, suppose that there is a non root of unity  $\lambda_0$  in  $k^*$  such that  $h_0.X = \lambda_0 X$ .*

Given the conditions of this hypothesis, we are going to show that there is a bijection between the  $\mathcal{H}$ -ideals of  $A$  and the  $\mathcal{H}$ -ideals of  $A[X^{\pm 1}; \sigma]$ , and, consequently, there is a bijection between  $\mathcal{H}\text{-Spec}(A)$  and those  $\mathcal{H}$ -primes of  $A[X; \sigma]$  that do not contain  $X$ .

**Lemma 2.2** *Assume Hypothesis 2.1 and let  $I$  be an  $\mathcal{H}$ -invariant ideal of  $A[X^{\pm 1}; \sigma]$ . Suppose that  $x = a_1X^{k_1} + \cdots + a_nX^{k_n} \in I$ , with  $a_i \in A$  and  $k_i$  all distinct. Then, each  $a_i \in I \cap A$ . Consequently,  $I = \bigoplus_{i \in \mathbb{Z}} (I \cap A)X^i$ .*

*Proof.* The proof is by induction on  $n$ . If  $n = 1$  the result is trivial, since  $X$  is invertible. Suppose now that  $n > 1$ . Since  $I$  is an  $\mathcal{H}$ -ideal, the element  $Xx - \lambda_0^{-k_n}h_0(x)X$  belongs to  $I$ . However,

$$\begin{aligned} Xx - \lambda_0^{-k_n}h_0(x)X &= \sum_{i=1}^n h_0(a_i)X^{k_i+1} - \sum_{i=1}^n \lambda_0^{k_i-k_n}h_0(a_i)X^{k_i+1} \\ &= \sum_{i=1}^{n-1} (1 - \lambda_0^{(k_i-k_n)})h_0(a_i)X^{k_i+1}; \end{aligned}$$

so that  $\sum_{i=1}^{n-1} (1 - \lambda_0^{(k_i-k_n)})h_0(a_i)X^{k_i+1} \in I$ . By the induction hypothesis, we see that  $(1 - \lambda_0^{(k_i-k_n)})h_0(a_i) \in I$  for each  $1 \leq i \leq n-1$ . The elements  $(1 - \lambda_0^{(k_i-k_n)})$  are nonzero, since  $\lambda_0$  is not a root of unity and the  $k_i$  are distinct. Thus, each  $h_0(a_i)$  is in the  $\mathcal{H}$ -ideal  $I \cap A$ , and so each  $a_i \in I \cap A$  for  $1 \leq i \leq n-1$ . Finally,  $a_nX^{k_n} = x - a_1X^{k_1} - \cdots - a_{n-1}X^{k_{n-1}} \in I$ ; and so  $a_n \in I \cap A$  also.  $\square$

The next result follows easily from this lemma.

**Theorem 2.3** *Assume Hypothesis 2.1. Then there is an inclusion preserving bijection from the set of  $\mathcal{H}$ -ideals of  $A$  to the set of  $\mathcal{H}$ -ideals of  $A[X^{\pm 1}; \sigma]$  given by  $I \mapsto \bigoplus_{i \in \mathbb{Z}} IX^i$ ; its inverse is defined by  $J \mapsto J \cap A$ . Furthermore, these bijections induce order preserving bijections between  $\mathcal{H}\text{-Spec}(A)$  and  $\mathcal{H}\text{-Spec}A[X^{\pm 1}; \sigma]$ .*

Let  $\mathcal{H}$  be a group acting by automorphisms on a noetherian ring  $R$  and suppose that  $X$  is a normal  $\mathcal{H}$ -eigenvector. Then there is a bijective correspondence between the  $\mathcal{H}$ -prime ideals of  $R$  that do not contain  $X$  and the  $\mathcal{H}$ -prime ideals of  $R[X^{-1}]$ , cf [1, Exercise II.1.J]. Using this fact, the next corollary follows easily.

**Corollary 2.4** *Assume Hypothesis 2.1. Then contraction  $P \mapsto P \cap A$  and extension  $P \mapsto \bigoplus_{i \geq 0} PX^i$  provide inverse order preserving bijections between the  $\mathcal{H}$ -prime ideals of  $A[X; \sigma]$  that do not contain  $X$  and  $\mathcal{H}\text{-Spec}(A)$ .*

**Definition 2.5** Let  $A$  be a domain that is a noetherian  $k$ -algebra and let  $R = A[X; \sigma, \delta]$  be a skew polynomial extension of  $A$ . We say that  $R = A[X; \sigma, \delta]$  is a *Cauchon Extension* provided that



- $\sigma$  is a  $k$ -algebra automorphism of  $A$  and  $\delta$  is a  $k$ -linear locally nilpotent  $\sigma$ -derivation of  $A$ . Moreover we assume that there exists  $q \in k^*$  which is not a root of unity such that  $\sigma \circ \delta = q\delta \circ \sigma$ .
- There exists an abelian group  $\mathcal{H}$  which acts on  $R$  by  $k$ -algebra automorphisms such that  $X$  is an  $\mathcal{H}$ -eigenvector and  $A$  is  $\mathcal{H}$ -stable.
- $\sigma$  coincides with the action on  $A$  of an element  $h_0 \in \mathcal{H}$ .
- Since  $X$  is an  $\mathcal{H}$ -eigenvector and since  $h_0 \in \mathcal{H}$ , there exists  $\lambda_0 \in k^*$  such that  $h_0.X = \lambda_0 X$ . We assume that  $\lambda_0$  is not a root of unity.
- Every  $\mathcal{H}$ -prime ideal of  $A$  is completely prime.

Note that the conditions of [1, II.5.3] are satisfied by any Cauchon extension; and so, for example, every  $\mathcal{H}$ -prime of  $R$  is also completely prime, by [1, Proposition II.5.11].

In a Cauchon extension  $R = A[X; \sigma, \delta]$  the set  $S = \{X^n \mid n \in \mathbb{N}\}$  is a right and left Ore set in  $R$ , [2, Lemme 2.1]; and so we can form the Ore localization  $\widehat{R} := RS^{-1} = S^{-1}R$ .

For each  $a \in A$ , set

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]_q!} \delta^n \circ \sigma^{-n}(a) X^{-n} \in \widehat{R}$$

Note that  $\theta(a)$  is a well-defined element of  $\widehat{R}$ , since  $\delta$  is locally nilpotent,  $q$  is not a root of unity, and  $0 \neq 1 - q \in k$ .

The following facts are established in [2, Section 2]. The map  $\theta : A \longrightarrow \widehat{R}$  is a  $k$ -algebra monomorphism. Let  $A[Y; \sigma]$  be a skew polynomial extension. Then  $\theta$  extends to a monomorphism  $\theta : A[Y; \sigma] \longrightarrow \widehat{R}$  with  $\theta(Y) = X$ . Set  $B = \theta(A)$  and  $T = \theta(A[Y; \sigma]) \subseteq \widehat{R}$ . Then  $T = B[X; \alpha]$ , where  $\alpha$  is the automorphism of  $B$  defined by  $\alpha(\theta(a)) = \theta(\sigma(a))$ .

The element  $X$  is a normal element in  $T$ , and so the set  $S$  is an Ore set in  $T$  and Cauchon shows that  $TS^{-1} = S^{-1}T = \widehat{R}$ .

Since  $X$  is an  $\mathcal{H}$ -eigenvector, it follows from [1, Exercise II.1.J] that  $\mathcal{H}$  also acts by automorphisms on  $\widehat{R}$ . Moreover, the following result shows that the group  $\mathcal{H}$  also acts by automorphisms on  $T$  and  $B$  by restriction.

Note, for later use, that, since each element of  $B = \theta(A)$  is of the form  $\theta(a) = \sum_{i=0}^n a_i X^{-i}$  for some  $a_i \in A$ , and each element of  $R$  is of the form  $\sum_{i=0}^n c_i X^i$  for some  $c_i \in A$ , it follows that  $B \cap R \subseteq A$ .

The next result shows that the action of  $\mathcal{H}$  can be transferred to  $B$  via  $\theta$ . This result is essentially a generalisation of [2, Proposition 2.1].

**Lemma 2.6** *Let  $R = A[X; \sigma, \delta]$  be a Cauchon extension and let  $h \in \mathcal{H}$ . Then  $h.\theta(a) = \theta(h.a)$  for each  $a \in A$ .*

*Proof.* We start by showing inductively that  $h.\delta^n(a) = \lambda_h^n \delta^n(h.a)$  for all  $n \in \mathbb{N}$ ,  $a \in A$  and  $h \in \mathcal{H}$ , where  $\lambda_h$  denotes the  $\mathcal{H}$ -eigenvalue associated to the  $\mathcal{H}$ -eigenvector  $X$ .

If  $n = 0$ , there is nothing to prove. Now we assume that  $n \geq 1$ . Then, since  $\delta^n(a) = X\delta^{n-1}(a) - \sigma \circ \delta^{n-1}(a)X = X\delta^{n-1}(a) - q^{n-1}\delta^{n-1}(\sigma(a))X$ , we deduce from the induction hypothesis that

$$h.\delta^n(a) = \lambda_h X \lambda_h^{n-1} \delta^{n-1}(h.a) - q^{n-1} \lambda_h^{n-1} \delta^{n-1}(h.\sigma(a)) \lambda_h X.$$

Since  $h.\sigma(a) = hh_0.a = h_0h.a = \sigma(h.a)$ , this leads to

$$\begin{aligned} h.\delta^n(a) &= \lambda_h^n [X\delta^{n-1}(h.a) - q^{n-1}\delta^{n-1}(\sigma(h.a))X] \\ &= \lambda_h^n [X\delta^{n-1}(h.a) - \sigma \circ \delta^{n-1}(h.a)X] = \lambda_h^n \delta^n(h.a). \end{aligned}$$

This achieves the induction.

Now, let  $a \in A$ . Then, using the notations of [2], we have

$$\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{-n}.$$

Hence we get

$$h.\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} h. [\delta^n \circ \sigma^{-n}(a)] h.X^{-n}.$$

Then the previous study shows that

$$h.\theta(a) = \sum_{n=0}^{+\infty} \lambda_h^n \frac{(1-q)^{-n}}{[n]!_q} \delta^n(h.\sigma^{-n}(a)) \lambda_h^{-n} X^{-n}.$$

Now, since  $\sigma$  coincide with the action of  $h_0 \in \mathcal{H}$  on  $A$ , we have  $h.\sigma^{-n}(a) = hh_0^{-n}.a = h_0^{-n}h.a = \sigma^{-n}(h.a)$ , so that

$$h.\theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(h.a) X^{-n},$$

that is,  $h.\theta(a) = \theta(h.a)$  as desired.

□

Note that for  $b \in B$  with  $b = \theta(a)$ , we have  $\alpha(b) = \alpha(\theta(a)) = \theta(\sigma(a)) = \theta(h_0.a) = h_0.\theta(a) = h_0.b$ ; so that the action of  $\alpha$  on  $B$  coincides with the action of  $h_0$ .

The above lemma shows that the action of  $\mathcal{H}$  on  $\widehat{R}$  by automorphisms induces an action of  $\mathcal{H}$  on  $B$  by automorphisms. Further, since  $T = B[X; \alpha]$  and since  $X$  is an  $\mathcal{H}$ -eigenvector, this observation also proves that the action of  $\mathcal{H}$  on  $\widehat{R}$  by automorphisms induces an action of  $\mathcal{H}$  on  $T$  by automorphisms. Moreover, since every  $\mathcal{H}$ -prime ideal of  $A$  is completely prime, we deduce that every  $\mathcal{H}$ -prime ideal of  $B = \theta(A)$  is completely prime. Then, it follows from [1, Proposition II.5.11] that every  $\mathcal{H}$ -prime ideal of  $T = B[X; \alpha]$  is also completely prime.

Let  $b \in B$  be an  $\mathcal{H}$ -eigenvector, say  $h.b = \lambda_h b$  for  $\lambda_h \in k$ , and suppose that  $b = \theta(a)$ . Then  $\theta(h.a - \lambda_h a) = h.\theta(a) - \lambda_h \theta(a) = h.b - \lambda_h b = 0$ ; so  $h.a = \lambda_h a$  and  $a$  is an  $\mathcal{H}$ -eigenvector with the same action of  $\mathcal{H}$  on  $a$  as on  $b$ .

**Definition 2.7** Suppose that  $A$  is a noetherian domain that is a  $k$ -algebra and suppose that  $\mathcal{H}$  is a group acting on  $A$  via  $k$ -automorphisms. Then  $A$  is an  $\mathcal{H}$ -UFD if each nonzero  $\mathcal{H}$ -prime  $Q$  of  $A$  contains a nonzero normal  $\mathcal{H}$ -eigenvector  $x$  such that the  $\mathcal{H}$ -ideal  $xA = Ax$  is completely prime. `label:def-h-ufd`

**Remark 2.8** In particular, in an  $\mathcal{H}$ -UFD, all  $\mathcal{H}$ -primes of height one as  $\mathcal{H}$ -primes have height one as ordinary prime ideals, by the principal ideal theorem. Thus, an ideal is an  $\mathcal{H}$ -prime of height one as an  $\mathcal{H}$ -prime if and only if it is a prime  $\mathcal{H}$ -ideal of height one as an ordinary prime ideal. Also, in an  $\mathcal{H}$ -UFD, the  $\mathcal{H}$ -primes of height one are principal, generated by a normal element, and completely prime.

**Proposition 2.9** Let  $R = A[X; \sigma, \delta]$  be a Cauchon extension. Suppose that  $A$  is an  $\mathcal{H}$ -UFD. Then  $R$  is an  $\mathcal{H}$ -UFD.

*Proof.* Since  $B$  is isomorphic to  $A$  via  $\theta$  and  $\theta$  preserves the  $\mathcal{H}$ -action, we know that every non-zero  $\mathcal{H}$ -prime of  $B$  contains a non-zero normal  $\mathcal{H}$ -eigenvector  $b$  such that  $bB = Bb$  is a completely prime ideal; that is,  $B$  is an  $\mathcal{H}$ -UFD. We start by showing that such an element  $b$  of  $B$  can be used to produce, in a natural way, an element of  $R$  with similar properties.

Note, that every  $\mathcal{H}$ -prime ideal of  $A$  and  $B$  is completely prime, since this is one of the properties of  $A$  being part of a Cauchon extension and  $B \cong A$  via a map compatible with the  $\mathcal{H}$ -actions.

Let  $b \in B$ . Then  $b \in B \subseteq T \subseteq \widehat{R} = RS^{-1}$ ; and so there exists  $n \geq 0$  with  $bX^n \in R$ .

Now, suppose that  $0 \neq b \in B$  is a normal  $\mathcal{H}$ -eigenvector such that  $bB = Bb$  is a completely prime ideal. Choose  $s \geq 0$  minimal such that  $x := bX^s \in R$ . We will show that  $x$  is a normal  $\mathcal{H}$ -eigenvector in  $R$  such that  $xR = Rx$  is a completely prime ideal.

First, note that  $x$  is an  $\mathcal{H}$ -eigenvector, since each of  $b$  and  $X$  is an  $\mathcal{H}$ -eigenvector. Next,

$$Xb = \alpha(b)X = h_0.bX = \eta bX$$

for some  $0 \neq \eta \in k$ , since  $b$  is an  $\mathcal{H}$ -eigenvector.

Hence,  $b$  is normal in  $T = B[X; \alpha]$ . Also,  $bT = Tb$  is an  $\mathcal{H}$ -invariant completely prime ideal of  $T$ .

It follows that  $b\widehat{R} = \widehat{R}b$  is an  $\mathcal{H}$ -invariant completely prime ideal of  $\widehat{R}$ . However,  $x\widehat{R} = bX^s\widehat{R} = b\widehat{R}$ ; and so  $x\widehat{R}$  is an  $\mathcal{H}$ -invariant completely prime ideal of  $\widehat{R}$ . Thus,  $I := x\widehat{R} \cap R = b\widehat{R} \cap R$  is an  $\mathcal{H}$ -invariant completely prime ideal of  $R$ . We will show that  $I = Rx$ .

It is obvious that  $Rx \subseteq I$ . For the reverse inclusion, let  $y \in I$ . Then  $y \in b\widehat{R}$  and so there exists  $u \geq 0$  such that  $yX^u \in bT = Tb$ . Thus, there exists  $c \in T$  such that  $yX^u = cb$ . Next, since  $c \in T \subseteq RS^{-1}$ , there exists  $v \geq 0$  such that  $cX^v \in R$ . Set  $r := \eta^{-v}cX^v \in R$ . Then, by using the fact that  $Xb = \eta bX$ , we get  $yX^{u+v+s} = cbX^{v+s} = \eta^{-v}cX^vbX^s = rx$ ; and so there exists  $t \geq 0$  such that  $yX^t = rx$  with  $r \in R$ . Choose such a  $t$  minimal.

Assume that  $t \geq 1$ . Express  $r$ ,  $y$  and  $x$  as elements in the Ore extension  $R = A[X; \sigma, \delta]$ , say,

$$r = \sum_{i=0}^d r_i X^i, \quad y = \sum_{i=0}^d y_i X^i \quad \text{and} \quad x = \sum_{i=0}^d x_i X^i,$$

where  $d \geq 0$  and  $r_i, y_i, x_i \in A$  for all  $0 \leq i \leq d$ . If  $s = 0$ , then  $x = b \in B \cap R \subseteq A$ , and so  $x_0 = b \neq 0$ . If  $s \geq 1$  then  $x_0 = 0$  would give

$$bX^{s-1} = (bX^s)X^{-1} = xX^{-1} = \sum_{i=1}^d x_i X^{i-1} \in R,$$

contradicting the minimality of  $s$ . Thus,  $x_0 \neq 0$  whatever the value of  $s \geq 0$ .

Recall that  $Xb = \eta bX$ , so that

$$rx = \sum_{i=0}^d r_i X^i b X^s = \sum_{i=0}^d \eta^i r_i b X^{i+s} = \sum_{i=0}^d \eta^i r_i x X^i,$$

that is,

$$rx = \sum_{i,j=0}^d \eta^i r_i x_j X^{i+j}.$$

Also,  $rx = yX^t = \sum_{i=0}^d y_i X^{i+t}$ ; and so we obtain the following equality

$$\sum_{i,j=0}^d \eta^i r_i x_j X^{i+j} = \sum_{i=0}^d y_i X^{i+t} \tag{1}$$

in  $R = A[X; \sigma, \delta]$ .

Since  $t \geq 1$ , the term of degree 0 in the left hand side of (1) must be zero; that is,  $r_0x_0 = 0$ . Since  $x_0 \neq 0$ , this gives  $r_0 = 0$ . Hence  $r = \sum_{i=1}^d r_i X^i = wX$  with  $w = \sum_{i=1}^d r_i X^{i-1} \in R$ . Consequently, the equality  $yX^t = rx$  can be rewritten as

$$yX^t = wXx = wXbX^s = \eta w b X^{s+1} = \eta w x X.$$

It follows that  $yX^{t-1} = \eta w x$ , with  $\eta w \in R$ , contradicting the minimality of  $t$ .

Hence  $t = 0$  and  $y = rx$  with  $r \in R$ ; so that  $y \in Rx$ , as required.

To sum up, we have established that  $I = Rx$ .

It remains to show that  $xR = I$ . First, note that, since  $Xb = \eta bX$ , we have  $x = \eta^{-s} X^s b \in R$  and it is clear that  $\min\{i \in \mathbb{N} \mid X^i b \in R\} = \min\{i \in \mathbb{N} \mid bX^i \in R\} = s$ . Now by writing elements of  $R$  as polynomials with coefficients on the right, a very similar calculation (which we omit) to that done above shows that  $xR = I$ . Thus,  $x = bX^s$  is a nonzero  $\mathcal{H}$ -eigenvector of  $R$  such that  $I = xR = Rx$  is a completely prime ideal. This finishes the first part of the proof.

Now, let  $J$  be any nonzero  $\mathcal{H}$ -prime ideal of  $R$ , and note that  $J$  is completely prime.

First, assume that  $X \notin J$ . Then  $JS^{-1} \cap T$  is a nonzero  $\mathcal{H}$ -invariant prime ideal of  $T$  and it follows that  $JS^{-1} \cap B$  is a nonzero  $\mathcal{H}$ -invariant prime ideal of  $B$ , by Corollary 2.4. Thus, there exists  $0 \neq b \in JS^{-1} \cap B$  such that  $b$  is a normal  $\mathcal{H}$ -eigenvector and  $bB = Bb$  is a completely prime ideal of  $B$ . As in the earlier part of the proof, set  $x := bX^s$ , where  $s$  is minimal such that  $bX^s \in R$ . Note that  $x \in JS^{-1} \cap R = J$ , and that  $x$  is a nonzero normal  $\mathcal{H}$ -eigenvector of  $R$  such that  $xR = Rx$  is a completely prime ideal of  $R$ .

Next, assume that  $X \in J$ . If  $\delta = 0$  then  $X$  is a nonzero normal  $\mathcal{H}$ -eigenvector such that  $XR = RX$  is completely prime (since  $A$  is a domain), as required. Thus, we may assume that  $\delta \neq 0$ .

Choose  $c \in A$  such that  $\delta(c) \neq 0$ , and note that  $0 \neq \delta(c) = Xc - \sigma(c)X \in J$ ; and so  $J \cap A \neq 0$ . It is clear that the map  $b \mapsto \theta^{-1}(b) + J$  defines a homomorphism from  $B$  to  $R/J$ , and this homomorphism extends to a homomorphism  $g$  from  $T$  to  $R/J$  such that  $g(X) = 0$ . This map, given by  $g(\sum b_i X^i) = \theta^{-1}(b_0) + J$ , commutes with the action of  $\mathcal{H}$ . Set  $J' = \ker(g)$ ; so that  $J'$  is a completely prime  $\mathcal{H}$ -ideal of  $T$ . With  $c \in A$  as above, note that  $g(\theta(\delta(c))) = \delta(c) + J = 0_{R/J}$ . Thus,  $J' \cap B$  is a nonzero  $\mathcal{H}$ -prime ideal of  $B$ . Thus, there is a nonzero normal  $\mathcal{H}$ -eigenvector  $b \in J' \cap B$  such that  $bB = Bb$  is a completely prime  $\mathcal{H}$ -ideal of  $B$ . Set  $x := bX^s$ , where  $s$  is minimal such that  $bX^s \in R$ . Then, as in the earlier part of the proof, we know that  $x$  is a nonzero normal  $\mathcal{H}$ -eigenvector of  $R$  such

that  $xR = Rx$  is a prime ideal. In order to finish this case, we will show that  $x \in J$ . Now,  $b = \theta(a)$  for some  $0 \neq a \in A$ . We use the explicit formula for  $\theta(a)$  to finish the calculation:

$$b = \theta(a) = \sum_{n=0}^{+\infty} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{-n}.$$

(The sum on the right hand side exists since  $\delta$  is locally nilpotent). Since  $\delta$  is locally nilpotent, there exists  $d \in \mathbb{N}$  such that  $\delta^d(a) \neq 0$  and  $\delta^{d+1}(a) = 0$ . Then, since  $q\delta \circ \sigma = \sigma \circ \delta$ , we have

$$b = \theta(a) = \sum_{n=0}^d \frac{(1-q)^{-n}}{[n]!_q} q^{n^2} \sigma^{-n} \circ \delta^n(a) X^{-n},$$

and so the smallest integer  $i$  such that  $bX^i \in R$  is equal to  $d$ . In other words,  $s = d$  and  $x = bX^d = \sum_{n=0}^d \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{d-n}$ , that is:

$$x = \frac{(1-q)^{-d}}{[d]!_q} \delta^d \circ \sigma^{-d}(a) + \left( \sum_{n=0}^{d-1} \frac{(1-q)^{-n}}{[n]!_q} \delta^n \circ \sigma^{-n}(a) X^{d-1-n} \right) X.$$

Since  $X \in J$ , in order to prove that  $x \in J$ , it is so sufficient to prove that  $\delta^d \circ \sigma^{-d}(a)$  belongs to  $J$ .

Observe that, since  $b \in J'$ , we have  $a = g(b) \in g(J') \subseteq J$ . Hence, if  $d = 0$ , then  $x = b = a$ , and so  $x \in J$  as desired. Assume now that  $d \geq 1$ . Then  $\delta^d \circ \sigma^{-d}(a) = \delta(\delta^{d-1} \circ \sigma^d(a))$ . Set  $e := \delta^{d-1} \circ \sigma^d(a) \in A$ . Then  $\delta^d \circ \sigma^{-d}(a) = \delta(e) = Xe - \sigma(e)X \in J$ , since  $X \in J$ . This was what we needed to conclude that  $x \in J$ , as required.

□

### 3 CGL extensions

In this section, we develop a suitable context in which to apply the results of the previous section to establish that certain iterated skew polynomial extensions are  $\mathcal{H}$ -UFDs. The next problem is to use this information, the Goodearl-Letzter stratification theory and the noncommutative version of Nagata's lemma that we have established, Proposition 1.6, to deduce that these extensions are, infact, noetherian UFDs

The next definition contains all of the conditions that are necessary for this programme to succeed. The definition is unwieldy, but is justified by the fact that many of the quantum algebras that we wish to study satisfy all of these conditions.

**Definition 3.1** An iterated skew polynomial extension

$$A = k[x_1][x_2; \sigma_2, \delta_2] \dots [x_n; \sigma_n, \delta_n]$$

is said to be a *CGL extension* (after Cauchon, Goodearl and Letzter) provided that the following list of conditions is satisfied:

- With  $A_j := k[x_1][x_2; \sigma_2, \delta_2] \dots [x_j; \sigma_j, \delta_j]$  for each  $1 \leq j \leq n$ , each  $\sigma_j$  is a  $k$ -automorphism of  $A_{j-1}$ , each  $\delta_j$  is a locally nilpotent  $k$ -linear  $\sigma_j$ -derivation of  $A_{j-1}$ , and there exist nonroots of unity  $q_j \in k^*$  with  $\sigma_j \delta_j = q_j \delta_j \sigma_j$ ;
- For each  $i < j$  there exists a  $\lambda_{ji}$  such that  $\sigma_j(x_i) = \lambda_{ji} x_i$ ;
- There is a torus  $\mathcal{H} = (k^*)^r$  acting rationally on  $A$  by  $k$ -algebra automorphisms;
- The  $x_i$  for  $1 \leq i \leq n$  are  $\mathcal{H}$ -eigenvectors;
- There exist elements  $h_1, \dots, h_n \in \mathcal{H}$  such that  $h_j(x_i) = \sigma_j(x_i)$  for  $j > i$  and such that the  $h_j$ -eigenvalue of  $x_j$  is not a root of unity.

If, in addition, the subgroup of  $k^*$  generated by the  $\lambda_{ji}$  is torsionfree then we will say that  $A$  is a *torsionfree CGL extension*.

For a discussion of rational actions of tori, see [1, Chapter II.2].

Note that any CGL extension will be a noetherian domain with finite GK dimension, cf. [1, Lemma II.9.7]; and so will satisfy the descending chain condition on prime ideals, as mentioned earlier.

Notice that, if  $A$  is a CGL extension, then the action of  $\mathcal{H}$  on  $k[x_1]$  is such that  $h_1 \cdot x_1 = \lambda x_1$ , where  $\lambda \in k^*$  is not a root of unity. From this, it follows easily that the only nonzero  $\mathcal{H}$ -prime of  $k[x_1]$  is  $\langle x_1 \rangle$ , which is (completely) prime. Using [1, II.5.11], we deduce that, if  $A$  is a CGL extension then each of the extensions  $A_j = A_{j-1}[x_j; \sigma_j, \delta_j]$  is a Cauchon extension; so the results of the previous section are available. Also, any CGL extension satisfies the conditions of [1, II.5.1] and so there are only finitely many  $\mathcal{H}$ -primes in  $A$  and they are all completely prime, by [1, Theorem II.5.12]. Further, if  $A$  is a torsionfree CGL extension, then all prime ideals of  $A$  are completely prime, by [1, Theorem II.6.9]. In particular, if such an  $A$  is a noetherian UFR then it is a noetherian UFD.

**Proposition 3.2** *Let  $A$  be a CGL extension. Then  $A$  is an  $\mathcal{H}$ -UFD; that is, each nonzero  $\mathcal{H}$ -prime  $Q$  of  $A$  contains a nonzero normal  $\mathcal{H}$ -eigenvector  $a$  such that the  $\mathcal{H}$ -ideal  $P := aA = Aa$  is completely prime.*

*Proof.* As already mentioned, the only nonzero  $\mathcal{H}$ -prime of  $k[x_1]$  is  $\langle x_1 \rangle$  and it follows immediately that  $k[x_1]$  is an  $\mathcal{H}$ -UFD. Now, each of the extensions  $A_j = A_{j-1}[x_j; \sigma_j, \delta_j]$  is a Cauchon extension; so apply Proposition 2.9 repeatedly.  $\square$

The main aim in this section is to show that any CGL extension is in fact a noetherian UFR. It then follows that any torsionfree CGL extension is a noetherian UFD. Since a CGL extension  $A$  is an  $\mathcal{H}$ -UFD, the prime ideals of height one that are  $\mathcal{H}$ -ideals are principal, generated by elements that are normal and  $\mathcal{H}$ -eigenvectors. Also, as noted above, there are only finitely many  $\mathcal{H}$ -primes, by [1, Theorem II.5.12], and they are all completely prime. Thus, in order to show that such an extension is a noetherian UFD, we have to deal with the primes of height one that are not  $\mathcal{H}$ -primes. In the language of Goodearl and Letzter, these primes are in the *stratum* of the zero ideal; that is, if  $P$  is a prime ideal of height one that is not an  $\mathcal{H}$ -prime, then the largest  $\mathcal{H}$ -ideal contained in  $P$  is the zero ideal. The Goodearl-Letzter stratification theory enables us to deal with these primes. The idea is simple. The stratification theory shows that, once we invert all the regular  $\mathcal{H}$ -eigenvectors, the prime ideals in the stratum of the zero ideal become centrally generated. In fact, the height one primes in the zero stratum become principal, generated by a central element in this localisation; this shows this localisation is a noetherian UFR. However, Proposition 1.6 is valid only when we are inverting a multiplicative set generated by finitely many normal elements. To deal with this point, it turns out, and this is what we show first, that it is enough to invert the multiplicative set generated by the finitely many generators of the  $\mathcal{H}$ -primes of height one in order to get a picture similar to that of the stratification theory.

**Lemma 3.3** *Let  $I$  be an  $\mathcal{H}$ -ideal in a CGL extension  $A$ . Then the prime ideals minimal over  $I$  are all  $\mathcal{H}$ -prime ideals.*

*Proof.* Since  $A$  is noetherian, there are finitely many primes minimal over  $I$ . Let  $Q$  be a prime minimal over  $I$ . The  $\mathcal{H}$ -orbit of  $Q$  consists of primes minimal over  $I$  and hence is finite. Now, [1, II.2.9] shows that  $Q$  is an  $\mathcal{H}$ -ideal.  $\square$

**Corollary 3.4** *Suppose that  $A$  is a CGL extension and that  $P_i = a_i A$  for  $1 \leq i \leq m$  are the prime ideals of height one that are  $\mathcal{H}$ -primes, where the  $a_i$  are normal  $\mathcal{H}$ -eigenvectors. Then, each nonzero  $\mathcal{H}$ -ideal of  $A$  contains a product of the  $a_i$  (repetitions allowed).*

*Proof.* Let  $I$  be a nonzero  $\mathcal{H}$ -ideal of  $A$ . Since  $A$  is noetherian, there are only a finite number of prime ideals that are minimal over  $I$ ; denote these primes by  $Q_1, \dots, Q_s$ . By



the previous lemma, these are all  $\mathcal{H}$ -primes. Since  $A$  is noetherian, the ideal  $I$  contains a product of the  $Q_i$ . However, each  $Q_i$  contains some  $P_j$ , by Proposition 3.2; and so  $I$  contains a product of the  $P_i$ , hence a product of the  $a_i$ .  $\square$

Set  $T$  to be the localisation of  $A$  with respect to the multiplicatively closed set generated by the normal  $\mathcal{H}$ -eigenvectors  $a_i$ . Then the rational action of  $\mathcal{H}$  on  $A$  extends to an action of  $\mathcal{H}$  on the localisation  $T$  by  $k$ -algebra automorphisms, since we are localising with respect to  $\mathcal{H}$ -eigenvectors, and this action of  $\mathcal{H}$  on  $T$  is also rational, by using [1, II.2.7]. We have the following proposition.

**Proposition 3.5** *The ring  $T$  is  $\mathcal{H}$ -simple; that is, the only  $\mathcal{H}$ -ideals of  $T$  are 0 and  $T$ .*

*Proof.* Let  $J$  be an  $\mathcal{H}$ -ideal of  $T$  and let  $I = J \cap A$ . Clearly,  $I$  is an  $\mathcal{H}$ -ideal of  $A$ . In addition,  $J = IT$ , by [17, 2.1.16]. If  $I = 0$ , then  $J = 0$ . Otherwise,  $J = T$ , by the previous corollary.  $\square$

We are now in position to show that the CGL extension  $A$  is a noetherian UFR.

**Theorem 3.6** *Let  $A = k[x_1][x_2; \sigma_2, \delta_2] \dots [x_n; \sigma_n, \delta_n]$  be a CGL extension. Then  $A$  is a noetherian UFR.*

*Proof.* By Proposition 1.6, it is enough to prove that the localisation  $T$  is a noetherian UFR. Now, as proved in Proposition 3.5,  $T$  is an  $\mathcal{H}$ -simple ring. Thus, using [1, II.3.9], it is a noetherian UFR, as required.  $\square$

**Theorem 3.7** *Let  $A$  be a torsionfree CGL-extension. Then  $A$  is a noetherian UFD.*

*Proof.* Use Theorem 3.6 and the fact that all prime ideals are completely prime in a torsionfree CGL-extension.  $\square$

This theorem applies to many quantum algebras. A selection of such algebras of current interest is given in the following corollary. For exact definitions of those of the algebras that are not explicitly defined in this paper, consult [8] or [2, Section 6.2]

**Corollary 3.8** *The following algebras are noetherian UFDs:*

- *The algebra of quantum matrices  $\mathcal{O}_q(M_{m,n})$ , with  $q$  not a root of unity, (see also the next section for more information about  $\mathcal{O}_q(M_{m,n})$ ), and, more generally, the multiparameter version  $\mathcal{O}_{\lambda, \mathbf{p}}(M_{m,n}(k))$ , with  $\lambda$  not a root of unity and the group  $\langle \lambda, p_{ij} \rangle$  torsionfree.*

- The quantized enveloping algebra  $U_q(\mathfrak{n}^+)$ , with  $q$  not a root of unity, of the nilpotent subalgebra  $\mathfrak{n}^+$  of a complex semisimple Lie algebra  $\mathfrak{g}$ .
- The quantized enveloping algebra  $U_q(\mathfrak{b}^+)$ , with  $q$  not a root of unity, of the Borel subalgebra  $\mathfrak{b}^+$  of a complex semisimple Lie algebra  $\mathfrak{g}$ .
- The quantum affine space  $\mathcal{O}_{\mathbf{q}}(k^n)$ , with  $\langle q_{ij} \rangle$  torsionfree.
- The quantized Weyl algebra  $A_n^{\mathcal{Q},\Gamma}(k)$  with each  $q_i$  not a root of unity and  $\langle q_i, \gamma_{ij} \rangle$  torsionfree.
- The quantum grassmannian  $\mathcal{G}_q(m, n)$ , with  $q$  not a root of unity.

*Proof.* The algebras  $\mathcal{O}_q(M_{m,n})$ ,  $\mathcal{O}_{\lambda, \mathbf{p}}(M_{m,n}(k))$ ,  $\mathcal{O}_{\mathbf{q}}(k^n)$ ,  $A_n^{\mathcal{Q},\Gamma}(k)$  are described in [8] as iterated skew polynomial extensions with appropriate torus actions, and can easily be checked to be torsionfree CGL-extensions. (The only awkward point is to check that the first condition holds, and, in particular, to check that the  $\delta_i$  involved all act locally nilpotently. The lemma below, which is easy to prove, helps deal with this point.)

The algebra  $U_q(\mathfrak{n}^+)$  is described in [2, Section 6.2] and is easily seen to be a CGL-extension.

The algebra  $U_q(\mathfrak{b}^+)$  is described in [8] as a localisation of an algebra that is an iterated skew polynomial extension with a torus action. This algebra is easily checked to be a CGL-extension.

The algebra  $\mathcal{G}_q(m, n)$  is shown to be a noetherian UFD in the last section of this paper.  $\square$

**Lemma 3.9** *Let  $R$  be a  $k$ -algebra,  $\tau$  a  $k$ -algebra automorphism,  $\delta$  a left  $\tau$ -derivation, which we assume to be  $k$ -linear and set  $S = R[x; \tau, \delta]$ . In addition, let  $X \subseteq R$  be a generating set of the  $k$ -algebra  $R$ . Then, the following holds.*

- (i) *Assume that there exists  $q \in k$  such that, for all  $x \in X$ ,  $\delta\tau(x) = q\tau\delta(x)$ , then  $\delta\tau = q\tau\delta$ .*
- (ii) *Assume that there exists  $q \in k$  such that  $\delta\tau = q\tau\delta$ . If, for all  $x \in X$ , there exists  $d \in \mathbb{N}^*$  such that  $\delta^d(x) = 0$ , then  $\delta$  is locally nilpotent.*

## 4 Height one $\mathcal{H}$ -primes in $\mathcal{O}_q(M_{m,n})$

In this section, we identify generators for each of the height one primes which are  $\mathcal{H}$ -ideals of the algebra of quantum matrices, in the generic case.

Throughout,  $k$  is a field and  $q$  is a nonzero element of  $k$  that is not a root of unity. Let  $m, n$  be positive integers. Recall that the algebra of  $m \times n$  quantum matrices,  $\mathcal{O}_q(M_{m,n})$ , is the  $k$ -algebra generated by  $mn$  indeterminates  $x_{ij}$ , with  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , subject to the relations

$$\begin{aligned} x_{ij}x_{il} &= qx_{il}x_{ij}, & (j < l); \\ x_{ij}x_{kj} &= qx_{kj}x_{ij}, & (i < k); \\ x_{ij}x_{kl} &= x_{kl}x_{ij}, & (i < k, j > l); \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj}, & (i < k, j < l). \end{aligned}$$

In the case that  $m = n$ , we write  $\mathcal{O}_q(M_n)$  for  $\mathcal{O}_q(M_{m,n})$ .

In view of the restriction that  $q$  is not a root of unity, we refer to  $\mathcal{O}_q(M_{m,n})$  as the algebra of *generic quantum matrices*.

Let  $\mathcal{H}$  be the  $(m+n)$ -torus  $(k^*)^m \times (k^*)^n$ . The torus  $\mathcal{H}$  acts on  $\mathcal{O}_q(M_{m,n})$  by  $k$ -algebra automorphisms in the following way:

$$(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \cdot x_{ij} := \alpha_i \beta_j x_{ij}.$$

The algebra  $\mathcal{O}_q(M_{m,n})$  can be presented as an iterated skew polynomial extension with the variables added in lexicographical order. With this presentation, and with the group  $\mathcal{H}$  above acting,  $\mathcal{O}_q(M_{m,n})$  is a torsionfree CGL extension; and so is a noetherian UFD by the results of the previous section. There are only finitely many height one prime ideals which are  $\mathcal{H}$ -primes, and the purpose of this section is to identify these  $\mathcal{H}$ -primes.

In the literature, many results are only stated for  $\mathcal{O}_q(M_n)$  but are easily translated to  $\mathcal{O}_q(M_{m,n})$ , by using arguments based on the following easy observations. First, if  $I$  is a set of row indices and  $J$  is a set of column indices then the subalgebra of  $\mathcal{O}_q(M_n)$  or  $\mathcal{O}_q(M_{m,n})$  generated by the  $x_{ij}$  with  $i \in I$  and  $j \in J$  is isomorphic to another quantum matrix algebra in a natural way. Secondly, let  $A = \mathcal{O}_q(M_n)$ , and let  $B = \mathcal{O}_q(M_{m,n})$ , with  $m \leq n$ , be the quantum matrix algebra generated by generators in the first  $m$  rows of  $A$ , then there is an algebra epimorphism  $\pi : A \rightarrow B$  defined by the projection given by  $x_{ij} \mapsto x_{ij}$  if  $i \leq m$  and  $x_{ij} \mapsto 0$  otherwise. By using the first observation, we may think of  $\mathcal{O}_q(M_{m,n})$  and  $\mathcal{O}_q(M_{n,m})$  being embedded in a common  $\mathcal{O}_q(M_n)$ . Then, there is an isomorphism between  $\mathcal{O}_q(M_{m,n})$  and  $\mathcal{O}_q(M_{n,m})$  given by transposition of the generators in  $\mathcal{O}_q(M_n)$ ; that is,  $x_{ij} \mapsto x_{ji}$ , see [18, Proposition 3.7.1]. For this reason, we will assume that  $m \leq n$ . In view of the restriction that  $q$  is not a root of unity, we will refer to  $\mathcal{O}_q(M_{m,n})$  as a *generic quantum matrix algebra*.

The algebra  $\mathcal{O}_q(M_n)$  has a special element,  $\det_q$ , the *quantum determinant*, defined by

$$\det_q := \sum_{\sigma} (-q)^{l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where the sum is taken over the permutations of  $\{1, \dots, n\}$  and  $l(\sigma)$  is the usual length function on such permutations. The quantum determinant is a central element of  $\mathcal{O}_q(M_n)$ , see, for example, [18, Theorem 4.6.1]. If  $I$  is a  $t$ -element subset of  $\{1, \dots, m\}$  and  $J$  is a  $t$ -element subset of  $\{1, \dots, n\}$  then the quantum determinant of the subalgebra of  $\mathcal{O}_q(M_{m,n})$  generated by  $\{x_{ij}\}$ , with  $i \in I$  and  $j \in J$ , is denoted by  $[I \mid J]$ . The elements  $[I \mid J]$  are the *quantum minors* of  $\mathcal{O}_q(M_{m,n})$ . They are not in general central; however, they do possess good commutation properties: in particular, in what follows, we will identify several quantum minors that are normal elements. Two elements  $a, b$  are said to *q-commute* if there is an integer  $s$  such that  $ab = q^s ba$ . An element that *q-commutes* with each of the generators of a quantum matrix algebra is easily seen to be normal, and this is a standard way to demonstrate normality. In many sources, such commutation relations are established for  $\mathcal{O}_q(M_n)$ . Usually, it is easy to transfer such results to  $\mathcal{O}_q(M_{m,n})$ , by including this quantum matrix algebra as a subalgebra of a suitable  $\mathcal{O}_q(M_n)$  by including extra rows or columns of generators: obviously, if an element *q-commutes* with each of the generators in this larger algebra then it *q-commutes* with the generators of the original algebra. In addition, we will use the transposition isomorphism to derive further *q-commutation* results, with little comment.

Cauchon's theory of deleting derivations, [2, 3], has been applied to quantum matrices with great success. In fact, in [3], Cauchon works with  $\mathcal{O}_q(M_n)$ ; however, the methods extend to  $\mathcal{O}_q(M_{m,n})$  and the details are worked out in [15]. Let  $w$  denote an  $m \times n$  array of square boxes in which each box is coloured either black or white. A *Cauchon diagram* is such an array with the following property: if a square is coloured black then either every square to the left of this square is also coloured black, or every square above this square is also coloured black. Cauchon [3] and Launois [15] prove that the  $\mathcal{H}$ -prime ideals of  $\mathcal{O}_q(M_{m,n})$  are in bijection with the  $m \times n$  Cauchon diagrams. In addition, if  $P$  is an  $\mathcal{H}$ -prime, then the height of  $P$  (as a prime ideal) is equal to the number of black boxes in the corresponding diagram, by [3], Théorème 6.3.3 (which is easily adapted to the rectangular case), and [15], Proposition 1.3.2.2. (Recall that, by [1, II.2.9], any  $\mathcal{H}$ -prime is prime.)

For  $1 \leq i \leq m$ , let  $c_i$  denote the  $i \times i$  quantum minor  $[m - i + 1, \dots, m \mid 1, \dots, i]$  of  $\mathcal{O}_q(M_{m,n})$  and let  $b_i$  denote the  $i \times i$  quantum minor  $[1, \dots, i \mid n - i + 1, \dots, n]$  of  $\mathcal{O}_q(M_{m,n})$ , while for  $m < i \leq n$ , let  $b_i$  denote the quantum minor  $[1, \dots, m \mid n - i + 1, \dots, n + m - i]$ . Note that  $c_m = b_n$ ; in particular, note that in  $\mathcal{O}_q(M_n)$  we have  $b_n = c_n = \det_q$ . For orientation, note that the  $b_i$  are the minors coming from the top right of the matrix of generators of  $\mathcal{O}_q(M_{m,n})$ , while the  $c_i$  come from the bottom left.

The quantum minors are  $\mathcal{H}$ -eigenvectors; and so, for example, the ideals generated by each of the elements  $b_i$  and  $c_i$ , defined above, are  $\mathcal{H}$ -ideals. We will show below that they are  $\mathcal{H}$ -prime ideals.

**Lemma 4.1** *The elements  $b_i$ , with  $1 \leq i \leq n$ , and  $c_i$ , with  $1 \leq i \leq m$ , are normal elements of  $\mathcal{O}_q(M_{m,n})$ .*

*Proof.* Let  $1 \leq i \leq m$ ; it follows easily from [9, Corollary A.2] that  $c_i$   $q$ -commute with each generator of  $\mathcal{O}_q(M_{m,n})$  and, using the transpose automorphism, that the same is true for  $b_i$ . For  $b_i$ , with  $m < i \leq n$ , a slightly more complicated argument is required. Fix an  $i$  with  $m < i \leq n$ . Consider a generator  $x_{kl}$ . If  $l \leq n + m - i$  then  $x_{kl}$  and  $b_i$  belong to the quantum matrix algebra  $\mathcal{O}_q(M_{m,n+m-i})$  obtained from the generators in the first  $n + m - i$  columns of  $\mathcal{O}_q(M_{m,n})$ . In fact,  $b_i$  is  $b_m$  in this subalgebra, and so  $x_{kl}$  and  $b_i$   $q$ -commute. If  $l > n + m - i$  then  $x_{kl}$  and  $b_i$  belong to the quantum matrix algebra  $\mathcal{O}_q(M_{m,i})$  obtained from the generators in the last  $i$  columns of  $\mathcal{O}_q(M_{m,n})$ . In this case,  $b_i$  is  $c_m$  in this subalgebra and so again we see that  $x_{kl}$  and  $b_i$   $q$ -commute. Thus  $b_i$   $q$ -commutes with each of the generators of  $\mathcal{O}_q(M_{m,n})$  and so is a normal element in this algebra.  $\square$

**Proposition 4.2** *There are precisely  $m + n - 1$  height one primes that are  $\mathcal{H}$ -primes in the generic quantum matrix algebra  $\mathcal{O}_q(M_{m,n})$ . They are the ideals generated by  $b_1, \dots, b_n$  and  $c_1, \dots, c_{m-1}$  (recall that  $c_m = b_n$ ).*

*Proof.* It is easily seen that the elements  $b_1, \dots, b_n, c_1, \dots, c_{m-1}$  generate pairwise distinct ideals.

The height one primes that are  $\mathcal{H}$ -primes are in bijection with the Cauchon diagrams with precisely one black box. Such Cauchon diagrams arise by filling in one box either in the first row of the array, or the first column. There are  $m + n - 1$  ways of doing this; and so there are  $m + n - 1$  height one primes that are  $\mathcal{H}$ -primes.

That the ideals specified are  $\mathcal{H}$ -ideals is due to the fact that the  $b_i$  and  $c_i$  are  $\mathcal{H}$ -eigenvectors. That the ideals are prime comes about in the following way. If we restrict to the quantum submatrix algebra  $A$ , say, specified by the rows and columns of a  $b_i$  or  $c_i$ , then when we factor out  $b_i$  or  $c_i$  from  $A$  we are factoring out the quantum determinant of  $A$ , and so the factor  $A/b_iA$  or  $A/c_iA$  is a domain, see, for example, [9, Theorem 2.5]. Since the  $b_i$  or  $c_i$   $q$ -commute with the remaining  $x_{ij}$  we can add the remaining  $x_{ij}$  in such a way that at any stage if we have reached a subalgebra  $B$  then  $B/b_iB$ , say, is an iterated skew polynomial algebra over  $A/b_iA$  and so is a domain. For example, if we are in the case that  $m < i \leq n$ , then we can add the  $x_{ij}$  to the left of the rows and columns used by

$b_i$  by moving from right to left along each row, starting with the bottom row and moving upwards row by row. We then can add the  $x_{ij}$  to the right of the rows and columns used by  $b_i$  in lexicographic order. Thus each  $\mathcal{O}_q(M_{m,n})/b_i\mathcal{O}_q(M_{m,n})$  and  $\mathcal{O}_q(M_{m,n})/c_i\mathcal{O}_q(M_{m,n})$  is a domain and so each ideal of  $\mathcal{O}_q(M_{m,n})$  generated by a  $b_i$  or  $c_i$  is a completely prime ideal. Since these ideals are  $\mathcal{H}$ -ideals, they are also  $\mathcal{H}$ -primes. Since we have precisely  $m + n - 1$  elements  $b_i$  or  $c_i$  this gives all of the height one primes that are  $\mathcal{H}$ -primes.  $\square$

## 5 Generic quantum grassmannians are UFD

Recall that the *quantum grassmannian* subalgebra,  $\mathcal{G}_q(m, n)$ , of  $\mathcal{O}_q(M_{m,n})$  is the subalgebra generated by the  $m \times m$  maximal quantum minors of  $\mathcal{O}_q(M_{m,n})$  (recall that we are assuming that  $m \leq n$ ). The algebra  $\mathcal{G}_q(m, n)$  is a noetherian domain, see, for example, [13, Theorem 1.1]. Our usual restriction that  $q$  is not a root of unity applies in this section; so we refer to  $\mathcal{G}_q(m, n)$  at the *generic quantum grassmannian*.

In view of the fact that each of the quantum minors that generates  $\mathcal{G}_q(m, n)$  is of the form  $[1, \dots, m \mid J]$  we will denote such a minor by  $[J]$ . The two extreme quantum minors,  $[1, \dots, m]$  and  $[n - m + 1, \dots, n]$  are normal in  $\mathcal{G}_q(m, n)$ , see, for example, [13, Corollary 1.1, Lemma 1.1].

We will use the fact that generic quantum matrices are noetherian UFD, and the dehomogenisation isomorphism

$$\mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi] \longrightarrow \mathcal{G}_q(m, n)[[n - m + 1, \dots, n]^{-1}]$$

of [13, Corollary 4.1] to show that  $\mathcal{G}_q(m, n)$  is a noetherian UFD. Note that the automorphism  $\phi$  used in the dehomogenisation isomorphism acts on generators via  $\phi(x_{ij}) = q^{-1}x_{ij}$ , see [13, Corollary 4.1].

To show that  $\mathcal{G}_q(m, n)$  is a noetherian UFD, we proceed as follows. First, we show that the problem reduces to proving that the localisation  $\mathcal{G}_q(m, n)[[n - m + 1, \dots, n]^{-1}]$  is a noetherian UFD. Once this is done, by the dehomogenisation theorem, the problem transfers to showing that  $\mathcal{O}_q(M_{m,n-m})[y, y^{-1}; \phi]$  is a noetherian UFD, and this is the second step.

The first step is easy and essentially amounts to proving Lemma 5.1 below.

**Lemma 5.1** *The ideal of  $\mathcal{G}_q(m, n)$  generated by  $[n - m + 1, \dots, n]$  is a completely prime ideal.*

*Proof.* Let  $R = \mathcal{G}_q(m, n)$ . The isomorphism discussed immediately before [13, Lemma 1.1] shows that the result follows provided that we show that  $a := [1, \dots, m]$  generates a completely prime ideal in  $\mathcal{G}_q(m, n)$ .

Note that  $u := [n - m + 1, \dots, n]$  is left regular modulo  $aR$ , see the proof of Theorem 6.1 of [13]. Hence, it is enough to prove that  $a$  generates a completely prime ideal in the localisation  $R[u^{-1}]$ . We use the dehomogenisation isomorphism introduced above. Set  $v := [1, \dots, t | 1, \dots, t]$  with  $t = m$  if  $m \leq n - m$  and  $t = n - m$  otherwise. By [16, Lemma 3.5.1], it is enough to show that  $v$  generates a completely prime ideal of  $\mathcal{O}_q(M_{m, n-m})[y, y^{-1}; \phi]$ . However,  $v$  generates a completely prime ideal of  $\mathcal{O}_q(M_{m, n-m})$  that is left invariant by  $\phi$ , since  $\phi(v)$  is a scalar multiple of  $v$ . Thus  $v$  generates a completely prime ideal of  $\mathcal{O}_q(M_{m, n-m})[y, y^{-1}; \phi]$ , as required.  $\square$

To achieve the second step, we observe first that  $\mathcal{O}_q(M_{m, n-m})[y; \phi]$  is a torsionfree CGL-extension (in  $s := m(n - m) + 1$  steps) as follows. The torus  $\mathcal{H} = (k^*)^n = (k^*)^m \times (k^*)^{n-m}$  acts on  $\mathcal{O}_q(M_{m, n-m})$  as defined at the beginning of the previous section, and we have already observed that this makes  $\mathcal{O}_q(M_{m, n-m})$  a CGL-extension (in  $s - 1 = m(n - m)$  steps). In order to deal with the last step (extension by  $y$ ) we proceed as follows. We extend this action of  $\mathcal{H}$  to  $\mathcal{O}_q(M_{m, n-m})[y; \phi]$  by setting  $(a_1, \dots, a_n).y = a_1 \dots a_n y$ . The element  $h_s$  needed for the final extension is given by  $h_s := (q^{-1}, \dots, q^{-1}, 1, \dots, 1) \in \mathcal{H}$  (with  $m$  occurrences of  $q^{-1}$ ), since we require that  $h_s(x_{ij}) = \phi(x_{ij}) = q^{-1}x_{ij}$ . Moreover  $h_s.y = q^{-m}y$ , and  $q^{-m}$  is not a root of unity, since  $q$  is not. With this information provided, it is easy to check the remaining conditions and conclude that  $\mathcal{O}_q(M_{m, n-m})[y; \phi]$  is a torsionfree CGL-extension.

**Theorem 5.2** *Suppose that  $q \in k^*$  is not a root of unity. Then  $\mathcal{O}_q(M_{m, n-m})[y, y^{-1}; \phi]$  is a noetherian UFD.*

*Proof.* That  $\mathcal{O}_q(M_{m, n-m})[y; \phi]$  is a noetherian UFD follows from Theorem 3.7, since  $\mathcal{O}_q(M_{m, n-m})[y; \phi]$  is a torsionfree CGL-extension. It follows that  $\mathcal{O}_q(M_{m, n-m})[y, y^{-1}; \phi]$  is a noetherian UFD.  $\square$

**Theorem 5.3** *The generic quantum grassmannian,  $\mathcal{G}_q(m, n)$ , is a noetherian UFD.*

*Proof.* The previous result shows that  $\mathcal{G}_q(m, n)[[n - m + 1, \dots, n]^{-1}]$  is a noetherian UFD, by using the dehomogenisation isomorphism. By Lemma 5.1, we know that  $[n - m + 1, \dots, n]$  generates a completely prime ideal. Thus the result follows from Lemma 1.4.  $\square$

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