Citation for published version

DOI
https://doi.org/10.1081/AGB-200051150

Link to record in KAR
http://kar.kent.ac.uk/7410/

Document Version
UNSPECIFIED

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Rank $t$ $\mathcal{H}$-primes in quantum matrices.

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Abstract

Let $\mathbb{K}$ be a (commutative) field and consider a nonzero element $q$ in $\mathbb{K}$ which is not a root of unity. In [5], Goodearl and Lenagan have shown that the number of $\mathcal{H}$-primes in $R = O_q(\mathcal{M}_n(\mathbb{K}))$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors is a perfect square. The aim of this paper is to make precise their result: we prove that this number is equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$. This result was conjectured by Goodearl, Lenagan and McCammond. The proof involves some closed formulas for the poly-Bernoulli numbers that were established in [10] and [1].


1 Introduction.

Fix a (commutative) field $\mathbb{K}$ and an integer $n$ greater than or equal to 2, and choose an element $q$ in $\mathbb{K}^* := \mathbb{K}\setminus\{0\}$ which is not a root of unity. Denote by $R = O_q(\mathcal{M}_n(\mathbb{K}))$ the quantization of the ring of regular functions on $n \times n$ matrices with entries in $\mathbb{K}$ and by $(Y_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$ the matrix of its canonical generators. The bialgebra structure of $R$ gives us an action of the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ on $R$ by $\mathbb{K}$-automorphisms (See [5]) via:

$$(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_ib_{\alpha}Y_{i,\alpha} \quad ((i, \alpha) \in [1,n]^2).$$

In [9], Goodearl and Letzter have shown that $R$ has only finitely many $\mathcal{H}$-invariant prime ideals (See [9], 5.7. (i)) and that, in order to calculate the prime and primitive spectra of $R$, it is enough to determine the $\mathcal{H}$-invariant prime ideals of $R$ (See [9], Theorem 6.6). Next, using the theory of deleting derivations, Cauchon has found a formula for the exact number of $\mathcal{H}$-invariant prime ideals in $R$ (See [4], Proposotion 3.3.2). In this paper, we investigate these ideals.

In [12] (See also [13]), we have proved, assuming that $\mathbb{K} = \mathbb{C}$ (the field of complex numbers) and $q$ is transcendental over $\mathbb{Q}$, that the $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{C}))$ are generated by quantum minors, as conjectured by Goodearl and Lenagan (See [5] and [6]). Next, using this result together with Cauchon’s description for the set of $\mathcal{H}$-invariant prime ideals of $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [4], Théoréme 3.2.1), we have constructed an algorithm which provides an explicit generating set of quantum minors for each $\mathcal{H}$-invariant prime ideal in $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [11] or [13]).
On the other hand, Goodearl and Lenagan have shown (in the general case where \( q \in \mathbb{K}^* \) is not a root of unity) that, in order to obtain descriptions of all the \( \mathcal{H} \)-invariant prime ideals of \( R \), we just need to determine the \( \mathcal{H} \)-invariant prime ideals of certain "localized step-triangular factors" of \( R \), namely the algebras

\[
R^+_t := \frac{R}{\langle Y_{t,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle} \left[ Y_{r_1,1}, \ldots, Y_{r_t,t}^{-1} \right]
\]

and

\[
R^-_t := \frac{R}{\langle Y_{t,\alpha} \mid i > t \text{ or } \alpha < c_i \rangle} \left[ Y_{1,c_1}, \ldots, Y_{t,c_t}^{-1} \right],
\]

where \( t \in [0, n] \) and where \( r = (r_1, \ldots, r_t) \) and \( c = (c_1, \ldots, c_t) \) are strictly increasing sequences of integers in the range \( 1, \ldots, n \) (See \[3\], Theorem 3.5). Using this result, Goodearl and Lenagan have computed the \( \mathcal{H} \)-invariant prime ideals of \( O_q(\mathcal{M}_n(\mathbb{K})) \) (See \[5\]) and \( O_q(\mathcal{M}_3(\mathbb{K})) \) (See \[3\]).

The aims of this paper are to provide a description for the set \( \mathcal{H} \text{-} Spec(R^+_t) \) of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \) and to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) (\( t \in [0, n] \)), that is those \( \mathcal{H} \)-invariant prime ideals of \( R \) which contain all \( (t+1) \times (t+1) \) quantum minors but not all \( t \times t \) quantum minors. In \[5\], the authors have shown that the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) is a perfect square. More precisely, they have established (See \[5\], 3.6) that, for any \( t \in [0, n] \):

\[
| \mathcal{H} \text{-} Spec^{[t]}(R) | = \left( \sum_{1 \leq r_1 < \cdots < r_t \leq n} \left| \mathcal{H} \text{-} Spec(R^+_t) \right| \right)^2
\]  

(1)

where \( \mathcal{H} \text{-} Spec^{[t]}(R) \) denotes the set of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) and where \( \mathcal{H} \text{-} Spec(R^+_t) \) denotes the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \). The above relation (1) opens a potential route to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \); if we can compute the number of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \), then we will be able to count the rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \).

So, to compute the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \), the first step is to study the \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \). Since this algebra is induced from \( R \) by factor and localization, we first construct (See Section 2), by using the deleting derivations theory (See \[4\]), \( \mathcal{H} \)-invariant prime ideals of \( R \) that provide, after factor and localization, \( 2^{r_{t-1}} \cdots t^{r_t-t_{t-1}}(t+1)^{n-r_t} \) quantum minors but not \( (t+1) \times (t+1) \) quantum minors. In \[2\], we are able to show that the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \) is greater than or equal to \( (t!)^2 S(n+1, t+1)^2 \), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \) (See Proposition 8.9).

Finally, after observing that the number of \( \mathcal{H} \)-invariant prime ideals of \( R \) is equal to the poly-Bernoulli number \( B_n^{(-n)} \) (See Proposition 2.7), we use a closed formula for the poly-Bernoulli number \( B_n^{(-n)} \) (See \[1\], Theorem 2) in order to prove our main result: the number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals of \( R \) is actually equal to \((t!)^2 S(n+1, t+1)^2\). This result was conjectured by Goodearl, Lenagan and McCammond. As a corollary, we obtain a description for the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \) (See Section 3.4).

2 \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(\mathbb{K})) \).

Throughout this paper, we use the following conventions:
If $I$ is a finite set, $|I|$ denotes its cardinality.
- $\mathbb{K}$ denotes a (commutative) field and we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
- $q \in \mathbb{K}^*$ is not a root of unity.
- $n$ denotes a positive integer with $n \geq 2$.
- $R = O_q(M_n(\mathbb{K}))$ denotes the quantization of the ring of regular functions on $n \times n$ matrices with entries in $\mathbb{K}$; it is the $\mathbb{K}$-algebra generated by the $n \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i, \alpha \leq n$, subject to the following relations:

If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any $2 \times 2$ sub-matrix of $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$, then

1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.

2. $tx = xt - (q - q^{-1})yz$.

These relations agree with the relations used in [1], [2], [4], [5], [6], [12], and [11], but they differ from those of [13] and [2] by an interchange of $q$ and $q^{-1}$. It is well known that $R$ can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $Y_{i,\alpha}$ adjoined in lexicographic order. Thus the ring $R$ is a Noetherian domain. We denote by $F$ its skew-field of fractions. Moreover, since $q$ is not a root of unity, it follows from [7, Theorem 3.2] that all prime ideals of $R$ are completely prime.

- It is well known that the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ acts on $R$ by $\mathbb{K}$-algebra automorphisms via:

$$
(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_n Y_{i,\alpha} \quad \forall (i, \alpha) \in [1,n]^2.
$$

An $\mathcal{H}$-eigenvector $x$ of $R$ is a nonzero element $x \in R$ such that $h(x) \in \mathbb{K}^* x$ for each $h \in \mathcal{H}$. An ideal $I$ of $R$ is said to be $\mathcal{H}$-invariant if $h(I) = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}$-$\text{Spec}(R)$ the set of $\mathcal{H}$-invariant prime ideals of $R$.

The aim of this paragraph is to construct $\mathcal{H}$-invariant prime ideals of $R$ that, after factor and localization, will provide $\mathcal{H}$-invariant prime ideals of $R^+_t$ (See the introduction for the definition of this algebra). In order to do this, we use the description of the set $\mathcal{H}$-$\text{Spec}(R)$ that Cauchon has obtained by applying the theory of deleting derivations (See [4]).

### 2.1 Standard deleting derivations algorithm and description of $\mathcal{H}$-$\text{Spec}(R)$.

In this section, we provide the background definitions and notations for the standard deleting derivations algorithm (See [4, 12, 11]) and we recall the description of the set $\mathcal{H}$-$\text{Spec}(R)$ that Cauchon has obtained by using this algorithm (See [4]).

#### Notations 2.1

- We denote by $\leq_s$ the lexicographic ordering on $\mathbb{N}^2$. We often call it the standard ordering on $\mathbb{N}^2$. Recall that $(i, \alpha) \leq_s (j, \beta) \iff ([i < j] \text{ or } [i = j \text{ and } \alpha \leq \beta])$.

- We set $E_s = ([1,n]^2 \cup \{(n,n+1)\}) \setminus \{(1,1)\}$.

- Let $(j, \beta) \in E_s$. If $(j, \beta) \neq (n,n+1)$, $(j, \beta)^+$ denotes the smallest element (relatively to $\leq_s$) of the set $\{(i, \alpha) \in E_s \mid (j, \beta) <_s (i, \alpha)\}$.
In \[4\], Cauchon has shown that the theory of deleting derivations (See \[3\]) can be applied to the iterated Ore extension \( R = \mathbb{C}[Y_{1,1}] \ldots [Y_{n,n}; \sigma_{n,n}, \delta_{n,n}] \) (where the indices are increasing for \( \leq s \)). The corresponding deleting derivations algorithm is called the standard deleting derivations algorithm. It consists in the construction, for each \( r \in E_s \), of the family \((Y^{(r)}_{i,\alpha})_{(i,\alpha)\in[1,n]^2}\) of elements of \( F = \text{Fract}(R) \), defined as follows:

1. If \( r = (n, n+1) \), then \( Y^{(n,n+1)}_{i,\alpha} = Y_{i,\alpha} \) for all \((i, \alpha)\in[1,n]^2\).

2. Assume that \( r = (j, \beta) <_s (n, n+1) \) and that the \( Y^{(r^+)}_{i,\alpha} \) (\((i, \alpha)\in[1,n]^2\)) are already constructed. Then, it follows from \[3\], Théorème 3.2.1 that \( Y^{(r^+)}_{j,\beta} \neq 0 \) and, for all \((i, \alpha)\in[1,n]^2\), we have:

\[
Y^{(r)}_{i,\alpha} = \begin{cases} 
Y^{(r^+)}_{i,\alpha} & \text{if } i < j \text{ and } \alpha < \beta \\
Y^{(r^+)}_{i,\alpha} - Y^{(r^+)}_{i,\beta} (Y^{(r^+)}_{j,\beta})^{-1} Y^{(r^+)}_{j,\alpha} & \text{otherwise.}
\end{cases}
\]

**Notation 2.2**

Let \( r \in E_s \). We denote by \( R^{(r)} \) the subalgebra of \( F = \text{Fract}(R) \) generated by the \( Y^{(r)}_{i,\alpha} \) (\((i, \alpha)\in[1,n]^2\)), that is, \( R^{(r)} := \mathbb{C}(Y^{(r)}_{i,\alpha} \mid (i, \alpha)\in[1,n]^2) \).

**Notations 2.3**

We set \( R := R^{(1,2)} \) and \( T_{i,\alpha} := Y^{(1,2)}_{i,\alpha} \) for all \((i, \alpha)\in[1,n]^2\).

Let \((j, \beta) \in E_s \) with \((j, \beta) \neq (n, n+1) \). The theory of deleting derivations allows us to construct embeddings \( \varphi_{(j,\beta)} : \text{Spec}(R^{(j,\beta)^+}) \rightarrow \text{Spec}(R^{(j,\beta)}) \) (See \[3\], 4.3). By composition, we obtain an embedding \( \varphi : \text{Spec}(R) \rightarrow \text{Spec}(\overline{R}) \) which is called the canonical embedding. In \[4\], Cauchon has described the set \( \mathcal{H}-\text{Spec}(R) \) by determining its "canonical image" \( \varphi(\mathcal{H}-\text{Spec}(R)) \). To do this, he has introduced the following conventions and notations.

**Conventions 2.4**

- Let \( v = (l, \gamma) \in [1,n]^2 \).
  1. The set \( C_v := \{(i, \gamma) \mid 1 \leq i \leq l\} \subseteq [1,n]^2 \) is called the truncated column with extremity \( v \).
  2. The set \( L_v := \{(l, \alpha) \mid 1 \leq \alpha \leq \gamma\} \subseteq [1,n]^2 \) is called the truncated row with extremity \( v \).

- \( W \) denotes the set of all the subsets in \([1,n]^2\) which are a union of truncated rows and columns.

**Notation 2.5**

Given \( w \in W \), \( K_w \) denotes the ideal in \( \overline{R} \) generated by the \( T_{i,\alpha} \) such that \((i, \alpha)\in w\).

(Recall that \( K_w \) is a completely prime ideal in the quantum affine space \( \overline{R} \) (See \[3\], 2.1).)
The following description of the set \( \mathcal{H} \text{-Spec}(R) \) was obtained by Cauchon (See [4], Corollaire 3.2.1).

**Proposition 2.6**

1. Given \( w \in W \), there exists a (unique) \( \mathcal{H} \)-invariant (completely) prime ideal \( J_w \) in \( R \) such that \( \varphi(J_w) = K_w \).

2. \( \mathcal{H} \text{-Spec}(R) = \{ J_w \mid w \in W \} \).

### 2.2 Number of \( \mathcal{H} \)-invariant prime ideals in \( R \).

In [4], Cauchon has used his description of the set \( \mathcal{H} \text{-Spec}(R) \) in order to give a formula for the total number \( S(n) \) of \( \mathcal{H} \)-invariant prime ideals of \( R \). More precisely, he has established (See [4], Proposition 3.3.2) that:

\[
S(n) = (-1)^{n-1} \sum_{k=1}^{n} (k+1)^n \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} j^n,
\]

that is

\[
S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n \left( \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n \right).
\]

Recall (See [15], p. 34) that \( \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n \) is equal to the Stirling number of second kind \( S(n, k) \) (See, for example, [15] for more details on the Stirling numbers of second kind). Hence, we have:

\[
S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n S(n, k),
\]

that is

\[
S(n) = (-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k).
\]

On the other hand, it follows from [10, Theorem 1] that:

\[
(-1)^n \sum_{k=0}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k) = B_n^{(-n)},
\]

where \( B_n^{(-n)} \) denotes the poly-Bernoulli number associated to \( n \) and \( -n \) (See [10] for the definition of the poly-Bernoulli numbers). Observing that \( S(n, 0) = 0 \) (See [15]), we get:

\[
(-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k) = B_n^{(-n)},
\]

and thus, we deduce from (2) that:
Proposition 2.7

\[ |\mathcal{H}\text{-Spec}(R)| = B_n^{(-n)}. \]

This rewriting of Cauchon’s formula was first obtained by Goodearl and McCammond.

2.3 Vanishing and non-vanishing criteria for the entries of \(q\)-quantum matrices.

Let \(J_w (w \in W)\) be an \(\mathcal{H}\)-invariant prime ideal of \(R\) (See Proposition 2.6). In the next section, we will need to know which indeterminates \(Y_{i,\alpha}\) belong to \(J_w\), that is which \(y_{i,\alpha} := Y_{i,\alpha} + J_w\) are zero. This problem is dealt with in Proposition 2.12 and Proposition 2.16 where we respectively obtain a non-vanishing criterion and a vanishing criterion for the entries of \(q\)-quantum matrices.

For the remainder of this section, \(K\) denotes a \(K\)-algebra which is also a skew-field. Except otherwise stated, all the considered matrices have their entries in \(K\).

Definitions 2.8

Let \(M = (x_{i,\alpha})_{(i,\alpha) \in [1,n]^2}\) be a \(n \times n\) matrix and let \((j, \beta) \in E_s\).

- We say that \(M\) is a \(q\)-quantum matrix if the following relations hold between the entries of \(M\):
  
  If \(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\) is any \(2 \times 2\) sub-matrix of \(M\), then
  
  1. \(yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt\).
  2. \(tx = xt - (q - q^{-1})yz\).

- We say that \(M\) is a \((j, \beta)\)-\(q\)-quantum matrix if the following relations hold between the entries of \(M\):
  
  If \(\begin{pmatrix} x & y \\ z & t \end{pmatrix}\) is any \(2 \times 2\) sub-matrix of \(M\), then
  
  1. \(yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt\).
  2. If \(t = x_v\), then \(\begin{cases} v \geq_s (j, \beta) \implies tx = xt \\ v <_s (j, \beta) \implies tx = xt - (q - q^{-1})yz. \end{cases}\)

Conventions 2.9

Let \(M = (x_{i,\alpha})_{(i,\alpha) \in [1,n]^2}\) be a \(q\)-quantum matrix.

As \(r\) runs over the set \(E_s\), we define matrices \(M^{(r)} = (x^{(r)}_{i,\alpha})_{(i,\alpha) \in [1,n]^2}\) as follows:

1. If \(r = (n, n+1)\), then the entries of the matrix \(M^{(n, n+1)}\) are defined by \(x^{(n, n+1)}_{i,\alpha} := x_{i,\alpha}\) for all \((i, \alpha) \in [1,n]^2\).
2. Assume that \(r = (j, \beta) \in E_s \setminus \{(n, n+1)\}\) and that the matrix \(M^{(r+)}\) is already known. The entries \(x^{(r)}_{i,\alpha}\) of the matrix \(M^{(r)}\) are defined as follows:
(a) If \( x_{j,\beta}^{(r')} = 0 \), then \( x_{i,\alpha}^{(r')} = x_{i,\alpha}^{(r)} \) for all \((i, \alpha) \in [1, n]^2\).

(b) If \( x_{j,\beta}^{(r')} \neq 0 \) and \((i, \alpha) \in [1, n]^2\), then
\[
x_{i,\alpha}^{(r')} = \begin{cases}
    x_{i,\alpha}^{(r)} - x_{i,\beta}^{(r)} \left( x_{j,\beta}^{(r')} \right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\
    x_{i,\alpha}^{(r)} & \text{otherwise}.
\end{cases}
\]

We say that \( M^{(r)} \) is the matrix obtained from \( M \) by applying the standard deleting derivations algorithm at step \( r \).

3. If \( r = (1, 2) \), we set \( t_{i,\alpha} := x_{i,\alpha}^{(1,2)} \) for all \((i, \alpha) \in [1, n]^2\).

Observe that the formulas of Conventions \[2.9\] allow us to express the entries of \( M^{(r')} \) in terms of those of \( M^{(r)} \).

**Proposition 2.10 (Restoration algorithm)**

Let \( M = (x_{i,\alpha})_{(i, \alpha) \in [1, n]^2} \) be a \( q \)-quantum matrix and let \( r = (j, \beta) \in E_s \) with \( r \neq (n, n + 1) \).

1. If \( x_{j,\beta}^{(r')} = 0 \), then \( x_{i,\alpha}^{(r')} = x_{i,\alpha}^{(r)} \) for all \((i, \alpha) \in [1, n]^2\).

2. If \( x_{j,\beta}^{(r')} \neq 0 \) and \((i, \alpha) \in [1, n]^2\), then
\[
x_{i,\alpha}^{(r')} = \begin{cases}
    x_{i,\alpha}^{(r)} + x_{i,\beta}^{(r)} \left( x_{j,\beta}^{(r')} \right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\
    x_{i,\alpha}^{(r)} & \text{otherwise}.
\end{cases}
\]

Note that our definitions of \( q \)-quantum matrix and \((j, \beta)\)-\( q \)-quantum matrix slightly differ from those of \[2\] (See \[2\], Définitions III.1.1 and III.1.3). Because of this, we must interchange \( q \) and \( q^{-1} \) whenever carrying over result of \[2\].

**Lemma 2.11**

Let \((j, \beta) \in E_s \).

If \( M = (x_{i,\alpha})_{(i, \alpha) \in [1, n]^2} \) is a \( q \)-quantum matrix, then the matrix \( M^{(j,\beta)} \) is \((j, \beta)\)-\( q \)-quantum.

**Proof**: This lemma is proved in the same manner as \[2\, Proposition III.2.3.1\]. □

We deduce from the above Lemma \[2.11\] the following non-vanishing criterion for the entries of a \( q \)-quantum matrix.

**Proposition 2.12**

Let \( M = (x_{i,\alpha})_{(i, \alpha) \in [1, n]^2} \) be a \( q \)-quantum matrix and let \((i, \alpha) \in [1, n]^2\).

If \( t_{i,\alpha} \neq 0 \), then \( x_{i,\alpha} \neq 0 \). In other words, if \( x_{i,\alpha} = 0 \), then \( t_{i,\alpha} = 0 \).

**Proof**: Assume that \( x_{i,\alpha} = 0 \). We first prove that \( x_{i,\alpha}^{(j,\beta)} = 0 \) for all \((j, \beta) \in E_s \). To achieve this aim, we proceed by decreasing induction (for \( \leq_s \)) on \((j, \beta)\).

Since \( x_{i,\alpha}^{(n,n+1)} = x_{i,\alpha} \), the case \((j, \beta) = (n, n+1) \) is done. Assume now that \((j, \beta) <_s (n, n+1) \) and \( x_{i,\alpha}^{(j,\beta)^+} = 0 \). If \( x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)^+} \), we obviously have \( x_{i,\alpha}^{(j,\beta)} = 0 \). Next, if \( x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)^+} \), then
\[ x_{i,\beta}^{(j,\beta)^+} x_{i,\alpha}^{(j,\beta)^+} - x_{i,\alpha}^{(j,\beta)^+} x_{j,\beta}^{(j,\beta)^+} = -(q - q^{-1}) x_{i,\beta}^{(j,\beta)^+} x_{j,\beta}^{(j,\beta)^+}. \]

Since \( x_{i,\beta}^{(j,\beta)^+} = 0 \), we deduce from this equality that, in \( K \), \( x_{i,\beta}^{(j,\beta)^+} x_{j,\alpha}^{(j,\beta)^+} = 0 \). Thus, \( x_{i,\beta}^{(j,\beta)^+} = 0 \) or \( x_{j,\alpha}^{(j,\beta)^+} = 0 \). On the other hand, since \( i < j \) and \( \alpha < \beta \), we have \( x_{i,\beta}^{(j,\beta)^+} = x_{i,\beta}^{(j,\beta)^+} - x_{i,\beta}^{(j,\beta)^+} \left( x_{j,\beta}^{(j,\beta)^+} \right)^{-1} x_{j,\beta}^{(j,\beta)^+} \). Now it follows from the induction hypothesis that \( x_{i,\beta}^{(j,\beta)^+} = 0 \). Hence, we have

\[ x_{i,\beta}^{(j,\beta)^+} = -x_{i,\beta}^{(j,\beta)^+} \left( x_{j,\beta}^{(j,\beta)^+} \right)^{-1} x_{j,\beta}^{(j,\beta)^+}. \]

Finally, since \( x_{i,\beta}^{(j,\beta)^+} = 0 \) or \( x_{j,\beta}^{(j,\beta)^+} = 0 \), we get \( x_{i,\beta}^{(j,\beta)^+} = 0 \), as desired. This achieves the induction.

In particular, we have shown that \( x_{i,\alpha}^{(1,2)} = 0 \), that is \( t_{i,\alpha} = 0 \). \( \square \)

Proposition 2.12 furnishes a non-vanishing criterion for the entries of a \( q \)-quantum matrix. In order to construct, in the next section, \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localization, \( \mathcal{H} \)-invariant prime ideals of \( R_r^+ := \left\{ \frac{R}{(Y_{i,\alpha} \ | \ \alpha > t \text{ or } i < r_{\alpha})} \left[ Y_{r_1,1}, \ldots, Y_{r_t,1} \right] \right\} \) \((r = (r_1, \ldots, r_t) \text{ with } 1 \leq r_1 < \cdots < r_t \leq n)\), we also need to get a vanishing criterion for the entries \( x_{i,\alpha}, \alpha > t \text{ or } i < r_{\alpha} \), of a \( q \)-quantum matrix. This is what we do now.

**Notation 2.13**

If \( t \) denotes an element of \([0, n]\), we set:

\[ R_t := \{ (r_1, \ldots, r_t) \in \mathbb{N} \ | \ 1 \leq r_1 < \cdots < r_t \leq n \}. \]

(If \( t = 0 \), then \( R_0 = \emptyset \).)

For the remainder of this section, we fix \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t \), and we denote by \( w_r \) the subset of \([1, n]^2 \) corresponding to indeterminates \( Y_{i,\alpha} \) that have been set equal to zero in \( R_r^+ \), that is, we set:

\[ w_r := \bigcup_{\alpha \in [1, t]} \left[ [1, r_\alpha - 1] \times \{ \alpha \} \right] \bigcup [1, n] \times [t + 1, n]. \]

For instance, if \( n = 3 \), \( t = 2 \) and \( r = (1, 3) \), we have:

\[ w_{(1,3)} = \begin{array}{c|c|c|c|c|c|c|c|c} \hline \hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline & & & & & & & & \\
\hline \end{array} \]

where the black boxes symbolize the elements of \( w_{(1,3)} \).

Note that \( w_r \) is a union of truncated columns, so that:

**Remark 2.14**

\( w_r \) belongs to \( W \).
Observation 2.15
Let $(i, \alpha) \in w_R$. If $\beta \in [\alpha, n]$, then $(i, \beta) \in w_R$.

Proof: We distinguish two cases.
- If $(i, \alpha) \in [1, n] \times [t+1, n]$, then $\alpha \geq t + 1$. Hence $\beta \geq \alpha \geq t + 1$ and thus, we have $(i, \beta) \in [1, n] \times [t+1, n] \subseteq w_R$, as required.
- Assume now that $(i, \alpha) \in \bigcup_{\gamma \in [1, t]} [1, r_\gamma - 1] \times \{\gamma\}$, so that we have $\alpha \leq t$ and $i \leq r_\alpha - 1$. If $\beta > t$, we conclude as in the previous case that $(i, \beta) \in w_R$. So we assume that $\beta \leq t$. Since $i \leq r_\alpha - 1$ and since $\alpha \leq \beta \leq t$, we have $i \leq r_\alpha - 1 \leq r_\beta - 1$. Hence, $(i, \beta) \in [1, r_\beta - 1] \times \{\beta\} \subseteq w_R$, as desired. \[ \]

This observation allows us to prove the following vanishing criterion:

Proposition 2.16
Let $M = (x_{i, \alpha})_{(i, \alpha) \in [1, n]^2}$ be a $q$-quantum matrix.
If $t_{i, \alpha} = 0$ for all $(i, \alpha) \in w_R$, then $x_{i, \alpha} = 0$ for all $(i, \alpha) \in w_R$.

Proof: Assume that $t_{i, \alpha} = 0$ for all $(i, \alpha) \in w_R$. We first prove by induction on $(j, \beta)$ (with respect of $\leq_s$) that $x_{i, \alpha}^{(j, \beta)} = 0$ for all $(i, \alpha) \in w_R$ and $(j, \beta) \in E_s$.

If $(j, \beta) = (1, 2)$, then $x_{i, \alpha}^{(1, 2)} = t_{i, \alpha} = 0$ for all $(i, \alpha) \in w_R$, as required. Assume now that $(j, \beta) <_s (n, n + 1)$ and that $x_{i, \alpha}^{(j, \beta)} = 0$ for all $(i, \alpha) \in w_R$. Let $(i, \alpha) \in w_R$. If $x_{i, \alpha}^{(j, \beta)} = x_{i, \alpha}^{(j, \beta)}$, the desired result follows from the induction hypothesis. Next, if $x_{i, \alpha}^{(j, \beta)} \neq x_{i, \alpha}^{(j, \beta)}$, it follows from Proposition 2.10 that $x_{i, \alpha}^{(j, \beta)} \neq 0$, $i < j$, $\alpha < \beta$ and $x_{i, \alpha}^{(j, \beta)} = x_{i, \alpha}^{(j, \beta)} + x_{i, \beta}^{(j, \beta)} (x_{j, \beta}^{(j, \beta)})^{-1} x_{j, \alpha}^{(j, \beta)}$.

Since $(i, \alpha) \in w_R$, we deduce from the induction hypothesis that $x_{i, \alpha}^{(j, \beta)} = 0$, so that $x_{i, \alpha}^{(j, \beta)} = x_{i, \beta}^{(j, \beta)} (x_{j, \beta}^{(j, \beta)})^{-1} x_{j, \alpha}^{(j, \beta)}$. Moreover, since $(i, \alpha) \in w_R$ and $\alpha < \beta$, it follows from Observation 2.15 that $(i, \beta) \in w_R$. Then, we deduce from the induction hypothesis that $x_{i, \beta}^{(j, \beta)} = 0$, so that $x_{i, \alpha}^{(j, \beta)} = x_{i, \beta}^{(j, \beta)} (x_{j, \beta}^{(j, \beta)})^{-1} x_{j, \alpha}^{(j, \beta)} = 0$. This achieves the induction.

In particular, we have proved that $x_{i, \alpha} = x_{i, \alpha}^{(n, n+1)} = 0$ for all $(i, \alpha) \in w_R$. \[ \]

2.4 $\mathcal{H}$-invariant prime ideals $J_w$ with $w_R \subseteq w$.

As in the previous section, we fix $t \in [0, n]$ and $r = (r_1, \ldots, r_t) \in \mathbb{R}_t$, and we set:

$$w_R := \bigcup_{\alpha \in [1, t]} [1, r_\alpha - 1] \times \{\alpha\} \bigcup [1, n] \times [t+1, n].$$

Recall (See Proposition 2.6) that, if $w \in W$, there exists a (unique) $\mathcal{H}$-invariant prime ideal of $R$ associated to $w$ (See Proposition 2.16) and that the $J_w$ ($w \in W$) are exactly the $\mathcal{H}$-invariant prime ideals in $R$. This section is devoted to the $\mathcal{H}$-invariant prime ideals $J_w$ ($w \in W$) of $R$ with $w_R \subseteq w$. More precisely, we want to know which indeterminates $Y_{i, \alpha}$ belong to these ideals.
Notations 2.17
Let \( w \in W \).

1. Set \( R_w := \frac{R}{J_w} \). It follows from Lemma 5.3.3 that, using the notations of Section 2.2, \( R_w \) and \( \frac{R}{K_w} \) are two Noetherian algebras with no zero-divisors, which have the same skew-field of fractions. We set \( F_w := \text{Fract}(R_w) = \text{Fract}\left(\frac{R}{K_w}\right) \).

2. If \( (i, \alpha) \in [1, n]^2 \), \( y_{i,\alpha} \) denotes the element of \( R_w \) defined by \( y_{i,\alpha} := Y_{i,\alpha} + J_w \).

3. We denote by \( M_w \) the matrix, with entries in the \( K \)-algebra \( F_w \), defined by:

\[
M_w := (y_{i,\alpha})_{(i,\alpha)\in[1,n]^2}.
\]

Let \( w \in W \). Since \( Y = (Y_{i,\alpha})_{(i,\alpha)\in[1,n]^2} \) is a \( q \)-quantum matrix, \( M_w \) is also a \( q \)-quantum matrix. Thus, we can apply the standard deleting derivations algorithm to \( M_w \) (See Conventions 2.9 with \( K = F_w \)) and if we still denote \( t_{i,\alpha} := y_{i,\alpha}^{(1,2)} \) for \( (i, \alpha) \in [1, n]^2 \), we get:

Proposition 2.18
\( t_{i,\alpha} = 0 \) if and only if \( (i, \alpha) \in w \).

Proof: By Propositions 5.4.1 and 5.4.2, there exists a \( K \)-algebra homomorphism \( f_{(1,2)} : R \rightarrow F_w \) such that \( f_{(1,2)}(T_{i,\alpha}) = t_{i,\alpha} \) for \( (i, \alpha) \in [1, n]^2 \). Its kernel is \( K_w \) and its image is the subalgebra of \( F_w \) generated by the \( t_{i,\alpha} \) with \( (i, \alpha) \in [1, n]^2 \). Hence, \( t_{i,\alpha} = 0 \) if and only if \( T_{i,\alpha} \in K_w \), that is, if and only if \( (i, \alpha) \in w \). □

Consider now an element \( w \) in \( W \) with \( w_r \subseteq w \) and denote by \( J_w \) the (unique) \( \mathcal{H} \)-invariant prime ideal of \( R \) associated to \( w \) (See Proposition 2.6). Since \( w_r \subseteq w \), we deduce from Proposition 2.18 that \( t_{i,\alpha} = 0 \) for all \( (i, \alpha) \in w_r \). Hence, we can apply Proposition 2.16 to the \( q \)-quantum matrix \( M_w \) and we obtain that \( y_{i,\alpha} = 0 \) for all \( (i, \alpha) \in w_r \), that is, \( Y_{i,\alpha} \in J_w \) for all \( (i, \alpha) \in w_r \).

So we have just established:

Proposition 2.19
Let \( w \in W \) with \( w_r \subseteq w \). If \( (i, \alpha) \in w_r \), then \( Y_{i,\alpha} \) belongs to \( J_w \).

We will now add truncated rows to the "\( w_r \) diagram" in order to obtain \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \). We will see later (See Section 3.4) that the \( \mathcal{H} \)-invariant prime ideals of \( R \) obtained by adding truncated rows to the "\( w_r \) diagram" are the only \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \).

Notation 2.20
We set \( \Gamma_r := \{ (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \mid \gamma_k \in [0, l] \text{ if } k \in [r_l + 1, r_{l+1}] \} \). (Here \( r_0 = 0 \) and \( r_{l+1} = n \).)
For instance, if $n = 3$, $t = 2$ and $r = (1, 3)$, we have:

$$\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \ gamma_2 \leq 1 and \ gamma_3 \leq 1\}.$$

**Theorem 2.21**

Let $(\gamma_1, \ldots, \gamma_n) \in \Gamma_r$ and set $w_{r,(\gamma_1,\ldots,\gamma_n)} := w_r \bigcup \sum_{k \in [1,n]} \{k \times \llbracket 1, \gamma_k \rrbracket \}$. Then $w_{r,(\gamma_1,\ldots,\gamma_n)}$ belongs to $W$ and the $H$-invariant prime ideal $J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ of $R$ has the following properties:

1. $Y_{i,\alpha} \in J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ for all $(i, \alpha) \in w_r$.
2. $Y_{r,k} \notin J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ for all $k \in [1,t]$.

**Proof:** Since $w_r$ is a union of truncated columns and since $\bigcup_{k \in [1,n]} \{k \times \llbracket 1, \gamma_k \rrbracket \}$ is a union of truncated rows, $w_{r,(\gamma_1,\ldots,\gamma_n)}$ is a union of truncated rows and columns, so that $w_{r,(\gamma_1,\ldots,\gamma_n)} \in W$.

Since $w_r \subseteq w_{r,(\gamma_1,\ldots,\gamma_n)}$, we deduce from Proposition 2.19 that $Y_{i,\alpha} \in J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ for all $(i, \alpha) \in w_r$.

Now we want to prove that $Y_{r_k,k} \notin J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ for all $k \in [1,t]$. Assume this is not the case, that is, assume that there exists $k \in [1,t]$ with $Y_{r_k,k} \in J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$. Then, $y_{r_k,k} = 0$ and it follows from Proposition 2.12 that $y_{r_k,k}^{(1,2)} = t_{r_k,k} = 0$. Thus, we deduce from Proposition 2.18 that $(r_k,k) \in w_{r,(\gamma_1,\ldots,\gamma_n)}$.

Observe now that, since $k \leq t$, $(r_k,k) \notin [1,n] \times [t+1,n]$. Further, it is obvious that $(r_k,k) \notin \bigcup_{\alpha \in [1,t]} \llbracket 1, r_{\alpha} - 1 \rrbracket \times \{\alpha\}$. Hence, $(r_k,k) \notin w_r$.

All this together shows that $(r_k,k) \in w_{r,(\gamma_1,\ldots,\gamma_n)} \setminus w_r = \bigcup_{t \in [1,n]} \{t \times \llbracket 1, \gamma_t \rrbracket \}$, so that $k \leq \gamma_{r_k}$.

However, since $(\gamma_1,\ldots,\gamma_n) \in \Gamma_r$, we have $\gamma_{r_k} \leq k - 1$. This is a contradiction and thus we have proved that $Y_{r_k,k} \notin J_{w_{r,(\gamma_1,\ldots,\gamma_n)}}$ for all $k \in [1,t]$.

Let us now give an example for the elements $w_{r,(\gamma_1,\ldots,\gamma_n)}$ $((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r)$ of Theorem 2.21.

If $n = 3$, $t = 2$ and $r = (1,3)$, we have already noted that

$$\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \ gamma_2 \leq 1 and \ gamma_3 \leq 1\},$$

so that the elements $w_{r,(\gamma_1,\ldots,\gamma_n)}$ $((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r)$ of Theorem 2.21 are:

$$w_{(1,3),(0,0,0)} = w_{(1,3)} = \begin{array}{cc}
\checkmark & \checkmark \\
\checkmark & \checkmark \\
\end{array}$$

$$w_{(1,3),(0,1,0)} = \begin{array}{cc}
\checkmark & \checkmark \\
\checkmark & \checkmark \\
\end{array}$$

$$w_{(1,3),(0,0,1)} = \begin{array}{cc}
\checkmark & \checkmark \\
\checkmark & \checkmark \\
\end{array}$$

$$w_{(1,3),(0,1,1)} = \begin{array}{cc}
\checkmark & \checkmark \\
\checkmark & \checkmark \\
\end{array}$$
(As previously, if \( w \in W \), the black boxes symbolize the elements of \( w \).)

3 Number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(K)) \).

In this paragraph, using the previous section, we begin by constructing \( \mathcal{H} \)-invariant prime ideals of the algebra \( R_t^+ := \frac{O_q(\mathcal{M}_n(K))}{(Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha)} [Y_{r_1,1}, \ldots, Y_{r_t,t}] \), where \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \) is a strictly increasing sequence of integers in the range \( 1, \ldots, n \). Next, following the route sketched in the introduction, we establish our main result: the number \( |\mathcal{H}-\text{Spec}^{[t]}(R)| \) of \( \mathcal{H} \)-invariant prime ideals of \( R = O_q(\mathcal{M}_n(K)) \) which contain all \( (t+1) \times (t+1) \) quantum minors but not all \( t \times t \) quantum minors is equal to \( (t!)^2 S(n+1, t+1)^2 \), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \). From this result, we derive a description of the set of \( \mathcal{H} \)-invariant prime ideals of \( R_t^+ \).

3.1 \( \mathcal{H} \)-invariant prime ideals in \( R_{t,0}^+ \).

Throughout this section, we fix \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t \), and we define \( w_r \) as in the previous section.

As in \([5, 2.1]\), we set \( R_{t,0}^+ = \frac{R}{(Y_{i,\alpha} \mid (i, \alpha) \in w_r)} \).

Recall (See \([5, 2.1]\)) that \( R_{t,0}^+ \) can be written as an iterated Ore extension over \( K \). Thus, \( R_{t,0}^+ \) is a Noetherian domain. Moreover, since \( q \) is not a root of unity, it follows from \([7, \text{Theorem 3.2}] \) that all primes of \( R \) are completely prime and thus, since this property survives in factors, all primes in the algebra \( R_{t,0}^+ \) are completely prime.

Observe now that, since the indeterminates \( Y_{i,\alpha} \) are \( \mathcal{H} \)-eigenvectors, \( (Y_{i,\alpha} \mid (i, \alpha) \in w_r) \) is an \( \mathcal{H} \)-invariant ideal of \( R \). Hence, the action of \( \mathcal{H} \) on \( R \) induces an action of \( \mathcal{H} \) on \( R_{t,0}^+ \) by automorphisms. As usually, an \( \mathcal{H} \)-eigenvector \( x \) of \( R_{t,0}^+ \) is a nonzero element \( x \in R_{t,0}^+ \) such that \( h(x) \in K^* x \) for each \( h \in \mathcal{H} \), and an ideal \( I \) of \( R_{t,0}^+ \) is said to be \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \). Further, we denote by \( \mathcal{H}-\text{Spec}(R_{t,0}^+) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R_{t,0}^+ \).

Notations 3.1

- We denote by \( \pi_{t,0}^+ : R \rightarrow R_{t,0}^+ \) the canonical surjective \( K \)-algebra homomorphism.
- If \( (i, \alpha) \in [1, n]^2 \), \( Y_{i,\alpha} \) denotes the element of \( R_{t,0}^+ \) defined by \( Y_{i,\alpha} := \pi_{t,0}^+(Y_{i,\alpha}) \).

Let \( (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \) (See Notation \([2.20]\)) and define \( w_{r,(\gamma_1, \ldots, \gamma_n)} \) as in Theorem \([2.21]\). Recall (See Theorem \([2.21]\)) that \( w_{r,(\gamma_1, \ldots, \gamma_n)} \) is an element of \( W \) and that the \( \mathcal{H} \)-invariant prime ideal \( J_{w_{r,(\gamma_1, \ldots, \gamma_n)}} \) of \( R \) contains the indeterminates \( Y_{i,\alpha} \) with \( (i, \alpha) \in w_r \), so that \( (Y_{i,\alpha} \mid (i, \alpha) \in w_r) \subseteq J_{w_{r,(\gamma_1, \ldots, \gamma_n)}} \). Thus, \( \pi_{t,0}^+(J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}) \) is a (completely) prime ideal of \( R_{t,0}^+ \). More precisely, we have:
We have already explained that \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is a \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which does not contain the \( \overline{Y}_{r,0} := \pi^+_{r,0} Y_{r,0} \) (\( k \in [1,t] \)).

**Proof:** We have already explained that \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is a (completely) prime ideal of \( R^+_{r,0} \). Moreover, since \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is \( \mathcal{H} \)-invariant, it is easy to check that \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) is also \( \mathcal{H} \)-invariant.

Finally, since \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) does not contain the indeterminates \( Y_{r,k} \) with \( k \in [1,t] \) (See Theorem 2.21), \( J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} \) does not contain the \( \overline{Y}_{r,k} = \pi^+_{r,0}(Y_{r,k}) \) with \( k \in [1,t] \). ■

### 3.2 \( \mathcal{H} \) -invariant prime ideals in \( R^+_r \)

As in the previous section, we fix \( t \in [0,n] \) and \( r = (r_1,\ldots,r_t) \in \mathbb{R}_t \). In [2, 2.1], Goodearl and Lenagan have observed that the \( \overline{Y}_{r,k} \) with \( k \in [1,t] \) are regular normal elements in \( R^+_{r,0} \), so that we can form the Ore localization:

\[
R^+_r := R^+_{r,0}S_r^{-1},
\]

where \( S_r \) denotes the multiplicative system of \( R^+_{r,0} \) generated by the \( \overline{Y}_{r,k} \) with \( k \in [1,t] \).

In the previous section, we have noted that all the primes of \( R^+_{r,0} \) are completely prime. Since this property survives in localization, all the primes of \( R^+_r \) are also completely prime.

Observe now that, since the \( \overline{Y}_{r,k} \) with \( k \in [1,t] \) are \( \mathcal{H} \)-eigenvectors of \( R^+_{r,0} \), the action of \( \mathcal{H} \) on \( R^+_{r,0} \) extends to an action of \( \mathcal{H} \) on \( R^+_r \) by automorphisms. We say that an ideal \( I \) of \( R^+_r \) is \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \) and we denote by \( \mathcal{H} \)-Spec\((R^+_r) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_r \). Observe now that contraction and extension provide inverse bijections between the set \( \mathcal{H} \)-Spec\((R^+_r) \) and the set of those \( \mathcal{H} \)-invariant prime ideals of \( R^+_{r,0} \) which are disjoint from \( S_r \).

Let \( (\gamma_1,\ldots,\gamma_n) \in \Gamma_r \) (See Notation 2.20) and define \( u_{r,(\gamma_1,\ldots,\gamma_n)} \) as in Theorem 2.21 By Proposition 3.2, \( J^+_{u_r,(\gamma_1,\ldots,\gamma_n)} := \pi^+_{r,0} \left(J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}\right) \) is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which does not contain the \( \overline{Y}_{r,k} \) (\( k \in [1,t] \)). Since \( S_r \) is generated by the \( \overline{Y}_{r,k} \) (\( k \in [1,t] \)), \( J^+_{u_r,(\gamma_1,\ldots,\gamma_n)} \) is an \( \mathcal{H} \)-invariant (completely) prime ideal of \( R^+_{r,0} \) which is disjoint from \( S_r \). Thus, we have the following statement:

**Proposition 3.3**

\[
J^+_{u_r,(\gamma_1,\ldots,\gamma_n)}S_r^{-1} \text{ is an } \mathcal{H} \text{-invariant (completely) prime ideal of } R^+_r.
\]

We will prove later (See Section 3.4) that the \( J^+_{u_r,(\gamma_1,\ldots,\gamma_n)}S_r^{-1} ((\gamma_1,\ldots,\gamma_n) \in \Gamma_r) \) are exactly the \( \mathcal{H} \)-invariant prime ideals of \( R^+_r \).

We deduce from the above Proposition 3.3 that:

**Corollary 3.4**

\( R^+_r \) has at least \( 1^r, 2^{r_2-r_1}, \ldots, t^{r_1-r_{t-1}} (t+1)^{n-r_t} \) \( \mathcal{H} \)-invariant prime ideals.
Proof: It follows from Proposition 3.3 that \( R^+_\Gamma \) has at least \( |\Gamma| \) \( H \)-invariant prime ideals, and it is obvious that \( |\Gamma| = 1^r_1 2^r_2 \ldots t^r_t (t + 1)^n-n-1 \).

### 3.3 Number of rank \( t \) \( H \)-invariant prime ideals in \( O_q(M_n(\mathbb{K})) \).

For convenience, we recall the following definitions (See [14]):

**Definitions 3.5**

- Let \( m \) be a positive integer and let \( M = (x_{i,\alpha})_{(i,\alpha)\in[1,m]^2} \) be a square \( q \)-quantum matrix. The quantum determinant of \( M \) is defined by:

\[
det_q(M) := \sum_{\sigma \in S_m} (-q)^{l(\sigma)} x_{1,\sigma(1)} \ldots x_{m,\sigma(m)},
\]

where \( S_m \) denotes the group of permutations of \([1,m] \) and \( l(\sigma) \) denotes the length of the \( m \)-permutation \( \sigma \).

- Let \( Y := (Y_{i,\alpha})_{(i,\alpha)\in[1,n]^2} \) be the \( q \)-quantum matrix of the canonical generators of \( R \). The quantum determinant of a square sub-matrix of \( Y \) is called a quantum minor.

We can now define the rank \( t \) \( H \)-invariant prime ideals of \( R \), as follows:

**Definition 3.6**

Let \( t \in [0,n] \). An \( H \)-invariant prime ideal \( J \) of \( R = O_q(M_n(\mathbb{K})) \) has rank \( t \) if \( J \) contains all \( (t + 1) \times (t + 1) \) quantum minors but not all \( t \times t \) quantum minors.

As in [5, 3.6], we denote by \( \mathcal{HSpec}^t(R) \) the set of rank \( t \) \( H \)-invariant prime ideals of \( R \).

Note that there is only one element in \( \mathcal{HSpec}^0(R) \): \( \langle Y_{i,\alpha} \mid (i, \alpha) \in [1,n]^2 \rangle \), the augmentation ideal of \( R \). Further, Goodearl and Lenagan have observed (See [5, 3.6]) that \( |\mathcal{HSpec}^1(R)| = (2^n - 1)^2 \) and \( |\mathcal{HSpec}^n(R)| = (n!)^2 \).

**Observation 3.7**

The sets \( \mathcal{HSpec}^t(R) \) \((t \in [0,n])\) partition the set \( \mathcal{HSpec}(R) \).

**Proof:** Let \( P \) be an \( H \)-invariant prime ideal of \( R \). Let \( t \in [0,n] \) be maximal such that \( P \) does not contain all \( t \times t \) quantum minors. Then \( P \) clearly belongs to \( \mathcal{HSpec}^t(R) \). Hence, we have proved that \( \mathcal{HSpec}(R) = \bigcup_{t \in [0,n]} \mathcal{HSpec}^t(R) \). Since this union is obviously disjoint, we get

\[ \mathcal{HSpec}(R) = \bigsqcup_{t \in [0,n]} \mathcal{HSpec}^t(R), \]

as desired. \( \blacksquare \)

In [8], the authors have established the following result that will be our starting point to compute the cardinality of \( \mathcal{HSpec}^t(R) \):

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14
**Proposition 3.8 (See [5], 3.6)**

For all \( t \in [0, n] \), we have \(|\mathcal{H}-\text{Spec}^{[t]}(R)| = \left( \sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| \right)^2\).

Before computing \(|\mathcal{H}-\text{Spec}^{[t]}(R)|\), we first give a lower bound for \( \sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| \).

**Proposition 3.9**

For any \( t \in [0, n] \), we have

\[
\sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| \geq t!S(n + 1, t + 1),
\]

where \( S(n + 1, t + 1) \) denotes the Stirling number of second kind associated to \( n + 1 \) and \( t + 1 \) (See, for instance, [15] for the definition of \( S(n + 1, t + 1) \)).

**Proof:** First, we deduce from Corollary 3.3 the following inequality:

\[
\sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| \geq \sum_{r \in R_t} 1^{r_1}2^{r_2-r_1-1} \cdots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t}.
\]  (3)

On the other hand, we know (See [15], Exercise 16 p46) that:

\[
S(n + 1, t + 1) = \sum_{a_1 + \cdots + a_{t+1} = n+1} 1^{a_1-1}2^{a_2-1} \cdots (t+1)^{a_{t+1}-1}.
\]  (4)

Observe now that the map \( f : \{(a_1, \ldots, a_{t+1}) \in (\mathbb{N}^*)^{t+1} \mid a_1 + \cdots + a_{t+1} = n+1\} \to \{(r_1, \ldots, r_t) \in (\mathbb{N})^t \mid 1 \leq r_1 < \cdots < r_t \leq n\} = R_t \) defined by \( f(a_1, \ldots, a_{t+1}) = (a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_t) \) is a bijection and that its inverse \( f^{-1} \) is defined by \( f^{-1}(r_1, \ldots, r_t) = (r_1, r_2-r_1, \ldots, r_t-r_{t-1}, n+1-r_t) \) for all \((r_1, \ldots, r_t) \in R_t \). Thus, by means of the change of variables \((a_1, \ldots, a_{t+1}) = f^{-1}(r_1, \ldots, r_t)\), the above equality (4) is transformed to

\[
S(n + 1, t + 1) = \sum_{1 \leq r_1 < \cdots < r_t \leq n} 1^{r_1-1}2^{r_2-r_1-1} \cdots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t},
\]

so that

\[
t!S(n + 1, t + 1) = \sum_{(r_1, \ldots, r_t) \in R_t} 1^{r_1}2^{r_2-r_1} \cdots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t}.
\]

Thus, we deduce from inequality (3) that:

\[
\sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| \geq t!S(n + 1, t + 1),
\]

as desired. \(\square\)

**Remark 3.10**

The proof of the above Proposition shows that, if there exists \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t \) such that \(|\mathcal{H}-\text{Spec}(R_t^+)| > t!\cdot 2^{r_2-r_1} \cdots t^{r_t-r_{t-1}-1}(t+1)^{n-r_t}\), then

\[
\sum_{r \in R_t} |\mathcal{H}-\text{Spec}(R_t^+)| > t!S(n + 1, t + 1).
\]
We can now prove our main result which was conjectured by Goodearl, Lenagan and Mc-Cammond:

**Theorem 3.11**

If \( t \in [0, n] \), then \(|\mathcal{H} \cdot \text{Spec}^{[t]}(R)| = (t!S(n + 1, t + 1))^2\).

**Proof:** First, since the sets \( \mathcal{H} \cdot \text{Spec}^{[t]}(R) \) \( (t \in [0, n]) \) partition \( \mathcal{H} \cdot \text{Spec}(R) \) (See Observation 3.7), we have:

\[
|\mathcal{H} \cdot \text{Spec}(R)| = \sum_{t=0}^{n} |\mathcal{H} \cdot \text{Spec}^{[t]}(R)|.
\]

Recall now (See Proposition 2.7) that \( |\mathcal{H} \cdot \text{Spec}(R)| \) is equal to the poly-Bernoulli number \( B_n^{(-n)} \). Thus, we deduce from the above equality that:

\[
B_n^{(-n)} = \sum_{t=0}^{n} |\mathcal{H} \cdot \text{Spec}^{[t]}(R)|.
\]

Further, by [1, Theorem 2], \( B_n^{(-n)} \) can also be written as follows:

\[
B_n^{(-n)} = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2.
\]

Hence, we have:

\[
\sum_{t=0}^{n} |\mathcal{H} \cdot \text{Spec}^{[t]}(R)| = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2,
\]

that is:

\[
\sum_{t=0}^{n} \left( |\mathcal{H} \cdot \text{Spec}^{[t]}(R)| - (t!S(n + 1, t + 1))^2 \right) = 0. \tag{5}
\]

On the other hand, recall (See [5], 3.6) that \( |\mathcal{H} \cdot \text{Spec}^{[t]}(R)| = \left( \sum_{r \in R_t^+) |\mathcal{H} \cdot \text{Spec}(R_t^+)| \right)^2 \).

Thus, since \( \sum_{r \in R_t^+} |\mathcal{H} \cdot \text{Spec}(R_t^+)| \geq t!S(n + 1, t + 1) \) (See Proposition 3.9), we have:

\[
|\mathcal{H} \cdot \text{Spec}^{[t]}(R)| \geq (t!S(n + 1, t + 1))^2.
\]

In other words, each of the terms which appears in the sum on the left hand side of (5) is non-negative. Since this sum is equal to zero, each term of this sum must be zero, that is, for all \( t \in [0, n] \), we have:

\[
|\mathcal{H} \cdot \text{Spec}^{[t]}(R)| = (t!S(n + 1, t + 1))^2. \quad \blacksquare
\]

**Remark 3.12**

The cases \( t = 0, t = 1 \) and \( t = n \) were already known (See [5], 3.6).
3.4 Description of the set $\mathcal{H}\text{-}Spec(R^+_t)$.

Throughout this section, we fix $t \in [0, n]$ and $r = (r_1, \ldots, r_t) \in \mathbb{R}_t$. We now use the above Theorem 3.11 to obtain a description of the set $\mathcal{H}\text{-}Spec(R^+_t)$. More precisely, we show that the only $\mathcal{H}$-invariant prime ideals of $R^+_t$ are those obtained in Proposition 3.3, that is, in the notations of Section 3.2.

Theorem 3.13

$$\mathcal{H}\text{-}Spec(R^+_t) = \{ J_{w_r, (\gamma_1, \ldots, \gamma_n)}^+ S_r^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$  

Proof: We already know (See Proposition 3.3) that

$$\mathcal{H}\text{-}Spec(R^+_t) \supseteq \{ J_{w_r, (\gamma_1, \ldots, \gamma_n)}^+ S_r^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$  

Assume now that

$$\mathcal{H}\text{-}Spec(R^+_t) \supsetneq \{ J_{w_r, (\gamma_1, \ldots, \gamma_n)}^+ S_r^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$  

Then we have $| \mathcal{H}\text{-}Spec(R^+_t) | > | \Gamma_r |$. Since $| \Gamma_r | = 1^{r_1}2^{r_2-\cdots-r_1} \cdots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$, we get $| \mathcal{H}\text{-}Spec(R^+_t) | > 1^{r_1}2^{r_2-\cdots-r_1} \cdots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$. Thus, it follows from Remark 3.10 that

$$\sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | > t!S(n+1, t+1).$$  

Hence we have

$$\left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | \right)^2 > (t!S(n+1, t+1))^2.$$  

Recall now (See [3, 3.6]) that

$$| \mathcal{H}\text{-}Spec^{[t]}(R) | = \left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-}Spec(R^+_t) | \right)^2.$$  

All this together shows that $| \mathcal{H}\text{-}Spec^{[t]}(R) | > (t!S(n+1, t+1))^2$.

However, it follows from Theorem 3.11 that $| \mathcal{H}\text{-}Spec^{[t]}(R) | = (t!S(n+1, t+1))^2$. This is a contradiction and thus we have proved that $\mathcal{H}\text{-}Spec(R^+_t) = \{ J_{w_r, (\gamma_1, \ldots, \gamma_n)}^+ S_r^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}$.  

Acknowledgments.

I thank T.H. Lenagan for very helpful conversations, and K.R. Goodearl for useful comments.
References


