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Rank \( t \mathcal{H} \)-primes in quantum matrices.

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Abstract

Let \( K \) be a (commutative) field and consider a nonzero element \( q \) in \( K \) which is not a root of unity. In [5], Goodearl and Lenagan have shown that the number of \( \mathcal{H} \)-primes in \( R = O_q (\mathcal{M}_n (K)) \) which contain all \((t+1) \times (t+1)\) quantum minors but not all \( t \times t \) quantum minors is a perfect square. The aim of this paper is to make precise their result: we prove that this number is equal to \( (t!)^2 S(n+1, t+1)^2 \), where \( S(n+1, t+1) \) denotes the Stirling number of second kind associated to \( n+1 \) and \( t+1 \). This result was conjectured by Goodearl, Lenagan and McCammond. The proof involves some closed formulas for the poly-Bernoulli numbers that were established in [10] and [1].


1 Introduction.

Fix a (commutative) field \( K \) and an integer \( n \) greater than or equal to 2, and choose an element \( q \) in \( K^* := K \setminus \{0\} \) which is not a root of unity. Denote by \( R = O_q (\mathcal{M}_n (K)) \) the quantization of the ring of regular functions on \( n \times n \) matrices with entries in \( K \) and by \((Y_{i, \alpha})_{(i, \alpha) \in [1, n]^2} \) the matrix of its canonical generators. The bialgebra structure of \( R \) gives us an action of the group \( \mathcal{H} := (\mathbb{C}^*)^2n \) on \( R \) by \( K \)-automorphisms (See [5]) via:

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i, \alpha} = a_i b_\alpha Y_{i, \alpha} \quad ((i, \alpha) \in [1, n]^2).
\]

In [9], Goodearl and Letzter have shown that \( R \) has only finitely many \( \mathcal{H} \)-invariant prime ideals (See [9], 5.7. (i)) and that, in order to calculate the prime and primitive spectra of \( R \) (See [9], Theorem 6.6). Next, using the theory of deleting derivations, Cauchon has found a formula for the exact number of \( \mathcal{H} \)-invariant prime ideals in \( R \) (See [4], Proposotion 3.3.2). In this paper, we investigate these ideals.

In [12] (See also [13]), we have proved, assuming that \( K = \mathbb{C} \) (the field of complex numbers) and \( q \) is transcendental over \( \mathbb{Q} \), that the \( \mathcal{H} \)-invariant prime ideals in \( O_q (\mathcal{M}_n (\mathbb{C})) \) are generated by quantum minors, as conjectured by Goodearl and Lenagan (See [5] and [6]). Next, using this result together with Cauchon’s description for the set of \( \mathcal{H} \)-invariant prime ideals of \( O_q (\mathcal{M}_n (\mathbb{C})) \) (See [4], Théorème 3.2.1), we have constructed an algorithm which provides an explicit generating set of quantum minors for each \( \mathcal{H} \)-invariant prime ideal in \( O_q (\mathcal{M}_n (\mathbb{C})) \) (See [11] or [13]).
On the other hand, Goodearl and Lenagan have shown (in the general case where $q \in \mathbb{K}^*$ is not a root of unity) that, in order to obtain descriptions of all the $\mathcal{H}$-invariant prime ideals of $R$, we just need to determine the $\mathcal{H}$-invariant prime ideals of certain "localized step-triangular factors" of $R$, namely the algebras

$$R^+_t := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle} \left[ \prod_{r_{1,1}, \ldots, r_{t,t}}^{-1}, \prod_{r_{1,1}, \ldots, r_{t,t}} \right]$$

and

$$R^c_t := \frac{R}{\langle Y_{i,\alpha} \mid i > t \text{ or } \alpha < c_i \rangle} \left[ \prod_{1, c_1, \ldots, t, c_1}^{-1}, \prod_{1, c_1, \ldots, t, c_1}^{-1} \right],$$

where $t \in [0, n]$ and where $r = (r_1, \ldots, r_t)$ and $c = (c_1, \ldots, c_t)$ are strictly increasing sequences of integers in the range $1, \ldots, n$ (See [4], Theorem 3.5). Using this result, Goodearl and Lenagan have computed the $\mathcal{H}$-invariant prime ideals of $O_q(\mathcal{M}_2(\mathbb{K}))$ (See [5]) and $O_q(\mathcal{M}_3(\mathbb{K}))$ (See [6]).

The aims of this paper are to provide a description for the set $\mathcal{H} \text{-Spec}(R^+_t)$ of $\mathcal{H}$-invariant prime ideals of $R^+_t$ and to count the rank $t$ $\mathcal{H}$-invariant prime ideals of $R$ (for $t \in [0, n]$), that is those $\mathcal{H}$-invariant prime ideals of $R$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. In [5], the authors have shown that the number of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$ is a perfect square. More precisely, they have established (See [5], 3.6) that, for any $t \in [0, n]$:

$$| \mathcal{H} \text{-Spec}^{[t]}(R) | = \left( \sum_{1 \leq r_1 < \cdots < r_t \leq n} | \mathcal{H} \text{-Spec}(R^+_t) | \right)^2$$

(1)

where $\mathcal{H} \text{-Spec}^{[t]}(R)$ denotes the set of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$ and where $\mathcal{H} \text{-Spec}(R^+_t)$ denotes the set of rank $t$ $\mathcal{H}$-invariant prime ideals of $R^+_t$. The above relation (1) opens a potential route to count the rank $t$ $\mathcal{H}$-invariant prime ideals of $R$; if we can compute the number of $\mathcal{H}$-invariant prime ideals of $R^+_t$, then we will be able to count the rank $t$ $\mathcal{H}$-invariant prime ideals of $R$.

So, to compute the number of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$, the first step is to study the $\mathcal{H}$-invariant prime ideals of $R^+_t$. Since this algebra is induced from $R$ by factor and localization, we first construct (See Section 2), by using the deleting derivations theory (See [4]), $\mathcal{H}$-invariant prime ideals of $R$ that provide, after factor and localization, $2^{r_2-r_1} \cdots t^{r_t-r_{t-1}}(t+1)^{n-r_t} \mathcal{H}$-invariant prime ideals of $R^+_t$ (See Section 3.2). Next, by using (1), we are able to show that the number of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$ is greater than or equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$ (See Proposition 2.9). Finally, after observing that the number of $\mathcal{H}$-invariant prime ideals of $R$ is equal to the poly-Bernoulli number $B_n^{(-n)}$ (See Proposition 2.7), we use a closed formula for the poly-Bernoulli number $B_n^{(-n)}$ (See [1], Theorem 2) in order to prove our main result: the number of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$ is actually equal to $(t!)^2 S(n+1, t+1)^2$. This result was conjectured by Goodearl, Lenagan and McCammond. As a corollary, we obtain a description for the set of $\mathcal{H}$-invariant prime ideals of $R^+_t$ (See Section 3.3).

## 2 $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

Throughout this paper, we use the following conventions:
• If \( I \) is a finite set, \(|I|\) denotes its cardinality.
• \( \mathbb{K} \) denotes a (commutative) field and we set \( \mathbb{K}^* := \mathbb{K} \setminus \{0\} \).
• \( q \in \mathbb{K}^* \) is not a root of unity.
• \( n \) denotes a positive integer with \( n \geq 2 \).
• \( R = O_q(\mathcal{M}_n(\mathbb{K})) \) denotes the quantization of the ring of regular functions on \( n \times n \) matrices with entries in \( \mathbb{K} \); it is the \( \mathbb{K} \)-algebra generated by the \( n \times n \) indeterminates \( Y_{i,\alpha} \), \( 1 \leq i, \alpha \leq n \), subject to the following relations:

If \( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) is any \( 2 \times 2 \) sub-matrix of \( Y := (Y_{i,\alpha})_{(i,\alpha)\in[1,n]^2} \), then

1. \( yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad yz = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt. \)

2. \( tx = xt - (q - q^{-1})yz. \)

These relations agree with the relations used in [4], [5], [6], [12] and [11], but they differ from those of [14] and [2] by an interchange of \( q \) and \( q^{-1} \). It is well known that \( R \) can be presented as an iterated Ore extension over \( \mathbb{K} \), with the generators \( Y_{i,\alpha} \) adjoined in lexicographic order. Thus the ring \( R \) is a Noetherian domain. We denote by \( F \) its skew-field of fractions. Moreover, since \( q \) is not a root of unity, it follows from [7, Theorem 3.2] that all prime ideals of \( R \) are completely prime.

• It is well known that the group \( \mathcal{H} := (\mathbb{C}^*)^{2n} \) acts on \( \mathbb{K} \)-algebra automorphisms via:

\[
(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_n Y_{i,\alpha} \quad \forall (i, \alpha) \in [1,n]^2.
\]

An \( \mathcal{H} \)-eigenvector \( x \) of \( R \) is a nonzero element \( x \in R \) such that \( h(x) \in \mathbb{K}^* x \) for each \( h \in \mathcal{H} \).

An ideal \( I \) of \( R \) is said to be \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \). We denote by \( \mathcal{H} \text{-Spec}(R) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R \).

The aim of this paragraph is to construct \( \mathcal{H} \)-invariant prime ideals of \( R \) that, after factor and localization, will provide \( \mathcal{H} \)-invariant prime ideals of \( R^+ \) (See the introduction for the definition of this algebra). In order to do this, we use the description of the set \( \mathcal{H} \text{-Spec}(R) \) that Cauchon has obtained by applying the theory of deleting derivations (See [4]).

2.1 Standard deleting derivations algorithm and description of \( \mathcal{H} \text{-Spec}(R) \).

In this section, we provide the background definitions and notations for the standard deleting derivations algorithm (See [4, 12, 11]) and we recall the description of the set \( \mathcal{H} \text{-Spec}(R) \) that Cauchon has obtained by using this algorithm (See [4]).

Notations 2.1

• We denote by \( \leq_s \) the lexicographic ordering on \( \mathbb{N}^2 \). We often call it the standard ordering on \( \mathbb{N}^2 \). Recall that \( (i, \alpha) \leq_s (j, \beta) \iff [(i < j) \text{ or } (i = j \text{ and } \alpha \leq \beta)] \).

• We set \( E_s = ([1,n]^2 \cup \{(n,n+1)\}) \setminus \{(1,1)\} \).

• Let \( (j, \beta) \in E_s \). If \( (j, \beta) \neq (n,n+1) \), \( (j, \beta)^+ \) denotes the smallest element (relatively to \( \leq_s \)) of the set \( \{(i, \alpha) \in E_s \mid (j, \beta) <_s (i, \alpha)\} \).
In [4], Cauchon has shown that the theory of deleting derivations (See [3]) can be applied to the iterated Ore extension \( R = \mathbb{C}[Y_{1,1}, \ldots, Y_{n,n}; \sigma_{n,n}, \delta_{n,n}] \) (where the indices are increasing for \( \leq s \)). The corresponding deleting derivations algorithm is called the standard deleting derivations algorithm. It consists in the construction, for each \( r \in E_s \), of the family \((Y_{i,\alpha}^{(r)})_{(i,\alpha) \in [1,n]^2} \) of elements of \( F = \text{Fract}(R) \), defined as follows:

1. If \( r = (n,n+1) \), then \( Y_{i,\alpha}^{(n,n+1)} = Y_{i,\alpha} \) for all \((i, \alpha) \in [1,n]^2\).

2. Assume that \( r = (j, \beta) <_s (n, n+1) \) and that the \( Y_{i,\alpha}^{(r)} \) \((i, \alpha) \in [1,n]^2\) are already constructed. Then, it follows from [3, Théorème 3.2.1] that \( Y_{j,\beta}^{(r)} \neq 0 \) and, for all \((i, \alpha) \in [1,n]^2\), we have:

\[
Y_{i,\alpha}^{(r)} = \begin{cases} 
Y_{i,\alpha}^{(r)} - Y_{i,\beta}^{(r)} \left( Y_{j,\beta}^{(r)} \right)^{-1} Y_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\
Y_{i,\alpha}^{(r)} & \text{otherwise}.
\end{cases}
\]

**Notation 2.2**

Let \( r \in E_s \). We denote by \( R^{(r)} \) the subalgebra of \( F = \text{Fract}(R) \) generated by the \( Y_{i,\alpha}^{(r)} \) \((i, \alpha) \in [1,n]^2\), that is, \( R^{(r)} := \mathbb{C}\langle Y_{i,\alpha}^{(r)} \mid (i, \alpha) \in [1,n]^2 \rangle \).

**Notations 2.3**

We set \( R := R^{(1,2)} \) and \( T_{i,\alpha} := Y_{i,\alpha}^{(1,2)} \) for all \((i, \alpha) \in [1,n]^2\).

Let \( (j, \beta) \in E_s \) with \( (j, \beta) \neq (n, n+1) \). The theory of deleting derivations allows us to construct embeddings \( \varphi_{(j,\beta)} : \text{Spec}(R^{(j,\beta)}) \rightarrow \text{Spec}(R^{(j,\beta)}) \) (See [3, 4.3]). By composition, we obtain an embedding \( \varphi : \text{Spec}(R) \rightarrow \text{Spec}(\overline{R}) \) which is called the canonical embedding. In [4], Cauchon has described the set \( \mathcal{H}-\text{Spec}(R) \) by determining its "canonical image" \( \varphi(\mathcal{H}-\text{Spec}(R)) \). To do this, he has introduced the following conventions and notations.

**Conventions 2.4**

- Let \( v = (l, \gamma) \in [1,n]^2 \).

  1. The set \( C_v := \{ (i, \gamma) \mid 1 \leq i \leq l \} \subset [1,n]^2 \) is called the truncated column with extremity \( v \).

  2. The set \( L_v := \{ (l, \alpha) \mid 1 \leq \alpha \leq \gamma \} \subset [1,n]^2 \) is called the truncated row with extremity \( v \).

- \( W \) denotes the set of all the subsets in \([1,n]^2\) which are a union of truncated rows and columns.

**Notation 2.5**

Given \( w \in W \), \( K_w \) denotes the ideal in \( \overline{R} \) generated by the \( T_{i,\alpha} \) such that \((i, \alpha) \in w \).

(Recall that \( K_w \) is a completely prime ideal in the quantum affine space \( \overline{R} \) (See [3, 2.1]).)
The following description of the set $\mathcal{H} \text{-Spec}(R)$ was obtained by Cauchon (See [4], Corollaire 3.2.1).

**Proposition 2.6**

1. Given $w \in W$, there exists a (unique) $\mathcal{H}$-invariant (completely) prime ideal $J_w$ in $R$ such that $\varphi(J_w) = K_w$.

2. $\mathcal{H} \text{-Spec}(R) = \{J_w \mid w \in W\}$.

### 2.2 Number of $\mathcal{H}$-invariant prime ideals in $R$.

In [4], Cauchon has used his description of the set $\mathcal{H} \text{-Spec}(R)$ in order to give a formula for the total number $S(n)$ of $\mathcal{H}$-invariant prime ideals of $R$. More precisely, he has established (See [4], Proposition 3.3.2) that:

$$S(n) = (-1)^{n-1} \sum_{k=1}^{n} (k+1)^n \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} j^n,$$

that is

$$S(n) = (-1)^{n} \sum_{k=1}^{n} (-1)^k k!(k+1)^n \left( \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n \right).$$

Recall (See [15], p. 34) that $\frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^n$ is equal to the Stirling number of second kind $S(n, k)$ (See, for example, [15] for more details on the Stirling numbers of second kind). Hence, we have:

$$S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n S(n, k),$$

that is

$$S(n) = (-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k). \quad (2)$$

On the other hand, it follows from [10, Theorem 1] that:

$$(-1)^n \sum_{k=0}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k) = B_n^{(-n)},$$

where $B_n^{(-n)}$ denotes the poly-Bernoulli number associated to $n$ and $-n$ (See [10] for the definition of the poly-Bernoulli numbers). Observing that $S(n, 0) = 0$ (See [15]), we get:

$$(-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^n} S(n, k) = B_n^{(-n)},$$

and thus, we deduce from (2) that:
Proposition 2.7

| \mathcal{H}-\text{Spec}(R) | = B_n^{-n}.

This rewriting of Cauchon’s formula was first obtained by Goodearl and McCammond.

2.3 Vanishing and non-vanishing criteria for the entries of $q$-quantum matrices.

Let $J_w (w \in W)$ be an $\mathcal{H}$-invariant prime ideal of $R$ (See Proposition 2.6). In the next section, we will need to know which indeterminates $Y_{i,\alpha}$ belong to $J_w$, that is which $y_{i,\alpha} := Y_{i,\alpha} + J_w$ are zero. This problem is dealt with in Proposition 2.12 and Proposition 2.16 where we respectively obtain a non-vanishing criterion and a vanishing criterion for the entries of $q$-quantum matrices.

For the remainder of this section, $K$ denotes a $K$-algebra which is also a skew-field. Except otherwise stated, all the considered matrices have their entries in $K$.

Definitions 2.8

Let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be a $n \times n$ matrix and let $(j, \beta) \in E_s$.

- We say that $M$ is a $q$-quantum matrix if the following relations hold between the entries of $M$:
  If \( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) is any $2 \times 2$ sub-matrix of $M$, then
  1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
  2. $tx = xt - (q - q^{-1})yz$.

- We say that $M$ is a $(j, \beta)$-q-quantum matrix if the following relations hold between the entries of $M$:
  If \( \begin{pmatrix} x & y \\ z & t \end{pmatrix} \) is any $2 \times 2$ sub-matrix of $M$, then
  1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
  2. If $t = x_v$, then \( \begin{cases} v \geq_s (j, \beta) \implies tx = xt \\ v <_s (j, \beta) \implies tx = xt - (q - q^{-1})yz. \end{cases} \)

Conventions 2.9

Let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be a $q$-quantum matrix.

As $r$ runs over the set $E_s$, we define matrices $M^{(r)} = (x^{(r)}_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ as follows:

1. If $r = (n, n+1)$, then the entries of the matrix $M^{(n,n+1)}$ are defined by $x^{(n,n+1)}_{i,\alpha} := x_{i,\alpha}$ for all $(i, \alpha) \in [1,n]^2$.

2. Assume that $r = (j, \beta) \in E_s \setminus \{(n,n+1)\}$ and that the matrix $M^{(r^+)}$ is already known. The entries $x^{(r)}_{i,\alpha}$ of the matrix $M^{(r)}$ are defined as follows:
(a) If $x_{j,\beta}^{(r)} = 0$, then $x_{i,\alpha}^{(r)} = x_{i,\alpha}^{(r)}$ for all $(i, \alpha) \in [1, n]^2$.

(b) If $x_{j,\beta}^{(r)} \neq 0$ and $(i, \alpha) \in [1, n]^2$, then

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r)} - x_{i,\beta}^{(r)} \left( x_{j,\beta}^{(r)} \right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r)} & \text{otherwise.} \end{cases}$$

We say that $M^{(r)}$ is the matrix obtained from $M$ by applying the standard deleting derivations algorithm at step $r$.

3. If $r = (1, 2)$, we set $t_{i,\alpha} := x_{i,\alpha}^{(1,2)}$ for all $(i, \alpha) \in [1, n]^2$.

Observe that the formulas of Conventions 2.9 allow us to express the entries of $M^{(r)}$ in terms of those of $M^{(r)}$.

**Proposition 2.10 (Restoration algorithm)**

Let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be a $q$-quantum matrix and let $r = (j, \beta) \in E_s$ with $r \neq (n, n + 1)$.

1. If $x_{j,\beta}^{(r)} = 0$, then $x_{i,\alpha}^{(r)} = x_{i,\alpha}^{(r)}$ for all $(i, \alpha) \in [1, n]^2$.

2. If $x_{j,\beta}^{(r)} \neq 0$ and $(i, \alpha) \in [1, n]^2$, then

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r)} + x_{i,\beta}^{(r)} \left( x_{j,\beta}^{(r)} \right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r)} & \text{otherwise.} \end{cases}$$

Note that our definitions of $q$-quantum matrix and $(j, \beta)$-$q$-quantum matrix slightly differ from those of [2] (See [2], Définitions III.1.1 and III.1.3). Because of this, we must interchange $q$ and $q^{-1}$ whenever carrying over result of [2].

**Lemma 2.11**

Let $(j, \beta) \in E_s$.

If $M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ is a $q$-quantum matrix, then the matrix $M^{(j,\beta)}$ is $(j, \beta)$-$q$-quantum.

**Proof:** This lemma is proved in the same manner as [2] Proposition III.2.3.1. ■

We deduce from the above Lemma 2.11 the following non-vanishing criterion for the entries of a $q$-quantum matrix.

**Proposition 2.12**

Let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be a $q$-quantum matrix and let $(i, \alpha) \in [1, n]^2$.

If $t_{i,\alpha} \neq 0$, then $x_{i,\alpha} \neq 0$. In other words, if $x_{i,\alpha} = 0$, then $t_{i,\alpha} = 0$.

**Proof:** Assume that $x_{i,\alpha} = 0$. We first prove that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(j, \beta) \in E_s$. To achieve this aim, we proceed by decreasing induction (for $\leq_s$) on $(j, \beta)$.

Since $x_{i,\alpha}^{(n, n+1)} = x_{i,\alpha}$, the case $(j, \beta) = (n, n+1)$ is done. Assume now that $(j, \beta) <_s (n, n+1)$ and $x_{i,\alpha}^{(j,\beta)^+} = 0$. If $x_{i,\alpha}^{(j,\beta)^-} = x_{i,\alpha}^{(j,\beta)^+}$, we obviously have $x_{i,\alpha}^{(j,\beta)} = 0$. Next, if $x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)^+}$, then...
\( i < j \) and \( \alpha < \beta \). Hence, it follows from Lemma 2.11 that the matrix 
\[
\begin{pmatrix}
 x_{i,\alpha} \quad x_{i,\beta} \\
 x_{j,\alpha} \quad x_{j,\beta}
\end{pmatrix}
\]
is \( q \)-quantum, so that
\[
x_{j,\beta} x_{i,\alpha} - x_{i,\alpha} x_{j,\beta} = -(q - q^{-1})x_{i,\beta} x_{j,\alpha}.
\]

Since \( x_{i,\alpha} = 0 \), we deduce from this equality that, in \( K \), \( x_{i,\beta} x_{j,\alpha} = 0 \). Thus, \( x_{i,\beta} = 0 \) or \( x_{j,\alpha} = 0 \). On the other hand, since \( i < j \) and \( \alpha < \beta \), we have \( x_{i,\alpha} = x_{i,\beta} - x_{j,\alpha} (x_{j,\beta})^{-1} x_{j,\alpha} \). Now it follows from the induction hypothesis that \( x_{i,\beta} = 0 \). Hence, we have
\[
x_{i,\alpha} = -x_{i,\beta} (x_{j,\beta})^{-1} x_{j,\alpha}.
\]
Finally, since \( x_{i,\beta} = 0 \) or \( x_{j,\alpha} = 0 \), we get \( x_{i,\alpha} = 0 \), as desired. This achieves the induction.

In particular, we have shown that \( x_{i,\alpha}^{(1,2)} = 0 \), that is \( t_{i,\alpha} = 0 \). ■

Proposition 2.12 furnishes a non-vanishing criterion for the entries of a \( q \)-quantum matrix. In order to construct, in the next section, \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localization, \( \mathcal{H} \)-invariant prime ideals of \( R_{\mathcal{R}} := \frac{R}{(Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha})} \left[ Y_{r_{1},1}, \ldots, Y_{r_{t},t}^{-1} \right] \) \((r = (r_{1}, \ldots, r_{t}) \text{ with } 1 \leq r_{1} < \cdots < r_{t} \leq n)\), we also need to get a vanishing criterion for the entries \( x_{i,\alpha}, \alpha > t \text{ or } i < r_{\alpha} \), of a \( q \)-quantum matrix. This is what we do now.

**Notation 2.13**

*If \( t \) denotes an element of \([0, n]\), we set:
\[
R_{t} := \{(r_{1}, \ldots, r_{t}) \in \mathbb{N} \mid 1 \leq r_{1} < \cdots < r_{t} \leq n\}.
\]
*(If \( t = 0 \), then \( R_{0} = \emptyset \).)*

For the remainder of this section, we fix \( t \in \mathbb{N} \) and \( r = (r_{1}, \ldots, r_{t}) \in R_{t} \), and we denote by \( w_{r} \) the subset of \([1, n]^{2}\) corresponding to indeterminates \( Y_{i,\alpha} \) that have been set equal to zero in \( R_{r}^{+} \), that is, we set:
\[
w_{r} := \bigcup_{\alpha \in [1, t]} \left[ 1, r_{\alpha} - 1 \right] \times \{\alpha\} \bigcup [1, n] \times [t + 1, n].
\]

For instance, if \( n = 3, t = 2 \) and \( r = (1, 3) \), we have:
\[
w_{(1,3)} = \begin{array}{c|c|c|c|c|c|c|c}
\end{array}
\]
, where the black boxes symbolize the elements of \( w_{(1,3)} \).

Note that \( w_{r} \) is a union of truncated columns, so that:

**Remark 2.14**

*\( w_{r} \) belongs to \( W \).*
Recall (See Proposition 2.6) that, if \( w \in H_{2.4} \) associated to prime ideals in \( i, \beta \), we conclude as in the previous case that \((i, \alpha) \leq t\). Hence, \((i, \beta) \leq t \) and \( i \leq r_\alpha - 1 \). If \( \beta > t \), we conclude as in the previous case that \((i, \beta) \in w_R\). So we assume that \( \beta \leq t \). Since \( i \leq r_\alpha - 1 \) and since \( \alpha \leq \beta \leq t \), we have \( i \leq r_\alpha - 1 \leq \beta \). Hence, \((i, \beta) \in [1, r_\beta - 1] \times \{ \beta \} \subseteq w_R\), as desired. ■

This observation allows us to prove the following vanishing criterion:

**Proposition 2.16**

Let \( M = (x_{i,\alpha})_{(i,\alpha) \in [1,n]^2} \) be a q-quantum matrix.

If \( t_{i,\alpha} = 0 \) for all \((i, \alpha) \in w_R\), then \( x_{i,\alpha} = 0 \) for all \((i, \alpha) \in w_R\).

**Proof**: Assume that \( t_{i,\alpha} = 0 \) for all \((i, \alpha) \in w_R\). We first prove by induction on \((j, \beta)\) (with respect of \( \leq_s \)) that \( x_{i,\alpha}^{(j,\beta)} = 0 \) for all \((i, \alpha) \in w_R\) and \((j, \beta) \in E_s\).

If \((j, \beta) = (1, 2)\), then \( x_{i,\alpha}^{(1,2)} = t_{i,\alpha} = 0 \) for all \((i, \alpha) \in w_R\), as required. Assume now that \((j, \beta) <_s (n, n + 1)\) and that \( x_{i,\alpha}^{(j,\beta)} = 0 \) for all \((i, \alpha) \in w_R\). Let \((i, \alpha) \in w_R\). If \( x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)}\), the desired result follows from the induction hypothesis. Next, if \( x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)}\), it follows from Proposition 2.10 that \( x_{i,\alpha}^{(j,\beta)} \neq 0\), \( i < j\), \( \alpha < \beta\) and \( x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)} + x_{i,\alpha}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{i,\alpha}^{(j,\beta)}\). Since \((i, \alpha) \in w_R\), we deduce from the induction hypothesis that \( x_{i,\alpha}^{(j,\beta)} = 0\), so that \( x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{i,\alpha}^{(j,\beta)}\). Moreover, since \((i, \alpha) \in w_R\) and \( \alpha < \beta\), it follows from Observation 2.15 that \((i, \beta) \in w_R\). Then, we deduce from the induction hypothesis that \( x_{i,\beta}^{(j,\beta)} = 0\), so that \( x_{i,\alpha}^{(j,\beta)} = x_{i,\beta}^{(j,\beta)} x_{j,\beta}^{(j,\beta)} x_{i,\alpha}^{(j,\beta)} = 0\). This achieves the induction.

In particular, we have proved that \( x_{i,\alpha} = x_{i,\alpha}^{(n,n+1)} = 0\) for all \((i, \alpha) \in w_R\). ■

### 2.4 \( H \)-invariant prime ideals \( J_w \) with \( w_R \subseteq w \).

As in the previous section, we fix \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t\), and we set:

\[
\begin{aligned}
w_R := \bigcup_{\alpha \in [1,t]} [1, r_\alpha - 1] \times \{ \alpha \} \cup [1, n] \times [t + 1, n].
\end{aligned}
\]

Recall (See Proposition 2.6) that, if \( w \in W\), there exists a (unique) \( H \)-invariant prime ideal of \( R \) associated to \( w \) (See Proposition 2.6) and that the \( J_w (w \in W) \) are exactly the \( H \)-invariant prime ideals in \( R \). This section is devoted to the \( H \)-invariant prime ideals \( J_w (w \in W) \) of \( R \) with \( w_R \subseteq w \). More precisely, we want to know which indeterminates \( Y_{i,\alpha} \) belong to these ideals.
Let \( w \in W \).

1. Set \( R_w := \frac{R}{J_w} \). It follows from \([3, \text{Lemme 5.3.3}]\) that, using the notations of Section 2.1, \( R_w \) and \( \frac{R}{K_w} \) are two Noetherian algebras with no zero-divisors, which have the same skew-field of fractions. We set \( F_w := \text{Fract}(R_w) = \text{Fract}\left(\frac{R}{K_w}\right) \).

2. If \((i, \alpha) \in [1, n]^2\), \( y_{i, \alpha} \) denotes the element of \( R_w \) defined by \( y_{i, \alpha} := Y_{i, \alpha} + J_w \).

3. We denote by \( M_w \) the matrix, with entries in the \( K \)-algebra \( F_w \), defined by:

\[
M_w := (y_{i, \alpha})_{(i, \alpha) \in [1, n]^2}.
\]

Let \( w \in W \). Since \( Y = (Y_{i, \alpha})_{(i, \alpha) \in [1, n]^2} \) is a \( q \)-quantum matrix, \( M_w \) is also a \( q \)-quantum matrix. Thus, we can apply the standard deleting derivations algorithm to \( M_w \) (See Conventions 2.9 with \( K = F_w \)) and if we still denote \( t_{i, \alpha} := y_{i, \alpha}^{(1, 2)} \) for \((i, \alpha) \in [1, n]^2\), we get:

**Proposition 2.18**

\( t_{i, \alpha} = 0 \) if and only if \((i, \alpha) \in w \).

**Proof:** By \([3, \text{Propositions 5.4.1 and 5.4.2}]\), there exists a \( K \)-algebra homomorphism \( f_{(1, 2)} : \overline{R} \to F_w \) such that \( f_{(1, 2)}(T_{i, \alpha}) = t_{i, \alpha} \) for \((i, \alpha) \in [1, n]^2\). Its kernel is \( K_w \) and its image is the subalgebra of \( F_w \) generated by the \( t_{i, \alpha} \) with \((i, \alpha) \in [1, n]^2\). Hence, \( t_{i, \alpha} = 0 \) if and only if \( T_{i, \alpha} \in K_w \), that is, if and only if \((i, \alpha) \in w \). \( \blacksquare \)

Consider now an element \( w \in W \) with \( w_r \subseteq w \) and denote by \( J_w \) the (unique) \( \mathcal{H} \)-invariant prime ideal of \( R \) associated to \( w \) (See Proposition 2.18). Since \( w_r \subseteq w \), we deduce from Proposition 2.18 that \( t_{i, \alpha} = 0 \) for all \((i, \alpha) \in w_r \). Hence, we can apply Proposition 2.16 to the \( q \)-quantum matrix \( M_w \) and we obtain that \( y_{i, \alpha} = 0 \) for all \((i, \alpha) \in w_r \), that is, \( Y_{i, \alpha} \in J_w \) for all \((i, \alpha) \in w_r \). So we have just established:

**Proposition 2.19**

Let \( w \in W \) with \( w_r \subseteq w \). If \((i, \alpha) \in w_r \), then \( Y_{i, \alpha} \) belongs to \( J_w \).

We will now add truncated rows to the "\( w_r \) diagram" in order to obtain \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( \mathcal{H} \)-invariant prime ideals of \( R^+_l \). We will see later (See Section 3.4) that the \( \mathcal{H} \)-invariant prime ideals of \( R \) obtained by adding truncated rows to the "\( w_r \) diagram" are the only \( \mathcal{H} \)-invariant prime ideals of \( R \) that will provide, after factor and localisation, \( \mathcal{H} \)-invariant prime ideals of \( R^+_l \).

**Notation 2.20**

We set \( \Gamma_r := \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n \mid \gamma_k \in [0, t] \text{ if } k \in [r_l + 1, r_{l+1}] \} \). (Here \( r_0 = 0 \) and \( r_{l+1} = n \).)
For instance, if $n = 3$, $t = 2$ and $r = (1, 3)$, we have:

$$
\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \ \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\}.
$$

**Theorem 2.21**

Let $(\gamma_1, \ldots, \gamma_n) \in \Gamma_r$ and set $w_{r,(\gamma_1, \ldots, \gamma_n)} := w_r \cup \bigcup_{k \in [1, n]} \{k\} \times [1, \gamma_k]$. Then $w_{r,(\gamma_1, \ldots, \gamma_n)}$ belongs to $W$ and the $H$-invariant prime ideal $J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ of $R$ has the following properties:

1. $Y_{i,\alpha} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ for all $(i, \alpha) \in w_r$.
2. $Y_{r_k,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ for all $k \in [1, t]$.

**Proof:** Since $w_r$ is a union of truncated columns and since $\bigcup_{k \in [1, n]} \{k\} \times [1, \gamma_k]$ is a union of truncated rows, $w_{r,(\gamma_1, \ldots, \gamma_n)}$ is a union of truncated rows and columns, so that $w_{r,(\gamma_1, \ldots, \gamma_n)} \in W$.

Since $w_r \subseteq w_{r,(\gamma_1, \ldots, \gamma_n)}$, we deduce from Proposition 2.19 that $Y_{i,\alpha} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ for all $(i, \alpha) \in w_r$.

Now we want to prove that $Y_{r_k,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ for all $k \in [1, t]$. Assume this is not the case, that is, assume that there exists $k \in [1, t]$ with $Y_{r_k,k} \in J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$. Then, $y_{r_k,k} = 0$ and it follows from Proposition 2.12 that $y_{r_k,k}^{(1,2)} = t_{r_k,k} = 0$. Thus, we deduce from Proposition 2.18 that $(r_k,k) \in \Gamma_{r,(\gamma_1, \ldots, \gamma_n)}$.

Observe now that, since $k \leq t$, $(r_k,k) \notin [1, n] \times [t + 1, n]$. Further, it is obvious that $(r_k,k) \notin \bigcup_{\alpha \in [1, t]} \{1, r_{\alpha} - 1\} \times \{\alpha\}$. Hence, $(r_k,k) \notin w_r$.

All this together shows that $(r_k,k) \in w_{r,(\gamma_1, \ldots, \gamma_n)} \setminus w_r = \bigcup_{l \in [1, n]} \{l\} \times [1, \gamma_l]$, so that $k \leq \gamma_{r_k}$.

However, since $(\gamma_1, \ldots, \gamma_n) \in \Gamma_r$, we have $\gamma_{r_k} \leq k - 1$. This is a contradiction and thus we have proved that $Y_{r_k,k} \notin J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}$ for all $k \in [1, t]$. $$

Let us now give an example for the elements $w_{r,(\gamma_1, \ldots, \gamma_n)} ((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r)$ of Theorem 2.21.

If $n = 3$, $t = 2$ and $r = (1, 3)$, we have already note that

$$
\Gamma_r = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \ \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\},
$$

so that the elements $w_{r,(\gamma_1, \ldots, \gamma_n)} ((\gamma_1, \gamma_2, \gamma_3) \in \Gamma_r)$ of Theorem 2.21 are:

- $w_{(1,3),(0,0,0)} = w_{(1,3)}$
- $w_{(1,3),(0,1,0)}$
- $w_{(1,3),(0,0,1)}$
- $w_{(1,3),(0,1,1)}$
(As previously, if \( w \in W \), the black boxes symbolize the elements of \( w \).)

3 Number of rank \( t \) \( \mathcal{H} \)-invariant prime ideals in \( O_q(\mathcal{M}_n(\mathbb{K})) \).

In this paragraph, using the previous section, we begin by constructing \( \mathcal{H} \)-invariant prime ideals of the algebra \( R^+_t := \frac{O_q(\mathcal{M}_n(\mathbb{K}))}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_{\alpha} \rangle \left[ Y_{r_{1,1}}, \ldots, Y_{r_{t,t}} \right] \} \), where \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \) is a strictly increasing sequence of integers in the range 1, \ldots, \( n \). Next, following the route sketched in the introduction, we establish our main result: the number \( |\mathcal{H}-\text{Spec}^t(R)| \) of \( \mathcal{H} \)-invariant prime ideals of \( R = O_q(\mathcal{M}_n(\mathbb{K})) \) which contain all \((t + 1) \times (t + 1)\) quantum minors but not all \( t \times t \) quantum minors is equal to \((t!)^2 S(n + 1, t + 1)^2\), where \( S(n + 1, t + 1) \) denotes the Stirling number of second kind associated to \( n + 1 \) and \( t + 1 \). From this result, we derive a description of the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_t \).

3.1 \( \mathcal{H} \)-invariant prime ideals in \( R^+_{t,0} \).

Throughout this section, we fix \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t \), and we define \( w_r \) as in the previous section.

As in [5, 2.1], we set \( R^+_{t,0} = \frac{R}{\langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle} \).

Recall (See [5, 2.1]) that \( R^+_{t,0} \) can be written as an iterated Ore extension over \( \mathbb{K} \). Thus, \( R^+_{t,0} \) is a Noetherian domain. Moreover, since \( q \) is not a root of unity, it follows from [7, Theorem 3.2] that all primes of \( R \) are completely prime and thus, since this property survive in factors, all primes in the algebra \( R^+_{t,0} \) are completely prime.

Observe now that, since the indeterminates \( Y_{i,\alpha} \) are \( \mathcal{H} \)-eigenvectors, \( \langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle \) is an \( \mathcal{H} \)-invariant ideal of \( R \). Hence, the action of \( \mathcal{H} \) on \( R \) induces an action of \( \mathcal{H} \) on \( R^+_{t,0} \) by automorphisms. As usually, an \( \mathcal{H} \)-eigenvector \( x \) of \( R^+_{t,0} \) is a nonzero element \( x \in R^+_{t,0} \) such that \( h(x) \in \mathbb{K}^*x \) for each \( h \in \mathcal{H} \), and an ideal \( I \) of \( R^+_{t,0} \) is said to be \( \mathcal{H} \)-invariant if \( h(I) = I \) for all \( h \in \mathcal{H} \). Further, we denote by \( \mathcal{H}-\text{Spec}(R^+_{t,0}) \) the set of \( \mathcal{H} \)-invariant prime ideals of \( R^+_{t,0} \).

Notations 3.1

- We denote by \( \pi^+_{t,0} : R \to R^+_{t,0} \) the canonical surjective \( \mathbb{K} \)-algebra homomorphism.

- If \((i, \alpha) \in [1, n]^2\), \( Y_{i,\alpha} \) denotes the element of \( R^+_{t,0} \) defined by \( Y_{i,\alpha} := \pi^+_{t,0}(Y_{i,\alpha}) \).

Let \((\gamma_1, \ldots, \gamma_n) \in \Gamma_r \) (See Notation 2.20) and define \( w_{r,(\gamma_1, \ldots, \gamma_n)} \) as in Theorem 2.21. Recall (See Theorem 2.22) that \( w_{r,(\gamma_1, \ldots, \gamma_n)} \) is an element of \( W \) and that the \( \mathcal{H} \)-invariant prime ideal \( J_{w_{r,(\gamma_1, \ldots, \gamma_n)}} \) of \( R \) contains the indeterminates \( Y_{i,\alpha} \) with \((i, \alpha) \in w_r \), so that \( \langle Y_{i,\alpha} \mid (i, \alpha) \in w_r \rangle \subseteq J_{w_{r,(\gamma_1, \ldots, \gamma_n)}} \). Thus, \( \pi^+_{t,0}(J_{w_{r,(\gamma_1, \ldots, \gamma_n)}}) \) is a (completely) prime ideal of \( R^+_{t,0} \). More precisely, we have:
We have already explained that $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} := \pi^+_r \left( J_{w_r,(\gamma_1,\ldots,\gamma_n)} \right)$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{r,0}$ which does not contain the $\mathcal{Y}_{r_k,k} (k \in [1,t])$.

**Proof:** We have already explained that $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ is a (completely) prime ideal of $R^+_{r,0}$. Moreover, since $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ is $\mathcal{H}$-invariant, it is easy to check that $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ is also $\mathcal{H}$-invariant. Finally, since $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ does not contain the indeterminates $Y_{r_k,k}$ with $k \in [1,t]$ (See Theorem 2.21), $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ does not contain the $\mathcal{Y}_{r_k,k} = \pi^+_r (Y_{r_k,k})$ with $k \in [1,t]$.

### 3.2 $\mathcal{H}$-invariant prime ideals in $R^+_r$.

As in the previous section, we fix $t \in [0,n]$ and $r = (r_1,\ldots,r_t) \in R_t$. In [5, 2.1], Goodearl and Lenagan have observed that the $\mathcal{Y}_{r_k,k}$ with $k \in [1,t]$ are regular normal elements in $R^+_r$, so that we can form the Ore localization:

$$R^+_r := R^+_{r,0} S_r^{-1},$$

where $S_r$ denotes the multiplicative system of $R^+_{r,0}$ generated by the $\mathcal{Y}_{r_k,k}$ with $k \in [1,t]$.

In the previous section, we have noted that all the primes of $R^+_{r,0}$ are completely prime. Since this property survives in localization, all the primes of $R^+_r$ are also completely prime.

Observe now that, since the $\mathcal{Y}_{r_k,k}$ with $k \in [1,t]$ are $\mathcal{H}$-eigenvectors of $R^+_{r,0}$, the action of $\mathcal{H}$ on $R^+_{r,0}$ extends to an action of $\mathcal{H}$ on $R^+_r$ by automorphisms. We say that an ideal $I$ of $R^+_r$ is $\mathcal{H}$-invariant if $h(I) = I$ for all $h \in \mathcal{H}$ and we denote by $\mathcal{H}-\text{Spec}(R^+_r)$ the set of $\mathcal{H}$-invariant prime ideals of $R^+_r$. Observe now that contraction and extension provide inverse bijections between the set $\mathcal{H}-\text{Spec}(R^+_r)$ and the set of those $\mathcal{H}$-invariant prime ideals of $R^+_{r,0}$ which are disjoint from $S_r$.

Let $(\gamma_1,\ldots,\gamma_n) \in \Gamma_r$ (See Notation 2.20) and define $w_r,(\gamma_1,\ldots,\gamma_n)$ as in Theorem 2.21. By Proposition 3.2, $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} := \pi^+_r \left( J_{w_r,(\gamma_1,\ldots,\gamma_n)} \right)$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{r,0}$ which does not contain the $\mathcal{Y}_{r_k,k} (k \in [1,t])$. Since $S_r$ is generated by the $\mathcal{Y}_{r_k,k} (k \in [1,t])$, $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)}$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{r,0}$ which is disjoint from $S_r$. Thus, we have the following statement:

**Proposition 3.3**

$J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} S_r^{-1}$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_r$.

We will prove later (See Section 3.3) that the $J^+_{w_r,(\gamma_1,\ldots,\gamma_n)} S_r^{-1} ((\gamma_1,\ldots,\gamma_n) \in \Gamma_r)$ are exactly the $\mathcal{H}$-invariant prime ideals of $R^+_r$.

We deduce from the above Proposition 3.3 that:

**Corollary 3.4**

$R^+_r$ has at least $1^t \cdot 2^{r_2-r_1} \ldots t^{r_1-r_{t-1}} (t+1)^{n-r_t}$ $\mathcal{H}$-invariant prime ideals.
Proof: It follows from Proposition 3.3 that $R^+_r$ has at least $|\Gamma_r|$ $\mathcal{H}$-invariant prime ideals, and it is obvious that $|\Gamma_r| = 1^{t_1}2^{t_2-\cdots}t^{t_t-1}(t+1)^{n-\cdots}$.

3.3 Number of rank $t$ $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

For convenience, we recall the following definitions (See [14]):

Definitions 3.5

- Let $m$ be a positive integer and let $M = (x_{i,\alpha})_{(i,\alpha)\in[1,m]^2}$ be a square $q$-quantum matrix. The quantum determinant of $M$ is defined by:

$$\det_q(M) := \sum_{\sigma \in S_m} (-q)^{l(\sigma)}x_{1,\sigma(1)}\cdots x_{m,\sigma(m)},$$

where $S_m$ denotes the group of permutations of $[1,m]$ and $l(\sigma)$ denotes the length of the $m$-permutation $\sigma$.

- Let $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha)\in[1,n]^2}$ be the $q$-quantum matrix of the canonical generators of $R$. The quantum determinant of a square sub-matrix of $\mathcal{Y}$ is called a quantum minor.

We can now define the rank $t$ $\mathcal{H}$-invariant prime ideals of $R$, as follows:

Definition 3.6

Let $t \in [0,n]$. An $\mathcal{H}$-invariant prime ideal $J$ of $R = O_q(\mathcal{M}_n(\mathbb{K}))$ has rank $t$ if $J$ contains all $(t+1)\times(t+1)$ quantum minors but not all $t\times t$ quantum minors.

As in [5, 3.6], we denote by $\mathcal{H}$-Spec$^{[t]}(R)$ the set of rank $t$ $\mathcal{H}$-invariant prime ideals of $R$.

Note that there is only one element in $\mathcal{H}$-Spec$^{[t]}(R)$: $\langle Y_{i,\alpha} \mid (i,\alpha) \in [1,n]^2 \rangle$, the augmentation ideal of $R$. Further, Goodearl and Lenagan have observed (See [5, 3.6]) that $|\mathcal{H}$-Spec$^{[1]}(R)| = (2^n - 1)^2$ and $|\mathcal{H}$-Spec$^{[n]}(R)| = (n!)^2$.

Observation 3.7

The sets $\mathcal{H}$-Spec$^{[t]}(R)$ ($t \in [0,n]$) partition the set $\mathcal{H}$-Spec$^{[t]}(R)$.

Proof: Let $P$ be an $\mathcal{H}$-invariant prime ideal of $R$. Let $t \in [0,n]$ be maximal such that $P$ does not contain all $t\times t$ quantum minors. Then $P$ clearly belongs to $\mathcal{H}$-Spec$^{[t]}(R)$. Hence, we have proved that $\mathcal{H}$-Spec$(R) = \bigcup_{t \in [0,n]} \mathcal{H}$-Spec$^{[t]}(R)$. Since this union is obviously disjoint, we get $\mathcal{H}$-Spec$(R) = \bigcup_{t \in [0,n]} \mathcal{H}$-Spec$^{[t]}(R)$, as desired.

In [5], the authors have established the following result that will be our starting point to compute the cardinality of $\mathcal{H}$-Spec$^{[t]}(R)$:
Proposition 3.8 (See [5], 3.6)
For all \( t \in [0, n] \), we have \(|\mathcal{H}\text{-Spec}^{(t)}(R)| = \left( \sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)| \right)^2\).

Before computing \(|\mathcal{H}\text{-Spec}^{(t)}(R)|\), we first give a lower bound for \( \sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)|\).

Proposition 3.9
For any \( t \in [0, n] \), we have
\[
\sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)| \geq t! S(n + 1, t + 1),
\]
where \( S(n + 1, t + 1) \) denotes the Stirling number of second kind associated to \( n + 1 \) and \( t + 1 \) (See, for instance, [15] for the definition of \( S(n + 1, t + 1) \)).

Proof: First, we deduce from Corollary 3.3 the following inequality:
\[
\sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)| \geq \sum_{r \in R_t} 1^{r_1} 2^{r_2 - r_1} \cdots t^{r_t - r_{t-1}} (t + 1)^{n - r_t}. \tag{3}
\]

On the other hand, we know (See [15], Exercise 16 p46) that:
\[
S(n + 1, t + 1) = \sum_{a_1 + \cdots + a_{t+1} = n+1} 1^{a_1-1} 2^{a_2-1} \cdots (t+1)^{a_{t+1}-1}. \tag{4}
\]

Observe now that the map \( f : \{(a_1, \ldots, a_{t+1}) \in (\mathbb{N}^+)^{t+1} | a_1 + \cdots + a_{t+1} = n+1\} \to \{(r_1, \ldots, r_t) \in (\mathbb{N}^+)^t | 1 \leq r_1 < \cdots < r_t \leq n\} = R_t \) defined by \( f(a_1, \ldots, a_{t+1}) = (a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_t) \) is a bijection and that its inverse \( f^{-1} \) is defined by \( f^{-1}(r_1, \ldots, r_t) = (r_1, r_2 - r_1, \ldots, r_t - r_{t-1}, n + 1 - r_t) \) for all \((r_1, \ldots, r_t) \in R_t\). Thus, by means of the change of variables \((a_1, \ldots, a_{t+1}) = f^{-1}(r_1, \ldots, r_t)\), the above equality (4) is transformed to
\[
S(n + 1, t + 1) = \sum_{1 \leq r_1 < \cdots < r_t \leq n} 1^{r_1-1} 2^{r_2 - r_1} \cdots t^{r_t - r_{t-1} - 1} (t + 1)^{n - r_t},
\]
so that
\[
t! S(n + 1, t + 1) = \sum_{(r_1, \ldots, r_t) \in R_t} 1^{r_1} 2^{r_2 - r_1} \cdots t^{r_t - r_{t-1} - 1} (t + 1)^{n - r_t}.
\]

Thus, we deduce from inequality (3) that:
\[
\sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)| \geq t! S(n + 1, t + 1),
\]
as desired. \( \blacksquare \)

Remark 3.10
The proof of the above Proposition shows that, if there exists \( t \in [0, n] \) and \( r = (r_1, \ldots, r_t) \in R_t \) such that \(|\mathcal{H}\text{-Spec}(R^+_t)| > 1^{r_1} 2^{r_2 - r_1} \cdots t^{r_t - r_{t-1} - 1} (t + 1)^{n - r_t} \), then
\[
\sum_{r \in R_t} |\mathcal{H}\text{-Spec}(R^+_t)| > t! S(n + 1, t + 1).
\]

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We can now prove our main result which was conjectured by Goodearl, Lenagan and McCammond:

**Theorem 3.11**

If $t \in [0, n]$, then $| \mathcal{H} \text{-} \text{Spec}^t(R) | = (t!S(n + 1, t + 1))^2$.

**Proof:** First, since the sets $\mathcal{H} \text{-} \text{Spec}^t(R)$ ($t \in [0, n]$) partition $\mathcal{H} \text{-} \text{Spec}(R)$ (See Observation 3.7), we have:

$$| \mathcal{H} \text{-} \text{Spec}(R) | = \sum_{t=0}^{n} | \mathcal{H} \text{-} \text{Spec}^t(R) | .$$

Recall now (See Proposition 2.7) that $| \mathcal{H} \text{-} \text{Spec}(R) |$ is equal to the poly-Bernoulli number $B_n^{(-n)}$. Thus, we deduce from the above equality that:

$$B_n^{(-n)} = \sum_{t=0}^{n} | \mathcal{H} \text{-} \text{Spec}^t(R) | .$$

Further, by Theorem 2), $B_n^{(-n)}$ can also be written as follows:

$$B_n^{(-n)} = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2 .$$

Hence, we have:

$$\sum_{t=0}^{n} | \mathcal{H} \text{-} \text{Spec}^t(R) | = \sum_{t=0}^{n} (t!S(n + 1, t + 1))^2 ,$$

that is:

$$\sum_{t=0}^{n} \left( | \mathcal{H} \text{-} \text{Spec}^t(R) | - (t!S(n + 1, t + 1))^2 \right) = 0 . \tag{5}$$

On the other hand, recall (See 3.6) that $| \mathcal{H} \text{-} \text{Spec}^t(R) | = \left( \sum_{r \in R^t} | \mathcal{H} \text{-} \text{Spec}(R^t) | \right)^2$. Thus, since $\sum_{r \in R^t} | \mathcal{H} \text{-} \text{Spec}(R^t) | \geq t!S(n + 1, t + 1)$ (See Proposition 3.9), we have:

$$| \mathcal{H} \text{-} \text{Spec}^t(R) | \geq (t!S(n + 1, t + 1))^2 .$$

In other words, each of the terms which appears in the sum on the left hand side of (5) is non-negative. Since this sum is equal to zero, each term of this sum must be zero, that is, for all $t \in [0, n]$, we have:

$$| \mathcal{H} \text{-} \text{Spec}^t(R) | = (t!S(n + 1, t + 1))^2 .$$

**Remark 3.12**

The cases $t = 0$, $t = 1$ and $t = n$ were already known (See 3.6).
3.4 Description of the set $\mathcal{H}\text{-Spec}(R^+_r)$.

Throughout this section, we fix $t \in \{0, n\}$ and $r = (r_1, \ldots, r_t) \in \mathbb{R}_t$. We now use the above Theorem 3.11 to obtain a description of the set $\mathcal{H}\text{-Spec}(R^+_r)$. More precisely, we show that the only $\mathcal{H}$-invariant prime ideals of $R^+_r$ are those obtained in Proposition 3.3, that is, in the notations of Section 3.2:

Theorem 3.13

$$\mathcal{H}\text{-Spec}(R^+_r) = \{ J^+_{w_r,(\gamma_1, \ldots, \gamma_n)} S_r^{-1} | (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$ 

Proof: We already know (See Proposition 3.3) that

$$\mathcal{H}\text{-Spec}(R^+_r) \supseteq \{ J^+_{w_r,(\gamma_1, \ldots, \gamma_n)} S_r^{-1} | (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$ 

Assume now that

$$\mathcal{H}\text{-Spec}(R^+_r) \supsetneq \{ J^+_{w_r,(\gamma_1, \ldots, \gamma_n)} S_r^{-1} | (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}.$$ 

Then we have $| \mathcal{H}\text{-Spec}(R^+_r) | > | \Gamma_r |$. Since $| \Gamma_r | = 1^{r_1} 2^{r_2-r_1} \ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$, we get $| \mathcal{H}\text{-Spec}(R^+_r) | > 1^{r_1} 2^{r_2-r_1} \ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$. Thus, it follows from Remark 3.10 that

$$\sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-Spec}(R^+_r) | > t! S(n+1, t+1).$$

Hence we have

$$\left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-Spec}(R^+_r) | \right)^2 > (t! S(n+1, t+1))^2.$$

Recall now (See [3, 3.6]) that

$$| \mathcal{H}\text{-Spec}^{[t]}(R) | = \left( \sum_{r \in \mathbb{R}_t} | \mathcal{H}\text{-Spec}(R^+_r) | \right)^2.$$

All this together shows that $| \mathcal{H}\text{-Spec}^{[t]}(R) | > (t! S(n+1, t+1))^2$.

However, it follows from Theorem 3.11 that $| \mathcal{H}\text{-Spec}^{[t]}(R) | = (t! S(n+1, t+1))^2$. This is a contradiction and thus we have proved that $\mathcal{H}\text{-Spec}(R^+_r) = \{ J^+_{w_r,(\gamma_1, \ldots, \gamma_n)} S_r^{-1} | (\gamma_1, \ldots, \gamma_n) \in \Gamma_r \}$.

$\blacksquare$

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References


