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Rank t \mathcal{H} -primes in quantum matrices.

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Abstract

Let \mathbb{K} be a (commutative) field and consider a nonzero element q in \mathbb{K} which is not a root of unity. In [5], Goodearl and Lenagan have shown that the number of \mathcal{H} -primes in $R = O_q(\mathcal{M}_n(\mathbb{K}))$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors is a perfect square. The aim of this paper is to make precise their result: we prove that this number is equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$. This result was conjectured by Goodearl, Lenagan and McCammond. The proof involves some closed formulas for the poly-Bernoulli numbers that were established in [10] and [1].

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1 Introduction.

Fix a (commutative) field \mathbb{K} and an integer n greater than or equal to 2, and choose an element q in $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ which is not a root of unity. Denote by $R = O_q(\mathcal{M}_n(\mathbb{K}))$ the quantization of the ring of regular functions on $n \times n$ matrices with entries in \mathbb{K} and by $(Y_{i,\alpha})_{(i,\alpha) \in \llbracket 1,n \rrbracket^2}$ the matrix of its canonical generators. The bialgebra structure of R gives us an action of the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ on R by \mathbb{K} -automorphisms (See [5]) via:

$$(a_1, \dots, a_n, b_1, \dots, b_n).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad ((i, \alpha) \in \llbracket 1, n \rrbracket^2).$$

In [9], Goodearl and Letzter have shown that R has only finitely many \mathcal{H} -invariant prime ideals (See [9], 5.7. (i)) and that, in order to calculate the prime and primitive spectra of R , it is enough to determine the \mathcal{H} -invariant prime ideals of R (See [9], Theorem 6.6). Next, using the theory of deleting derivations, Cauchon has found a formula for the exact number of \mathcal{H} -invariant prime ideals in R (See [4], Proposition 3.3.2). In this paper, we investigate these ideals.

In [12] (See also [13]), we have proved, assuming that $\mathbb{K} = \mathbb{C}$ (the field of complex numbers) and q is transcendental over \mathbb{Q} , that the \mathcal{H} -invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{C}))$ are generated by quantum minors, as conjectured by Goodearl and Lenagan (See [5] and [6]). Next, using this result together with Cauchon's description for the set of \mathcal{H} -invariant prime ideals of $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [4], Théorème 3.2.1), we have constructed an algorithm which provides an explicit generating set of quantum minors for each \mathcal{H} -invariant prime ideal in $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [11] or [13]).

On the other hand, Goodearl and Lenagan have shown (in the general case where $q \in \mathbb{K}^*$ is not a root of unity) that, in order to obtain descriptions of all the \mathcal{H} -invariant prime ideals of R , we just need to determine the \mathcal{H} -invariant prime ideals of certain "localized step-triangular factors" of R , namely the algebras

$$R_{\mathbf{r}}^+ := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle} \left[\bar{Y}_{r_1,1}^{-1}, \dots, \bar{Y}_{r_t,t}^{-1} \right]$$

and

$$R_{\mathbf{c}}^- := \frac{R}{\langle Y_{i,\alpha} \mid i > t \text{ or } \alpha < c_i \rangle} \left[\bar{Y}_{1,c_1}^{-1}, \dots, \bar{Y}_{t,c_t}^{-1} \right],$$

where $t \in \llbracket 0, n \rrbracket$ and where $\mathbf{r} = (r_1, \dots, r_t)$ and $\mathbf{c} = (c_1, \dots, c_t)$ are strictly increasing sequences of integers in the range $1, \dots, n$ (See [5], Theorem 3.5). Using this result, Goodearl and Lenagan have computed the \mathcal{H} -invariant prime ideals of $O_q(\mathcal{M}_2(\mathbb{K}))$ (See [5]) and $O_q(\mathcal{M}_3(\mathbb{K}))$ (See [6]).

The aims of this paper are to provide a description for the set $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)$ of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$ and to count the rank t \mathcal{H} -invariant prime ideals of R ($t \in \llbracket 0, n \rrbracket$), that is those \mathcal{H} -invariant prime ideals of R which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. In [5], the authors have shown that the number of rank t \mathcal{H} -invariant prime ideals of R is a perfect square. More precisely, they have established (See [5], 3.6) that, for any $t \in \llbracket 0, n \rrbracket$:

$$|\mathcal{H}\text{-Spec}^{[t]}(R)| = \left(\sum_{\substack{\mathbf{r}=(r_1,\dots,r_t) \\ 1 \leq r_1 < \dots < r_t \leq n}} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| \right)^2 \quad (1)$$

where $\mathcal{H}\text{-Spec}^{[t]}(R)$ denotes the set of rank t \mathcal{H} -invariant prime ideals of R and where $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)$ denotes the set of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$. The above relation (1) opens a potential route to count the rank t \mathcal{H} -invariant prime ideals of R : if we can compute the number of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$, then we will be able to count the rank t \mathcal{H} -invariant prime ideals of R .

So, to compute the number of rank t \mathcal{H} -invariant prime ideals of R , the first step is to study the \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$. Since this algebra is induced from R by factor and localization, we first construct (See Section 2), by using the deleting derivations theory (See [4]), \mathcal{H} -invariant prime ideals of R that provide, after factor and localization, $2^{r_2-r_1} \dots t^{r_t-r_{t-1}} (t+1)^{n-r_t}$ \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$ (See Section 3.2). Next, by using (1), we are able to show that the number of rank t \mathcal{H} -invariant prime ideals of R is greater than or equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$ (See Proposition 3.9). Finally, after observing that the number of \mathcal{H} -invariant prime ideals of R is equal to the poly-Bernoulli number $B_n^{(-n)}$ (See Proposition 2.7), we use a closed formula for the poly-Bernoulli number $B_n^{(-n)}$ (See [1], Theorem 2) in order to prove our main result: the number of rank t \mathcal{H} -invariant prime ideals of R is actually equal to $(t!)^2 S(n+1, t+1)^2$. This result was conjectured by Goodearl, Lenagan and McCammond. As a corollary, we obtain a description for the set of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$ (See Section 3.4).

2 \mathcal{H} -invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

Throughout this paper, we use the following conventions:

- If I is a finite set, $|I|$ denotes its cardinality.
- \mathbb{K} denotes a (commutative) field and we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
- $q \in \mathbb{K}^*$ **is not a root of unity**.
- n denotes a positive integer with $n \geq 2$.
- $R = O_q(\mathcal{M}_n(\mathbb{K}))$ denotes the quantization of the ring of regular functions on $n \times n$ matrices with entries in \mathbb{K} ; it is the \mathbb{K} -algebra generated by the $n \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i, \alpha \leq n$, subject to the following relations:

If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$, then

1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
2. $tx = xt - (q - q^{-1})yz$.

These relations agree with the relations used in [4], [5], [6], [12] and [11], but they differ from those of [14] and [2] by an interchange of q and q^{-1} . It is well known that R can be presented as an iterated Ore extension over \mathbb{K} , with the generators $Y_{i,\alpha}$ adjoined in lexicographic order. Thus the ring R is a Noetherian domain. **We denote by F its skew-field of fractions**. Moreover, since q is not a root of unity, it follows from [7, Theorem 3.2] that all prime ideals of R are completely prime.

- It is well known that the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ acts on R by \mathbb{K} -algebra automorphisms via:

$$(a_1, \dots, a_n, b_1, \dots, b_n) \cdot Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \forall (i, \alpha) \in [1, n]^2.$$

An \mathcal{H} -**eigenvector** x of R is a nonzero element $x \in R$ such that $h(x) \in \mathbb{K}^*x$ for each $h \in \mathcal{H}$. An ideal I of R is said to be \mathcal{H} -**invariant** if $h(I) = I$ for all $h \in \mathcal{H}$. We denote by $\underline{\mathcal{H}\text{-Spec}(R)}$ the set of \mathcal{H} -invariant prime ideals of R .

The aim of this paragraph is to construct \mathcal{H} -invariant prime ideals of R that, after factor and localization, will provide \mathcal{H} -invariant prime ideals of $R_{\mathfrak{r}}^+$ (See the introduction for the definition of this algebra). In order to do this, we use the description of the set $\mathcal{H}\text{-Spec}(R)$ that Cauchon has obtained by applying the theory of deleting derivations (See [4]).

2.1 Standard deleting derivations algorithm and description of $\mathcal{H}\text{-Spec}(R)$.

In this section, we provide the background definitions and notations for the standard deleting derivations algorithm (See [4, 12, 11]) and we recall the description of the set $\mathcal{H}\text{-Spec}(R)$ that Cauchon has obtained by using this algorithm (See [4]).

Notations 2.1

- We denote by \leq_s the lexicographic ordering on \mathbb{N}^2 . We often call it **the standard ordering on \mathbb{N}^2** . Recall that $(i, \alpha) \leq_s (j, \beta) \iff [(i < j) \text{ or } (i = j \text{ and } \alpha \leq \beta)]$.
- We set $E_s = ([1, n]^2 \cup \{(n, n+1)\}) \setminus \{(1, 1)\}$.
- Let $(j, \beta) \in E_s$. If $(j, \beta) \neq (n, n+1)$, $(j, \beta)^+$ denotes the smallest element (relatively to \leq_s) of the set $\{(i, \alpha) \in E_s \mid (j, \beta) <_s (i, \alpha)\}$.

In [4], Cauchon has shown that the theory of deleting derivations (See [3]) can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}] \dots [Y_{n,n}; \sigma_{n,n}, \delta_{n,n}]$ (where the indices are increasing for \leq_s). The corresponding deleting derivations algorithm is called **the standard deleting derivations algorithm**. It consists in the construction, for each $r \in E_s$, of the family $(Y_{i,\alpha}^{(r)})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ of elements of $F = \text{Fract}(R)$, defined as follows:

1. If $r = (n, n + 1)$, then $Y_{i,\alpha}^{(n, n+1)} = Y_{i,\alpha}$ for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.
2. Assume that $r = (j, \beta) <_s (n, n + 1)$ and that the $Y_{i,\alpha}^{(r^+)}$ ($(i, \alpha) \in \llbracket 1, n \rrbracket^2$) are already constructed. Then, it follows from [3, Théorème 3.2.1] that $Y_{j,\beta}^{(r^+)} \neq 0$ and, for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$, we have:

$$Y_{i,\alpha}^{(r)} = \begin{cases} Y_{i,\alpha}^{(r^+)} - Y_{i,\beta}^{(r^+)} \left(Y_{j,\beta}^{(r^+)} \right)^{-1} Y_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta \\ Y_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

Notation 2.2

Let $r \in E_s$. We denote by $R^{(r)}$ the subalgebra of $F = \text{Fract}(R)$ generated by the $Y_{i,\alpha}^{(r)}$ ($(i, \alpha) \in \llbracket 1, n \rrbracket^2$), that is, $R^{(r)} := \mathbb{C}\langle Y_{i,\alpha}^{(r)} \mid (i, \alpha) \in \llbracket 1, n \rrbracket^2 \rangle$.

Notations 2.3

We set $\overline{R} := R^{(1,2)}$ and $T_{i,\alpha} := Y_{i,\alpha}^{(1,2)}$ for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.

Let $(j, \beta) \in E_s$ with $(j, \beta) \neq (n, n + 1)$. The theory of deleting derivations allows us to construct embeddings $\varphi_{(j,\beta)} : \text{Spec}(R^{(j,\beta)^+}) \longrightarrow \text{Spec}(R^{(j,\beta)})$ (See [3], 4.3). By composition, we obtain an embedding $\varphi : \text{Spec}(R) \longrightarrow \text{Spec}(\overline{R})$ which is called **the canonical embedding**. In [4], Cauchon has described the set $\mathcal{H}\text{-Spec}(R)$ by determining its "canonical image" $\varphi(\mathcal{H}\text{-Spec}(R))$. To do this, he has introduced the following conventions and notations.

Conventions 2.4

- Let $v = (l, \gamma) \in \llbracket 1, n \rrbracket^2$.
 1. The set $C_v := \{(i, \gamma) \mid 1 \leq i \leq l\} \subset \llbracket 1, n \rrbracket^2$ is called the **truncated column with extremity** v .
 2. The set $L_v := \{(l, \alpha) \mid 1 \leq \alpha \leq \gamma\} \subset \llbracket 1, n \rrbracket^2$ is called the **truncated row with extremity** v .
- W denotes the set of all the subsets in $\llbracket 1, n \rrbracket^2$ which are a union of truncated rows and columns.

Notation 2.5

Given $w \in W$, K_w denotes the ideal in \overline{R} generated by the $T_{i,\alpha}$ such that $(i, \alpha) \in w$. (Recall that K_w is a completely prime ideal in the quantum affine space \overline{R} (See [8], 2.1).)

The following description of the set $\mathcal{H}\text{-Spec}(R)$ was obtained by Cauchon (See [4], Corollaire 3.2.1).

Proposition 2.6

1. Given $w \in W$, there exists a (unique) \mathcal{H} -invariant (completely) prime ideal J_w in R such that $\varphi(J_w) = K_w$.
2. $\mathcal{H}\text{-Spec}(R) = \{J_w \mid w \in W\}$.

2.2 Number of \mathcal{H} -invariant prime ideals in R .

In [4], Cauchon has used his description of the set $\mathcal{H}\text{-Spec}(R)$ in order to give a formula for the total number $S(n)$ of \mathcal{H} -invariant prime ideals of R . More precisely, he has established (See [4], Proposition 3.3.2) that:

$$S(n) = (-1)^{n-1} \sum_{k=1}^n (k+1)^n \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} j^n,$$

that is

$$S(n) = (-1)^n \sum_{k=1}^n (-1)^k k! (k+1)^n \left(\frac{(-1)^k}{k!} \sum_{j=1}^k (-1)^j \binom{k}{j} j^n \right).$$

Recall (See [15], p. 34) that $\frac{(-1)^k}{k!} \sum_{j=1}^k (-1)^j \binom{k}{j} j^n = \frac{(-1)^k}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} j^n$ is equal to the Stirling number of second kind $S(n, k)$ (See, for example, [15] for more details on the Stirling numbers of second kind). Hence, we have:

$$S(n) = (-1)^n \sum_{k=1}^n (-1)^k k! (k+1)^n S(n, k),$$

that is

$$S(n) = (-1)^n \sum_{k=1}^n \frac{(-1)^k k!}{(k+1)^{-n}} S(n, k). \tag{2}$$

On the other hand, it follows from [10, Theorem 1] that:

$$(-1)^n \sum_{k=0}^n \frac{(-1)^k k!}{(k+1)^{-n}} S(n, k) = B_n^{(-n)},$$

where $B_n^{(-n)}$ denotes the poly-Bernoulli number associated to n and $-n$ (See [10] for the definition of the poly-Bernoulli numbers). Observing that $S(n, 0) = 0$ (See [15]), we get:

$$(-1)^n \sum_{k=1}^n \frac{(-1)^k k!}{(k+1)^{-n}} S(n, k) = B_n^{(-n)},$$

and thus, we deduce from (2) that:

Proposition 2.7

$$|\mathcal{H}\text{-Spec}(R)| = B_n^{(-n)}.$$

This rewriting of Cauchon's formula was first obtained by Goodearl and McCammond.

2.3 Vanishing and non-vanishing criteria for the entries of q -quantum matrices.

Let J_w ($w \in W$) be an \mathcal{H} -invariant prime ideal of R (See Proposition 2.6). In the next section, we will need to know which indeterminates $Y_{i,\alpha}$ belong to J_w , that is which $y_{i,\alpha} := Y_{i,\alpha} + J_w$ are zero. This problem is dealt with in Proposition 2.12 and Proposition 2.16 where we respectively obtain a non-vanishing criterion and a vanishing criterion for the entries of q -quantum matrices.

For the remainder of this section, K denotes a \mathbb{K} -algebra which is also a skew-field. Except otherwise stated, all the considered matrices have their entries in K .

Definitions 2.8

Let $M = (x_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$ be a $n \times n$ matrix and let $(j, \beta) \in E_s$.

- We say that M is a **q -quantum matrix** if the following relations hold between the entries of M :

If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of M , then

1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
2. $tx = xt - (q - q^{-1})yz$.

- We say that M is a **(j, β) - q -quantum matrix** if the following relations hold between the entries of M :

If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of M , then

1. $yx = q^{-1}xy$, $zx = q^{-1}xz$, $zy = yz$, $ty = q^{-1}yt$, $tz = q^{-1}zt$.
2. If $t = x_v$, then $\begin{cases} v \geq_s (j, \beta) & \implies tx = xt \\ v <_s (j, \beta) & \implies tx = xt - (q - q^{-1})yz. \end{cases}$

Conventions 2.9

Let $M = (x_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$ be a q -quantum matrix.

As r runs over the set E_s , we define matrices $M^{(r)} = (x_{i,\alpha}^{(r)})_{(i,\alpha) \in [1,n]^2}$ as follows:

1. If $r = (n, n+1)$, then the entries of the matrix $M^{(n, n+1)}$ are defined by $x_{i,\alpha}^{(n, n+1)} := x_{i,\alpha}$ for all $(i, \alpha) \in [1, n]^2$.
2. Assume that $r = (j, \beta) \in E_s \setminus \{(n, n+1)\}$ and that the matrix $M^{(r^+)}$ is already known. The entries $x_{i,\alpha}^{(r)}$ of the matrix $M^{(r)}$ are defined as follows:

(a) If $x_{j,\beta}^{(r^+)} = 0$, then $x_{i,\alpha}^{(r)} = x_{i,\alpha}^{(r^+)}$ for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.

(b) If $x_{j,\beta}^{(r^+)} \neq 0$ and $(i, \alpha) \in \llbracket 1, n \rrbracket^2$, then

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r^+)} - x_{i,\beta}^{(r^+)} \left(x_{j,\beta}^{(r^+)} \right)^{-1} x_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

We say that $M^{(r)}$ is the matrix obtained from M by applying the standard deleting derivations algorithm at step r .

3. If $r = (1, 2)$, we set $t_{i,\alpha} := x_{i,\alpha}^{(1,2)}$ for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.

Observe that the formulas of Conventions 2.9 allow us to express the entries of $M^{(r^+)}$ in terms of those of $M^{(r)}$.

Proposition 2.10 (Restoration algorithm)

Let $M = (x_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ be a q -quantum matrix and let $r = (j, \beta) \in E_s$ with $r \neq (n, n+1)$.

1. If $x_{j,\beta}^{(r)} = 0$, then $x_{i,\alpha}^{(r^+)} = x_{i,\alpha}^{(r)}$ for all $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.

2. If $x_{j,\beta}^{(r)} \neq 0$ and $(i, \alpha) \in \llbracket 1, n \rrbracket^2$, then

$$x_{i,\alpha}^{(r^+)} = \begin{cases} x_{i,\alpha}^{(r)} + x_{i,\beta}^{(r)} \left(x_{j,\beta}^{(r)} \right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r)} & \text{otherwise.} \end{cases}$$

Note that our definitions of q -quantum matrix and (j, β) - q -quantum matrix slightly differ from those of [2] (See [2], Définitions III.1.1 and III.1.3). Because of this, we must interchange q and q^{-1} whenever carrying over result of [2].

Lemma 2.11

Let $(j, \beta) \in E_s$.

If $M = (x_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ is a q -quantum matrix, then the matrix $M^{(j,\beta)}$ is (j, β) - q -quantum.

Proof : This lemma is proved in the same manner as [2, Proposition III.2.3.1]. ■

We deduce from the above Lemma 2.11 the following non-vanishing criterion for the entries of a q -quantum matrix.

Proposition 2.12

Let $M = (x_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ be a q -quantum matrix and let $(i, \alpha) \in \llbracket 1, n \rrbracket^2$.

If $t_{i,\alpha} \neq 0$, then $x_{i,\alpha} \neq 0$. In other words, if $x_{i,\alpha} = 0$, then $t_{i,\alpha} = 0$.

Proof : Assume that $x_{i,\alpha} = 0$. We first prove that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(j, \beta) \in E_s$. To achieve this aim, we proceed by decreasing induction (for \leq_s) on (j, β) .

Since $x_{i,\alpha}^{(n,n+1)} = x_{i,\alpha}$, the case $(j, \beta) = (n, n+1)$ is done. Assume now that $(j, \beta) <_s (n, n+1)$ and $x_{i,\alpha}^{(j,\beta)^+} = 0$. If $x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)^+}$, we obviously have $x_{i,\alpha}^{(j,\beta)} = 0$. Next, if $x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)^+}$, then

$i < j$ and $\alpha < \beta$. Hence, it follows from Lemma 2.11 that the matrix $\begin{pmatrix} x_{i,\alpha}^{(j,\beta)^+} & x_{i,\beta}^{(j,\beta)^+} \\ x_{j,\alpha}^{(j,\beta)^+} & x_{j,\beta}^{(j,\beta)^+} \end{pmatrix}$ is q -quantum, so that

$$x_{j,\beta}^{(j,\beta)^+} x_{i,\alpha}^{(j,\beta)^+} - x_{i,\alpha}^{(j,\beta)^+} x_{j,\beta}^{(j,\beta)^+} = -(q - q^{-1}) x_{i,\beta}^{(j,\beta)^+} x_{j,\alpha}^{(j,\beta)^+}.$$

Since $x_{i,\alpha}^{(j,\beta)^+} = 0$, we deduce from this equality that, in K , $x_{i,\beta}^{(j,\beta)^+} x_{j,\alpha}^{(j,\beta)^+} = 0$. Thus, $x_{i,\beta}^{(j,\beta)^+} = 0$ or $x_{j,\alpha}^{(j,\beta)^+} = 0$. On the other hand, since $i < j$ and $\alpha < \beta$, we have $x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)^+} - x_{i,\beta}^{(j,\beta)^+} \left(x_{j,\beta}^{(j,\beta)^+} \right)^{-1} x_{j,\alpha}^{(j,\beta)^+}$. Now it follows from the induction hypothesis that $x_{i,\alpha}^{(j,\beta)^+} = 0$. Hence, we have $x_{i,\alpha}^{(j,\beta)} = -x_{i,\beta}^{(j,\beta)^+} \left(x_{j,\beta}^{(j,\beta)^+} \right)^{-1} x_{j,\alpha}^{(j,\beta)^+}$. Finally, since $x_{i,\beta}^{(j,\beta)^+} = 0$ or $x_{j,\alpha}^{(j,\beta)^+} = 0$, we get $x_{i,\alpha}^{(j,\beta)} = 0$, as desired. This achieves the induction.

In particular, we have shown that $x_{i,\alpha}^{(1,2)} = 0$, that is $t_{i,\alpha} = 0$. ■

Proposition 2.12 furnishes a non-vanishing criterion for the entries of a q -quantum matrix. In order to construct, in the next section, \mathcal{H} -invariant prime ideals of R that will provide, after factor and localization, \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+ := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle} [\overline{Y}_{r_1,1}^{-1}, \dots, \overline{Y}_{r_t,t}^{-1}]$ ($\mathbf{r} = (r_1, \dots, r_t)$ with $1 \leq r_1 < \dots < r_t \leq n$), we also need to get a vanishing criterion for the entries $x_{i,\alpha}$, $\alpha > t$ or $i < r_\alpha$, of a q -quantum matrix. This is what we do now.

Notation 2.13

If t denotes an element of $\llbracket 0, n \rrbracket$, we set:

$$\mathbf{R}_t := \{(r_1, \dots, r_t) \in \mathbb{N} \mid 1 \leq r_1 < \dots < r_t \leq n\}.$$

(If $t = 0$, then $\mathbf{R}_0 = \emptyset$.)

For the remainder of this section, we fix $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$, and we denote by $w_{\mathbf{r}}$ the subset of $\llbracket 1, n \rrbracket^2$ corresponding to indeterminates $Y_{i,\alpha}$ that have been set equal to zero in $R_{\mathbf{r}}^+$, that is, we set:

$$w_{\mathbf{r}} := \left[\bigcup_{\alpha \in \llbracket 1, t \rrbracket} \llbracket 1, r_\alpha - 1 \rrbracket \times \{\alpha\} \right] \cup \llbracket 1, n \rrbracket \times \llbracket t + 1, n \rrbracket.$$

For instance, if $n = 3$, $t = 2$ and $\mathbf{r} = (1, 3)$, we have:

$$w_{(1,3)} = \begin{array}{|c|c|c|} \hline \square & \blacksquare & \blacksquare \\ \hline \square & \blacksquare & \blacksquare \\ \hline \square & \square & \blacksquare \\ \hline \end{array}, \text{ where the black boxes symbolize the elements of } w_{(1,3)}.$$

Note that $w_{\mathbf{r}}$ is a union of truncated columns, so that:

Remark 2.14

$w_{\mathbf{r}}$ belongs to W .

Observation 2.15

Let $(i, \alpha) \in w_{\mathbf{r}}$. If $\beta \in \llbracket \alpha, n \rrbracket$, then $(i, \beta) \in w_{\mathbf{r}}$.

Proof : We distinguish two cases.

- If $(i, \alpha) \in \llbracket 1, n \rrbracket \times \llbracket t + 1, n \rrbracket$, then $\alpha \geq t + 1$. Hence $\beta \geq \alpha \geq t + 1$ and thus, we have $(i, \beta) \in \llbracket 1, n \rrbracket \times \llbracket t + 1, n \rrbracket \subseteq w_{\mathbf{r}}$, as required.
- Assume now that $(i, \alpha) \in \bigcup_{\gamma \in \llbracket 1, t \rrbracket} \llbracket 1, r_{\gamma} - 1 \rrbracket \times \{\gamma\}$, so that we have $\alpha \leq t$ and $i \leq r_{\alpha} - 1$. If $\beta > t$, we conclude as in the previous case that $(i, \beta) \in w_{\mathbf{r}}$. So we assume that $\beta \leq t$. Since $i \leq r_{\alpha} - 1$ and since $\alpha \leq \beta \leq t$, we have $i \leq r_{\alpha} - 1 \leq r_{\beta} - 1$. Hence, $(i, \beta) \in \llbracket 1, r_{\beta} - 1 \rrbracket \times \{\beta\} \subseteq w_{\mathbf{r}}$, as desired. ■

This observation allows us to prove the following vanishing criterion:

Proposition 2.16

Let $M = (x_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ be a q -quantum matrix.

If $t_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$, then $x_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$.

Proof : Assume that $t_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$. We first prove by induction on (j, β) (with respect of \leq_s) that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$ and $(j, \beta) \in E_s$.

If $(j, \beta) = (1, 2)$, then $x_{i,\alpha}^{(1,2)} = t_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$, as required. Assume now that $(j, \beta) <_s (n, n + 1)$ and that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$. Let $(i, \alpha) \in w_{\mathbf{r}}$. If $x_{i,\alpha}^{(j,\beta)+} = x_{i,\alpha}^{(j,\beta)}$, the desired result follows from the induction hypothesis. Next, if $x_{i,\alpha}^{(j,\beta)+} \neq x_{i,\alpha}^{(j,\beta)}$, it follows from Proposition 2.10 that $x_{j,\beta}^{(j,\beta)} \neq 0$, $i < j$, $\alpha < \beta$ and $x_{i,\alpha}^{(j,\beta)+} = x_{i,\alpha}^{(j,\beta)} + x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)} \right)^{-1} x_{j,\alpha}^{(j,\beta)}$. Since $(i, \alpha) \in w_{\mathbf{r}}$, we deduce from the induction hypothesis that $x_{i,\alpha}^{(j,\beta)} = 0$, so that $x_{i,\alpha}^{(j,\beta)+} = x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)} \right)^{-1} x_{j,\alpha}^{(j,\beta)}$. Moreover, since $(i, \alpha) \in w_{\mathbf{r}}$ and $\alpha < \beta$, it follows from Observation 2.15 that $(i, \beta) \in w_{\mathbf{r}}$. Then, we deduce from the induction hypothesis that $x_{i,\beta}^{(j,\beta)} = 0$, so that $x_{i,\alpha}^{(j,\beta)+} = x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)} \right)^{-1} x_{j,\alpha}^{(j,\beta)} = 0$. This achieves the induction.

In particular, we have proved that $x_{i,\alpha} = x_{i,\alpha}^{(n,n+1)} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$. ■

2.4 \mathcal{H} -invariant prime ideals J_w with $w_{\mathbf{r}} \subseteq w$.

As in the previous section, we fix $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$, and we set:

$$w_{\mathbf{r}} := \left[\bigcup_{\alpha \in \llbracket 1, t \rrbracket} \llbracket 1, r_{\alpha} - 1 \rrbracket \times \{\alpha\} \right] \cup \llbracket 1, n \rrbracket \times \llbracket t + 1, n \rrbracket.$$

Recall (See Proposition 2.6) that, if $w \in W$, there exists a (unique) \mathcal{H} -invariant prime ideal of R associated to w (See Proposition 2.6) and that the J_w ($w \in W$) are exactly the \mathcal{H} -invariant prime ideals in R . This section is devoted to the \mathcal{H} -invariant prime ideals J_w ($w \in W$) of R with $w_{\mathbf{r}} \subseteq w$. More precisely, we want to know which indeterminates $Y_{i,\alpha}$ belong to these ideals.

Notations 2.17

Let $w \in W$.

1. Set $R_w := \frac{R}{J_w}$. It follows from [3, Lemme 5.3.3] that, using the notations of Section 2.1, R_w and $\frac{\overline{R}}{K_w}$ are two Noetherian algebras with no zero-divisors, which have the same skew-field of fractions. We set $F_w := \text{Fract}(R_w) = \text{Fract}\left(\frac{\overline{R}}{K_w}\right)$.
2. If $(i, \alpha) \in \llbracket 1, n \rrbracket^2$, $y_{i,\alpha}$ denotes the element of R_w defined by $y_{i,\alpha} := Y_{i,\alpha} + J_w$.
3. We denote by M_w the matrix, with entries in the \mathbb{K} -algebra F_w , defined by:

$$M_w := (y_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}.$$

Let $w \in W$. Since $\mathcal{Y} = (Y_{i,\alpha})_{(i,\alpha) \in \llbracket 1, n \rrbracket^2}$ is a q -quantum matrix, M_w is also a q -quantum matrix. Thus, we can apply the standard deleting derivations algorithm to M_w (See Conventions 2.9 with $K = F_w$) and if we still denote $t_{i,\alpha} := y_{i,\alpha}^{(1,2)}$ for $(i, \alpha) \in \llbracket 1, n \rrbracket^2$, we get:

Proposition 2.18

$t_{i,\alpha} = 0$ if and only if $(i, \alpha) \in w$.

Proof : By [3, Propositions 5.4.1 and 5.4.2], there exists a \mathbb{K} -algebra homomorphism $f_{(1,2)} : \overline{R} \rightarrow F_w$ such that $f_{(1,2)}(T_{i,\alpha}) = t_{i,\alpha}$ for $(i, \alpha) \in \llbracket 1, n \rrbracket^2$. Its kernel is K_w and its image is the subalgebra of F_w generated by the $t_{i,\alpha}$ with $(i, \alpha) \in \llbracket 1, n \rrbracket^2$. Hence, $t_{i,\alpha} = 0$ if and only if $T_{i,\alpha} \in K_w$, that is, if and only if $(i, \alpha) \in w$. ■

Consider now an element w in W with $w_{\mathbf{r}} \subseteq w$ and denote by J_w the (unique) \mathcal{H} -invariant prime ideal of R associated to w (See Proposition 2.6). Since $w_{\mathbf{r}} \subseteq w$, we deduce from Proposition 2.18 that $t_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$. Hence, we can apply Proposition 2.16 to the q -quantum matrix M_w and we obtain that $y_{i,\alpha} = 0$ for all $(i, \alpha) \in w_{\mathbf{r}}$, that is, $Y_{i,\alpha} \in J_w$ for all $(i, \alpha) \in w_{\mathbf{r}}$. So we have just established:

Proposition 2.19

Let $w \in W$ with $w_{\mathbf{r}} \subseteq w$. If $(i, \alpha) \in w_{\mathbf{r}}$, then $Y_{i,\alpha}$ belongs to J_w .

We will now add truncated rows to the " $w_{\mathbf{r}}$ diagram" in order to obtain \mathcal{H} -invariant prime ideals of R that will provide, after factor and localisation, \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$. We will see later (See Section 3.4) that the \mathcal{H} -invariant prime ideals of R obtained by adding truncated rows to the " $w_{\mathbf{r}}$ diagram" are the only \mathcal{H} -invariant prime ideals of R that will provide, after factor and localisation, \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$.

Notation 2.20

We set $\Gamma_{\mathbf{r}} := \{(\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n \mid \gamma_k \in \llbracket 0, l \rrbracket \text{ if } k \in \llbracket r_l + 1, r_{l+1} \rrbracket\}$. (Here $r_0 = 0$ and $r_{t+1} = n$.)

For instance, if $n = 3$, $t = 2$ and $\mathbf{r} = (1, 3)$, we have:

$$\Gamma_{\mathbf{r}} = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\}.$$

Theorem 2.21

Let $(\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}$ and set $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)} := w_{\mathbf{r}} \cup \left(\bigcup_{k \in [1, n]} \{k\} \times \llbracket 1, \gamma_k \rrbracket \right)$.

Then $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ belongs to W and the \mathcal{H} -invariant prime ideal $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ of R has the following properties:

1. $Y_{i, \alpha} \in J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ for all $(i, \alpha) \in w_{\mathbf{r}}$.
2. $Y_{r_k, k} \notin J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ for all $k \in \llbracket 1, t \rrbracket$.

Proof : Since $w_{\mathbf{r}}$ is a union of truncated columns and since $\bigcup_{k \in [1, n]} \{k\} \times \llbracket 1, \gamma_k \rrbracket$ is a union of truncated rows, $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ is a union of truncated rows and columns, so that $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)} \in W$.

Since $w_{\mathbf{r}} \subseteq w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$, we deduce from Proposition 2.19 that $Y_{i, \alpha} \in J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ for all $(i, \alpha) \in w_{\mathbf{r}}$.

Now we want to prove that $Y_{r_k, k} \notin J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ for all $k \in \llbracket 1, t \rrbracket$. Assume this is not the case, that is, assume that there exists $k \in \llbracket 1, t \rrbracket$ with $Y_{r_k, k} \in J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$. Then, $y_{r_k, k} = 0$ and it follows from Proposition 2.12 that $y_{r_k, k}^{(1, 2)} = t_{r_k, k} = 0$. Thus, we deduce from Proposition 2.18 that $(r_k, k) \in w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$.

Observe now that, since $k \leq t$, $(r_k, k) \notin \llbracket 1, n \rrbracket \times \llbracket t + 1, n \rrbracket$. Further, it is obvious that $(r_k, k) \notin \bigcup_{\alpha \in [1, t]} \llbracket 1, r_{\alpha} - 1 \rrbracket \times \{\alpha\}$. Hence, $(r_k, k) \notin w_{\mathbf{r}}$.

All this together shows that $(r_k, k) \in w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)} \setminus w_{\mathbf{r}} = \bigcup_{l \in [1, n]} \{l\} \times \llbracket 1, \gamma_l \rrbracket$, so that $k \leq \gamma_{r_k}$.

However, since $(\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}$, we have $\gamma_{r_k} \leq k - 1$. This is a contradiction and thus we have proved that $Y_{r_k, k} \notin J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ for all $k \in \llbracket 1, t \rrbracket$. ■

Let us now give an example for the elements $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ ($(\gamma_1, \gamma_2, \gamma_3) \in \Gamma_{\mathbf{r}}$) of Theorem 2.21. If $n = 3$, $t = 2$ and $\mathbf{r} = (1, 3)$, we have already note that

$$\Gamma_{\mathbf{r}} = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0, \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\},$$

so that the elements $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ ($(\gamma_1, \gamma_2, \gamma_3) \in \Gamma_{\mathbf{r}}$) of Theorem 2.21 are:

$$\begin{aligned} w_{(1,3),(0,0,0)} = w_{(1,3)} &= \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} & w_{(1,3),(0,1,0)} &= \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \\ w_{(1,3),(0,0,1)} &= \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} & w_{(1,3),(0,1,1)} &= \begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} \end{aligned}$$

(As previously, if $w \in W$, the black boxes symbolize the elements of w .)

3 Number of rank t \mathcal{H} -invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

In this paragraph, using the previous section, we begin by constructing \mathcal{H} -invariant prime ideals of the algebra $R_{\mathbf{r}}^+ := \frac{O_q(\mathcal{M}_n(\mathbb{K}))}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle} [\bar{Y}_{r_1,1}^{-1}, \dots, \bar{Y}_{r_t,t}^{-1}]$, where $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t)$ is a strictly increasing sequence of integers in the range $1, \dots, n$. Next, following the route sketched in the introduction, we establish our main result: the number $|\mathcal{H}\text{-Spec}^{[t]}(R)|$ of \mathcal{H} -invariant prime ideals of $R = O_q(\mathcal{M}_n(\mathbb{K}))$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors is equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$. From this result, we derive a description of the set of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$.

3.1 \mathcal{H} -invariant prime ideals in $R_{\mathbf{r},0}^+$.

Throughout this section, we fix $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$, and we define $w_{\mathbf{r}}$ as in the previous section.

As in [5, 2.1], we set $R_{\mathbf{r},0}^+ = \frac{R}{\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle}$.

Recall (See [5], 2.1) that $R_{\mathbf{r},0}^+$ can be written as an iterated Ore extension over \mathbb{K} . Thus, $R_{\mathbf{r},0}^+$ is a Noetherian domain. Moreover, since q is not a root of unity, it follows from [7, Theorem 3.2] that all primes of R are completely prime and thus, since this property survives in factors, all primes in the algebra $R_{\mathbf{r},0}^+$ are completely prime.

Observe now that, since the indeterminates $Y_{i,\alpha}$ are \mathcal{H} -eigenvectors, $\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle$ is an \mathcal{H} -invariant ideal of R . Hence, the action of \mathcal{H} on R induces an action of \mathcal{H} on $R_{\mathbf{r},0}^+$ by automorphisms. As usually, an \mathcal{H} -eigenvector x of $R_{\mathbf{r},0}^+$ is a nonzero element $x \in R_{\mathbf{r},0}^+$ such that $h(x) \in \mathbb{K}^*x$ for each $h \in \mathcal{H}$, and an ideal I of $R_{\mathbf{r},0}^+$ is said to be \mathcal{H} -invariant if $h(I) = I$ for all $h \in \mathcal{H}$. Further, we denote by $\underline{\mathcal{H}\text{-Spec}}(R_{\mathbf{r},0}^+)$ the set of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r},0}^+$.

Notations 3.1

- We denote by $\pi_{\mathbf{r},0}^+ : R \rightarrow R_{\mathbf{r},0}^+$ the canonical surjective \mathbb{K} -algebra homomorphism.
- If $(i,\alpha) \in \llbracket 1, n \rrbracket^2$, $\bar{Y}_{i,\alpha}$ denotes the element of $R_{\mathbf{r},0}^+$ defined by $\bar{Y}_{i,\alpha} := \pi_{\mathbf{r},0}^+(Y_{i,\alpha})$.

Let $(\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}$ (See Notation 2.20) and define $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ as in Theorem 2.21. Recall (See Theorem 2.21) that $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ is an element of W and that the \mathcal{H} -invariant prime ideal $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ of R contains the indeterminates $Y_{i,\alpha}$ with $(i,\alpha) \in w_{\mathbf{r}}$, so that $\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle \subseteq J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$. Thus, $\pi_{\mathbf{r},0}^+(J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}})$ is a (completely) prime ideal of $R_{\mathbf{r},0}^+$. More precisely, we have:

Proposition 3.2

$J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+ := \pi_{\mathbf{r},0}^+ \left(J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}} \right)$ is an \mathcal{H} -invariant (completely) prime ideal of $R_{\mathbf{r},0}^+$ which does not contain the $\overline{Y}_{r_k,k}$ ($k \in \llbracket 1, t \rrbracket$).

Proof : We have already explained that $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+$ is a (completely) prime ideal of $R_{\mathbf{r},0}^+$. Moreover, since $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ is \mathcal{H} -invariant, it is easy to check that $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+$ is also \mathcal{H} -invariant. Finally, since $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}$ does not contain the indeterminates $Y_{r_k,k}$ with $k \in \llbracket 1, t \rrbracket$ (See Theorem 2.21), $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+$ does not contain the $\overline{Y}_{r_k,k} = \pi_{\mathbf{r},0}^+(Y_{r_k,k})$ with $k \in \llbracket 1, t \rrbracket$. ■

3.2 \mathcal{H} -invariant prime ideals in $R_{\mathbf{r}}^+$.

As in the previous section, we fix $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$. In [5, 2.1], Goodearl and Lenagan have observed that the $\overline{Y}_{r_k,k}$ with $k \in \llbracket 1, t \rrbracket$ are regular normal elements in $R_{\mathbf{r},0}^+$, so that we can form the Ore localization:

$$R_{\mathbf{r}}^+ := R_{\mathbf{r},0}^+ S_{\mathbf{r}}^{-1},$$

where $S_{\mathbf{r}}$ denotes the multiplicative system of $R_{\mathbf{r},0}^+$ generated by the $\overline{Y}_{r_k,k}$ with $k \in \llbracket 1, t \rrbracket$.

In the previous section, we have noted that all the primes of $R_{\mathbf{r},0}^+$ are completely prime. Since this property survives in localization, all the primes of $R_{\mathbf{r}}^+$ are also completely prime.

Observe now that, since the $\overline{Y}_{r_k,k}$ with $k \in \llbracket 1, t \rrbracket$ are \mathcal{H} -eigenvectors of $R_{\mathbf{r},0}^+$, the action of \mathcal{H} on $R_{\mathbf{r},0}^+$ extends to an action of \mathcal{H} on $R_{\mathbf{r}}^+$ by automorphisms. We say that an ideal I of $R_{\mathbf{r}}^+$ is **\mathcal{H} -invariant** if $h(I) = I$ for all $h \in \mathcal{H}$ and we denote by $\underline{\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)}$ the set of \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$. Observe now that contraction and extension provide inverse bijections between the set $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)$ and the set of those \mathcal{H} -invariant prime ideals of $R_{\mathbf{r},0}^+$ which are disjoint from $S_{\mathbf{r}}$.

Let $(\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}$ (See Notation 2.20) and define $w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}$ as in Theorem 2.21. By Proposition 3.2, $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+ := \pi_{\mathbf{r},0}^+ \left(J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}} \right)$ is an \mathcal{H} -invariant (completely) prime ideal of $R_{\mathbf{r},0}^+$ which does not contain the $\overline{Y}_{r_k,k}$ ($k \in \llbracket 1, t \rrbracket$). Since $S_{\mathbf{r}}$ is generated by the $\overline{Y}_{r_k,k}$ ($k \in \llbracket 1, t \rrbracket$), $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+$ is an \mathcal{H} -invariant (completely) prime ideal of $R_{\mathbf{r},0}^+$ which is disjoint from $S_{\mathbf{r}}$. Thus, we have the following statement:

Proposition 3.3

$J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+ S_{\mathbf{r}}^{-1}$ is an \mathcal{H} -invariant (completely) prime ideal of $R_{\mathbf{r}}^+$.

We will prove later (See Section 3.4) that the $J_{w_{\mathbf{r},(\gamma_1, \dots, \gamma_n)}}^+ S_{\mathbf{r}}^{-1}$ ($(\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}$) are exactly the \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$.

We deduce from the above Proposition 3.3 that:

Corollary 3.4

$R_{\mathbf{r}}^+$ has at least $1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n-r_t}$ \mathcal{H} -invariant prime ideals.

Proof : It follows from Proposition 3.3 that $R_{\mathbf{r}}^+$ has at least $|\Gamma_{\mathbf{r}}|$ \mathcal{H} -invariant prime ideals, and it is obvious that $|\Gamma_{\mathbf{r}}| = 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$. ■

3.3 Number of rank t \mathcal{H} -invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

For convenience, we recall the following definitions (See [14]):

Definitions 3.5

- Let m be a positive integer and let $M = (x_{i,\alpha})_{(i,\alpha) \in [1,m]^2}$ be a square q -quantum matrix. The quantum determinant of M is defined by:

$$\det_q(M) := \sum_{\sigma \in S_m} (-q)^{l(\sigma)} x_{1,\sigma(1)} \dots x_{m,\sigma(m)},$$

where S_m denotes the group of permutations of $[1, m]$ and $l(\sigma)$ denotes the length of the m -permutation σ .

- Let $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha) \in [1,n]^2}$ be the q -quantum matrix of the canonical generators of R . The quantum determinant of a square sub-matrix of \mathcal{Y} is called a quantum minor.

We can now define the rank t \mathcal{H} -invariant prime ideals of R , as follows:

Definition 3.6

Let $t \in [0, n]$. An \mathcal{H} -invariant prime ideal J of $R = O_q(\mathcal{M}_n(\mathbb{K}))$ has rank t if J contains all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. As in [5, 3.6], we denote by $\mathcal{H}\text{-Spec}^{[t]}(R)$ the set of rank t \mathcal{H} -invariant prime ideals of R .

Note that there is only one element in $\mathcal{H}\text{-Spec}^{[0]}(R)$: $\langle Y_{i,\alpha} \mid (i,\alpha) \in [1,n]^2 \rangle$, the augmentation ideal of R . Further, Goodearl and Lenagan have observed (See [5], 3.6) that $|\mathcal{H}\text{-Spec}^{[1]}(R)| = (2^n - 1)^2$ and $|\mathcal{H}\text{-Spec}^{[n]}(R)| = (n!)^2$.

Observation 3.7

The sets $\mathcal{H}\text{-Spec}^{[t]}(R)$ ($t \in [0, n]$) partition the set $\mathcal{H}\text{-Spec}^{[t]}(R)$.

Proof : Let P be an \mathcal{H} -invariant prime ideal of R . Let $t \in [0, n]$ be maximal such that P does not contain all $t \times t$ quantum minors. Then P clearly belongs to $\mathcal{H}\text{-Spec}^{[t]}(R)$. Hence, we have proved that $\mathcal{H}\text{-Spec}(R) = \bigcup_{t \in [0, n]} \mathcal{H}\text{-Spec}^{[t]}(R)$. Since this union is obviously disjoint, we get

$$\mathcal{H}\text{-Spec}(R) = \bigsqcup_{t \in [0, n]} \mathcal{H}\text{-Spec}^{[t]}(R), \text{ as desired. } \blacksquare$$

In [5], the authors have established the following result that will be our starting point to compute the cardinality of $\mathcal{H}\text{-Spec}^{[t]}(R)$:

Proposition 3.8 (See [5], 3.6)

For all $t \in \llbracket 0, n \rrbracket$, we have $|\mathcal{H}\text{-Spec}^{[t]}(R)| = \left(\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | \right)^2$.

Before computing $|\mathcal{H}\text{-Spec}^{[t]}(R)|$, we first give a lower bound for $\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) |$.

Proposition 3.9

For any $t \in \llbracket 0, n \rrbracket$, we have

$$\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | \geq t!S(n+1, t+1),$$

where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$ (See, for instance, [15] for the definition of $S(n+1, t+1)$).

Proof : First, we deduce from Corollary 3.4 the following inequality:

$$\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | \geq \sum_{\mathbf{r} \in \mathbf{R}_t} 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n-r_t}. \quad (3)$$

On the other hand, we know (See [15], Exercise 16 p46) that:

$$S(n+1, t+1) = \sum_{a_1 + \dots + a_{t+1} = n+1} 1^{a_1 - 1} 2^{a_2 - 1} \dots (t+1)^{a_{t+1} - 1}. \quad (4)$$

Observe now that the map $f : \{(a_1, \dots, a_{t+1}) \in (\mathbb{N}^*)^{t+1} \mid a_1 + \dots + a_{t+1} = n+1\} \rightarrow \{(r_1, \dots, r_t) \in (\mathbb{N}^*)^t \mid 1 \leq r_1 < \dots < r_t \leq n\} = \mathbf{R}_t$ defined by $f(a_1, \dots, a_{t+1}) = (a_1, a_1 + a_2, \dots, a_1 + \dots + a_t)$ is a bijection and that its inverse f^{-1} is defined by $f^{-1}(r_1, \dots, r_t) = (r_1, r_2 - r_1, \dots, r_t - r_{t-1}, n+1 - r_t)$ for all $(r_1, \dots, r_t) \in \mathbf{R}_t$. Thus, by means of the change of variables $(a_1, \dots, a_{t+1}) = f^{-1}(r_1, \dots, r_t)$, the above equality (4) is transformed to

$$S(n+1, t+1) = \sum_{1 \leq r_1 < \dots < r_t \leq n} 1^{r_1 - 1} 2^{r_2 - r_1 - 1} \dots t^{r_t - r_{t-1} - 1} (t+1)^{n-r_t},$$

so that

$$t!S(n+1, t+1) = \sum_{(r_1, \dots, r_t) \in \mathbf{R}_t} 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n-r_t}.$$

Thus, we deduce from inequality (3) that:

$$\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | \geq t!S(n+1, t+1),$$

as desired. ■

Remark 3.10

The proof of the above Proposition 3.9 shows that, if there exists $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$ such that $|\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | > 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n-r_t}$, then

$$\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) | > t!S(n+1, t+1).$$

We can now prove our main result which was conjectured by Goodearl, Lenagan and Mc-Cammond:

Theorem 3.11

If $t \in \llbracket 0, n \rrbracket$, then $|\mathcal{H}\text{-Spec}^{[t]}(R)| = (t!S(n+1, t+1))^2$.

Proof : First, since the sets $\mathcal{H}\text{-Spec}^{[t]}(R)$ ($t \in \llbracket 0, n \rrbracket$) partition $\mathcal{H}\text{-Spec}(R)$ (See Observation 3.7), we have :

$$|\mathcal{H}\text{-Spec}(R)| = \sum_{t=0}^n |\mathcal{H}\text{-Spec}^{[t]}(R)|.$$

Recall now (See Proposition 2.7) that $|\mathcal{H}\text{-Spec}(R)|$ is equal to the poly-Bernoulli number $B_n^{(-n)}$. Thus, we deduce from the above equality that:

$$B_n^{(-n)} = \sum_{t=0}^n |\mathcal{H}\text{-Spec}^{[t]}(R)|.$$

Further, by [1, Theorem 2], $B_n^{(-n)}$ can also be written as follows:

$$B_n^{(-n)} = \sum_{t=0}^n (t!S(n+1, t+1))^2.$$

Hence, we have:

$$\sum_{t=0}^n |\mathcal{H}\text{-Spec}^{[t]}(R)| = \sum_{t=0}^n (t!S(n+1, t+1))^2,$$

that is:

$$\sum_{t=0}^n \left(|\mathcal{H}\text{-Spec}^{[t]}(R)| - (t!S(n+1, t+1))^2 \right) = 0. \tag{5}$$

On the other hand, recall (See [5], 3.6) that $|\mathcal{H}\text{-Spec}^{[t]}(R)| = \left(\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| \right)^2$.

Thus, since $\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| \geq t!S(n+1, t+1)$ (See Proposition 3.9), we have:

$$|\mathcal{H}\text{-Spec}^{[t]}(R)| \geq (t!S(n+1, t+1))^2.$$

In other words, each of the terms which appears in the sum on the left hand side of (5) is non-negative. Since this sum is equal to zero, each term of this sum must be zero, that is, for all $t \in \llbracket 0, n \rrbracket$, we have:

$$|\mathcal{H}\text{-Spec}^{[t]}(R)| = (t!S(n+1, t+1))^2. \blacksquare$$

Remark 3.12

The cases $t = 0$, $t = 1$ and $t = n$ were already known (See [5], 3.6).

3.4 Description of the set $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)$.

Throughout this section, we fix $t \in \llbracket 0, n \rrbracket$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$. We now use the above Theorem 3.11 to obtain a description of the set $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)$. More precisely, we show that the only \mathcal{H} -invariant prime ideals of $R_{\mathbf{r}}^+$ are those obtained in Proposition 3.3, that is, in the notations of Section 3.2:

Theorem 3.13

$$\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) = \{J_{w_{\mathbf{r}}, (\gamma_1, \dots, \gamma_n)}^+ S_{\mathbf{r}}^{-1} \mid (\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Proof : We already know (See Proposition 3.3) that

$$\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) \supseteq \{J_{w_{\mathbf{r}}, (\gamma_1, \dots, \gamma_n)}^+ S_{\mathbf{r}}^{-1} \mid (\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Assume now that

$$\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) \not\supseteq \{J_{w_{\mathbf{r}}, (\gamma_1, \dots, \gamma_n)}^+ S_{\mathbf{r}}^{-1} \mid (\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Then we have $|\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| > |\Gamma_{\mathbf{r}}|$. Since $|\Gamma_{\mathbf{r}}| = 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$, we get $|\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| > 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$. Thus, it follows from Remark 3.10 that

$$\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| > t! S(n+1, t+1).$$

Hence we have

$$\left(\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| \right)^2 > (t! S(n+1, t+1))^2.$$

Recall now (See [5, 3.6]) that

$$|\mathcal{H}\text{-Spec}^{[t]}(R)| = \left(\sum_{\mathbf{r} \in \mathbf{R}_t} |\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+)| \right)^2.$$

All this together shows that $|\mathcal{H}\text{-Spec}^{[t]}(R)| > (t! S(n+1, t+1))^2$.

However, it follows from Theorem 3.11 that $|\mathcal{H}\text{-Spec}^{[t]}(R)| = (t! S(n+1, t+1))^2$. This is a contradiction and thus we have proved that $\mathcal{H}\text{-Spec}(R_{\mathbf{r}}^+) = \{J_{w_{\mathbf{r}}, (\gamma_1, \dots, \gamma_n)}^+ S_{\mathbf{r}}^{-1} \mid (\gamma_1, \dots, \gamma_n) \in \Gamma_{\mathbf{r}}\}$. ■

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