Rank t $\mathcal{H}$-primes in quantum matrices.

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Abstract

Let $\mathbb{K}$ be a (commutative) field and consider a nonzero element q in $\mathbb{K}$ which is not a root of unity. In [5], Goodearl and Lenagan have shown that the number of $\mathcal{H}$-primes in $R = O_q(\mathcal{M}_n(\mathbb{K}))$ which contain all $(t+1)\times(t+1)$ quantum minors but not all $t \times t$ quantum minors is a perfect square. The aim of this paper is to make precise their result: we prove that this number is equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$. This result was conjectured by Goodearl, Lenagan and McCammond. The proof involves some closed formulas for the poly-Bernoulli numbers that were established in [10] and [1].

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1 Introduction.

Fix a (commutative) field $\mathbb{K}$ and an integer n greater than or equal to 2, and choose an element q in $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ which is not a root of unity. Denote by $R = O_q(\mathcal{M}_n(\mathbb{K}))$ the quantization of the ring of regular functions on $n \times n$ matrices with entries in $\mathbb{K}$ and by $(Y_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ the matrix of its canonical generators. The bialgebra structure of R gives us an action of the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ on R by $\mathbb{K}$-automorphisms (See [5]) via:

$$(a_1, \dots, a_n, b_1, \dots, b_n).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad ((i,\alpha) \in [\![1,n]\!]^2).$$

In [9], Goodearl and Letzter have shown that R has only finitely many $\mathcal{H}$-invariant prime ideals (See [9], 5.7. (i)) and that, in order to calculate the prime and primitive spectra of R, it is enough to determine the $\mathcal{H}$-invariant prime ideals of R (See [9], Theorem 6.6). Next, using the theory of deleting derivations, Cauchon has found a formula for the exact number of $\mathcal{H}$-invariant prime ideals in R (See [4], Proprosition 3.3.2). In this paper, we investigate these ideals.

In [12] (See also [13]), we have proved, assuming that $\mathbb{K} = \mathbb{C}$ (the field of complex numbers) and q is transcendental over $\mathbb{Q}$, that the $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{C}))$ are generated by quantum minors, as conjectured by Goodearl and Lenagan (See [5] and [6]). Next, using this result together with Cauchon's description for the set of $\mathcal{H}$-invariant prime ideals of $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [4], Théorème 3.2.1), we have constructed an algorithm which provides an explicit generating set of quantum minors for each $\mathcal{H}$-invariant prime ideal in $O_q(\mathcal{M}_n(\mathbb{C}))$ (See [11] or [13]).

On the other hand, Goodearl and Lenagan have shown (in the general case where $q \in \mathbb{K}^*$ is not a root of unity) that, in order to obtain descriptions of all the $\mathcal{H}$-invariant prime ideals of R, we just need to determine the $\mathcal{H}$-invariant prime ideals of certain "localized step-triangular factors" of R, namely the algebras

$$R_{\mathbf{r}}^{+} := \frac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle} \left[\overline{Y}_{r_1,1}^{-1}, \ldots, \overline{Y}_{r_t,t}^{-1} \right]$$

and

$$R_{\mathbf{c}}^{-} := \frac{R}{\langle Y_{i,\alpha} \mid i > t \text{ or } \alpha < c_i \rangle} \left[\overline{Y}_{1,c_1}^{-1}, \ldots, \overline{Y}_{t,c_t}^{-1} \right],$$

where $t \in [\![0, n]\!]$ and where $\mathbf{r} = (r_1, \ldots, r_t)$ and $\mathbf{c} = (c_1, \ldots, c_t)$ are strictly increasing sequences of integers in the range $1, \ldots, n$ (See [5], Theorem 3.5). Using this result, Goodearl and Lenagan have computed the $\mathcal{H}$-invariant prime ideals of $O_q(\mathcal{M}_2(\mathbb{K}))$ (See [5]) and $O_q(\mathcal{M}_3(\mathbb{K}))$ (See [6]).

The aims of this paper are to provide a description for the set $\mathcal{H}$-$Spec(R_{\mathbf{r}}^{+})$ of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$ and to count the rank t $\mathcal{H}$-invariant prime ideals of R ($t \in [\![0, n]\!]$), that is those $\mathcal{H}$-invariant prime ideals of R which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors. In [5], the authors have shown that the number of rank t $\mathcal{H}$-invariant prime ideals of R is a perfect square. More precisely, they have established (See [5], 3.6) that, for any $t \in [\![0, n]\!]$:

$$\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = \left(\sum_{\substack{\mathbf{r}=(r_1,\ldots,r_t) \\ 1 \le r_1 < \cdots < r_t \le n}} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid \right)^2 \tag{1}$$

where $\mathcal{H}$-$Spec^{[t]}(R)$ denotes the set of rank t $\mathcal{H}$-invariant prime ideals of R and where $\mathcal{H}$-$Spec(R_{\mathbf{r}}^{+})$ denotes the set of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$. The above relation (1) opens a potential route to count the rank t $\mathcal{H}$-invariant prime ideals of R: if we can compute the number of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$, then we will be able to count the rank t $\mathcal{H}$-invariant prime ideals of R.

So, to compute the number of rank t $\mathcal{H}$-invariant prime ideals of R, the first step is to study the $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$. Since this algebra is induced from R by factor and localization, we first construct (See Section 2), by using the deleting derivations theory (See [4]), $\mathcal{H}$-invariant prime ideals of R that provide, after factor and localization, $2^{r_2-r_1} \ldots t^{r_t-r_{t-1}}(t+1)^{n-r_t}$ $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$ (See Section 3.2). Next, by using (1), we are able to show that the number of rank t $\mathcal{H}$-invariant prime ideals of R is greater than or equal to $(t!)^2 S(n+1, t+1)^2$, where $S(n+1, t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$ (See Proposition 3.9). Finally, after observing that the number of $\mathcal{H}$-invariant prime ideals of R is equal to the poly-Bernoulli number $B_n^{(-n)}$ (See Proposition 2.7), we use a closed formula for the poly-Bernoulli number $B_n^{(-n)}$ (See [1], Theorem 2) in order to prove our main result: the number of rank t $\mathcal{H}$-invariant prime ideals of R is actually equal to $(t!)^2 S(n+1, t+1)^2$. This result was conjectured by Goodearl, Lenagan and McCammond. As a corollary, we obtain a description for the set of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$ (See Section 3.4).

2 $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

Throughout this paper, we use the following conventions:

• If I is a finite set, $|I|$ denotes its cardinality.
• $\mathbb{K}$ denotes a (commutative) field and we set $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.
• $q \in \mathbb{K}^*$ **is not a root of unity**.
• n denotes a positive integer with $n \geq 2$.
• $R = O_q(\mathcal{M}_n(\mathbb{K}))$ denotes the quantization of the ring of regular functions on $n \times n$ matrices with entries in $\mathbb{K}$; it is the $\mathbb{K}$-algebra generated by the $n \times n$ indeterminates $Y_{i,\alpha}$, $1 \leq i, \alpha \leq n$, subject to the following relations:
If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha) \in [\![1,n]\!]^2}$, then

1. $yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt.$

2. $tx = xt - (q - q^{-1})yz.$

These relations agree with the relations used in [4], [5], [6], [12] and [11], but they differ from those of [14] and [2] by an interchange of q and q^{-1}. It is well known that R can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $Y_{i,\alpha}$ adjoined in lexicographic order. Thus the ring R is a Noetherian domain. **We denote by F its skew-field of fractions**. Moreover, since q is not a root of unity, it follows from [7, Theorem 3.2] that all prime ideals of R are completely prime.
• It is well known that the group $\mathcal{H} := (\mathbb{C}^*)^{2n}$ acts on R by $\mathbb{K}$-algebra automorphisms via:

$$(a_1, \ldots, a_n, b_1, \ldots, b_n).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha} \quad \forall\, (i,\alpha) \in [\![1,n]\!]^2.$$

An **$\mathcal{H}$-eigenvector** x of R is a nonzero element $x \in R$ such that $h(x) \in \mathbb{K}^* x$ for each $h \in \mathcal{H}$. An ideal I of R is said to be **$\mathcal{H}$-invariant** if $h(I) = I$ for all $h \in \mathcal{H}$. We denote by $\underline{\mathcal{H}\text{-}Spec(R)}$ the set of $\mathcal{H}$-invariant prime ideals of R.

The aim of this paragraph is to construct $\mathcal{H}$-invariant prime ideals of R that, after factor and localization, will provide $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^+$ (See the introduction for the definition of this algebra). In order to do this, we use the description of the set $\mathcal{H}$-$Spec(R)$ that Cauchon has obtained by applying the theory of deleting derivations (See [4]).

2.1 Standard deleting derivations algorithm and description of $\mathcal{H}$-$Spec(R)$.

In this section, we provide the background definitions and notations for the standard deleting derivations algorithm (See [4, 12, 11]) and we recall the description of the set $\mathcal{H}$-$Spec(R)$ that Cauchon has obtained by using this algorithm (See [4]).

Notations 2.1

- *We denote by $\leq_s$ the lexicographic ordering on $\mathbb{N}^2$. We often call it **the standard ordering on** $\mathbb{N}^2$. Recall that $(i,\alpha) \leq_s (j,\beta) \Longleftrightarrow [(i < j)$ or $(i = j$ and $\alpha \leq \beta)]$.*

- *We set $E_s = \big([\![1,n]\!]^2 \cup \{(n, n+1)\}\big) \setminus \{(1,1)\}$.*

- *Let $(j,\beta) \in E_s$. If $(j,\beta) \neq (n,n+1)$, $(j,\beta)^+$ denotes the smallest element (relatively to $\leq_s$) of the set $\{(i,\alpha) \in E_s \mid (j,\beta) <_s (i,\alpha)\}$.*

In [4], Cauchon has shown that the theory of deleting derivations (See [3]) can be applied to the iterated Ore extension $R = \mathbb{C}[Y_{1,1}] \dots [Y_{n,n}; \sigma_{n,n}, \delta_{n,n}]$ (where the indices are increasing for $\leq_s$). The corresponding deleting derivations algorithm is called **the standard deleting derivations algorithm**. It consists in the construction, for each $r \in E_s$, of the family $(Y^{(r)}_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ of elements of $F = Fract(R)$, defined as follows:

1. <u>If $r = (n, n+1)$</u>, then $Y^{(n,n+1)}_{i,\alpha} = Y_{i,\alpha}$ for all $(i, \alpha) \in [\![1, n]\!]^2$.

2. <u>Assume that $r = (j, \beta) <_s (n, n+1)$</u> and that the $Y^{(r^+)}_{i,\alpha}$ ($(i, \alpha) \in [\![1, n]\!]^2$) are already constructed. Then, it follows from [3, Théorème 3.2.1] that $Y^{(r^+)}_{j,\beta} \neq 0$ and, for all $(i, \alpha) \in [\![1, n]\!]^2$, we have:

$$Y^{(r)}_{i,\alpha} = \begin{cases} Y^{(r^+)}_{i,\alpha} - Y^{(r^+)}_{i,\beta} \left(Y^{(r^+)}_{j,\beta}\right)^{-1} Y^{(r^+)}_{j,\alpha} & \text{if } i < j \text{ and } \alpha < \beta \\ Y^{(r^+)}_{i,\alpha} & \text{otherwise.} \end{cases}$$

Notation 2.2
Let $r \in E_s$. We denote by $R^{(r)}$ the subalgebra of $F = Fract(R)$ generated by the $Y^{(r)}_{i,\alpha}$ ($(i,\alpha) \in [\![1, n]\!]^2$), that is, $R^{(r)} := \mathbb{C}\langle Y^{(r)}_{i,\alpha} \mid (i, \alpha) \in [\![1, n]\!]^2\rangle$.

Notations 2.3
We set $\overline{R} := R^{(1,2)}$ and $T_{i,\alpha} := Y^{(1,2)}_{i,\alpha}$ for all $(i, \alpha) \in [\![1, n]\!]^2$.

Let $(j, \beta) \in E_s$ with $(j, \beta) \neq (n, n + 1)$. The theory of deleting derivations allows us to construct embeddings $\varphi_{(j,\beta)} : Spec(R^{(j,\beta)^+}) \longrightarrow Spec(R^{(j,\beta)})$ (See [3], 4.3). By composition, we obtain an embedding $\varphi : Spec(R) \longrightarrow Spec(\overline{R})$ which is called **the canonical embedding**. In [4], Cauchon has described the set $\mathcal{H}$-$Spec(R)$ by determining its "canonical image" $\varphi(\mathcal{H}$-$Spec(R))$. To do this, he has introduced the following conventions and notations.

Conventions 2.4

- *Let $v = (l, \gamma) \in [\![1, n]\!]^2$.*
  1. *The set $C_v := \{(i, \gamma) \mid 1 \leq i \leq l\} \subset [\![1, n]\!]^2$ is called the **truncated column with extremity** v.*
  2. *The set $L_v := \{(l, \alpha) \mid 1 \leq \alpha \leq \gamma\} \subset [\![1, n]\!]^2$ is called the **truncated row with extremity** v.*
- *W denotes the set of all the subsets in $[\![1, n]\!]^2$ which are a union of truncated rows and columns.*

Notation 2.5
*Given $w \in W$, K_w denotes the ideal in $\overline{R}$ generated by the $T_{i,\alpha}$ such that $(i, \alpha) \in w$.
(Recall that K_w is a completely prime ideal in the quantum affine space $\overline{R}$ (See [8], 2.1).)*

The following description of the set $\mathcal{H}$-$Spec(R)$ was obtained by Cauchon (See [4], Corollaire 3.2.1).

Proposition 2.6

1. *Given $w \in W$, there exists a (unique) $\mathcal{H}$-invariant (completely) prime ideal J_w in R such that $\varphi(J_w) = K_w$.*
2. *$\mathcal{H}$-$Spec(R) = \{J_w \mid w \in W\}$.*

2.2 Number of $\mathcal{H}$-invariant prime ideals in R.

In [4], Cauchon has used his description of the set $\mathcal{H}$-$Spec(R)$ in order to give a formula for the total number $S(n)$ of $\mathcal{H}$-invariant prime ideals of R. More precisely, he has established (See [4], Proposition 3.3.2) that:

$$S(n) = (-1)^{n-1} \sum_{k=1}^{n} (k+1)^n \sum_{j=1}^{k} (-1)^{j-1} \begin{pmatrix} k \\ j \end{pmatrix} j^n,$$

that is

$$S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n \left(\frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} j^n \right).$$

Recall (See [15], p. 34) that $\displaystyle\frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} j^n = \frac{(-1)^k}{k!} \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} j^n$ is equal to the Stirling number of second kind $S(n,k)$ (See, for example, [15] for more details on the Stirling numbers of second kind). Hence, we have:

$$S(n) = (-1)^n \sum_{k=1}^{n} (-1)^k k!(k+1)^n S(n,k),$$

that is

$$S(n) = (-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^{-n}} S(n,k). \tag{2}$$

On the other hand, it follows from [10, Theorem 1] that:

$$(-1)^n \sum_{k=0}^{n} \frac{(-1)^k k!}{(k+1)^{-n}} S(n,k) = B_n^{(-n)},$$

where $B_n^{(-n)}$ denotes the poly-Bernoulli number associated to n and $-n$ (See [10] for the definition of the poly-Bernoulli numbers). Observing that $S(n,0) = 0$ (See [15]), we get:

$$(-1)^n \sum_{k=1}^{n} \frac{(-1)^k k!}{(k+1)^{-n}} S(n,k) = B_n^{(-n)},$$

and thus, we deduce from (2) that:

Proposition 2.7

$$|\ \mathcal{H}\text{-}Spec(R)\ | = B_n^{(-n)}.$$

This rewriting of Cauchon's formula was first obtained by Goodearl and McCammond.

2.3 Vanishing and non-vanishing criteria for the entries of q-quantum matrices.

Let J_w $(w \in W)$ be an $\mathcal{H}$-invariant prime ideal of R (See Proposition 2.6). In the next section, we will need to know which indeterminates $Y_{i,\alpha}$ belong to J_w, that is which $y_{i,\alpha} := Y_{i,\alpha} + J_w$ are zero. This problem is dealt with in Proposition 2.12 and Proposition 2.16 where we respectively obtain a non-vanishing criterion and a vanishing criterion for the entries of q-quantum matrices.

For the remainder of this section, K denotes a $\mathbb{K}$-algebra which is also a skew-field. Except otherwise stated, all the considered matrices have their entries in K.

Definitions 2.8
Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ be a $n \times n$ matrix and let $(j,\beta) \in E_s$.

- *We say that M is a q-**quantum matrix** if the following relations hold between the entries of M:*
  If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of M, then
  1. *$yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt.$*
  2. *$tx = xt - (q - q^{-1})yz.$*
- *We say that M is a (j,β)-q-**quantum matrix** if the following relations hold between the entries of M:*
  If $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ is any 2×2 sub-matrix of M, then
  1. *$yx = q^{-1}xy, \quad zx = q^{-1}xz, \quad zy = yz, \quad ty = q^{-1}yt, \quad tz = q^{-1}zt.$*
  2. *If $t = x_v$, then $\begin{cases} v \geq_s (j,\beta) & \Longrightarrow \quad tx = xt \\ v <_s (j,\beta) & \Longrightarrow \quad tx = xt - (q - q^{-1})yz. \end{cases}$*

Conventions 2.9
Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ be a q-quantum matrix.
As r runs over the set E_s, we define matrices $M^{(r)} = (x_{i,\alpha}^{(r)})_{(i,\alpha)\in[\![1,n]\!]^2}$ as follows:

1. *<u>If $r = (n, n+1)$, then the entries of the matrix $M^{(n,n+1)}$ are defined by $x_{i,\alpha}^{(n,n+1)} := x_{i,\alpha}$ for all $(i,\alpha) \in [\![1,n]\!]^2$.</u>*
2. *<u>Assume that $r = (j,\beta) \in E_s \setminus \{(n,n+1)\}$</u> and that the matrix $M^{(r^+)}$ is already known. The entries $x_{i,\alpha}^{(r)}$ of the matrix $M^{(r)}$ are defined as follows:*

(a) *If* $x_{j,\beta}^{(r^+)} = 0$, *then* $x_{i,\alpha}^{(r)} = x_{i,\alpha}^{(r^+)}$ *for all* $(i,\alpha) \in [\![1,n]\!]^2$.

(b) *If* $x_{j,\beta}^{(r^+)} \neq 0$ *and* $(i,\alpha) \in [\![1,n]\!]^2$, *then*

$$x_{i,\alpha}^{(r)} = \begin{cases} x_{i,\alpha}^{(r^+)} - x_{i,\beta}^{(r^+)} \left(x_{j,\beta}^{(r^+)}\right)^{-1} x_{j,\alpha}^{(r^+)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r^+)} & \text{otherwise.} \end{cases}$$

We say that $M^{(r)}$ ***is the matrix obtained from*** M ***by applying the standard deleting derivations algorithm at step*** r.

3. *If* $r = (1,2)$, *we set* $t_{i,\alpha} := x_{i,\alpha}^{(1,2)}$ *for all* $(i,\alpha) \in [\![1,n]\!]^2$.

Observe that the formulas of Conventions 2.9 allow us to express the entries of $M^{(r^+)}$ in terms of those of $M^{(r)}$.

Proposition 2.10 (Restoration algorithm)
Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ *be a* q*-quantum matrix and let* $r = (j,\beta) \in E_s$ *with* $r \neq (n,n+1)$.

1. *If* $x_{j,\beta}^{(r)} = 0$, *then* $x_{i,\alpha}^{(r^+)} = x_{i,\alpha}^{(r)}$ *for all* $(i,\alpha) \in [\![1,n]\!]^2$.

2. *If* $x_{j,\beta}^{(r)} \neq 0$ *and* $(i,\alpha) \in [\![1,n]\!]^2$, *then*

$$x_{i,\alpha}^{(r^+)} = \begin{cases} x_{i,\alpha}^{(r)} + x_{i,\beta}^{(r)} \left(x_{j,\beta}^{(r)}\right)^{-1} x_{j,\alpha}^{(r)} & \text{if } i < j \text{ and } \alpha < \beta \\ x_{i,\alpha}^{(r)} & \text{otherwise.} \end{cases}$$

Note that our definitions of q-quantum matrix and (j,β)-q-quantum matrix slightly differ from those of [2] (See [2], Définitions III.1.1 and III.1.3). Because of this, we must interchange q and q^{-1} whenever carrying over result of [2].

Lemma 2.11
Let $(j,\beta) \in E_s$.
If $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ *is a* q*-quantum matrix, then the matrix* $M^{(j,\beta)}$ *is* (j,β)*-*q*-quantum.*

Proof : This lemma is proved in the same manner as [2, Proposition III.2.3.1]. ■

We deduce from the above Lemma 2.11 the following non-vanishing criterion for the entries of a q-quantum matrix.

Proposition 2.12
Let $M = (x_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ *be a* q*-quantum matrix and let* $(i,\alpha) \in [\![1,n]\!]^2$.
If $t_{i,\alpha} \neq 0$, *then* $x_{i,\alpha} \neq 0$. *In other words, if* $x_{i,\alpha} = 0$, *then* $t_{i,\alpha} = 0$.

Proof : Assume that $x_{i,\alpha} = 0$. We first prove that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(j,\beta) \in E_s$. To achieve this aim, we proceed by decreasing induction (for $\leq_s$) on (j,β).

Since $x_{i,\alpha}^{(n,n+1)} = x_{i,\alpha}$, the case $(j,\beta) = (n,n+1)$ is done. Assume now that $(j,\beta) <_s (n,n+1)$ and $x_{i,\alpha}^{(j,\beta)^+} = 0$. If $x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)^+}$, we obviously have $x_{i,\alpha}^{(j,\beta)} = 0$. Next, if $x_{i,\alpha}^{(j,\beta)} \neq x_{i,\alpha}^{(j,\beta)^+}$, then

$i < j$ and $\alpha < \beta$. Hence, it follows from Lemma 2.11 that the matrix $\begin{pmatrix} x_{i,\alpha}^{(j,\beta)+} & x_{i,\beta}^{(j,\beta)+} \\ x_{j,\alpha}^{(j,\beta)+} & x_{j,\beta}^{(j,\beta)+} \end{pmatrix}$ is q-quantum, so that

$$x_{j,\beta}^{(j,\beta)+} x_{i,\alpha}^{(j,\beta)+} - x_{i,\alpha}^{(j,\beta)+} x_{j,\beta}^{(j,\beta)+} = -(q-q^{-1}) x_{i,\beta}^{(j,\beta)+} x_{j,\alpha}^{(j,\beta)+}.$$

Since $x_{i,\alpha}^{(j,\beta)+} = 0$, we deduce from this equality that, in K, $x_{i,\beta}^{(j,\beta)+} x_{j,\alpha}^{(j,\beta)+} = 0$. Thus, $x_{i,\beta}^{(j,\beta)+} = 0$ or $x_{j,\alpha}^{(j,\beta)+} = 0$. On the other hand, since $i < j$ and $\alpha < \beta$, we have $x_{i,\alpha}^{(j,\beta)} = x_{i,\alpha}^{(j,\beta)+} - x_{i,\beta}^{(j,\beta)+} \left(x_{j,\beta}^{(j,\beta)+}\right)^{-1} x_{j,\alpha}^{(j,\beta)+}$. Now it follows from the induction hypothesis that $x_{i,\alpha}^{(j,\beta)+} = 0$. Hence, we have
$x_{i,\alpha}^{(j,\beta)} = -x_{i,\beta}^{(j,\beta)+} \left(x_{j,\beta}^{(j,\beta)+}\right)^{-1} x_{j,\alpha}^{(j,\beta)+}$. Finally, since $x_{i,\beta}^{(j,\beta)+} = 0$ or $x_{j,\alpha}^{(j,\beta)+} = 0$, we get $x_{i,\alpha}^{(j,\beta)} = 0$, as desired. This achieves the induction.

In particular, we have shown that $x_{i,\alpha}^{(1,2)} = 0$, that is $t_{i,\alpha} = 0$. ■

Proposition 2.12 furnishes a non-vanishing criterion for the entries of a q-quantum matrix. In order to construct, in the next section, $\mathcal{H}$-invariant prime ideals of R that will provide, after factor and localization, $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^+ := \dfrac{R}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle}\left[\overline{Y}_{r_1,1}^{-1}, \dots, \overline{Y}_{r_t,t}^{-1}\right]$ ($\mathbf{r} = (r_1, \dots, r_t)$ with $1 \leq r_1 < \cdots < r_t \leq n$), we also need to get a vanishing criterion for the entries $x_{i,\alpha}$, $\alpha > t$ or $i < r_\alpha$, of a q-quantum matrix. This is what we do now.

Notation 2.13
If t denotes an element of $[\![0, n]\!]$, we set:

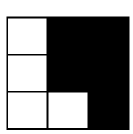
$$\mathbf{R}_t := \{(r_1, \dots, r_t) \in \mathbb{N} \mid 1 \leq r_1 < \cdots < r_t \leq n\}.$$

(If $t = 0$, then $\mathbf{R}_0 = \emptyset$.)

For the remainder of this section, we fix $t \in [\![0, n]\!]$ and $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$, and we denote by $w_{\mathbf{r}}$ the subset of $[\![1, n]\!]^2$ corresponding to indeterminates $Y_{i,\alpha}$ that have been set equal to zero in $R_{\mathbf{r}}^+$, that is, we set:

$$w_{\mathbf{r}} := \left[\bigcup_{\alpha \in [\![1,t]\!]} [\![1, r_\alpha - 1]\!] \times \{\alpha\}\right] \bigcup [\![1, n]\!] \times [\![t+1, n]\!].$$

For instance, if $n = 3$, $t = 2$ and $\mathbf{r} = (1, 3)$, we have:

$w_{(1,3)} =$  , where the black boxes symbolize the elements of $w_{(1,3)}$.

Note that $w_{\mathbf{r}}$ is a union of truncated columns, so that:

Remark 2.14
$w_{\mathbf{r}}$ belongs to W.

Observation 2.15
Let $(i,\alpha) \in w_{\mathbf{r}}$. If $\beta \in [\![\alpha, n]\!]$, then $(i,\beta) \in w_{\mathbf{r}}$.

Proof : We distinguish two cases.
• If $(i,\alpha) \in [\![1,n]\!] \times [\![t+1,n]\!]$, then $\alpha \geq t+1$. Hence $\beta \geq \alpha \geq t+1$ and thus, we have $(i,\beta) \in [\![1,n]\!] \times [\![t+1,n]\!] \subseteq w_{\mathbf{r}}$, as required.
• Assume now that $(i,\alpha) \in \bigcup_{\gamma \in [\![1,t]\!]} [\![1, r_\gamma - 1]\!] \times \{\gamma\}$, so that we have $\alpha \leq t$ and $i \leq r_\alpha - 1$. If $\beta > t$, we conclude as in the previous case that $(i,\beta) \in w_{\mathbf{r}}$. So we assume that $\beta \leq t$. Since $i \leq r_\alpha - 1$ and since $\alpha \leq \beta \leq t$, we have $i \leq r_\alpha - 1 \leq r_\beta - 1$. Hence, $(i,\beta) \in [\![1, r_\beta - 1]\!] \times \{\beta\} \subseteq w_{\mathbf{r}}$, as desired. ■

This observation allows us to prove the following vanishing criterion:

Proposition 2.16
Let $M = (x_{i,\alpha})_{(i,\alpha) \in [\![1,n]\!]^2}$ be a q-quantum matrix.
If $t_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$, then $x_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$.

Proof : Assume that $t_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$. We first prove by induction on (j,β) (with respect of $\leq_s$) that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$ and $(j,\beta) \in E_s$.

If $(j,\beta) = (1,2)$, then $x_{i,\alpha}^{(1,2)} = t_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$, as required. Assume now that $(j,\beta) <_s (n,n+1)$ and that $x_{i,\alpha}^{(j,\beta)} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$. Let $(i,\alpha) \in w_{\mathbf{r}}$. If $x_{i,\alpha}^{(j,\beta)^+} = x_{i,\alpha}^{(j,\beta)}$, the desired result follows from the induction hypothesis. Next, if $x_{i,\alpha}^{(j,\beta)^+} \neq x_{i,\alpha}^{(j,\beta)}$, it follows from Proposition 2.10 that $x_{j,\beta}^{(j,\beta)} \neq 0$, $i < j$, $\alpha < \beta$ and $x_{i,\alpha}^{(j,\beta)^+} = x_{i,\alpha}^{(j,\beta)} + x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)}\right)^{-1} x_{j,\alpha}^{(j,\beta)}$. Since $(i,\alpha) \in w_{\mathbf{r}}$, we deduce from the induction hypothesis that $x_{i,\alpha}^{(j,\beta)} = 0$, so that $x_{i,\alpha}^{(j,\beta)^+} = x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)}\right)^{-1} x_{j,\alpha}^{(j,\beta)}$. Moreover, since $(i,\alpha) \in w_{\mathbf{r}}$ and $\alpha < \beta$, it follows from Observation 2.15 that $(i,\beta) \in w_{\mathbf{r}}$. Then, we deduce from the induction hypothesis that $x_{i,\beta}^{(j,\beta)} = 0$, so that $x_{i,\alpha}^{(j,\beta)^+} = x_{i,\beta}^{(j,\beta)} \left(x_{j,\beta}^{(j,\beta)}\right)^{-1} x_{j,\alpha}^{(j,\beta)} = 0$. This achieves the induction.

In particular, we have proved that $x_{i,\alpha} = x_{i,\alpha}^{(n,n+1)} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$. ■

2.4 $\mathcal{H}$-invariant prime ideals J_w with $w_{\mathbf{r}} \subseteq w$.

As in the previous section, we fix $t \in [\![0,n]\!]$ and $\mathbf{r} = (r_1, \ldots, r_t) \in \mathbf{R}_t$, and we set:

$$w_{\mathbf{r}} := \left[\bigcup_{\alpha \in [\![1,t]\!]} [\![1, r_\alpha - 1]\!] \times \{\alpha\} \right] \bigcup [\![1,n]\!] \times [\![t+1,n]\!].$$

Recall (See Proposition 2.6) that, if $w \in W$, there exists a (unique) $\mathcal{H}$-invariant prime ideal of R associated to w (See Proposition 2.6) and that the J_w ($w \in W$) are exactly the $\mathcal{H}$-invariant prime ideals in R. This section is devoted to the $\mathcal{H}$-invariant prime ideals J_w ($w \in W$) of R with $w_{\mathbf{r}} \subseteq w$. More precisely, we want to know which indeterminates $Y_{i,\alpha}$ belong to these ideals.

Notations 2.17
Let $w \in W$.

1. *Set $R_w := \dfrac{R}{J_w}$. It follows from [3, Lemme 5.3.3] that, using the notations of Section 2.1, R_w and $\dfrac{\overline{R}}{K_w}$ are two Noetherian algebras with no zero-divisors, which have the same skew-field of fractions. We set $F_w := Fract\,(R_w) = Fract\left(\dfrac{\overline{R}}{K_w}\right)$.*

2. *If $(i,\alpha) \in [\![1,n]\!]^2$, $y_{i,\alpha}$ denotes the element of R_w defined by $y_{i,\alpha} := Y_{i,\alpha} + J_w$.*

3. *We denote by M_w the matrix, with entries in the $\mathbb{K}$-algebra F_w, defined by:*

$$M_w := (y_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}\,.$$

Let $w \in W$. Since $\mathcal{Y} = (Y_{i,\alpha})_{(i,\alpha)\in[\![1,n]\!]^2}$ is a q-quantum matrix, M_w is also a q-quantum matrix. Thus, we can apply the standard deleting derivations algorithm to M_w (See Conventions 2.9 with $K = F_w$) and if we still denote $t_{i,\alpha} := y_{i,\alpha}^{(1,2)}$ for $(i,\alpha) \in [\![1,n]\!]^2$, we get:

Proposition 2.18
$t_{i,\alpha} = 0$ if and only if $(i,\alpha) \in w$.

Proof : By [3, Propositions 5.4.1 and 5.4.2], there exists a $\mathbb{K}$-algebra homomorphism $f_{(1,2)} : \overline{R} \to F_w$ such that $f_{(1,2)}\,(T_{i,\alpha}) = t_{i,\alpha}$ for $(i,\alpha) \in [\![1,n]\!]^2$. Its kernel is K_w and its image is the subalgebra of F_w generated by the $t_{i,\alpha}$ with $(i,\alpha) \in [\![1,n]\!]^2$. Hence, $t_{i,\alpha} = 0$ if and only if $T_{i,\alpha} \in K_w$, that is, if and only if $(i,\alpha) \in w$. $\blacksquare$

Consider now an element w in W with $w_{\mathbf{r}} \subseteq w$ and denote by J_w the (unique) $\mathcal{H}$-invariant prime ideal of R associated to w (See Proposition 2.6). Since $w_{\mathbf{r}} \subseteq w$, we deduce from Proposition 2.18 that $t_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$. Hence, we can apply Proposition 2.16 to the q-quantum matrix M_w and we obtain that $y_{i,\alpha} = 0$ for all $(i,\alpha) \in w_{\mathbf{r}}$, that is, $Y_{i,\alpha} \in J_w$ for all $(i,\alpha) \in w_{\mathbf{r}}$. So we have just established:

Proposition 2.19
Let $w \in W$ with $w_{\mathbf{r}} \subseteq w$. If $(i,\alpha) \in w_{\mathbf{r}}$, then $Y_{i,\alpha}$ belongs to J_w.

We will now add truncated rows to the "$w_{\mathbf{r}}$ diagram" in order to obtain $\mathcal{H}$-invariant prime ideals of R that will provide, after factor and localisation, $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$. We will see later (See Section 3.4) that the $\mathcal{H}$-invariant prime ideals of R obtained by adding truncated rows to the "$w_{\mathbf{r}}$ diagram" are the only $\mathcal{H}$-invariant prime ideals of R that will provide, after factor and localisation, $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$.

Notation 2.20
We set $\Gamma_{\mathbf{r}} := \{(\gamma_1,\ldots,\gamma_n) \in \mathbb{N}^n \mid \gamma_k \in [\![0,l]\!]$ if $k \in [\![r_l+1, r_{l+1}]\!]\}$. (Here $r_0 = 0$ and $r_{t+1} = n$.)

For instance, if $n=3$, $t=2$ and $\mathbf{r}=(1,3)$, we have:

$$\Gamma_{\mathbf{r}} = \{(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0,\ \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\}.$$

Theorem 2.21

Let $(\gamma_1,\dots,\gamma_n) \in \Gamma_{\mathbf{r}}$ *and set* $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)} := w_{\mathbf{r}} \bigcup \left(\bigcup_{k\in[\![1,n]\!]} \{k\} \times [\![1,\gamma_k]\!] \right)$.
Then $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$ *belongs to* W *and the* $\mathcal{H}$*-invariant prime ideal* $J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ *of* R *has the following properties:*

1. $Y_{i,\alpha} \in J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ *for all* $(i,\alpha) \in w_{\mathbf{r}}$.
2. $Y_{r_k,k} \notin J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ *for all* $k \in [\![1,t]\!]$.

<u>*Proof :*</u> Since $w_{\mathbf{r}}$ is a union of truncated columns and since $\bigcup_{k\in[\![1,n]\!]} \{k\} \times [\![1,\gamma_k]\!]$ is a union of truncated rows, $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$ is a union of truncated rows and columns, so that $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)} \in W$.

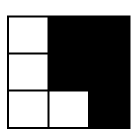
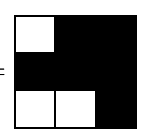
Since $w_{\mathbf{r}} \subseteq w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$, we deduce from Proposition 2.19 that $Y_{i,\alpha} \in J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ for all $(i,\alpha) \in w_{\mathbf{r}}$.

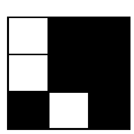
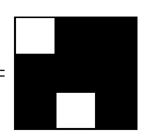
Now we want to prove that $Y_{r_k,k} \notin J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ for all $k \in [\![1,t]\!]$. Assume this is not the case, that is, assume that there exists $k \in [\![1,t]\!]$ with $Y_{r_k,k} \in J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$. Then, $y_{r_k,k} = 0$ and it follows from Proposition 2.12 that $y^{(1,2)}_{r_k,k} = t_{r_k,k} = 0$. Thus, we deduce from Proposition 2.18 that $(r_k,k) \in w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$.
Observe now that, since $k \leq t$, $(r_k,k) \notin [\![1,n]\!] \times [\![t+1,n]\!]$. Further, it is obvious that $(r_k,k) \notin \bigcup_{\alpha\in[\![1,t]\!]} [\![1,r_\alpha - 1]\!] \times \{\alpha\}$. Hence, $(r_k,k) \notin w_{\mathbf{r}}$.
All this together shows that $(r_k,k) \in w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)} \setminus w_{\mathbf{r}} = \bigcup_{l\in[\![1,n]\!]} \{l\} \times [\![1,\gamma_l]\!]$, so that $k \leq \gamma_{r_k}$.
However, since $(\gamma_1,\dots,\gamma_n) \in \Gamma_{\mathbf{r}}$, we have $\gamma_{r_k} \leq k-1$. This is a contradiction and thus we have proved that $Y_{r_k,k} \notin J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ for all $k \in [\![1,t]\!]$. ■

Let us now give an example for the elements $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$ ($(\gamma_1,\gamma_2,\gamma_3) \in \Gamma_{\mathbf{r}}$) of Theorem 2.21. If $n=3$, $t=2$ and $\mathbf{r}=(1,3)$, we have already note that

$$\Gamma_{\mathbf{r}} = \{(\gamma_1,\gamma_2,\gamma_3) \in \mathbb{N}^3 \mid \gamma_1 = 0,\ \gamma_2 \leq 1 \text{ and } \gamma_3 \leq 1\},$$

so that the elements $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$ ($(\gamma_1,\gamma_2,\gamma_3) \in \Gamma_{\mathbf{r}}$) of Theorem 2.21 are:

$w_{(1,3),(0,0,0)} = w_{(1,3)} =$  $w_{(1,3),(0,1,0)} =$ 

$w_{(1,3),(0,0,1)} =$  $w_{(1,3),(0,1,1)} =$ 

(As previously, if $w \in W$, the black boxes symbolize the elements of w.)

3 Number of rank t $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

In this paragraph, using the previous section, we begin by constructing $\mathcal{H}$-invariant prime ideals of the algebra $R_{\mathbf{r}}^{+} := \frac{O_q(\mathcal{M}_n(\mathbb{K}))}{\langle Y_{i,\alpha} \mid \alpha > t \text{ or } i < r_\alpha \rangle}\left[\overline{Y}_{r_1,1}^{-1}, \ldots, \overline{Y}_{r_t,t}^{-1}\right]$, where $t \in [\![0,n]\!]$ and $\mathbf{r} = (r_1, \ldots, r_t)$ is a strictly increasing sequence of integers in the range $1, \ldots, n$. Next, following the route sketched in the introduction, we establish our main result: the number $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid$ of $\mathcal{H}$-invariant prime ideals of $R = O_q(\mathcal{M}_n(\mathbb{K}))$ which contain all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors is equal to $(t!)^2 S(n+1,t+1)^2$, where $S(n+1,t+1)$ denotes the Stirling number of second kind associated to $n+1$ and $t+1$. From this result, we derive a description of the set of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$.

3.1 $\mathcal{H}$-invariant prime ideals in $R_{\mathbf{r},0}^{+}$.

Throughout this section, we fix $t \in [\![0,n]\!]$ and $\mathbf{r} = (r_1, \ldots, r_t) \in \mathbf{R}_t$, and we define $w_{\mathbf{r}}$ as in the previous section.

As in [5, 2.1], we set $R_{\mathbf{r},0}^{+} = \frac{R}{\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle}$.

Recall (See [5], 2.1) that $R_{\mathbf{r},0}^{+}$ can be written as an iterated Ore extension over $\mathbb{K}$. Thus, $R_{\mathbf{r},0}^{+}$ is a Noetherian domain. Moreover, since q is not a root of unity, it follows from [7, Theorem 3.2] that all primes of R are completely prime and thus, since this property survives in factors, all primes in the algebra $R_{\mathbf{r},0}^{+}$ are completely prime.

Observe now that, since the indeterminates $Y_{i,\alpha}$ are $\mathcal{H}$-eigenvectors, $\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle$ is an $\mathcal{H}$-invariant ideal of R. Hence, the action of $\mathcal{H}$ on R induces an action of $\mathcal{H}$ on $R_{\mathbf{r},0}^{+}$ by automorphisms. As usually, an **$\mathcal{H}$-eigenvector** x of $R_{\mathbf{r},0}^{+}$ is a nonzero element $x \in R_{\mathbf{r},0}^{+}$ such that $h(x) \in \mathbb{K}^* x$ for each $h \in \mathcal{H}$, and an ideal I of $R_{\mathbf{r},0}^{+}$ is said to be **$\mathcal{H}$-invariant** if $h(I) = I$ for all $h \in \mathcal{H}$. Further, we denote by $\underline{\mathcal{H}\text{-}Spec(R_{\mathbf{r},0}^{+})}$ the set of $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r},0}^{+}$.

Notations 3.1

- *We denote by $\pi_{\mathbf{r},0}^{+} : R \to R_{\mathbf{r},0}^{+}$ the canonical surjective $\mathbb{K}$-algebra homomorphism.*
- *If $(i,\alpha) \in [\![1,n]\!]^2$, $\overline{Y}_{i,\alpha}$ denotes the element of $R_{\mathbf{r},0}^{+}$ defined by $\overline{Y}_{i,\alpha} := \pi_{\mathbf{r},0}^{+}(Y_{i,\alpha})$.*

Let $(\gamma_1, \ldots, \gamma_n) \in \Gamma_{\mathbf{r}}$ (See Notation 2.20) and define $w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}$ as in Theorem 2.21. Recall (See Theorem 2.21) that $w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}$ is an element of W and that the $\mathcal{H}$-invariant prime ideal $J_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}}$ of R contains the indeterminates $Y_{i,\alpha}$ with $(i,\alpha) \in w_{\mathbf{r}}$, so that $\langle Y_{i,\alpha} \mid (i,\alpha) \in w_{\mathbf{r}} \rangle \subseteq J_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}}$. Thus, $\pi_{\mathbf{r},0}^{+}\left(J_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}}\right)$ is a (completely) prime ideal of $R_{\mathbf{r},0}^{+}$. More precisely, we have:

Proposition 3.2
$J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}} := \pi^+_{\mathbf{r},0}\left(J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}\right)$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{\mathbf{r},0}$ which does not contain the $\overline{Y}_{r_k,k}$ ($k \in [\![1,t]\!]$).

Proof : We have already explained that $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ is a (completely) prime ideal of $R^+_{\mathbf{r},0}$. Moreover, since $J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ is $\mathcal{H}$-invariant, it is easy to check that $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ is also $\mathcal{H}$-invariant. Finally, since $J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ does not contain the indeterminates $Y_{r_k,k}$ with $k \in [\![1,t]\!]$ (See Theorem 2.21), $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ does not contain the $\overline{Y}_{r_k,k} = \pi^+_{\mathbf{r},0}(Y_{r_k,k})$ with $k \in [\![1,t]\!]$. ■

3.2 $\mathcal{H}$-invariant prime ideals in $R^+_{\mathbf{r}}$.

As in the previous section, we fix $t \in [\![0,n]\!]$ and $\mathbf{r} = (r_1,\dots,r_t) \in \mathbf{R}_t$. In [5, 2.1], Goodearl and Lenagan have observed that the $\overline{Y}_{r_k,k}$ with $k \in [\![1,t]\!]$ are regular normal elements in $R^+_{\mathbf{r},0}$, so that we can form the Ore localization:

$$R^+_{\mathbf{r}} := R^+_{\mathbf{r},0} S_{\mathbf{r}}^{-1},$$

where $S_{\mathbf{r}}$ denotes the multiplicative system of $R^+_{\mathbf{r},0}$ generated by the $\overline{Y}_{r_k,k}$ with $k \in [\![1,t]\!]$.

In the previous section, we have noted that all the primes of $R^+_{\mathbf{r},0}$ are completely prime. Since this property survives in localization, all the primes of $R^+_{\mathbf{r}}$ are also completely prime.

Observe now that, since the $\overline{Y}_{r_k,k}$ with $k \in [\![1,t]\!]$ are $\mathcal{H}$-eigenvectors of $R^+_{\mathbf{r},0}$, the action of $\mathcal{H}$ on $R^+_{\mathbf{r},0}$ extends to an action of $\mathcal{H}$ on $R^+_{\mathbf{r}}$ by automorphisms. We say that an ideal I of $R^+_{\mathbf{r}}$ is **$\mathcal{H}$-invariant** if $h(I) = I$ for all $h \in \mathcal{H}$ and we denote by $\mathcal{H}$-*Spec*$(R^+_{\mathbf{r}})$ the set of $\mathcal{H}$-invariant prime ideals of $R^+_{\mathbf{r}}$. Observe now that contraction and extension provide inverse bijections between the set $\mathcal{H}$-*Spec*$(R^+_{\mathbf{r}})$ and the set of those $\mathcal{H}$-invariant prime ideals of $R^+_{\mathbf{r},0}$ which are disjoint from $S_{\mathbf{r}}$.

Let $(\gamma_1,\dots,\gamma_n) \in \Gamma_{\mathbf{r}}$ (See Notation 2.20) and define $w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}$ as in Theorem 2.21. By Proposition 3.2, $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}} := \pi^+_{\mathbf{r},0}\left(J_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}\right)$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{\mathbf{r},0}$ which does not contain the $\overline{Y}_{r_k,k}$ ($k \in [\![1,t]\!]$). Since $S_{\mathbf{r}}$ is generated by the $\overline{Y}_{r_k,k}$ ($k \in [\![1,t]\!]$), $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}}$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{\mathbf{r},0}$ which is disjoint from $S_{\mathbf{r}}$. Thus, we have the following statement:

Proposition 3.3
$J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}} S_{\mathbf{r}}^{-1}$ is an $\mathcal{H}$-invariant (completely) prime ideal of $R^+_{\mathbf{r}}$.

We will prove later (See Section 3.4) that the $J^+_{w_{\mathbf{r},(\gamma_1,\dots,\gamma_n)}} S_{\mathbf{r}}^{-1}$ ($(\gamma_1,\dots,\gamma_n) \in \Gamma_{\mathbf{r}}$) are exactly the $\mathcal{H}$-invariant prime ideals of $R^+_{\mathbf{r}}$.

We deduce from the above Proposition 3.3 that:

Corollary 3.4
$R^+_{\mathbf{r}}$ has at least $1^{r_1} 2^{r_2-r_1} \dots t^{r_t-r_{t-1}} (t+1)^{n-r_t}$ $\mathcal{H}$-invariant prime ideals.

Proof : It follows from Proposition 3.3 that $R_{\mathbf{r}}^{+}$ has at least $\mid \Gamma_{\mathbf{r}} \mid$ $\mathcal{H}$-invariant prime ideals, and it is obvious that $\mid \Gamma_{\mathbf{r}} \mid = 1^{r_1} 2^{r_2 - r_1} \dots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$. ■

3.3 Number of rank t $\mathcal{H}$-invariant prime ideals in $O_q(\mathcal{M}_n(\mathbb{K}))$.

For convenience, we recall the following definitions (See [14]):

Definitions 3.5

- *Let m be a positive integer and let $M = (x_{i,\alpha})_{(i,\alpha) \in [\![1,m]\!]^2}$ be a square q-quantum matrix. The quantum determinant of M is defined by:*

$$det_q(M) := \sum_{\sigma \in S_m} (-q)^{l(\sigma)} x_{1,\sigma(1)} \dots x_{m,\sigma(m)},$$

  where S_m denotes the group of permutations of $[\![1,m]\!]$ and $l(\sigma)$ denotes the length of the m-permuation σ.

- *Let $\mathcal{Y} := (Y_{i,\alpha})_{(i,\alpha) \in [\![1,n]\!]^2}$ be the q-quantum matrix of the canonical generators of R. The quantum determinant of a square sub-matrix of $\mathcal{Y}$ is called a quantum minor.*

We can now define the rank t $\mathcal{H}$-invariant prime ideals of R, as follows:

Definition 3.6
Let $t \in [\![0,n]\!]$. An $\mathcal{H}$-invariant prime ideal J of $R = O_q(\mathcal{M}_n(\mathbb{K}))$ has rank t if J contains all $(t+1) \times (t+1)$ quantum minors but not all $t \times t$ quantum minors.
As in [5, 3.6], we denote by $\mathcal{H}$-$Spec^{[t]}(R)$ the set of rank t $\mathcal{H}$-invariant prime ideals of R.

Note that there is only one element in $\mathcal{H}$-$Spec^{[0]}(R)$: $\langle Y_{i,\alpha} \mid (i,\alpha) \in [\![1,n]\!]^2 \rangle$, the augmentation ideal of R. Further, Goodearl and Lenagan have observed (See [5], 3.6) that $\mid \mathcal{H}$-$Spec^{[1]}(R) \mid = (2^n - 1)^2$ and $\mid \mathcal{H}$-$Spec^{[n]}(R) \mid = (n!)^2$.

Observation 3.7
The sets $\mathcal{H}$-$Spec^{[t]}(R)$ ($t \in [\![0,n]\!]$) partition the set $\mathcal{H}$-$Spec^{[t]}(R)$.

Proof : Let P be an $\mathcal{H}$-invariant prime ideal of R. Let $t \in [\![0,n]\!]$ be maximal such that P does not contain all $t \times t$ quantum minors. Then P clearly belongs to $\mathcal{H}$-$Spec^{[t]}(R)$. Hence, we have proved that $\mathcal{H}$-$Spec(R) = \bigcup_{t \in [\![0,n]\!]} \mathcal{H}$-$Spec^{[t]}(R)$. Since this union is obviously disjoint, we get $\mathcal{H}$-$Spec(R) = \bigsqcup_{t \in [\![0,n]\!]} \mathcal{H}$-$Spec^{[t]}(R)$, as desired. ■

In [5], the authors have established the following result that will be our starting point to compute the cardinality of $\mathcal{H}$-$Spec^{[t]}(R)$:

Proposition 3.8 (See [5], 3.6)
For all $t \in [\![0, n]\!]$, *we have* $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = \left(\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid \right)^2$.

Before computing $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid$, we first give a lower bound for $\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid$.

Proposition 3.9
For any $t \in [\![0, n]\!]$, *we have*

$$\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid \geq t!S(n+1, t+1),$$

where $S(n+1, t+1)$ *denotes the Stirling number of second kind associated to* $n+1$ *and* $t+1$ *(See, for instance, [15] for the definition of* $S(n+1, t+1)$*).*

<u>*Proof :*</u> First, we deduce from Corollary 3.4 the following inequality:

$$\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid \geq \sum_{\mathbf{r}\in\mathbf{R}_t} 1^{r_1} 2^{r_2-r_1} \dots t^{r_t-r_{t-1}} (t+1)^{n-r_t}. \tag{3}$$

On the other hand, we know (See [15], Exercise 16 p46) that:

$$S(n+1, t+1) = \sum_{a_1+\cdots+a_{t+1}=n+1} 1^{a_1-1} 2^{a_2-1} \dots (t+1)^{a_{t+1}-1}. \tag{4}$$

Observe now that the map $f : \{(a_1, \dots, a_{t+1}) \in (\mathbb{N}^*)^{t+1} \mid a_1+\cdots+a_{t+1} = n+1\} \to \{(r_1, \dots, r_t) \in (\mathbb{N}^*)^t \mid 1 \leq r_1 < \cdots < r_t \leq n\} = \mathbf{R}_t$ defined by $f(a_1, \dots, a_{t+1}) = (a_1, a_1 + a_2, \dots, a_1 + \cdots + a_t)$ is a bijection and that its inverse f^{-1} is defined by $f^{-1}(r_1, \dots, r_t) = (r_1, r_2 - r_1, \dots, r_t - r_{t-1}, n + 1 - r_t)$ for all $(r_1, \dots, r_t) \in \mathbf{R}_t$. Thus, by means of the change of variables $(a_1, \dots, a_{t+1}) = f^{-1}(r_1, \dots, r_t)$, the above equality (4) is transformed to

$$S(n+1, t+1) = \sum_{1\leq r_1<\cdots<r_t\leq n} 1^{r_1-1} 2^{r_2-r_1-1} \dots t^{r_t-r_{t-1}-1} (t+1)^{n-r_t},$$

so that

$$t!S(n+1, t+1) = \sum_{(r_1,\dots,r_t)\in\mathbf{R}_t} 1^{r_1} 2^{r_2-r_1} \dots t^{r_t-r_{t-1}} (t+1)^{n-r_t}.$$

Thus, we deduce from inequality (3) that:

$$\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid \geq t!S(n+1, t+1),$$

as desired. ■

Remark 3.10
The proof of the above Proposition 3.9 shows that, if there exists $t \in [\![0, n]\!]$ *and* $\mathbf{r} = (r_1, \dots, r_t) \in \mathbf{R}_t$ *such that* $\mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid > 1^{r_1} 2^{r_2-r_1} \dots t^{r_t-r_{t-1}} (t+1)^{n-r_t}$, *then*

$$\sum_{\mathbf{r}\in\mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R^+_{\mathbf{r}}) \mid > t!S(n+1, t+1).$$

We can now prove our main result which was conjectured by Goodearl, Lenagan and McCammond:

Theorem 3.11
If $t \in [\![0,n]\!]$, then $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = (t!S(n+1,t+1))^2$.

Proof : First, since the sets $\mathcal{H}\text{-}Spec^{[t]}(R)$ ($t \in [\![0,n]\!]$) partition $\mathcal{H}\text{-}Spec(R)$ (See Observation 3.7), we have :

$$\mid \mathcal{H}\text{-}Spec(R) \mid = \sum_{t=0}^{n} \mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid .$$

Recall now (See Proposition 2.7) that $\mid \mathcal{H}\text{-}Spec(R) \mid$ is equal to the poly-Bernoulli number $B_n^{(-n)}$. Thus, we deduce from the above equality that:

$$B_n^{(-n)} = \sum_{t=0}^{n} \mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid .$$

Further, by [1, Theorem 2], $B_n^{(-n)}$ can also be written as follows:

$$B_n^{(-n)} = \sum_{t=0}^{n} (t!S(n+1,t+1))^2 .$$

Hence, we have:

$$\sum_{t=0}^{n} \mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = \sum_{t=0}^{n} (t!S(n+1,t+1))^2 ,$$

that is:

$$\sum_{t=0}^{n} \left(\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid \; - (t!S(n+1,t+1))^2 \right) = 0. \tag{5}$$

On the other hand, recall (See [5], 3.6) that $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = \left(\sum_{\mathbf{r} \in \mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid \right)^2$.
Thus, since $\sum_{\mathbf{r} \in \mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid \geq t!S(n+1,t+1)$ (See Proposition 3.9), we have:

$$\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid \geq (t!S(n+1,t+1))^2 .$$

In other words, each of the terms which appears in the sum on the left hand side of (5) is non-negative. Since this sum is equal to zero, each term of this sum must be zero, that is, for all $t \in [\![0,n]\!]$, we have:

$$\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = (t!S(n+1,t+1))^2 . \blacksquare$$

Remark 3.12
The cases $t=0$, $t=1$ and $t=n$ were already known (See [5], 3.6).

3.4 Description of the set $\mathcal{H}$-$Spec(R_{\mathbf{r}}^{+})$.

Throughout this section, we fix $t \in [\![0, n]\!]$ and $\mathbf{r} = (r_1, \ldots, r_t) \in \mathbf{R}_t$. We now use the above Theorem 3.11 to obtain a description of the set $\mathcal{H}$-$Spec(R_{\mathbf{r}}^{+})$. More precisely, we show that the only $\mathcal{H}$-invariant prime ideals of $R_{\mathbf{r}}^{+}$ are those obtained in Proposition 3.3, that is, in the notations of Section 3.2:

Theorem 3.13

$$\mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) = \{J^{+}_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}} S_{\mathbf{r}}^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Proof : We already know (See Proposition 3.3) that

$$\mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \supseteq \{J^{+}_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}} S_{\mathbf{r}}^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Assume now that

$$\mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \supsetneqq \{J^{+}_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}} S_{\mathbf{r}}^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_{\mathbf{r}}\}.$$

Then we have $\mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid > \mid \Gamma_{\mathbf{r}} \mid$. Since $\mid \Gamma_{\mathbf{r}} \mid = 1^{r_1} 2^{r_2 - r_1} \ldots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$, we get $\mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid > 1^{r_1} 2^{r_2 - r_1} \ldots t^{r_t - r_{t-1}} (t+1)^{n - r_t}$. Thus, it follows from Remark 3.10 that

$$\sum_{\mathbf{r} \in \mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid > t! S(n+1, t+1).$$

Hence we have

$$\left(\sum_{\mathbf{r} \in \mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid \right)^2 > \left(t! S(n+1, t+1) \right)^2.$$

Recall now (See [5, 3.6]) that

$$\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = \left(\sum_{\mathbf{r} \in \mathbf{R}_t} \mid \mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) \mid \right)^2.$$

All this together shows that $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid > (t! S(n+1, t+1))^2$.

However, it follows from Theorem 3.11 that $\mid \mathcal{H}\text{-}Spec^{[t]}(R) \mid = (t! S(n+1, t+1))^2$. This is a contradiction and thus we have proved that $\mathcal{H}\text{-}Spec(R_{\mathbf{r}}^{+}) = \{J^{+}_{w_{\mathbf{r},(\gamma_1,\ldots,\gamma_n)}} S_{\mathbf{r}}^{-1} \mid (\gamma_1, \ldots, \gamma_n) \in \Gamma_{\mathbf{r}}\}$.
■

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