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USING LIE GROUP INTEGRATORS TO SOLVE TWO DIMENSIONAL VARIATIONAL PROBLEMS WITH SYMMETRY

M. Zadra and E.L. Mansfield *

SMSAS, University of Kent
Canterbury, CT2 7FS, UK

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Abstract. The theory of moving frames has been used successfully to solve one dimensional (1D) variational problems invariant under a Lie group symmetry [7, 8, 9]. Unlike in the 1D case, where Noether's laws give first integrals of the Euler–Lagrange equations, in higher dimensional problems the conservation laws do not enable the exact integration of the Euler–Lagrange system. In this paper we use a moving frame to solve, numerically, a two dimensional (2D) variational problem, invariant under a projective action of $SL(2)$. In order to find a solution to the variational problem, we may solve a related 2D system of linear, first order, coupled ODEs for the moving frame, evolving on $SL(2)$. We demonstrate that Lie group integrators [12] may be used in this context, by showing that such systems are also numerically compatible, up to order 5, that is, the result is independent of the order of integration. This compatibility is a testament to the level of geometry built into the Lie group integrators.

1. Introduction.

One dimensional (1D) variational problems with Lie group symmetries have been solved exactly, by making use of the moving frame theory [7, 8, 9]. The idea behind the method is to define a moving frame for the Lie group action, find a generating set of differential invariants, and then to rewrite the Lagrangian in terms of the generating differential invariants and their derivatives. Using the results of [7, 8, 9], one obtains directly the invariantised Euler–Lagrange (E–L) equations, as well as a set of conservation laws given in terms of the frame. Once the E–L equations are solved for the invariants, the frame can be used to find the solution in terms of the original variables. For a 1D problem, the laws yield algebraic equations for the frame and these can be used to ease the integration problem for the minimising solution. For two dimensional (2D) and higher dimensional problems, the laws do not in general lend themselves to finding exact solutions.

In this paper we reduce the problem of finding the minimiser, to that of solving the Euler–Lagrange equations for the invariants and then the compatible 2D system of ODEs

$$\begin{cases} \rho_x = \mathcal{Q}^x \rho \\ \rho_y = \mathcal{Q}^y \rho \\ \rho(x_0, y_0) = \rho_0 \end{cases} \tag{1}$$

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* Corresponding author: E.L. Mansfield.

for the frame ρ, where G is the Lie group, $\rho \in G$ is the moving frame, $\mathfrak{g}$ is the Lie algebra of G, and $\mathcal{Q}^x, \mathcal{Q}^y \in \mathfrak{g}$ are the so–called curvature matrices. The system (1) is compatible in the sense that $\rho_{xy} = \rho_{yx}$, that is,

$$\frac{d}{dx}\mathcal{Q}^y - \frac{d}{dy}\mathcal{Q}^x - [\mathcal{Q}^x, \mathcal{Q}^y] = 0. \tag{2}$$

The curvature matrices depend on the invariants, which are known as functions of the independent variables as soon as the E–L equations have been solved. We solve (1) by showing that the Magnus expansion–based Lie group integrator for a single such equation, can be applied in a well-defined way, provided the compatibility condition (2) is satisfied, at least to order 5 in the discretised step sizes.

In section 2 we present the basic concepts of the theory of moving frames which we will use in our application. We give necessary conditions to define a moving frame, as well as differential invariants, syzygies and curvature matrices. Examples will be given for a projective action of SL(2) on a Lagrangian. Further details about moving frame theory and its application to the calculus of variations can be found in [7, 8, 9, 16, 17].

Section 3 gives a summary of the main results concerning Lie group integrators, developed for a matrix ODE system evolving on a Lie group (see [1, 4, 12] for surveys on the topic, [6] for numerical software).

We then present the main result of this paper: that Lie group integrators can be used to solve the compatible 2D ODE system (1) at least to order 5. We do this by showing that the numerical integration is independent of the order in which the ODEs are solved. In fact our result shows more, that a set of N pairwise compatible equations of the form $\rho_{x_i} = \mathcal{Q}^i\rho$, $i = 1, \ldots, N$, with $\rho \in G$ and $\mathcal{Q}^i \in \mathfrak{g}$, may be solved numerically using a Lie group integrator, at least to order 5.

Our application is to find the minimiser of a 2D variational problem which is invariant under a projective action of SL(2). Section 4 contains some numerical tests which confirm our theoretical findings.

We conclude with a conjecture, that compatibility of the system (1) implies the numerical compatibility of the Lie group integrator, to all orders.

2. Moving frames.

The aim of this section is to present the basics of the modern moving frames Lie group–based theory, and how it can be applied to solve some problems in the calculus of variations. We show that once the E–L equations are solved for the invariants, then the minimiser can be obtained by solving for the frame.

Given an (open) domain $X \subset \mathbb{R}^p$ of independent variables with coordinates $\mathbf{x} = (x_1, .., x_p)$ and a domain $U \subset \mathbb{R}^q$ of dependent variables with coordinates $\mathbf{u} = (u^1, .., u^q)$, we consider the n–th jet bundle $J^n(X \times U)$. Let $K = (k_1, .., k_p) \in \mathbb{N}^p$ and $|K| = \sum_i k_i$, then we can introduce the following notation for the derivatives of the variables in U with respect to the variables in X:

$$u^\alpha_K = \frac{\partial^{|K|} u^\alpha}{\partial^{k_1} x_1 \cdots \partial^{k_p} x_p} \tag{3}$$

A point $\mathbf{z}$ in $J^n(X \times U)$ has coordinates $\mathbf{z} = (x_1, .., x_p, u^1, .., u^q, \ldots u^1_K, ..)$ where $|K| \leq n$.

Definition 2.1. Given a Lie group G, a *left action* of G on $M = X \times U$ is a map

$$G \times M \to M, \qquad (g, z) \mapsto g \cdot z = \tilde{z}$$

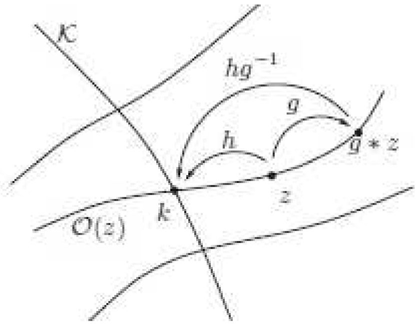

FIGURE 1. Geometric construction of a moving frame. Here g, $h \in G$, $\mathcal{O}$(z) is the orbit through z, and $h \cdot z = k \in \mathcal{K}$.

such that $e \cdot z = z$ where $e \in G$ is the identity element, and

$$g \cdot (h \cdot z) = (gh) \cdot z \qquad \forall h \in G.$$

We now extend the action to $M = J^n(X \times U)$. The operator $\frac{\partial}{\partial x_i}$ extends to the *total differentiation operator* D_i acting on the algebra of the smooth functions on $J^n(X \times U)$ as

$$D_i = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{K} u^{\alpha}_{Ki} \frac{\partial}{u^{\alpha}_{Ki}}$$

In the case of invariant independent variables (which is the case of our main example), an action can be induced on the derivatives of u^α as

$$g \cdot u^{\alpha}_{K} = \widetilde{u^{\alpha}_{K}} = (D_1)^{k_1} \cdots (D_p)^{k_p} \widetilde{u^{\alpha}}$$

For the general case of independent variables that are not invariant, see [17].

Definition 2.2. A moving frame for a given action, $G \times M \to M$, is an equivariant map $\rho : M \to G$. For a right frame and a left action, the equivariance takes the form,

$$\rho(g \cdot z) = \rho(z) g^{-1}$$

We now construct a right moving frame. For any $\mathbf{z} \in \mathcal{U}$, we denote the group orbit passing through z as $\mathcal{O}(z) = \{g \cdot z \,|\, g \in G\}$. We assume the following to be true on some domain $\mathcal{U}$ of M:

1. The group orbits all have the dimension of the group and foliate $\mathcal{U}$,
2. There is a surface $\mathcal{K} \subset \mathcal{U}$ which crosses the group orbits transversally and for which the intersection of a given group orbit with $\mathcal{K}$ is a single point. The surface $\mathcal{K}$ is called the *cross section*,
3. For any $z \in \mathcal{U}$, the group element h taking z to $\{k\} = \mathcal{O}(z) \cap \mathcal{K}$ is unique.

Under these assumptions, the moving frame $\rho : \mathcal{U} \to G$ is defined by $\rho(z) = h$ where h is the unique element of G satisfying $h \cdot z = k$, see Figure 1. The equivariance, $\rho(g \cdot z) = \rho(z) g^{-1}$ follows immediately from assumption 3, as both elements of G take $g \cdot z$ to $\mathcal{K}$.

For our main example we will consider two real independent variables, i.e. $X = \{(x, y) : x, y \in \mathbb{R}\}$ and one real dependent variable, i.e $U = \{u \in \mathbb{R}\}$. We are interested in studying Lagrangians defined on a suitable domain in $J^n(X \times U)$

and which are invariant under the following action of the Lie group $SL(2)$: if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \in SL(2)$, so that $ad - bc = 1$, then

$$g \cdot x = x, \qquad g \cdot y = y, \qquad g \cdot u = \frac{au + b}{cu + d} \tag{4}$$

For the action to be well defined on all of $\mathbb{R}$, it is usual to add a point at infinity, and to define $g \cdot \infty = a/c$ and $g \cdot (-d/c) = \infty$. In order to have conditions 1–3 above, satisfied, we prolong the action to the derivatives of u. In this case we need two more derivatives and we choose to prolong the action on the derivatives in the x–direction. The prolongation of action (4) is

$$\begin{aligned} g \cdot u_x &= \frac{u_x}{(cu + d)^2} \\ g \cdot u_{xx} &= \frac{u_{xx}(cu + d) - 2cu_x^2}{(cu + d)^3} \end{aligned}$$

We follow [8] and consider the cross section $\mathcal{K}$ given by

$$\begin{cases} u = 0 \\ u_x = 1 \\ u_{xx} = 0 \end{cases} \tag{5}$$

Then a moving frame for the action (4), on a domain defined by $u_x > 0$, can be found by solving the system of equations

$$\begin{cases} g \cdot u = 0 \\ g \cdot u_x = 1 \\ g \cdot u_{xx} = 0 \end{cases} \tag{6}$$

for the group element g. The solution to (6) is the moving frame ρ given by

$$\rho = \begin{pmatrix} \frac{1}{\sqrt{u_x}} & -\frac{u}{\sqrt{u_x}} \\ \frac{u_x x}{2u_x^{3/2}} & \frac{2u_x^2 - 2uu_x}{2u_x^{3/2}} \end{pmatrix} \tag{7}$$

A frame on the domain defined by $u_x < 0$ is obtained from the cross–section defined by $u = 0$, $u_x = -1$ and $u_{xx} = 0$.

Moving frames can be used to find the generating differential invariants of the action at hand, as we can see in the following lemma.

Lemma 2.3 ([17])**.** *Given a moving frame and $z \in \mathcal{U}$ we have*

$$\iota(z) = \rho(z) \cdot z$$

is invariant.

Proof. For a right moving frame and a left action we have

$$\iota(g \cdot z) = \rho(g \cdot z) \cdot (g \cdot z) = (\rho(z)g^{-1}) \cdot (g \cdot z) = (\rho(z)g^{-1}g) \cdot z = \rho(z) \cdot z = \iota(z)$$

□

In fact, any invariant can be written in terms of the components of $\iota(z)$. Indeed, if $F(z)$ is an invariant so that $F(z) = F(g \cdot z)$ for all $g \in G$, then setting $g = \rho(z)$ yields $F(z) = F(\rho(z) \cdot z) = F(\iota(z))$. We say that the components of $\iota(z)$ are generating invariants as they generate the algebra of invariant functions.

In [17] it is shown that the generating differential invariants related to the frame (7) are

$$\sigma = \rho \cdot u_{xxx} = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} \qquad \kappa = \rho \cdot u_y = \frac{u_y}{u_x} \tag{8}$$

The differential invariant σ is also known in the literature as *Schwarzian derivative*, $\{u; x\}$. Any Lagrangian which is invariant under the projective action of $SL(2)$ can be written in terms of σ, κ and their derivatives.

2.1. **Curvature Matrix.** In the context of $SL(2)$ acting on surfaces as (4), we can define two *curvature matrices*, one for each independent variable.

Definition 2.4. The curvature matrices for the moving frame (7) are defined as

$$\mathcal{Q}^i = (\mathcal{D}_i \rho)\rho^{-1}, \qquad i = 1, .., p$$

The components of the curvature matrices can be expressed as a function of the generating differential invariants and their derivatives. This is the first part of the next result.

Proposition 1 ([17])**.** *Let $\mathfrak{g}$ be the Lie algebra associated to the Lie group G, then for ρ a smooth moving frame defined on $\mathcal{U} \subset J^n(X \times U)$ and for $i = 1, .., p$:*

1. *The components of $(\mathcal{D}_i \rho)\,\rho^{-1}$ are invariant, and*
2. $(\mathcal{D}_i \rho)\,\rho^{-1} \,:\, J^n(X \times U) \to \mathfrak{g}$

The curvature matrices can be computed knowing only the cross section $\mathcal{K}$ and the group action, without the need of solving for the frame [17]. In the case of action (4) and the equations (6) for the frame, the curvature matrices are

$$\mathcal{Q}^x = \begin{pmatrix} 0 & -1 \\ \frac{1}{2}\sigma & 0 \end{pmatrix} \qquad \mathcal{Q}^y = \begin{pmatrix} -\frac{1}{2}\kappa_x & -\kappa \\ \frac{1}{2}(\kappa_{xx} + \sigma\kappa) & \frac{1}{2}\kappa_x \end{pmatrix}$$

Theorem 2.5 ([17])**.** *The curvature matrices satisfy*

$$\mathcal{D}_j(\mathcal{Q}^i) - \mathcal{D}_i(\mathcal{Q}^j) = ([\mathcal{D}_j, \mathcal{D}_i]\rho)\rho^{-1} + [\mathcal{Q}^j, \mathcal{Q}^i], \qquad i, j = 1, .., p \tag{9}$$

where $[\mathcal{D}_j, \mathcal{D}_i] = \mathcal{D}_j\mathcal{D}_i - \mathcal{D}_i\mathcal{D}_j$.

As we are dealing with standard commutative differential operators, equation (9) simplifies to

$$\mathcal{D}_j(\mathcal{Q}^i) - \mathcal{D}_i(\mathcal{Q}^j) - [\mathcal{Q}^j \mathcal{Q}^i] = 0 \tag{10}$$

The non–zero components of equation (10) yields differential relations, called syzygies, between the generating differential invariants. In [8] it is shown that for the action (4) with invariants σ and κ, there is a single syzygy, namely

$$\sigma_y = \kappa_{xxx} + 2\sigma\kappa_x + \sigma_x\kappa \tag{11}$$

Suppose now we have a variational problem defined by a Lagrangian, which is invariant under a Lie group action of $SL(2)$. It is possible to write the Lagrangian in terms of the generating differential invariants and their derivatives, namely

$$\mathcal{L} = \int L(\sigma, \kappa, \sigma_x, \sigma_y, \kappa_x, ...)\, dx\, dy \tag{12}$$

In order to effect the variation in our variational problems, it is expedient to introduce a new independent and invariant dummy variable τ. This generates a new invariant $\rho \cdot u_\tau = \frac{u_\tau}{u_x} = \delta$ and it is possible to express [8] the derivatives of σ, κ with respect to τ as

$$\frac{\partial}{\partial \tau}\begin{pmatrix}\sigma \\ \kappa\end{pmatrix} = \mathcal{H}\delta = \begin{pmatrix}\mathcal{D}_x^3 + 2\sigma\mathcal{D}_x + \sigma_x \\ \mathcal{D}_y - \kappa\mathcal{D}_x + \kappa_x\end{pmatrix}\delta \tag{13}$$

The operator $\mathcal{H}$ is known as the *syzygy operator* and is always linear. It can be derived using (10) first with respect to the independent variables τ and x and then with respect to τ and y.

Define the E–L operator as

$$E^v(L) = \sum_K (-1)^{|K|}\frac{\partial^{|K|}}{\partial x^{k_1}\partial y^{k_2}}\frac{\partial L}{\partial v_K}$$

then it is proved in [8] that

$$E^u(L) = \mathcal{H}^*\begin{pmatrix}E^\sigma(L) \\ E^\kappa(L)\end{pmatrix} \tag{14}$$

where $\mathcal{H}^*$ is the adjoint of $\mathcal{H}$ in (13). Equation (14) provides us with a second differential relation between the two generating differential invariants. Once we have solved (11)–(14) for σ and κ, we can compute the curvature matrices $\mathcal{Q}^x, \mathcal{Q}^y$. Finally, we have the differential system for ρ,

$$\begin{cases}\rho_x = \mathcal{Q}^x\rho \\ \rho_y = \mathcal{Q}^y\rho \\ \rho(x_0, y_0) = \rho_0\end{cases} \tag{15}$$

where $\rho(x_0, y_0) = \rho_0$ represents an initial condition for the system. The compatibility of this system is guaranteed by Equation (10).

If we can solve the Euler–Lagrange equations for the invariants, so that the curvature matrices are known as functions of the independent variables, and then solve (15) for ρ, we can write down the minimiser, as we now explain. We set

$$\rho(x,y) = \begin{pmatrix}a(x,y) & b(x,y) \\ c(x,y) & d(x,y)\end{pmatrix} \tag{16}$$

and consider the inverse of the moving frame, acting on surfaces as

$$\rho^{-1}\cdot u = \frac{du - b}{-cu + a}. \tag{17}$$

From the first equation in (6), $\rho \cdot u = 0$, if we can solve (15) for ρ, then we obtain that the solution to the variational problem we are studying is given by

$$u(x,y) = \rho^{-1}\cdot 0 = \frac{d(x,y)\cdot 0 - b(x,y)}{-c(x,y)\cdot 0 + a(x,y)} = -\frac{b(x,y)}{a(x,y)} \tag{18}$$

3. **Lie group integrators.** In the previous section we saw how the moving frame (7) is the solution of the compatible system

$$\begin{cases}\rho_x = \mathcal{Q}^x\rho & (19) \\ \rho_y = \mathcal{Q}^y\rho & (20) \\ \rho(x_0, y_0) = \rho_0\end{cases}$$

where the compatibility condition $\frac{d}{dx}\mathcal{Q}^y - \frac{d}{dy}\mathcal{Q}^x - [\mathcal{Q}^x, \mathcal{Q}^y] = 0$ is guaranteed to hold.

Equations (19)–(20) are linear coupled ODEs which evolve on a Lie group. It is possible to solve each of them numerically using numerical schemes developed to solve ODEs on Lie groups: the so–called 'Lie group integrators'. In the following subsection we review some aspects of the theory Lie group integrators. In–depth surveys on this can be found in [1, 4, 12].

3.1. **Matrix ODEs.** As our focus is on a specific $SL(2)$ action, we will assume we are dealing with matrix Lie groups.

Definition 3.1. Suppose we have a Lie group G with Lie algebra $\mathfrak{g}$. An initial value problem on G is the system

$$\begin{cases} Y'(t) = A(t,Y)Y(t) \\ Y(0) = Y_0 \\ t \geq 0 \end{cases} \tag{21}$$

where $Y \in G$ and $A \in \mathfrak{g}$ is an element of the (matrix) Lie algebra associated to G.

To solve the initial value problem (21) it is necessary to extend the exponential function to $M_n(\mathbb{C})$, i.e. the set of $n \times n$ matrices with coefficients in $\mathbb{C}$.

Definition 3.2 ([11])**.** If $A \in \mathfrak{g}$ is a $n \times n$ matrix, the *matrix exponential* is defined as

$$\mathrm{expm} : \mathfrak{g} \to G, \qquad A \mapsto \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{22}$$

It can be proved [11], that if we take a matrix that is sufficiently close to the identity matrix, not only does the series in (22) converge, but it can also be inverted. The inverse function is known as the *matrix logarithm* and denoted as logm. Otherwise, $\mathrm{expm}(A)$ is always convergent (but not in general invertible), if $A \in M_n(\mathbb{C})$.

Definition 3.3 ([12])**.** Suppose $A(t) \in \mathfrak{g}$ is an $n \times n$ matrix. Then the differential of $\mathrm{expm}(A)$, denoted by dexp, is given by

$$\frac{d}{dt}\mathrm{expm}(A(t)) = \mathrm{dexp}_{A(t)}(A'(t))\,\mathrm{expm}(A(t))$$

Consider the function, for fixed $A \in \mathfrak{g}$,

$$\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}, \qquad Y \mapsto [A, Y].$$

Then it can be proved [22], that dexp_A is an analytic function of ad_A, namely

$$\mathrm{dexp}_A(B) = \sum_{i=0}^{\infty} \frac{\mathrm{ad}_A^i B}{(i+1)!} = \frac{\mathrm{expm}(\mathrm{ad}_A(B)) - I}{\mathrm{ad}_A(B)} \tag{23}$$

where I stands for the identity matrix, and we used the notation

$$\mathrm{ad}_A^i B = \underbrace{[A, [A, [A, .., [A}_{i-1\,\text{times}}, [A, B]]]] \quad \text{for } i \in \mathbb{N}$$

We follow [12] and read the ratio in the second equality of (23) in the sense of the power series

$$\frac{e^x - 1}{x} = \sum_{i=0}^{\infty} \frac{x^j}{(j+1)!}$$

where x is replaced by ad_A and exp by expm. As dexp is an analytic function, we can invert it and obtain

$$\mathrm{dexp}_A^{-1} = (\mathrm{ad}_A)\,(\mathrm{expm}(\mathrm{ad}_A) - I)^{-1}$$

This last equation should also be read as a power series, recalling that

$$\frac{x}{\exp(x)-1}=\sum_{i=0}^{\infty}\frac{B_i}{i!}x^i$$

where B_i is the ith–Bernoulli number [5, Eq. 24.2.1]. Hence

$$\operatorname{dexp}_A^{-1}(B)=\sum_{i=0}^{\infty}\frac{B_i}{i!}\operatorname{ad}_A^i(B) \tag{24}$$

We now state the fundamental lemma that lies behind the theory of the Lie group integrators.

Lemma 3.4 ([12])**.** *There exists a $t_0\in\mathbb{R}$ such that the solution of (21) in $[0,t_0]$ is given by*

$$Y(t)=\operatorname{expm}(\Theta(t))Y_0$$

and $\Theta(t)\in\mathfrak{g}$ is the solution of

$$\begin{cases}\Theta(t)'=\operatorname{dexp}_{\Theta(t)}^{-1}(A(t,Y))\\ \Theta(0)=\mathbf{O}\end{cases} \tag{25}$$

3.2. **The Magnus expansion.** We are interested in using a class of numerical methods, that goes under the name of 'Magnus expansion' [13]. This is a particular case of the Runge–Kutta–Munthe–Kaas methods developed in [18],[19],[20] and [21]. We follow [12] and restrict our focus to linear ODEs where $A(t,Y)=A(t)$, which is the case arising in our class of application. In order to solve (25), the method of Picard iterations is used, which relies on the concept of *uniformly Liepschitz continuous function* [10].

Definition 3.5 ([10])**.** A function f: $\mathbb{R}^m\to\mathbb{R}^n$ is said to be *uniformly Liepschitz continuous* if there exists a constant $L\geq 0$, such that, for every $x,y\in\mathbb{R}^m$ it holds

$$||f(x)-f(y)||_{\mathbb{R}^n}\leq L||x-y||_{\mathbb{R}^m}$$

This definition plays a central role in the Picard–Lindelof theorem:

Theorem 3.6 (Picard–Lindelof, [10])**.** *Consider the initial value problem given by*

$$\begin{cases}y'(t)=f(t,y(t))y(t)\\ y(t_0)=y_0\end{cases} \tag{26}$$

If $f(t,y(t))$ is uniformly Liepschitz continuous in y and continuous in t, then there exists $\epsilon>0$ such that there exists a unique solution to (26) on the interval $[t_0-\epsilon,t_0+\epsilon]$. Further, this solution is the limit of the Picard iterations.

As seen in (24), the inverse of dexp can be written as a series involving the ad operator. Applying the Picard iterations to (25) yields

$$\begin{cases}\Theta^{[0]}=\mathbf{O}\\ \Theta^{[m+1]}=\int_0^t\operatorname{dexp}_{\Theta^{[m]}(\xi)}^{-1}A(\xi)\,d\xi=\sum_{i=0}^{\infty}\frac{B_i}{i!}\int_0^t\operatorname{ad}_{\Theta^{[m]}(\xi)}^{i}A(\xi)\,d\xi\end{cases}$$

for $m=0,1,2,...$ As the function $\operatorname{dexp}_{\Theta(t)}^{-1}A(t)$ has no dependence on Y and is assumed to be continuous in t, Picard's theorem can be applied, to yield a unique local solution to (25), namely $\Theta(t) = \lim_{m\to\infty} \Theta^{[m]}(t)$. It can be seen [12], that it is possible to rearrange the terms in Θ as

$$\Theta(t) = \sum_{i=0}^{\infty} H_i(t) \tag{27}$$

where every $H_i(t)$ is made by precisely i commutators and $i+1$ integrals. The expression defined in (27) is called *Magnus expansion*.

3.3. **Magnus expansion and coupled systems of ODEs.** We are interested in applying the theory of Lie group integrators based on the Magnus expansion to solve 2D problems. Let us recall the system we want to solve in order to find the moving frame ρ:

$$\begin{cases} \rho_x = \mathcal{Q}^x \rho & (28)\\ \rho_y = \mathcal{Q}^y \rho & (29)\\ \rho(x_0, y_0) = \rho_0 \end{cases}$$

Recall that $\rho \in G$ and $\mathcal{Q}^x, \mathcal{Q}^y \in \mathfrak{g}$. System (28)–(29) is a system of two linear matrix ODEs to be solved in a suitable domain of $\mathbb{R}^2$ and we want the solution to belong to $SL(2)$ at every point where it is defined. We also recall the compatibility condition (10) for (28)–(29) to have a solution. We denote this condition by R, that is,

$$R = \frac{d}{dx}\mathcal{Q}^y - \frac{d}{dy}\mathcal{Q}^x - [\mathcal{Q}^x, \mathcal{Q}^y] \tag{30}$$

which must be identically zero for the system to be compatible. We apply Lemma (3.4) to equations (28)–(29), obtaining the coupled system of ODEs

$$\begin{cases} \Theta_x(x,y) = \mathrm{dexp}^{-1}_{\Theta(x,y)} \mathcal{Q}^x(x,y) \\ \Theta_y(x,y) = \mathrm{dexp}^{-1}_{\Theta(x,y)} \mathcal{Q}^y(x,y) \end{cases} \tag{31}$$

which is a coupled system of ODEs in $\mathbb{R}^2$. Proceeding in the spirit of the Lie group integrators based on the Magnus expansion, the method of Picard iterations is applied to (31) to yield,

$$\begin{cases} \Theta^H_{[0]} = \mathbf{O} \\ \Theta^V_{[0]} = \mathbf{O} \\ \Theta^H_{[n+1]} = \sum_{i=0}^{\infty} \frac{B_i}{i!} \int_0^t \mathrm{ad}^i_{\Theta^h_{[n]}(\xi,y)} Q^x(\xi,y)\, d\xi \\ \Theta^V_{[n+1]} = \sum_{i=0}^{\infty} \frac{B_i}{i!} \int_0^t \mathrm{ad}^i_{\Theta^v_{[n]}(x,\xi)} Q^y(x,\xi)\, d\xi \end{cases}$$

for $N = 0, 1, 2, \ldots$, where the iterations of Θ^H and Θ^V solve the equation for Θ_x and Θ_y in (31) respectively. We use the superscripts H and V to denote the horizontal and vertical integrations with respect to the standard representation of the (x, y) plane. As in (27), we rearrange terms such that

$$\Theta^H(y) = \sum_{i=0}^{\infty} M^x_i(y) \tag{32}$$

$$\Theta^V(x) = \sum_{i=0}^{\infty} M^y_i(x) \tag{33}$$

where M^x_i, M^y_i both contain exactly i commutators and $i+1$ integrals.

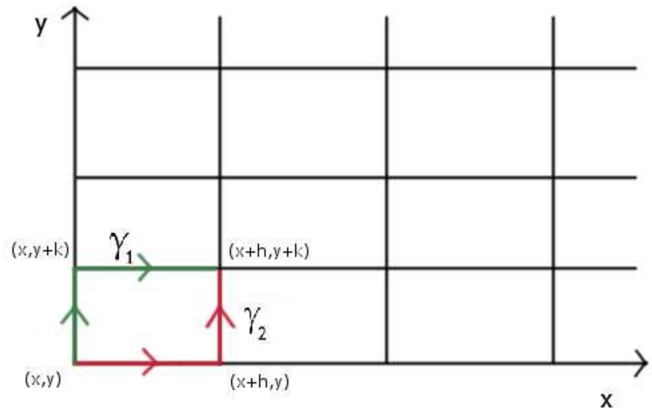

FIGURE 2. The two different paths γ_1,γ_2.

3.4. **Lie group integrators based on the Magnus expansions commute up to order 5.** We now show that the Magnus expansion yields a well defined integration method for a system of the form (28)–(29).

In our calculations, we will make strong use of the Baker–Campbell–Hausdorff (BCH) formula which shows how two matrix exponentials may be multiplied to obtain a single matrix exponential. Although we will use a truncated BCH expansion up to order 5, a recursive formula to determine every term has been proved by Dynkin.

Theorem 3.7 (BCH formula, [15])**.**

$$\log(\mathrm{expm}(X)\mathrm{expm}(Y)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_1+s_1>0 \\ \vdots \\ r_n+s_n>0}} \frac{[X^{r_1}Y^{r_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}]}{\sum_{i=1}^{n}(r_i+s_i)\Pi_{i=1}^{n} r_i! s_i!}$$

where

$$[X^{r_1}Y^{r_1}X^{r_2}Y^{s_2}\cdots X^{r_n}Y^{s_n}]$$
$$= [\underbrace{X,[X,\cdots[X}_{r_1},\underbrace{[Y,[Y,\cdots[Y}_{s_1},\cdots\underbrace{[X,[X,\cdots[X}_{r_n},\underbrace{[Y,[Y,\cdots Y}_{s_n}]]\cdots]$$

Theorem 3.8. *The Magnus expansion yields a well–defined, path–independent integrator for the compatible system* (28)*–*(29)*, to order at least 5.*

Proof. We discretise our domain with constant step sizes h, k for the x–discretisation and y–discretisation respectively. To have a well defined integration method, we need to prove that if we start from the initial data $\rho_0 = \rho(x_0, y_0)$, then we obtain a unique expression for $\rho(x_0 + h, y_0 + k)$, regardless of the order of integration, that is, regardless of whether we integrate first horizontally or vertically.

Let us consider two paths, say γ_1 and γ_2, such that they both start at (x_0, y_0) and end at $(x_0+h, y_0+k) = (x_1, y_1)$, but γ_1 first goes vertically to (x_0, y_1) and then horizontally to (x_1, y_1), while γ_2 travels first horizontally to (x_1, y_0) before going upwards to (x_1, y_1) (see Figure 2). We compute the solution $\rho(x_1, y_1)$ along the two paths, and compare the two results. We call $\rho^{\gamma_1}(x_1, y_1)$ and $\rho^{\gamma_2}(x_1, y_1)$ the solution $\rho(x_1, y_1)$ obtained along γ_1 and γ_2 respectively. To make the calculations tractable, we will approximate the solutions ρ^{γ_1} and ρ^{γ_2} up to order five. However, as we will see in the numerical examples in the next section, it seems reasonable to conjecture that the result holds up to every order.

Using Lemma (3.4) we compute $\rho^{\gamma_1}(x_1, y_1), \rho^{\gamma_2}(x_1, y_1)$ in two steps. First we obtain the solution of

$$\begin{cases} \rho^{\gamma_1}_y = \mathcal{Q}^y \rho \\ \rho^{\gamma_1}(x_0, y_0) = \rho_0 \\ (x, y) \in \{x_0\} \times [y_0, y_1] \end{cases} \qquad \begin{cases} \rho^{\gamma_2}_x = \mathcal{Q}^x \rho \\ \rho^{\gamma_2}(x_0, y_0) = \rho_0 \\ (x, y) \in [x_0, x_1] \times \{y_0\} \end{cases}$$

as

$$\begin{aligned} \rho^{\gamma_1}(x_0, y_1) &= \mathrm{expm}(\Theta^V(x_0))\rho_0 \\ \rho^{\gamma_2}(x_1, y_0) &= \mathrm{expm}(\Theta^H(y_0))\rho_0 \end{aligned}$$

Then the following step is to solve the systems

$$\begin{cases} \rho^{\gamma_1}_x = \mathcal{Q}^x \rho \\ \rho^{\gamma_1}(x_0, y_1) = \mathrm{expm}(\Theta^V(x_0))\rho_0 \\ (x, y) \in [x_0, x_1] \times \{y_1\} \end{cases} \qquad \begin{cases} \rho^{\gamma_2}_y = \mathcal{Q}^y \rho \\ \rho^{\gamma_2}(x_1, y_0) = \mathrm{expm}(\Theta^H(y_0))\rho_0 \\ (x, y) \in \{x_1\} \times [y_0, y_1] \end{cases}$$

and we obtain the two solutions that we want to compare, namely

$$\begin{aligned} \rho^{\gamma_1}(x_1, y_1) &= \mathrm{expm}(\Theta^H(y_1))\mathrm{expm}(\Theta^V(x_0))\rho_0 \\ \rho^{\gamma_2}(x_1, y_1) &= \mathrm{expm}(\Theta^V(x_1))\mathrm{expm}(\Theta^H(y_0))\rho_0 \end{aligned}$$

Therefore, we consider

$$\begin{aligned} \mathrm{logm}(\rho^{\gamma_1}\rho_0^{-1}) - \mathrm{logm}(\rho^{\gamma_2}\rho_0^{-1}) = {} & \mathrm{logm}(\mathrm{expm}(\Theta^H(y_1))\mathrm{expm}(\Theta^V(x_0))) \\ & - \mathrm{logm}(\mathrm{expm}(\Theta^V(x_1))\mathrm{expm}(\Theta^H(y_0))). \end{aligned} \tag{34}$$

We will show that the right hand side is zero to order 5 in h, k.

Suppose now the functions $\Theta^H(y)$ and $\Theta^V(x)$ are continuously differentiable at least 4 times in both the x and y directions. We begin applying the BCH formula to the RHS of (34). We truncate the expansion at order 5, so the terms that are relevant for our result are

$$
\begin{aligned}
\operatorname{logm}(\rho^{\gamma_1}(x_1,y_1)\rho_0^{-1}) &= \operatorname{logm}(\operatorname{expm}(\Theta^H(y_1)\operatorname{expm}(\Theta^V(x_0)))) \qquad (35)\\
&= \Theta^H(y_1)+\Theta^V(x_0)+\frac{1}{2}[\Theta^H(y_1),\Theta^V(x_0)]\\
&+\frac{1}{2}\left([\Theta^H(y_1),[\Theta^H(y_1),\Theta^V(x_0)]]+[\Theta^V(x_0),[\Theta^V(x_0),\Theta^H(y_1)]]\right)\\
&-\frac{1}{24}[\Theta^V(x_0),[\Theta^V(x_0),[\Theta^H(y_1),\Theta^V(x_0)]]]\\
&-\frac{1}{720}[\Theta^V(x_0),[\Theta^V(x_0),[\Theta^V(x_0),[\Theta^V(x_0),\Theta^H(y_1)]]]]\\
&-\frac{1}{720}[\Theta^H(y_1),[\Theta^H(y_1),[\Theta^H(y_1),[\Theta^H(y_1),\Theta^V(x_0)]]]]\\
&+\frac{1}{360}[\Theta^H(y_1),[\Theta^V(x_0),[\Theta^V(x_0),[\Theta^v(x_0),\Theta^H(y_1)]]]]\\
&+\frac{1}{360}[\Theta^V(x_0),[\Theta^H(y_1),[\Theta^H(y_1),[\Theta^H(y_1),\Theta^V(x_0)]]]]\\
&+\frac{1}{120}[\Theta^V(x_0),[\Theta^V(x_0),[\Theta^H(y_1),[\Theta^V(x_0),\Theta^H(y_1)]]]]\\
&+\frac{1}{120}[\Theta^H(y_1),[\Theta^H(y_1),[\Theta^V(x_0),[\Theta^H(y_1),\Theta^V(x_0)]]]]+\text{h.o.t.}
\end{aligned}
$$

where 'h.o.t' stands for higher order terms. The expression for $\operatorname{logm}(\rho^{\gamma_2}(x_1,y_1)\rho_0^{-1})$ is analogous.

We compute $\rho^{\gamma_1}(x_1,y_1)-\rho^{\gamma_2}(x_1,y_1)$ and we expand both Θ^H and Θ^V around (x_0,y_0) as Taylor series. The first thing to note is that the terms of order 0 in h, k disappear.

The terms we need for the Magnus expansion of $\Theta^H(y_0)$ are,

$$
\begin{aligned}
\Theta^H(y_0) &= \int_{x_0}^{x_1}\mathcal{Q}^x(\xi,y_0)\,d\xi-\frac{1}{2}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0)\,d\xi_2,\,\mathcal{Q}^x(\xi_1,y_0)\right]d\xi_1\\
&+\frac{1}{12}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0)\,d\xi_2,\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0),\mathcal{Q}^x(\xi_1,y_0)\right]\right]d\xi_1\\
&+\frac{1}{4}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\left[\mathcal{Q}^x(\xi_3,y_0)\,\xi_3,\,\mathcal{Q}^h(\xi_2,y_0)\right]d\xi_2,\,\mathcal{Q}^x(\xi_1,y_0)\right]d\xi_1\\
&-\frac{1}{24}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\left[\int_{x_0}^{\xi_2}\mathcal{Q}^x(\xi_3,y_0)\,d\xi_3,\left[\int_{x_0}^{\xi_2}\mathcal{Q}^x(\xi_3,y_0)\,d\xi_3,\,\mathcal{Q}^x(\xi_2,y_0)\right]\right]d\xi_2,\right.\\
&\qquad\left.\mathcal{Q}^x(\xi_1,y_0)\right]d\xi_1
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{24}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\left[\int_{x_0}^{\xi_2}\mathcal{Q}^x(\xi_3,y_0)\,d\xi_3,\mathcal{Q}^x(\xi_2,y_0)\right]d\xi_2,\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0)\,d\xi_2,\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.\left.\mathcal{Q}^x(\xi_1,y_0)\right]\right]d\xi_1\\
&- \frac{1}{24}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0),\left[\int_{x_0}^{\xi_1}\left[\int_{x_0}^{\xi_2}\mathcal{Q}^x(\xi_3,y_0)\,d\xi_3,\mathcal{Q}^x(\xi_2,y_0)\right]d\xi_2,\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\left.\left.\mathcal{Q}^x(\xi_1,y_0)\right]\right]d\xi_1\\
&- \frac{1}{8}\int_{x_0}^{x_1}\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0),\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0)\,d\xi_2,\left[\int_{x_0}^{\xi_1}\mathcal{Q}^x(\xi_2,y_0),\mathcal{Q}^x(\xi_1,y_0)\right]\right]\right]d\xi_1
\end{aligned}
$$

The expression for $\Theta^V(x_0)$ is analogous.

Using these to expand the right hand side of (34) around (x_0, y_0) as a Taylor polynomial up to order 5 in h, k, it becomes trivial to integrate the internal expressions as they are polynomial in ξ, ξ_1 and ξ_2. We obtain a polynomial expression in h and k, whose coefficients depend on $\mathcal{Q}^x$ and its partial derivatives at (x_0, y_0). The final step is to simplify with respect to the compatibility expression R defined in (30) and its partial derivatives (evaluated at the arbitrary point (x_0, y_0)). We summarise the result in the table below, noting that the coefficient of $h^n k^m$ can be obtained from that of $h^m k^n$ by interchanging x and y. It can be seen that every coefficient is a linear expression in R which is identically zero, and hence the right hand side of (34) is zero. This ends the proof.

Order	Monomial	Coefficient
2	hk	R
3	h^2k	$\frac{1}{2}\mathcal{D}_x R$
4	h^3k	$\frac{1}{6}\mathcal{D}_x^2 R - \frac{1}{12}\operatorname{ad}_{\mathcal{Q}^x}(\mathcal{D}_x R) + \frac{1}{12}\operatorname{ad}_{\mathcal{D}_x\mathcal{Q}^x}(R)$
	h^2k^2	$\frac{1}{4}\mathcal{D}_x\mathcal{D}_y R - \frac{1}{6}\operatorname{ad}_{\operatorname{ad}_{\mathcal{Q}^x}(\mathcal{Q}^y)}(R) - \frac{1}{12}\operatorname{ad}_{\mathcal{Q}^y}(\operatorname{ad}_{\mathcal{Q}^x}(R))$
5	h^4k	$\frac{1}{24}\mathcal{D}_x^3 R - \frac{1}{24}\operatorname{ad}_{\mathcal{Q}^x}(\mathcal{D}_x^2 R) + \frac{1}{24}\operatorname{ad}_{\mathcal{D}_x^2\mathcal{Q}^x}(R)$
	h^3k^2	$\frac{1}{12}\mathcal{D}_x^2\mathcal{D}_y R - \frac{1}{24}\operatorname{ad}_{\mathcal{Q}^x}(\mathcal{D}_x\mathcal{D}_y R) - \frac{1}{24}\operatorname{ad}_{\mathcal{D}_x\mathcal{Q}^x}(\mathcal{D}_y R)$ $-\frac{1}{24}\operatorname{ad}_{\mathcal{Q}^x}(\operatorname{ad}_{\mathcal{Q}^y}(\mathcal{D}_x R)) - \frac{1}{12}\operatorname{ad}_R(\mathcal{D}_x R) + \frac{1}{2}\operatorname{ad}_{\mathcal{D}_x\mathcal{Q}^y}(\mathcal{D}_x R)$ $+\frac{1}{6}\operatorname{ad}_{\mathcal{D}_x^2\mathcal{Q}^y}(R) - \frac{1}{24}\operatorname{ad}_{\mathcal{Q}^y}(\operatorname{ad}_{\mathcal{D}_y\mathcal{Q}^x}(R)) - \frac{1}{24}\operatorname{ad}_{\mathcal{D}_x\mathcal{Q}^y}(\operatorname{ad}_{\mathcal{Q}^x}(R))$ $+\frac{1}{8}\operatorname{ad}_{[\mathcal{D}_x\mathcal{Q}^y,\mathcal{Q}^x]}(R) + \frac{1}{8}\operatorname{ad}_{[\mathcal{Q}^y,\mathcal{D}_x\mathcal{Q}^x]}(R)$

□

4. **Numerical examples.**

We showed in the previous section that the Lie group integrators based on the Magnus expansion commute at least up to order 5, and we now show some numerical examples. These not only confirm the result, but hint that the Lie group integrators based on the Magnus expansion will commute for every order. We consider two variational problems and, in order to solve the system of coupled matrix ODEs for the frame, we use a sixth-order Magnus series method which is included in the Matlab package *DiffMan* ([6], Algorithm A.2.5). This note this algorithm is cost efficient [2, 3, 14], which means that not all the terms in the Magnus expansion are used in the calculations. This does not appear to affect the numerical compatibility that we demonstrate here.

Recall we are interested in the Lie group $SL(2)$, given by

$$SL(2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}$$

and we let it act projectively on surfaces as

$$g \cdot x = x, \qquad g \cdot y = y, \qquad g \cdot u = \frac{au + b}{cu + d} \tag{36}$$

Given the frame ρ as in (7), the generating differential invariants are:

$$\begin{aligned} \kappa(x, y) &= \rho \cdot u_y = \frac{u_y}{u_x} \\ \sigma(x, y) &= \rho \cdot u_{xxx} = \frac{u_{xxx}}{u_x} - \frac{2u_{xx}^2}{2u_x^2}. \end{aligned}$$

The two curvature matrices are

$$\mathcal{Q}^x = \begin{pmatrix} 0 & -1 \\ \frac{1}{2}\sigma & 0 \end{pmatrix} \quad \mathcal{Q}^y = \begin{pmatrix} -\frac{1}{2}\kappa_x & -\kappa \\ \frac{1}{2}\left(\kappa_{xx} + \sigma\kappa\right) & \frac{1}{2}\kappa_x \end{pmatrix}$$

and to obtain the moving frame ρ, we need to solve the 2D system

$$\begin{cases} \rho_x = \mathcal{Q}^x \rho \\ \rho_y = \mathcal{Q}^y \rho \\ \rho(x_0, y_0) = \rho_0 \\ (x, y) \in [x_0, x_n] \times [y_0, y_n] \end{cases} \tag{37}$$

In the following we will numerically solve some variational problems using two different methods:

1 integrating first **vertically** along the line $x = x_0$, and then, for $j = 0, .., n$, use the points $\rho(x_j, y_0)$ as initial condition for the solution found integrating horizontally along the line $y = y_j$.

2 integrating first **horizontally** along the line $y = y_0$, and then, for $j = 0, .., n$, use the points $\rho(x_0, y_j)$ as initial condition for the solution found integrating vertically along the line $x = x_j$.

and we will compare the solutions obtained. Finally, we use (18) to plot the minimiser, given the frame.

4.0.1. *Example 1.* Consider the Lagrangian given by

$$\mathcal{L} = \int_D \kappa^2(x, y) \quad dx\, dy \tag{38}$$

where D is the square $[6, 7] \times [6, 7]$ and we choose a step size equal in both directions $h = k = 0.05$. The E–L equation in (14) becomes

$$\kappa_y = 3\kappa\kappa_x$$

and if we add a boundary condition as $\kappa(x, 1) = x$, then a solution is

$$\kappa(x, y) = -\frac{x}{3y - 4} \tag{39}$$

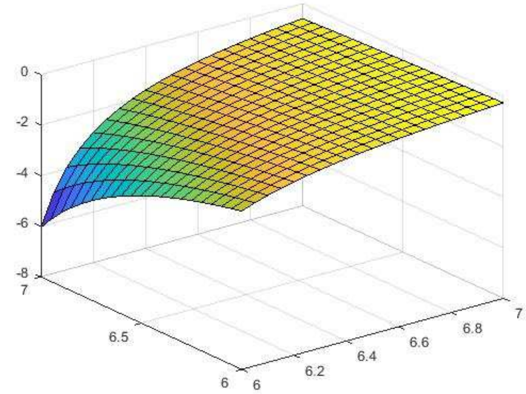

FIGURE 3. Plots of solutions to the variational problem defined by (38), computed integrating the two different ways; the plots look identical to the naked eye.

Setting κ into the sygyzy equation in (11), we obtain an equation for σ,

$$\sigma_y = -2\frac{\sigma}{(3y-4)} - \frac{x\sigma_x}{(3y-4)}$$

and if we impose that $\sigma(1,y) = y$, we obtain the solution

$$\sigma(x,y) = \frac{4x^3 + 3y - 4}{3x^5} \tag{40}$$

Inserting (39) and (40) into (37), adding an initial condition

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and integrating as we described using the two methods above, we obtain two surfaces, identical to the naked eye, shown in Figure 3. A plot of the absolute difference between the two surfaces is shown in Figure 4. We can see in this case, that the point–wise difference of the two surfaces plotted in Figure 3 is of order at least 7 in h, k.

4.0.2. *Example 2.* Consider the Lagrangian given by

$$\mathcal{L} = \int_D \sigma_x^2(x,y) \quad dx\,dy \tag{41}$$

where D is the square $[2,3]\times[1,2]$ and we choose a step size equal in both directions $h = k = 0.05$. The E–L equation in (14) becomes

$$\sigma_{xxxxx} + 2\sigma\sigma_{xxx} + \sigma_x\sigma_{xx} = 0$$

and we notice that all the terms in the differential equation above involve at least the second derivative in x. So a solution is

$$\sigma(x,y) = x - y \tag{42}$$

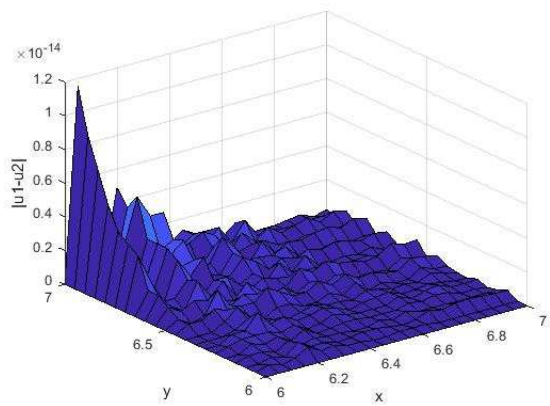

Figure 4. Absolute value of the difference between the two surfaces in Figure 3.

Now we can substitute the expression for σ into the sygyzy equation (11), obtaining an equation for κ

$$\kappa_{xxx} + (2x - 2y)\kappa_x + \kappa + 1 = 0 \tag{43}$$

and if we impose that

$$\begin{cases} \kappa(0,y) = y \\ \kappa_x(0,y) = 0 \\ \kappa_{xx}(0,y) = \frac{1}{y} \end{cases} \tag{44}$$

we obtain a solution in terms of the Airy functions of first and second kind (and their first derivative). Inserting (42) and the solution to (43)–(44) into (37), adding an initial condition

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and integrating as we described in 1–2 above, we obtain the two surfaces shown in Figure 5. A plot of the absolute difference between the two surfaces is given in Figure 6. In this example we obtain that the difference between the two surfaces is of order greater than 5.

5. **Conclusion.** In this paper, we have shown that Lie group integrators can be used to solve, numerically, the system of equations for a moving frame, (1), which evolves on a Lie group, in the case where the base space has two dimensions. In fact, our result extends immediately to the system of equations for a moving frame on an N-dimensional base space, as these equations are pairwise compatible. We have applied our result to find a minimiser for a variational problem which is invariant under the projective action of $SL(2)$. Our method can, in principle, be applied to any variational problem with a Lie group symmetry which can be described and analysed using a Lie group based moving frame.

Cost efficient Lie group integrators [2, 3, 14] reduce the number of commutators involved in the numerical computation, and the implementation we have used, [6],

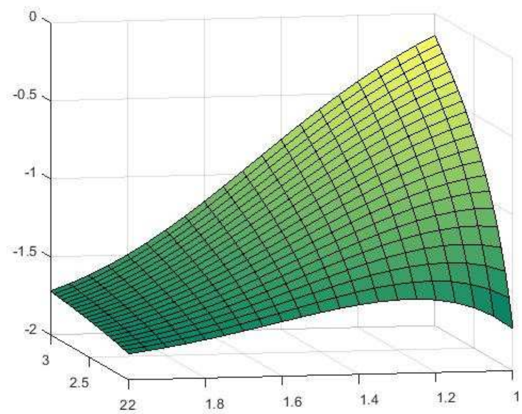

FIGURE 5. Plots of solutions to the variational problem defined by (41), computed integrating the two different ways; the plots look identical to the naked eye.

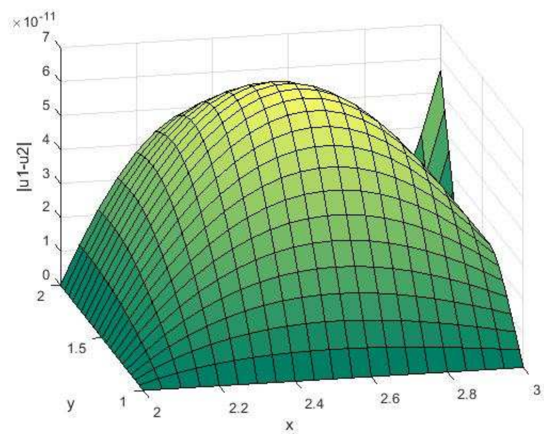

FIGURE 6. Absolute value of the difference between the two surfaces in Figure 5.

takes advantage of these ideas. The interplay between compatibility and efficiency is a topic for further study.

While we have shown that the Lie group integrators are compatible to order 5, it is clear that our proof of the compatibility (34) could have continued to higher and higher orders, and our numerical results demonstrate this. However, the calculations become less and less tractable, and there is no clear, discernible, recursive pattern. Hence the general result is likely to need a different approach. We conclude by stating the general result as a conjecture.

Conjecture 1. *Lie group integrators are compatible to all orders, that is, the right–hand side of* (34) *is identically zero to all orders of* h, k.

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E-mail address: E.L.Mansfield@kent.ac.uk
E-mail address: mz233@kent.ac.uk