Citation for published version

Luo, Ming and Wu, Shaomin (2018) A comprehensive analysis of warranty claims and optimal policies. European Journal of Operational Research . ISSN 0377-2217.

DOI

https://doi.org/10.1016/j.ejor.2018.12.034

Link to record in KAR

Document Version

Author's Accepted Manuscript

A comprehensive analysis of warranty claims and optimal policies

Ming Luo

Huddersfield Business School, University of Huddersfield,
Huddersfield HD1 3DH, United Kingdom

Shaomin Wu [1]

Kent Business School, University of Kent,
Canterbury, Kent CT2 7FS, United Kingdom

Abstract

Nowadays many products, such as 3C products (Computer, Communication and Consumer Electronics) and cars, consist of software and hardware. The causes of warranty claims of such products may be attributed to software specific failures, hardware specific failures, software-hardware interaction failures and human errors. Apparently, those causes may be dependent. For example, one may claim warranty due to the malfunction of the embedded software in a product item and then the entire item may be replaced. Nevertheless, the existing research on warranty management studies mainly concentrates on warranty analysis of hardware subsystems, assuming that the warranty claims are statistically independent of those caused by the failures of software subsystems or human factors, that is, the interactions between those causes are neglected.

This paper investigates warranty costs incurred due to those three subsystems with a focus on their interactions. It estimates the costs due to different cause, develops integrated warranty cost models and optimises warranty policies considering the above possible combinations. Numerical examples are given to illustrate the proposed models.

Keywords: Manufacturing, Warranty, compound Poisson process, human factor, integrated warranty model.

---

[1] Corresponding author. Email: s.m.wu@kent.ac.uk. Telephone: 0044 1227 827940.

Suggested citation: M. Luo, S. Wu, A comprehensive analysis of warranty claims and optimal policies, European Journal of Operational Research, DOI: 10.1016/j.ejor.2018.12.034

1 Introduction

A warranty is a contractual obligation incurred by a manufacturer in connection with the sale of a product. In broad terms, the purpose of warranty is to establish liability in the event of a premature failure of an item or the inability of the item to perform its intended function (Blischke & Murthy, 1992). For products sold with warranty, manufacturers bear additional cost incurred due to warranty servicing. Such cost, often referred to as warranty servicing cost, is generally substantial. For example, according to WarrantyWeek (2016), Apple paid $1.25 billion or more for warranty claims during a single quarter in the second half of 2015; and during the same time period, HP Inc. paid around $300 million per quarter. Therefore, accurately estimating warranty cost is indispensable to the manufacturers.

Nowadays, many products, such as 3C products (Computer, Communication and Consumer Electronics) and cars consist of two subsystems, hardware and embedded software. The designed functions of the products are performed based on the reliable collaboration of their hardware and software subsystems. That is, a hardware failure or/and a software failure may cause a warranty claim. For example, Fig 1 shows four warranty claims of four air-conditioners, which are four real cases collected from a Chinese air-conditioner manufacturer. The warranty claim No 3, as interpreted in English in Fig 2, is due to the failure of the control software, which is embedded in the control board. As a result, the entire control board is replaced.

NO	Date	Customer	Barcode	Reason of claim	Repair
1	08/12/ 2014	王XX	8603005xxxxxxx	水箱连接截止阀出油，确认为水箱截止阀有漏点，而且是水箱内部	更换水箱
2	04/12/ 2014	姚XX	8206003xxxxxxx	客户想机组退厂，翻喷钣金	更换物料
3	12/12/ 2014	胡XX	8202004xxxxxxx	主机程序有问题，设定好温度调停后，还要手动按开关才能开机，设定还会自动变	更换主板、线控器
4	09/12/ 2014	谢XX		天御款采暖机未设计冷凝	补冷凝电加热带

Fig. 1. Four warranty claims, in which No 3 is due to a software failure and its host hardware is replaced.

It should also be noted that warranty claims are not always triggered by the failures of product items, some users' behaviours (human factors) may also contribute warranty claims (Wu, 2011).

NO	Date	Customer	Barcode	Reason of claim	Repair
1	08/12 /2014	XX Wang	8603005xxxxxxx	Oil leakage from the slam-shut valve in the water tank	Replace the water tank
2	04/12 /2014	XX Yao	8206003xxxxxxx	The customer wanted to return the entire air-conditioner for repainting	Replace the shell
3	12/12 /2014	XX Hu	8202004xxxxxxx	Something is wrong with the control software. Once the temperature is manually set, the machine does not switch on automatically and a pre-specified temperature value may swift	Replace the control board
4	09/12 /2014	XX Xie		The air-conditioner is not equipped with a cooling component	Install a cooling component

Fig. 2. Figure 1 in English

1.1 *Related work*

1.1.1 *Warranty policy optimisation*

In the literature, many methods aiming to optimise the warranty price and the warranty length of an individual product have been proposed. Fig. 3 illustrates the evolution of the research in warranty policy optimisation, which shows that the research evolves from simple and unrealistic assumptions to more complex and realistic ones.

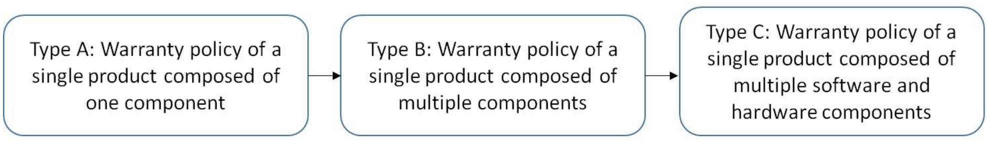


Fig. 3. Evolution of warranty policy optimisation

At the early stage, many researchers attempt to find the optimal price and warranty length, assuming that the product is composed of only one component. Some other factors, such as production rate, market competition and demand, etc., may also be considered. Ladany and Shore (2007) address a method to determine the optimal warranty period with considering the products lifetime and market demand. Lin, Wang, and Chin (2009) optimise the price, warranty length and production rate of a one component system dynamically. Wu, Chou, and Huang (2009) develop a decision model to determine the optimal price, the length of warranty and the production rate to maximise profit based on the pre-determined life cycle in a static demand market. Dai, Zhou, and Xu (2012) indicate warranty costs are incurred by both the supplier and the manufacturer, and provide the structural properties of the equilibrium strategies with considering warranty length in warranty management. Ding, Rusmevichientong, and Topaloglu (2014) investigate the relationship between the sales revenues and the repair costs under warranty coverage with considering the partial information about product reliability. Yazdian, Shahanaghi, and Makui (2016) jointly optimises the acquisition price, re-manufacturing degree, selling price and the length of warranty of re-manufacturing products under linear and non-linear demand functions.

The assumption that a product is composed of only one component is too simplistic and even unrealistic. Researchers then consider the assumption that a product is composed of multiple components. Huang, Liu, and Murthy (2007) develop a model to determine the optimal product reliability, price and warranty strategy to achieve the maximum total integrated profit for a general repairable multi-component product sold under a free replacement-repair warranty strategy. Matis, Jayaraman, and Rangan (2008) explore the optimal price and pro rate warranty length for a multi-component product with considering the different repair options on the components. Liu, Wu, and Xie (2015) analysed cost for a multi-component system with failure interaction under renewing free-replacement warranty, Ahmadi (2016) addresses an optimal replacement problem for complex multi-component systems by determining an optimal operating time which balances income and cost to maximizes the expected profit over a cycle. Chen, Lo, and Weng (2017) seek to maximize the total profit per item of a multi-component product through optimally determine the production run length and the warranty period. Additionally, some researchers have noticed that, in the real world, a manufacturer may produce more than one product. The warranty claims of different products produced by the same manufacturer can be statistically dependent, because these products share similar design, same parts, same production line and other common causes. Such dependence should be considered by collective warranty models (Luo & Wu, 2018b, 2018a). Somboonsavatdee and Sen (2015) focuses on the analysis of repairable systems that are subject to multiple sources of recurrence, which assumes that dependence among the cause-specific recurrent processes is induced via a shared frailty structure.

All of the work mentioned simply considers hardware subsystems. However, currently, as many product consists of hardware and software components, the difference and interplay between these two different types of components should be considered in warranty policy.

Hardware failures under warranty are usually rectified by the manufacturer, with no fee or partial fee to the consumer, based on the type of warranty policy used (Murthy & Djamaludin, 2002).

Software subsystems plays a vitally important role in many products. In this paper, we chiefly discuss software subsystems embedded in hardware subsystems. In spite of great advancements in software reliability/quality assurance, potential faults may still be introduced into the software during its development process (Kimura, Toyota, & Yamada, 1999; Williams, 2007). Software failures are usually caused by incorrect logic, incorrect statements, incorrect input data, and what not. In order to satisfy the reliability requirement and/or reduce the operating cost, software testing actions are normally performed to detect and remove software faults before the software is released. Software reliability can be improved by increasing the testing effort and by correcting detected faults. Therefore, in terms of software management, the determination of the optimal software release time, i.e. the optimal testing time, may be an important decision problem, which is called the optimal software release problem in the literature. The reader is referred to Kimura et al. (1999) for more details on this problem.

In addition to hardware and software subsystems, product users, may be considered as another essential sub-system in many situations. Warranty claims are not always triggered by hardware or software failures, they may also be due to human factors. According to Wu (2011), there are at least two types of human factors in the context of warranty management. (1) consumers might not be bothered to claim warranty for failed items that are still under warranty, which is called failed-but-not-reported (FBNR) phenomenon; (2) consumers may conduct a fraudulent warranty claim, or claim failure due to misuse or many other human factors, which is referred to as non-failed but reported (NFBR) claims. The first type of human factor relaxes a common assumption in warranty literature — all failures may cause warranty claims whereas the second type relaxes the assumption — all claims reported are due to product failures. Furthermore, after the updates of the software released by the manufacturer, whether and when to download and install the updates are decided by the customers. This implies the software update adoption rate, which can affect the software's reliability, may also be influenced by human factors.

Apparently, a product with lower price can enhance its sales volume; but reduce the unit profit of the product. Regarding the effect of warranty on manufacturer's profit, Wu et al. (2009) state that, in practice, consumers may predict the quality of a product based on its warranty, and a satisfactory warranty will certainly enhance consumers' purchase willingness, which is the well-known warranty's *signalling theory.* Products with longer warranty length may increase the total warranty cost to the manufacturer (Dai et al., 2012). Hence, it is important to trade-off the price and warranty length of a product in practice.

According to the literature, the failures of a product item consisting of hardware and software subsystems can be divided into three categories: hardware specific, software specific and hardware-software interaction failures (Fernandez & Stol, 2017; Roy, Murthy, & Mohanta, 2015; Teng, Pham, & Jeske, 2006). In order to estimate the warranty cost more accurately, this paper proposes a new model that considers hardware and software failures of a product and even its user's behaviour integrally.

1.1.2 The sales volume

The sales volume of a product is affected by two critical marketing variables: the selling price and the warranty length (Chen et al., 2017). For example, these two variables, selling price P

and warranty length T, can influence the sales volume, M, and profit, ω. The sales volume of a product is negatively related to its selling price and positively related to its warranty length. The profit of product k in this paper is the revenue deducting the warranty cost, i.e. $\omega = MP - S(T)$, where $S(T)$ is the aggregated warranty cost of the product within T.

In the literature, the sales volume M, is expressed by a function of product price P and length of warranty T in different forms, including linear (Lin et al., 2009; Yazdian et al., 2016) and non-linear (Huang et al., 2007; Ladany & Shore, 2007) ones. For simplicity, a linearity form, introduced by (Yazdian et al., 2016), is used in this paper. The sales volume is defined by

$$M = A - \beta P + \eta T, \tag{1}$$

where $A(> 0)$ is a constant relating to the market size of the product, and $\beta(> 0)$ and $\eta(> 0)$ are the price and length of warranty elasticities, respectively.

1.2 Novelty and contributions

The existing research on warranty management focuses on hardware subsystems (Ye & Murthy, 2016), software subsystems (Pham & Zhang, 1999) or human factors (Wu, 2011) separately. Little research, however, has been devoted to investigate the warranty claims due to the interplay of those three sub-subsystems.

This is the first paper that takes a holistic consideration of warranty claims caused by different factors: hardware failure, software failure and user behaviours. The interplays between hardware and software failures are investigated for five different aspects.

The paper has important managerial implications. In warranty management, optimising warranty policy and forecasting warranty claims are two of the most important activities. This requires analysts to understand the interplay of different warranty claim causes in order to make a precise forecasting and warranty optimisation. The methods proposed in this paper, offers a better way than existing ones. The methods therefore advance the state-of-the-art in warranty claim forecasting and policy optimisation, and offer warranty managers theoretically established methods that can be used in their projects.

1.3 Summary

The rest part of this paper is structured as follows. Section 2 gives assumptions and notations that will be used in this paper. Section 3 categories the routes of warranty claims into different situations, derives warranty cost models assuming that warranty claims due to hardware and software failures are statistically independent, and optimises the warranty policies through maximising the expected total profit from a manufacturer perspective. Section 4 derives cost models when the interplay of different subsystems is considered. Section 5 integrates the cost models derived from its preceding sections. Section 6 gives numerical examples illustrating the derived models. Section 7 concludes the paper.

2 Assumptions and Notation

In this paper, we analyse warranty cost from a manufacturer's perspective. The following assumptions are made:

(i) Products are new at $t = 0$ when they are sold.
(ii) Non-renewing free replacement warranty (NFRW) policy is offered to protect hardware failures. Under this policy, the manufacturer provides its customers with repair or replacement on hardware failures at no cost within the warranty period, the original warranty is not altered upon a failed item, and the manufacturer only guarantees satisfactory service on the item within the original warranty period.
(iii) Hardware failures require rectification to restore the products to an operating state.
(iv) Repair on hardware failures is assumed to be minimal repair, i.e. an item with a hardware failure is restored to the operating state that is exactly before it failed. Compared with the warranty duration, the repair time is so short that it is negligible.
(v) Software failures can be fixed through the removal of problems by debugging errors.
(vi) When a software failure occurs, the manufacturer can detect the fault which and remove it, failures due to this error may not occur again.
(vii) An individual consumer makes at most one non-failed but reported (NFBR) claim. Upon a NFBR claim, only administration cost is incurred to the manufacturer.
(viii) The hardware and software of a product have the same warranty period.

The notations used in this study are presented in Table 1.

3 Independent profit analysis

3.1 Possible warranty claim routes

A typical warranty claim process is shown in Figure 4. This process starts from the time when the item is thought to be failed and ends in five different routes.

Table 1
Notations

T	length of the warranty period
P	price of the product
M	total sales volume of the product
$S(T)$	total warranty cost of the product until time T
ω	expected total profit of the product
$\lambda_1(t)$	expected number of hardware failures at time t
$\lambda_2(t)$	expected number of software failures at time t
$\lambda_3(t)$	expected number of software failures when the default bugs are not removed in operating
$\Lambda_1(t)$	expected total number of hardware failures until time t
$\Lambda_2(t)$	expected total number of software failures until time t
$C_h(T)$	total warranty cost of hardware until time T
c_{h1}	expected cost of a warranty claim due to a hardware failure
c_{h2}	expected cost of a hardware components replacement
ω_h	expected total profit of a hardware product over warranty period
c_{s1}	expected cost of a claim due to a software failure
c_{s2}	expected cost of producing and releasing a software update/patch
c_{s3}	expected cost of developing software to resolve a hardware caused product failure
$C_{s1}(T)$	total warranty cost of a software product with Type I policy over warranty period
$C_{s2}(T)$	total warranty cost of a software product with Type II policy over warranty period
τ_i	time of ith software patches/updates release, $\tau_0 = 0$
$\Delta\tau$	interval between software patches/updates releases
$\lambda_{2,i}$	expected number of software failures after ith release installed, $\lambda_{2,0} = \lambda_2(0)$
θ	impact factor of patches/updates installation on expected number of software failures
n	number of patches/updates releases over warranty period
ω_{s1}	expected total profit of software product with Type I policy over warranty period
ω_{s2}	expected total profit of software product with Type II policy over warranty period
$H_1(t)$	cumulative distribution function of time to a NFBR claim
C_{hu1}	cost of NFBR claim when the warranty will not be ceased
C_{hu2}	cost of NFBR claim when the warranty will be ceased
c_{hu1}	administration cost per NFBR claim
c_{hu2}	expected cost on fixing the cause of an NFBR claim.
$q_1(t)$	probability of consumers being inclined to claim warranty
$q_2(t)$	probability of consumers installing updates
$N_h(T)$	total number of warranty claims due to hardware failure
$N_{s1}(T)$	total number of warranty claims due to software failure with Type I policy
$N_{s2}(T)$	total number of warranty claims due to software failure with Type II policy
$N_{s3}(T)$	total number of warranty claims due to software failure when the default bugs are kept
$p(t)$	probability of a software failure leads to hardware failure
$N_{hs1}(T)$	total number of claims due to hardware failure with software impacts under Type I policy
$N_{hs2}(T)$	total number of claims due to hardware failure with software impacts under Type II policy
$N_{hs3}(T)$	total number of warranty claims due to software failure with hardware impacts
$N_{hr}(T)$	total number of hardware components replacements due to software failure
$C_{01}(T)$	total warranty cost in Scenario 0 with Type I policy
$C_{02}(T)$	total warranty cost in Scenario 0 with Type II policy
ω_{01}	expected total profit in Scenario 0 with Type I software warranty policy
ω_{02}	expected total profit in Scenario 0 with Type I software warranty policy
$C_{int11}(T)$	total hardware warranty cost with considering human factors
$C_{int21}(T)$	total software warranty cost with considering human factors
$C_{int31}(T)$	total warranty cost based on the hybrid model

Route 1 If a user reports a failure to the manufacturer (or the warranty servicing agent of her area), and the failure is diagnosed as a hardware failure covered by the warranty policy, the manufacturer may offer the user free repair or replacement of the item. Then, the process ends (End 1 in Figure 4). The cost of the manufacturer on this event consists of the hardware repair/replacement cost and the related management cost.

Route 2 If a user does not report a failure to the manufacturer, this process ends (End 2 in Figure 4). This phenomenon is called as the *failed-but-not-reported* (FBNR) event, which may be due to various reasons, for example, an item is not expensive so that the user is not bothered to claim warranty (Wu, 2011). This event does not incur any cost to the manufacturer.

Route 3 If a user reports a failure to the manufacturer, and the failure is not covered by the warranty policy or the item is not really failed, this process ends (End 3 in Figure 4). This phenomenon is named as *not-failed but reported* (NFBR) claim, which may be due to misuse, fraud, etc (Wu, 2011). The manufacturer may pay the related management cost, such as diagnosis fee, caused by this event.

Route 4 If a user reports a failure to the manufacturer, and the failure is diagnosed as a software failure covered by the warranty policy, the manufacturer should offer the user free repair of the software system. If the software is not connected to the internet, this process ends (End 4 in Figure 4). The cost of the manufacturer on this event consists of the software debugging and repairing cost and the related management cost.

Route 5 If a user reports a failure to the manufacturer, and the failure is diagnosed as a software failure covered by the warranty policy, the manufacturer should offer the user free repair of the software system. If the software is connected to the internet, the manufacturer may develop and release the related update/patch on-line. Then, this process ends (End 5 in Figure 4). The cost of the manufacturer on this event consists of the software debugging and repairing cost, the update/patch developing and releasing cost and the related management cost.

The above five routes and their associated costs should be considered by the manufacturer to support precise warranty management. Sometimes, Route 3 and Route 4 or Route 5 may occur concurrently, because some interactions may exist between hardware and software subsystems. This paper aims to optimise warranty policies comprehensively to maximise the manufacturer's profit with considering the issues related to the hardware, software and user of a products.

3.2 Warranty claims due to hardware failure

A hardware failure during the operating phase may be rectified through repair or replacement, and the warranty may be non-renewing or renewing. More specifically, hardware failures under warranty are usually rectified by the manufacturer, with no fee or partial fee to the consumer, based on the type of warranty policy used (Murthy & Djamaludin, 2002).

Suppose the manufacturer takes non-renewing warranty policy, minimal repair on hardware failures is performed and the time on repair is negligible. Denote $\lambda_1(t)$ as the failure intensity function and $\Lambda_1(t)$ as the cumulative failure intensity function. Then the expected warranty

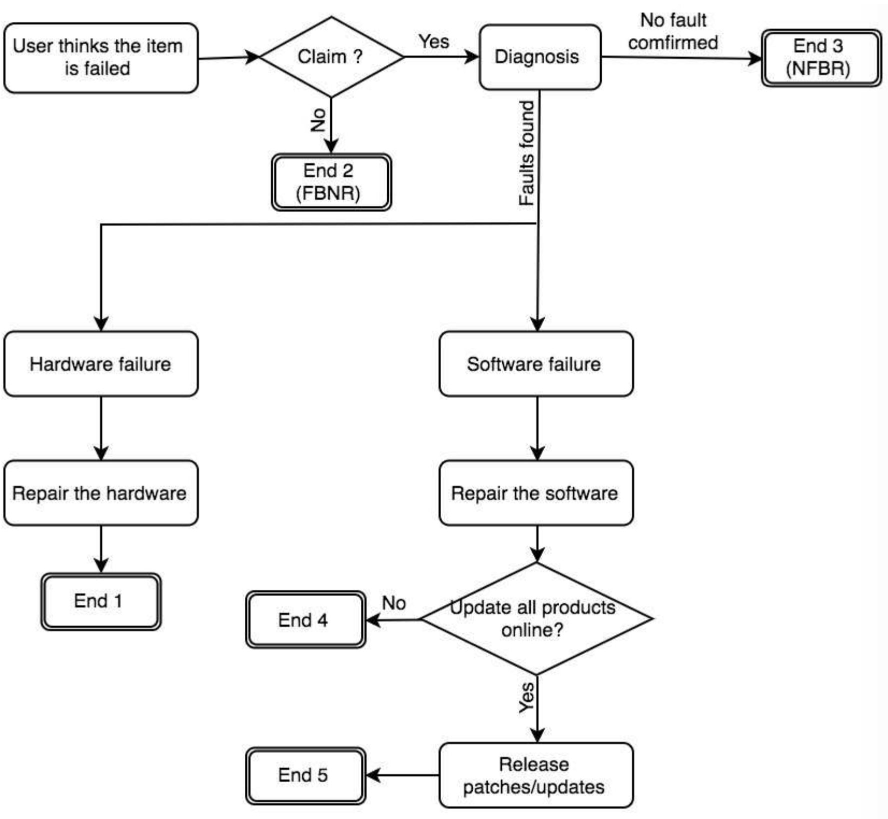


Fig. 4. Warranty claim process

cost of hardware failures is

$$\mathrm{E}[C_h(T)] = Mc_{h1}\Lambda_1(T) = Mc_{h1}\int_0^T \lambda_1(t)dt, \tag{2}$$

where T is the length of warranty, c_{h1} is the expected cost of each warranty claim due to hardware failure. The expected profit of a hardware product is

$$\omega_h = MP - \mathrm{E}[C_h(T)] = (A - \beta P + \eta T)(P - c_{h1}\Lambda_1(T)). \tag{3}$$

Eq. (3) presents that the expected profit is a function of two variables P and T. To maximise ω_h, the optimal P and T should be found. It is easy to prove that when T is known, an optimal P exists to maximize ω_h, even the form of failure intensity function $\lambda_1(t)$ is unknown.

Meanwhile, the relationship between ω_h and T is determined by the forms of failure intensity function $\lambda_1(t)$ and cumulative failure intensity function $\Lambda_1(T)$. Suppose that the arrival process of warranty claims due to hardware failures following a Non-Homogeneous Poisson Process (NHPP) with failure intensity function $\lambda_1(t)$ and the cumulative failure intensity function $\Lambda_1(t) = \int_0^t \lambda_1(u)du$. Assuming the NHPP follows a power law intensity in this paper, the intensity and cumulative intensity functions of an item are $\lambda_1(t) = a_1 b_1 t^{b_1-1}$ and $\Lambda_1(t) = a_1 t^{b_1}$, where $a_1 > 0$ indicates the initial intensity and $b_1 > 1$ means that the hardware reliability is decreasing over time. Then, the expected profit of hardware product is

$$\omega_h = MP - \mathrm{E}[C_h(T)] = (A - \beta P + \eta T)(P - c_{h1} a_1 T^{b_1}). \tag{4}$$

According to Eqs. (3) and (4), we have the following Proposition 1. The proof is presented in the Appendix.

Proposition 1 *(1) If P is decision variable and T is known, the optimal solution, which maximizes the expected profit of a hardware product ω_h, exists. (2) If T is decision variable, P is known, and the arrival process of warranty claims due to hardware failures following a power law NHPP, the optimal solution, which maximizes the expected profit of a hardware product ω_h, exists.*

3.3 Warranty claims due to software failure

Potential faults or causes of failures are introduced into the software during its development process. Once a fault is diagnosed and removed, some of the software's errors may be debugged and the total number of potential faults are reduced, which results in a growing reliability of software. In recent times, there is a trend that software patches are provided during early software release and updating. To satisfy customers concern of reliable software, manufacturers may provide warranty on the embedded software. Within the warranty period, the manufacturer provides assurance to the customers that the software may work properly and if any defect is found, the manufacturer may either repair or replace the software without charging the customer (Kansal, Singh, Kumar, & Kapur, 2016; Singh, Kapur, Shrivastava, & Kumar, 2015).

The software warranty policies can be divided into two types:

Type I The consumer is entitled to return the software, and the manufacturer should provide support to bring the software back up to its operating mode. Every product is repaired independently under this type of warranty policy.

Type II The manufacturers may collect error reports via the internet, and then debug the errors and release updates or patches online according to the reports. All consumers who buy the products can download and install the updates free of charge.

The cumulative number of software failures can be represented by a counting process with failure intensity $\lambda_2(t)$ and cumulative failure intensity $\Lambda_2(t) = \int_0^t \lambda_2(t)dt$. The failure intensity decreases with time as the initial faults will be detected and removed in operating. Such model is named by the software reliability growth model (SRGM).

If the manufacturer takes Type I software warranty policy, the expected warranty cost on software failures is

$$\mathrm{E}[C_{s1}(T)] = Mc_{s1}\Lambda_2(T) = Mc_{s1}\int_0^T \lambda_2(t)dt, \tag{5}$$

where c_{s1} is the expected warranty cost of each claim on software failure under Type I software warranty policy, and M is the number of product sold at $t = 0$.

If the manufacturer takes Type II software warranty policy, to model the warranty cost we have the following assumptions:

(i) if a software system failure occurs before the corresponding update is executed, the software is brought back to the operating mode as the same version, i.e. the error isn't removed; and
(ii) the manufacturer releases the updates/ patches based on a pre-specified time schedule, for example, releasing updates online in every 6 months.

Under this type of warranty policy, the expected warranty cost on software failures is:

$$\mathrm{E}[C_{s2}(T)] = nc_{s2} + Mc_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + Mc_{s1}\int_{\tau_n}^{T} \lambda_{2,n}(t)dt, \tag{6}$$

where $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is the number of releasing patches; M is the number of product sold at $t = 0$; τ_i is the time of releasing ith patch/update under time-based policy, $\tau_i - \tau_{i-1} = \Delta\tau$ and $\tau_0 = 0$; $\lambda_{2,i}$ is the failures intensity after ith patches/updates released, $\lambda_{2,i} = \theta\lambda_{2,i-1}$ and $\lambda_{2,0}(t) = \lambda_2(t)$; and c_{s2} is the expected total cost of releasing an update. In this model, $\Delta\tau$ is a pre-specified time length, such as 1 month, 6 months, etc.; and $\theta(> 0)$ is the impact factor of patch/update on the software failure intensity, and θ can be estimated based on lab data or field data in practice.

Regarding the expected profit of software product, under Type I software warranty, according to Eq. (5), the expected profit of a software product is

$$\omega_{s1} = MP - \mathrm{E}[C_{s1}(T)] = (A - \beta P + \eta T)(P - c_{s1}\Lambda_2(T)). \tag{7}$$

If T is known as a constant, the right hand side of Eq. (7) becomes to a parabolic function, and the coefficient of P^2 item is negative, hence ω_{s1} has global maxima.

If P is known as a constant, the relationship between ω_{s1} and warranty length T is determined by the form of SRGM. To demonstrate the relationship, the intensity and cumulative intensity can be constructed according to the most well-known NHPP-based SRGM, Goel-Okumoto (G-O) model (Wang, Wu, Shu, & Zhang, 2015). The intensity is $\lambda_2(t) = a_2 b_2 e^{-b_2 t - 1}$, and cumulative intensity is $\Lambda_2(t) = a_2(1 - e^{-b_2 t})$. The parameters of $\lambda_2(t)$ and $\Lambda_2(t)$, a_2 and b_2 are positive, which indicate the expected number of failures is decreasing with t and the expected total number of failures until time t has a upper limit a_2. Then, the expected profit of a software product is

$$\omega_{s1} = MP - \mathrm{E}[C_{s1}(T)] = (A - \beta P + \eta T)(P - c_{s1}a_2(1 - e^{-b_2 T})). \tag{8}$$

When P is known, the first order derivative of ω_{s1} on T is $\frac{d\omega_{s1}}{dT} = c_{s1}a_2 e^{-b_2T-1}(Ab_2 + \beta Ab_2P + \eta - \eta Tb_2) + \eta P - \eta c_{s1}a_2$. Thus, we have the following result.

Proposition 2 *(1) If P is decision variable and T is known, the optimal solution, which maximizes the expected profit of a software product ω_{s1} under Type I software warranty, exists. (2) If T is decision variable and P is known, and the software failures follow G-O SRGM; the optimal solution, which maximizes the expected profit of a software product ω_{s1} under Type I software warranty, only exists on the boundaries.*

When the manufacturer takes Type II software warranty, according to Eq. (6), the expected profit of a software product is

$$\begin{aligned}\omega_{s2} &= MP - \mathrm{E}[C_{s2}(T)] \\ &= MP - [nc_{s2} + Mc_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + Mc_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt],\end{aligned} \tag{9}$$

There are three variables in this situation, which are: product price P, warranty length T and updating interval $\Delta\tau$. Then we have the following two propositions.

Proposition 3 *If $\Delta\tau$ is pre-specified, P is decision variable and T is known, the optimal solution, which maximizes the expected profit of a software product ω_{s2} under Type II software warranty, exists.*

Proposition 4 *If both P and T are known and $\Delta\tau$ is a decision variable, the optimal solution, which maximizes the expected profit of a software product ω_{s2} under Type II software warranty, exists.*

3.4 Warranty claims due to users

The NFBR and FBNR events should be considered in warranty cost analysis, however there is another human factor, the *adoption rate*, which should not be ignored either. The adoption rate is the percentage of the users who have downloaded and installed the updates/patches of the software embedded in the product. Sometimes users may not download and install software patches/updates immediately after they are released. The adoption rate may be a function of time, for example, the adoption rate of the software embedded iPhone may increase over time.

According to Wu (2011), it is reasonable to assume that a consumer makes at most one NFBR claim. Wu (2011) proposes three models to estimate the expected warranty cost when both

NFBR claims and FBNR phenomenon are considered. Manufacturers responses to NFBR claims may be different: (1) some manufacturers may cease the warranty contract for consumers with NFBR claims, and (2) some may not cease the warranty contract, as it is not easy to tell if a NFBF claim is intentionally or unintentionally committed. However, both responses incur costs to the manufacturers, and therefore should be considered in estimating warranty cost. Following (Wu, 2011), we assume that time to a NFBR claims is a random variable Z with cumulative distribution function $H_1(t) = 1 - e^{-(t/\alpha_1)^{\alpha_2}}$, $\alpha_1, \alpha_2 > 0$. Then, we consider the following two scenarios.

(i) A NFBR claim may not cause warranty to be ceased, the manufacturer may check and return the item. Then the expected warranty cost is given by

$$\mathrm{E}[C_{hu1}(T)] = Mc_{hu1}H_1(T), \tag{10}$$

where c_{hu1} is the administration cost per NFBR claim.

(ii) A NFBR claim may cause warranty to be ceased, the manufacturer may fix and return the item. Once the warranty ceases, there are no further costs to the manufacturer. Then the expected warranty cost is given by

$$\mathrm{E}[C_{hu2}(T)] = Mc_{hu2}H_1(T), \tag{11}$$

where c_{hu2} is the expected cost on fixing the cause of a NFBR claim

Regarding the FBNR phenomenon, the consumers' willingness to claim warranty may diminish with time, then the probability of consumers being inclined to claim warranty is assumed to be

$$q_1(t) = e^{-\gamma_1 - \gamma_2 t}, \tag{12}$$

which is called a warranty execution function (WEF) (Wu, 2011), where $\gamma_1, \gamma_2 > 0$.

Regarding the effects of delayed updating behaviour on warranty cost, the probability of consumers installing the update, i.e. adoption rate, $q_2(\Delta t)$, increases with time after releasing. $\Delta t = t - \tau_i$, where t is the current time and τ_i is the time when the ith update is released. The delayed updating behaviour does not incur new cost directly, but it can affect the cost $\mathrm{E}[C_{s2}(T)]$.

4 Comprehensive profit analysis and optimization

In literature, most of the researchers use the Markov process to model the interaction between hardware and software based on the physic structures of products (Roy et al., 2015; Teng et al., 2006), in this paper, briefly, the hardware-software interaction failures are modelled under two different situations.

(i) the interaction between hardware and software failures can be categorised into two categories, hardware-failure-caused software failure and software-failure-caused hardware failure; and
(ii) the causes of hardware-software interaction failures cannot be determined.

Below we discuss the above two situations.

The scenarios of product failure are presented in Table 2. Scenario 0 is the basic one, in which the hardware and software failures are independent, and a warranty claim can be caused by either hardware or software. In Scenario 1, beyond the claims in Scenario 0, some claims are caused both hardware and software failures at the same time, as some software failures can lead to hardware failures. In Scenario 2, the same type of claims in Scenario 0 also occurs, and besides some claims are caused by hardware-software (h-s) interplay problems which can be resolved through replacing hardware parts by some other types. The causes of warranty claims in Scenario 3 is similar to those in Scenario 2, but the problems can be resolved by developing software instead of replacing hardware. Scenario 4 is a predicted scenario with considering auto-programming technology based on Artificial Intelligence (AI), in which the faults of software are not only generated in developing stage before release but also introduced by hardware failures in operation.

Table 2
Scenarios of product failure

Scenarios	Interplay	Solution
0	Hardware and software failures are independents.	Repair hardware/software independently.
1	Software failure can lead to hardware failure.	Repair hardware and software.
2	Problems of h-s interplay lead to product failure.	Replace hardware parts to improve reliability.
3	Problems of h-s interplay lead to product failure.	Develop software to improve reliability.
4	Hardware failure can lead to software failure.	Repair hardware and software.

The costs and profits in these scenarios should be modelled in different ways, which are discussed below.

In this section, the cost of the ith warranty claim due to hardware failure is denoted by $c_{h,i}$, with $\mathrm{E}(c_{h,i}) = c_{h1}$, and the cost of jth warranty claim due to software failure is denoted by $c_{s,j}$, with $\mathrm{E}(c_{s,j}) = c_{s1}$, where $c_{h,i}$ and $c_{s,j}$ follow non-negative continuous probability distributions. In practice, a product may be composed of many different hardware subsystems, which may be controlled by one or more software subsystems. To investigate the interplay between software and hardware subsystems, we consider the following 5 scenarios.

Scenario 0 The occurrences of hardware and software failures are statistically independent. If the software system cannot be updated online, the total warranty cost of M sold items of a product during the warranty period is

$$C_{01}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{s1}(T)} c_{s,j}, \tag{13}$$

where $N_h(T)$ is the total number of warranty claims of M items due to hardware failures during the warranty period T, and $N_{s1}(T)$ is the total number of warranty claims of M items due to software failures during the warranty period T. $N_h(T)$ and $N_{s1}(T)$ have cumulative intensities $\Lambda_1(T)$ and $\Lambda_2(T)$, respectively.

The expected total cost in this situation is

$$\mathrm{E}[C_{01}(T)] = Mc_{h1}\Lambda_1(T) + Mc_{s1}\Lambda_2(T) \tag{14}$$

and the expected total profit in this scenario is

$$\omega_{01} = MP - \mathrm{E}[C_{01}(T)] = M\left[P - c_{h1}\Lambda_1(T) - c_{s1}\Lambda_2(T)\right]. \tag{15}$$

Assume that the software can be updated online. Once a fault is reported, confirmed and repaired, the software that is embedded all of the sold items of this type of product will be repaired. Then, the total warranty cost of a product during the warranty period is

$$C_{02}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{16}$$

where n is the number of software updates during the warranty period, and $c_{u,k}$ is the cost of the kth debugging and updating, $\mathrm{E}(c_{u,k}) = c_{s2}$. $N_{s2}(T)$ is more complicated than $N_{s1}(T)$ because the intensity function of $N_{s2}(T)$ is influenced by software updating activities.

$$\mathrm{E}[N_{s2}(T)] = M\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + M\int_{\tau_n}^{T} \lambda_{2,n}(t)dt, \tag{17}$$

The parameters in Eq. (17) are defined the same as those in Eq. (6). Then the expected total cost is

$$\mathrm{E}[C_{02}(T)] = Mc_{h1}a_1T^{b_1} + Mc_{s1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right] + nc_{s2}, \tag{18}$$

and the expected total profit in this scenario is

$$\omega_{02} = MP - \mathrm{E}[C_{02}(T)]. \tag{19}$$

Scenario 1 The occurrence of hardware and software failures are statistical dependent; but no failure is due to the physical interaction of the hardware-software subsystems. A software failure can lead to hardware failures of the M sold items with probability $p(t)$ at time t, where $p(t)$ can be estimated based on historical data. If the software cannot be updated online, the total warranty cost of the M sold items of a product during the warranty period is given by

$$C_{11}(T) = \sum_{i=1}^{N_{hs1}(T)} c_{h,i} + \sum_{j=1}^{N_{s1}(T)} c_{s,j}, \tag{20}$$

where $N_{hs1}(T) = N_h(T) + N_{s1}(T)\int_0^T p(t)dt$. Then, the expected total warranty cost is

$$\mathrm{E}[C_{11}(T)] = Mc_{h1}\left[\Lambda_1(T) + \int_0^T \lambda_2(t)p(t)dt\right] + Mc_{s1}\Lambda_2(T). \tag{21}$$

Then the expected total profit is

$$\omega_{11} = MP - \mathrm{E}[C_{11}(T)] = M\left\{P - c_{h1}\left[\Lambda_1(T) + \int_0^T \lambda_2(t)p(t)dt\right] - c_{s1}\Lambda_2(T)\right\}. \tag{22}$$

If the software can be updated online, once a fault is reported, confirmed and repaired, all of the sold items' embedded software will be repaired. Then, the total warranty cost of a product during the warranty period is

$$C_{12}(T) = \sum_{i=1}^{N_{hs2}(T)} c_{h,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{23}$$

where $N_{hs2}(T) = N_h(T) + N_{s2}(T)\int_0^T p(t)dt$. Then, the expected total warranty cost is

$$\begin{aligned}\mathrm{E}[C_{12}(T)] = {} & Mc_{h1}a_1T^{b_1} + Mc_{h1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\int_0^T p(t)dt \\ & + Mc_{s1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right] + nc_{s2},\end{aligned} \tag{24}$$

and the expected total profit in this scenario is

$$\omega_{12} = MP - \mathrm{E}[C_{12}(T)]. \tag{25}$$

The updating time interval, $\Delta\tau$, is pre-specified and decided by the manufacturer.

Scenario 2 The occurrence of hardware and software failures are statistical dependent; and the failures of the product are not only caused by pure hardware/software factors but also by some design problems related to hardware-software interaction. In this scenario, repairing the software is impossible or the software repairing cost is huge, the manufacturer may decide to replace the hardware subsystems of all items to reduce the failure rate of the product. Then, the total warranty cost of the M sold items of a product during the warranty period is

$$C_{21}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{hr}(T)} M c_{hr,j}, \tag{26}$$

where $c_{hr,j}$ is the cost of replacing hardware components for the jth software-caused hardware failure, $\mathrm{E}(c_{hr,j}) = c_{h2}$. In this case, the software bugs would not be removed, and the number of software failure following a counting process with failure intensity $\lambda_3(t)$. Then $N_{hr}(T) = N_{s3}(T) \int_0^T p(t)dt$. The expected total warranty cost is

$$\mathrm{E}[C_{21}(T) = M \left\{ c_{h1}\Lambda_1(T) + c_{h2} \int_0^T \lambda_3(t)p(t)dt \right\} \tag{27}$$

The expected total profit is

$$\omega_{21} = M \left\{ P - \{c_{h1}\Lambda_1(T) - c_{h2} \int_0^T \lambda_3(t)p(t)dt \right\}. \tag{28}$$

Scenario 3 The hardware and software are interplaying; and the failures of the product are not only caused by pure hardware/software factors but also by some problems relating to hardware-software interplay (eg. bad design). Different from Scenario 2, in this scenario, it is impossible to merely replace the hardware subsystem. The manufacturer may decide to develop/improve the software to reduce the failure rate of the product. Then, if the product cannot be updated online, the total warranty cost of M sold items of a product during the warranty period is

$$C_{31}(T) = \sum_{i=1}^{N_h(T)} c_{sh,i} + \sum_{j=1}^{N_{s1}(T)} c_{s,j}, \tag{29}$$

where $\mathrm{E}(c_{sh,i}) = c_{s3}$ is the expected cost of developing software to resolve a hardware-caused product failure.

Then the expected cost is

$$\mathrm{E}[C_{31}] = M\left[c_{s3}\Lambda_1(T) + c_{s1}\Lambda_2(T)\right], \tag{30}$$

and the expected total profit is

$$\omega_{31} = M\left\{P - c_{s3}\Lambda_1(T) - c_{s1}\Lambda_2(T)\right\}. \tag{31}$$

If the product can be updated online, then the total warranty cost is

$$C_{32}(T) = \sum_{i=1}^{N_h(T)} c_{sh,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{32}$$

Then, the expected total warranty cost is

$$\mathrm{E}[C_{32}(T)] = M\left\{c_{s3}a_1T^{b_1} + c_{s1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\right\} + nc_{s2}, \tag{33}$$

and the expected total profit in this scenario is

$$\omega_{32} = MP - \mathrm{E}[C_{32}(T)]. \tag{34}$$

The updating time interval, $\Delta\tau$, is pre-specified and decided by the manufacturer.

Scenarios 4 This scenarios describes a potential situation in the near future. The Artificial Intelligence (AI) technology allows software subsystems themselves to program automatically to deal the dynamic state of the product, which implies that faults may be introduced into the software not only during the developing process before released but also during the operating phase. Then, the total warranty cost in this scenario is

$$C_{41}(T) = \sum_{i=1}^{N_{hs1}(T)} c_{h,i} + \sum_{j=1}^{N_{hs3}(T)} c_{s,j}, \tag{35}$$

where $N_{hs1}(T) = N_h(T) + N_{s1}(T)\int_0^T p(t)dt$ and $N_{hs3}(T) = N_h(T)\int_0^T q(t,M)dt + N_{s1}(T)$. $\int_0^T q(t,M)dt$ is the probability that a hardware failure leads to software failure at time T. Then the expected cost is

$$\begin{aligned}\mathrm{E}[C_{41}(T)] = &Mc_h\left[\Lambda_1(T) + \int_0^T \lambda_2(t)p(t)dt\right]\\ &+Mc_{s1}\left[\int_0^T \lambda_1(t)q(t)dt + \Lambda_2(T)\right],\end{aligned} \tag{36}$$

and the expected total profit in this scenario is

$$\omega_{41} = MP - \mathrm{E}[C_{41}(T)]. \tag{37}$$

According to Proposition 1 and Proposition 2, in all of the above 5 scenarios, we have the following two propositions.

Proposition 5 *If the software cannot be updated online, P is decision variable and T is known, the optimal solution, which maximizes the expected profit of the product, exists.*

Proposition 6 *If the software cannot be updated online, P is decision variable and T and $\Delta\tau$ are known, the optimal solution, which maximizes the expected profit of the product, exists. If*

$\Delta\tau$ is decision variable and T and P are known, the optimal solution, which maximizes the expected profit of the product, exists.

5 Integrated warranty models

The integrated warranty models are built in three situations. In the first situation, the warranty claims of a hardware product and the human factors are integrated. In the second situation, the warranty claims of a software product and the human factors are integrated. In the third situation, the warranty claims of a product consisting of hardware and software and the human factors are integrated.

5.1 Hardware warranty with considering human factors

If a product is pure hardware, the NFBR and FBNR phenomena can influence the total warranty cost of it. Routes 1, 2 and 3 of warranty claim process discussed in Section 3.1 can occur.

If the NFBR may not cause warranty to be ceased, according to Eqs. (2), (10) and (12), the expected total cost is

$$\begin{aligned} \mathrm{E}[C_{int11}(T)] &= \mathrm{E}[C_h(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\ &= Mc_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + Mc_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{38}$$

Then, the expected profit of the product is

$$\begin{aligned} \omega_{int11} &= MP - \mathrm{E}[C_{int11}(T)] \\ &= (A-\beta P+\eta T)(P - c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} - c_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{39}$$

If the NFBR may cause warranty to be ceased, according to Eqs. (2), (11) and (12), the expected total cost is

$$\begin{aligned} \mathrm{E}[C_{int12}(T)] &= [\mathrm{E}[C_h(T)]q_1(T) + Mc_{hu2}]\, H_1(T) + \mathrm{E}[C_h(T)]q_1(T)(1-H_1(T)) \\ &= M\left[c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1-e^{-(t/\alpha_1)^{\alpha_2}}) \\ &\quad + Mc_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}. \end{aligned} \tag{40}$$

Then, the expected profit of the product is

$$\begin{aligned}\omega_{int12} = & MP - \mathrm{E}[C_{int12}(T)] \\ = & MP - M\left[c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1-e^{-(t/\alpha_1)^{\alpha_2}}) \\ & +c_{h1}a_1MT^{b1}e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}).\end{aligned} \tag{41}$$

5.2 *Software warranty with considering human factors*

If only the software warranty is considered and the software cannot be updated online, the NFBR and FBNR phenomena may influence the total warranty cost of it, but the delayed updating is not applied on this situation. Routes 1, 2 and 4 of warranty claim process discussed in Section 3.1 can occur.

If the NFBR may not cause warranty to be ceased, according to Eqs. (5), (10) and (12), the expected total cost is

$$\begin{aligned}\mathrm{E}[C_{int21}(T)] = & \mathrm{E}[C_{s1}(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\ = & Mc_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T} + Mc_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}).\end{aligned} \tag{42}$$

Then, the expected profit of the product is

$$\begin{aligned}\omega_{int21} = & MP - \mathrm{E}[C_{int21}(T)] \\ = & (A-\beta P+\eta T)(P - c_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T} - c_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}).\end{aligned} \tag{43}$$

If the NFBR may cause warranty to be ceased, according to Eqs. (5), (11) and (12), the expected total cost is

$$\begin{aligned}\mathrm{E}[C_{int22}(T)] = & [\mathrm{E}[C_{s1}(T)]q_1(T) + Mc_{hu2}]\,H_1(T) + \mathrm{E}[C_{s1}(T)]q_1(T)(1-H_1(T)) \\ = & M\left[c_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1-e^{-(t/\alpha_1)^{\alpha_2}}) \\ & +Mc_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}.\end{aligned} \tag{44}$$

Then, the expected profit of the product is

$$\begin{aligned}\omega_{int22} = & MP - \mathrm{E}[C_{int22}(T)] \\ = & MP - M\left[c_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1-e^{-(t/\alpha_1)^{\alpha_2}}) \\ & -Mc_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}).\end{aligned} \tag{45}$$

If the software can be updated online, the NFBR, FBNR and delayed updating phenomena all can influence the total warranty cost of it. Routes 1, 2 and 5 of warranty claim process discussed in Section 3.1 can occur.

Assume the proportion of users installed the update, i.e. adoption rate, is $q_2(\Delta t) = 1 - e^{-d\Delta\tau}$ where $d > 0$, and the FBNR is considered then Eq. (6) can be modified to

$$\mathrm{E}[C'_{s2}(T)] = nc_{s2} + q_2(\Delta\tau)c_{s1}\sum_{i=1}^{n} q_1(\tau_i)\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + q_2(T-\tau_n)q_1(T)c_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt. \quad (46)$$

Take the NFBR and FBNR into account, if the NFBR may not cause warranty to be ceased, according to Eqs. (10), (12) and (46), the expected total cost is

$$\mathrm{E}[C_{int23}(T)] = \mathrm{E}[C'_{s2}(T)] + \mathrm{E}[C_{hu1}(T)], \quad (47)$$

then, the expected profit of the product is

$$\begin{aligned}\omega_{int23} =& MP - \mathrm{E}[C_{int23}(T)]\\ =& MP - nc_{s2} - q_1(\Delta\tau)q_2(\Delta\tau)c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right]\\ &-q_1(T-\tau_n)q_2(T-\tau_n)c_{s1}\theta^n\left[\Lambda_2(T) - \Lambda_2(\tau_n)\right] - Mc_{hu1}(1 - e^{-(t/\alpha_1)^{\alpha_2}}). \quad (48)\end{aligned}$$

If the NFBR may cause warranty to be ceased, according to Eqs. (11), (12) and (46), the expected total cost is

$$\mathrm{E}[C_{int24}(T)] = \left(\mathrm{E}[C'_{s2}(T)] + Mc_{hu2}\right)H_1(T) + \mathrm{E}[C'_{s2}(T)](1 - H_1(T)), \quad (49)$$

then, the expected profit of the product is

$$\omega_{int24} = MP - \mathrm{E}[C_{int24}(T)]. \quad (50)$$

5.3 Hybrid warranty with considering human factors

If a product is composed of hardware and software subsystems, the warranty cost of such product is influenced by the warranty claims on both hardware and software failures, the three human factors and the interplay between hardware and software subsystems. In Section 4.1, 5 different scenarios of hardware and software subsystems interplay are discussed, in this section, the hybrid model is constructed based on the interplay described in Scenario 1, i.e. a software failure can lead to hardware failure with probability $p(t)$ at time t within the M sold items of a product.

If the software cannot be updated online, the NFBR and FBNR phenomenon may affect the total warranty cost, but the delayed updating is not applied in this situation. If the NFBR may not cause warranty to be ceased, according to Eqs. (10), (12) and (21), the expected total warranty cost is

$$
\begin{aligned}
\mathrm{E}[C_{int31}(T)] =& \mathrm{E}[C_{11}(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\
=& Mq_1(T)\left\{c_{h1}\left[a_1 T^{b_1} + a_2(1-e^{-b_2 T})\int_0^T p(t)dt\right] + c_{s1}a_2(1-e^{-b_2 T})\right\} \\
&+ Mc_{hu1}(1-e^{-(T/\alpha_1)^{\alpha_2}}).
\end{aligned} \tag{51}
$$

Then the expected total profit is

$$
\omega_{int31} = MP - \mathrm{E}[C_{int31}(T)]. \tag{52}
$$

If the NFBR may cause warranty to be ceased, according to Eqs. (11), (12) and (21), the expected total warranty cost is

$$
\mathrm{E}[C_{int32}(T)] = [\mathrm{E}[C_{11}(T)]q_1(T) + Mc_{hu2}]\, H_1(T) + \mathrm{E}[C_{11}(T)]q_1(T)(1 - H_1(T)). \tag{53}
$$

Then the expected total profit is

$$
\omega_{int32} = MP - \mathrm{E}[C_{int32}(T)]. \tag{54}
$$

If the software can be updated online, the delayed updating effect is applied. Then, the expected warranty cost considering delayed updating and FBNR is

$$
\begin{aligned}
\mathrm{E}[C'_{12}(T)] =& Mq_1(T)c_{h1}a_1 T^{b_1} + nc_{s2} \\
&+ c_{h1}\left[q_1(\Delta\tau)q_2(\Delta\tau)\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + q_1(T-\tau_n)q_2(T-\tau_n)\int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\int_0^T p(t)dt \\
&+ c_{s1}\left[q_1(\Delta\tau)q_2(\Delta\tau)\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + q_1(T-\tau_n)q_2(T-\tau_n)\int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right].
\end{aligned} \tag{55}
$$

If the NFBR may not cause warranty to be ceased, according to Eqs. (10), (12) and (55), the expected total warranty cost is

$$
\mathrm{E}[C_{int33}(T)] = \mathrm{E}[C'_{12}(T)] + \mathrm{E}[C_{hu1}(T)], \tag{56}
$$

and the expected total profit in this scenario is

$$
\omega_{int33} = MP - \mathrm{E}[C_{int33}(T)]. \tag{57}
$$

If the NFBR may cause warranty to be ceased, according to Eqs. (11), (12) and (55), the expected total warranty cost is

$$
\mathrm{E}[C_{int34}(T)] = [\mathrm{E}[C'_{12}(T)] + Mc_{hu2}]\, H_1(T) + \mathrm{E}[C'_{12}(T)](1 - H_1(T)), \tag{58}
$$

and the expected total profit is

$$
\omega_{int34} = MP - \mathrm{E}[C_{int34}(T)]. \tag{59}
$$

The effect of the variables P and T on then expected profits ω_{int31}, ω_{int32}, ω_{int33} and ω_{int34} are explored by the corresponding numerical examples.

6 Maintenance policies

Considering maintenance policy optimisation has been one of the focuses in the warranty related research. For example, recently, Y.-S. Huang, Huang, and Ho (2017) proposes a customized extended warranty policy, in which different preventive maintenance schedules are applied. For more detailed discussion on maintenance models, the reader is referred to Shafiee and Chukova (2013) for maintenance models in warranty and Peng, Liu, Zhai, and Wang (2017); Wu (2018b); Zhao, He, and Xie (2018) for maintenance models in general.

Assume that a product composed of software and hardware subsystems has a warranty period T. Preventive maintenance on the hardware subsystem will be performed at time points $T_i = i\frac{T}{N}$ for $i = 1, 2, ..., N$, which means $N - 1$ preventive maintenance (PM) actions are performed. A question is how the value N, i.e., the number of PM actions, can be determined to minimise the expected total cost during the warranty period $(0, T)$, from a manufacturer's perspective. To this end, we make the following assumptions.

(1) The PM is performed at planned times $T_i = i\frac{T}{N}$ $(i = 1, 2, ..., N)$. The interval from T_{i-1} to T_i is called the i-th PM period, where $T_0 = 0$.
(2) If a hardware failure occurs during the i-th PM period, the warranty of the entire product item is reported and then a minimal repair is conducted on the hardware subsystem.
(3) The failure rate of the hardware subsystem after the ith PM becomes $\nu^{i-1}\lambda_1(t)$ during the ith PM period, where $t \in (0, \frac{T}{N})$ and $\nu \geq 1$.
(4) During the ith PM period, the probability $p(t)$, which is the probability of the occurrence of software failures causing the hardware to fail, becomes $\mu^{i-1}p(t)$, with $\mu < 1$.
(5) The failure rate $\lambda_1(t)$ is strictly increasing.
(6) The cost on each minimal repair is c_r, and the cost of each PM is c_p.
(7) The times on repair and PM are negligible.

Assumption (4) implies that a PM on the hardware subsystem makes the dependence between the hardware and software subsystems weaker. c_r in Assumption (6) may include the repair cost of the failure and the other cost associated with the failure and warranty claim.

Suppose that PM does not influence the users' behaviour towards warranty claims. The the expected total cost during the warranty period is given by

$$
\begin{aligned}
C(N) =& c_r \sum_{i=1}^{N} \left(\nu^{i-1} \int_0^{\frac{T}{N}} \lambda_1(t)dt + \mu^{i-1} \int_0^{\frac{T}{N}} \lambda_2(t)p(t)dt \right) + (N-1)c_p \\
=& c_r \frac{\nu^N - 1}{\nu - 1} \Lambda_1(\frac{T}{N}) + c_r \frac{1-\mu^N}{1-\mu} \Lambda_2'(\frac{T}{N}) + (N-1)c_p
\end{aligned} \tag{60}
$$

where $\Lambda_2'(\frac{T}{N}) = \int_0^{\frac{T}{N}} \lambda_2(t)p(t)dt$. If $\lambda_1(t) = a_1 b_1 t^{b_1 - 1}$, $\lambda_2(t) = a_2 b_2 e^{-b_2 t - 1}$, and $p(t) = \delta e^{-\delta t}$, where $b_1, b_2 > 1$, then

$$
\begin{aligned}
C(N) =& c_r \sum_{i=1}^{N} \left(\nu^{i-1} \int_0^{\frac{T}{N}} a_1 b_1 t^{b_1 - 1} dt + \mu^{i-1} \int_0^{\frac{T}{N}} a_2 b_2 e^{-b_2 t - 1} \delta e^{-\delta t} dt \right) + (N-1)c_p \\
=& c_r a_1 \left(\frac{T}{N} \right)^{b_1} \left(\frac{\nu^N - 1}{\nu - 1} \right) + \frac{c_r a_2 b_2 \delta e^{-1}}{b_2 + \delta} \left(\frac{1-\mu^N}{1-\mu} \right) \left[1 - e^{-(b_2+\delta)\frac{T}{N}} \right] + (N-1)c_p
\end{aligned} \tag{61}
$$

One may then seek number N^* that minimises $C(N)$ in Eq. (61). To find an N^* that minimises $C(N)$ we need $C(N+1) \geq C(N)$ and $C(N) < C(N-1)$, which implies $D(N) \geq 0$ and $D(N-1) < 0$, where

$$
\begin{aligned}
D(N) =& a_1 \left(\frac{T}{N+1} \right)^{b_1} \left(\frac{\nu^{N+1} - 1}{\nu - 1} \right) + \frac{a_2 b_2 \delta e^{-1}}{b_2 + \delta} \left[\mu^N - \left(\frac{1-\mu^{N+1}}{1-\mu} \right) e^{-(b_2+\delta)\frac{T}{N+1}} \right] \\
&- a_1 \left(\frac{T}{N} \right)^{b_1} \left(\frac{\nu^N - 1}{\nu - 1} \right) + \frac{a_2 b_2 \delta e^{-1}}{b_2 + \delta} \left(\frac{1-\mu^N}{1-\mu} \right) e^{-(b_2+\delta)\frac{T}{N}} + \frac{c_p}{c_r}.
\end{aligned} \tag{62}
$$

Under some conditions, N^* can be sought. For example, if

$$
\begin{aligned}
D(1) =& a_1 \left(\frac{T}{2} \right)^{b_1} (\nu + 1) + \frac{a_2 b_2 \delta e^{-1}}{b_2 + \delta} \mu \\
&+ \frac{a_2 b_2 \delta e^{-1}}{b_2 + \delta} e^{-(b_2+\delta)T} + \frac{c_p}{c_r} - a_1 T^{b_1} - \frac{a_2 b_2 \delta e^{-1}}{b_2 + \delta} (\mu + 1) e^{-(b_2+\delta)\frac{T}{2}} < 0,
\end{aligned} \tag{63}
$$

then N^* exists because it can be easy to prove $D(\infty) \to \infty$, or $C(N+1) \geq C(N)$ holds.

7 Numeric examples

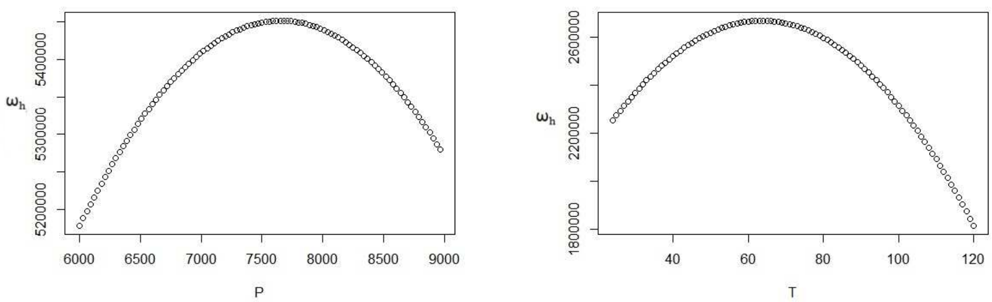
In this section, the models of product profit considering warranty costs due to hardware failures and software failure, and the integrated models are illustrated through numeric examples. The sales volume parameters are set: market size parameter, $A = 1000$, coefficient of product price, $\beta = 0.1$, and coefficient of warranty length, $\eta = 21$.

If the hardware is under a non-renewing warranty, the values of the parameters of Eq. (3) are set: Expected cost per hardware claim $c_{h1} = 100$, Power law parameter, $a_1 = 0.1$, Power law parameter, $b_1 = 1.04$.

Then, the expected total product profit only considering warranty cost due to hardware failures is

$$\omega_h(P, T) = (1000 - 0.1P + 21T)(P - 100 \times 0.1 \times T^{1.04}). \tag{64}$$

According to Proposition 1, if the warranty length is known, for example, set $T = 24$ months; then the expected total profit can be maximised at $P = 7656.27$, and the maximum expected total profit is $\omega_h = 5,451,953$. The relationship between ω_h and P is reflected by the Figure 5a. When the product price is known, for example, set $P = 2000$; then the expected total profit can be maximised at $T = 64$, and the maximum expected total profit is $\omega_h = 2,667,529$. The relationship between ω_h and T is showed by the Figure 5b.



(a) The expected total profit ω_h against product price P, $T = 24$.

(b) The expected total profit ω_h against warranty length T, $P = 2,000$.

Fig. 5. The expected total profit ω_h.

If the manufacturer takes Type I software warranty policy, the parameters of software warranty cost and the software reliability growth model (SRGM) are set as: $c_{s1} = 100$, $a_2 = 50$, and $b_2 = 0.05$.

Then, the cumulative intensity function is

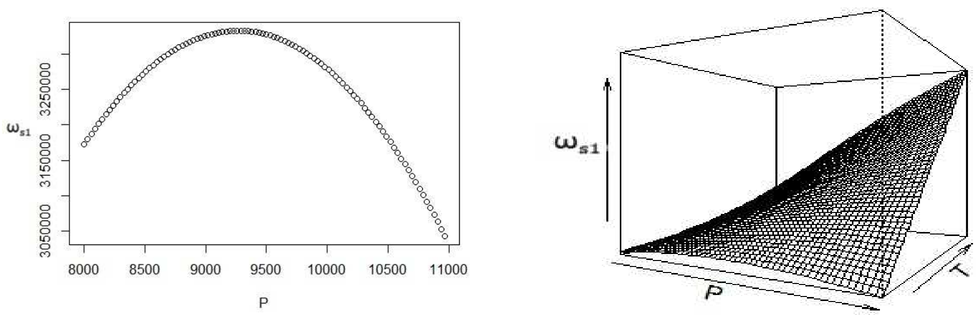
$$\Lambda_2(T) = 50(1 - e^{-0.05T}),$$

and the expected total profit of a software product is

$$\omega_{s1}(P, T) = (1000 - 0.1P + 21T)(P - 100 \times 50 \times (1 - e^{-0.05T})). \tag{65}$$

When the warranty length is known, for example, set $T = 24$ month, then, the expected total profit of a software product can be maximised at $P = 9267.44$, and the maximum expected total profit is $3,332,737$. The following Figure 6a presents the expected total profit against the product price P.

However, if P is known and $T \geq 0$, the first order derivative of Eq. (7) cannot be zero, i.e. the Eq. (7) does not have any maxima or minima. The surface in Figure 6b presents the expected total profit of a software product against P and T. According to this surface, we can find that when warranty length T is approaching positive infinity, the expected profit can also be positive infinity. Because, in SRGM, the expected cumulative number of software failures has an upper limit, i.e. the expected total warranty cost has a maximum value. However, in the real world, this phenomenon will not occur for two reasons: (1) a software has a limited length of service life; and (2) the linear sales volume function of M is only valid for a limited length of warranty in literature. Hence, the surface in Figure 6b only presents the expected total profit of a software product against P and T within a limited range of time.



(a) The expected total profit of a software product when $T = 24$.

(b) The expected total profit of a software product against P and T.

Fig. 6. The expected total profit of a software product.

If the manufacturer takes Type II software warranty policy, the related parameters are set in the following Table 3. Then, the expected total profit is

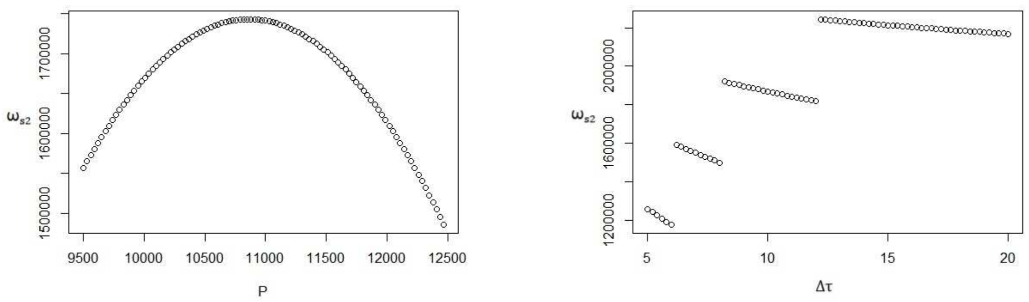
Table 3
Parameters for software warranty under Type II policy.

Software claim cost	SRGM parameter		Update cost	The changes after update
c_{s1}	a_2	b_2	c_{s2}	θ
100	50	0.05	500	90%

$$\omega_{s2}(P, T, \Delta\tau) = MP - [500n + 100 \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + 100 \int_{\tau_n}^{T} \lambda_{2,n}(t)dt], \quad (66)$$

where $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is the number of releasing patches; M is the number of product sold at $t = 0$; τ_i is the time of ith patches/updates release under time-based policy, $\tau_i = \tau_{i-1} + \Delta\tau$ and $\tau_0 = 0$; $\lambda_{2,i}$ is the failures intensity after ith patches/updates release, $\lambda_{2,i} = 90\% \times \lambda_{2,i-1}$ and $\lambda_{2,0}(t) = \lambda_2(t)$.

Then, if the warranty length is 2-year, i.e. $T = 24$, and the software's update is released quarterly, i.e. $\Delta\tau = 3$; the maximum expected total profit of this product $\omega_{s2} = 1,958,190$ can be achieved at $P = 10,614.86$. The Figure 7a presents the relationship between the expected total profit and product price under Type II software warranty policy if T and $\Delta\tau$ are known.



(a) The expected total profit ω_{s2} against P, when $T = 24$ and $\Delta\tau = 3$.

(b) The expected total profit ω_{s2} against $\Delta\tau$, when $P = 6500$ and $T = 24$.

Fig. 7. The expected total profit ω_{s2}.

If the warranty length is still 2-year, and the market price of this product is $P = 6500$; then the optimal update releasing interval is $\Delta\tau = 13.6$ month, and the maximized expected total profit is $2,227,765$. The Figure 7b presents the relationship between the expected total profit and the update releasing interval under Type II software warranty policy if P and T are known. The curve in Figure 7b is not continuous because the times of update releasing $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is an integer.

7.2 Expected total profit considering hardware-software interactions

There are 5 scenarios of hardware-software interactions discussed, in this section, the numeric examples for Scenario 0 and Scenario 1 are provided. For the product consists of hardware and software subsystems, the parameters are set in the Table 4.

Table 4
Parameters for Scenario 0 and 1 of interaction.

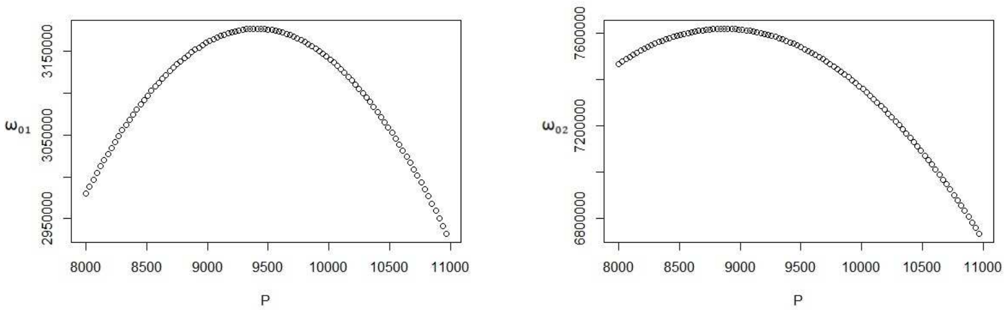
c_{h1}	a_1	b_1	c_{s1}	a_2	b_2	c_{s2}	θ
100	0.1	1.04	100	50	0.05	500	90%

7.2.1 Scenario 0

In Scenario 0, the product consists of hardware and software subsystems, but the failures of these two subsystems are assumed independently. The expected total profit under the first condition (software cannot be updated online) of Scenario 0 is

$$\omega_{01} = (1000 - 0.1P + 21T)\left[P - 10T^{1.04} - 5000(1 - e^{-0.05T})\right]. \tag{67}$$

When $T = 24$ is known, then the expected total profit is maximised at $P = 9403.55$, and the maximum expected total profit is $3,177,261$. The curve of expected total profit ω_{01} against product price P is displayed in Figure 8a.



(a) The expected total profit ω_{01} against P, when $T = 24$.

(b) The expected total profit ω_{02} against P, when $T = 24$.

Fig. 8. The expected total profit ω_{01} and ω_{02}, respectively.

If the software can be updated online and the pre-specified updating interval is $\Delta\tau = 6$months the expected total profit in Scenario 0 is

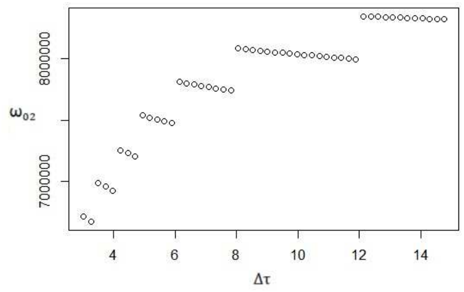
$$\begin{aligned}\omega_{02} = &(1000 - 0.1P + 21T)P - 10 \times (1000 - 0.1P + 21T)T^{1.04} \\ &+100 \times \left[\sum_{i=1}^{n} 0.9^{i-1}\left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right] + 0.9^n\left[\Lambda_2(T) - \Lambda_2(\tau_n)\right]\right] + 500n. \end{aligned}\tag{68}$$

If the warranty length is $T = 24$ months, then $n = 4$ the expected total profit is maximised at $P = 8,743.16$ and the maximum expected profit is $7,930,037$. The curve of expected total profit ω_{02} against product price P is displayed in Figure 8b.

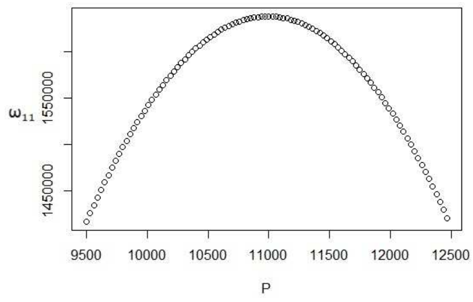
If the warranty length is $T = 24$ months and the product price $P = 8000$, the maximum expected total profit is $8,345,104$ achieved at $\Delta\tau = 12.25$. The curve of expected total profit ω_{02} against the updating interval $\Delta\tau$ is displayed in Figure 9a.

7.2.2 Scenario 1

In Scenario 1, the occurrence of hardware and software failures are statistical dependent: a software failure can lead to hardware failures with probability $p(t)$ at time t. Assume $p(t) = 0.1e^{-0.1t}$ in this case. If the software cannot be updated online, the expected total profit is Then



(a) The expected total profit ω_{02} against $\Delta\tau$, when $T = 24$ and $P = 8000$.



(b) The expected total profit ω_{11} against P, when $T = 24$.

Fig. 9. The expected total profit ω_{02} and ω_{11}, respectively.

the expected total profit is

$$\omega_{11} = (1000 - 0.1P + 21T)\left\{P - 100\left[0.1T^{1.04} + 50(1 - e^{-0.05T})\int_0^T 0.1e^{-0.1t}dt\right] - 100 \times 50(1 - e^{-0.05T})\right\}. \tag{69}$$

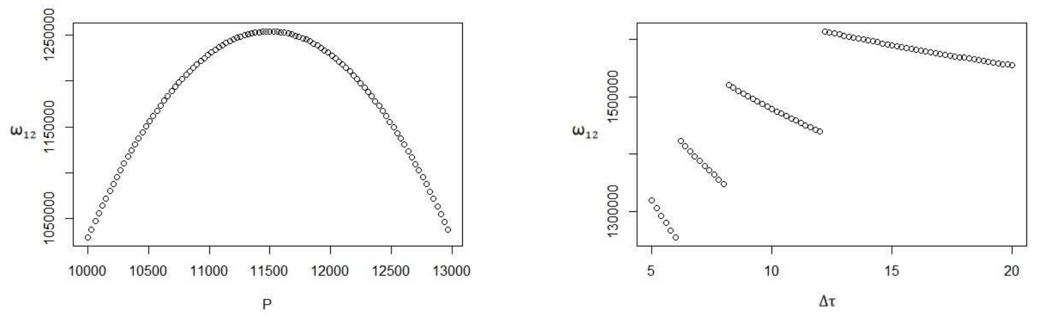
When $T = 24$ is known, then the maximum expected total profit $\omega_{11} = 1,638,784$ can be reached at $P = 10,991.81$. The curve of expected total profit ω_{11} against product price P is displayed in Figure 9b.

If the software can be updated online, then the expected total profit is

$$\begin{aligned}\omega_{12} &= (1000 - 0.1P + 21T)P - \mathrm{E}[C_{12}(T)] \\ &= (1000 - 0.1P + 21T)P - (1000 - 0.1P + 21T) \times 10T^{1.04} \\ &\quad -100\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T} \lambda_{2,n}(t)dt\right]\int_0^T 0.1e^{-0.1t}dt \\ &\quad -100\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T} \lambda_{2,n}(t)dt\right] - \lfloor\frac{T}{\Delta\tau}\rfloor \times 500.\end{aligned} \tag{70}$$

If the software is updated online in every 6 month, i.e. $\Delta\tau = 6$, the warranty length is $T = 24$ and the times of updating is $n = \lfloor\frac{T}{\Delta\tau}\rfloor = 4$; then, the maximum expected total profit $\omega_{12} = 1,336,010$ can be achieved at $P = 11,384.86$. The curve of expected total profit ω_{12} against product price P is displayed in Figure 10a.

If the warranty length is $T = 24$ months and the product price $P = 12,000$, the maximum expected total profit is $1,448,997$ achieved at $\Delta\tau = 13.03$. The curve of expected total profit ω_{12} against the updating interval $\Delta\tau$ is displayed in Figure 10b.



(a) The expected total profit ω_{12} against P, when $T = 24$ and $\Delta\tau = 6$.

(b) The expected total profit ω_{12} against $\Delta\tau$, when $T = 24$ and $P = 12,000$.

Fig. 10. The expected total profit ω_{12}, respectively.

7.3 Expected total profit considering hardware-software interactions and human factors

If a product is composed of hardware and software subsystems, the warranty cost of such product is influenced by the warranty claims on both hardware and software failures, the three human factors and the interplay between hardware and software subsystems. In this case, the hardware and software warranty parameters are set as same as above examples and the human parameters are set in Table 5.

Table 5
Parameters for human factors.

NFBR parameters				FBNR parameters		Delayed updating parameter
α_1	α_2	c_{hu1}	c_{hu2}	γ_1	γ_2	d
60	2	50	80	0.01	0.018	0.9

7.3.1 Off-line situation

If Scenario 1 of hardware-software interplay occurs, the software cannot be updated online, the NFBR and FBNR phenomena affect the total warranty cost, and the NFBR does not cause warranty to be ceased, the expected total profit is

$$\begin{aligned}\omega_{int31} &= MP - \mathrm{E}[C_{int31}(T)] = MP - \mathrm{E}[C_{11}(T)]q_1(T) - \mathrm{E}[C_{hu1}(T)]\\ &= MP - Me^{-0.01-0.018T}\left\{100\left[0.1T^{1.04} + 50(1-e^{-0.05T})\int_0^T 0.1e^{-0.1t}dt\right] + 5000(1-e^{-0.05T})\right\}\\ &\quad -50M(1-e^{-(T/60)^2}).\end{aligned} \tag{71}$$

If the warranty length is $T = 24$ months, the maximum expected total profit $\omega_{int31} = 2,786,918$ can be achieved at $P = 10,000$. The curve of expected total profit ω_{int31} against the price P

is displayed in Figure 11 by the line curve; in this figure, the point-curve, which is under the line curve, represents the total profit ω_{int31} against the price P without considering the human factors. Figure 11 indicates that if the human factors are not taken into account, the expected

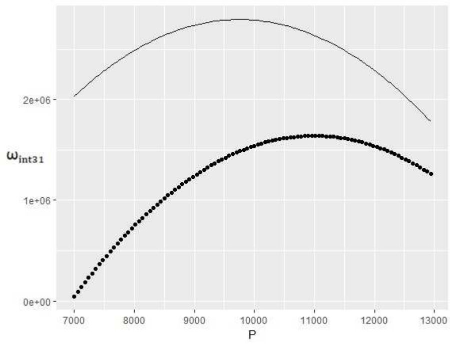


Fig. 11. The expected total profit ω_{int31} against P with and without considering human factors, when $T = 24$.

total profit is undervalued in this case.

If the NFBR may cause warranty to be ceased, the expected total profit is

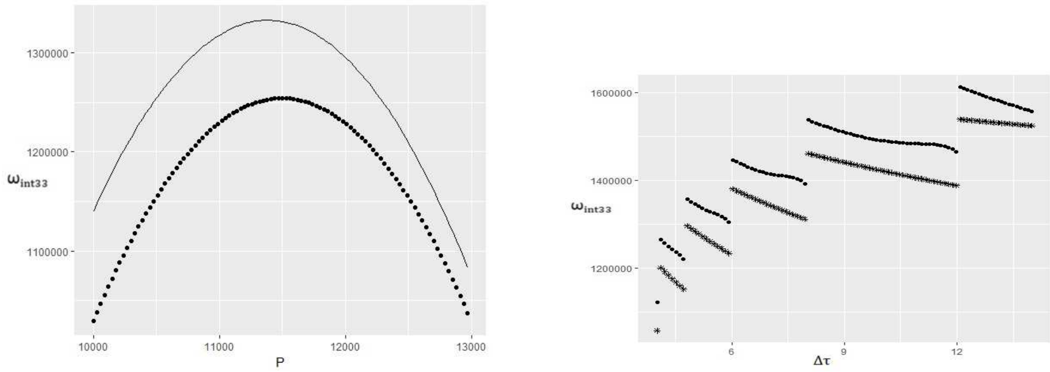
$$\begin{aligned}\omega_{int32} &= MP - \mathrm{E}[C_{int32}(T)] \\ &= MP - [\mathrm{E}[C_{11}(T)]q_1(T) + Mc_{hu2}]\, H_1(T) - \mathrm{E}[C_{11}(T)]q_1(T)(1 - H_1(T)).\end{aligned} \tag{72}$$

If the warranty length is $T = 24$ months, the maximum expected total profit $\omega_{int32} = 2,784,683$ can be achieved at $P = 9,999$. This result indicate that whether the NFBR event causes warranty to be ceased or not, the optimal value of price P is not influenced significantly.

7.3.2 Online situation

If the software can be updated online, the delayed updating effect is applied. Meanwhile, if the NFBR may not cause warranty to be ceased, the expected total profit is $\omega_{int33} = MP - \mathrm{E}[C_{int33}(T)]$, where $\mathrm{E}[C_{int33}(T)]$ consists of the cost of the NFBR event cost and the warranty cost modified with considering FBNR and delayed updating phenomena. According to Eqs. (53), (54) and (55), the optimal P and $\Delta\tau$ can be found by simulation. The simulation result shows that if the warranty length is $T = 24$ months and the updating interval is $\Delta\tau = 6$, the maximum expected total profit $\omega_{int33} = 1,521,743$ can be achieved at $P = 11,139.05$. The curve of expected total profit ω_{int33} against the price P is displayed in Figure 12a by the line curve; in this figure, the point-curve, which is under the line curve, represents the total profit ω_{int33} against the price P without considering the human factors.

If the warranty length $T = 24$ and the price $P = 12000$ are known, the optimal updating interval may also be determined by simulation. The result shows the maximum expected total



(a) The expected total profit ω_{int33} against P with and without considering human factors, when $T = 24$ and $\Delta\tau = 6$.

(b) The expected total profit ω_{int33} against $\Delta\tau$ with and without considering human factors, when $T = 24$ and $P = 12000$.

Fig. 12. The expected total profit ω_{int33}.

profit $\omega_{int33} = 1,584,178$ is achieved at $\Delta\tau = 13.03$. The plot of expected total profit ω_{int33} against the updating interval $\Delta\tau$ is displayed in Figure 12b by the points; in this figure, the stars, which is under the points, represent the total profit ω_{int33} against $\Delta\tau$ without considering the human factors.

Figure 12a and 12b indicate that if the human factors are not taken into account, the expected total profit is undervalued under this online situation.

8 Conclusions

The warranty cost of a product with embedded software system should be modelled in a manner that the failure of hardware, software and users. This paper developed the models of warranty costs incurred by hardware specific, software specific and hardware-software interaction failures and provided integrated models.

Future work in this area includes the form of the cumulative intensity function of hardware-software interaction failure and the relationships among the three types of failures in the second scenario. In the meantime, this paper assumes the repair upon a hardware failure is minimal. In our future work, we will consider imperfect repair using models such as the arithmetic reduction of age models, the arithmetic reduction of intensity models (Doyen & Gaudoin, 2004) or the doubly geometric process (Wu, 2018a).

In some scenarios in this paper, some scenarios assume that the repair upon failures is minimal, which may be too strong. Our future work will relax this assumption and consider more generic scenarios.

Acknowledgements

The authors are indebted to the reviewers for their comments that help improve the presentation.

References

Ahmadi, R. (2016). An optimal replacement policy for complex multi-component systems. *International Journal of Production Research*, *54*(17), 5303-5316.

Blischke, W., & Murthy, D. (1992). Product warranty management - i: A taxonomy for warranty policies. *European Journal of Operational Research*, *62*(2), 127-148.

Chen, C.-K., Lo, C.-C., & Weng, T.-C. (2017). Optimal production run length and warranty period for an imperfect production system under selling price dependent on warranty period. *European Journal of Operational Research*, *259*(2), 401-412.

Dai, Y., Zhou, S., & Xu, Y. (2012). Competitive and collaborative quality and warranty management in supply chains. *Production and Operations Management*, *21*(1), 129-144.

Ding, C., Rusmevichientong, P., & Topaloglu, H. (2014). Balancing revenues and repair costs under partial information about product reliability. *Production and Operations Management*, *23*(11), 1899-1918.

Doyen, L., & Gaudoin, O. (2004). Classes of imperfect repair models based on reduction of failure intensity or virtual age. *Reliability Engineering & System Safety*, *84*(1), 45–56.

Fernandez, A., & Stol, N. (2017). Economic, dissatisfaction, and reputation risks of hardware and software failures in pons. *IEEE/ACM Transactions on Networking*, *25*(2), 1119-1132.

Huang, H.-Z., Liu, Z.-J., & Murthy, D. (2007). Optimal reliability, warranty and price for new products. *IIE Transactions*, *39*(8), 819–827.

Huang, Y.-S., Huang, C.-D., & Ho, J.-W. (2017). A customized two-dimensional extended warranty with preventive maintenance. *European Journal of Operational Research*, *257*(3), 971–978.

Kansal, Y., Singh, G., Kumar, U., & Kapur, P. (2016). Optimal release and patching time of software with warranty. *International Journal of Systems Assurance Engineering and Management*, *7*(4), 462-468.

Kimura, M., Toyota, T., & Yamada, S. (1999). Economic analysis of software release problems with warranty cost and reliability requirement. *Reliability Engineering and System Safety*, *66*(1), 49-55.

Ladany, S., & Shore, H. (2007). Profit maximizing warranty period with sales expressed by a demand function. *Quality and Reliability Engineering International*, *23*(3), 291-301.

Lin, P.-C., Wang, J., & Chin, S.-S. (2009). Dynamic optimisation of price, warranty length and production rate. *International Journal of Systems Science*, *40*(4), 411–420.

Liu, B., Wu, J., & Xie, M. (2015). Cost analysis for multi-component system with failure interaction under renewing free-replacement warranty. *European Journal of Operational Research*, *243*(3), 874–882.

Luo, M., & Wu, S. (2018a). A mean-variance optimisation approach to collectively pricing warranty policies. *International Journal of Production Economics*, *196*, 101-112.

Luo, M., & Wu, S. (2018b). A value-at-risk approach to optimisation of warranty policy. *European Journal of Operational Research*, *267*(2), 513–522.

Matis, T. I., Jayaraman, R., & Rangan, A. (2008). Optimal price and pro rata decisions for combined warranty policies with different repair options. *IIE Transactions*, *40*(10), 984–991.

Murthy, D., & Djamaludin, I. (2002). New product warranty: A literature review. *International Journal of Production Economics*, *79*(3), 231-260.

Peng, R., Liu, B., Zhai, Q., & Wang, W. (2017). Optimal maintenance strategy for systems with two failure modes. *Reliability Engineering & System Safety*. doi: 10.1016/j.ress.2017.07.014

Pham, H., & Zhang, X. (1999). A software cost model with warranty and risk costs. *IEEE Transactions on Computers*, *48*(1), 71–75.

Roy, D., Murthy, C., & Mohanta, D. (2015). Reliability analysis of phasor measurement unit incorporating hardware and software interaction failures. *IET Generation, Transmission and Distribution*, *9*(2), 164-171.

Shafiee, M., & Chukova, S. (2013). Maintenance models in warranty: A literature review. *European Journal of Operational Research*, *229*(3), 561–572.

Singh, O., Kapur, P., Shrivastava, A., & Kumar, V. (2015). Release time problem with multiple constraints. *International Journal of Systems Assurance Engineering and Management*, *6*(1), 83-91.

Somboonsavatdee, A., & Sen, A. (2015). Parametric inference for multiple repairable systems under dependent competing risks. *Applied Stochastic Models in Business and Industry*, *31*(5), 706–720.

Teng, X., Pham, H., & Jeske, D. (2006). Reliability modeling of hardware and software interactions, and its applications. *IEEE Transactions on Reliability*, *55*(4), 571-577.

Wang, J., Wu, Z., Shu, Y., & Zhang, Z. (2015). An imperfect software debugging model considering log-logistic distribution fault content function. *Journal of Systems and Software*, *100*, 167-181.

WarrantyWeek. (2016, Dec). *Apple's warranties & service contracts.* http://www.warrantyweek.com/archive/ww20161215.html.

Williams, D. (2007). Study of the warranty cost model for software reliability with an imperfect debugging phenomenon. *Turkish Journal of Electrical Engineering and Computer*

Sciences, *15*(3), 369-381.
Wu, C.-C., Chou, C.-Y., & Huang, C. (2009). Optimal price, warranty length and production rate for free replacement policy in the static demand market. *Omega*, *37*(1), 29-39.
Wu, S. (2011). Warranty claim analysis considering human factors. *Reliability Engineering and System Safety*, *96*(1), 131-138.
Wu, S. (2018a). Doubly geometric processes and applications. *Journal of the Operational Research Society*, *69*(1), 66–77.
Wu, S. (2018b). A failure process model with the exponential smoothing of intensity functions. *European Journal of Operational Research*. doi: 10.1016/j.ejor.2018.11.045
Yazdian, S., Shahanaghi, K., & Makui, A. (2016). Joint optimisation of price, warranty and recovery planning in remanufacturing of used products under linear and non-linear demand, return and cost functions. *International Journal of Systems Science*, *47*(5), 1155-1175.
Ye, Z.-S., & Murthy, D. P. (2016). Warranty menu design for a two-dimensional warranty. *Reliability Engineering & System Safety*, *155*, 21–29.
Zhao, X., He, S., & Xie, M. (2018). Utilizing experimental degradation data for warranty cost optimization under imperfect repair. *Reliability Engineering & System Safety*, *177*, 108–119.

Appendix

Proof of Proposition 1: The expected profit of a hardware product is

$$\omega_h = MP - \mathrm{E}[C_h(T)] = (A - \beta P + \eta T)(P - c_{h1}a_1T^{b_1}).$$

If T is known, the first order derivatives of ω_h is $\frac{d\omega_h}{dP} = A - 2\beta P + \eta T + \beta c_{h1}a_1T^{b_1}$, and the second order derivatives of ω_h is $\frac{d^2\omega_h}{dP^2} = -2\beta < 0$. Then, ω_h is maximised at $P = \frac{A+\eta T+\beta c_{h1}a_1T^{b_1}}{2\beta}$.

If P is known, the first order derivatives of ω_h is $\frac{d\omega_h}{dT} = -\eta c_{h1}a_1(b_1+1)T^{b_1}+(\beta P-A)c_{h1}a_1b_1T^{b_1-1}+\eta P$, and the second order derivatives of ω_h is $\frac{d^2\omega_h}{dT^2} = -\eta c_{h1}a_1(b_1+1)b_1T^{b_1-1}+(\beta P-A)c_{h1}a_1b_1(b_1-1)T^{b_1-2}$.

Because $M = A - \beta P + \eta T \geq 0$, $T \geq 0$ and $b_1 > 1$, then, $\frac{d^2\omega_h}{dT^2} < 0$ and the optimal solution of T, maximising ω_h, exists. □

Proof of Proposition 2: The expected profit of a software product is

$$\omega_{s1} = MP - \mathrm{E}[C_{s1}(T)] = (A - \beta P + \eta T)(P - c_{s1}a_2(1 - e^{-b_2T})),$$

If T is known, the above function becomes to a parabolic function, it means the optimal P, which maximises ω_{s1}, exists.

If P is known, the first order derivative of ω_{s1} on T is $\frac{d\omega_{s1}}{dT} = c_{s1}a_2e^{-b_2T}(Ab_2+\beta Ab_2P+\eta-\eta Tb_2)+\eta P-\eta c_{s1}a_2$, then the second order derivative of ω_{s1} on T is $\frac{d^2\omega_{s1}}{dT^2} = b_2c_{s1}a_2e^{-b_2T}\left[b_2(A-\beta P+\eta T)-2\eta\right] > 0$, hence, the optimal T, which maximises ω_{s1}, does not exist. □

Proof of Proposition 3: If the manufacturer takes Type II software warranty, according to Eq. (6), the expected profit of a software product is

$$\omega_{s2} = MP - [nc_{s2} + c_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + c_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt].$$

If $\Delta\tau = \tau_i - \tau_{i-1}$ and T are known, the number of release n is also known, then the expected profit is

$$\omega_{s2} = MP - \left[nc_{s2} + c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i)-\Lambda_2(\tau_{i-1})\right] + c_{s1}\theta^n\left[\Lambda_2(T)-\Lambda_2(\tau_n)\right]\right];$$

it is equal to

$$\omega_{s2} = (A-\beta P+\eta T)(P-B) - nc_{s2},$$

where $B = c_{s1}a_2\sum_{i=1}^{n}\theta^{i-1}(e^{-b_2\tau_{i-1}} - e^{-b_2\tau_i}) + c_{s1}a_2\theta^n(e^{-b_2\tau_n} - e^{-b_2T})$ is a constant.

Obviously, the above function also is a parabolic function, then the optimal P, which maximises ω_{s2}, exists. □

Proof of Proposition 4: If the manufacturer takes Type II software warranty, according to Eq. (6), the expected profit of a software product is

$$\omega_{s2} = MP - \left[nc_{s2} + c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i)-\Lambda_2(\tau_{i-1})\right] + c_{s1}\theta^n\left[\Lambda_2(T)-\Lambda_2(\tau_n)\right]\right].$$

If P and T are known, this function is equal to

$$\omega_{s2} = (A-\beta P+\eta T)(P-c_{s1}a_2D) - nc_{s2},$$

where $D = \sum_{i=1}^{n}\theta^{i-1}(e^{-b_2\tau_{i-1}} - e^{-b_2\tau_i}) + \theta^n(e^{-b_2\tau_n} - e^{-b_2T})$ and $-nc_{s2}$ are the non-constant terms.

As $n = \lfloor\frac{T}{\Delta\tau}\rfloor$ is an integer greater than 0, and ω_{s2} decreases with n, then, when $n = 1$ t, ω_{s2} can be maximised. When $n = 1$, $\frac{T}{2} \leq \Delta\tau \leq T$. Then $D = (1-e^{-b_2\tau_1}) + \theta(e^{-b_2\tau_1} - e^{-b_2T})$, a local optimal $\Delta\tau$, which maximises ω_{s2}, exists. □