Determining the symmetries of difference equations

Pavlos Xenitidis

September 30, 2018

Abstract

We derive the determining equations for the N-th order generalised symmetries of partial difference equations defined on d consecutive quadrilaterals on the lattice using the theory of integrability conditions. We provide their algebraic formulation and develop the necessary theoretical framework for their analysis along with a systematic method for solving functional equations of the form $\mathcal{T}(f) + Af + B = 0$. Our approach is algorithmic and can be easily implemented in symbolic computations. We demonstrate our approach by deriving the symmetries of various equations and discuss certain applications and extensions of the theory.

1 Introduction

The theory of the symmetries of differential equations is well developed, its relation to integrability is known and there is a plethora of corresponding results and applications, see for instance [16, 23, 13]. Moreover the theory provides us the means to compute the symmetries of a given equation in an algorithmic way and, most importantly, to implement it in symbolic computations, see for instance the Mathematica package Sym [3].

The corresponding theory for difference equations is relatively new and several methods and approaches have been proposed recently for the computation of generalised symmetries of partial difference equations [5, 11, 12, 14, 15, 17, 20]. Levi and Yamilov developed the so-called generalised symmetry method for difference equations [11, 12] which has been used in a classification problem [8], and for the computation of symmetries, e.g. in [19]. Moreover based on this method, the authors of [5] proposed the use of the characteristic vector fields for the derivation of determining equations and the computation of symmetries. A different approach is offered by the theory of integrability conditions developed in [14, 15] which can be used either for the classification of integrable equations or to determine the symmetries of a given equation. Finally, other approaches have also been used in [20, 17].

All these methods deal with partial difference equations defined on an elementary quadrilateral of the square lattice (quad equations) and lead essentially to the same conditions for the existence of symmetries of order one or two. But there do exist difference equations defined on d consecutive quadrilaterals (d-quad equations),

$$Q(u_{n,m}, u_{n+1,m}, \ldots, u_{n+d,m}, u_{n,m+1}, u_{n+1,m+1}, \ldots, u_{n+d,m+1}) = 0, \quad d \in \mathbb{N}^*,$$

admitting generalised symmetries of order N, see for instance [1, 2, 4] and references therein. Therefore our aim here is to *consider d-quad equations* and systematically study their *generalised symmetries of any order*.

To achieve that we extend the theory of integrability conditions to include d-quad equations and derive corresponding conditions for the existence of symmetries of order N in the n direction. These conditions can then be used for the derivation of symmetries. Specifically, the relation between integrability conditions and symmetries stems from the fact that the latter (viewed as differential-difference equations) and partial difference equations share the same recursion operator [14]. This allows us to interpret the integrability conditions for the existence of a recursion operator of order N as *determining equations for symmetries* of the same order by identifying certain coefficients of the formal recursion operator $\mathfrak{R}$ with the first order derivatives of the symmetry generator F [15]. As we have two different ways to replace a pseudo-difference operator with a formal series (either Taylor or Laurent), we are able to construct a set of $2N$ *linear functional equations* with unknowns all the first order derivatives of F (except $\partial_{u_{n,m}} F$). But the most important fact is that these determining equations can be *derived algebraically* for any d-quad equation and its N-th order symmetry.

Since the determining equations are functional relations of the form $\mathcal{T}(f) + Af + B = 0$, where A, B are known functions, operator $\mathcal{T}$ is the shift in the m direction, and $f = f(n, m, u_{n-N,m}, \ldots, u_{n+N,m})$, we develop the necessary framework and tools to solve them. This amounts to defining two sets of dynamical variables based on the arguments of f and $\mathscr{T}(f)$, respectively; two elimination maps to eliminate one set of dynamical variables in favour of the other; and appropriate differential operators (chain rule) related to those sets of dynamical variables. Using this machinery, we propose a general algorithmic method to solve functional equations of the above form.

The main advantage of our approach is that all the key elements, like determining equations, differential operators and elimination maps, can be easily defined for any d-quad equation in a certain class and its N-th order symmetry and subsequently employed for solving functional equations in any software suitable for symbolic computations. It should be also emphasised that even though the integrability conditions were originally developed for autonomous quad equations and their autonomous symmetries, in their interpretation as determining equations they can also be used for the *study of non-autonomous equations and their autonomous and non-autonomous symmetries*, e.g. see [22], as well as [6] for a similar interpretation of the generalised symmetry method for quadrilateral equations.

The paper is organised as follows. Section 2 presents our notation and framework and gives the algebraic formulation of formal series and that of the integrability conditions. Section 3 deals with the dynamical variables and elimination maps necessary for dealing with functional equations and Section 4 presents our strategy for solving such equations. Section 5 implements the proposed method by deriving the symmetries of three difference equations and in the concluding section 6 we discuss further applications and extensions of the theory.

2 Symmetries and their determining equations

We start this section by introducing our notation. In order to make our presentation self-contained, we give a short review on symmetries of difference equations, pseudo-difference operators and integrability conditions based on [14, 15].We present the necessary extensions of the theory to cover scalar d-quad equations and derive algebraic formulae for the computation of formal series. Finally we present the integrability conditions, or in our terminology determining equations, in a purely algebraic form.

2.1 Notation and the class of difference equations

In what follows we consider scalar partial difference equations for a function u of two independent discrete variables n and m. The dependence of u on those variables is denoted in the standard way with indices, i.e. $u(n+i,m+j) = u_{n+i,m+j}$. The shift operators in the n and m direction are denoted by $\mathscr{S}$ and $\mathscr{T}$, respectively, and their action is defined as $\mathscr{S}^k(u_{n,m}) = u_{n+k,m}$ and $\mathscr{T}^\ell(u_{n,m}) = u_{n,m+\ell}$, respectively

The equations we are going to consider are partial difference equations defined on d consecutive quadrilaterals on the lattice, i.e. equations of the form

$$Q(u_{n,m}, u_{n+1,m}, \ldots, u_{n+d,m}, u_{n,m+1}, u_{n+1,m+1}, \ldots, u_{n+d,m+1}) = 0, \quad d \in \mathbb{N}^*. \tag{1}$$

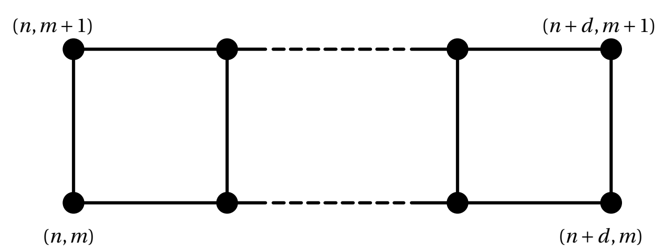

Figure 1: d consecutive quadrilaterals where equation (1) is defined.

For simplicity in our notation we will denote the derivatives of Q as

$$Q_{i,j} := \frac{\partial Q}{\partial u_{n+i,m+j}}. \tag{2}$$

Throughout our analysis we assume that function Q depends explicitly on the values of u at the corners of the quadrilateral on which the equation is defined. We can state this as

Requirement 1. *The defining function Q of equation (1) is such that*

$$Q_{0,0}Q_{d,0}Q_{0,1}Q_{d,1} \neq 0. \tag{3}$$

Moreover, we assume that

Requirement 2. *Function Q cannot be factored and represented as a product of functions depending on the same or a smaller number of variables.*

Requirement 3. *Equation (1) can be solved uniquely with respect to any of the corner values $u_{n,m}$, $u_{n+d,m}$, $u_{n,m+1}$ and $u_{n+d,m+1}$.*

Finally, we want to exclude the case that equation (1) degenerates to one defined on $d' < d$ quadrilaterals by a point transformation of the independent variable n. For instance, we want to exclude equations of the forms

$$Q(u_{n,m}, u_{n+k,m}, \ldots, u_{n+sk,m}, u_{n,m+1}, u_{n+k,m+1}, \ldots, u_{n+sk,m+1}) = 0 \quad \text{and} \quad Q(u_{n,m}, u_{n+d,m}, u_{n,m+1}, u_{n+d,m+1}) = 0,$$

which clearly can be written as

$$Q(u_{n',m}, u_{n'+1,m}, \ldots, u_{n'+s,m}, u_{n',m+1}, u_{n'+1,m+1}, \ldots, u_{n'+s,m+1}) = 0 \quad \text{and} \quad Q(u_{n',m}, u_{n'+1,m}, u_{n',m+1}, u_{n'+1,m+1}) = 0,$$

respectively, after an appropriate change of the independent variable n. So we have to assume that Q depends explicitly on at least one of the intermediate shifts of u. We can formulate this as

Requirement 4. *Let $k > 1$ denote the divisors of $d > 1$. We require that for every divisor k at least one of the derivatives $Q_{i\,k+\ell,j}$ is not identically zero, with $i = 0, \ldots, \frac{d}{k} - 1$, $\ell = 1, \ldots, k-1$ and $j = 0, 1$. Equivalently, for every divisor $k > 1$ of d we require that*

$$\sum_{i=0}^{d/k-1} \sum_{\ell=1}^{k-1} \sum_{j=0}^{1} Q^2_{i\,k+\ell,j} \neq 0. \tag{4}$$

Example 2.1. It is easy to check that the quadrilateral equation [19]

$$u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}(u_{n,m} + u_{n+1,m+1} + 1) + \chi = 0, \tag{5}$$

satisfies the first three requirements. As in this case $d = 1$, the last requirement does not apply to (5).

On the other hand, the two-quad equation

$$u_{n,m}u_{n+2,m+1}\left(u_{n+1,m+1}(u_{n+1,m} + u_{n,m+1}) + u_{n+1,m}u_{n+2,m}\right) + \alpha = 0 \tag{6}$$

satisfies requirements 1–3 (unless $\alpha = 0$ in which case requirement 2 is not satisfied). For the fourth one, since it is $d = 2$ and $k = 2$, relation (4) becomes $Q^2_{1,0} + Q^2_{1,1} \neq 0$, which obviously holds. □

2.2 Symmetries and recursion operators

We collect here some necessary definitions to make our presentation self-contained. We also use the notation $f([u])$ to denote that function f depends on $u_{n,m}$ and a finite, but otherwise unspecified, number of shifted values of u.

Definition 2.1. *The Fréchet derivative of a function $f([u])$ is defined as*

$$\mathrm{D}_f := \sum_{i,j} \frac{\partial f}{\partial u_{n+i,m+j}} \mathcal{S}^i \mathcal{T}^j. \tag{7}$$

Using the notion of Fréchet derivative, we can define the symmetries of a difference equation in the following way.

Definition 2.2. *A function $F(n,m,[u])$ is a symmetry of equation $Q(n,m,[u])=0$ if*

$$\mathrm{D}_Q(F)=0 \tag{8}$$

holds on solutions of $Q(n,m,[u])=0$.

It is also useful to interpret a symmetry as a differential-difference equation compatible with the difference equation. More precisely,

Definition 2.3. *Assume that $u_{n,m}$ depends also on a continuous variable t. Then, the differential-difference equation*

$$\partial_t u_{n,m} = F(n,m,[u]) \tag{9}$$

defines a symmetry of the difference equation $Q(n,m,[u])=0$ if $D_t(Q)=0$ on solutions of the difference equation.

It is not difficult to verify that the requirement $D_t(Q)=0$ in view of (9) is equivalent to (8).

In what follows we discuss *generalised symmetries* of equation (1), and more precisely generalised symmetries in the n direction which are of the form $F(n,m,u_{n-N,m},\ldots,u_{n+N,m})$. We call the positive integer N the *order (or length) of symmetry F*. We restrict our analysis to this kind of symmetries because they can be computed systematically. However, and as far as we are aware, the symmetries in the other lattice direction are of the general form

$$G = G(u_{n,m-M},\ldots,u_{n,m+M},\ldots,u_{n+d-1,m-M},\ldots,u_{n+d-1,m+M}), \quad M\in\mathbb{N},$$

for which, at the moment, there is no systematic way to compute them when $d>1$.

The existence of an infinite hierarchy of generalised symmetries of increasing order serves as a definition or criterion of *integrability*. In fact to prove integrability in this context is sufficient to find a *recursion operator* $\mathfrak{R}$ which maps symmetries to (higher order) symmetries. For the symmetries we are interested in recursion operators are $\mathscr{S}$-pseudo-difference operators [14] which can be formally represented either by their corresponding *formal Taylor series*

$$\mathfrak{R}_T = \sum_{i=-N}^{\infty} \hat{r}_i \mathscr{S}^i = \hat{r}_{-N}\mathscr{S}^{-N} + \hat{r}_{-N+1}\mathscr{S}^{-N+1} + \cdots + \hat{r}_{-1}\mathscr{S}^{-1} + \hat{r}_0 + \hat{r}_1\mathscr{S} + \cdots,$$

or by their *formal Laurent series*

$$\mathfrak{R}_L = \sum_{i=-N}^{\infty} \tilde{r}_{-i} \mathscr{S}^{-i} = \tilde{r}_{N}\mathscr{S}^{N} + \tilde{r}_{N-1}\mathscr{S}^{N-1} + \cdots + \tilde{r}_{1}\mathscr{S} + \tilde{r}_0 + \tilde{r}_{-1}\mathscr{S}^{-1} + \cdots.$$

The positive integer N is also referred to as *the order* of the recursion operator.

Example 2.2. Consider the discrete potential KdV (or H1) equation

$$(u_{n,m}-u_{n+1,m+1})(u_{n+1,m}-u_{n,m+1}) = \alpha-\beta. \tag{10}$$

Its lowest symmetry in the n direction is of length one and has the form

$$\partial_t u_{n,m} = F_{n,m} = \frac{1}{u_{n+1,m}-u_{n-1,m}}. \tag{11}$$

The corresponding recursion operator is [14]

$$\mathfrak{R} = F_{n,m}\,(\mathscr{S}+1)\,(\mathscr{S}-1)^{-1}\,F_{n,m}\left(\mathscr{S}-\mathscr{S}^{-1}\right).$$

To represent this operator as a formal series, we have to replace $(\mathscr{S}-1)^{-1}$ with a formal series and then expand the resulting expression. If we represent $(\mathscr{S}-1)^{-1}$ with its formal Taylor series $(\mathscr{S}-1)^{-1} = -1-\mathscr{S}-\mathscr{S}^2-\cdots$, then we can write the recursion operator as

$$\mathfrak{R}_T = F_{n,m}^2\mathscr{S}^{-1} + 2F_{n,m}F_{n+1,m} + F_{n,m}(2F_{n+2,m}-F_{n,m})\mathscr{S} + \cdots. \tag{12}$$

If we have employed the formal Laurent series $(\mathscr{S}-1)^{-1}=\mathscr{S}^{-1}+\mathscr{S}^{-2}+\mathscr{S}^{-3}+\cdots$, then we would have ended up with

$$\mathfrak{R}_L = F_{n,m}^2\mathscr{S} + 2F_{n,m}F_{n-1,m} + F_{n,m}(2F_{n-2,m}-F_{n,m})\mathscr{S}^{-1}+\cdots. \tag{13}$$

In both cases we can easily check that $(\mathscr{S}-1)^{-1}\circ(\mathscr{S}-1)=(\mathscr{S}-1)\circ(\mathscr{S}-1)^{-1}=1$ hold when we replace operator $(\mathscr{S}-1)^{-1}$ with either of its formal series.

It should be noted that H1 admits one more symmetry of the same order [20], namely

$$\partial_s u_{n,m} = \frac{n}{u_{n+1,m}-u_{n-1,m}} + \frac{u_{n,m}}{2(\alpha-\beta)},$$

which can be derived using our approach based on the integrability conditions and the formal recursion operator. However the action of $\mathfrak{R}$ on the above symmetry cannot be defined locally and leads to non-local symmetries [21]. □

2.3 Determining equations

In this section we exploit recursion operators to derive necessary *integrability conditions* for equation (1) which also serve as *determining equations* for the symmetries of the same equation. More precisely, our approach extents the corresponding theory for quadrilateral equations developed in [14, 15] and employs the first few of these conditions as equations to determine the symmetries of the equation.

We start be presenting the necessary generalisation of Theorem 1 in [14].

Theorem 2.4. *Consider the difference equation (1).*

1. *If there exist $\mathscr{S}$-pseudo-difference operators $\mathfrak{R}$ and $\mathfrak{P}$ such that*

$$\mathrm{D}_Q\circ\mathfrak{R} = \mathfrak{P}\circ\mathrm{D}_Q, \tag{14}$$

where

$$\mathrm{D}_Q = \sum_{i=0}^{d} Q_{i,0}\mathscr{S}^i + \sum_{i=0}^{d} Q_{i,1}\mathscr{S}^i\mathscr{T}, \tag{15}$$

then $\mathfrak{R}$ is a recursion operator for equation (1).

2. *Relation (14) is satisfied if and only if*

$$\mathscr{T}(\mathfrak{R}) = \mathscr{B}^{-1}\circ\mathscr{A}\circ\mathfrak{R}\circ\mathscr{A}^{-1}\circ\mathscr{B}, \tag{16}$$

and the operator $\mathfrak{P}$ satisfies

$$\mathfrak{P} = \mathscr{A}\circ\mathfrak{R}\circ\mathscr{A}^{-1}, \tag{17}$$

where

$$\mathscr{A} := \sum_{i=0}^{d} Q_{i,0}\mathscr{S}^i \quad \text{and} \quad \mathscr{B} := \sum_{i=0}^{d} Q_{i,1}\mathscr{S}^i, \tag{18}$$

Proof. The proof is omitted here as it is similar to the one given in [14]. □

Equation (16) is obviously satisfied if we replace $\mathscr{A}^{-1}$, $\mathscr{B}^{-1}$ and $\mathfrak{R}$ with their respective formal series. And we can do that using either Taylor or Laurent formal series. Even though these two options lead to equivalent integrability conditions and conservation laws [14], they provide us with two *inequivalent sets of determining equations for the symmetry F* as they involve *different derivatives of F*.

To see that we have to take into account that difference and differential-difference equations share the same recursion operator. To be more precise, if $\partial_t u_{n,m}=F$ is a symmetry of equation (1) and $\mathfrak{R}$ is the corresponding recursion operator, then $\mathfrak{R}$ is a recursion operator for this differential-difference equation and the following relation holds.

$$\partial_t\mathfrak{R} = [\mathrm{D}_F,\mathfrak{R}] \tag{19}$$

If we replace the recursion operator in the above relation with its formal series, then we can derive the connection among the derivatives of F and the coefficients in the formal series of $\mathfrak{R}$. Specifically, if we employ the Laurent series

$$\mathfrak{R}_L = \tilde{r}_N \mathscr{S}^N + \tilde{r}_{N-1}\mathscr{S}^{N-1} + \cdots + \tilde{r}_1 \mathscr{S} + \tilde{r}_0 + \tilde{r}_{-1}\mathscr{S}^{-1} + \ldots \tag{20}$$

or the Taylor series

$$\mathfrak{R}_T = \hat{r}_{-N}\mathscr{S}^{-N} + \hat{r}_{-N+1}\mathscr{S}^{-N+1} + \cdots + \hat{r}_{-1}\mathscr{S}^{-1} + \hat{r}_0 + \hat{r}_1\mathscr{S} + \ldots, \tag{21}$$

then relation (19) implies that $\tilde{r}_i$ and $\hat{r}_{-i}$, with $i = 1,\ldots,N$, are proportional to first order derivatives of F. However, as we are interested in determining symmetry F and *not* the formal recursion operator, it is sufficient to consider that

$$\tilde{r}_i = \frac{\partial F}{\partial u_{n+i,m}}, \quad \hat{r}_{-i} = \frac{\partial F}{\partial u_{n-i,m}}, \quad i = 1,\ldots,N. \tag{22}$$

So our next target is to extract these determining equations from (16) by exploiting Taylor and Laurent series and formulate them *algebraically*.

Example 2.3. Starting with equation (10), its symmetry (11) and the formal recursion operators given in Example 2.2 we can readily verify that (i) The leading term in the Taylor series (12) is indeed the derivative of $F_{n,m}$ (11) with respect to $u_{n-1,m}$, and (ii) The leading term in the Laurent series (13) is equal to $-\partial_{u_{n+1,m}} F_{n,m}$. □

2.4 Formal series and their algebraic formulation

For the algebraic formulation of (16) we need to compute formal series for pseudo-difference operators. For this purpose we first define two matrices.

Definition 2.5. *For any $K \in \mathbb{N}^*$, we define the $K \times K$ matrices $\mathrm{L}(\boldsymbol{a})$ and $\mathrm{T}(\boldsymbol{a})$, where $\boldsymbol{a} \in \mathbb{R}^K$, as*

$$(\mathrm{L}(\boldsymbol{a}))_{i,j} = \begin{cases} \mathscr{S}^{1-j}\left((\boldsymbol{a})_{i-j+1}\right), & i \geq j \\ 0 & i < j \end{cases} \quad \text{and} \quad (\mathrm{T}(\boldsymbol{a}))_{i,j} = \begin{cases} \mathscr{S}^{j-1}\left((\boldsymbol{a})_{i-j+1}\right), & i \geq j \\ 0 & i < j \end{cases}, \tag{23}$$

respectively, where $(\boldsymbol{a})_k$ denotes the k^{th} entry of $\boldsymbol{a}$.

Moreover, with any difference operator Φ of order d, i.e. $\Phi = \phi_d \mathscr{S}^d + \ldots + \phi_1 \mathscr{S} + \phi_0$ where $d \in \mathbb{N}^*$ and $\phi_0\phi_d \neq 0$, we associate two vectors $\boldsymbol{\phi}_L$, $\boldsymbol{\phi}_T$ in $\mathbb{R}^K$ according to the following rules.

$$\boldsymbol{\phi}_L = \begin{cases} (\phi_d \cdots \phi_{d+1-K})^\intercal, & \text{if } K \leq d+1 \\ (\phi_d \cdots \phi_0\, 0 \cdots 0)^\intercal, & \text{if } K > d+1 \end{cases}, \quad \text{and} \quad \boldsymbol{\phi}_T = \begin{cases} (\phi_0 \cdots \phi_{K-1})^\intercal, & \text{if } K \leq d+1 \\ (\phi_0 \cdots \phi_{d+1}\, 0 \cdots 0)^\intercal, & \text{if } K > d+1 \end{cases} \tag{24}$$

Using the matrices in Definition 2.5 and the vectors in (24), we can determine algebraically the coefficients of the formal series of the inverse of operator Φ.

Theorem 2.6. *Consider the difference operator $\Phi = \phi_d \mathscr{S}^d + \ldots + \phi_1 \mathscr{S} + \phi_0$, where $d \in \mathbb{N}^*$ and $\phi_0\phi_d \neq 0$, and its associated vectors $\boldsymbol{\phi}_L$, $\boldsymbol{\phi}_T$ given in (24). Moreover, let $\boldsymbol{e}_1$ denote vector $(1\,0 \cdots)^\intercal$ of $\mathbb{R}^K$.*

1. *The formal Laurent series of the inverse of Φ can be written as*

$$\Phi_L^{-1} = \sum_{j=d}^{\infty} \tilde{\phi}_j \mathscr{S}^{-j} = \tilde{\phi}_d \mathscr{S}^{-d} + \tilde{\phi}_{d+1}\mathscr{S}^{-d-1} + \ldots, \tag{25}$$

with the first K coefficients given by

$$\tilde{\boldsymbol{\phi}}_L = (\tilde{\phi}_d \cdots \tilde{\phi}_{d+K-1})^\intercal = \mathscr{S}^{-d}\left(\mathrm{L}(\boldsymbol{\phi}_L)^{-1}\right)\boldsymbol{e}_1. \tag{26}$$

2. *The formal Taylor series of the inverse of* Φ *can be written as*

$$\Phi_T^{-1} = \sum_{j=0}^{\infty} \hat{\phi}_j \mathscr{S}^j = \hat{\phi}_0 + \hat{\phi}_1 \mathscr{S} + \hat{\phi}_2 \mathscr{S}^2 + \dots, \tag{27}$$

with the first K coefficients given by

$$\hat{\boldsymbol{\phi}}_T = (\hat{\phi}_0 \cdots \hat{\phi}_{K-1})^\top = \mathrm{T}(\boldsymbol{\phi}_T)^{-1} \boldsymbol{e}_1. \tag{28}$$

Proof. Using Φ and the Laurent series (25) along with $\Phi^{-1} \circ \Phi = 1$, we end up with

$$\sum_{k=0}^{\infty} \left(\sum_{i=d}^{d+k} \tilde{\phi}_i \mathscr{S}^{-i}(\phi_{i-k}) \right) \mathscr{S}^{-k} = \tilde{\phi}_d \mathscr{S}^{-d}(\phi_d) + \left(\tilde{\phi}_d \mathscr{S}^{-d}(\phi_{d-1}) + \tilde{\phi}_{d-1} \mathscr{S}^{-d-1}(\phi_d) \right) \mathscr{S}^{-1} + \dots = 1.$$

We can collect the first K terms in the above relation and write them as

$$\begin{pmatrix} \mathscr{S}^{-d}(\phi_d) & 0 & \cdots & 0 & 0 \\ \mathscr{S}^{-d}(\phi_{d-1}) & \mathscr{S}^{-d-1}(\phi_d) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathscr{S}^{-d}(\phi_{d-K+1}) & \mathscr{S}^{-d-1}(\phi_{d-K+2}) & \cdots & \cdots & \mathscr{S}^{-d-K+1}(\phi_d) \end{pmatrix} \begin{pmatrix} \tilde{\phi}_d \\ \tilde{\phi}_{d+1} \\ \vdots \\ \tilde{\phi}_{d+K-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\phi_d \neq 0$, the lower triangular matrix of the system is invertible and thus we can express the first K coefficients in the Laurent series (25) in terms of the components of operator Φ. Taking into account the definition (23) of matrix L we can write the solution to the above system as in (26). Working in the same way but employing Taylor series (27), we end up with a similar linear system for the first K components of the series (27), the solution to which is (28). ☐

Using Theorem 2.6, we can now compute the composition of an $\mathscr{S}$-operator and a formal series according to

Theorem 2.7. *Consider the difference operators* $\Phi = \phi_d \mathscr{S}^d + \dots + \phi_1 \mathscr{S} + \phi_0$ *and* $\Psi = \psi_d \mathscr{S}^d + \dots + \psi_1 \mathscr{S} + \psi_0$ *with* $d \in \mathbb{N}^*$ *and* $\phi_0 \phi_d \psi_0 \psi_d \neq 0$. *Let* $\boldsymbol{\phi}_L$, $\boldsymbol{\phi}_T$ *and* $\boldsymbol{\psi}_L$, $\boldsymbol{\psi}_T$ *be their associated vectors (24). Then*

1. *The first K coefficients of the formal Laurent series*

$$\Phi^{-1} \circ \Psi = \tilde{c}_0 + \tilde{c}_1 \mathscr{S}^{-1} + \tilde{c}_2 \mathscr{S}^{-2} + \dots$$

are given by

$$(\tilde{c}_0 \cdots \tilde{c}_{K-1})^\top = \mathscr{S}^{-d} \left(\mathrm{L}(\boldsymbol{\psi}_L) \mathrm{L}(\boldsymbol{\phi}_L)^{-1} \right) \boldsymbol{e}_1. \tag{29}$$

2. *The first K coefficients of the formal Taylor series*

$$\Phi^{-1} \circ \Psi = \hat{c}_0 + \hat{c}_1 \mathscr{S} + + \hat{c}_2 \mathscr{S}^2 + \dots$$

are given by

$$(\hat{c}_0 \cdots \hat{c}_{K-1})^\top = \mathrm{T}(\boldsymbol{\psi}_T) \mathrm{T}(\boldsymbol{\phi}_T)^{-1} \boldsymbol{e}_1. \tag{30}$$

Proof. Using Theorem 2.6 we replace operator Φ^{-1} with its formal series. Then we expand the composition $\Phi^{-1} \circ \Psi$ and collect coefficients of different powers of the shift operator which yields the above algebraic relations. ☐

2.5 Algebraic formulation of the determining equations

We may now replace in (16) all the pseudo-difference operators with their formal series and use Theorem 2.7 accordingly to derive the sought algebraic form of the determining equations.

Theorem 2.8 (Laurent series)**.** *Consider relation (16) with operators $\mathscr{A}$, $\mathscr{B}$ given in (18) and $\mathfrak{R}$ being an $\mathscr{S}$-pseudo-difference operator of order N. Let $\boldsymbol{q_{L_0}}$ and $\boldsymbol{q_{L_1}}$ be the vectors associated with $\mathscr{A}$ and $\mathscr{B}$, respectively, defined as $\left(\boldsymbol{q_{L_j}}\right)_i = Q_{d+1-i,j}$, $i=1,\ldots,K$ and $j=0,1$. If*

$$\mathfrak{R}_L = \tilde{r}_N \mathscr{S}^N + \ldots + \tilde{r}_0 + \tilde{r}_{-1}\mathscr{S}^{-1} + \ldots$$

is the formal Laurent series of $\mathfrak{R}$, then the first K integrability conditions following from (16) can be written as

$$\mathscr{T}(\tilde{\boldsymbol{r}}) = \mathscr{S}^N\Big(\mathrm{L}(\tilde{\boldsymbol{x}})\Big)\,\mathrm{L}(\tilde{\boldsymbol{r}})\,\mathscr{S}^{-d}\Big(\mathrm{L}(\boldsymbol{q_{L_0}})\,\mathrm{L}(\boldsymbol{q_{L_1}})^{-1}\Big)\,\boldsymbol{e}_1, \tag{31}$$

where $\tilde{\boldsymbol{r}} = (\tilde{r}_N\, \tilde{r}_{N-1} \cdots \tilde{r}_{N-K+1})^\intercal$ and $\tilde{\boldsymbol{x}} = (\tilde{x}_0 \cdots \tilde{x}_{K-1})^\intercal := \mathscr{S}^{-d}\left(\mathrm{L}(\boldsymbol{q_{L_1}})\mathrm{L}(\boldsymbol{q_{L_0}})^{-1}\right)\boldsymbol{e}_1$.

Proof. If we replace $\mathscr{A}^{-1}\circ\mathscr{B}$ and $\mathscr{B}^{-1}\circ\mathscr{A}$ in (16) according to Theorem 2.7 and $\mathfrak{R}$ with its formal Laurent series, then relation (31) follows by employing twice Theorem 2.7. □

Using formal Taylor series we end up with

Theorem 2.9 (Taylor series)**.** *Consider relation (16) with operators $\mathscr{A}$, $\mathscr{B}$ given in (18) and $\mathfrak{R}$ being an $\mathscr{S}$-pseudo-difference operator of order N. Let $\boldsymbol{q_{T_0}}$ and $\boldsymbol{q_{T_1}}$ be the vectors associated with $\mathscr{A}$ and $\mathscr{B}$, respectively, defined as $\left(\boldsymbol{q_{T_j}}\right)_i = Q_{i-1,j}$, $i=1,\ldots,K$ and $j=0,1$. If*

$$\mathfrak{R}_T = \hat{r}_{-N}\mathscr{S}^{-N} + \ldots + \hat{r}_0 + \hat{r}_1\mathscr{S} + \ldots$$

is the formal Taylor series of $\mathfrak{R}$, then the first K integrability conditions following from (16) can be written as

$$\mathscr{T}(\hat{\boldsymbol{r}}) = \mathscr{S}^{-N}\Big(\mathrm{T}(\hat{\boldsymbol{x}})\Big)\,\mathrm{T}(\hat{\boldsymbol{r}})\,\mathrm{T}(\boldsymbol{q_{T_0}})\,\mathrm{T}(\boldsymbol{q_{T_1}})^{-1}\,\boldsymbol{e}_1, \tag{32}$$

where $\hat{\boldsymbol{r}} = (\hat{r}_{-N}\,\hat{r}_{-N+1}\cdots\hat{r}_{K-N-1})^\intercal$ and $\hat{\boldsymbol{x}} := (\hat{x}_0\cdots\hat{x}_{K-1})^\intercal = \mathrm{T}(\boldsymbol{q_{T_1}})\mathrm{T}(\boldsymbol{q_{T_0}})^{-1}\boldsymbol{e}_1$.

Proof. It is similar to the proof of Theorem 2.8. □

Remark 2.1. Writing equation (16) in the equivalent form $\mathfrak{R} = \mathscr{A}^{-1}\circ\mathscr{B}\circ\mathscr{T}(\mathfrak{R})\circ\mathscr{B}^{-1}\circ\mathscr{A}$ we can express vectors $\tilde{\boldsymbol{r}}$, $\hat{\boldsymbol{r}}$ in terms of their $\mathscr{T}$ shifts. In particular the use of formal Laurent series leads to

$$\tilde{\boldsymbol{r}} = \mathscr{S}^N\Big(\mathrm{L}(\tilde{\boldsymbol{y}})\Big)\,\mathrm{L}(\mathscr{T}(\tilde{\boldsymbol{r}}))\,\mathscr{S}^{-d}\Big(\mathrm{L}(\boldsymbol{q_{L_1}})\,\mathrm{L}(\boldsymbol{q_{L_0}})^{-1}\Big)\,\boldsymbol{e}_1, \quad \tilde{\boldsymbol{y}} := \mathscr{S}^{-d}\left(\mathrm{L}(\boldsymbol{q_{L_0}})\mathrm{L}(\boldsymbol{q_{L_1}})^{-1}\right)\boldsymbol{e}_1, \tag{33}$$

while the use of Taylor series results to

$$\hat{\boldsymbol{r}} = \mathscr{S}^{-N}\Big(\mathrm{T}(\hat{\boldsymbol{y}})\Big)\,\mathrm{T}(\mathscr{T}(\hat{\boldsymbol{r}}))\,\mathrm{T}(\boldsymbol{q_{T_1}})\,\mathrm{T}(\boldsymbol{q_{T_0}})^{-1}\,\boldsymbol{e}_1, \quad \hat{\boldsymbol{y}} := \mathrm{T}(\boldsymbol{q_{T_0}})\mathrm{T}(\boldsymbol{q_{T_1}})^{-1}\boldsymbol{e}_1. \tag{34}$$

The positive integer K appearing in Theorems 2.8 and 2.9 is arbitrary. However, for the derivation of a symmetry of order N, it is sufficient to choose $K=N$ because of relations (22). Thus, *functional equations (31-34) with $K=N$ and in view of relations (22), along with equation (8), are the determining equations for the N-th order generalised symmetry F in the n direction of the difference equation (1).*

Remark 2.2. We are interested in determining *the lowest order generalised symmetries in the n direction.* As there is no criterion for the initial choice of N, we have to start with $N=1$ and work successively until we find a non-trivial generalised symmetry. All the examples we have at our disposal suggest that the lowest order generalised symmetries admitted by equation (1) are of order $N \le d+1$. □

Now the next step is to try to solve the functional equations (31-34) by reducing them into a system of *partial differential equations* for the coefficients $\tilde{r}$ and $\hat{r}$ of the formal recursion operator. For this purpose, we have first to introduce a few concepts and tools.

3 Dynamical variables, elimination maps and differentiation

In this section we present the tools we are going to use to solve functional equations like the determining equations (31), (32). More precisely, we define two different sets of *dynamical variables*, corresponding *elimination maps*, and appropriate *total derivatives* for the analysis of functional equations. We define these concepts in a way suitable for symbolic computations.

3.1 Dynamical variables

The defining equation (8) for symmetries and the integrability condition (16) hold on solutions of the corresponding difference equation. This means that we must use the latter and its shifts to eliminate some of the values of u appearing in the former equations. Since Requirements 1 and 3 allow us to solve equation (1) uniquely for any of the corner values $u_{n,m}$, $u_{n,m+1}$, $u_{n+d,m}$ and $u_{n+d,m+1}$, we choose to eliminate always values of u which lie on the same horizontal line. More precisely,

- We use equation (1) and its shifts to eliminate variables $u_{n+s,m+1}$ with $s \geq d$ or $s < 0$. Then, any expression involving those variables, like (31) and (32), will become an expression depending only on $\{u_{n+i,m}\}_{i\in\mathbb{Z}}$ and $\{u_{n+j,m+1}\}_{j=0}^{d-1}$.
- Alternatively, we employ equation (1) and its shifts to eliminate variables $u_{n+s,m}$ with $s \geq d$ or $s < 0$. In this case, any expression involving these variables will be reduced to one depending only on $\{u_{n+i,m+1}\}_{i\in\mathbb{Z}}$ and $\{u_{n+j,m}\}_{j=0}^{d-1}$.

These are the two different sets of *dynamical variables* which we are going to use in our analysis of functional equations and we denote them as

$$\mathrm{U}_0 = \left\{\{u_{n+i,m}\}_{i\in\mathbb{Z}} \cup \{u_{n+j,m+1}\}_{j=0}^{d-1}\right\}, \quad \mathrm{U}_1 = \left\{\{u_{n+i,m+1}\}_{i\in\mathbb{Z}} \cup \{u_{n+j,m}\}_{j=0}^{d-1}\right\}. \tag{35}$$

We also denote the sets of the *eliminated variables* as

$$\begin{aligned} \mathrm{V}_0 = \mathrm{U}_0 \setminus \mathrm{U}_1 = \left\{u_{n+s,m} : s \geq d \text{ or } s < 0\right\}, \\ \mathrm{V}_1 = \mathrm{U}_1 \setminus \mathrm{U}_0 = \left\{u_{n+s,m+1} : s \geq d \text{ or } s < 0\right\}. \end{aligned} \tag{36}$$

In applications we always deal with relations and equations depending on a finite number of the dynamical variables. Thus, in what follows when we say that a relation depends on U_0 or U_1 or $\mathrm{U}_0 \cup \mathrm{U}_1$, we mean that it depends on a finite, but otherwise unspecified, subset of the corresponding set of variables.

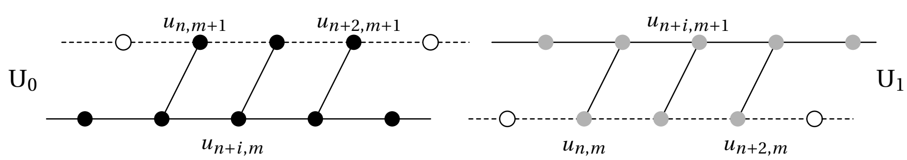

Figure 2: *The two sets of dynamical variables* U_0 *(black dots) and* U_1 *(grey dots) with* $d = 3$. *In each case two of the eliminated values, from* V_1 *and* V_0 *respectively, are denoted with circles.*

3.2 Elimination maps

The concept of the elimination map was introduced in [14] for quadrilateral equations. In a similar way we introduce here two elimination maps $\mathscr{E}_0$ and $\mathscr{E}_1$ which are adapted to our considerations of the dynamical variables U_0 and U_1 and the eliminated ones V_0 and V_1.

3.2.1 Elimination map $\mathscr{E}_0$

The elimination of variables V_0 requires us to solve equation (1) for $u_{n+d,m}$ or $u_{n,m}$, shift the result appropriately and then replace all variables V_0 recursively. Specifically, if we denote

$$\begin{aligned} u_{n+d,m} = X(u_{n,m}, \ldots, u_{n+d-1,m}, u_{n,m+1}, \ldots, u_{n+d,m+1}), \\ u_{n,m} = Y(u_{n+1,m}, \ldots, u_{n+d,m}, u_{n,m+1}, \ldots, u_{n+d,m+1}), \end{aligned} \tag{37}$$

then we can describe the elimination of these variables as follows.

Definition 3.1. *The elimination map* $\mathscr{E}_0 : \mathrm{U}_0 \cup \mathrm{U}_1 \to \mathrm{U}_1$ *is defined recursively as*

$$\textit{for } 0 \le i < d: \quad \mathscr{E}_0(u_{n+i,m}) = u_{n+i,m}; \tag{38a}$$

$$\textit{for all } i: \quad \mathscr{E}_0(u_{n+i,m+1}) = u_{n+i,m+1}; \tag{38b}$$

$$\textit{for } i \ge d:$$

$$\mathscr{E}_0(u_{n+i,m}) = X(\mathscr{E}_0(u_{n+i-d,m}),\dots,\mathscr{E}_0(u_{n+i-1,m}), u_{n+i-d,m+1},\dots,u_{n+i,m+1}); \tag{38c}$$

$$\textit{for } i < 0:$$

$$\mathscr{E}_0(u_{n+i,m}) = Y(\mathscr{E}_0(u_{n+i+1,m}),\dots,\mathscr{E}_0(u_{n+d+i,m}), u_{n+i,m+1},\dots,u_{n+d+i,m+1}). \tag{38d}$$

3.2.2 Elimination map $\mathscr{E}_1$

The elimination of variables V_1 in favour of variables U_0 can be done in a similar way. Now we solve equation (1) for $u_{n+d,m+1}$ or $u_{n,m+1}$, i.e.

$$\begin{aligned} u_{n+d,m+1} &= Z(u_{n,m},\dots,u_{n+d,m}, u_{n,m+1},\dots,u_{n+d-1,m+1}),\\ u_{n,m+1} &= W(u_{n,m},\dots,u_{n+d,m}, u_{n+1,m+1},\dots,u_{n+d,m+1}), \end{aligned} \tag{39}$$

and then we describe this process as follows.

Definition 3.2. *The elimination map* $\mathscr{E}_1 : \mathrm{U}_0 \cup \mathrm{U}_1 \to \mathrm{U}_0$ *is defined recursively as*

$$\textit{for all } i: \quad \mathscr{E}_1(u_{n+i,m}) = u_{n+i,m}; \tag{40a}$$

$$\textit{for } 0 \le i < d: \quad \mathscr{E}_1(u_{n+i,m+1}) = u_{n+i,m+1}; \tag{40b}$$

$$\textit{for } i \ge d:$$

$$\mathscr{E}_1(u_{n+i,m+1}) = Z(u_{n+i-d,m},\dots,u_{n+i,m},\mathscr{E}_1(u_{n+i-d,m+1}),\dots,\mathscr{E}_1(u_{n+i-1,m+1})); \tag{40c}$$

$$\textit{for } i < 0:$$

$$\mathscr{E}_1(u_{n+i,m+1}) = W(u_{n+i,m},\dots,u_{n+d+i,m},\mathscr{E}_1(u_{n+i+1,m+1}),\dots,\mathscr{E}_1(u_{n+d+i,m+1})). \tag{40d}$$

3.3 Differentiation

Our requirements for the defining function Q of equation (1) allow us to express variables V_ℓ as functions of variables $\mathrm{U}_{1-\ell}$. In this context, using implicit differentiation, we compute the derivatives of variables $u_{n+r,m+\ell} \in \mathrm{V}_\ell$ with respect to $u_{n+k,m+\ell} \in \mathrm{U}_0 \cap \mathrm{U}_1$, where $k \in I = \{0,\dots,d-1\}$, $r \notin I$ and $\ell = 0,1$. We also use these expressions to define differential operators which annihilate any function depending on $\mathrm{V}_{1-\ell} \cup \{u_{n,m+1-\ell},\dots,u_{n+d-1,m+1-\ell}\}$.

Proposition 3.3. *The derivatives of* $u_{n+r,m+\ell}$, $d \le r \le d+N$, *with respect to* $u_{n+k,m+\ell}$, *with* $0 \le k < d$ *and* $\ell = 0$ *or* 1, *are the solutions of the system*

$$\tilde{\mathrm{A}}_\ell\, \tilde{\boldsymbol{u}}_{(k,\ell)} = \tilde{\boldsymbol{v}}_{(k,\ell)}, \tag{41a}$$

where the entries of the $(N+1)\times(N+1)$ *matrix* $\tilde{\mathrm{A}}_\ell$ *are given by*

$$\left(\tilde{\mathrm{A}}_\ell\right)_{i,j} = \mathscr{S}^{i-1}(Q_{d+j-i,\ell}), \tag{41b}$$

and the vectors $\tilde{\boldsymbol{u}}_{(k,\ell)}$, $\tilde{\boldsymbol{v}}_{(k,\ell)}$ *are defined as*

$$\left(\tilde{\boldsymbol{u}}_{(k,\ell)}\right)_i = \frac{\partial u_{n+d+i-1,m+\ell}}{\partial u_{n+k,m+\ell}}, \quad \left(\tilde{\boldsymbol{v}}_{(k,\ell)}\right)_i = -\mathscr{S}^{i-1}(Q_{k-i+1,\ell}), \quad i = 1,\dots,N+1. \tag{41c}$$

Proof. Consider equation (1) and its positive $\mathscr{S}$-shifts. Our requirements for function Q imply that we can solve uniquely all these equations for variables $u_{n+r,m+\ell}$, with $r \ge d$ and $\ell = 0$ or 1, and express them as functions of the remaining dynamical variables. Using implicit differentiation, we differentiate all the difference equations $\{\mathscr{S}^i(Q) = 0\}_{i=0}^N$ with respect to $u_{n+k,m+\ell}$, with $0 \le k < d$, to find

$$\partial_{u_{n+k,m+\ell}}\left(\mathscr{S}^i(Q)\right) + \sum_{j=d}^{d+i} \partial_{u_{n+j,m+\ell}}\left(\mathscr{S}^i(Q)\right)\frac{\partial u_{n+j,m+\ell}}{\partial u_{n+k,m+\ell}} = 0, \quad i = 0,\dots,N$$

which can be written also as

$$\sum_{j=0}^{i} \mathscr{S}^i(Q_{d+j-i,\ell}) \frac{\partial u_{n+d+j,m+\ell}}{\partial u_{n+k,m+\ell}} = -\mathscr{S}^i(Q_{k-i,\ell}), \quad i = 0,\ldots,N.$$

These relations can be clearly cast in the form of system (41). □

Proposition 3.4. *The derivatives of $u_{n+r,m+\ell}$, $-N \le r \le -1$, with respect to $u_{n+k,m+\ell}$, with $0 \le k < d$ and $\ell = 0$ or 1, are the solutions of the system*

$$\hat{\mathrm{A}}_\ell \, \hat{\boldsymbol{u}}_{(k,\ell)} = \hat{\boldsymbol{v}}_{(k,\ell)}, \tag{42a}$$

where the entries of the $N \times N$ matrix $\hat{\mathrm{A}}_\ell$ are given by

$$\left(\hat{\mathrm{A}}_\ell\right)_{i,j} = \mathscr{S}^{-i}(Q_{i-j,\ell}), \tag{42b}$$

and the vectors $\hat{\boldsymbol{u}}_{(k,\ell)}$, $\hat{\boldsymbol{v}}_{(k,\ell)}$ are defined as

$$\left(\hat{\boldsymbol{u}}_{(k,\ell)}\right)_i = \frac{\partial u_{n-i,m+\ell}}{\partial u_{n+k,m+\ell}}, \quad \left(\hat{\boldsymbol{v}}_{(k,\ell)}\right)_i = -\mathscr{S}^{-i}(Q_{k+i,\ell}), \quad i = 1,\ldots,N. \tag{42c}$$

Proof. Now we consider all the negative $\mathscr{S}$-shifts of equation (1). Our requirements for function Q imply that we can solve uniquely all these equations for variables $u_{n+r,m+\ell}$ with $r < 0$ and $\ell = 0$ or 1, and express them as functions of the remaining dynamical variables. Using implicit differentiation, we differentiate all the difference equations $\{\mathscr{S}^{-i}(Q) = 0\}_{i=1}^N$ with respect to $u_{n+k,m+\ell}$, with $0 \le k < d$, to find

$$\partial_{u_{n+k,m+\ell}} \left(\mathscr{S}^{-i}(Q)\right) + \sum_{j=1}^{i} \partial_{u_{n-j,m+\ell}} \left(\mathscr{S}^{-i}(Q)\right) \frac{\partial u_{n-j,m+\ell}}{\partial u_{n+k,m+\ell}} = 0, \quad i = 1,\ldots,N,$$

which can be written also as

$$\sum_{j=1}^{i} \mathscr{S}^{-i}(Q_{i-j,\ell}) \frac{\partial u_{n-j,m+\ell}}{\partial u_{n+k,m+\ell}} = -\mathscr{S}^{-i}(Q_{k+i,\ell}), \quad i = 1,\ldots,N,$$

which can be written as system (42). □

Using the above Propositions we define the differential operators which we employ in our strategy for solving functional equations in the following section.

Definition 3.5. *We define the derivative operators $\mathscr{D}_{(k,\ell)}$ as*

$$\mathscr{D}_{(k,\ell)} = \partial_{u_{n+k,m+\ell}} + \left(\tilde{\mathrm{A}}_\ell^{-1} \, \tilde{\boldsymbol{v}}_{(k,\ell)}\right) \cdot \Delta_\ell + \left(\hat{\mathrm{A}}_\ell^{-1} \, \hat{\boldsymbol{v}}_{(k,\ell)}\right) \cdot \nabla_\ell, \quad 0 \le k < d, \quad \ell = 0, 1, \tag{43a}$$

where the matrices and the vectors involved are given in Propositions 3.3 and 3.4, and

$$\Delta_\ell = (\partial_{u_{n+d,m+\ell}} \cdots \partial_{u_{n+d+N,m+\ell}})^\intercal, \quad \nabla_\ell = (\partial_{u_{n-1,m+\ell}} \cdots \partial_{u_{n-N,m+\ell}})^\intercal, \tag{43b}$$

and the $\cdot$ denotes the usual scalar product of vectors. Moreover, we define the vector operator

$$\mathscr{D}_{\boldsymbol{\ell}} = \left(\mathscr{D}_{(0,\ell)} \cdots \mathscr{D}_{(d-1,\ell)}\right)^\intercal. \tag{44}$$

Remark 3.1. When $d = 1$ there exist only two operators, namely $\mathscr{D}_{(0,0)}$ and $\mathscr{D}_{(0,1)}$, and they have been used previously in [20] and in the equivalent form $\mathscr{E}_{1-i}\left(\mathscr{D}_{(0,i)}\right)$, $i = 0, 1$, in [5].

4 Solving functional equations

In this section we present an algorithmic method for solving functional equations, like (31-34), which can be easily implemented in symbolic computations. More precisely we consider equations

$$E(f) := \mathscr{T}(f) + A f + B = 0 \tag{45}$$

which must hold on solutions of (1), where $f = f(n, m, u_{n+a,m}, \ldots, u_{n+b,m})$ is the unknown function, $\mathscr{T}(f)$ is the shift of f in the m-direction, i.e. $\mathscr{T}(f) = f(n, m+1, u_{n+a,m+1}, \ldots, u_{n+b,m+1})$, and the integers a, b are such that $a < b$. For our presentation it is convenient to denote the set of values of u appearing as arguments of f and $\mathscr{T}(f)$ with $\mathrm{U}_f = \{u_{n+a,m}, \ldots, u_{n+b,m}\}$ and $\mathrm{U}_{\mathscr{T}(f)} = \mathscr{T}(\mathrm{U}_f) = \{u_{n+a,m+1}, \ldots, u_{n+b,m+1}\}$ and consider them as subsets of U_0 and U_1, respectively. Also, $A \neq 0$, B are given functions of n, m and of variables $\mathrm{U}_0 \cup \mathrm{U}_1$.

Our aim is, starting from equation (45), to derive differential equations involving only either f or $\mathscr{T}(f)$ since they depend on different sets of variables. This last observation implies that we can apply vector operator $\mathscr{D}_{\mathbf{0}}$ to equation (45) to eliminate $\mathscr{T}(f)$ because it does not depend on variables $u_{n,m}, \ldots, u_{n+d-1,m}$. This results to a set of d equations for the first order derivatives of f, namely

$$\mathscr{D}_{\mathbf{0}}(E(f)) = A\mathscr{D}_{\mathbf{0}}(f) + \mathscr{D}_{\mathbf{0}}(A) f + \mathscr{D}_{\mathbf{0}}(B) = 0 \tag{46}$$

Since f depends only on $\mathrm{U}_f \subset \mathrm{U}_0$, and all the other functions in (46) depend on variables from $\mathrm{U}_0 \cup \mathrm{U}_1$, we apply elimination map $\mathscr{E}_1$ to remove all variables belonging in $\mathrm{V}_1 = \mathrm{U}_1 \setminus \mathrm{U}_0$.

$$\mathscr{E}_1\left(\mathscr{D}_{\mathbf{0}}(E(f))\right) = \mathscr{E}_1\left(A\mathscr{D}_{\mathbf{0}}\right)(f) + \mathscr{E}_1\left(\mathscr{D}_{\mathbf{0}}(A)\right) f + \mathscr{E}_1\left(\mathscr{D}_{\mathbf{0}}(B)\right) = 0 \tag{47}$$

But f now depends on U_f, a subset of U_0, thus variables $\mathrm{U}_0 \setminus \mathrm{U}_f$ can be used as separation variables. In this way, from (47) we end up with a system R_0 for the first order derivatives of f.

$$R_0 := \left\{\text{Coeffients}\left(\mathscr{E}_1\left(\mathscr{D}_{\mathbf{0}}(E(f))\right), \mathrm{U}_0 \setminus \mathrm{U}_f\right) = 0\right\} \tag{48}$$

Now we consider equation $E(f)/A$, i.e.

$$E'(f) := \frac{1}{A}\mathscr{T}(f) + f + \frac{B}{A} = 0. \tag{49}$$

Since $\mathscr{D}_{\mathbf{1}}(f) = 0$, the application of $\mathscr{D}_{\mathbf{1}}$ to (49) yields a system of differential equations only for $\mathscr{T}(f)$, namely

$$\mathscr{D}_{\mathbf{1}}(E'(f)) = A^{-1}\mathscr{D}_{\mathbf{1}}\left(\mathscr{T}(f)\right) + \mathscr{D}_{\mathbf{1}}(A^{-1})\mathscr{T}(f) + \mathscr{D}_{\mathbf{1}}\left(A^{-1}B\right) = 0.$$

In these equations, $\mathscr{T}(f)$ depends only on $\mathrm{U}_{\mathscr{T}(f)} \subset \mathrm{U}_1$, and all the other functions depend on variables from $\mathrm{U}_0 \cup \mathrm{U}_1$. Thus we apply elimination map $\mathscr{E}_0$ to remove variables belonging in $\mathrm{V}_0 = \mathrm{U}_0 \setminus \mathrm{U}_1$.

$$\mathscr{E}_0\left(\mathscr{D}_{\mathbf{1}}(E'(f))\right) = \mathscr{E}_0\left(A^{-1}\mathscr{D}_{\mathbf{1}}\right)\left(\mathscr{T}(f)\right) + \mathscr{E}_0\left(\mathscr{D}_{\mathbf{1}}(A^{-1})\right)\mathscr{T}(f) + \mathscr{E}_0\left(\mathscr{D}_{\mathbf{1}}\left(A^{-1}B\right)\right) = 0$$

Proceeding in a similar fashion, we split the above equations using $\mathrm{U}_1 \setminus \mathrm{U}_{\mathscr{T}(f)}$ as separation variables. In this way, we derive a system for the first order derivatives of $\mathscr{T}(f)$. The resulting system then can be shifted backwards in the m direction leading to another system R_1 for f and its derivatives.

$$R_1 := \left\{\text{Coeffients}\left(\mathscr{T}^{-1}\left(\mathscr{E}_0\left(\mathscr{D}_{\mathbf{1}}(E'(f))\right)\right), \mathscr{T}^{-1}\left(\mathrm{U}_1 \setminus \mathrm{U}_{\mathscr{T}(f)}\right)\right) = 0\right\}. \tag{50}$$

Having derived two linear first-order systems of partial differential equations for f, namely (48) and (50), it is necessary to check the compatibility among the equations constituting them. If R_0 and R_1 are inconsistent, then equation (45) does not have any solution. Otherwise, we have to include any consistency conditions into $R_0 \cup R_1$ and solve the resulting extended linear system for f. If the system admits a unique solution then we have to check that our original equation (45) is also satisfied.

5 Examples

In this section we consider specific quad ($d = 1$) and two-quad ($d = 2$) equations and derive their generalised symmetries. We employ the corresponding determining equations following from (31-34) and apply our strategy from Section 4 to solve them. Then, we use relations (22) to determine symmetry F (up to an arbitrary function of $u_{n,m}$) and substitute it back into (8) to determine the dependence of F on $u_{n,m}$. It should be noted that for quad equations ($d = 1$), we can easily rearrange all the formulae so that to compute symmetries in the m direction. Specifically we have to interchange indices $(u_{n+i,m+j}, Q_{i,j}) \to (u_{n+j,m+i}, Q_{j,i})$, as well as the shift operators $\mathscr{S}$ and $\mathscr{T}$, in the definitions (23) and the determining equations (31-34).

Example 5.1. Our first illustrative example is about the symmetries of equation

$$u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}(u_{n,m} + u_{n+1,m+1} + 1) + \chi = 0, \tag{51}$$

which was first given in [19]. We start our derivations, according to Remark 2.2, with the choice $N = d = 1$. In this case (31) and (32) become

$$\mathscr{T}(\tilde{r}_1) - \frac{\left(u_{n-1,m} + u_{n-1,m+1} + u_{n-1,m}u_{n-1,m+1} + u_{n-1,m+1}u_{n,m+1}\right)\left(u_{n,m+1} + u_{n,m+1}u_{n+1,m}\right)}{u_{n-1,m+1}\left(1 + u_{n,m}\right)\left(u_{n,m} + u_{n,m+1} + u_{n,m}u_{n,m+1} + u_{n,m+1}u_{n+1,m+1}\right)}\tilde{r}_1 = 0, \tag{52}$$

$$\mathscr{T}(\hat{r}_{-1}) - \frac{\left(u_{n,m} + u_{n-1,m}u_{n,m} + u_{n,m+1} + u_{n,m}u_{n,m+1}\right)\left(u_{n+1,m} + u_{n,m+1}u_{n+1,m}\right)}{\left(1 + u_{n-1,m+1}\right)u_{n,m}\left(u_{n+1,m+1} + u_{n+1,m}\left(1 + u_{n,m} + u_{n+1,m+1}\right)\right)}\hat{r}_{-1} = 0. \tag{53}$$

To make contact with the set notation of the previous sections, $\mathrm{U}_f = \{u_{n-1,m}, u_{n,m}, u_{n+1,m}\}$ and it is sufficient to consider $\mathrm{U}_0 = \{u_{n-1,m}, u_{n,m}, u_{n+1,m}, u_{n,m+1}\}$ and $\mathrm{U}_1 = \{u_{n-1,m+1}, u_{n,m+1}, u_{n+1,m+1}, u_{n,m}\}$.

Following our strategy in the previous section, we apply to (52) operator $\mathscr{D}_0$, which in this case is just

$$\mathscr{D}_{(0,0)} = \partial_{u_{n,m}} - \frac{u_{n+1,m}(u_{n,m+1} + 1)}{u_{n,m} + u_{n,m+1}(u_{n,m} + u_{n+1,m+1} + 1)}\partial_{u_{n+1,m}} - \frac{u_{n-1,m} + u_{n-1,m+1}(u_{n-1,m} + u_{n,m+1} + 1)}{u_{n,m}(u_{n-1,m+1} + 1)}\partial_{u_{n-1,m}},$$

then the elimination map $\mathscr{E}_1$, which boils down to replacing $u_{n+1,m+1}$ and $u_{n-1,m+1}$ using relations

$$u_{n+1,m+1} = -\frac{u_{n,m}u_{n+1,m} + u_{n+1,m}u_{n,m+1}(u_{n,m} + 1) + \chi}{u_{n,m+1}(u_{n+1,m} + 1)}, \quad u_{n-1,m+1} = -\frac{u_{n-1,m}u_{n,m} + \chi}{u_{n,m+1} + u_{n,m}(u_{n-1,m} + u_{n,m+1} + 1)},$$

and finally we use $\mathrm{U}_0 \setminus \mathrm{U}_f = \{u_{n,m+1}\}$ as a separation variable to find

$$\begin{gathered}(u(2 + 3u + 2(1 + u)x) - (1 + 2u)\chi)\tilde{r}_1 + (1 + u)\left(u(-u + \chi)\tilde{r}_{1_u} + ux(1 + x)\tilde{r}_{1_x} - (1 + y)\chi\tilde{r}_{1_y}\right) = 0,\\ (1 + u(3 + 2x))\tilde{r}_1 - u(1 + u)\tilde{r}_{1_u} + ux(1 + x)\tilde{r}_{1_x} + (y - \chi)\tilde{r}_{1_y} = 0,\end{gathered}$$

where $y = u_{n-1,m}$, $u = u_{n,m}$, $x = u_{n+1,m}$ and $\tilde{r}_1 = \tilde{r}_1(y, u, x)$.

Next we solve equation (52) for $\tilde{r}_1$ (or alternatively we may use (33)). We apply operator $\mathscr{D}_1$, which according to Remark 3.1 becomes

$$\mathscr{D}_{(0,1)} = \partial_{u_{n,m+1}} - \frac{u_{n+1,m+1} + u_{n+1,m}(u_{n,m} + u_{n+1,m+1} + 1)}{u_{n,m+1}(u_{n+1,m} + 1)}\partial_{u_{n+1,m+1}} - \frac{u_{n-1,m+1}(u_{n,m} + 1)}{u_{n,m+1} + u_{n,m}(u_{n-1,m} + u_{n,m+1} + 1)}\partial_{u_{n-1,m+1}},$$

to the resulting equation, then the elimination map $\mathscr{E}_0$, i.e. replace $u_{n+1,m}$ and $u_{n-1,m}$ using relations

$$u_{n+1,m} = -\frac{u_{n,m+1}u_{n+1,m+1} + \chi}{u_{n,m} + u_{n,m+1}(u_{n,m} + u_{n+1,m+1} + 1)}, \quad u_{n-1,m} = -\frac{u_{n-1,m+1}u_{n,m} + u_{n-1,m+1}u_{n,m+1}(u_{n,m} + 1) + \chi}{u_{n,m}(u_{n-1,m+1} + 1)},$$

and finally we shift backwards in the m direction, i.e. we apply $\mathscr{T}^{-1}$. After that we use $\mathscr{T}^{-1}\left(\mathrm{U}_1 \setminus \mathrm{U}_{\mathscr{T}(f)}\right) = \{u_{n,m-1}\}$ as separation variable and this yields the following two equations.

$$\begin{gathered}(-u + \chi)\tilde{r}_1 + u(u - \chi)\tilde{r}_{1_u} + (1 + x)\chi\tilde{r}_{1_x} - uy(1 + y)\tilde{r}_{1_y} = 0,\\ (\chi + u(\chi + y(2 + u + \chi)))\tilde{r}_1 + (uy + \chi)\left(-u(1 + u)\tilde{r}_{1_u} + (x - \chi)\tilde{r}_{1_x} + uy(1 + y)\tilde{r}_{1_y}\right) = 0.\end{gathered}$$

So far we have derived a homogeneous system of four linear equations for $\tilde{r}_1$ and its three first-order derivatives. The matrix of this system is invertible which implies that the only solution is $\tilde{r}_1 = 0$. Hence the symmetry cannot depend on $u_{n+1,m}$ (see relation (22)).

Then we proceed to the second determining equation (53). Following exactly the same procedure as we did above we conclude that $\hat{r}_{-1} = 0$ as well, and consequently that the symmetry is independent of $u_{n-1,m}$. Hence we have shown that the equation does not admit any symmetry of order one (i.e. depending only on the first order shifts of u), and we have to proceed to the $N = 2$ case.

Now the first two of the four determining equations (equations (31) and (32) with $d = 1$ and $N = 2$) involve only $\tilde{r}_2$ and $\hat{r}_{-2}$, respectively, and have the following forms.

$$\mathcal{T}(\tilde{r}_2) - \frac{\left(u_{n-1,m} + u_{n-1,m+1}\left(1 + u_{n-1,m} + u_{n,m+1}\right)\right) - \left(u_{n+1,m+1} + u_{n+1,m+1}u_{n+2,m}\right)\tilde{r}_2}{\left(u_{n-1,m+1} + u_{n-1,m+1}u_{n,m}\right)\left(u_{n+1,m} + u_{n+1,m+1}\left(1 + u_{n+1,m} + u_{n+2,m+1}\right)\right)} = 0 \tag{54}$$

$$\mathcal{T}(\hat{r}_{-2}) - \frac{\left(u_{n-1,m+1} + u_{n-1,m}\left(1 + u_{n-2,m} + u_{n-1,m+1}\right)\right)\left(u_{n+1,m} + u_{n,m+1}u_{n+1,m}\right)\hat{r}_{-2}}{\left(u_{n-1,m} + u_{-2+n,1+m}u_{n-1,m}\right)\left(u_{n+1,m+1} + u_{n+1,m}\left(1 + u_{n,m} + u_{n+1,m+1}\right)\right)} = 0. \tag{55}$$

In this case $\mathrm{U}_f = \{u_{n-2,m}, u_{n-1,m}, u_{n,m}, u_{n+1,m}, u_{n+2,m}\}$ and $\mathrm{U}_0 = \mathrm{U}_f \cup \{u_{n,m+1}\}$ and $\mathrm{U}_1 = \mathcal{T}\left(\mathrm{U}_f\right) \cup \{u_{n,m}\}$.

Starting with equation (54) and applying the same procedure, we end up with the following linear system of four equations for $\tilde{r}_2$ and its first-order derivatives.

$$\begin{aligned}
&y_1(\chi + u(-1 + u(-2 + x_1) + x_1 + 2(1+u)x_1x_2 + 2\chi))\tilde{r}_2 + \\
&(1+u)y_1\left(u(u-\chi)\tilde{r}_{2_u} - ux_1(1+x_1)\tilde{r}_{2_{x_1}} + ux_1x_2(1+x_2)\tilde{r}_{2_{x_2}} + (1+y_1)\chi\tilde{r}_{2_{y_1}}\right) - \\
&((y_2-\chi)\chi + u(\chi + y_2(y_1 + \chi + y_1\chi)))\tilde{r}_{2_{y_2}} = 0,
\end{aligned} \tag{56}$$

$$\begin{aligned}
&y_1\left(u^2x_1(x_1 + 2x_1x_2 - 2) - \chi - 2u(x_1 + x_1x_2 + \chi)\right)\tilde{r}_2 + ux_1(ux_1 - 1)x_2(1+x_2)y_1\tilde{r}_{2_{x_2}} \\
&+(ux_1+\chi)\left(uy_1\left((1+u)\tilde{r}_{2_u} - x_1(1+x_1)\tilde{r}_{2_{x_1}}\right) + y_1(\chi - y_1)\tilde{r}_{2_{y_1}} - (1+y_2)\chi\tilde{r}_{2_{y_2}}\right) = 0,
\end{aligned} \tag{57}$$

$$\begin{aligned}
&(\chi + u(\chi + x_1(2 + u + \chi)))\tilde{r}_2 + ((x_2-\chi)\chi + u(\chi + x_2(x_1 + \chi + x_1\chi)))\tilde{r}_{2_{x_2}} - \\
&(1+u)x_1\left((1+x_1)\chi\tilde{r}_{2_{x_1}} + u\left((u-\chi)\tilde{r}_{2_u} - y_1(1+y_1)\tilde{r}_{2_{y_1}} + y_1y_2(1+y_2)\tilde{r}_{2_{y_2}}\right)\right) = 0,
\end{aligned} \tag{58}$$

$$\begin{aligned}
&\left(u(1+u)x_1y_1 + u(x_1 + y_1 + x_1y_1)\chi + \chi^2\right)\tilde{r}_2 + ux_1y_1(1 - uy_1)y_2(1+y_2)\tilde{r}_{2_{y_2}} - \\
&(uy_1+\chi)\left((-1-x_2)\chi\tilde{r}_{2_{x_2}} + x_1\left(u(1+u)\tilde{r}_{2_u} + (-x_1+\chi)\tilde{r}_{2_{x_1}} - uy_1(1+y_1)\tilde{r}_{2_{y_1}}\right)\right) = 0,
\end{aligned} \tag{59}$$

where $y_i = u_{n-i,m}$, $x_i = u_{n+i,m}$, with $i = 1, 2$, and $u = u_{n,m}$.

The above system can be solved for four of the five first-order derivatives of $\tilde{r}_2$, and we have chosen to solve them for $\tilde{r}_{2_{x_2}}$, $\tilde{r}_{2_{x_1}}$, $\tilde{r}_{2_u}$ and $\tilde{r}_{2_{y_1}}$. Then one more equation arises from the compatibility of these equations, which is $\tilde{r}_{2_{y_2}} = 0$. Taking into account the last equation, the resulting system is consistent and has a unique solution. In this way we determine $\tilde{r}_2$ up to an arbitrary function of n and m only. Then we substitute this solution into (54) to find that the arbitrary function must be independent of m. Hence we can write function $\tilde{r}_2$ as

$$\tilde{r}_2 = \frac{u_{n,m}(u_{n,m}+1)u_{n+1,m}(u_{n-1,m}u_{n,m} + \chi)(u_{n,m}u_{n+1,m} + \chi)}{(u_{n-1,m}u_{n,m}u_{n+1,m} - \chi)(u_{n,m}u_{n+1,m}u_{n+2,m} - \chi)^2}\tilde{R}_n.$$

Then we focus on equation (55). We can solve this equation in exactly the same way and determine $\hat{r}_{-2}$ up to an arbitrary function of n. More precisely we find that

$$\hat{r}_{-2} = \frac{u_{n,m}(u_{n,m}+1)u_{n-1,m}(u_{n-1,m}u_{n,m} + \chi)(u_{n,m}u_{n+1,m} + \chi)}{(u_{n-1,m}u_{n,m}u_{n+1,m} - \chi)(u_{n-2,m}u_{n-1,m}u_{n,m} - \chi)^2}\hat{R}_n.$$

Finally, using relations (22) and the above expressions for $\tilde{r}_2$ and $\hat{r}_{-2}$, we can completely determine the dependence of the symmetry on variables $u_{n+2,m}$ and $u_{n-2,m}$. Specifically we find that

$$F = \frac{(u_{n,m}+1)(u_{n-1,m}u_{n,m}+\chi)(u_{n,m}u_{n+1,m}+\chi)\left(\tilde{R}_n G_{n-1,m} + \hat{R}_n G_{n+1,m}\right)}{G_{n-1,m}G_{n,m}G_{n+1,m}} + H, \tag{60}$$

where $G_{n,m} = u_{n-1,m}u_{n,m}u_{n+1,m} - \chi$ and $H = H(n,m,u_{n-1,m},u_{n,m},u_{n+1,m})$ is an arbitrary function.

Next we proceed to the second set of determining equations using the general form of function F (60) and relations (22). Our strategy to analyse these equations is the same and leads to $\tilde{R}_n = -\hat{R}_n = c \in \mathbb{R}$ and $H \equiv 0$. Hence the lowest order symmetry of (51) is[1]

$$\partial_t u_{n,m} = \frac{u_{n,m}\left(1+u_{n,m}\right)\left(\chi + u_{n-1,m}u_{n,m}\right)\left(\chi + u_{n,m}u_{n+1,m}\right)\left(u_{n-2,m}u_{n-1,m} - u_{n+1,m}u_{n+2,m}\right)}{\left(u_{n-2,m}u_{n-1,m}u_{n,m} - \chi\right)\left(u_{n-1,m}u_{n,m}u_{n+1,m} - \chi\right)\left(u_{n,m}u_{n+1,m}u_{n+2,m} - \chi\right)}. \tag{61}$$

Finally, in view of our comments at the beginning of this section, we can easily compute the symmetries in the m direction for equation (51) which actually follow from (61) by changing shifts $u_{n+i,m}$ to $u_{n,m+i}$ and then change $(u_{n,m},\chi)$ to $(-u_{n,m}-1,-\chi-1)$. □

Example 5.2. Our second example is provided by the quadrilateral equation[2]

$$\gamma\left(u_{n,m}u_{n+1,m+1}-1\right)\left(u_{n+1,m}u_{n,m+1}+1\right) + \left(u_{n,m}u_{n,m+1}+1\right)\left(u_{n+1,m}u_{n+1,m+1}+1\right) = 0, \quad \gamma \in \mathbb{R}_+. \tag{62}$$

The symmetry analysis in the m direction reveals that (62) admits no symmetries of order one. With $N = 2$, the determining equations (31-34) lead to

$$F = \frac{G_{n,m}G_{n,m+1}}{K_{n,m}}\left(\frac{r_m}{P_{n,m+1}} - \frac{r_{m-1}}{P_{n,m}}\right) - \hat{\beta}\frac{r_m - r_{m-1}}{4}\,\frac{\hat{\alpha}u_{n,m}(u_{n,m+1}+u_{n,m-1})+2}{K_{n,m}} + H(n,m,u_{n,m}), \tag{63a}$$

where $\hat{\alpha} := 1+\gamma$, $\hat{\beta} := 1-\gamma$, H is an arbitrary function and

$$G_{n,m} := (1+u_{n,m-1}u_{n,m})(\hat{\beta}+\hat{\alpha}u_{n,m-1}u_{n,m}), \tag{63b}$$

$$P_{n,m} := 2\hat{\alpha}\,u_{n,m-2}u_{n,m-1}u_{n,m}u_{n,m+1} + \hat{\alpha}\hat{\beta}(u_{n,m-2}+u_{n,m})(u_{n,m-1}+u_{n,m+1}) + 2\hat{\beta}, \tag{63c}$$

$$K_{n,m} := 2u_{n,m-1}u_{n,m}u_{n,m+1} + \hat{\beta}(u_{n,m+1}+u_{n,m-1}). \tag{63d}$$

Then, using equation (8) we find that H corresponds to point symmetries (hence we can choose $H = 0$) and $r_{m+1} = r_{m-1}$. Thus, equation (62) admits two generalised symmetries of order two in the m direction generated by

$$\partial_{s^1} u_{n,m} = \frac{G_{n,m}G_{n,m+1}}{P_{n,m}P_{n,m+1}}\left(u_{n,m+2} - u_{n,m-2}\right), \tag{64}$$

$$\partial_{\sigma^1} u_{n,m} = (-1)^m \frac{G_{n,m}G_{n,m+1}}{K_{n,m}}\left(\frac{1}{P_{n,m}} + \frac{1}{P_{n,m+1}} + \frac{\hat{\beta}}{2}\,\frac{\hat{\alpha}u_{n,m}(u_{n,m+1}+u_{n,m-1})+2}{G_{n,m}G_{n,m+1}}\right). \tag{65}$$

On the hand, the lowest order generalised symmetries in the n-direction of equation (62) are generated by

$$\partial_{t^1} u_{n,m} = (-1)^m \frac{u_{n,m}^2 + u_{n+1,m}u_{n-1,m}}{u_{n+1,m}+u_{n-1,m}} \quad \text{and} \quad \partial_{t^2} u_{n,m} = \frac{\left(u_{n,m}^2 - u_{n-1,m}^2\right)\left(u_{n,m}^2 - u_{n+1,m}^2\right)\left(u_{n+2,m} - u_{n-2,m}\right)}{\left(u_{n-2,m}+u_{n,m}\right)\left(u_{n+1,m}+u_{n-1,m}\right)^2\left(u_{n,m}+u_{n+2,m}\right)}, \tag{66}$$

respectively. See also [8, 7]. □

[1] The Miura transformation $v_{n,m} = (u_{n-1,m}u_{n,m}+\chi)u_{n+1,m}/(u_{n-1,m}u_{n,m}u_{n+1,m}-\chi)$ maps symmetry (61) to the Bogoyavlensky lattice, [15], $\partial_t v_{n,m} = v_{n,m}(v_{n,m}+1)(v_{n+2,m}v_{n+1,m} - v_{n-1,m}v_{n-2,m})$.

[2] Up to point transformations and renaming of the parameters, this equation was given in [18]. It should also be noted that the particular equation with $\gamma = -1$ and its symmetries were studied in [8, 7]. Here we restrict to $\gamma > 0$ since the equations with opposite values of γ are related by the reciprocal transformation $u_{n,m} \to u_{n,m}^{-1}$.

Example 5.3. The two-quad equation

$$\frac{u_{n+1,m+1}-u_{n,m}}{u_{n+1,m+1}(u_{n+1,m+1}u_{n,m+1}-u_{n+1,m}u_{n,m})}-\frac{u_{n+2,m+1}-u_{n+1,m}}{u_{n+1,m}(u_{n+2,m+1}u_{n+1,m+1}-u_{n+2,m}u_{n+1,m})}=0 \tag{67}$$

was derived in [10]. Here we apply our method to compute the first order symmetries of (67), the determining equations of which are

$$\mathscr{T}(\tilde{r}_1)-\frac{(u_{n+1,m}-u_{n,m+1})(u_{n,m+1}-u_{n-1,m})}{(u_{n+1,m+1}-u_{n,m})(u_{n,m}-u_{n-1,m+1})}\tilde{r}_1=0, \qquad \mathscr{T}(\hat{r}_{-1})-\frac{(u_{n+1,m}-u_{n,m+1})(u_{n,m+1}-u_{n-1,m})}{(u_{n+1,m+1}-u_{n,m})(u_{n,m}-u_{n-1,m+1})}\hat{r}_{-1}=0. \tag{68}$$

In these functional equations, functions $\tilde{r}_1$ and $\hat{r}_1$ depend on $\mathrm{U}_f=\{u_{n-1,m},u_{n,m},u_{n+1,m}\}$, whereas the dynamical variables are $\mathrm{U}_0=\{u_{n-2,m},\dots,u_{n+2,m}\}\cup\{u_{n,m+1},u_{n+1,m+1}\}$ and $\mathrm{U}_1=\{u_{n-2,m+1},\dots,u_{n+2,m+1}\}\cup\{u_{n,m},u_{n+1,m}\}$, since we are dealing with a two-quad equation ($d=2$).

Applying $\mathscr{D}_0$ to the first determining equation in (68) and then $\mathscr{E}_1$, we end up with two rational expressions. The coefficients of $u_{n,m+1}$ and $u_{n+1,m+1}$ in the numerators of these expressions lead to $\partial_{u_{n+1,m}}\tilde{r}_1=\partial_{u_{n-1,m}}\tilde{r}_1=0$ and $\partial_{u_{n,m}}\tilde{r}_1=2\tilde{r}_1/u_{n,m}$, which clearly imply that $\tilde{r}_1=a_{n,m}u_{n,m}^2$. Substituting back into the first equation in (68) we find $a_{n,m+1}=a_{n,m}$. Hence $\tilde{r}_1=a_n u_{n,m}^2$. In the same fashion, the second equation in (68) yields $\hat{r}_1=b_n u_{n,m}^2$. Finally, taking into account (22), we conclude that the first order symmetries of (67) must be of the form

$$F=u_{n,m}^2(a_n u_{n+1,m}+b_n u_{n-1,m})+f(n,m,u_{n,m}).$$

Then we substitute the above form of the symmetry into (8) and after the use of elimination maps we find that f corresponds to point symmetries, $a_{n+1}-2a_n+a_{n-1}=0$ and $b_n=-a_{n-2}$. As we are interested in generalised symmetries we choose $f=0$, whereas the solution of the remaining two difference equations can be written as $a_n=c_0+c_1(n+1)$, $b_n=-c0-c_1(n-1)$. This implies that equation (67) admits two symmetries of order one, the modified Volterra equation, $\partial_t u_{n,m}=u_{n,m}^2(u_{n+1,m}-u_{n-1,m})$, [10], and its master symmetry, $\partial_\tau u_{n,m}=u_{n,m}^2((n+1)u_{n+1,m}-(n-1)u_{n-1,m})$. □

Example 5.4. The last equation to present is

$$u_{n,m}u_{n+2,m+1}\left(u_{n+1,m+1}(u_{n+1,m}+u_{n,m+1})+u_{n+1,m}u_{n+2,m}\right)+\alpha=0, \tag{69}$$

which, as far as we are aware, is new. Starting with symmetries of order one ($N=1$), the analysis of the corresponding determining equations shows that there exist no such symmetries. The same is true for $N=2$. It follows then that the lowest order symmetries in the n direction are of order three, and they are generated by

$$\partial_t u_{n,m}=\frac{u_{n,m}^2(u_{n+3,m}u_{n+2,m}u_{n+1,m}-u_{n-1,m}u_{n-2,m}u_{n-3,m})}{\prod_{k=0}^{3}\mathscr{S}^k\left(u_{n-3,m}u_{n-2,m}u_{n-1,m}u_{n,m}-\alpha\right)}. \tag{70}$$

6 Applications, extensions and discussion

We presented a systematic and algorithmic way to compute generalised symmetries of difference equations. Our approach exploits the theory of integrability conditions, employs Laurent and Taylor formal series of pseudo-difference operators and formulates algebraically the determining equations. We also presented a strategy to solve certain classes of functional equations. The main advantage of our approach is that all the necessary equations and tools are given in terms of the defining function Q of the difference equation (1) and the two invertible and lower triangular matrices L, T, defined in (23), all of which can be easily implemented in a computer algebra software for symbolic computations.

Integrability conditions provide us also the means to construct conservation laws. More precisely, if we determine the first K coefficients of the N-th order formal recursion operator using equations (31) with $K>N$, then we can employ all these coefficients to derive higher order conserved densities by considering powers of the recursion operator and computing their residues [14, 15]. To be more specific, if $\mathfrak{R}^p=\tilde{r}_{pN}^{(p)}\mathscr{S}^{pN}+\dots+\tilde{r}_0^{(p)}+\tilde{r}_{-1}^{(p)}\mathscr{S}^{-1}+\dots$ is the p-th power of the formal recursion operator $\mathfrak{R}$, with $p=1,\dots,K$, then the vectors $\tilde{\boldsymbol{r}}^{(p)}=\left(\tilde{r}_{pN}^{(p)}\cdots\tilde{r}_{(p-K)N}^{(p)}\right)^{\intercal}$ are determined recursively by

$$\tilde{\boldsymbol{r}}^{(p)}=\mathscr{S}^{pN}\left(\mathrm{L}\left(\tilde{\boldsymbol{r}}^{(1)}\right)\right)\cdot\tilde{\boldsymbol{r}}^{(p-1)}, \quad p=2,\dots,K, \tag{71}$$

and the residue of $\mathfrak{R}^p$, i.e. the function $\tilde{r}_0^{(p)}$, is a conserved density.

We can also use the strategy we presented in Section 4 to compute first integrals $\mathscr{T}(J) = J$ of equation (1), where $J = J(n, m, u_{n,m}, \ldots, u_{n+L,m})$ and $L \geq d$. Indeed, to find J we have to solve the functional equation $\mathscr{T}(J) - J = 0$ which is the particular case of (45) with $A = -1$, $B = 0$ and $a = 0$, $b = L$. Since there is no criterion for the choice of L, we have to start with $L = d$ and work successively as we have done with symmetries (see also Remark 2.2 [3]). However it would be interesting to derive necessary and sufficient criteria for the existence of first integrals for d-quad equations.

Here we considered only equations satisfying Requirement 3. This assumption can be relaxed in order to consider difference equations which can be solved uniquely for at least two values of u not lying on the same horizontal line and not necessarily at the corners of the stencil on which the equation is defined. In this case we can easily adjust the definitions of dynamical variables, elimination maps and differentiation to study equations of this kind. However we cannot remove the second requirement completely as this will lead to difficulties with the elimination of variables.

Finally our framework can be extended straightforwardly to systems of difference equations of the form

$$Q^{(i)}(\mathbf{u}_{n,m}, \mathbf{u}_{n+1,m}, \mathbf{u}_{n,m+1}, \mathbf{u}_{n+1,m+1}) = 0, \quad i = 0, \ldots, d-1, \tag{72}$$

where $\mathbf{u}_{n,m} = (u_{n,m}^{(0)}, \ldots, u_{n,m}^{(d-1)})$, which satisfy conditions

$$\det \mathrm{J}_{k,l} = \det\left(\partial_{u^{(j)}_{n+k,m+l}} Q^{(i)}\right) \neq 0, \quad \text{for all } (k,l) \in \{(0,0),(1,0),(0,1),(1,1)\},$$

and can be solved uniquely with respect to all values of $\mathbf{u}$ involved in (72). Such systems and the d-quad equation (1) satisfying Requirements 1–4 are related in the following way. Starting with (1) we can set $\mathscr{S}^j(Q) = Q^{(j)}$ and then apply the transformation $u_{n+j+pd,m+q} \mapsto u^{(j)}_{n'+p,m+q}$, for $j = 0, \ldots, d-1$ and $p, q \in \mathbb{Z}$. Our requirements for function Q guarantee that the corresponding matrices $\mathrm{J}_{k,l}$ will be invertible and the system can be solved uniquely with respect to all values of $\mathbf{u}$. This connection readily provides a way to compute symmetries in the m direction for (1) by studying the corresponding symmetries of the related d component quad system. It would be also interesting to consider quadrilateral systems which do not necessarily satisfy all these requirements (see for instance [10]), derive necessary integrability conditions and employ them in the computation of symmetries and conservation laws.

Data accessibility

This work does not have any experimental data.

Competing interests

We have no competing interests.

Acknowledgements

The author would like to thank Alexander Mikhailov and Willy Hereman for useful discussions and comments.

Funding

This research was supported by a Royal Society Research Grant (Grant No RG150494).

Ethic Statement

This work did not involve any active collection of human data.

[3]For $d = 1$ a similar method has been proposed in [9] and implemented in a classification problem for the first time in [8].

References

[1] V. E. Adler and V. V. Postnikov (2014) On discrete 2D integrable equations of higher order *J. Phys. A: Math. Theor.* **47** 045206 (16pp) doi:10.1088/1751-8113/47/4/045206

[2] G. Berkeley, A. V. Mikhailov, and P. Xenitidis (2016) Darboux transformations with tetrahedral reduction group and related integrable systems *J Mathl Phys* **57** 092701; doi: 10.1063/1.4962803

[3] S. Dimas, D. Tsoubelis (2005) SYM : A new symmetry - finding package for Mathematica, in *The 10th International Conference in MOdern GRoup ANalysis*, N.H.Ibragimov, C.Sophocleous, and P.A.Damianou (eds), Nicosia, University of Cyprus, pp 64 – 70

[4] A. Fordy, P. Xenitidis (2017) $\mathbb{Z}_N$ graded discrete Lax pairs and discrete integrable systems *J Phys A: Math Theor* **50** 165205 (30pp) doi:10.1088/1751-8121/aa639a

[5] R. N. Garifullin, E. V. Gudkova and I. T. Habibullin (2011) Method for searching higher symmetries for quad-graph equations, *J. Phys. A: Math. Theor.* **44** 325202, doi.org/10.1088/1751-8113/44/32/325202

[6] R. N. Garifullin, I. T. Habibullin and R. I. Yamilov (2015) Peculiar symmetry structure of some known discrete nonautonomous equations *J. Phys. A: Math. Theor.* **48** 235201, doi.org/10.1088/1751-8113/48/23/235201

[7] R. N. Garifullin, A. V. Mikhailov, and R. I. Yamilov (2014) Discrete equation on a square lattice with a nonstandard structure of generalized symmetries *Theor. Math. Phys.* **180** 765–780

[8] R. N. Garifullin, R. I. Yamilov (2012) Generalized symmetry classification of discrete equations of a class depending on twelve parameters, *J. Phys. A: Math. Theor.* **45** 345205, doi.org/10.1088/1751-8113/45/34/345205

[9] I.T. Habibullin (2005) Characteristic Algebras of Fully Discrete Hyperbolic Type Equations *SIGMA* **1** 023, doi.org/10.3842/SIGMA.2005.023

[10] S. Konstantinou-Rizos, A. Mikhailov, P. Xenitidis P (2015) Reduction groups and related integrable difference systems of nonlinear Schrödinger type *J. Math. Phys.* **56** 082701 doi:10.1063/1.4928048

[11] D. Levi, R. I. Yamilov (2009) The generalized symmetry method for discrete equations, *J. Phys. A: Math. Theor.* **42** 454012, doi.org/10.1088/1751-8113/42/45/454012

[12] D. Levi, R. I. Yamilov (2011) Generalized symmetry integrability test for discrete equations on the square lattice *J. Phys. A: Math. Theor.* **44** 145207 (22pp), doi:10.1088/1751-8113/44/14/145207

[13] A.V. Mikhailov (ed.) (2009) *Integrability*. Lecture Notes in Physics **767**, Springer

[14] A.V. Mikhailov, J.P. Wang and P. Xenitidis (2011) Recursion operators, conservation laws and integrability conditions for difference equations *Theor. Math. Phys.* **167** 421-443

[15] A.V. Mikhailov, P. Xenitidis (2014) Second order integrability conditions for difference equations. An integrable equation *Lett. Math. Phys.* **104** 431-450 doi:10.1007/s111005-013-0668-8

[16] P. J. Olver (1993) *Applications of Lie Groups to Differential Equations*. Grad. Texts in Math. **107**, Springer

[17] O.G. Rasin, P.E. Hydon (2007) Symmetries of Integrable Difference Equations on the Quad-Graph, *Studiels in Applied Mathematics* **119** 253–269 doi:10.1111/j.1467-9590.2007.00385.x

[18] J. A. G. Roberts, D. T. Tran (2015) Signatures over finite fields of growth properties for lattice equations *J. Phys. A: Math. Theor.* **48** 085201 (20pp) doi:10.1088/1751-8113/48/8/085201

[19] C. Scimiterna, M. Hay and D. Levi (2014) On the integrability of a new lattice equation found by multiple scale analysis *J. Phys. A: Math. Theor.* **47** 265204 (16pp) doi:10.1088/1751-8113/47/26/265204

[20] A. Tongas, D. Tsoubelis and P. Xenitidis (2007) Affine linear and D_4 symmetric lattice equations: symmetry analysis and reductions *J. Phys. A: Math. Theor.* **40** 13353–13384

[21] P. Xenitidis (2011) Symmetries and conservation laws of the ABS equations and corresponding differential-difference equations of Volterra type *J. Phys. A: Math. Theor.* **44** 435201 doi:10.1088/1751-8113/44/43/435201

[22] P. Xenitidis (2018) Deautonomizations of integrable equations and their reductions *Journal of Integrable Systems* **3** xyy009, doi.org/10.1093/integr/xyy009

[23] V.E. Zakharov (ed.) (1991) *What is integrability?* Springer-Verlag, Berlin.