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Abstract: We consider a two-component Hamiltonian system of partial differential equations with quadratic nonlinearities introduced by Popowicz, which has the form of a coupling between the Camassa-Holm and Degasperis-Procesi equations. Despite having reductions to these two integrable partial differential equations, the Popowicz system itself is not integrable. Nevertheless, as one of the authors showed with Irle, it admits distributional solutions of peaked soliton (peakon) type, with the dynamics of N peakons being determined by a Hamiltonian system on a phase space of dimension $3N$. As well as the trivial case of a single peakon ($N=1$), the case $N=2$ is Liouville integrable. We present the explicit solution for the two-peakon dynamics, and describe some of the novel features of the interaction of peakons in the Popowicz system.

Response to Reviewers: PLA-D-18-01785

We are grateful to both reviewers for their careful reading of the paper. In addition to the amendments detailed below, in the revised version we have included a second figure, further illustrating the scattering of two peakons with the contour plot of $v(x,t)$ for the same set of initial values.

Reviewer #1: No changes requested.

Reviewer #2: We have addressed the three points raised as follows:
1. The comment that the given form (1.14) of the operator $\hat{\mathbf{B}}$, involving fractional powers, does not make sense when m, n are a linear combination of Dirac delta measures is a very good one, and although it has been addressed in the cited literature, it is important to discuss it here. We have added a comment at the bottom of page 4, just before Theorem 1, to show how this operator can be rewritten in terms of m, n, m_x, n_x, without any fractional powers.
2. For arbitrary N, Lemma 2 says that $\mathrm{d}J/\mathrm{d}t = 0$. Since Hamilton's equations mean that $\mathrm{d}f/\mathrm{d}t = \{ f, h \}$ for any function f on phase space, J being a conserved quantity of the motion is equivalent to it being in involution with h with respect to the Poisson bracket. Although this is a well known fact about Hamiltonian systems, we have added $\{ h, J \} = 0$ to the statement of Lemma 2, as well as a brief comment at the end of the proof, to make it clearer for the reader.
3. The main point of Remark 5 was to observe briefly that, when the amplitudes are all positive, Gronwall's equality can be used to give the same sort of upper and lower bounds on b_j as were found for p_j in reference [7] in the case of the b-family. In the original submission, we included another sentence about the argument used in [7], and the fact that a priori it cannot be used to assert that the amplitudes remain positive if they are positive initially: this line of argument is definitely incorrect for the Popowicz peakons, but in fact it seems to be invalid even for the b-family, despite what is stated in [7] (or at least, the argument given there is incomplete). In the resubmission, we have removed this additional sentence, because it is unclear as it stands, and irrelevant to the case at hand. Nevertheless, since the reviewer asked about it, below we give a full explanation of this point. As it is essentially a rebuttal of an argument made in [7], about a different system, we think it is perhaps best not to mention it in our paper (although we may contact the authors of [7] privately to point out their error).

Proposition 2.4 in [7] states the analogue of Lemma 4, namely that if $p_j(0)$ are all positive then $p_j(t) > 0$ for all t. The b-family peakon amplitudes satisfy the ODEs

$$\dot{p}_i = -(\mathrm{b}-1)p_i \sum_j p_j G'(x_i - x_j), \tag{1}$$

and there are analogous coupled ODEs for the positions x_j. Actually, (1) gives a generalized version, referred to as "pulsons" in ref. [14], involving an even symmetric kernel $G(x)$ (so $G'(x)$ is odd). The peakon case corresponds to

$$G(x) = \frac{1}{2}\exp(-|x|), \tag{2}$$

up to rescaling. In [7] it is shown directly that

$$M_0 = \sum_j p_j(t) = \sum_j p_j(0) \tag{3}$$

is independent of t. (This also follows immediately from the reduction of the Hamiltonian structure (1.4) described in [14]: up to a factor of $\mathrm{b}-1$, H in (1.5) reduces to M_0.) Recall Gronwall's inequality, namely that, for a constant K,

$$\dot{f} \le Kf \implies f(t) \le f(0)\exp(Kt) \quad \text{for} \quad t \ge 0$$

(or more generally, if $K = K(t)$ then one has the exponential of the integral of K on the right-hand side), and there is the analogous statement with a lower bound. In [7], in the proof given for Proposition 2.4, it is stated that for the b-family with $\mathrm{b} > 1$, the bounds

$$p_i(0)\exp(-Kt) \leq p_i(t) \leq p_i(0)\exp(Kt) \tag{4}$$

hold for $t \geq 0$, with

$$K = (\mathrm{b} - 1)M_0||G'||_\infty > 0.$$

The bounds (4) clearly *imply* that the p_i are all positive if they are so at $t = 0$. Moreover, it is also clear from (1) and (3), that *if* the p_i are all positive (or even just non-negative) then we have

$$-Kp_i \leq \dot{p}_i \leq Kp_i, \tag{5}$$

so by Gronwall, (4) must hold for all i. However, if the amplitudes p_i have mixed signs then this need not be the case. To see this, note that $G'(0) = 0$ since G' is odd, so the term corresponding to $j = i$ in the sum (1) is empty; and as a concrete example, let us take $N = 2$, and fix the Green's function to be (2), which in this case gives $||G'||_\infty = \lim_{x\to 0}|G'(x)| = 1/2$. Now at some t, to have different signs let us be even more concrete and choose $p_1(t) = 3$, $p_2(t) = -2$, so $M_0 = 1 \implies K = (\mathrm{b} - 1)/2$; and we can always choose $x_1(t)$ and $x_2(t)$ to be as close as we wish, so $G'(x_1 - x_2) = 1/2 - \epsilon = -G'(x_2 - x_1)$. Then by (1), at time t,

$$\dot{p}_1 = -(\mathrm{b} - 1)p_1p_2G'(x_1 - x_2) = (\mathrm{b} - 1)(1 - 2\epsilon)p_1 > Kp_1.$$

Thus the bounds (5) do not hold for p_1, and we cannot infer (4). Therefore it seems that the argument offered as a proof of Proposition 2.4 is circular: in order to obtain the bounds (4), it must be assumed that the p_i are all positive, which is what one is trying to prove! (A correct argument for the b-family can be given by adapting the proof of Lemma 4 in our paper, using the first integrals M_0 and P, which are the analogues of h and J.)

If the reviewer thinks that it is important to discuss this in detail in our paper, then it would probably best go into an appendix, but otherwise we would prefer to omit it.

Highlights: PLA-D-18-01785

- The Popowicz system, which couples the integrable Camassa-Holm and Degasperis-Procesi equations together, is not integrable, but it has peaked soliton (peakon) solutions which are conservative.
- A proof of the reduction of the Hamiltonian structure of the Popowicz system to the submanifold of N-peakon solutions is given.
- The dynamics of two peakons ($N = 2$) is Liouville integrable. We explicitly integrate the equations of motion and describe the interaction of two peakons.

Dynamics of conservative peakons in a system of Popowicz

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Abstract

We consider a two-component Hamiltonian system of partial differential equations with quadratic nonlinearities introduced by Popowicz, which has the form of a coupling between the Camassa-Holm and Degasperis-Procesi equations. Despite having reductions to these two integrable partial differential equations, the Popowicz system itself is not integrable. Nevertheless, as one of the authors showed with Irle, it admits distributional solutions of peaked soliton (peakon) type, with the dynamics of N peakons being determined by a Hamiltonian system on a phase space of dimension $3N$. As well as the trivial case of a single peakon ($N = 1$), the case $N = 2$ is Liouville integrable. We present the explicit solution for the two-peakon dynamics, and describe some of the novel features of the interaction of peakons in the Popowicz system.

1 Introduction

For the past 25 years there has been a huge amount of interest in partial differential equations (PDEs) which admit peaked soliton solutions, known as peakons, with a discontinuous first derivative at the peaks. This began with the work of Camassa and Holm [4], who found the integrable PDE

$$u_t + 2\kappa u_x - u_{xxt} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0 \tag{1.1}$$

in the context of shallow water wave theory. In fact this was a rediscovery, since the integrability of the latter equation had already been recognized in the work of Fokas and Fuchssteiner on hereditary symmetries and recursion operators [13]. However, the pioneering contribution of Camassa and Holm was their analysis of the remarkable properties of the solutions of (1.1), and in particular the fact that in the absence of linear dispersion ($\kappa = 0$) it has multipeakon solutions of the form

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\, e^{-|x-q_j(t)|}, \tag{1.2}$$

as well as displaying wave breaking, and also (for $\kappa > 0$) smooth solitons vanishing at spatial infinity.

The equation (1.1) with $\kappa = 0$ is the case $\mathrm{b} = 2$ of the 1-parameter family

$$m_t + u\, m_x + \mathrm{b}\, u_x\, m = 0, \qquad m = u - u_{xx}, \tag{1.3}$$

introduced in [10] after it was shown that the case $\mathrm{b} = 3$, identified by Degasperis and Procesi [9], is also integrable (linear dispersion can always be removed by a combination of a shift $u \to u+\mathrm{const}$ and a

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Galilean transformation). With the inclusion of linear dispersion, the whole b-family of equations (1.3) was subsequently derived via shallow water approximations [11, 8]. All of the equations in the family have at least one Hamiltonian structure, given by

$$m_t = \mathbf{B}\,\frac{\delta H}{\delta m}, \tag{1.4}$$

where (subject to appropriate modifications for $\mathrm{b} = 0, 1$)

$$H = \frac{1}{\mathrm{b}-1}\int m\,\mathrm{d}x, \qquad \text{with} \quad \mathbf{B} = -\mathrm{b}^2\, m^{1-1/\mathrm{b}}\partial_x m^{1/\mathrm{b}}\mathcal{L}^{-1}m^{1/\mathrm{b}}\partial_x m^{1-1/\mathrm{b}}, \quad \mathcal{L} = \partial_x - \partial_x^3 \tag{1.5}$$

and admit multipeakon solutions of the form (1.2). However, $\mathrm{b} = 2, 3$ are the only values for which there is a bi-Hamiltonian structure, and these correspond to the integrable cases, in the sense that the equation (1.3) has infinitely many local symmetries for these values of b alone [20].

Due to the discontinuous derivatives at the peaks, it is necessary to specify in what sense (1.2) is a solution of (1.3). The shape of the peakons corresponds to the fact that $\frac{1}{2}e^{-|x|}$ is the Green's function of the one-dimensional Helmholtz operator $1 - \partial_x^2$, so for N peakons the quantity m is given by a sum of Dirac delta functions,

$$m(x,t) = 2\sum_{j=1}^{N} p_j(t)\,\delta(x - q_j(t)), \tag{1.6}$$

with support at each of the peak positions $x = q_j(t)$ at time t. Thus it is necessary to interpret (1.3) as an equation for distributions. The problem is then how to make sense of the nonlinear terms, which include products of distributions with common support. An ad hoc solution to this problem is to interpret the product $u_x m$ as being $< u_x > m$, where

$$< f(x) > := \frac{1}{2}\lim_{\epsilon\to 0}\Big(f(x+\epsilon) + f(x-\epsilon)\Big) \tag{1.7}$$

is the average of the left and right limits. However, a more satisfying solution, which turns out to yield equivalent results, is the following weak formulation of (1.3), presented in [17]:

$$E[u(x,t)] := (1-\partial_x^2)u_t + (\mathrm{b}+1-\partial_x^2)\partial_x\Big(\frac{1}{2}u^2\Big) + \partial_x\Big(\frac{3-\mathrm{b}}{2}u_x^2\Big) = 0. \tag{1.8}$$

With the above, $u(x,t)$ is said to be a weak solution if

$$\int E[u(x,t)]\,\phi(x)\,\mathrm{d}x = 0 \tag{1.9}$$

for all compactly supported test functions $\phi \in C^\infty(\mathbb{R})$, with ∂_x in (1.8) being viewed as a distributional derivative, where it is further required that, for each fixed t, u_t is a continuous linear functional, and also $u \in H^1_{\mathrm{loc}}(\mathbb{R})$, so that u^2 and u_x^2 define continuous linear functionals as well.

The preceding requirements entail that (1.2) is a weak solution of (1.3) if and only if $(q_j, p_j)_{j=1,\ldots,N}$ satisfy the system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}q_j}{\mathrm{d}t} = \frac{\partial\tilde{H}}{\partial p_j}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}t} = -(b-1)\frac{\partial\tilde{H}}{\partial q_j}, \qquad j = 1,\ldots N, \tag{1.10}$$

with

$$\tilde{H} = \frac{1}{2}\sum_{j,k=1,\ldots N} p_j p_k e^{-|q_j - q_k|}. \tag{1.11}$$

In the case $\mathrm{b} = 2$, the ODEs (1.10) form a canonical Hamiltonian system, with $\tilde{H}$ being the Hamiltonian. For all other values of b, $\tilde{H}$ is not a conserved quantity; nevertheless, for all b the equations (1.10) are

Hamiltonian with respect to a non-canonical Poisson bracket derived by restriction of the bracket defined by the operator $\mathbf{B}$ in (1.5) to the finite-dimensional submanifold of N-peakon solutions [14].

The pioneering work of Camassa and Holm inspired the search for integrable analogues of (1.1) with two or more components, starting with [6, 12]. The subject of this article is the two-component system of PDEs given by

$$\begin{array}{rcll} m_t + (2u+v)\,m_x + 3(2u_x+v_x)\,m &=& 0, & m = u - u_{xx}, \\ n_t + (2u+v)\,n_x + 2(2u_x+v_x)\,n &=& 0, & n = v - v_{xx}, \end{array} \tag{1.12}$$

which was derived by Popowicz via Dirac reduction of a Hamiltonian operator depending on three fields [21]. With $\mathbf{m} = (m,n)^T$, the Hamiltonian structure of (1.12) is given by

$$\mathbf{m}_t = \hat{\mathbf{B}}\,\frac{\delta H_0}{\delta \mathbf{m}}, \tag{1.13}$$

with

$$H_0 = \int (m+n)\,\mathrm{d}x, \qquad \hat{\mathbf{B}} = -\begin{pmatrix} 9m^{2/3}\partial_x m^{1/3}\mathcal{L}^{-1}m^{1/3}\partial_x m^{2/3} & 6m^{2/3}\partial_x m^{1/3}\mathcal{L}^{-1}n^{1/2}\partial_x n^{1/2} \\ 6n^{1/2}\partial_x n^{1/2}\mathcal{L}^{-1}m^{1/3}\partial_x m^{2/3} & 4n^{1/2}\partial_x n^{1/2}\mathcal{L}^{-1}n^{1/2}\partial_x n^{1/2} \end{pmatrix}. \tag{1.14}$$

The Hamiltonian operator $\hat{\mathbf{B}}$ admits two one-parameter families of Casimir functionals, given by

$$\begin{array}{rcl} H_1 &=& \int (nm^{-2/3})^{\lambda} m^{1/3}\,\mathrm{d}x, \\ H_2 &=& \int (nm^{-2/3})^{\lambda}(-9n_x^2 n^{-2} m^{1/3} + 12 n_x m_x n^{-1} m^{-4/3} - 4m_x^2 m^{-7/3})\,\mathrm{d}x, \end{array}$$

where λ is arbitrary. The system (1.12) is a coupling between the Camassa-Holm and Degasperis-Procesi equations, that is the cases $\mathrm{b} = 2, 3$ of (1.3), to which it reduces when $u = 0$, $v = 0$, respectively, and this led Popowicz to speculate that it should be integrable. However, a combination of a reciprocal transformation together with Painlevé analysis, applied by one of us in work with Irle [16], provides strong evidence of the non-integrability of the coupled system (1.12).

Despite its apparent non-integrability, it is nevertheless the case that the Popowicz system admits multipeakon solutions, given by the ansatz

$$u(x,t) = \sum_{j=1}^{N} a_j(t)\,e^{-|x-q_j(t)|}, \qquad v(x,t) = \sum_{j=1}^{N} b_j(t)\,e^{-|x-q_j(t)|}, \tag{1.15}$$

whose properties were outlined in [16]. The purpose of this article is to describe more precisely in what sense these are distributional solutions of (1.12), and provide some details of the dynamics of the peakons, which behave somewhat differently from those that appear in the Camassa-Holm and Degasperis-Procesi equations. Due to the Hamiltonian properties of the solutions (1.15), which are inherited from those of the PDE system, we refer to them as conservative peakons, following [1], where peakons with analogous properties were considered for a family of peakon equations derived from the bi-Hamiltonian structure of the nonlinear Schrödinger hierarchy.

2 Conservative peakons

The first thing to observe about the system (1.12) is that it does not admit a weak formulation suitable for multipeakon solutions of the form (1.15), analogous to the formulation (1.8) for the b-family of equations. If the two coupled equations are denoted by $E_j[u(x,t), v(x,t)] = 0$, $j = 1, 2$, then a bona fide weak solution should be one for which

$$\int (E_1[u(x,t),v(x,t)]\,\phi_1(x) + E_2[u(x,t),v(x,t)]\,\phi_2(x))\,\mathrm{d}x = 0,$$

for an arbitrary pair of compactly supported test functions $\phi_1, \phi_2 \in C^\infty(\mathbb{R})$, with sufficiently many derivatives in E_1, E_2 being interpreted as distributional derivatives. For peakons, products of u, v and u_x, v_x define continuous linear functionals, and all higher derivatives should be viewed in the sense of distributions. If we start by considering E_1, then we can use (1.8) to write this as

$$E_1 = (1-\partial_x^2)u_t + 2(4-\partial_x^2)\partial_x\Big(\frac{1}{2}u^2\Big) + R,$$

where

$$R = (u_x - u_{xxx})v + 3(u-u_{xx})v_x. \tag{2.1}$$

There are four mixed products of u, v and their first derivatives, so we need to be able to write the terms in R with a total of three x derivatives in the form

$$A\partial_x^3(uv) + \partial_x^2(Buv_x + Cu_xv) + D\partial_x(u_xv_x), \tag{2.2}$$

for some constants A, B, C, D, where ∂_x above should be regarded as a distributional derivative. (Strictly speaking, isolated products uv_x and u_xv have discontinuities in the case of peakons, but we write these terms separately for the sake of completeness.) Upon comparing the coefficients of $u_{xxx}v$ and $u_{xx}v_x$ (and the absent terms u_xv_{xx}, uv_{xxx}) in (2.1) with (2.2), we find the linear system

$$\begin{array}{rcl} A+C &=& -1, \\ 3A+B+2C+D &=& -3, \\ 3A+2B+C+D &=& 0, \\ A+B &=& 0, \end{array}$$

which has no solution, so there can be no weak formulation suitable for peakons.

Despite the fact that the Popowicz system does not admit a weak formulation for multipeakons, one can select a distributional interpretation, by using the average (1.7), in such a way that the Hamiltonian properties of the PDE system are inherited by these solutions. By taking the common prefactor in $(2u_x+v_x)m$ and $(2u_x+v_x)n$ to mean the average $< 2u_x+v_x >$, one finds a system of equations for distributions, namely

$$\begin{array}{rcl} m_t + \partial_x\Big((2u+v)\,m\Big) + 2 < 2u_x+v_x > \, m &=& 0, \\ n_t + \partial_x\Big((2u+v)\,n\Big) + < 2u_x+v_x > \, n &=& 0, \end{array} \tag{2.3}$$

with ∂_x being the distributional derivative. For peakons, the main upshot of this averaging procedure is that, to include the situation where it appears in front of a delta function with the same support, the derivative of $e^{-|x|}$ should interpreted as $-\mathrm{sgn}(x)e^{-|x|}$, where the signum function is defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & \text{if } x>0; \\ 0, & \text{if } x=0; \\ -1, & \text{if } x<0. \end{cases}$$

The quantities m, n are defined as above, so that for multipeakons of the form (1.15) they are given by

$$m(x,t) = 2\sum_{j=1}^N a_j(t)\,\delta(x-q_j(t)), \qquad n(x,t) = 2\sum_{j=1}^N b_j(t)\,\delta(x-q_j(t)), \tag{2.4}$$

Making use of the nomenclature from [1], it is appropriate to refer to the multipeakons which satisfy (2.3), in the sense of distributions, as conservative peakons, since (in particular) the Hamiltonian functional H_0 is conserved by these solutions. Furthermore, by rewriting the first order differential operators appearing in (1.14) as

$$m^{2/3}\partial_x m^{1/3} = m\partial_x + \frac{m_x}{3}, \quad m^{1/3}\partial_x m^{2/3} = m\partial_x + \frac{2m_x}{3}, \quad n^{1/2}\partial_x n^{1/2} = n\partial_x + \frac{n_x}{2},$$

to remove the fractional powers (which do not make sense for distributions), the Poisson structure defined by $\hat{\mathbf{B}}$ can be reduced to these multipeakon solutions. The following result was stated without proof in [16].

Theorem 1. *With the formulation (2.3), the Popowicz system admits N-peakon solutions of the form (1.15), where the amplitudes a_j, b_j and positions q_j satisfy the dynamical system*

$$\begin{array}{rcl}\dot{a}_j &=& 2a_j \sum_{k=1}^N (2a_k+b_k)\mathrm{sgn}(q_j-q_k)e^{-|q_j-q_k|},\\ \dot{b}_j &=& b_j \sum_{k=1}^N (2a_k+b_k)\mathrm{sgn}(q_j-q_k)e^{-|q_j-q_k|},\\ \dot{q}_j &=& \sum_{k=1}^N (2a_k+b_k)e^{-|q_j-q_k|},\end{array} \tag{2.5}$$

for $j=1,\ldots,N$. These equations are in Hamiltonian form, that is

$$\dot{a}_j = \{\, a_j, h\,\}, \qquad \dot{b}_j = \{\, b_j, h\,\}, \qquad \dot{q}_j = \{\, q_j, h\,\},$$

with the Hamiltonian

$$h = 2\sum_{j=1}^N (a_j+b_j) \tag{2.6}$$

and the Poisson bracket

$$\begin{array}{rcl}\{\, a_j, a_k\,\} &=& 2a_ja_k\,\mathrm{sgn}(q_j-q_k)e^{-|q_j-q_k|},\\ \{\, b_j, b_k\,\} &=& \frac{1}{2}b_jb_k\,\mathrm{sgn}(q_j-q_k)e^{-|q_j-q_k|},\\ \{\, q_j, q_k\,\} &=& \frac{1}{2}\mathrm{sgn}(q_j-q_k)(1-e^{-|q_j-q_k|}),\\ \{\, q_j, a_k\,\} &=& a_k\, e^{-|q_j-q_k|},\\ \{\, q_j, b_k\,\} &=& \frac{1}{2}b_k\, e^{-|q_j-q_k|},\\ \{\, a_j, b_k\,\} &=& a_jb_k\,\mathrm{sgn}(q_j-q_k)e^{-|q_j-q_k|},\end{array} \tag{2.7}$$

where the latter has N Casimirs given by

$$C_j = \frac{a_j}{b_j^2}, \qquad \text{for } b_j \neq 0, \qquad j=1,\ldots,N. \tag{2.8}$$

Proof. By substituting (1.15) and (2.4) into (2.3) and integrating against test functions ϕ_1, ϕ_2 with support in a small neighbourhood of $x=q_j$, such that $\phi_1(q_j)=1$, $\phi_1'(q_j)=0$ and $\phi_2(q_j)=0$, $\phi_2'(q_j)=1$, one obtains the equations

$$\dot{a}_j + 2a_j\Big(< 2u_x(q_j) > + < v_x(q_j) > \Big) = 0, \qquad \dot{b}_j + b_j\Big(< 2u_x(q_j) > + < v_x(q_j) > \Big) = 0,$$

$$\dot{q}_j - \Big(2u(q_j)+v(q_j)\Big) = 0,$$

which yield (2.5). The Hamiltonian (2.6) is obtained by inserting (2.4) into the functional H_0 and integrating. The derivation of the Poisson brackets (2.7) follows the same steps as applied to the case of the b-family peakons in [14]: one starts from the expressions for local brackets between fields, defined by the Hamiltonian operator $\hat{\mathbf{B}}$ for the PDE, as in (1.14). For instance, with $G(x)=\frac{1}{2}\mathrm{sgn}(x)(1-e^{-|x|})$ one has

$$\{\, m(x), m(y)\,\} = m_x(x)m_x(y)G(x-y) + 3(m(x)m_x(y)-m_x(x)m(y))G(x-y) - 9m(x)m(y)G(x-y),$$

and then substituting in (2.4) on both sides and integrating against pairs of test functions of x and y, of the same of form as ϕ_1, ϕ_2 above, with support at $x=q_j$ and $y=q_k$ for each pair j,k, produces the brackets $\{\, a_j, a_k\,\}$, $\{\, q_j, a_k\,\}$ and $\{\, q_j, q_k\,\}$, while the other brackets are derived from the expressions for $\{\, m(x), n(y)\,\}$ and $\{\, n(x), n(y)\,\}$ given in [16]; further details of this calculation can be found in [18]. It is straightforward to check that, away from where the b_j vanish, each C_j is a Casimir for the bracket specified by (2.7), and this is a complete set of Casimirs since the bracket has rank $2N$. □

In [7] it was shown that, in addition to the quantity $\tilde{h} = \sum_{j=1}^{N} p_j$ (which, up to rescaling, corresponds to the restriction of the functional H in (1.5) to the multipeakon solutions), the equations (1.10) for the peakons in the b-family, ordered so that $q_j < q_{j+1}$ for all j, admit the first integral

$$P = \left(\prod_{j=1}^{N} p_j\right) \prod_{k=1}^{N-1} (1 - e^{-|q_k - q_{k+1}|})^{\mathrm{b}-1}.$$

An analogous result holds for the peakons in the Popowicz system, as was noted in [16] in the case $N = 2$.

Lemma 2. *In addition to the Hamiltonian h and the Casimirs C_j, $j = 1, \ldots, N$, the ODEs (2.5) for the peakons in the Popowicz system admit the first integral*

$$J = \left(\prod_{j=1}^{N} b_j\right) \prod_{k=1}^{N-1} \left(1 - e^{-|q_k - q_{k+1}|}\right), \tag{2.9}$$

so that $\{ h, J \} = 0$, where the peaks are ordered as follows:

$$q_1 < q_2 < \cdots < q_N. \tag{2.10}$$

Proof. Taking the logarithm of (2.9) and differentiating gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \log J = \sum_{j=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} \log b_j + \sum_{k=1}^{N-1} \frac{(\dot{q}_k - \dot{q}_{k+1})\mathrm{sgn}(q_k - q_{k+1}) E_{k,k+1}}{1 - E_{k,k+1}},$$

where we have introduced the convenient notation

$$E_{j,k} = E_{k,j} = e^{-|q_j - q_k|}.$$

Substituting for the time derivatives from (2.5) yields

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \log J &= \textstyle\sum_{j,k=1}^{N} (2a_k + b_k)\mathrm{sgn}(q_j - q_k) E_{jk} + \sum_{k=1}^{N-1} \sum_{\ell=1}^{N} (2a_\ell + b_\ell)\mathrm{sgn}(q_k - q_{k+1}) \tfrac{(E_{k,\ell} - E_{k+1,\ell})E_{k,k+1}}{1 - E_{k,k+1}} \\ &= \textstyle\sum_{k=1}^{N} (2a_k + b_k)\, S_k, \end{aligned}$$

where, with the ordering (2.10),

$$S_k = -\sum_{j=1}^{k-1} E_{jk} + \sum_{j=k+1}^{N} E_{jk} - \sum_{\ell=1}^{N-1} \frac{(E_{\ell,k} - E_{\ell+1,k})E_{\ell,\ell+1}}{1 - E_{\ell,\ell+1}}.$$

Then the properties of the exponential, together with the assumed ordering of the peakons, produce the identity

$$\frac{(E_{\ell,k} - E_{\ell+1,k})E_{\ell,\ell+1}}{1 - E_{\ell,\ell+1}} = \begin{cases} -E_{\ell,k}, & \text{for } 1 \le \ell \le k; \\ E_{\ell+1,k}, & \text{for } k \le \ell \le N-1. \end{cases}$$

Thus $S_k = 0$ for all k, and the result follows. Note that $\mathrm{d}f/\mathrm{d}t = \{ f, h \}$ for any function on phase space, so J Poisson commutes with h. ☐

In the case of a single peakon ($N = 1$), the ODE system (2.5) is trivially integrable, and the fields u, v take the form

$$u(x,t) = a e^{-|x - ct - x_0|}, \qquad v(x,t) = b e^{-|x - ct - x_0|}, \qquad \text{with} \quad c = 2a + b, \tag{2.11}$$

where a, b, x_0 are arbitrary constants. For the case $N = 2$, upon restricting to four-dimensional symplectic leaves $C_1 = \text{const}$, $C_2 = \text{const}$, by the preceding result there remain the two independent first integrals h, J with $\{ h, J \} = 0$, hence we have the following

Corollary 3. *The Hamiltonian system (2.5) is Liouville integrable for $N = 1$ and $N = 2$.*

For the b-family, in each of the special cases b $= 2, 3$ there is a linear system (Lax pair) which can be used to construct N independent first integrals for the corresponding ODE system (1.10) [4, 10], and can be further employed to develop a spectral theory for the peakons, leading to an explicit solution for all N [3, 19]. However, for $N > 2$ there is no reason to expect that the system (2.5) has any first integrals other than h, J and the N Casimirs. Thus, while Liouville's theorem guarantees that the solution for $N = 2$ can be found by quadratures, which we explicitly derive in the next section, for larger N this does not seem possible.

3 Explicit dynamics of two peakons

For arbitrary N, the $3N$-dimensional system (2.5) can always be reduced to $2N$-dimensional symplectic leaves by fixing the values of the N Casimirs C_j (away from $b_j = 0$); in particular, one can eliminate the variables a_j to leave $2N$ equations for $(q_i, b_i)_{i=1,...,N}$. Here we consider the Liouville integrable case $N = 2$, for which there are the two first integrals

$$h = 2(a_1 + a_2 + b_1 + b_2), \qquad J = b_1 b_2 (1 - e^{-|q_1 - q_2|}), \tag{3.1}$$

in addition to the two Casimirs C_1, C_2, and show how to explicitly integrate the equations of motion. For the sake of concreteness, we restrict to the situation where $a_1, a_2 > 0$, so that the values of the Casimirs are positive, and fix these to be constant values:

$$C_i = k_i^2, \qquad k_i > 0, \qquad i = 1, 2. \tag{3.2}$$

We further assume that $b_1, b_2 > 0$ (at least at time $t = 0$), meaning that in this case both fields u and v initially consist of peakons, with positive amplitudes (rather than anti-peakons, with negative amplitudes); as we shall see, this implies that the amplitudes b_i remain positive for all time. Thus we can reduce the solution of (2.5) for $N = 2$ to solving the system

$$\begin{array}{rcl} \dot{b}_1 & = & b_1 b_2 (2k_2^2 b_2 + 1)\mathrm{sgn}(q_1 - q_2) e^{-|q_1 - q_2|}, \\ \dot{b}_2 & = & b_1 b_2 (2k_1^2 b_1 + 1)\mathrm{sgn}(q_2 - q_1) e^{-|q_1 - q_2|}, \\ \dot{q}_1 & = & b_1 (2k_1^2 b_1 + 1) + b_2 (2k_2^2 b_2 + 1) e^{-|q_1 - q_2|}, \\ \dot{q}_2 & = & b_2 (2k_2^2 b_2 + 1) + b_1 (2k_1^2 b_1 + 1) e^{-|q_1 - q_2|}, \end{array} \tag{3.3}$$

and then the amplitudes a_i are determined by $a_i = k_i^2 b_i^2$, $i = 1, 2$.

Following [4], it is convenient to set

$$q = q_1 - q_2, \qquad Q = q_1 + q_2,$$

and we will assume that the peaks are ordered so that

$$q_1 < q_2 \implies q < 0; \tag{3.4}$$

if the latter condition holds initially, then it will continue to hold as long as the peakons do not overlap (this possibility will be considered in due course). In that case, the system (3.3) is equivalent to

$$\begin{array}{rcl} \dot{b}_1 & = & -b_1 b_2 (2k_2^2 b_2 + 1) e^q, \\ \dot{b}_2 & = & b_1 b_2 (2k_1^2 b_1 + 1) e^q, \\ \dot{q} & = & \Big(b_1 (2k_1^2 b_1 + 1) - b_2 (2k_2^2 b_2 + 1)\Big)(1 - e^q), \\ \dot{Q} & = & \Big(b_1 (2k_1^2 b_1 + 1) + b_2 (2k_2^2 b_2 + 1)\Big)(1 + e^q). \end{array} \tag{3.5}$$

In order to integrate the above equations explicitly, it is useful to note that, substituting for a_1, a_2 in terms of b_1, b_2 and fixing the Hamiltonian h to a constant value defines an ellipse in the (b_1, b_2) plane, i.e.

$$h = 2b_1(k_1^2 b_1 + 1) + 2b_2(k_2^2 b_2 + 1) = \text{const}, \tag{3.6}$$

which can be specified parametrically in terms of an angle $\theta \in (-\pi, \pi]$, so that b_1 and b_2 are given by

$$b_1 = \frac{\lambda}{k_1}\sin\theta - \frac{1}{2k_1^2}, \qquad b_2 = \frac{\lambda}{k_2}\cos\theta - \frac{1}{2k_2^2}, \qquad \text{where} \quad \lambda^2 = \frac{h}{2} + \frac{1}{4k_1^2} + \frac{1}{4k_2^2}, \tag{3.7}$$

with θ measured clockwise from the vertical, that is, the positive b_2 axis.

Lemma 4. *If the initial amplitudes $b_1(0)$, $b_2(0)$ are positive, then they remain positive for all time t, and the peakons do not overlap.*

Proof. Since $(b_1(t), b_2(t))$ lies on the ellipse $h = \text{const}$ defined by (3.6), the amplitudes b_j are bounded for all t. From the formula for J in (3.1), the assumption $q(0) = q_1(0) - q_2(0) < 0$ implies $J > 0$. However, $q(t) = 0$ for some t would imply $J = 0$, contradicting the fact that J is a first integral, so the two peaks cannot overlap and $q(t) < 0$ for all t. Thus

$$J = b_1 b_2 (1 - e^q) < b_1 b_2, \tag{3.8}$$

so $(b_1(t), b_2(t)) \in \mathbb{R}^2$ lies in the positive quadrant above the upper branch of the hyperbola $b_1 b_2 = J$. □

Remark 5. *Adapting an argument used in [7], Proposition 2.4, in the case where the amplitudes are positive we may write*

$$-hb_1 < -(2k_2^2 b_2^2 + b_2)b_1 \le \dot{b}_1 \le (2k_2^2 b_2^2 + b_2)b_1 < hb_1,$$

and similarly for b_2, so that both an upper and a lower bound is obtained for $b_j(t)$, namely

$$b_j(0)\, e^{-ht} \le b_j(t) \le b_j(0)\, e^{ht}, \qquad j = 1, 2$$

holds for all $t \ge 0$, by Gronwall's inequality.

The first two equations in (3.5) both yield the same equation for the time derivative of θ, that is

$$\dot{\theta} = -2k_1 k_2 b_1 b_2 e^q = 2k_1 k_2 (J - b_1 b_2), \tag{3.9}$$

(where the second equality comes from fixing the first integral $J = \text{const}$, as in (3.1), to eliminate $q = q_1 - q_2$). Then replacing b_1, b_2 with their parametric forms (3.7) in terms of θ leads to an autonomous equation for θ alone, namely

$$\dot{\theta} = 2\Big(Jk_1 k_2 - \frac{1}{4k_1 k_2} + \frac{\lambda}{2k_1}\cos\theta + \frac{\lambda}{2k_2}\sin\theta - \lambda^2 \sin\theta\cos\theta\Big) \equiv f(\theta) \tag{3.10}$$

At this stage we have already shown that the $N = 2$ peakon equations can be completely reduced to quadratures (as is guaranteed by Liouville's theorem [2]). To see this, observe that by performing the quadrature

$$\int \frac{d\theta}{f(\theta)} = t + \text{const}$$

we obtain $\theta = \theta(t)$ from (3.10), and then b_1, b_2 are specified as functions of t by (3.7); hence $q(t)$ is found from (3.8), so that

$$q = \log\left(1 - \frac{J}{b_1 b_2}\right) < 0 \tag{3.11}$$

by the initial assumption on $\mathrm{sgn}(q)$. Finally, having specified the right-hand side of (109) as functions of t, an additional quadrature with respect to t yields $Q = Q(t)$.

In order to carry out the integration explicitly, it is convenient to make use of the standard T-substitution, to convert (3.10) into a rational differential equation for the variable

$$T = \tan\frac{\theta}{2}.$$

Thus (3.10) is transformed to

$$\dot{T} = F(T), \qquad \text{where} \qquad F(T) = \frac{P(T)}{T^2+1}, \tag{3.12}$$

with the quartic polynomial

$$P(T) = \left(Jk_1k_2 - \frac{1}{4k_1k_2}\right)(T^2+1)^2 - \frac{\lambda}{2k_1}(T^4-1) + \frac{\lambda}{k_2}T(T^2+1) + 2\lambda^2 T(T^2-1). \tag{3.13}$$

Then applying a partial fraction decomposition, we find

$$\frac{1}{F(T)} = \frac{T^2+1}{P(T)} = K^{-1}\sum_{j=1}^{4}\frac{(T_j^2+1)e_j}{T-T_j},$$

where the quartic $P(T)$ is factorized as

$$P(T) = K\prod_{j=1}^{4}(T-T_j), \qquad K = Jk_1k_2 - \frac{1}{4k_1k_2} - \frac{\lambda}{2k_1}, \qquad \text{and} \quad e_j = \prod_{1\leq k\leq 4, k\neq j}(T_j-T_k)^{-1}.$$

Thus the general solution of (3.12) is given implicitly by

$$K^{-1}\sum_{j=1}^{4}(T_j^2+1)e_j\log(T-T_j) = t + \text{const.} \tag{3.14}$$

The above form of the solution is valid for complex values of T (and t), where the constant of integration should also be allowed to be complex; but since we are interested in real values of $T = \tan(\theta/2)$ for real t, in the case where the coefficients of $P(T)$ are all real, the solution may need to be specified in different forms according to the combinations of real/complex roots of this quartic. For example, if the four roots T_j are all real then for real T the solution can be written as

$$K^{-1}\sum_{j=1}^{4}(T_j^2+1)e_j\log|T-T_j| = t - t_0 \tag{3.15}$$

with a real constant of integration t_0. If, on the other hand, $P(T)$ has two real roots and a complex conjugate pair, then (for real T) two of the logarithms in (3.14) can be combined into an arctangent. Note that the roots of $P(T)$ correspond precisely to the points in the (b_1, b_2) plane where the ellipse (3.6) intersects with the hyperbola $b_1b_2 = J$, and the proof of Lemma 4 guarantees that there are two such points in the positive quadrant, hence $P(T)$ has at least two real roots. Having obtained $T(t)$ implicitly, from (3.7) we then find b_1, b_2 as

$$b_1 = \frac{2\lambda T}{k_1(1+T^2)} - \frac{1}{2k_1^2}, \qquad b_2 = \frac{\lambda(1-T^2)}{k_2(1+T^2)} - \frac{1}{2k_2^2}, \tag{3.16}$$

and hence q is found from (3.11).

For the second quadrature, to find $Q(t)$ from the last equation in (3.5), it is convenient to write

$$\dot{Q} = \frac{dQ}{dT}\,\dot{T},$$

replace the right-hand side of (3.5) by the corresponding expressions in terms of T, and then obtain $Q = Q(T)$ by integrating with respect to T (instead of t). This leads to the equation

$$\frac{dQ}{dT} = 2\lambda R(T)\left(\frac{1}{P(T)} - \frac{1}{\hat{P}(T)}\right), \tag{3.17}$$

given in terms of two additional polynomials, one quadratic and the other quartic, namely

$$R(T) = \lambda(T^2+1) - \frac{T}{k_1} + \frac{T^2-1}{2k_2}, \qquad \text{and} \qquad \hat{P}(T) = Jk_1k_2(T^2+1)^2 - P(T).$$

Then writing

$$\hat{P}(T) = \hat{K}\prod_{j=1}^{4}(T-\hat{T}_j),$$

the general solution of (109) can be written in terms of $T = T(t)$ as

$$Q = 2\lambda\Big(K^{-1}\sum_{j=1}^{4} R(T_j)e_j\log(T-T_j) - \hat{K}^{-1}\sum_{j=1}^{4} R(\hat{T}_j)\hat{e}_j\log(T-\hat{T}_j)\Big) + \text{const}, \tag{3.18}$$

with

$$\hat{e}_j = \prod_{1\leq k\leq 4, k\neq j}(\hat{T}_j - \hat{T}_k)^{-1}.$$

The preceding formulae can be used to describe the scattering of two peakons, in terms of their asymptotic behaviour as $t \to \pm\infty$. For certain values of the parameters/initial data, the behaviour of two peakons in the Popowicz system appears to be qualitatively similar to that of peakons in integrable PDEs such as the Camassa-Holm and Degasperis-Procesi equations: for large negative/positive times the two peaks are well separated and asymptotically move with constant velocities and constant amplitudes. However, there is one main difference: unlike those integrable single component equations, in which the two peakons asymptotically switch their velocities and amplitudes, resulting only in a phase shift in the resulting trajectories before/after interaction, the Popowicz peakons exchange different amounts of velocity and amplitude during the interaction (with the amplitudes being different for the two components u, v), so that generically the pair of peakon velocities is different before and after.

The asymptotic form of the two-peakon solution for the Popowicz system is controlled by the first order ODE (3.12) for T. This equation has fixed points at the roots of $F(T)$, i.e. at $T = T_k$ where the roots of the quartic polynomial $P(T)$ lie. Near to a fixed point, the local behaviour is

$$T \sim T_k + A_k e^{F'(T_k)t},$$

where A_k is a constant, and we have

$$F'(T_k) = \frac{K}{(1+T_k^2)e_k} \tag{3.19}$$

compared with the coefficients in (3.14). The initial data for the peakon system at $t = 0$ determines an initial point on the ellipse $h = \text{const}$ in the (b_1, b_2) plane, and hence an initial angle $\theta(0)$ and corresponding value $T(0) = \tan(\theta(0)/2)$. Given that $T(0)$ lies between two real roots of F (or equivalently of P), the fact that $J < b_1b_2$ and the assumption $q < 0$ implies from (3.9) that $\dot{\theta} < 0$, hence $\dot{T} < 0$. We denote the two adjacent roots by $T_\pm$ with

$$T_+ < T(0) < T_-,$$

and the asymptotic behaviour is then given by

$$T \sim T_\pm \pm e^{F'(T_\pm)t+\delta_\pm} \qquad \text{as} \qquad t \to \pm\infty, \tag{3.20}$$

where the constant $\delta_\pm$ depends on the terms in (3.14) that are regular at $T = T_\pm$, as well as the integration constant; so in particular, when there are four real roots, $\delta_\pm$ depends on the arbitrary constant t_0 in (3.15). Moreover, the given assumptions imply that T_+ is a stable fixed point of (3.12), and T_- is unstable, so

$$F'(T_+) < 0 < F'(T_-).$$

From (3.16) we can immediately read off the asymptotic amplitudes of the two peakons in the field $v(x,t)$, that is

$$b_1 \to b_1^\pm = \frac{2\lambda T_\pm}{k_1(1+T_\pm^2)} - \frac{1}{2k_1^2}, \qquad b_2 \to b_2^\pm = \frac{\lambda(1-T_\pm^2)}{k_2(1+T_\pm^2)} - \frac{1}{2k_2^2}, \qquad \text{as} \quad t \to \pm\infty. \tag{3.21}$$

The corresponding amplitudes for the field $u(x,t)$ are then obtained from the formula $a_j = C_j b_j^2 = k_j^2 b_j^2$. Note that the points $(b_1^\pm, b_2^\pm) \in \mathbb{R}^2$ are precisely the two intersections of the ellipse (3.6) with the upper branch of the hyperbola $b_1 b_2 = J$, which must always exist if the initial amplitudes are positive, by the proof of Lemma 4. The asymptotic behaviour of the positions is more complicated. For the difference $q = q_1 - q_2$ we have from (3.9) and (3.11) that

$$\begin{array}{rcl} q & = & \log(-\dot{T}) - \log(k_1 k_2 b_1 b_2 (1+T^2)) \\ & \sim & F'(T_\pm)t + \delta_\pm + \log(\mp F'(T_\pm)) - \log(k_1 k_2 J(1+T_\pm^2)) \quad \text{as} \quad t \to \pm\infty, \end{array} \tag{3.22}$$

where we used (3.20) and the fact that $b_1 b_2 \to J$ as $|t| \to \infty$. The sum $Q = q_1 + q_2$ is determined from (3.18), which near $T = T_\pm$ gives

$$Q \sim \frac{2\lambda e_\pm R(T_\pm) F'(T_\pm)}{K} t + \text{const},$$

hence from (3.19) we have

$$Q \sim \frac{2\lambda R(T_\pm)}{1+T_\pm^2} t + \text{const} \qquad \text{as} \qquad t \to \pm\infty, \tag{3.23}$$

where the constant depends on $\delta_\pm$, as well as the arbitrary constant of integration and the other terms in (3.18).

It is worth comparing these results with the corresponding asymptotic formulae for Camassa-Holm peakons [4, 5]: in that case, if the leftmost peak at position q_1 has asymptotic velocity c_1 (which is the same as its amplitude) for large negative times, and the peak at $q_2 > q_1$ has asymptotic velocity c_2, with $c_1 > c_2$ so that they collide, then for large positive times these asymptotic velocities (and amplitudes) are switched. In terms of the difference q this corresponds to having

$$q \sim \mp(c_1 - c_2)t + \text{const} \qquad \text{as} \quad t \to \pm\infty,$$

while for the sum Q the leading order behaviour is the same in both asymptotic regimes, that is

$$Q \sim (c_1 + c_2)t + \text{const} \qquad \text{as} \quad t \to \pm\infty;$$

the next to leading order (constant) terms determine the phase shifts, i.e. the changes in the relative position of each soliton that result from their interaction. For the Popowicz peakons, in contrast, such switching of asymptotic velocities is far from being generic behaviour: from (3.22) and (3.23), asymptotic switching would require that both

$$P'(T_+)/(T_+^2 + 1) = -P'(T_-)/(T_-^2 + 1)$$

(the asymptotic velocity of q changes sign between $t \to \pm\infty$) and

$$\frac{2\lambda R(T_+)}{1+T_+^2} = \frac{2\lambda R(T_-)}{1+T_-^2}$$

(Q has the same asymptotic velocity for $t \to \pm\infty$) should hold, which puts constraints on the parameters/initial values.

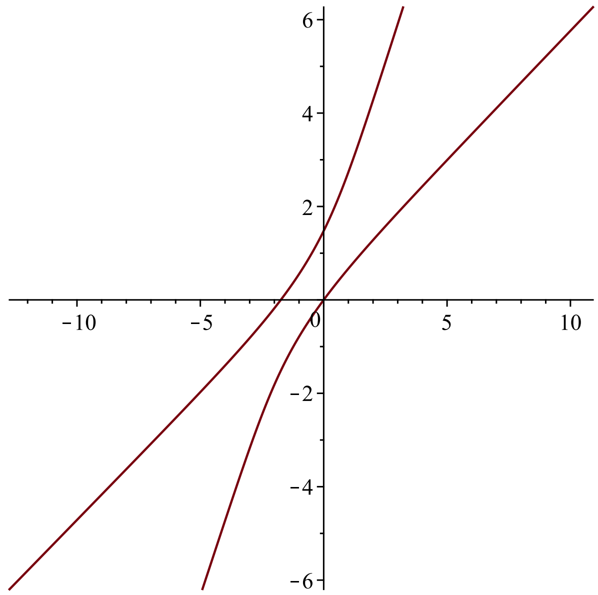

Figure 1: Scattering of two peakons with initial data (3.24) as a spacetime plot (time is the vertical axis).

To illustrate these results, it is instructive to consider a particular numerical example. Upon choosing the initial data and parameters for (3.3) as

$$b_1(0) = \frac{94417}{165416},\ b_2(0) = \frac{103298821}{164092672},\ q_1(0) = \log\left(\frac{2890857125}{16447158149}\right),\ q_2(0) = 0,\ k_1 = 1,\ k_2 = \frac{496}{593}, \tag{3.24}$$

the values of the first integrals are found to be

$$h = \frac{58672865}{16267808}, \qquad J = \frac{298710111}{1008604096}.$$

Then $\lambda = 35425/22816$, and

$$P(T) = -\frac{3365375}{4066952}\left(T + \frac{111}{152}\right)\left(T - \frac{3}{10}\right)\left(T - \frac{1}{2}\right)(T - 8),$$

so that $T(0) = 2/5$ lies between $T_+ = 3/10$ and $T_- = 1/2$, and also

$$\hat{P}(T) = \frac{6133}{5704}\left(T^2 - \frac{35425}{5704}T + 1\right)\left(T^2 - \frac{10893}{24532}\right),$$

where the latter quartic also has four real roots, namely $T = \frac{35425}{5704} \pm \sqrt{\frac{1124788161}{32535616}}, \pm\sqrt{\frac{10893}{24532}}$. Substituting the roots of P into (3.15) and setting $T = 2/5$ at $t = 0$ yields $t_0 \approx 0.2686597887$, and similarly the integration constant in (3.18) can be fixed, so that $q_1 = (Q + q)/2$ and $q_2 = (Q - q)/2$ are completely determined parametrically in terms of $t(T)$. Figure 1 is a plot of their trajectories, with q_1 being the topmost curve. Figure 2 shows the same scattering process in the form of a contour plot of $v = v(x, t)$ viewed from above; the dark band visible around $t = 0$ is an artefact of the overlap between two different parametric plots, which were required to deal with larger positive/negative t values separately. The asymptotic amplitudes are

$$(b_1^-, b_2^-) = \left(\frac{4233}{5704}, \frac{70567}{176824}\right) \approx (0.7421107994, 0.3990804416) \qquad \text{for} \quad t \to -\infty,$$

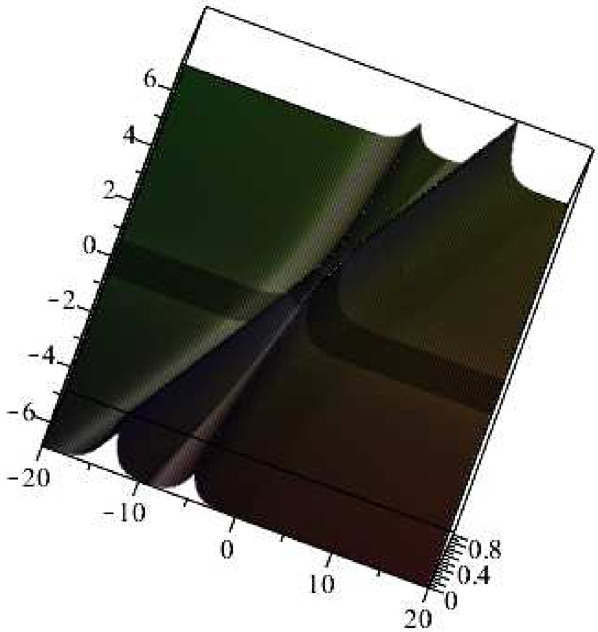

Figure 2: Contour plot of $v(x,t)$ in the box $-20 \le x \le 20$, $-6 \le t \le 6$, $0 \le v \le 0.8$.

and

$$(b_1^+, b_2^+) = \left(\frac{2023}{5704}, \frac{147657}{176824}\right) \approx (0.3546633941, 0.8350506719) \qquad \text{for} \quad t \to \infty.$$

Then combining (3.22) and (3.23), we find that the asymptotic positions of the two peakons are given by

$$q_j \sim c_j^{\pm} t + \text{const}, \qquad \text{as} \quad t \to \pm\infty, \qquad j = 1, 2,$$

where for $t \to -\infty$ the asymptotic velocities are

$$c_1^- = \frac{29990805}{16267808} \approx 1.8435676767, \qquad c_2^- = \frac{2529345}{4066952} \approx 0.6219264452,$$

and for $t \to \infty$ they are

$$c_1^+ = \frac{9862125}{16267808} \approx 0.6062356404, \qquad c_2^+ = \frac{7364175}{4066952} \approx 1.8107356566.$$

Hence in this case the two peakons do not exactly exchange their asymptotic velocities and amplitudes, but only approximately so.

4 Conclusions

We have considered the dynamics of peakon solutions in the non-integrable coupled system (1.12). In the absence of a weak formulation appropriate for these solutions, they are interpreted as distributional solutions in such a way that the peakons inherit the Hamiltonian properties of the PDE system, so their dynamics is conservative. The two-peakon dynamics is Liouville integrable, and we have explicitly integrated the equations of motion and described the interaction of the peakons in the case when all the amplitudes are positive. The case where the amplitudes have mixed sign (peakon-antipeakon interaction) is more subtle: in the Camassa-Holm case, it involves a head-on collision, with overlapping peaks [4, 5]; while in this case, if the peakons overlap ($q = 0$) then the form of first integral J, as in (3.8), implies that at least one of the amplitudes must diverge to infinity.

For three or more peakons, we do not expect that the dynamics of peakons is integrable. Nevertheless, in the case where all the peakons have positive amplitudes, the qualitative features of their interaction should be similar to the two-peakon case. In particular, it is not hard to see that the analogue of Lemma 4 holds for all N: upon fixing $a_j = k_j^2 b_j^2 > 0$, the fixed energy hypersurface

$$h = 2\sum_{j=1}^{N}(k_j^2 b_j^2 + b_j) = \text{const}$$

is a compact quadric in $\mathbb{R}^N$, so from the form of (2.9) there can be no overlap $q_k = q_{k+1}$ between initially adjacent peaks; but then, from the ordering (2.10), peaks that are initially non-adjacent cannot overlap without first passing through their nearest neighbours, which cannot happen, $0 < J < \prod_{j=1}^{N} b_j$ holds, and the result follows. Furthermore, it seems reasonable that the overall dynamics of three or more peakons should be determined approximately by the local interaction between each pair of peaks, at least when they are well separated from the rest.

It would be interesting to carry out numerical studies of the peakon ODEs (2.5) for $N > 2$, and to perform a numerical integration of the full PDE system (1.12) to see whether peakons emerge naturally from generic initial data, as is the case for the b-family in the parameter range $\mathrm{b} > 1$ [15]. However, the PDE integration is likely to be at least as challenging as for scalar peakon equations, which are already known to be difficult (for instance, see [7] and references). Before embarking on such a study, it is worth noting that all of the considerations in this paper admit a natural generalization to a vector **b**-family of PDEs, given by

$$\mathbf{m}_t = \hat{\mathbf{B}}\,\frac{\delta H_0}{\delta \mathbf{m}},$$

with

$$\mathbf{m} = (m_1, m_2, \ldots, m_d)^T, \qquad \mathbf{b} = (\mathrm{b}_1, \mathrm{b}_2, \ldots, \mathrm{b}_d)^T$$

being a d-component vector of fields and a corresponding vector of parameters, and

$$H_0 = \int \sum_{j=1}^{d} m_j \,\mathrm{d}x, \qquad \hat{\mathbf{B}} = \mathbf{w}\,\mathcal{L}^{-1}\,\mathbf{w}^\dagger,$$

where $\mathbf{w} = (w_1, w_2, \ldots, w_d)^T$ is a vector operator with components

$$w_j = \mathrm{b}_j\, m_j^{1-1/\mathrm{b}_j}\,\partial_x\, m_j^{1/\mathrm{b}_j}, \qquad j = 1, \ldots, d.$$

So the original b-family (1.3) is just the case $d = 1$, while the Popowicz system corresponds to $d = 2$ with fields $(m_1, m_2) = (m, n)$ and parameters $(\mathrm{b}_1, \mathrm{b}_2) = (3, 2)$.
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References

[1] S.C. Anco, X. Chang and J. Szmigielski, `arXiv:1711.01429`

[2] V.I. Arnold, Mathematical Methods of Classical Mechanics, 2nd edition, Springer, 1997.

[3] R. Beals, D.H. Sattinger and J. Szmigielski, Adv. Math. **154** (2000) 229–257.

[4] R. Camassa and D.D. Holm, Phys. Rev. Lett. **71** (1993) 1661–1664.

[5] R. Camassa, D.D. Holm and J.M. Hyman, Adv. Appl. Mech. **31** (1994) 1–33.

[6] M. Chen, S.-Q. Liu and Y. Zhang, Lett. Math. Phys. **75** (2006) 115.

[7] A. Chertock, J.-G. Liu and T. Pendleton, SIAM J. Numer. Anal. **50** (2012) 1–21.

[8] A. Constantin and D. Lannes, Arch. Ration. Mech. Anal. **192** (2009) 165–186.

[9] A. Degasperis and M. Procesi, Asymptotic integrability, in *Symmetry and Perturbation Theory*, eds. Degasperis and G. Gaeta, River Edge, NJ: World Scientific (1999) 23–37.

[10] A. Degasperis, D.D. Holm and A.N.W. Hone, Theoret. Math. Phys. **133** (2002) 1461–1472.

[11] H.R. Dullin, G.A. Gottwald and D.D. Holm, Physica D **190** (2004) 1–14.

[12] G. Falqui, J. Phys. A: Math. Gen. **39** (2006) 327–342.

[13] B. Fuchssteiner and A.S. Fokas, Physica D **4** (1981) 47–66.

[14] D.D. Holm and A.N.W. Hone, J. Nonlin. Math. Phys. **12** (2005) 380–394.

[15] D.D. Holm and M. Staley, Phys. Lett. A **308** (2003) 437–444.

[16] A.N.W. Hone and M.V. Irle, On the non-integrability of the Popowicz peakon system, in *Dynamical Systems and Differential Equations, Proc. 7th AIMS International Conference*, Disc. Cont. Dyn. Sys. Supplement (2009) 359–366.

[17] A.N.W. Hone, H. Lundmark and J. Szmigielski, Dyn. Partial Differ. Equ. **6** (2009) 253–289.

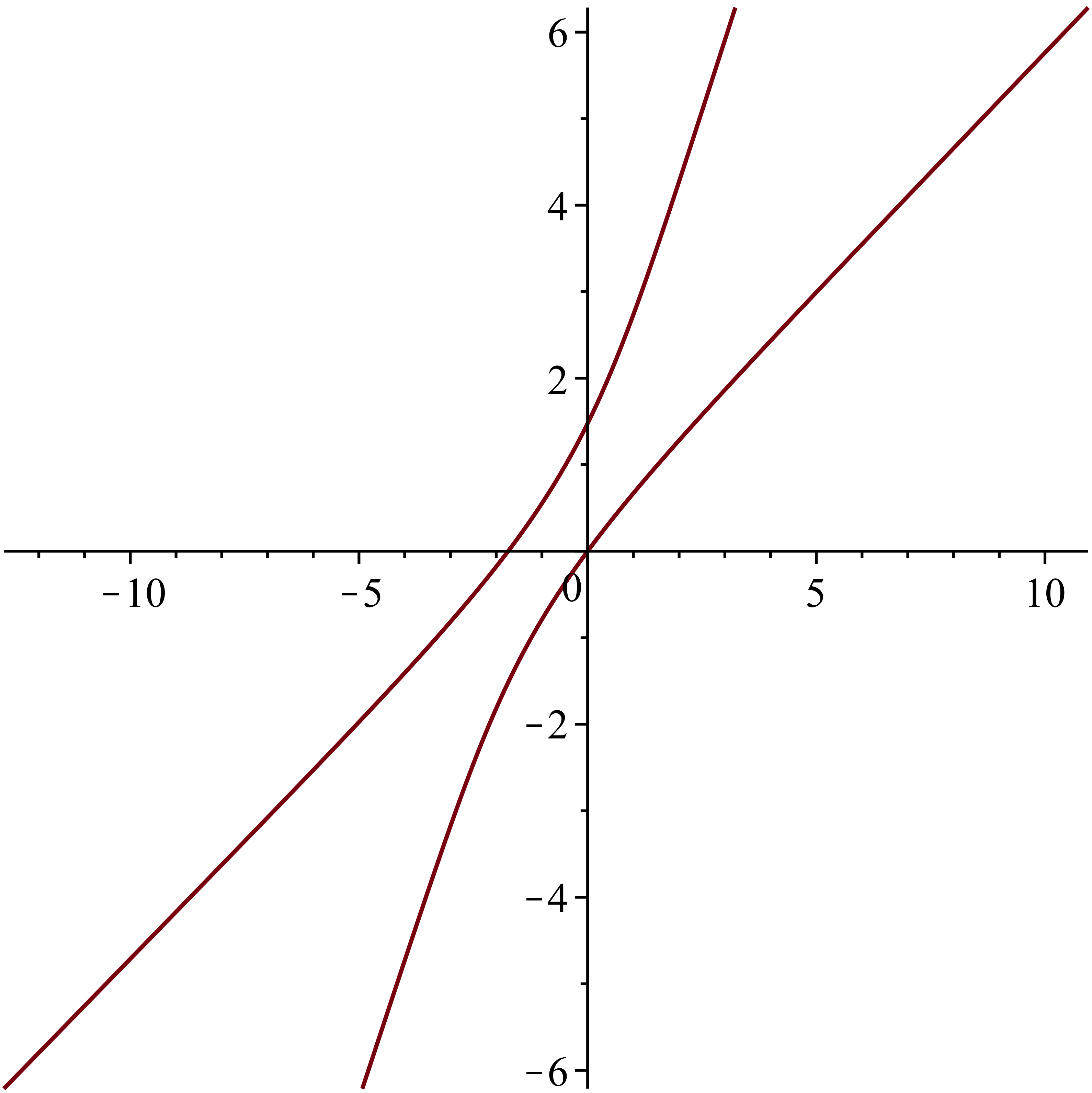
[18] M.V. Irle, *Solitons, peakons and Hamiltonian structure of particular partial differential equations*, MSc thesis, University of Kent, 2010.

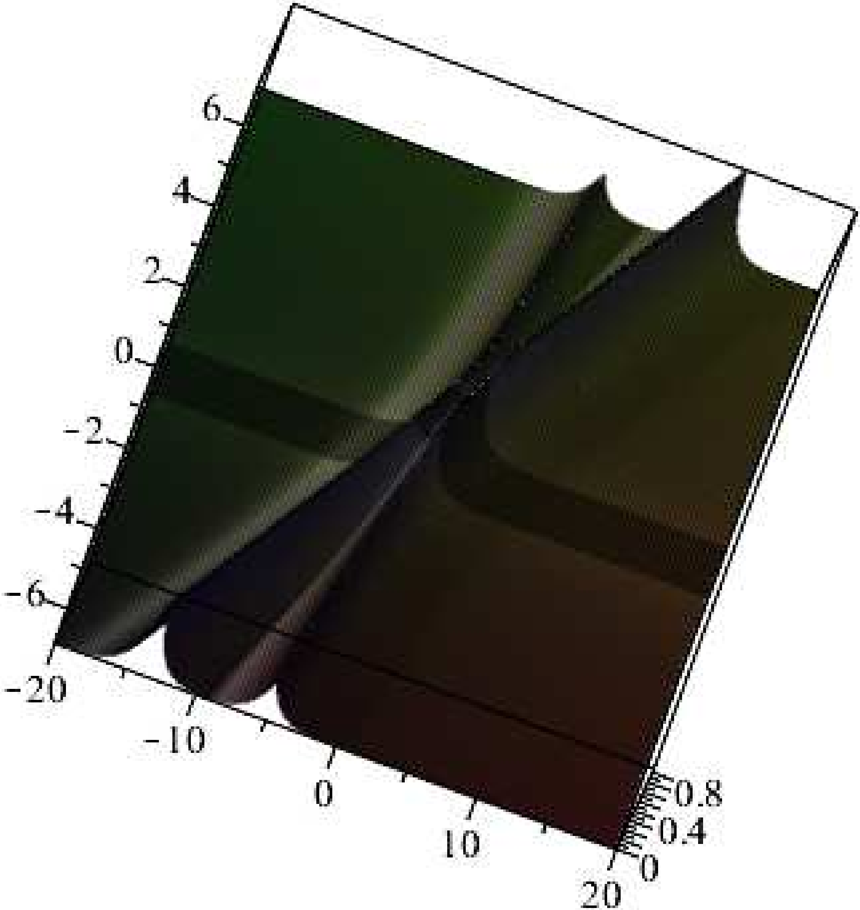
[19] H. Lundmark and J. Szmigielski, Int. Math. Res. Pap. IMRP **2005** (2005) 53–116.

[20] A.V. Mikhailov and V.S. Novikov, J. Phys. A: Math. Gen. **35** (2002) 4775–4790.

[21] Z. Popowicz, J. Phys. A: Math. Gen. **39** (2006) 13717–13726.

Figure





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