A Hirota bilinear equation for Painlevé transcendents P_{IV}, P_{II} and P_I.

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Abstract

We present some observations on the tau-function for the fourth Painlevé equation. By considering a Hirota bilinear equation of order four for this tau-function, we describe the general form of the Taylor expansion around an arbitrary movable zero. The corresponding Taylor series for the tau-functions of the first and second Painlevé equations, as well as that for the Weierstrass sigma function, arise naturally as special cases, by setting certain parameters to zero.

1 Introduction

The six Painlevé equations (denoted $P_I - P_{VI}$) can be considered as nonlinear analogues of the classical functions: they admit a Hamiltonian representation [15], all of them (apart from P_I) possess Bäcklund transformations [2], and they each arise as a compatibility condition for an associated isomonodromy problem [13]. General solutions of Painlevé equations have asymptotics in terms of elliptic functions, which was originally obtained (for P_I and P_{II}) by Boutroux [1]. It is also known that through a limiting procedure, usually called the coalescence cascade, it is possible to obtain all the equations $P_V - P_I$ just from equation P_{VI} (see e.g. [12]). Furthermore, the equations P_I, P_{II} and P_{IV} share the property that all their local solutions are meromorphic and possess a meromorphic continuation in the whole complex plane [11].

The Hamiltonian functions for P_I-P_{VI} are polynomials $h_j = h_j(q, p, z)$ in the canonically conjugate phase space variables q, p, and are rational in the independent variable z. Letting a prime denote differentiation with respect to z, the Hamiltonian formulation allows each of the Painlevé equations to be formulated as a first order system,

$$q' = \frac{\partial h_j}{\partial p}, \qquad p' = -\frac{\partial h_j}{\partial q}, \qquad j = I, ..., VI. \tag{1}$$

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The functions h_j themselves, as functions of the time z, solve certain differential equations; these functions, defined by $\sigma_j(z) = h_j(q(z), p(z), z)$, where $q(z), p(z)$ satisfy (1), are usually called "sigma functions" [13, 15]. Every solution of the Painlevé equation can be written in terms of the solution of a corresponding differential equation for σ_j, which is of second order and second degree. Moreover, the sigma function is given in terms of the logarithmic derivative of a tau-function. In some sense, one can view the sigma function or the tau-function as being more fundamental than the solution of the Painlevé equation, since in applications (such as in the theory of random matrices [7]) these are usually the main objects of interest.

In recent work [10] we have shown how the recursive formula for the coefficients in the Laurent series expansion of solutions of the first Painlevé equation can be considered as an extension of the analogous formula for the Weierstrass $\wp$ function. In addition, the recursive formulae for the Taylor expansion of the tau-function around one of its zeros lead to natural extensions of the expressions found by Weierstrass [18] for the elliptic sigma function (not to be confused with the sigma function of the Painlevé equations). The key to these recursive formulae was the use of a Hirota bilinear equation for the tau-function, amenable to the same method that was applied to the elliptic sigma function in [3].

The purpose of this short article is to derive recursive formulae for the expansion of the tau-function of the fourth Painlevé equation around a movable zero. Bilinear equations for P_{IV} tau-functions have been derived previously, either as a system of two equations relating two tau-functions [9], or as a symmetric system involving three tau-functions (see e.g. Theorem 3.5 in [14]). However, by starting from the equation for the sigma function σ_{IV}, we can use a single Hirota bilinear equation of fourth order to obtain the Taylor series expansion of the P_{IV} tau-function around a zero. By exploiting the freedom in the definition of σ_{IV}, we introduce additional parameters into the sigma function equation, and show how the corresponding series solutions for both P_{II} and P_I arise directly from the same bilinear equation as degenerate special cases, by setting suitable parameters to zero, while all of these series can be viewed as natural extensions of the elliptic case treated in [3].

In the next section we briefly review the Hamiltonian formulation of the fourth Painlevé equation and the corresponding sigma equation, before introducing a "shifted" sigma equation (given by (11) below), which is suitable for studying series expansions around movable poles, as well as the degeneration to P_{II}, P_I and elliptic functions. Section 3 is concerned with the properties of the tau-function for P_{IV}, the corresponding bilinear equation, and the presentation of the main result, namely the recursion for the Taylor coefficients (Theorem 3.2). The fourth section is devoted to a numerical application of the main result, using it to calculate approximations to the zeros of a particular tau-function for P_{II}, and we end with some conclusions and suggestions for future work.

2 Hamiltonian and sigma equation for P_{IV}

The fourth Painlevé equation can be derived from the Hamiltonian function

$$h_{IV}(q,p,z) = \zeta(qp^2 - q^2p) + \zeta^{-1}\Big((e_2-e_3)p + (e_3-e_1)q\Big) + (e_3 - \zeta^2 qp)z, \quad (2)$$

where $\zeta \neq 0$ and e_j for $j = 1,2,3$ are parameters. The corresponding Hamilton's equations (1) are given explicitly by (hereafter a prime denotes a derivative with respect to z)

$$q' = \zeta q(2p-q) + \zeta^{-1}(e_2-e_3) - \zeta^2 zq, \qquad p' = \zeta p(2q-p) + \zeta^{-1}(e_1-e_3) + \zeta^2 zp. \quad (3)$$

The ordinary differential equation of second order satisfied by q arises by eliminating p from (3), to yield

$$q'' = \frac{(q')^2}{2q} + \frac{3}{2}\zeta^2 q^3 + 2\zeta^3 zq^2 + \Big(\frac{1}{2}\zeta^4 z^2 - \alpha\Big)q + \frac{\beta}{q}, \quad (4)$$

where

$$\alpha = e_2 + e_3 - 2e_1 + \zeta^2, \qquad \beta = -\frac{(e_2-e_3)^2}{2\zeta^2},$$

which (up to rescaling q and z) is just the fourth Painlevé equation P_{IV}. By symmetry, upon eliminating q from (3), it follows that p satisfies

$$p'' = \frac{(p')^2}{2p} + \frac{3}{2}\zeta^2 p^3 - 2\zeta^3 zp^2 + \Big(\frac{1}{2}\zeta^4 z^2 - \tilde{\alpha}\Big)p + \frac{\tilde{\beta}}{p}, \quad (5)$$

with

$$\tilde{\alpha} = e_1 + e_3 - 2e_2 - \zeta^2, \qquad \tilde{\beta} = -\frac{(e_1-e_3)^2}{2\zeta^2},$$

so that $-p$ satisfies the same form (4) of P_{IV} as q does, but for different values of the parameters α, β.

There is a certain amount of redundancy in the choice of parameters used above. Although the parameter ζ appears inessential, as (providing it is non-zero) it can always be removed by rescaling q, p and z, it will be needed in what follows. As for the three quantities e_j, $j = 1,2,3$, the solutions of P_{IV} only depend on the differences $e_j - e_k$, but the inclusion of the term $e_3 z$ in (2) shows that the sigma function

$$\sigma_{IV}(z) = h_{IV}(q(z), p(z), z)$$

also depends on the parameter

$$\mu^* = \frac{e_1 + e_2 + e_3}{3}. \quad (6)$$

Indeed, by taking derivatives of the Hamiltonian with respect to z, it follows that the sigma function satisfies the following equation of second order and second degree:

$$(\sigma''_{IV})^2 - \zeta^4(z\sigma'_{IV} - \sigma_{IV})^2 + 4(\sigma'_{IV} - e_1)(\sigma'_{IV} - e_2)(\sigma'_{IV} - e_3) = 0. \quad (7)$$

Moreover, q and p are given in terms of the solution of the latter equation by

$$q = \frac{\sigma''_{IV} - \zeta^2(z\sigma'_{IV} - \sigma_{IV})}{2\zeta(\sigma'_{IV} - e_1)}, \qquad p = \frac{\sigma''_{IV} + \zeta^2(z\sigma'_{IV} - \sigma_{IV})}{2\zeta(\sigma'_{IV} - e_2)}. \tag{8}$$

The freedom to permute e_1, e_2, e_3 shows that generically the same solution of (7) provides six different solutions of the equation (4), with different α, β; this is one manifestation of the affine A_2 symmetry for P_{IV} [15], which can be seen more easily from its symmetric form [14].

Henceforth we regard the sigma equation (7) as the fundamental object of interest, and proceed to consider the behaviour of solutions near singularities. Since $q(z)$ and $p(z)$ are both meromorphic for all $z \in \mathbb{C}$ (see e.g. [11] or [17]), it follows from (2) that $\sigma_{IV}(z)$ is also a globally meromorphic function, and it is straightforward to see that its only possible singularities are movable simple poles with a local Laurent expansion of the form

$$\sigma_{IV}(z) = \frac{1}{z - z_0} + B + O\Big((z - z_0)\Big), \tag{9}$$

where both the pole position z_0 and the quantity B (resonance parameter) are arbitrary. For fixed values of the coefficients e_j and ζ, any solution of the second order equation (7) is completely specified by a particular choice of the two values z_0, B in (9), which is then determined on the whole complex plane by analytic continuation.

In order to understand how the solution of (7) depends on the parameters z_0, B, it is convenient to shift

$$z \to z + z_0, \qquad \sigma_{IV} \to \sigma + B, \tag{10}$$

which leads to an equation of the form

$$(\Sigma'')^2 - \eta\,(z\Sigma' - \Sigma)^2 + 2(\kappa\Sigma' - \lambda)\,(z\Sigma' - \Sigma) + 4(\Sigma')^3 - g_2\Sigma' + g_3 = 0 \tag{11}$$

where

$$\eta = \zeta^4,$$

and for $\mu = \mu^* + \eta z_0^2/12$, the dependent variable Σ is given by

$$\Sigma(z) = \sigma(z) - \mu z, \tag{12}$$

with the parameters $\kappa, \lambda, g_2, g_3$ being polynomials in η, z_0, B and the e_j. Having fixed the pole to lie at $z = 0$, and shifted away the parameter B, the function $\Sigma(z)$ satisfying (11) depends only on the 5 parameters $\eta, \kappa, \lambda, g_2, g_3$, while $\sigma(z)$ depends on μ also.

Lemma 2.1. *For $\zeta \neq 0$, via translations of the form (10), there is a one-to-one correspondence between solutions of (7) with a pole at some $z_0 \in \mathbb{C}$, and functions*

$$\sigma(z) = \Sigma(z) + \mu z$$

with a pole at $z = 0$, where $\Sigma(z)$ is the solution of (11) specified by the local Laurent expansion

$$\Sigma(z) = \frac{1}{z} + O(z^2). \tag{13}$$

Remark 2.2. The above result applies to any solution of (7) with at least one pole; in particular, this excludes certain trivial solutions which are linear in z. If we scale (4) so that $\zeta = 1$, then all solutions of P_{IV} which are transcendental, meaning that they are neither rational nor can be reduced to solutions of a Riccati equation, have infinitely many simple poles with residue $+1$ and infinitely many with residue -1 [8]. The formula (8) shows that (for $\zeta = 1$) q has a pole with residue -1 at places where σ_{IV} has a simple pole, and q does not depend on the parameter μ, so its behaviour near such a pole is completely determined by a function Σ specified as above. Poles of q with residue $+1$ correspond to places where σ_{IV} has a zero with $\sigma'_{IV} \to e_1$; the behaviour at such poles can also be determined by using the well known observation of Okamoto [15] that when $\zeta = 1$ every solution of (4) can be written as the difference of two Hamiltonians, i.e.

$$q(z) = \tilde{\sigma}_{IV}(z) - \sigma_{IV}(z),$$

where $\tilde{\sigma}_{IV}$ satisfies (7) but with suitably shifted parameters.

For future reference, we record the equation of third order that results by taking the derivative of (11) and removing a factor of Σ'', that is

$$\Sigma''' + 6(\Sigma')^2 - z(\eta z - 2\kappa)\Sigma' + (\eta z - \kappa)\Sigma - \lambda z - \frac{1}{2}g_2 = 0. \tag{14}$$

Clearly the parameter g_3 in (11) is a first integral for the above equation.

We now consider the degenerate case $\eta = \zeta^4 = 0$, which is no longer related to P_{IV}.

Proposition 2.3. *If $\eta = 0$ and $\kappa \neq 0$, then*

$$v = 2\sigma' = 2(\Sigma' + \mu) \qquad \textit{with} \qquad \mu = -\lambda\kappa^{-1} \tag{15}$$

satisfies the P_{XXXIV} equation in the form

$$vv'' - \frac{1}{2}(v')^2 + 2v^3 + (\kappa z - 6\mu)v^2 + \frac{\ell^2}{2} = 0, \tag{16}$$

where

$$\ell^2 = 16\mu^3 - 4g_2\mu - 4g_3,$$

Thus

$$u = \frac{v' + \ell}{2v} \tag{17}$$

satisfies the second Painlevé equation P_{II} in the form

$$u'' = 2u^3 + (\kappa z - 6\mu)u + \ell - \frac{1}{2}\kappa, \tag{18}$$

and conversely v is given in terms of u and its first derivative by

$$v = -u' - u^2 - \frac{1}{2}(\kappa z - 6\mu). \tag{19}$$

Proof. If $\eta = 0$, then (11) reduces to the sigma equation for P_{II}, provided that $\kappa \neq 0$. Upon multiplying (14) by $v/2 = \Sigma' - \lambda\kappa^{-1}$ and subtracting off half of (11), the terms involving Σ are eliminated, and what remains is the equation (16) for v, which is referred to as P_{XXXIV} in [12]. Every solution of (16) gives a solution of (18), and vice-versa, according to the formulae (17) and (19). □

Remark 2.4. The relations (17) and (19) can be rewritten as

$$v' = \frac{\partial h_{II}}{\partial u}, \quad u' = -\frac{\partial h_{II}}{\partial v}, \qquad \text{with} \quad h_{II} = u^2 v - \ell u + \frac{1}{2}v^2 + \frac{1}{2}(\kappa z - 6\mu)v,$$

which is the Hamiltonian formulation of P_{II} found in [15]. The standard version of (16), or that of (18), has $\mu = 0$. However, the situation for $\eta = 0$, $\kappa \neq 0$ is completely analogous to that in Lemma 2.1: we can use expansions around $z = 0$ for Σ, of the form, (13) to obtain local Laurent expansions for the standard version of P_{XXXIV} (or P_{II}) around a pole in an arbitrary position z_0.

When $\eta = \kappa = 0$, then a further degeneration occurs.

Proposition 2.5. *If $\eta = \kappa = 0$ and $\lambda \neq 0$, then*

$$w = -\Sigma'$$

satisfies the first Painlevé equation P_I in the form

$$w'' = 6w^2 - \lambda z - \frac{1}{2}g_2, \tag{20}$$

while if $\eta = \kappa = \lambda = 0$, then the general solution of (11) is given in terms of the Weierstrass zeta function with invariants g_2, g_3 by

$$\Sigma(z) = \zeta(z - z_0; g_2, g_3) + B, \tag{21}$$

for z_0, B arbitrary, so $\Sigma' = -\wp(z - z_0; g_2, g_3)$ is an elliptic function of z.

Proof. Up to replacing $\lambda \to 6\lambda$, this coincides with the case considered in [10]. □

3 Tau-function and bilinear equation

For the sigma equation in the form (11), the tau-function $\tau(z)$ is defined by

$$\Sigma(z) = \frac{d}{dz}\log\tau(z). \tag{22}$$

Since Σ is meromorphic, with its only singularities being simple poles with residue $+1$, the above formula implies that $\tau(z)$ is holomorphic, but is only defined up to overall scaling $\tau \to A\tau$ for an arbitrary non-zero constant A. By substituting (22) into (11), an equation of third order which is homogeneous of degree four in τ results, that is

$$\begin{aligned} &\tau^2(\tau''')^2 - 6\tau\tau'\tau''\tau''' + 4(\tau')^3\tau''' + 4\tau(\tau'')^3 - 3(\tau'\tau'')^2 \\ &-z(\eta z - 2\kappa)\Big(\tau\tau'' - (\tau')^2\Big)^2 + 2(\eta z - \kappa)\Big(\tau^2\tau'\tau'' - \tau(\tau')^3\Big) \\ &+(2\lambda z - \eta + g_2)\tau^2(\tau')^2 - (2\lambda z + g_2)\tau^3\tau'' + 2\lambda\tau^3\tau' + g_3\tau^4 \quad = 0. \end{aligned} \tag{23}$$

Taylor expansions of (23) around a movable zero, which correspond to a movable simple pole in (11), take the form

$$\tau(z) = C_0(z - z_0) + C_1(z - z_0)^2 + C_2(z - z_0)^3 + \ldots, \qquad C_0 \neq 0,$$

where z_0 (the position of the zero) and C_0, C_1 are arbitrary, while all subsequent coefficients are determined uniquely in terms of these three parameters. By considering gauge transformations of the form

$$\tau(z) \to A\exp(Bz)\tau(z), \qquad A \neq 0, \tag{24}$$

the initial coefficient C_0 can be set to 1, and C_1 can be set to 0; in that case one can check that the next coefficient C_2 is also 0. The overall effect of the transformation (24) is to send

$$\Sigma \to \Sigma + B,$$

which results in changing the parameters in (11) and (23). However, this change does not affect the form of the equation, and thus we obtain an alternative version of Lemma 2.1, reformulated in terms of the tau-function.

Lemma 3.1. *For $\zeta \neq 0$, via translations of the form (10), there is a one-to-one correspondence between solutions of (7) with a pole at some $z_0 \in \mathbb{C}$, and functions*

$$\sigma(z) = \frac{d}{dz}\log\tau(z) + \mu z$$

with a pole at $z = 0$, where $\tau(z)$ is the solution of (23) specified by the local Taylor expansion

$$\tau(z) = z + O(z^4). \tag{25}$$

The degree four equation (23) is somewhat awkward for computing the coefficients in the local expansion

$$\tau(z) = \sum_{n=0}^{\infty} C_n z^{n+1} \tag{26}$$

around a zero at $z = 0$. It is much more convenient to take the derivative of (23), so that after removing an overall factor one finds the bilinear (degree two) equation

$$D_z^4\tau\cdot\tau - z(\eta z - 2\kappa)D_z^2\tau\cdot\tau + 2(\eta z - \kappa)\tau\tau' - (2\lambda z + g_2)\tau^2 = 0, \tag{27}$$

which has been written concisely in terms of the Hirota derivative defined by

$$D_z^n f\cdot g(z) = \left(\frac{d}{dz} - \frac{d}{dz'}\right)^n f(z)g(z')|_{z'=z}. \tag{28}$$

The equation (27) also follows immediately by making the substitution (22) in (14). The quantity g_3 in (23) also corresponds to a first integral of (27).

In order to describe the expansion of the tau-function around a zero, we use the bilinear equation (27), and note that the action of the Hirota operators D_z^2 and D_z^4 on monomials is given by

$$D_z^2 z^j \cdot z^k = a_{j,k} z^{j+k-2}, \quad D_z^4 z^j \cdot z^k = b_{j,k} z^{j+k-4},$$

where the multipliers appearing on the right-hand side are

$$a_{j,k} = 2! \sum_{\ell=0}^{2} (-1)^\ell \begin{pmatrix} j \\ \ell \end{pmatrix} \begin{pmatrix} k \\ 2-\ell \end{pmatrix}, \quad b_{j,k} = 4! \sum_{\ell=0}^{4} (-1)^\ell \begin{pmatrix} j \\ \ell \end{pmatrix} \begin{pmatrix} k \\ 4-\ell \end{pmatrix}.$$

The resulting recursion relation leaves the coefficients C_0, C_1 and C_6 undetermined. The freedom to chose C_6 in (27) corresponds to the value of the first integral g_3, so in order to match the term at order z^7 arising from (23), the correct value of C_6 must be inserted in the recursion. Before stating the result, it is convenient to define the shifted multipliers

$$\hat{a}_{j,k} = a_{j,k} - 2k, \qquad a^*_{jk} = a_{j,k} - k.$$

Theorem 3.2. *The coefficients C_n in the expansion (26) obey the recursion*

$$\begin{aligned} n(n^2-1)(n-6)C_n = \; & -\tfrac{1}{2} \sum_{j=1}^{n-1} b_{j+1,n+1-j} C_j C_{n-j} + \tfrac{1}{2}\eta \sum_{j=0}^{n-4} \hat{a}_{j+1,n-3-j} C_j C_{n-4-j} \\ & -\kappa \sum_{j=0}^{n-3} a^*_{j+1,n-2-j} C_j C_{n-3-j} + \tfrac{1}{2} g_2 \sum_{j=0}^{n-4} C_j C_{n-4-j} \\ & +\lambda \sum_{j=0}^{n-5} C_j C_{n-5-j}. \end{aligned} \tag{29}$$

To obtain the expansion in the form (25), the free coefficients must be fixed as

$$C_0 = 1, \qquad C_1 = 0, \qquad C_6 = \frac{1}{5040}\kappa^2 - \frac{g_3}{840}.$$

With the latter choice, each coefficient C_n is a weighted homogeneous polynomial of total degree n in $\mathbb{Q}[\kappa, \eta, g_2, \lambda, g_3]$ with weights 3, 4, 4, 5, 6 *respectively, so that*

$$C_n = \frac{\mathrm{P}_n(\kappa, \eta, g_2, \lambda, g_3)}{(n+1)!}, \tag{30}$$

where $\mathrm{P}_n(\xi^3\kappa, \xi^4\eta, \xi^4 g_2, \xi^5\lambda, \xi^6 g_3) = \xi^n \mathrm{P}_n(\kappa, \eta, g_2, \lambda, g_3)$ for all $\xi \in \mathbb{C}^*$.

This above result extends the analogous recursion for the Taylor series coefficients of the Weierstrass sigma function [3] and for the tau-function of the first Painlevé equation [10]. We record the first few polynomials P_n here:

$$\begin{array}{llll} \mathrm{P}_0 = 1, & \mathrm{P}_1 = \mathrm{P}_2 = 0, & \mathrm{P}_3 = -\kappa, & \mathrm{P}_4 = 2\eta - \frac{1}{2}g_2, \\ \mathrm{P}_5 = -6\lambda, & \mathrm{P}_6 = \kappa^2 - 6g_3, & \mathrm{P}_7 = -\kappa(11\eta + g_2), & \\ \mathrm{P}_8 = 12\eta^2 + 6g_2\eta + 51\lambda\kappa - \frac{9}{4}g_2^2, & \mathrm{P}_9 = 17\kappa^3 - 42(\eta + g_2)\lambda + 108 g_3\kappa. & & \end{array}$$

Computer calculations up to P_{100} suggest that, after suitable scaling of the variables g_2, g_3, these polynomials have integer coefficients. The form of the expression (30) implies that the Taylor series solution of (23) with leading order (25) can be written as a multiple sum

$$\tau(z) = \sum_{j,k,l,m,n\geq 0} A_{j,k,l,m,n}\, \kappa^j \eta^k \left(\frac{g_2}{2}\right)^l \lambda^m (6g_3)^n \frac{z^{3j+4k+4l+5m+6n+1}}{(3j+4k+4l+5m+6n+1)!} \tag{31}$$

where $A_{j,k,l,m,n} \in \mathbb{Q}$.

Conjecture 3.3. *The series (31) has*

$$A_{j,k,l,m,n} \in \mathbb{Z} \quad \forall j,k,l,m,n \geq 0.$$

Remark 3.4. In the case $\kappa = \eta = \lambda = 0$, Weierstrass [18] considered the series for the elliptic sigma function in the form (31), and Onishi proved that $2^l 24^n A_{0,0,l,0,n} \in \mathbb{Z}$ for all l, n [16], while in [10] we already found considerable numerical evidence to suggest that $A_{0,0,l,m,n} \in \mathbb{Z}$.

The tau-function transforms in a very specific way when it is expanded around another zero, at a location $\Omega \neq 0$.

Proposition 3.5. *Let $\tau(z) = \tau(z;\eta,\kappa,\lambda,g_2,g_3)$ denote the solution of (23) having the Taylor expansion (25) around $z = 0$, and suppose that this function also vanishes at $z = \Omega \neq 0$. Then*

$$\tau(z+\Omega;\eta,\kappa,\lambda,g_2,g_3) = A \exp\left(Bz + \frac{1}{2}\tilde{\mu} z^2\right)\tau(z;\eta,\tilde{\kappa},\tilde{\lambda},\tilde{g}_2,\tilde{g}_3), \tag{32}$$

where

$$A = \tau'(\Omega), \qquad B = \frac{\tau''(\Omega)}{2\tau'(\Omega)}, \qquad \tilde{\mu} = \frac{1}{12}\Omega(\Omega\eta - 2\kappa), \qquad \tilde{\kappa} = \kappa - \Omega\eta, \tag{33}$$

$$\tilde{\lambda} = \lambda - B\eta - \tilde{\kappa}\tilde{\mu}, \qquad \tilde{g}_2 = g_2 + 12\tilde{\mu}^2 + 2\Omega\lambda + 2B\tilde{\kappa}, \qquad \tilde{g}_3 = g_3 - \tilde{g}_2\tilde{\mu} + 4\tilde{\mu}^3 - B^2\eta + 2B\lambda.$$

Proof. Upon replacing $z \to z + \Omega$ in (23), and introducing

$$\tilde{\sigma}(z) = \frac{d}{dz}\log\tau(z+\Omega),$$

we see that $\tilde{\sigma}$ satisfies an equation of the general form (7), and has a pole at $z = 0$ because $\tau(\Omega) = 0$. If we now set

$$\tilde{\sigma}(z) = \tilde{\Sigma}(z) + B + \tilde{\mu} z, \tag{34}$$

with $\tilde{\mu}$ given by the expression in (33), then for any choice of B, $\tilde{\Sigma}(z)$ satisfies an equation of the canonical form (11), but with different coefficients $\tilde{\kappa}, \tilde{\lambda}, \tilde{g}_2, \tilde{g}_3$. Now we further require that

$$\tilde{\Sigma}(z) = \frac{d}{dz}\log\tilde{\tau}(z),$$

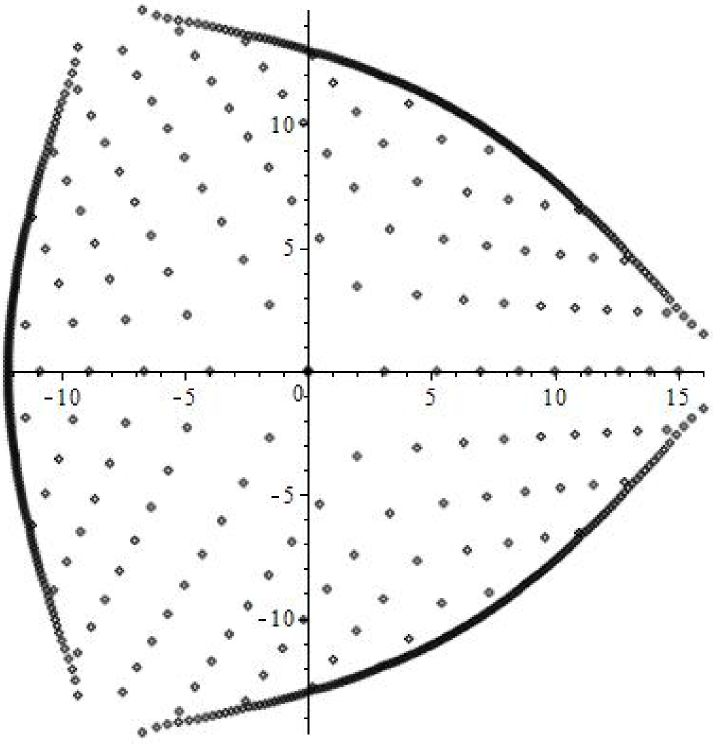

Figure 1: Approximation to the poles of the symmetrical solution of (37).

where $\tilde{\tau}$ has the Taylor expansion (25) around $z = 0$. By integrating both sides of (34) and exponentiating, we see that

$$\tau(z + \Omega) = A \exp\left(Bz + \frac{1}{2}\tilde{\mu}z^2\right)\tilde{\tau}(z),$$

for some $A \neq 0$. By performing a Taylor expansion on each side of the above relation up to terms of order z^3 , we obtain the expressions for A and B as in (33), as well as the equation

$$\tilde{\mu} = \frac{\tau'''(\Omega)}{3\tau'(\Omega)} - \frac{\tau''(\Omega)^2}{4\tau'(\Omega)^2}.$$

The latter formula can be seen to be consistent with the previous expression for $\tilde{\mu}$ by setting $z = \Omega$ in (23). Hence $\tilde{\tau}(z)$ satisfies the same equation (23) but with parameters $\tilde{\kappa}, \tilde{\lambda}, \tilde{g}_2, \tilde{g}_3$ found from the equation corresponding equation (11) for $\tilde{\Sigma}(z)$. □

Remark 3.6. The expression (32) is a generalization of the classical formula for transformation of the Weierstrass sigma function under shifting by a period (see e.g. §20.421 in [19]).

4 Numerical example: poles in P_{XXXIV}

The numerical evaluation of Painlevé transcendents and the structure of their pole fields is a very active research area (see e.g. [4]-[6] and references therein). The recursion relation in Theorem 3.2 is extremely convenient for computing numerical approximations to the tau-function close to the origin, by truncating the Taylor series for $\tau(z)$. The roots of the polynomials obtained by truncation provide approximations to the zeros of the tau-function lying near to $z = 0$, or equivalently the positions of the poles of the sigma equation.

As an example, we consider the case of parameters

$$\eta = \lambda = g_2 = g_3 = 0, \qquad \kappa = 1, \tag{35}$$

for which the function $\tau(z) = \tau(z; 0, 1, 0, 0, 0)$ is such that

$$v(z) = \frac{d^2}{dz^2} \log \tau(z) \tag{36}$$

satisfies the P_{XXXIV} equation in the canonical form

$$vv'' - \frac{1}{2}(v')^2 + 2v^3 + zv^2 = 0, \tag{37}$$

with parameter $\ell = 0$, while from (17) we have that

$$u = \frac{1}{2}\frac{d}{dz} \log v$$

satisfies P_{II} in the form

$$u'' = 2u^3 + zu - \frac{1}{2}. \tag{38}$$

For the parameter values (35), the equation (11) admits a trivial solution $\Sigma = \text{const}$, giving $v = 0$, but we have neglected such solutions here, by considering the generic situation where Σ has poles (cf. Lemma 2.1). (In fact, setting $v = 0$ in (19) yields a Riccati equation, corresponding to the special case that (38) is solved in Airy functions.) The tau-function $\tau(z) = \tau(z; 0, 1, 0, 0, 0)$ given by the Taylor series defined in Theorem 3.2 is such that the expansion (31) takes the special form

$$\tau(z) = \sum_{j=0}^{\infty} \frac{\hat{A}_j\, z^{3j+1}}{(3j+1)!}, \qquad \hat{A}_j = A_{j,0,0,0,0}, \tag{39}$$

which is invariant under the order 3 symmetry

$$z \to \omega z, \qquad \tau \to \omega^{-1} z, \qquad \omega = \exp(2\pi \mathrm{i}/3). \tag{40}$$

Hence the zeros of τ have the same symmetry: if $\Omega \neq 0$ is a zero of τ, then so are $\omega\Omega$ and $\omega^2\Omega$. These zeros of τ are the simple poles of Σ, and the double poles of v, i.e. the particular solution of (37) defined by (36). Similarly, the associated function u that satisfies the case (38) of P_{II} has simple poles with residue -1 at these same positions, as well as simple poles with residue $+1$ at the places where v vanishes, all of them symmetrically placed on triangles centred at 0.

For illustration, in Figure 1 we have plotted the approximate positions of some of the zeros of τ (or the equivalently the poles of Σ and v). To begin with, the first

201 non-zero terms of the series (39) were found. The first few coefficients have prime factorizations

$$\hat{A}_0 = 1, \hat{A}_1 = -1, \hat{A}_2 = 1, \hat{A}_3 = 17, \hat{A}_4 = -557, \hat{A}_5 = 59 \cdot 349, \hat{A}_6 = -1017719,$$
$$\hat{A}_7 = 5 \cdot 7^2 \cdot 59 \cdot 4391, \hat{A}_8 = -5 \cdot 13 \cdot 131 \cdot 550439, \hat{A}_9 = 5^2 \cdot 7 \cdot 2224640081,$$
$$\hat{A}_{10} = -5^2 \cdot 570919 \cdot 2406689, \hat{A}_{11} = 5^2 \cdot 41 \cdot 61 \cdot 46043405509, \ldots$$

and it appears to be the case that

$$\hat{A}_j = 0 \bmod 5 \quad \forall j \geq 7, \qquad \hat{A}_j = 0 \bmod 7 \quad \forall j \geq 14.$$

To prove either of these two statements seems not to be simple. However from the bilinear equation (27) it follows that the coefficients $\hat{A}_j$ are determined by the quadratic recurrence:

$$\begin{aligned}(n+1)\hat{A}_{n+3} = \; & \tfrac{9n^4+18n^3-85n^2-246n-144}{8} A_{n+2} + \\ & -\tfrac{1}{3}\textstyle\sum_{j=0}^{n} \tfrac{(3n+7)!}{(3n-3j+4)!(3j+4)!} a_{3n-\frac{3}{2}j+6,\frac{9}{2}j+6} \hat{A}_{n-j+1}\hat{A}_{j+1} \\ & +\tfrac{1}{6}\textstyle\sum_{j=1}^{n} \tfrac{(3n+7)!}{(3n-3j+7)!(3j+4)!} b_{3(n-j)+7,3j+4} \hat{A}_{n-j+2}\hat{A}_{j+1},\end{aligned}$$

subject to the initial conditions $\hat{A}_0 = 1$, $\hat{A}_1 = -1$ and $\hat{A}_2 = 1$. The values of $a_{j,k}$ and $b_{j,k}$ are defined by the action of the Hirota operators D^2 and D^4 on polynomials (see before Theorem 3.2).
Given these coefficients, we took the polynomial

$$\mathcal{P}_{601}(z) = \sum_{j=0}^{200} \frac{\hat{A}_j \, z^{3j+1}}{(3j+1)!},$$

and calculated its roots numerically in order to produce the figure. By comparing the values of the roots with those of the successive approximations $\mathcal{P}_{20}$, $\mathcal{P}_{40}$, $\mathcal{P}_{60}$, $\mathcal{P}_{80}$, etc. we were able to confirm that the values of the zeros closest to 0 were converging to a high degree of accuracy. For instance, the non-zero roots closest to the origin lie at $\Omega_1, \omega\Omega_1, \omega^2\Omega_1$, and the next closest roots are at $\Omega_2, \omega\Omega_2, \omega^2\Omega_2$, where

$$\Omega_1 \approx 3.10938452954168950042, \quad \Omega_2 \approx -3.97992802289816587870,$$

to 20 decimal places. The largest roots of the polynomial, which can be seen to coalesce on the boundary of the figure, are numerical artefacts; they do not provide good approximations to the zeros of the tau-function.

Remark 4.1. Due to the homogeneity of the parameters κ, g_3, all of the tau-functions $\tau(z; 0, \kappa, 0, 0, g_3)$ admit the symmetry (40).

Remark 4.2. Non-polynomial rational solutions of the sigma equation are also included in the formulation of Lemma 2.1. For example, for the parameter values

$$\eta = \lambda = g_2 = 0, \qquad \kappa = 1, \qquad g_3 = -\frac{9}{16},$$

the equation (11) has the rational solution

$$\Sigma(z) = \frac{1}{z} - \frac{z^2}{8},$$

corresponding to the tau-function

$$\tau(z; 0, 1, 0, 0, -9/16) = z \exp(-z^3/24),$$

which illustrates the symmetry (40) explicitly. This corresponds to

$$v = -\frac{2}{z^2} - \frac{z}{2}, \qquad u = -\frac{1}{z}$$

which are rational solutions of the P_{XXXIV} equation (16) and the P_{II} equation (18), respectively, with $\kappa = 1$, $\mu = 0$, $\ell = 3/2$.

5 Conclusions

Our analysis shows that the "shifted" sigma equation for P_{IV}, given by (11), is a fundamental object which contains not only the general solution of P_{IV}, but also that of P_{II}, P_I, and the Weierstrass $\wp$ function. Although the connection between Painlevé transcendents and elliptic functions has a long history at the level of asymptotic expansions [1], the results presented here show that from the viewpoint of the sigma function the Painlevé transcendents are multi-parameter extensions of elliptic functions. Furthermore, although there is a coalescence cascade $P_{IV} \to P_{II} \to P_I$, this requires taking asymptotic limits of both the dependent and independent variables [12], whereas at the level of the solution of (11) one has $P_{IV} \supset P_{II} \supset P_I$, with the inclusion denoting that a parameter has been set to zero. It would be interesting to see if this approach can be extended to the sigma function of P_{VI}, in which case all the other Painlevé equations would be included as special cases.

In future work we propose to consider Mittag-Leffler expansions of the solutions of (11), and the asymptotic behaviour of the coefficients in Laurent expansions for the solutions of the sigma equation, as well as the corresponding Painlevé equations. It would also be good to obtain precise a priori bounds on the growth of the polynomials P_n appearing in Theorem 3.2, as this would yield an independent proof that the tau-function is holomorphic (hence providing yet another proof of the Painlevé property for P_I, P_{II} and P_{IV}; cf. [17]). The arithmetic properties of the coefficients $A_{j,k,l,m,n}$ in (31) are also worthy of further study.

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