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What is in a Definition? Understanding Frege's Account

Edward Kanterian

Abstract

Joan Weiner (2007) has argued that Frege's definitions of numbers are linguistic stipulations, with no content-preserving or ontological point: they don't capture any determinate content of numerals, as these have none, and don't present numbers as pre-existing objects. I show that this view is based on exegetical and systematic errors. First, I demonstrate that Weiner misrepresents the Fregean notions of '*Foundations*-content', sense, reference, and truth. I then consider the role of definitions, demonstrating that they cannot be mere linguistic stipulations, since they have a content-preserving, ontological point, and a decompositional aspect; Frege's project of logical analysis and systematisation makes no sense without definitions so understood. The pivotal ontological role of elucidations is also explained. Next, three aspects of definition are distinguished, the informal versus the formal aspect, and the aspect of definition achieved through the entire process of systematisation, which encompasses the previous two and is little discussed in the literature. It is suggested that these insights can contribute to resolving some of the puzzles concerning the tension between the epistemological aim of logicism and Frege's presentation of definitions as arbitrary conventions. Finally, I stress the interdependence between the epistemological and ontological aspects of Frege's project of defining number.

1 Preliminary

Frege's logicism, supported by his logical calculus, had an indisputable epistemological aim: to prove that the truths of arithmetic are *a priori* and analytic, and thus deducible from the laws of logic. More problematic, and much debated, is

the status of his definitions.¹ Frege dismisses previous attempts at the definition of number, introducing new definitions. In addition, he sometimes describes definitions as arbitrary conventions. So Frege does not seem to want to capture the pre-existing meaning of arithmetical symbols, to tell us what numbers are and always have been, but to stipulate new arithmetical symbols, with new meanings. But how can this revisionism be reconciled with the epistemological aim? How can arbitrary conventions prove the logical status of the truths of arithmetic, i.e. of truths established prior to and independently of those conventions?

Joan Weiner (2007) has offered a new solution to this puzzle: Frege was more thoroughly revisionist than we tend to believe. His revisionism affected not only his conception of definition, but also of sense, reference and truth. Prior to his concept-script systematisation of arithmetic, numerals had no determinate sense and reference, and arithmetical statements were not strictly true. According to Weiner, this systematisation was not intended to unveil the true nature of numbers and the true referents of numerals, but to introduce stricter semantic and inferential constraints stipulating the sense and reference of numerals and of arithmetical statements within a rigorous scientific system. Systematisation was thus for Frege meant as a normative linguistic precisification to reach the epistemological aim, with no claim to semantical and ontological *discoveries* about pre-systematic arithmetic. Hence, no dilemma arises.

Weiner's solution is among the most intriguing and thought provoking offered in recent years. Nevertheless, it is unwarranted, involving exegetical errors and saddling Frege's theory with insuperable substantive difficulties. I give an overview over her argument in section 2, and show, in sections 3-5, that she misrepresents the notions of (what she calls) '*Foundations-content*', and of sense, reference and truth in Frege. Then I focus on Frege's definitions, demonstrating that they have, *pace* Weiner, not only a content-preserving, ontological point, but also a decompositional aspect, which shows that they are not mere stipulations (6-9). The important ontological role of elucidations is explained in this context (10-11), and a three-fold distinction between levels at which Fregean definitions operate emerges, which will help clarify the more general problem of Frege's puzzling remarks about definitions (12-14). Finally, I stress both the epistemological and the ontological aspects of Frege's project, and their interdependence (15).²

1. Cf. Benacerraf 1981, Weiner 1984, Picardi 1988, Dummett 1991c, Tappenden 1995, Kemp 1996, Shieh 2008.

2. The current paper is a much enlarged and improved successor to Kanterian 2010.

2 Weiner's argument

Weiner offers a wealth of substantive and exegetical considerations in favour of her view, focusing most explicitly on the role of definitions within Frege's work, especially in the *Foundations of Arithmetic*. She investigates the requirements a definition (of number, numerals etc.) must satisfy to qualify as adequate or faithful. An obvious requirement seems to be the following:

The Obvious Requirement: A definition of an expression must pick out the object to which the expression already refers or applies (680).³

Weiner rejects this requirement. Of course, she argues, definitions must be conservative in some sense, i.e. faithful to pre-systematic arithmetic in such a way that they don't transform it into some 'new and foreign science' (687). But this does not entail preservation of reference. She explains: 'Faithful definitions must be definitions on which those sentences that we take to express truths of arithmetic come out true and on which those series of sentences that we take to express correct inferences turn out to be enthymematic versions of gapless proofs in the logical system' (690). Systematisation thus preserves truth-related and inference-related content, what Weiner calls *Foundations-content*. For instance, a definition of '0' and '1' is unacceptable, if it presents as true in the system what is previously taken to be false, e.g. '0=1'. Thus faithful definitions must cover for what are taken to be the well known properties of numbers.⁴ Equally, the definition must preserve an inference previously taken to be valid, including in applied arithmetic, for example 'If Venus has zero moons and Earth one, then given that $0 < 1$, the Earth has more moons than Venus' (686).

Since *Foundations-content* is 'some sort of content connected with inferences' (692), it relates to *judgeable content* ('beurteilbarer Inhalt') – a notion developed by Frege in *Begriffsschrift* and designating that which has only 'significance for the inferential sequence' (Frege 1879: 6). However, according to Weiner, *Foundations-content* also anticipates the later notion of sense, *Sinn* (689ff.). First, the judgeable content of a term is not its reference (690), just as much as sense is not reference. Second, a term hitherto considered non-empty will not lose its *Foundations-content* if we discover it is empty, for the discovery will lead to a re-evaluation of pre-systematic beliefs and inferences, a re-evaluation still involving the term itself (690). Moreover, a fictional term like 'Hamlet' has *Foundations-content*, since some speakers think it enables them to express truths and correct inferences (691). Hence, it is not required for a term to have a referent in order to

3. Unless otherwise specified, all page references are to Weiner 2007.

4. Weiner 2007: 688. Cf. Frege 1879: §70.

have *Foundations*-content, and this brings *Foundations*-content in the vicinity of Sinn.

Thus, concept-script systematisation has a conservative side insofar as it preserves *Foundations*-content. But it also has a creative side. For ‘preservation’ should not be understood in the sense that it there is something outside of the system identical with something in the system. Systematisation involves the process of *proving in the logical system* as true pre-systematic sentences taken to express truths and as correct pre-system inferences taken to be correct. But proving ‘in the logical system’ is a highly normative process, guided essentially by *two precision requirements* that distinguish sharply system from pre-system: the gapless proof requirement, i.e. all proofs must be absolutely gapless, and the sharpness requirement, i.e. genuine concepts must have sharp boundaries.⁵ These constraints do not apply to pre-systematic language; they are the sharp dividing line between pre-system and system. ‘Frege’s task is to replace imprecise pre-systematic sentences with precise systematic sentences’ (710). ‘Preservation’ means therefore something like ‘constraint-guided transformation’. It aims at the creative precisification of indeterminate and vacillating pre-systematic language (697), as Frege apparently suggests about expressions quite generally (Frege 1906a: 302–3). Precisification is of course a semantic process. Hence, Frege intends to reach his epistemological aim by semantics.

Weiner’s view has intriguing corollaries, summarised as follows. (More details are presented later.)

(a) *Foundations*-content and sense/*Sinn* are related, but not identical. To have a determinate sense, an expression needs a definition satisfying the precision requirements. But pre-systematic expressions have no such definition, hence they have no sense and, by extension, no reference.⁶ Sense and reference (*Sinn* and *Bedeutung*) are only system-internal features of expressions.⁷ There is no contrary evidence in Frege, Weiner claims. He never says that pre-systematic terms have a reference or require one to have a use (706f.).⁸ This is only apparently absurd: fixing the sense and reference of a term is only the ideal end of a science, once it culminates in a system (709f.).⁹

5. See Weiner 2007: 701, also Frege 1884: §62, §74, 1903: §56.

6. The alternative of an indeterminate sense (and reference) is excluded for Frege: there is no such thing. See e.g. 1903: §56.

7. A similar claim has been advanced before, if for different reasons. See e.g. Stekeler-Weithofer 1986: 280.

8. Weiner’s argument comes close to a Wittgensteinian theory of meaning as use, although Wittgenstein is not mentioned.

9. See also Frege 1914: 242.

(b) If pre-systematic terms have no determinate reference, then given compositionality, pre-systematic sentences have no determinate reference, no truth-value. We only 'take them to express truths' (690f.). This does not mean there is nothing 'right' about them (710), only that their rightness does not satisfy systematisation-constraints. We must distinguish, like Frege, between *pre-systematic truth* and *strict truth* (709f.).¹⁰ Preserving pre-systematic truth is one of the aforementioned faithfulness requirements a definition must satisfy.

(c) Since pre-systematic arithmetical expressions have no determinate reference, ordinary arithmetical predicates like 'is a number' have no reference either. Hence, the concept of number is not already fixed prior to Frege's definitions (696). On the contrary, *Foundations* (§100) stresses the arbitrariness of definitions (695ff.), as does the important posthumous text "Logic in Mathematics" (1914), where a determination of sense is said to be either decompositional, and thus a self-evident axiom, or stipulative (a 'constructive definition'). Since Frege's *Foundations*-definitions are not self-evident (1884: §69), they must be stipulations, precisifying *Foundations*-content and thus transforming arithmetic into a scientific system.

(d) Some think that Frege was committed to the *ontological thesis* that numbers are pre-existing, language-independent objects, whose nature his definitions capture. Weiner denies this.¹¹ Frege makes no claim that numbers existed before his definitions, or else they would be discoveries about pre-existing objects. But they are linguistic stipulations, not ontological discoveries. Frege is not interested in claiming that numbers are really extensions (1884: §107), but only considers the linguistic question 'Are the assertions we make about extensions assertions we can make about numbers?', and answers it by means of a linguistic principle par excellence, the context principle (Weiner 2007: 698f.). As Frege writes: 'I attach no decisive importance to bringing the extensions of concepts into the matter at all' (1884: §107).

3 'Foundations-content' ?

My discussion can start with the notion of *Foundations*-content, on which Weiner bases her rejection of the Obvious Requirement. There is, in fact, no decisive evidence for a notion of content in the *Foundations* closely related, although not

10. 'Pre-systematic truth' is not Weiner's term, but my correlate to her phrase 'regarding a sentence as true' (ibid. 706, 709).

11. Weiner defends this anti-ontological stance in previous work, e.g. Weiner 1990: ch. 5.

identical to his later notion of *Sinn*. Frege uses the term ‘content’ loosely in *Foundations*. It variously means ‘sense of a sentence’, i.e. a judgement or thought (1884: x, §3, §70, §106), ‘sense of a recognition judgement’ (§106, §109), ‘judgeable content’ (§62, §74), ‘reference’ (§74fn.), but can also refer to conventional meaning and use (§60). ‘Content’ in the *Foundations* is not a technical term. Frege intended the book as a prolegomenon, an accessible introduction to his logicism. There is also no claim in the book that an expression could have a content without a referent, e.g. that ‘Hamlet’ has *Foundations*-content. Frege actually claims the opposite: ‘the largest proper fraction’ has no content or sense, because no object falls under it (§74). He makes a similar point about ‘the square root of -1’ (§97). Moreover, the philosophically significant role of ‘content’ in the *Foundations* lies in Frege’s repeated claims that we need to specify a content of arithmetical judgements so that they come to express identities, and thus to secure the objecthood of numbers, given that identity is an essential mark of objecthood (§62f.).¹² His aim to specify the content of equations, understood to be identities, as a step in proving the logicist thesis that numbers are objects, would make no sense, if sentential content (of equations/identities) was not essentially linked with numerals have a reference (namely numbers). Hence, there are strong reasons to assume that content in the *Foundations* is quite close to *Bedeutung*.

But is *Foundations*-content not related to *Begriffsschrift*-content and insofar only inferential, not referential? But there is no dichotomy here. Judgeable content is so intimately tied to reference that this affects the basic formation rules of the notation in *Begriffsschrift*. Frege stipulates that any expression following the content stroke must have a judgeable content. And the relation between that expression and its judgeable content is *designation*, which is visible from two things: (a) the expression of a judgeable content is a designator starting with the definite article, paradigmatically a nominalised proposition (‘the circumstance that there are houses’, ‘the murder of Archimedes at the capture of Syracuse’; Frege 1879: §2-3), and (b) the expression of a judgeable content can flank the sameness of content sign, i.e. the identity sign.¹³ Hence, the expression of a judgeable content is a name of the judgeable content and no formula in concept-script is syntactic if such a name fails to designate anything.

Weiner argues that since ‘Phosphorus = Phosphorus’ is derivable from the law of identity, while ‘Hesperus = Phosphorus’ is not, the two names cannot have the

12. See also Rumfitt 2003: 198.

13. Frege refers to the expression of a judgeable content flanking the identity sign explicitly as a name (Frege 1879: §8, *passim*). His proviso that in ‘ $A \equiv B$ ’ the names stand for themselves because an identity statement has metalinguistic content does not refute my point, for really they stand for themselves *as names of the same content*.

same *Begriffsschrift*-content (Weiner 2007: 690). But this example is uncongenial to Frege's concerns: 'Phosphorus' and 'Hesperus' are names of *unjudgeable* content, while in *Begriffsschrift* only names of judgeable content interest him, of assertible content. Thus we cannot distinguish between what a name of a judgeable content refers to and its inferential content. Therefore, once we establish (by a synthetic judgement) that two names name the same judgeable content, 'A' and 'B' are intersubstitutable for purposes of inference (' \equiv ' functions as an identity sign, but also as a stipulative sign; Frege 1879: §8, §23), as much as, vacuously, 'A' and 'A'. The difference between two names of the same judgeable content is neither a referential nor an inferential difference, only of a different mode of determining the same inferential content, as Frege states (§8). Claiming that *Foundations*-content is connected to *Begriffsschrift* content, while revising Frege's official view of the latter, is problematic.

4 Sense and reference

It is equally unwarranted to treat sense and reference (*Sinn* and *Bedeutung*) as system-internal features inapplicable to ordinary, pre-systematic expressions. To the contrary, Frege stresses that 'The moon is the reference of "the moon"' (1919: 255) and that 'If we claim that the sentence "Etna is higher than Vesuvius" is true, then the two proper names do not just have a sense [...], but also a reference: the real, external things that are designated'¹⁴. Again: 'If we say "Jupiter is larger than Mars", what are we talking about? About the heavenly bodies themselves, the references [*Bedeutungen*] of the proper names "Jupiter" and "Mars"' (1906b: 193). In *Grundgesetze* he notes that the notion of reference cannot be genuinely explained, since any explanation would presuppose knowledge that some terms have a reference (Frege 1893: §30); hence, reference predates any system. Since having a reference implies having a sense, ordinary expressions with a reference also have a sense. Neither feature is exclusively system-internal. Hence, even if we described Frege's definitional project as preserving '*Foundations*-content', this would still involve preserving pre-systematic content that has a referential component, and thus content not unlike that described in *Grundgesetze* as split into sense and reference (cf. Frege 1893: x). In addition, there is evidence in the *Foundations* that not only systematic expressions have a determinate content. For Frege writes that ' $1000^{(1000^{1000})}$ ', has 'a perfectly definite sense', which he connects to the fact that $1000^{(1000^{1000})}$ is an object with recognisable properties (1884: §104). This is indicative of Frege's ontological agenda, to which I return below.

14. Carnap 2004: 150.

5 Kinds of truth?

Weiner claims that Frege admits many notions of truth, including pre-systematic and strict truth. But Frege makes no such claim. He argues, in “Thoughts”, that truth is undefinable, because any definition must analyse truth into constituent elements (of the truth bearer), which must then be *true of* a particular truth-bearer if the definition is to be applicable (1918a: 60). This circularity suggests that truth is not only undefinable, but also simple, and thus univocal. Frege advances other points incompatible with Weiner’s views. He distinguishes between ‘holding a thought to be true’ (*Fürwahrhalten*) and ‘proving the true’ (*Beweis des Wahren*). The former arises from psychological laws, the latter from the laws of truth, the object of logic, not psychology (*ibid.* 58f.; see also 1893: xv). Since Weiner characterises pre-systematic truth in terms strongly resembling *Fürwahrhalten*, e.g. ‘what we take to express truth’ or ‘what we regard as true’, it would follow, absurdly, that Frege assigns truths of pre-systematic arithmetic to the psychological, and subsumes pre-systematic arithmetic under psychology. Equally, if pre-systematic truth is vacillating and vague, as Weiner holds, it would be a predicate coming in degrees. But truth does not allow for ‘more or less’ (1918a: 60). Also, Frege speaks repeatedly about truths of arithmetic or mathematics as such (e.g. 1884: §3, §11, §14, §17, §109), without qualifying them as merely pre-systematic, as opposed to systematic truths.

More generally, the truth of a thought is timeless for Frege (cf. 1884: §77, 1918a: 74). Hence, either a pre-systematic truth is also timeless, which hardly squares with its indeterminacy or vagueness, or a pre-systematic statement expresses no thought at all. Weiner may affirm just this: a pre-systematic statement only expresses *Foundations*-content, not a thought. But then *Foundations*-content is expressible, i.e. assertible, thus must be also thinkable, judgeable, negatable etc. However, Weiner’s *Foundations*-content can’t be really judgeable, since judging, for Frege, is ‘acknowledging the truth of a thought’ (1918a: 62). Judgeable is only that to which strict truth may apply. But then *Foundations*-content is not assertible either, for to assert is to make manifest a judgement (*ibid.*). And if it is not judgeable, *Foundations*-content lacks the essential association with judgeable content Weiner claims it has, and thus has no inferential character either, for, as Frege explains, to infer is to judge (1879/1891: 3). Pre-systematic arithmetical propositions would not be expressible, assertible, thinkable, judgeable, negatable, or occur in inferences. The distinction between pre-systematic and strict truth has catastrophic consequences.¹⁵ This is surely one of the reasons why Frege asserts,

15. For related criticism, see Kemp 1996: 178f. Incidentally, how should we take Weiner’s own arguments, formulated, as they are, not in concept-script, but in pre-systematic prose? Are they

late in his life: 'I cannot recognize different meanings of the word "true"' (1917: 20).

6 Objectivity of definitions: the material mode

Doubts about Weiner's interpretation also arise from what Frege says about, and does with, his definitions in the *Foundations*. Some passages stress the stipulative character of definitions, using expressions such as 'to fix' ('*festsetzen*') and 'stipulations' ('*Festsetzungen*') (e.g. 1884: §7, §65, §67f., §75, §104, §109). But his phrasing indicates an objective side of definition too, since he speaks of the need to ascertain or find out ('*feststellen*'), to explain the sense of an equation (1884: x, §62, §106), which Austin inaccurately translated as 'to fix/to define the sense', and to find or attend to ('*aufsuchen*') a judgeable content which can be transformed into an identity whose sides contain the new numbers (1884: §104).

This objectivity is also manifest in Frege's tendency to adopt the material mode and define *objects*, not mere expressions (1884: §7-10, §18, §67). Twice he speaks explicitly of 'the definition of an object' (1884: §67, §74). Weiner believes to find support for her thesis that Fregean definitions are arbitrary linguistic stipulations in §100 of *Foundations*, where Frege apparently denies that the meaning of 'the square root of -1' is fixed before our definitions. But here is the full context:

'We should be equally entitled to choose as further square roots of -1 a certain quantum of electricity, a certain surface area, and so on; but then we should naturally have to use *different symbols* to signify these *different roots*. That we are able, *apparently*, to create in this way as many square roots of -1 as we please, is not so astonishing when we reflect that the meaning of the square root of -1 is not something which was already unalterably fixed before we made these choices, but is decided for the first time by and along with them' (1884: §100; my italics).

Clearly, Frege discusses the putative definition of an object. This is why different *symbols* are supposed to be assigned to each square root of -1, *once* each square root of -1 has been chosen, i.e. defined. So the definitional choice here is not linguistic. Moreover, Frege's tone is ironical, as in the embedding discussion.¹⁶ He is not

valid? Are her conclusions strictly true? Both a 'yes' and a 'no' invalidate her theory.

16. In the same context he remarks: 'Let the Moon multiplied by itself be -1. This gives us a square root of -1 in the shape of the Moon' (1884: §100). This was hardly written with a straight face. See also his attack on Kossack (1884: §103): 'We are given no answer at all to the question,

saying that creative definition entitles us to introduce indefinitely many square roots of -1 , but is presenting his opponent's view ('apparently').¹⁷ The opponent is a reformed formalist, who has accepted Frege's previous anti-formalist arguments, conceding that signs alone will not bring complex numbers into existence. The opponent now tries to supplement his definition of a complex number with the assignment of a random object, to fix the meaning of the sign for a complex number, but Frege's reply is to spell out the absurd consequence: there can be any number of such assignments.¹⁸ Weiner's misunderstanding of this passage is twofold: Frege is not propounding his *own* view of definition of a linguistic *symbol* as arbitrary stipulation, but his *opponent's* view of definition of an *object* as a random ontic assignment. There is no evidence in *Foundations* that Frege takes definitions to be arbitrary, creative definitions.¹⁹

Weiner similarly misunderstands an eligibility condition of primitive truths. In "On Formal Theories of Arithmetic" Frege argues that every definition comes to an end, reaching undefinable primitives, the original building blocks of science (*Urbausteine*), expressed in axioms (1885: 96). Weiner takes this to be a semantic point: the eligibility condition 'is that the *expression* of the primitive truth should include only simple, undefinable *expressions*. For these simples are the ultimate building blocks of the discipline' (Weiner 2007: 682, my italics). But this is not Frege's point. Frege adopts the material, not the formal mode, concerned with defining the objects themselves, here geometrical objects: 'It will not be possible *to define an angle* without presupposing knowledge about the straight line. Of course, what a definition is based on might itself have been defined previously' (1885: 104, my italics). The primitives terminating a chain of definitions are not expressions, but undefinable objects 'whose properties are expressed in the axioms'.²⁰ It is also in the material mode that Frege proceeds to explain that the building blocks of arithmetic must be purely logical (to account for the universality of arithmetic), and that the terminological replacement of 'set' with 'concept' is not mere renaming, but important for the actual state of affairs (1885: 104f.). Similarly, in his polemic against Husserl, Frege insists that concerning the business of definition the mathematicians, unlike the psychologistic logicians, only care

what does $1 + i$ really mean? Is it the idea of an apple and a pear, or the idea of toothache and gout? Not both at once, at any rate, because then $1 + i$ would not be always identical with $1 + i$.'

17. Dummett also misunderstands this passage. See Dummett 1991a: 178f., 1991c: 34f., 40.

18. A few pages later Frege argues that the formalist origin of this reasoning leads to misconstruing the subject matter of arithmetic as synthetic and even as synthetic *a posteriori* (see Frege 1884: §103).

19. An arbitrary definition seems to be given of 'gleichzahlig' ('equinumerous'; 1884: §67), but Frege explains that the arbitrariness only concerns the symbol, not its meaning.

20. We can call the expressions designating such primitives also 'primitives', but only metonymically.

about the things themselves, the reference (*Bedeutung*) of expressions, in particular the extensions of concepts (1894: 319f.).

7 Objectivity of definitions: Platonic realism

Another challenge to Weiner is Frege's Platonism, which concerns arithmetical truth, arithmetical and logical objects, and appears in the importance existence proofs have for his definitions. Frege explains the definition of 1 by reference to the sempiternality and apriority of the truth of the propositions it helps to derive, e.g. '1 is the immediate successor of 0'; no physical occurrence, not even the 'subjective' constitution of our brains, could affect this truth (1884: §77). Linguistic precisification cannot explain this sempiternality; the truth-value of 'Tom is alive' still depends on contingent, empirical facts, however sharply we cut the boundaries of the embedded expressions.²¹ What explains sempiternality is the objectivity of the content of arithmetical propositions, the nature of arithmetical objects. This nature is grounded in the nature of logical objects, given that number-statements are ultimately about relations between logical objects (correlations between extensions of second-order concepts). Far from explaining logical objects by linguistic constraints and stipulations, he awards them ontological objectivity: they are simply there, ready to be discovered by us, independent of such 'external aids [...] as language, numerals, etc.' (1884: iii). Thus judgeable contents, the paradigmatic logical objects of the early work, are as objective as any mind- and language-independent object, e.g. the Sun (1879/1891: 7).²² Definitions could play no creative role at this ontic level, and don't play it in concept-script, where a definition is merely an abbreviation, stipulating that a simple sign have the same judgeable content as a complex sign (1879: §24). The existence of judgeable contents and their names precedes any definition in concept-script.

Frege claims the same kind of objectivity for numbers, i.e. independence of the mind, language, stipulations. Corresponding passages are legion (e.g. §26, §60, 62), and it is hard to see how they are to be read as merely specifying precision requirements imposed by system construction. Frege compares the objectivity of number with that of the North Sea, noting the independence of both from arbitrary

21. Frege's eternalist theory of truth, defended elsewhere, does not concern us here, since that theory concerns not the justification of the truth-value of a proposition, but the proper truth-bearer.

22. Elsewhere he says something similar of the relation between an object and its concept: 'To bring an object under a concept is merely to recognise a relation that already existed beforehand' (1894: 317). For similar remarks about the objectivity of thoughts see 1893, xvi, 1918a: 76f., 1918b: 151.

linguistic stipulations (§26).²³ Altering the meaning of ‘North Sea’ today would not falsify whatever true content (thought) has been expressed until now by ‘The North Sea is 290.000 square miles in area’. The North Sea is there, objective, ready to be discovered. Presumably, then, there are correct and incorrect definitions of the North Sea, if the North Sea is independent of the definition of ‘the North Sea’: a correct definition will pick out precisely the North Sea. *Pace* Weiner, this suggests that there is an element of discovery in at least some definitions: correct definitions pick out a pre-existing object in a precise and determinate manner. This is entirely compatible with Frege’s claims about numbers, namely that we discover them in the concepts (1884: §48, §58), and that mathematicians are like geographers: neither can create ‘things at will; [both] can only discover what is there and give it a name’ (§96).²⁴ It is unclear how Weiner’s interpretation can accommodate such passages. They are not mentioned in her article.

Frege’s rejection of empiricism in logic and mathematics shows that the objectivity he claims for logical and arithmetical entities is stronger than that of physical entities. Numbers are abstract objects, ready to be recognised by us, but without physical properties, including spatiality and temporality. Our recognition of abstracta is situated in time, but abstracta are not, as he explains using the example of the equator, which was not created, in the sense that nothing positive could be said about it before its alleged creation (1884: §26). Actually, the equator is a dependent abstractum: before the creation of Earth there was indeed nothing positive to say about the equator. But the point is valid for self-subsistent abstracta like numbers: there is something positive to say about them at all times (with appropriate linguistic usage in place). Hence temporalising the truth of statements concerning numbers (e.g. ‘“Numbers are extensions” is not a true statement before Frege formulated definitions in 1879’), as Weiner’s argument entails, is objectionable and not in line with Frege’s views.

Frege’s Platonism is manifest in another respect. In the *Foundations* he does not merely provide a definition of, say, 0, and content himself with its satisfying the sharpness requirement. Instead, he gives various *existence proofs*, e.g. that if 0 is the number of an empty concept F and an empty concept G , then there is a relation φ bringing F and G into one-one correlation (1884: §75), or that there is something which immediately succeeds 0 (§77). Equally, he justifies his diagnosis that the formalists have only introduced empty signs, not new number-words, by their failure to prove the existence of the new numbers (§92ff.). The importance of

23. Note, and ignore, that Frege’s discussion of the arbitrariness of stipulations is confusingly embedded in a discussion against psychologism.

24. Frege also compares the mathematician with the botanist who determines something objective when he determines the number and colour of a plant’s petals. See also 1893: xiii.

existence proofs emerges in his sharp contrast between defining a concept by the properties objects must have to fall under it and proving that something falls under it (1884: §74, 1893: xiv). Thus definitions, on his account, cannot bring objects into existence, only specify concepts hitherto lacking a designator. It is for this reason that he argues at length in *Grundgesetze* precisely against the possibility of creative definitions (1903: §141f.).

8 Numbers as extensions

Frege clearly has an ontological aim, to discover what is already there. Weiner claims that there is no argument in the *Foundations* that numbers are really extensions, citing §69 and §107. In §69, she suggests, Frege avoids the ontological question ‘Are numbers extensions?’, and asks a linguistic question instead, motivated by the context principle: ‘Are the assertions we make about extensions assertions we can make about numbers?’ But Frege does not believe that the linguistic question excludes the ontological question. His interest in language is actually only ontological: ‘There is no intention of saying anything about the symbols; no one wants to know anything about them, except insofar as some property of theirs directly mirrors some property in what they symbolise’ (1884: §24). The context principle, in the *Foundations*, is not deployed as an anti-ontological tool, but against the psychologistic prejudice that an expression designates an entity only when, in isolation, we associate a mental idea with it (§59). The principle actually serves the ontological agenda: numbers are self-subsistent entities because their expressions can be construed as singular terms, not in isolation, but at least in the sentential contexts of their paradigmatic arithmetical use, which is ontologically significant (equations understood as identities).

Nor does Frege’s assertion that he does not attach any decisive importance to bringing the extension of a concept into the matter (1884: §107) count against his ontologism. He introduces extensions in his ‘explicit’ definition of number. This definition responds to the ‘Julius Caesar’ problem: all recognition statements of the form ‘ $N_t F(t) = a$ ’ must have a sense, i.e. we must decide for any object a whether the statement is true or not (§§66-8, §107).²⁵ But of course, recognition statements are identity statements, and their truth is a criterion for the objecthood of the content of the signs flanking the identity sign (§62, §107). Hence, what Frege is most concerned with here is, again, an ontological issue: to specify a logicist criterion for the objecthood of numbers. He invokes extensions of concepts for

25. ‘ $N_t F(t)$ ’ stands for ‘the number belonging to the concept F ’.

this, which are logical objects on his account. His wariness in §107 is not about the need for *some* suitable objects to underpin his definition: such objects are essential to his definition. We can see this from the defence of his own suggestion that one could write ‘concept’ instead of ‘extension of the concept’ in his definition of number (1884: §68fn.): by substituting ‘concept’ for ‘extension of the concept’ in the definition ‘the Number which belongs to the concept F is the extension of the concept “equinumerous to the concept F”’ the word ‘concept’ would be preceded by the definite article, and the whole phrase (‘the concept’) would be thus still a singular term, determining numbers as objects.²⁶ Rather, the wariness in §107 concerns the fact that the phrase ‘extension of the concept’ is left undefined (‘presupposed’) in the *Foundations*, and hence that the ‘Julius Caesar’ problem remains; for to decide whether $N_t F(t) = a$ we must be able to decide whether a is a certain extension. So Frege’s wariness is ontological *par excellence*: it is concerned with securing a sharp objecthood criterion for numbers, which is not achieved just by employing the notion ‘extension of the concept’. This interpretation is confirmed once we look at Frege’s mature solution to the problem in *Grundgesetze*, where he brings in extensions as value-ranges not as *defined*, but as *primitive* objects.²⁷ This is obviously an ontological move, in fact one untouched by the stipulative role of definitions. This suggests that merely focusing on the stipulative aspect of definitions won’t help understand Frege’s project.

9 The decompositional character of definition: the 1914 argument

But what make of Frege’s conclusion in e.g. “Logic in Mathematics” (1914) that genuine definitions are indeed mere stipulations? Frege makes there a distinction between two ways to determine the sense of a sign, constructive definition and decompositional analysis (*Zerlegung*). A constructive definition introduces a new sign, previously without sense, to replace a group of signs that already have a sense (1914: 207f.). It is an arbitrary stipulation (1914: 211). Decompositional analysis starts with a sign A with an established use and sense, and attempts a logical analysis of the constituents of its sense. If it succeeds we find a complex expression designating the constituents of the sense of the simple sign, and the whole expression of the analysis is a self-evident axiom. Thus, if *Foundations*-definitions were decompositional, they would be self-evident. But, Weiner argues, *Foundations*-definitions are not self-evident, for we ‘think of the extensions of concepts

26. See Frege 1892: 48. See also Burge 1984: 274-84.

27. See Dummett 1991: 159.

as something quite different from numbers' (1884: §69). Hence, *Foundations*-definitions cannot be axioms, but must be constructive definitions, according to her.²⁸

This conclusion is premature. It faces difficulties when the whole 1914 argument and other passages, presenting a more complex account of definition, are taken into account. First, if *Foundations*-definitions had a stipulative, system-constructive role, they should take the form $A:=B$, where B represents a pre-systematic verbal expression and A a symbol in concept-script (like definitions in logic manuals, marking the transition from informal to formal language). But no *Foundations*-definition has this form, as it involves pre-systematic expressions on its *both* sides. Furthermore, while *Grundgesetze*-definitions are indeed presented as stipulations, they all involve only *concept-script* symbols on both sides. Frege actually specifies that for every newly introduced name we must be able to specify a co-referring complex name consisting only of the eight primitive names of *Grundgesetze* (1893: §33). Hence, definitions occur in *Grundgesetze* only inside the system. Neither *Foundations*-definitions nor *Grundgesetze*-definitions establish the transition from pre-system to system. Neither can be constructive in Weiner's sense.

Second, Frege's own understanding of constructive definitions poses a problem for Weiner. A constructive definition is an *abbreviation* introducing a new sign replacing some group of signs. This conception is already explained and employed in *Begriffsschrift* (1879: §24), and explicitly adopted in *Grundgesetze* (1893: vi, §27).²⁹ Abbreviations facilitate surveyability, especially in proofs. Such definitions are psychologically useful, if logically dispensable (1914: 208f.). They 'add nothing to the content' and are 'not essential to the system'. If a 'definition' were logically indispensable, a proof would be impossible without it; then it would be an axiom or theorem, not a definition (1914: 208). But on Weiner's interpretation the transition from pre-system to system is effected by definitions, so they must be logically essential to a system. To be fair, this remains a problem independently of Weiner, since in *Foundations* Frege does not suggest that his definitions are logically dispensable abbreviations. After defining number as extensions of

28. Actually, this argument is incomplete, since *Foundations*-definitions could be, if not axioms, at least theorems awaiting proof. But this possibility can be dismissed. If the definition of 0 were a theorem (or axiom), it would also assert a truth *about* 0. But then it would not be a *Foundations*-definition, for Frege writes that a definition does not assert 'anything about an object, but only lays down the meaning of a symbol' (1884: §67). Theorems don't establish the meaning of symbols, on Frege's account. Moreover, *Foundations*-definitions are not derivations at all, so cannot be theorems anyway.

29. It also occurs in "Über Grundlagen der Geometrie I" (1906: 320), where Frege insists, as Weiner notes, that definitions are arbitrary stipulations. (Other passages she mentions are less explicit, and one, in "On the Law of Inertia" (1891), is not relevant (and wrongly referenced, both in terms of page number and year of publication: see Weiner 2007: 697, fn. 27).)

concepts (1884: §68), he writes: ‘Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless’ (1884: §70). Frege also insists, against Kant, on the possibility and necessity of definitions which expand our knowledge by drawing new boundaries (1884: §88). Could it be that Frege means by a constructive definition in 1914 a *definition in concept-script*? This would explain why definitions in both *Begriffsschrift* and *Grundgesetze* are presented as abbreviations, but not in *Foundations*. This suggests that we should distinguish between *Foundations*-definitions, in the verbal language, and concept-script definitions, in the formula language, with only the latter being abbreviatory and ‘constructive’ à la 1914. Accordingly, *Foundations*-definitions could not be constructive definitions. Weiner does not seem to be aware of this distinction.³⁰

Third, being abbreviations constructive definitions are not creative. They stipulate that a new sign is to have the same sense as the signs abbreviated (1914: 207).³¹ Hence, a definition is a sense-conserving device. A *definiendum* is a receptacle into which we ‘cram’ a complex sense, to unpack it again whenever we need to. Note that Frege uses here his technical term ‘*Sinn*’,³² and redescribes a definition also as a bestowal of *Bedeutung* (1914: 208f.). Hence, definitions do not transform, by precisification, pre-systematic content into *Sinn* and *Bedeutung*. There must already be expressions with *Sinn* and *Bedeutung* for constructive definitions to be possible.

Another aspect of logical decomposition, as presented by Frege, reinforces this. Decomposition has the form $A=B+C$ (‘=’ indicates sameness of sense), which requires a complex expression $B+C$ whose constituents have the same sense as the constituents of A . If $A=B+C$ is self-evident, we obtain an axiom. If not, we infringe the sharpness requirement, since we don’t know the complete sense of A and can’t decide, for all contexts, whether A is substitutable with $B+C$, preserving the same thought (and truth-value). Hence, we go prescriptive: we introduce a new sign D , stipulated to have exactly the sense of the complex expression, i.e. $D:=B+C$, and operate with D , discarding the original *analysandum* A . Or we keep A , but as a ‘new’ sign, stipulated to have the same sense as the *analysans*, i.e. $A*:=B+C$ (1914: 211, Carnap 2004: 140). A , as an old sign, is discarded in any

30. Others are, but don’t make much of it. See e.g. Shieh 2008: 1005, who mentions, but denies that the distinction is relevant for making sense of Frege’s epistemological project. But see below for a contrasting view.

31. See also Frege 1906: 289.

32. This is also visible from the fact that sense is here subject to a compositionality principle at the level of thought: the sense of a part of a sentence is part of the thought expressed. That the sense of a sentence is a thought is of course a core aspect of Frege’s technical notion of sense.

case.³³ Note the Janus-faced character of A in this case. At any rate, we introduce constructive definitions when $A=B+C$ is not self-evident. Therefore, *constructive definitions involve logical decompositions*. For we started with a putative logical decomposition of A by means of $B+C$, and it is *this* complex expression $B+C$ which ends up on the right-hand side of the two possible constructive definitions ($D:=B+C$ or $A^*:=B+C$). The original symbol A led the way, as it were. Decompositions, however, are conservative, not creative: they presuppose some group of signs with an established sense.³⁴ Therefore, *constructive definitions are decompositional and content-preserving*, *pace* Weiner and *pace* some misleading passages in Frege, e.g. when he writes that in constructing a new system we take no account of anything prior to the system, since '[e]verything has to be made anew from the ground up', with none of our analytical activities making appearance in the new system (1914: 211).³⁵ If this were true, the very account of constructive definition he gives would be unintelligible, for where would $B+C$ come from? In reality, Frege's distinction between constructive definitions and logical decompositions does not exclude their close connection, as he explains in a lecture based on the 1914 text: '*Definitions* don't just *help* to construct, but also to *analyze* what is complex' (Carnap 2004: 140). Therefore our pre-systematic analytical activities (such as those offered in *Foundations*) do appear in the system (e.g. of *Grundgesetze*).

10 The undefinable basis of definition

We have established that, for Frege, there is no contradiction between a definition being stipulative, constructive, arbitrary, and its being decompositional and content-preserving. A definition is at best arbitrary regarding the selection of the *definiendum*-cum-sign, not the content assigned to it. While this undermines Weiner's denial of the Obvious Requirement, we must now probe further to locate the actual role and place of definitions in Frege's thought. Frege views every genuine definition as a stipulated identity³⁶, and this presupposes that with which the sense of the *definiendum* is identical: the sense of the *definiens*. Eventually, definitions must therefore come to an end; they are parasitic on indefinables (1885:

33. Hence, it is not quite correct to say, as Horty does (2007: 38), that the question of eliminability does not arise at all with decompositions (he calls them 'explicative definitions'). What is true is that A is not discarded in terms of an expression sharing exactly the same sense.

34. Cf. 'We can only allow something as a constituent of a complex expression if it has an established sense [*anerkannten Sinn*]' (1914: 210; translation amended).

35. The decompositional character of definitions is missed by other recent commentators as well, e.g. Shieh 2008: 1004.

36. See 1879: §23, 1884: §98, 1893: §27, §33, 1903a: 320.

96, 1906a: 301f.), hence can't contribute to the latter's constitution. Indefinables constitute the ontological foundation of a system of science: they contain 'like a seed the entire content of [a science]' (1885: 96). In the logicist project, definitions trace back the arithmetical to this realm of indefinables, 'the logical' (ibid.). The structure of *Grundgesetze* reflects this: first, eight *Urzeichen* are introduced as names of primitive logical objects such as value-ranges ($\hat{\varepsilon}\varphi(\varepsilon)$) or first-level functions ($\hat{\xi}$).³⁷ These names are subsequently employed to formulate six basic laws, self-evident axioms about the (inter-relations between the) primitive logical objects (1893: §§1-25). Only then are definitions introduced, expanding the stock of signs, which eventually include arithmetical symbols, '0', '1' etc. (1893: §§26ff.), or rather, their concept-script equivalents. By deriving the basic arithmetical laws with these symbols (§§53ff.), Frege purports to prove logicism. Consequently, definitions are only part of a wider process, and they don't establish the system alone.

Weiner mentions a passage in "On the Foundations of Geometry (Second Series)", where pre-systematic scientific expressions are described as too vacillating for strict science (1906a: 302f.). Frege then argues that definitions play a fundamental role in a system of science. Does this not contradict the 1914 view that definitions are logically superfluous, supporting Weiner's view that their system-constructive contribution is just precisification? No. Frege does not deny here that definitions *can* precisify, but insists that their role is not *exclusively* precisification. Definitions stipulate the content of a word, and if the *definiendum* is new, they must give it a precise content. If the *definiendum* is already in use, its use might be vacillating. Frege does not say how to proceed then, but presumably we could follow the 1914 recommendation and go prescriptive. However, Frege's focus in 1906 is not this case, but a putative *definiendum* satisfying 'the strictest requirements', when 'we might think that a definition is unnecessary' (1906a: 303). But this is not so, he argues, for while precise terms facilitate scientific communication, the 'real significance of definition lies in *the logical construction out of primitive elements*' (my emphasis), giving us the benefit of insight into logical structure (and the logical linkage between truths; see below). This is what makes it a 'constituent of the system of science'. The focus is here on 'primitive elements', not 'construction', for Frege immediately adds that definition can be generated by construction or decomposition, and which we employ does not matter. Therefore, there is no contradiction between the 1906 and 1914 arguments. In 1906 he stresses the reliance of

37. These are actually not eight names of eight logical entities (functions), but eight forms of names of eight kinds of logical entities (functions). Symbols like ' ζ ' are not themselves primitives of concept-script, but only introduced to exemplify, elucidate forms of concept-script symbols (see 1893: §1, fn. 1). I ignore complications arising from this detail, and treat ' ζ ' as a symbol of concept-script.

definition on primitive symbols, reached constructively or analytically. In 1914 he calls only a stipulative convention ('constructive definition') a genuine definition; but, as seen, even this involves decomposition. Construction and decomposition are two sides of the same coin.

How is then Frege's denial in 1914 that definition has logical significance to be understood? This is ambiguous. On the one hand, if Frege denies that formal definition is a necessary part of a system, he is right. Insofar as concept-script definitions abbreviate groups of signs reducible to primitive symbols, definitions are logically dispensable. Arithmetic could be reconstructed without e.g. definition Θ (1893: §41), without introducing a special symbol '0' for zero, using the *definiens* ' $\mathfrak{H}\hat{\varepsilon}(\top\varepsilon = \varepsilon)$ ' instead. Naturally, this will make the system much less surveyable and obscure its relationship to ordinary arithmetic. The cardinal operator ' \mathfrak{H} ' abbreviates itself a complex sign which in turn contains several occurrences of further abbreviations (the signs for elementhood, the extension of the reverse of a relation, etc.). Fully translated into the eight *Urzeichen* the sign for zero would be certainly a monster (e.g. containing over 30 quantifiers, real variables not counted). The resulting complexity is not a trivial matter, and without abbreviations we might not be even able to perform the proofs Frege offers and which advance our knowledge. This 'great importance for thinking as it actually takes place in human beings' (1914: 209) can therefore be seen as one aspect of the fruitfulness of stipulative definitions. The receptacle-function of definition is therefore not to be underestimated.³⁸ Nevertheless, this is still only of 'psychological', not logical significance. We could imagine a community of mathematicians performing pure and applied systematic arithmetic with the eight primitive symbols alone, every formula displaying the full logical structure of its *Bedeutung*. Formal definitions are therefore ultimately dispensable, as they are mere replacement rules. Nevertheless, when definitions *are* introduced, they do have logical significance: definition Θ , say, fixes the *Bedeutung* of a simple concept-script symbol for zero with an *analysis* in terms of a group of concept-script symbols designating, ultimately, indefinables, thus displaying, or at least intimating the full logical structure of zero.

On the other hand, if Frege meant that, even when introduced, definitions are logically irrelevant, he would be wrong, forsaking his 1906 insight into the relation between definitions and primitive symbols designating indefinables, a relation which proves the undeniably decompositional aspect of formal definition. To be sure, in 1914 Frege officially underplays the ontological role of definition by assigning only psychological usefulness to it. But, as seen, the 1914 argument implicitly

38. This aspect of definition is elaborated in detail in Horty 1993 and 2007, if to the detriment of elucidations and the ontological aspect of definition.

acknowledges the analytical nature of definition.³⁹ Both arguments allow, correctly, that definitions alone do not establish a system of science, despite their logical significance, when employed, rather require something prior and indefinable.

11 The missing link: elucidations

If not definitions, what does establish the system? How is the caesura between pre-systematic and systematic arithmetic to be understood and how is it bridged? It is bridged, Weiner says, by stipulative definitions meeting standards of precisification. But we have seen that Frege's definitions, as understood so far, do not link pre-system with system; they occur either in pre-systematic language, as in *Foundations*, or in concept-script, as in *Grundgesetze*. Moreover, *Grundgesetze* definitions only abbreviate groups of signs consisting, ultimately, of symbols designating *Urelemente*. These symbols cannot be introduced through definitions, since they designate what is logically simple and unanalysable (1892: 192, 1893: 4, 1914: 235). Definitions presuppose knowledge of the *Urelemente* and their *symbols* (1906a: 302). Knowledge of the former is *a priori* and unmediated (1884: §27f., §105), knowledge of the latter is established through *elucidations*. Elucidations are not part of a system, but belong to its propaedeutic, the 'antechamber' (1899: 36, 1906a: 301). They introduce symbols for the *Urelemente in* pre-systematic language (1914: 207), and elaborate the basic categories of a specific science.⁴⁰ They are essentially vague, and it is for this reason that they can't figure in proofs, and depend, like hints, 'on being met half-way by an intelligent guess' (1899: 37, 1906a: 302).⁴¹ Elucidations, and not definitions, are the true links between pre-system and system, and *pace* Weiner they do not answer to precision constraints. Only definitions do. Still, this leaves us with a problem, for if the precision constraints are employed in setting up the system, they precede the system, and if pre-systematic language is not yet subjected to them, then how are they made possible, if not by whatever is going on in the elucidatory transition? But elucidations are supposedly vague, and the resources we can employ to further clarify them are not essentially different from those employed in vacillating pre-systematic language. Circularity (1914: 207) and scepticism (1906a: 301) looms. Frege admits that in theory there is no way to overcome this difficulty. He simply assumes

39. And the 1914 account is more explicit in other respects, as seen above.

40. E.g. 'concept' (1892: 193), 'negation' (1893: §6), 'truth' (1918a: 60), 'judgement' (1918b: 126), 'point' (1906: 304f.), 'function' (1924/1925: 271).

41. See also 1892: 193, 1914: 235, 1893: 4. Russell and Whitehead have a similar account in *Principia Mathematica*. See Russell/Whitehead 1910: 91.

that in practice it is somehow overcome (1914: 207). But this is no solution. Elucidations are Frege's 'Münchhausen problem'.⁴²

In any case, it is noteworthy that in one passage Frege suggests that elucidations serve only a communicative purpose: a solitary scientist would need no elucidation (1906a: 301). But this clashes with Frege's claim that a system of science rests on undefinable *Urelemente*. Surely, even a solitary scientist establishing a system needs a concept-script, and since a concept-script differs from his pre-systematic language, the transition between the two would still require links similar to elucidations. Thus elucidations have not only a communicative, but also a system-constructive role.⁴³ Unlike definitions, they are neither part of the system nor logically indispensable nor subject to precision constraints. Their role is essentially *ontological*: in arithmetic they import primitive logical *objects*, towards which pre-systematic expressions vaguely gesture, into the system, where the structure and nature of those objects is fully revealed. These objects predate the system-language, since the elementary symbols are introduced only through elucidations, which require already meaningful, if (mostly) imprecise, expressions. But primitive logical objects also antedate pre-systematic, indeed any, language, given Frege's Platonism.

No matter how we solve the 'Münchhausen problem', elucidations are therefore necessary for system-construction, for they provide the system with its ontic foundation.⁴⁴ Weiner's exclusive focus on definitions, to the detriment of elucidations, is misleading.⁴⁵ Frege's conception of analysis, of a scientific system, depends on the contrast between elucidations and definitions. Pre-systematic arithmetical language is, to some extent, imprecise and vacillating, but not without ontic content. It is just this content that is dissected and displayed in the system.

42. That Frege continued to grapple with this problem is evident from notes towards the end of his life (see 1924: 266).

43. Alternatively, we could deny that the solitary scientist needs a concept-script, since that too has only a communicative, pragmatic purpose. But this conflicts with the epistemological and logical value Frege assigns a concept-script (see e.g. 1879: ixff., 1882). Or are we really supposed to think that the private language of the solitary scientist is itself already a concept-script?

44. In 1893: §35 Frege seems to contradict this, by saying that it is the definition of $\xi \circ \zeta$ ('elementhood'-function) which matters for subsequent proofs, not its elucidations, whose falsity would not invalidate the proofs. But $\xi \circ \zeta$ has a definition because it is a *complex* function, and its elucidation does not occur at the formative level of concept-script. The constituents of the definition, by contrast, must eventually consist of primitive symbols whose elucidations can't be false.

45. In a recent article on elucidations Weiner focuses mostly on the elucidation of categoricals like 'function', not on the primitive symbols of concept-script, and she does not address elucidation's ontological, but only its communicative role (Weiner 2005: 210), or rather its role of highlighting the incommunicable element in philosophy. This ontological omission also applies to Weiner's earlier discussion of elucidations (1990: 227ff.). For more criticism of that discussion see Kemp 1996: 175ff.

Otherwise, there would be nothing to analyse and systematize, and the sequence elucidations-axioms-definitions-proofs could not even begin. Since as elucidations introduce concept-script symbols for undefinable logical simples, there would be no basic vocabulary (1893: §31), if the pre-systematic element of an elucidation lacked content. An elucidation of a first-level function must be *of* that function, for which it introduces a symbol ('— ξ '). Otherwise '— ξ ' would have no *Bedeutung*, and '— ξ ' would infringe the requirement of *Grundgesetze* that all names, simple or complex, have a *Bedeutung* (1893: §§28-31). The pre-systematic element of an elucidation (the 'prose') must have a *Bedeutung* too (how acquired does not concern Frege), however vague. That pre-systematic expressions vacillate does not show that they lack a *Bedeutung*, but only that its precise expression needs a system. In conclusion, there is continuity of *Bedeutung* between pre-system and system, *pace* Weiner. Call this *the Continuity Thesis*.

12 The epistemic gap

While Frege believes that there is a gap between pre-system and system, it is not so wide that the question about the existence of arithmetical objects and the referents of arithmetical expressions only arises within the system. The gap is not ontological, but epistemic. This is suggested by the several passages. For instance, arguing against the attempt to study the nature of arithmetic and logic in historicist and empiricist terms, he distinguishes in *Foundations* between our knowledge of concepts, which is gradual and hazy, and the concepts themselves:

'What is known as the history of concepts is really a history either of our knowledge of concepts or of the meanings of words. Often it is only after immense intellectual effort, which may have continued over centuries, that humanity at last succeeds in achieving knowledge of a concept in its pure form, in stripping off the irrelevant accretions which veil it from the eyes of the mind. [...] Do the concepts, as we approach their supposed sources, reveal themselves in peculiar purity? Not at all; we see everything as through a mist, blurred and undifferentiated. It is as though everyone who wished to know about America were to try to put himself back in the position of Columbus, at the time when he caught the first dubious glimpse of his supposed India. Of course, a comparison like this proves nothing; but it should, I hope, make my point clear.' (1884: viif.)

The mist metaphor recurs in the 1914 text, in connexion with the question what to do if a proposed analysis of a simple sign A as $B+C$ is not self-evident (see above). Unlike Weiner, Frege readily distinguishes between the sign's sense and our incomplete grasp of it: 'If $[A:=B+C]$ is open to question [...] then the reason must lie in the fact that we do not have a clear grasp of the sense of the simple sign, but that its outlines are confused as if we saw it through a mist' (1914: 211).⁴⁶ Grasping incompletely the sense of A is surely different from grasping the incomplete sense of A . This fits the Continuity Thesis: given the ontological continuity between pre-system and system, mediated by elucidations, any imprecision can only be epistemic. Moreover, the 1914 passage just quoted in claims not just continuity of *Bedeutung*, but also of *Sinn*, which is also presented as objective and as the goal of precisification.⁴⁷ This is compatible with vacillating *use* of pre-systematic language, since Frege nowhere identifies sense with use.

Weiner mistakenly concludes from the acknowledged lack of self-evidence of Frege's *Foundations*-definitions (1884: §69) that numerals have no pre-systematic sense and reference, and that numbers are not really extensions and not discoverable to be so (Weiner 2007: 697f.). The mist-metaphor suggests just the opposite: there is a relative gap between *our* grasp of number and its real nature, a gap to be spanned by systematisation. The ontological objectivity of numbers contrasts with our subjective, vacillating access to it. The content of ' $2+3=5$ ' is as objective as the sun, although it may seem different to different people (1879/1891: 7). *Foundations* §69 is no counter-evidence to this contrast between appearance and reality,⁴⁸ for Frege's formulation is quite weak: 'That this definition [of number] is correct will *perhaps* be hardly evident *at first*. For do we not think of concepts as something quite different from numbers?' (my emphases). It is not that numbers don't have objective, self-evident properties, but only that these properties are not blindingly obvious: self-evidence does not entail immediate assent.⁴⁹ Of course, the inves-

46. See also 1914: 217. The metaphor also occurs in Carnap's notes of Frege's 1914 lectures. See Carnap 1914: 141. See also Horty 2007: 37 on this.

47. Continuity of *Sinn* entails continuity of *Bedeutung*, but Frege stresses only the former here, since a few pages before he was concerned with what a proof proves (the *Sinn* of a sentence, not a sentence) (1914: 206).

48. If indeed this passage is concerned with self-evidence. Two doubts arise: (a) if a definition says nothing about an object, i.e. if it is not a judgement (1884: §67), the evidential status of a definition does not arise; (b) contrary to the Austin translation, Frege does not speak of evidence. A better translation of the first sentence of *Foundations* §69 might be: 'That this explanation is appropriate will perhaps be little acceptable'.

49. Weiner admits this, in a footnote (2007: 682, fn. 6), but does not explain how this squares with her denial of an epistemic gap between the objective properties of number and our grasp of them. See Künne (2010: 672ff.) for a helpful discussion of the distinction between self-evidence and immediate assent. In his otherwise valuable discussion Kemp (1996: 181) seems to miss this point, which is just another aspect of Frege's rationalism. Incidentally, the aforementioned distinction is related to that between a proposition's being necessary and its being known to be

tigation of number in *Foundations* can't attain self-evidence for a simple reason: it offers no logical analysis, no definitions and no proofs *in concept-script*. Hence Frege's claim to have made the analyticity of arithmetic only *probable* in *Foundations* (1884: §87, §90, §109).⁵⁰ The aforementioned distinction between informal and formal definitions thus proves essential for understanding Frege's epistemological project, for only concept-script definitions really close the epistemic gap, not *Foundations*-definitions.⁵¹ These are still in verbal language, hence they still make us 'see everything as through a mist, blurred and undifferentiated', although much less so than previous definitions. 'Blurred and undifferentiated': i.e. *Foundations*-definitions, lacking the right multiplicity of symbols, still obscure, at least partly, the structure of their referents. We are getting a sense of what causes and what closes the epistemic gap: incomplete vs. complete articulation. I will return to this.

Weiner denies the epistemic gap. She repudiates Tyler Burge's defence of a similar gap (Weiner 2007: 693, fn. 22). Burge states: 'Strictly speaking, from Frege's point of view, no one had fully grasped and mastered the concept of number or *sense* (as distinguished from conventional significance/use) of "Number" by the time he wrote *Foundations* in 1884'⁵². The problem also applies to reference. If Burge were distinguishing strictly between the conventional significance or use and the sense and reference of a numeral, Weiner's misgivings would be understandable.⁵³ Not only did Frege not make any such strict distinction, but the distinction seemingly entails radical externalism about sense: the conventional use of a numeral, its 'nominal essence' (to use a Lockean phrase), could entirely diverge from its sense, its 'real essence'; a community might use symbols expressing senses not grasped by anybody (Weiner 2007: 693, fn. 22). Semantic scepticism threatens: how can we tell that we have left the conventional use behind and *now* grasp the sense?⁵⁴ Weiner's answer would be to say that pre-systematic numerals have no

necessary (see Kenny 1966: 135f.).

50. Remember that concept-script was developed by Frege to avoid the clandestine intrusion of anything intuitive into the chain of reasoning, in other words to articulate proofs in a 'gapless' manner (Frege 1879: preface). Since the propositions of *Foundations* are not thus formulated, we cannot be sure yet that something intuitive is not supporting the number-theoretic investigation. And if the presence of intuition cannot be excluded with certainty, then the claim of analyticity is merely probable, for intuition involves syntheticity, the opposite of analyticity.

51. Hence, we need to treat with great care passages in *Grundgesetze* in which Frege refers the reader, in the context of giving some formal definition, to the corresponding informal definition in *Grundlagen* (see e.g. 1893: §§41f.). The former are not simply formal translations of the latter, or else *Grundgesetze* must also be understood as proving the mere probability of the analyticity of arithmetic.

52. Burge 1984: 276; see also Burge 1990: 257, Baker & Hacker 1984: 176.

53. This is a counterfactual. Given Burge's careful formulation ('no one had fully grasped etc. '), the following should not be interpreted as criticising him.

54. Weiner's own explicit challenge to Burge is less convincing: 'Supposing that there are two

definite sense, only *Foundations*-content, but this was found wanting above. She invokes Frege's discussion of definition in *Grundgesetze* vol. II as additional evidence, suggesting that Frege criticises other mathematicians' definitions of arithmetical terms, because their definitions do not satisfy the sharpness requirement; therefore, the terms, before his system, lack genuine, complete definitions (1903: §62), hence definite sense and, by extension, reference, i.e. truth-value (Weiner 2007: 701). After all, Frege writes: 'would the sentence "any square root of 9 is odd" have a comprehensible sense at all if *square root of 9* were not a concept with a sharp boundary?' (1903: §56).

However, Frege's rhetoric here, indicated by the subjunctive, requires caution. He claims not that 'square root of 9', a pre-systematic term, has no comprehensible sense,⁵⁵ but only that, *if* the term had no complete definition, it would express no concept. For 'concepts' without sharp boundaries are pseudo-concepts (1893: §5, fn. 3, 1903: §56, §62). Sentences containing them would not just be lacking the allegedly system-internal feature of *Sinn*, but be nonsense.⁵⁶ But Frege never claimed, preposterously, that arithmetical propositions were nonsense before his definitions. His criticism only challenges mathematicians' official definitions. These definitions contrast with mathematicians' practice in their proofs, which, one passage suggests, is more correct than the official definitions (1903: §62, fn. 1)⁵⁷. The subjunctive only indicates a counterfactual situation. If arithmetical propositions *were* nonsense, even Weiner's precisifying definitions could not assign them sense and reference, for nonsense does not admit degrees, the highest degree of which is sense. Moreover, without sharp boundaries arithmetical predicates would not designate concepts, belying the *universal applicability* of numbers. Numbers, on Frege's view, apply universally because number-statements express judgements about concepts applicable to all objects, from all domains of the think-

distinct senses (say, of 'number'), each of which is entirely consistent with accepted mathematical thought. What is it that determines which of these two senses is the one we are currently expressing?' (ibid.). This supposition is not one Burge needs to accept, since it is incompatible with Frege's conception of the sense of a term.

55. Weiner might also be misled by Geach's inaccurate translation of a relevant passage in *Grundgesetze* §62. Frege does not say there that a complete definition (actually explanation, '*Erklärung*') of number is 'what is most lacking' (Geach/Black 1960: 165), but what is lacking frequently or most often ('*und daran fehlt es meist*').

56. One might object to this on the basis of *Begriffsschrift* §2 and §27, according to which 'Something is a heap' has content, despite 'is a heap' having no sharp boundary. But for this reason the content here is sometimes unjudgeable, as Frege stresses in *Begriffsschrift* §27, hence as a *general judgement* 'Something is a heap' will still be nonsense.

57. Frege's point in this passage is quite specific (concerning the failure of mathematicians to treat equality as identity in their official definitions), but the point can be generalised. In fact, Frege makes similar points elsewhere, e.g. when he explains that Weierstrass's sentences express true thoughts, *if we understand them correctly*, despite the fact that we would go astray if we were to follow his official definitions (1914: 222). See also Horty 2007: 39.

able (1882: 100, 1884: §14, §24). If pre-systematic predicates like ‘is the number of F ’ designated no concept, ‘ $2+2=4$ ’ would not be universally applicable, since its content contains the content of ‘is the number of F ’. But ‘ $2+2=4$ ’ is universally applicable. Frege sets out to explain the applicability of number, not to invent it. We see here why it is not only an apparently absurd view to claim, as Weiner does (2007: 706ff.), that it is no prerequisite for our pre-systematic terms to have a *Bedeutung*, but a truly absurd one.

13 Naming, defining, analysing

We must therefore avoid interpretations that make a travesty of Frege’s logical analysis, without assuming an unacceptable externalism about Fregean content. The gap in question is precisely not between pre-systematic conventional significance/use on the one hand, and sense and reference as features stipulated within the system on the other. Rather, the gap, or contrast, is between *incomplete and complete articulation* of one and the same thing, *judgeable content*, initially conceptualised as a unit, and later split up into sense and reference. This judgeable content is already designated or expressed by conventional significance, but only opaquely and impurely, since the language of the marketplace is not logically perfect (1879: xi), but made fully perspicuous only in concept-script.⁵⁸ This is confirmed by an early text, in which Frege writes that his concept-script is also a *lingua characteristica*, meant to render judgeable content ‘more exactly than is done by verbal language’, to ‘spell it out in full’ (1880/1881: 12f.). I will discuss this text shortly. As the ‘immense intellectual effort’ passage quoted above demonstrates, Frege is a rationalist, as suggested by Burge: mathematical practice has a deeper justification than may initially be visible, and it is only through unearthing the grounds of this justification by means of a system of science that genuine arithmetical knowledge, i.e. unshakeable, non-intuitive *a priori* knowledge of the most fundamental kind can be reached.⁵⁹ This unearthing certainly involves

58. Shieh 2008: 1007f. makes a related distinction to defend the epistemic gap, between expressing rule-governed sense (presumably something like conventional use) and grasping this sense fully. This distinction is underwritten by the idea of logical segmentation, as suggested by Thomas Ricketts. However, one problem with this idea is that it is developed on the basis of mere logico-linguistic principles (patterns of inferences etc.), ignoring Frege’s ontological agenda. This is obvious from the fact that Shieh’s otherwise valuable discussion ignores elucidations altogether. This applies to Horty 2007: 27f. as well.

59. See on this Burge (1984: 279f., 297f., 1990). For a similarly rationalist view of science in general, see Kant (1781: A 834). It is hard to see how one can deny Frege’s rationalism, as Kemp 1996: 182f. seems to do against Burge 1990, not considering, unlike Burge, a wider range of passages. What does Kemp think Frege has in mind when he speaks of the ‘immense effort’ stretching over centuries to recognise the essence of a concept? That humanity was ‘thinking

a more precise and refined language. But it is refined in a very specific respect, aiming only at the analysis of judgeable content relevant for proofs (1879: 6), i.e. the most important aspect of mathematical practice. An exhaustive account of conventional significance was never intended or covered by Frege's programme of logical analysis.⁶⁰ This is visible from Frege's very distinction between judgeable and unjudgeable contents (1879: §2), following which unjudgeable content, of e.g. 'house', is not analysable in concept-script.⁶¹ The later sense-reference distinction does not change this: since *Grundgesetze* still analyses only judgeable content, now split up into sense and reference (1893: x), not other features of declarative sentences. Hence, the *Begriffsschrift* contrast between judgeable and unjudgeable content remains, independently of how the sense-reference distinction applies to expressions of unjudgeable content.⁶² This contrast fits the above account of analysis: analysing is decomposing judgeable contents into contents ultimately no further analysable by concept-script, primitive logical objects with symbols introduced by elucidations. These objects are elementary unjudgeable contents. As in *Begriffsschrift*, concept-script in *Grundgesetze* is only a tool to analyse judgeable content – judgeable content shared by both pre-systematic and systematic expressions, and that decomposes into unjudgeable contents already signified, if

with the [sharp] sense', but was merely muddled when attempting 'to think *about* the sense, to give a definition' (Kemp 1996: 183)? How was such a persistent discrepancy possible?

60. This of course was argued by Weiner herself, in her 1990. See also Stekeler-Weithofer 1986: 200ff. for a similar thesis.

61. There are further substantiating points. *Begriffsschrift* acknowledges formal languages covering various domains, such as arithmetic or chemistry. Concept-script does not cover those, but is an additional domain, 'the central one, which borders on all the others' (1879: 7). This suggests that concept-script captures only the judgeable content of those pre-existing formal languages, not that it replaces them. This applies even to fundamental logical principles and rules (§13). Also, in the actual concept-script Frege allows for two sentences having the same judgeable content, but a different linguistic sense ('sense' should not be confused here with the later *Sinn*). Sense is individuated by the subject-predicate structure, judgeable content by function-argument decomposition (1879: §3). He also considers certain modal specifications to have merely grammatical significance, and argues that the difference between apodictic and assertoric forms of judgement does not entail difference in judgeable content (1879: §4). Furthermore, while concept-script symbols like '⊢' (definitional sign) or '⊃' (judgement stroke) are meaningful in concept-script, neither they nor any formulas containing them have judgeable content. Later, in *Foundations*, Frege claims that using a numeral as a predicate or attribute is to be excluded, which changes its meaning, but not that this changes the judgeable content of statements containing numerals, especially equations (1884: §60). Also, when, after splitting judgeable content into sense and reference (1893: x), Frege considers sentences containing indexicals, he seems to assign *Sinn* and *Bedeutung* to token-sentences, not type-sentences (1893: xvi-xvii). If *type*-sentences are the bearers of linguistic meaning, this too allows for a distinction between linguistic meaning and judgeable content (or rather the twin successors of the latter).

62. Notice that *Grundgesetze* does not include an analysis of even the most obvious expressions of unjudgeable content, elementary proper names, i.e. names that can figure as substitution instances in function-names. In fact, elementary proper names are not even elucidated in *Grundgesetze*; only the eight basic function-names are.

hazily, by pre-systematic language.⁶³ So there is no tension between conventional significance and content for Frege, since concept-script analysis only handles the latter, not other aspects of pre-systematic mathematical language.⁶⁴

Nevertheless, concerning the transition from pre-system to system, there is still the ‘Münchhausen problem’ mentioned in section 11. For Frege’s rationalism requires him to tell us how a potentially private and hazy pre-systematic acquaintance with arithmetical objects can be turned into a maximally objective, precise and perspicuous recognition thereof, if the means (concept-script) to achieve the latter is developed by means (elucidations) substantially not different from the resources of the pre-system⁶⁵. Suggesting that precisification does the job won’t do, for that requires an already sharp, non-circular, non-subjective concept and method of precisification, and this is precisely ‘what is most lacking’, for Frege admits that there are no theoretical reasons against scepticism and circularity (1914: 207). So a fundamental problem concerning the transition remains, even if we discount semantic externalism.

Leaving this problem to aside, I have claimed that systematisation amounts for Frege to complete articulation. Admittedly, there are not many passages explicitly discussing this. Frege was not much concerned with an explicit vis-à-vis comparison between pre-systematic mathematical language and its concept-script analysis, presumably because he took the relation between the two to be self-evident.⁶⁶ An important exception, not discussed by Weiner, is the early, long article presenting the advantages of his concept-script over Boole’s calculus (1880/1881); for to demonstrate these advantages Frege had to resort to a neutral ground of comparison, i.e. show how his logical analysis can discern features of pre-systematic arithmetical language *eluding* Boole’s notation. Frege first demonstrates how a judgeable content like $2^4=16$ can undergo functional analysis, decomposing it into, say, 2, 16 and $x^4=y$, by regarding 2 and 16 as replaceable or variable. We thus gain a certain concept, y being the fourth root of x , independently of the more rigid subject-predicate analysis. This replaceability and variability is essential for Frege’s understanding of fruitful concept-formation, of drawing ‘new boundary lines’, for it yields open sentences which can be subjected to quantification and

63. It was for this reason that Frege labelled his logical system ‘concept-script’ in *Begriffsschrift* (1879: 5f.), and still labels it ‘concept-script’ in *Grundgesetze* (1893: V).

64. Of course, Frege might be advancing a mistaken view about what counts as conventional significance versus judgeable content. See Kanterian 2012: 223, fn. 13 for one reason to assume this.

65. There is therefore agreement in principle, if not in all details, with Kemp 1996.

66. There are numerous juxtapositions of formulas with verbal paraphrases in *Begriffsschrift* and *Grundgesetze* (see e.g. 1893: 240ff.), but theoretical reflections on the relation between the two are missing.

thus generality (see 1880/1881: 16, 32ff.). Essentially, this amounts to the introduction of 'new' quantificational structure, providing not only insight into logical structure but also, as explained later, into the 'logical linkage of truths', thus reducing the number of axioms to an absolute minimum (1906a: 302, 1914: 209).⁶⁷ Of course, this procedure allows alternative analyses, yielding other concepts (e.g. $2^x=16$). But while such decomposition is meant to extend our knowledge, it is still conservative. For, as he will write later, while the conclusions drawn from fruitful definitions are not contained in the definitions 'as beams are contained in the house', they are nevertheless contained, namely 'as plants are contained in their seeds' (1884: §88). Hence, decomposition must start with a judgeable content, whose expression 'must already be structured in itself'⁶⁸. Frege continues:

'We may infer from this that at least the properties and relations which are not further analysable must have their own simple designations. But it doesn't follow from this that the ideas of these properties and relations are formed apart from objects [...]. Hence in the concept-script their designations never occur on their own, but always in combinations which express contents of possible judgement' (1880/1881: 17).⁶⁹

Frege next illustrates how to represent in concept-script arithmetical formulas, and also concepts and propositions of arithmetic hitherto expressed only by combining arithmetical symbols and prose. A good example is the concept 'the real function $\Phi(x)$ of a real variable x is continuous throughout the interval from A to B ', which he specifies in concept-script thus:⁷⁰

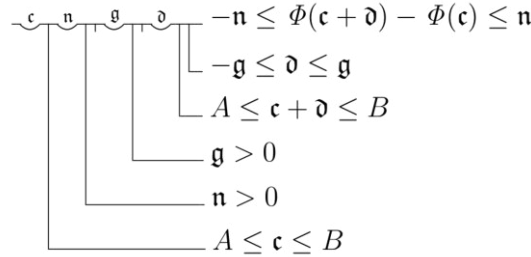
67. See Dummett 1991a: 38ff., Tappenden 1995: 427ff., Horty 2007: 39f.

68. This is a more accurate translation than the Long/White one as 'must already be itself articulated' (1880/1881: 17).

It should be noted that there is a major problem here with Frege's analytical method, which is undermined by two fundamentally incompatible paradigms of analysis, part-whole decomposition versus functional analysis. I have discussed this issue in detail in Kanterian 2012: 136ff., 184ff.

69. Notice how this passage anticipates the context-principle.

70. This characterisation of a concept is only the terminus of a longer chain of analysis, starting with a judgeable content. For Frege has noted before that 'we arrive at a concept by splitting up the content of a possible judgement' (1880/1881: 17).



One might think, following Weiner, that the differences between the words and the formula are radical enough to assume that it introduces something new, ‘content’ (or sense and reference) previously unavailable in pre-systematic language or even non-existent. But this is deceptive: ‘If in this case the formula seems longwinded by comparison with the verbal expression, you must always bear in mind that the former gives the definition of a concept which the latter only names’ (1880/1881: 24). This is a remarkable passage. Frege contrasts here *naming* with *defining*; the latter is analytical, decomposing the *definiendum* into indefinable elements (see section 8). This contrast confirms both the Obvious Requirement and the Continuity Thesis: the verbal expression already names the concept, and this presupposes the existence of the concept. The contrast is also consistent with the relative epistemic gap defended above, for naming an object does not entail knowing everything about it.

Frege describes the pre-systematic mathematician’s introduction of symbols repeatedly as one of naming entities that are already there. Recall the *Foundations*’ comparison of the mathematician with the geographer mentioned above, according to which both merely ‘discover what is there and give it a name’ (1884: §96). The same picture and argument are offered in *Grundgesetze*, where the mathematician’s definition is denied any creative power and is described as merely marking out something and giving it a name, similar to what a geographer’s definition of a sea accomplishes (1879: xiii). It is noteworthy that Frege identifies defining with naming in this later passage. Does this not contradict the contrast between naming and defining just drawn? One could reply that Frege means here by ‘definition’ the introduction of symbols in *pre-systematic* language (such as his own definitions given in *Foundations*), contrasting with definitions in *concept-script* (see section 8 above). And we would want to say only of the latter that they have a decompositional aspect, not of the former, which only name an object, but do not reveal its essence. So while pre-systematic definitions would only name a mathematical object or concept, definitions in concept-script would reveal its full essence (its atomic number, as it were). But in fact pre-systematic definitions

have a decompositional aspect as well. Take Frege's example above: the verbal expression, the 'name' of the concept under investigation is 'the real function $\Phi(x)$ of a real variable x is continuous throughout the interval from A to B ': this is not a mere tag, but already 'structured in itself', prefiguring some, although by far not all elements found in the final analysis in concept-script (e.g. the free, 'real' variables Φ , A , B). As seen, Frege observed the necessity of such structuredness for the possibility of functional analysis. Hence, even the pre-systematic name has an analytical aspect, and the contrast between the verbal expression and the concept-script formula is not absolute in this respect either. In other words, we can defend the Continuity Thesis not only with respect to there being a pre-given object or concept which the verbal expression names and the concept-script formula defines, but also in the sense that the verbal expression structurally anticipates, qua pre-systematic definition, at least some of the elements that the formula, qua definition in concept-script, will give a complete account of. So the difference between the verbal expression and the concept-script formula is a matter of degree, as is that between naming, defining and analysing, with the verbal expression as the beginning and the concept-script formula as the terminal point of the scale.⁷¹

This difference in degree is obscured by usual Frege's omission of intermediate steps between the verbal expression and the formula (including in the currently discussed example of the continuity of a real function; cf. Frege 1880/1881: 24). But such steps must exist: the formula is not reached just by glancing at the verbal expression, even with its semi-decompositionality. These steps were available in mathematics before Frege's concept-script definitions, and Frege uses them himself. For instance, before giving a concept-script specification of the concept 'the real function $\Phi(x)$ is continuous at $x = A$ ', he interpolates the informal paraphrase 'given any positive non-zero number n , there is a positive non-zero g such that any number d lying between $+g$ and $-g$ satisfies the inequality $-n \leq \Phi(A + d) - \Phi(A) \leq n$ ' (ibid.), which is a rephrased version of Weierstrass's 1859 definition.⁷² The only difference is that Frege's paraphrase is also elucidatory, containing concept-script symbols (free variables and German letters) in anticipation of the concept-script formula (see 1879: §11). Such intermediate, elucidatory steps are actually legion

71. Note that the contrast drawn here between verbal formulas and concept-script formulas allows for further intermediate steps, especially on the side of verbal formulas, also in the more context case of Frege's definition of number (which is not dealt with in the 1880/1881 article). It is one thing to use the phrase 'the number 0', and quite a different thing to use the definition of 0 proposed by Frege in *Foundations*, i.e. 'the extension of the concept "equal to the concept *not identical with itself*"' (1879: §68, §73). The latter goes a long way towards giving the analysis of 0, unlike 'the number 0' of course. But even 'the number 0' indicates something fundamental about 0: the fact that it is an object.

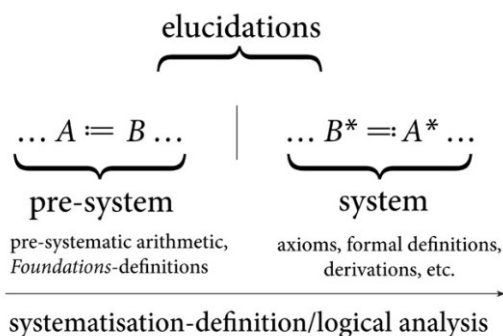
72. See Kline 1972: 952, also Tappenden 1995: 431, 458, fn. 14.

in *Begriffsschrift* and especially in the *Grundgesetze*'s 'Zerlegung' sections accompanying the actual proofs. While these are not part of the system (1879: v), they surely contribute to its constitution, as they are its 'antechamber' (as he would call them in 1914). In our example Frege's verbal paraphrase anticipates much of the formula. If Weiner were right, the verbal paraphrase should be nonsense, or at least lack system-internal content (understood in terms of the allegedly system-internal features of *Sinn* and *Bedeutung*). But for Frege verbal paraphrase and formula differ in degree, not content. He describes them as 'equivalent' in 1880/1881, but the formula is briefer and more perspicuous (see 1880/1881: 27). 'Equivalent' ('gleichwertig') very likely means more than material equivalence here, since the expressions involved are concept-words, not sentences. But even he means merely material equivalence, if an informal paraphrase is equivalent to a concept-script formula, the paraphrase must have content of the same *kind* as the formula, e.g. truth, and not just pre-systematic truth. In occasional, but for our purposes important passages *Grundgesetze* too says not that such paraphrases lack content, but that they are not *fully* precise and do not express the *whole* content, unlike the concept-script formulas (see e.g. 1903: 240, fn. 1, 242, fn. 1).

Altogether, we can now distinguish between at least four steps in the chain of analysis/definition. Specified to the example above, these are: (a) the verbal expression 'the real function $\Phi(x)$ is continuous at $x = A$ ', (b) Weierstrass's 'informal' definition without concept-script symbols, (c) Frege's informal paraphrase using concept-script symbols, and (d) Frege's concept-script definition (see 1880/1881: 24). The definition of 0 serves as another example. The four steps are here: (a) 'the number zero' or '0'; (b) Frege's *Foundations definiens*, i.e. 'the Number which belongs to the concept "not identical with itself"' (1884: §74); (c) Frege's informal paraphrase using concept-script symbols, 'the number of the $\hat{\varepsilon}(\neg\varepsilon = \varepsilon)$ -concept' (1893: §41); and (d) ' $\wp\hat{\varepsilon}(\neg\varepsilon = \varepsilon)$ ' (i.e. the *definiens* in definition Θ ; *ibid.*). In fact, these are only the more salient elements. To get anywhere close to a complete list and analysis of nought, we would have to include previous attempts to define mathematical concepts and objects (certainly not all mistaken), Frege's discussion of them, his own informal attempts, his elucidations and elaborations of logic and concept-script, and a version of 0 fully spelled out in concept-script, involving only its primitive terms. This gives us an inkling as to why 'only after immense intellectual effort, which may have continued over centuries, [...] humanity at last succeeds in achieving knowledge of a concept in its pure form' (1884: xixf.).

14 Systematisation-definition: Frege's goal

One last point remains: we have, so far, made an explicit distinction between definitions in pre-systematic science and definitions in concept-script, such that the *definiendum* and the *definiens* both belong to the same respective language. But this distinction cannot accommodate the definitional-analytical chains beginning with a pre-systematic verbal expression and ending with a concept-script formula. '*Definiendum*' and '*definiens*' here belong to different languages, leaving no canonical way to represent the definition. ' $\ulcorner \text{the number zero} \urcorner = \varepsilon$ ' would not be well-formed in concept-script.⁷³ Still, Frege describes the concept-script formula as defining *what* the verbal expression names.⁷⁴ And *that* is the object itself, zero, the common reference point throughout the whole chain. The ontological continuity between the two languages is thus crucial for analysis. The definition *in* concept-script, a mere abbreviation of a complex sign by a simple sign, would miss its point if it did not define zero, the object pre-systematic expressions like 'the number zero' or '0' name. This explains why in his concept-script definition Frege chooses a 'new' sign, the struck-out nought ('0') that 'happens' to resemble the original, pre-systematic numeral '0' (1893: §41). We must therefore distinguish a *third, wider type of definition*, definition of an object or concept, which is neither an informal nor a formal definition, and is achieved *through the whole process of analysis and systematisation*, stretching from the pre-systematic verbal expression to the eventual concept-script definition.⁷⁵ We can call this *systematisation-definition*. Here is a possible way to display it:



73. Not well-formed if 'the number zero' has its customary pre-systematic meaning. The definition would be well-formed if 'the number zero' is taken as a previously unused sign. But this would be just another definition *in* concept-script, not the special kind of definition discussed here.

74. On this problem see also Baker & Hacker 1984: 175ff.

75. Definition must be given an even wider sense, since, according to Frege, its validation involves the derivation of the truths of arithmetic within the whole system.

The asterisk indicates a concept-script ‘equivalent’ of a pre-systematic expression. Taking the analysis of 0 as an example, A is ‘0’, B is its *Foundations*-definition using verbal language, B^* the concept-script version of the latter, and A^* is the totally new symbol ‘0’, the concept-script equivalent of ‘0’. All expressions here have the same *Bedeutung*, but they don’t all share the same *Sinn*. In particular, the concept-script expressions do not have the same sense as their pre-systematic versions, and this is precisely the knowledge-increasing aspect of logical analysis. Whether A and B have the same *Sinn* depends on how we interpret the seed-metaphor, i.e. in what way the quantificational structure explicit in B is ‘contained’ in A as a ‘seed’. Of course, there may be no general way to determine this question, since A and B express their senses only ‘hazily’, on Frege’s view. What matters is that we end up with the right *Bedeutung* (1894: 319f.), and the rest is up to systematisation. A^* and B^* , on the one hand, do have the same *Sinn*, by definition (1893: §27). This is possible despite A^* not having the same syntactic richness as B^* ; for ‘sameness of sense’ must be understood dynamically here: the *definiendum* A^* is a receptacle into which the sense of B^* has been crammed.

Some interesting points arise. We see that systematisation-definition starts, from left to right, with a *definiendum* already in use, A , which is defined by a *definiens*, B , that introduces and unveils new quantificational structure, which in turn finds its fully perspicuous formal equivalent, after the elucidatory setup of the system, in B^* , which finally gets abbreviated by a new, simple symbol, A^* . In a sense we are turning full circle, with A at the beginning, and A^* at the end of the definitional chain.⁷⁶ This explains why the formal definition is a stipulation or abbreviation, as opposed to the informal one. It also explains why the formal definition is arbitrary: we can choose whatever new sign we want to abbreviate B^* (and Frege is quite inventive in his choice of *definienda*; for a sample, see 1893: appendix 2, “Tafel der Definitionen”). For the same reason, *Foundations*-definitions can’t be arbitrary; their *definienda* have been long in use. Consider that, say, the *Foundations*-definition of 0 is nothing even remotely like an abbreviation. Nevertheless, this does not mean, with respect to the 1914 argument, that only *Foundations*-definitions are analytical, while formal ones are merely constructive, stipulative, for we have seen that constructive definitions have an analytical aspect as well. And this fits with the place of formal definition in the above diagram, reading it from right to left: the formal definition presupposes a complex *definiens*, B^* , which in turn has a pre-systematic counterpart, B , which itself gave an analysis

76. This is visible even from the syntax of informal and formal definition respectively. In informal definitions, the *definiendum* is on the left-hand side, in formal definition on the right-hand side of the equation. See e.g. 1884: §74 and 1893: §41 respectively.

of A . Thus A^* is indeed an analysis of A , despite being merely an abbreviation of B^* . For in being so it shares not only the same *Bedeutung*, but also the same *Sinn* with B^* (in the sense just specified), and that *Sinn* is just the maximally perspicuous mode of presentation of the *Bedeutung* of A , mediated via the less perspicuous mode of presentation afforded by B , and the elucidations. ‘Decomposition’ (‘analysis’), ‘constructive definition’, ‘informal definition’, ‘elucidation’, ‘formal definition’ are all various moments in the broader chain of systematisation-definition. There is no simple way to describe this complex state of affairs. We can only understand it by comparing its various aspects with each other and their role within the broader context of Frege’s definitional project. Some problems in the literature arise, because priority is given to the individual aspects, and not the organic whole.

Frege gave no explicit account of systematisation-definition.⁷⁷ Nevertheless, it is what his project aimed for. In giving the formal definition Θ , say, Frege gave a systematisation-definition of ‘0’, the ordinary numeral, and thus of the complex logical object to which ‘0’ *really* refers, in terms of logical *Urelemente*. The Obvious Requirement is correct. So is the Continuity Thesis, for no matter how we conceive of the *Sinne* of the expressions in the chain of systematisation, they all share the same *Bedeutung*, on Frege’s conception of the matter. Of course, the worry raised by the ‘Münchhausen dilemma’ remains. And there are other worries.⁷⁸

At any rate, the essential role of systematisation-definition for Frege’s logicism refutes Weiner’s interpretation. While systematisation-definition involves precisification, it does not precisify in virtue of being a stipulation (meeting precision constraints). For it is not a stipulation, since it does not assume the canonical form ‘*definiendum* := *definiens*’. It does not have the form of what Frege calls definitions either in *Foundations* or *Grundgesetze*. Nor can it establish the systematic language, since this is the role of elucidations. But it is the very aim of Frege’s logical analysis of arithmetic.⁷⁹

77. For further elaboration, see Baker & Hacker 1984:175ff., Kanterian 2012: 118ff.

78. These relate to the precise nature of the relation between sense and reference. If *Sinn* is merely understood as ‘the mode of presentation of *Bedeutung*’, as Frege occasionally puts it, then we may conceive of one and the same *Bedeutung* being given, initially, ‘as through a mist’, and later on in a more perspicuous way. But if there is a much closer structural relation between Fregean sense and reference, with *both* having to be understood in function-theoretic terms, then Frege has a problem. For discussion, see Kanterian 2012: 179ff.

79. Cf. Baker & Hacker 1984: 176.

15 Epilogue: Frege's epistemological-ontological foundationalism

My discussion has aimed at clarifying the tension between Frege's epistemological aim and the strictures of systematisation. Clarifying the tension does not entail solving it. To solve that would involve solving the Münchhausen dilemma without giving up Frege's rationalism. Whatever the solution, the discussion has also shown that Fregean definitions have an ontological underpinning. *Pace* Weiner, his logicism is incomprehensible without the ontological agenda. I have also shown that Frege's explanations and employments of definition, early to late, are part of a more encompassing view than they appear. For instance, there is no clash between definition presented as analysis, in *Foundations*, and as stipulation, in concept-script. Definition in Frege's work is a richer and more multifaceted notion than often assumed, not being reducible to the notion of formal or informal definition.

One final remark may be permitted. While Weiner wrongly underplays Frege's ontological agenda, her stress on his epistemological aim is certainly correct. However, she believes that Frege pursues his epistemological aim *semantically*, by sharpening pre-systematic arithmetical language. But it is unclear how sharpening alone could satisfy Frege's Cartesian craving for certainty. *Foundations* has only established a probable thesis (1884: §87, §90), while he wants 'to place the truth of a proposition beyond all doubt' (§2), to gain 'absolute certainty that it contains no mistake and no gap' (§91), to raise the probability that arithmetical truths are analytic and *a priori* to certainty (§109), etc. Vagueness of concepts is not the only source of doubt and error; sharpened concepts do not guarantee certainty. 'X is a sharp concept, but it is uncertain whether y falls under X' is coherent. Nor does gapless proof alone yield certainty: we still need to reach the unshakeable ground supporting derived propositions, the axioms expressing *Urwahrheiten*, *Urgesetze* (§2ff.).⁸⁰ Frege wants not only conceptual and proof-technical rigour, to be achieved by mere stipulations, but genuinely reductive analysis: an arithmetical truth has found its epistemological classification if we trace its proof back to primitive truths (§4), whose number is reduced to a minimum (§2). Since primitive truths are truths evident without proof, they imply a source of indubitable *a priori* knowledge. Hence, Frege's epistemology has a foundationalist agenda. Moreover, this foundationalism does not exclude, but rather presupposes his ontologism. Truths about logical objects are self-evident by their nature: 'In arithmetic we are not concerned with objects which we come to know as something alien from without

80. Clearly, Frege's image, '*Unerschütterlichkeit eines Felsblockes*' (1884: §2), best translated as 'unshakeability of a boulder', lies in the metaphorical orbit of Descartes' *fundamentum inconcussum*.

through the medium of the senses, but with objects given directly to our reason and, as its nearest kin, utterly transparent to it. And yet, or rather for that very reason, these objects are not subjective fantasies. There is nothing more objective than the laws of arithmetic' (§105). The ultimate rationale for the epistemological agenda is Platonism: 'If there were nothing firm, eternal in the continual flux of all things, the world would cease to be knowable, and everything would be plunged in confusion' (vii)⁸¹. Ultimately, for Frege, epistemological puzzles are deeply intertwined with ontological questions concerns.

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Note: *Begriffsschrift*, *Foundations* and *Grundgesetze* quotations always refer to the sections of the books. Frege's other published writings are cited by the original paginations. Posthumous writings and letters are cited by the English translation in Gabriel et al. 1979 and 1980.

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