On the approximate periodicity of sequences attached to noncrystallographic root systems

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Abstract

We study Fomin-Zelevinsky's mutation rule in the context of noncrystallographic root systems. In particular, we construct approximately periodic sequences of real numbers for the noncrystallographic root systems of rank 2 by adjusting the exchange relation for cluster algebras. Moreover, we describe matrix mutation classes for type H_3 and H_4.

1 Introduction

Fomin and Zelevinsky have introduced cluster algebras in an impactful article [3, Definition 2.3]. In the last ten years diverse authors have found cluster algebra structures in various branches of mathematics such as representation theory, algebra and combinatorics. To define a cluster algebra, Fomin-Zelevinsky have defined *seeds* and *mutations of seeds*. Here, a seed (without frozen variables) is a pair $(\mathbf{x}, B)$ which consists of a *cluster* and a *mutation matrix*. The cluster $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a sequence of *cluster variables* and the mutation matrix B is a skew-symmetrizable integer $n \times n$ matrix. Given an initial seed, the cluster algebra is now defined to be generated by all cluster variables in all seeds that are obtained from the initial seed by a sequence of mutations. The natural number n is called the *rank* of the cluster algebra.

Some cluster algebras are of *finite type* and some cluster algebras are of *infinite type*. Here, we say a cluster algebra is of finite type if the mutation process yields only finitely many cluster variables. In another impactful article, Fomin-Zelevinsky [4, Theorem 1.4] have classified the cluster algebras of finite type via finite type root systems. The theorem implies that finite type cluster algebras (without frozen variables) are in bijection with *Dynkin diagrams* of type $A_n (n \geq 1)$, $B_n (n \geq 2)$, $C_n (n \geq 3)$, $D_n (n \geq 4)$, $E_n (n = 6, 7, 8)$, F_4, and G_2.

These Dynkin diagrams classify crystallographic root systems. In particular, such a diagram visualizes the Coxeter structure of the Weyl group of the corresponding root system. In the setup of cluster algebras, the crystallographic condition yields integer entries in the mutation matrix B. On the other hand, finite Coxeter groups are in bijection with *Coxeter-Dynkin* diagrams. Coxeter-Dynkin diagrams do not necessarily satisfy the crystallographic condition. Examples of non crystallographic Coxeter groups are dihedral groups (with Coxeter-Dynkin diagram $I_2(m)$ with $m = 5$ or $m \geq 7$) and the symmetry group of the icosahedron (with Coxeter-Dynkin diagram H_3).

The aim of this note is to generalize Fomin-Zelevinsky's matrix and seed mutation to noncrystallographic root systems. For every noncrystallographic root system of rank 2 the mutation class of the B-matrix contains two elements. Given two initial positive real numbers we define a sequence of real numbers by adjusting the exchange relations for cluster algebras to our setup. It turns out that the sequence is no longer a periodic sequence, but it is an almost periodic sequence meaning that it is approximately equal to a periodic sequence. For the noncrystallographic root system of type H_3 the mutation class of the B-matrix is finite, but we do not observe the phenomenon of almost periodicity. The question of approximate periodicity in this setup has also been touched by Reading-Speyer, see Armstrong [1, Problem 6.4].

2 Background

2.1 Fomin-Zelevinsky's cluster algebras

In this section we wish to recall the definition of Fomin-Zelevinsky's cluster algebras. We only consider coefficient-free cluster algebras without frozen variables over the field of rational numbers.

Let $n \geq 1$ be an integer and let $u_1, u_2, \ldots, u_n$ be algebraically independent variables over the field $\mathbb{Q}$ of rational numbers. The field $\mathcal{F} = \mathbb{Q}(u_1, u_2, \ldots, u_n)$ of rational functions is also called the *ambient field*. A *cluster* is a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ of algebraically independent elements. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called *skew-symmetrizable* if there exists a diagonal $n \times n$ matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with positive integer diagonal entries such that the matrix DB is skew-symmetric, i. e. $d_i b_{ij} = -d_j b_{ji}$ for all $1 \leq i, j \leq n$. In this case, the diagonal matrix D is called a *skew-symmetrizer* of B. A *seed* is a pair $(\mathbf{x}, B)$ formed by a cluster $\mathbf{x}$ and a skew-symmetrizable integer $n \times n$ matrix B. We denote the set of seeds by $\mathcal{S}$.

Let $k \in \{1, 2, \ldots, n\}$ be a natural number. A *mutation* in direction k is a map $\mu_k \colon \mathcal{S} \to \mathcal{S}$, $(\mathbf{x}, B) \mapsto \mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$, where $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n)$ is the sequence that we obtain from the cluster $\mathbf{x}$ by replacing the variable x_k with

$$x'_k = \frac{1}{x_k} \left(\prod_{b_{ik}>0} x_k^{b_{ik}} + \prod_{b_{ik}<0} x_k^{-b_{ik}} \right) \in \mathcal{F},$$

and keeping all other cluster variables $x'_i = x_i$ with $i \neq k$, and $B' = (b'_{ij})$ is the $n \times n$ matrix with

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

For every seed $(\mathbf{x}, B) \in \mathcal{S}$ the pair $\mu_k(\mathbf{x}, B) = (\mathbf{x}', B')$ is again seed, i. e. the elements of the sequence $\mathbf{x}'$ are also algebraically independent over the field of rational numbers and the matrix B' has integer entries and is skew-symmetrizable (with the same skew-symmetrizer D). Thus the map $\mu_k \colon \mathcal{S} \to \mathcal{S}$ is well-defined.

Fomin-Zelevinsky's mutation of seeds has many remarkable properties. Firstly, for every index k we have $\mu_k^2 = \mathrm{id}_{\mathcal{S}}$ so that the map μ_k is an involution. We declare two seeds $(\mathbf{x}, B), (\mathbf{x}', B') \in \mathcal{S}$ to be *mutation equivalent* if there exists a sequence $(k_1, k_2, \ldots, k_r)$ of indices such that $(\mathbf{x}, B) = (\mu_{k_1} \circ \mu_{k_2} \circ \ldots \circ \mu_{k_r})(\mathbf{x}', B')$. In this case we write $(\mathbf{x}, B) \simeq (\mathbf{x}', B')$. It follows that $\simeq$ is an equivalence relation on the set of all seeds.

Suppose that $(\mathbf{x}, B)$ is an initial seed. The cluster algebra $\mathcal{A}(\mathbf{x}, B) \subseteq \mathcal{F}$ is the $\mathbb{Q}$-subalgebra generated by all cluster variables x'_k in all seeds $(\mathbf{x}', B')$ that are mutation equivalent to $(\mathbf{x}, B)$. By construction we have $\mathcal{A}(\mathbf{x}, B) \subseteq \mathcal{F}$. More generally, Fomin-Zelevinsky's *Laurent phenomenon* [3, Theorem 3.1] asserts that $\mathcal{A}(\mathbf{x}, B) \subseteq \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. More generally, every cluster variable is an element in the ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

2.2 Cluster algebras of rank 2

A skew-symmetrizable integer 2×2 matrix has the form $B = \pm \left(\begin{smallmatrix} 0 & a \\ -b & 0 \end{smallmatrix} \right)$ for some natural numbers $a, b \geq 1$. Note that the two possible choices of the sign yield isomorphic cluster algebras which we will denote by $\mathcal{A}(a,b)$. We can parametrize the cluster variables in $\mathcal{A}(a,b)$ by the set of integers, so that we obtain cluster variables x_i, with $i \in \mathbb{Z}$, and clusters (x_{i-1}, x_i), with $i \in \mathbb{Z}$. The equation

$$x_{i-1}x_{i+1} = \begin{cases} x_i^a + 1, & \text{if } i \text{ is even;} \\ x_i^b + 1, & \text{if } i \text{ is odd;} \end{cases}$$

describes the mutation from the cluster (x_{i-1}, x_i) to the cluster (x_i, x_{i+1}). Fomin-Zelevinsky's classification theorem implies that the cluster algebra $\mathcal{A}(a,b)$ is of finite type if and only if $ab < 4$. In these cases, the sequence $(x_i)_{i\in\mathbb{Z}}$ is a periodic sequence. The period of the sequence is equal to 5, 6, or 8 when (a,b) is equal to $(1,1)$, $(2,1)$, or $(3,1)$.

Assume that $\mathcal{A}(a,b)$ is of finite type and let $i \in \mathbb{Z}$ be an integer. Due to the Laurent phenomenon we can write $x_i = f(x_1,x_2)/(x_1^{a_1}x_2^{a_2})$ for some polynomial $f \in \mathbb{Z}[x_1,x_2]$. It is easy to see that in these cases the constant term in the polynomial $f(x_1,x_2)$ is always equal to 1. Evaluation of the Laurent polynomial at the pair (x_1,x_2) yields a function $x_i \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. Motivated from tropical geometry, we could approximate $x_i \approx 1/(x_1^{a_1}x_2^{a_2})$. In this paper we wish to introduce an approximation, which is more accurate than the tropical approximation and also works in a more general (noncrystallographic) setup where we do not have a Laurent phenomenon.

3 Approximately periodic sequences attached to noncrystallographic root systems of rank 2

3.1 The definition of the almost periodic sequences for type I

The Dynkin diagrams attached to the cluster algebras $\mathcal{A}(1,1)$, $\mathcal{A}(2,1)$ and $\mathcal{A}(3,1)$ are A_2, B_2 and G_2, respectively. The corresponding Coxeter groups are the dihedral symmetry groups of the equilateral triangle, the square and the regular hexagon. More generally, the Coxeter-Dynkin diagram associated with the dihedral group of symmetries of the regular m-gon, for some $m \geq 3$, consists of two vertices that are joined by an edge of weight $a = 4\cos^2(\frac{\pi}{m})$. Note that $a \geq 1$. Generalizing the classical construction, the two possible orientations of the diagram yield two possible mutation matrices $B = \pm\left(\begin{smallmatrix} 0 & a \\ -1 & 0 \end{smallmatrix}\right)$. With this data we associate the following recursion. Let $\mathbf{x} = (x_1, x_2)$ be an initial cluster consisting of positive real numbers x_1 and x_2. Define a sequence $(x_i)_{i\in\mathbb{Z}}$ of positive real numbers by

$$x_{i-1}x_{i+1} = \begin{cases} x_i^a + 1, & \text{if } i \text{ is even;} \\ x_i + 1, & \text{if } i \text{ is odd.} \end{cases} \tag{1}$$

In contrast to the cases $m = 3,4,6$ the sequences are neither periodic nor do we notice the Laurent phenomenon. But we observe some approximate periodicity: in the case $m = 5$ (where we have $a = 4\cos^2(\frac{\pi}{5}) = \frac{1}{2}(3+\sqrt{5}) \approx 2.618033988$) we have randomly chosen starting values $x_1 = 0.829497$ and $x_2 = 0.363532$ from the open interval $(0,1)$, and computed the first few terms numerically, as the first two columns in Figure 1 illustrate. After 14 steps, we always get close to our starting values, e. g. $x_{-5} \approx x_9$ and $x_{-4} \approx x_{10}$. The same phenomenon also occurs for other values of m, and the number of steps is either $m+2$ or $2(m+2)$ depending on the parity of m. The aim of this section is to explain this phenomenon. From now on we assume that $m > 4$, because in the other cases we have exact periodicity. Note that $m > 4$ implies $a > 2$.

3.2 A recursion formula for a subsequence

It is enough to look at every other term of the sequence $(x_i)_{i\in\mathbb{Z}}$, because we can recover every term from the exchange relation (1) once we know its neighbors. To this end, let us define a sequence

n	x_n	$Y_{n/2}$	relative error
-6	0.935815	0.919721	0.017198
-5	0.136311		
-4	1.214248	1.170883	0.035714
-3	19.531300		
-2	16.908654	16.788570	0.007102
-1	84.093907		
0	5.032565	4.881875	0.029943
1	0.829497		
2	0.363532	0.363532	0.000000
3	1.290794		
4	6.301497	6.301497	0.000000
5	96.739925		
6	15.510588	15.228954	0.018158
7	13.546623		
8	0.937851	0.919721	0.019332
9	0.136223		
10	1.211518	1.170883	0.033541

Figure 1: An example of an approximately periodic sequence (x_n) with $m = 5$

$(y_i)_{i\in\mathbb{Z}}$ of positive real numbers by putting $y_i = x_{2i}$ for all integers $i \in \mathbb{Z}$. As above, we can view every element $y_i = y_i(x_1, x_2)$ as a function $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ in the initial values x_1, x_2. We will see that the sequence $(y_i)_{i\in\mathbb{Z}}$ is almost periodic; this will imply that the original sequence $(x_i)_{i\in\mathbb{Z}}$ is also almost periodic. The following proposition shows that there is a self-contained recursion formula for the elements of the sequence $(y_i)_{i\in\mathbb{Z}}$.

Proposition 3.1. Let $i \in \mathbb{Z}$ be an integer. Then the elements y_{i-1} y_i and y_{i+1} satisfy the equation $y_{i-1}y_iy_{i+1} = y_{i-1} + y_{i+1} + y_i^{a-1}$.

Proof. Let $i \in \mathbb{Z}$ be an integer. By construction we have $y_{i-1} = x_{2i-2}$, $y_i = x_{2i}$ and $y_{i+1} = x_{2i+2}$. Note that $y_{i-1}y_i - 1 = x_{2i-1}$ and $y_iy_{i+1} - 1 = x_{2i+1}$. The relation $(y_{i-1}y_i - 1)(y_iy_{i+1} - 1) = y_i^a + 1$ yields $y_{i-1}y_iy_{i+1} = y_{i-1} + y_{i+1} + y_i^{a-1}$. □

3.3 An approximation of the sequence

We define another sequence $(Y_i)_{i\in I}$. Moreover, we view the sequence $(Y_i)_{i\in I}$ as a numerical approximation of the sequence $(y_i)_{i\in I}$. The index set I is equal to $I = \{-2, -1, 0, 1, \ldots, \frac{m}{2}\}$ if m is even and to $I = \{-\frac{m+1}{2}, -\frac{m-1}{2}, \ldots, \frac{m+3}{2}, \frac{m+5}{2}\}$ if m is odd. In both cases, we put $Y_1 = y_1$ and $Y_2 = y_2$, and define the other elements in the sequence recursively. We put:

$$Y_0 = \frac{Y_2}{Y_1Y_2 - 1}; \qquad Y_{-1} = \frac{Y_0^{a-1}}{Y_0Y_1 - 1}; \qquad Y_{-2} = \frac{Y_{-1}^{a-1}}{Y_0Y_{-1}} = \frac{Y_{-1}^{a-2}}{Y_0};$$

$$Y_3 = \frac{Y_2^{a-1}}{Y_1Y_2 - 1}; \qquad Y_{i+1} = \frac{Y_i^{a-1}}{Y_{i-1}Y_i} = \frac{Y_i^{a-2}}{Y_{i-1}} \text{ for } 3 \leq i \leq \tfrac{m-1}{2}.$$

for all m. If m is odd, then we define the missing elements in the sequence as follows:

$$Y_{(m+3)/2} = \frac{Y_{(m+1)/2}^{a-1} + Y_{(m-1)/2}}{Y_{(m+1)/2}Y_{(m-1)/2}}, \qquad Y_{(m+5)/2} = \frac{Y_{(m+1)/2}}{Y_{(m+3)/2}Y_{(m+1)/2} - 1},$$

$$Y_{i-1} = \frac{Y_i^{a-1}}{Y_{i+1}Y_i} = \frac{Y_i^{a-2}}{Y_{i+1}} \quad \text{for} \quad -\tfrac{m-3}{2} \leq i \leq -2, \qquad Y_{-(m+1)/2} = \frac{Y_{-(m-1)/2}^{a-1} + Y_{-(m-3)/2}}{Y_{-(m-1)/2}Y_{-(m-3)/2}}.$$

Using the relations $Y_1 = x_2$ and $Y_2 = x_4 = \frac{1+x_1+x_2^a}{x_1x_2}$, we can view every element $Y_i = Y_i(x_1, x_2)$ as a real-valued function in the initial variables x_1, x_2 as above. Note that $x_3 = Y_1Y_2 - 1$. Moreover, $(Y_0Y_1 - 1)(Y_1Y_2 - 1) = 1$ implies $Y_0Y_1 - 1 = x_3^{-1}$. Thus we have $Y_2 = Y_0x_3$.

Next, we will give an explicit formula for the elements of the sequence $(Y_i)_{i\in\mathbb{Z}}$. To do so, we define a sequence $(g_i)_{i\geq 0}$ of integers by $g_0 = 0$, $g_1 = 1$ and $g_{i+1} = (a-2)g_i - g_{i-1}$ for $i \geq 1$. The following proposition relates the two sequences.

Proposition 3.2. Let $i \in I$. If $0 \leq i \leq \frac{m-3}{2}$, then we have $Y_{i+2} = Y_2^{g_i+g_{i+1}}x_3^{-g_i} = Y_0^{g_i+g_{i+1}}x_3^{g_{i+1}}$. Moreover, if $0 \leq i \leq \frac{m-1}{2}$, then we have $Y_{-i} = Y_2^{g_i+g_{i+1}}x_3^{-g_{i+1}} = Y_0^{g_i+g_{i+1}}x_3^{g_i}$.

Proof. The relation $Y_2 = Y_0x_3$ implies $Y_2^{g_i+g_{i+1}}x_3^{-g_i} = Y_0^{g_i+g_{i+1}}x_3^{g_{i+1}}$ and $Y_2^{g_i+g_{i+1}}x_3^{-g_{i+1}} = Y_0^{g_i+g_{i+1}}x_3^{g_i}$ for all i. Trivially, we have $Y_2 = Y_2^1x_3^0$ and $Y_0 = Y_0^1x_3^0$, so that the formulae hold true for $i = 0$. By definition, we have $Y_3 = Y_2^{a-1}x_3^{-1}$ and $Y_{-1} = Y_0^{a-1}x_3^1$ so that the formulae hold true for $i = 1$. The general case follows from the definition of the sequence $(Y_i)_{i\in\mathbb{Z}}$ by mathematical induction. ☐

Proposition 3.2 asserts that the sequence $(g_i)_{i\in\mathbb{Z}}$ controls the sequence $(Y_i)_{i\in\mathbb{Z}}$. The following Proposition states the main features of this sequence. We denote by $\omega = \exp(\frac{2\pi i}{m}) \in \mathbb{C}$ the root of unity and by $\overline{\omega} \in \mathbb{C}$ its complex conjugate.

Proposition 3.3. The sequence $(g_i)_{i\geq 0}$ is periodic. The period is equal to m, if m is odd, and equal to $\frac{m}{2}$, if m is even. Moreover, the following formula holds true for all natural numbers $i \geq 0$:

$$g_i = \frac{\omega^i - \overline{\omega}^i}{\omega - \overline{\omega}}. \tag{2}$$

Proof. Note that the sequence $(g_i)_{i\in\mathbb{Z}}$ is a homogeneous linear recurrence relation with characteristic polynomial $X^2 - (a-2)X + 1$. By elementary trigonometry we have $a - 2 = 4\cos^2(\frac{\pi}{m}) - 2 = 2\cos(\frac{2\pi}{m})$ so that the characteristic polynomial splits as $(X - \omega)(X - \overline{\omega})$. Therefore, the sequence $(g_i)_{i\in\mathbb{Z}}$ is a $\mathbb{C}$-linear combination of the sequences $(\omega^i)_{i\in\mathbb{N}}$ and $(\overline{\omega}^i)_{i\in\mathbb{N}}$. A comparison of coefficients for the initial values g_0 and g_1 yields equation (2). ☐

Note that the previous proposition implies that g_i is positive if $1 \leq i \leq \frac{m-1}{2}$. Similarly, g_i is negative if $-\frac{m-1}{2} \leq i \leq -1$. Moreover $|g_i| \geq 1$ unless $i \in \{\frac{m}{2}, \frac{0,m\pm 1}{2}\}$. The explicit formula implies that the following terms in the sequence $(Y_i)_{i\in\mathbb{N}}$ are equal.

Theorem 3.4. If m is even, then the equations $Y_{-2} = Y_{(m-2)/2}$ and $Y_{-1} = Y_{m/2}$ hold. If m is odd, then the equations $Y_{-(m+1)/2} = Y_{(m+3)/2}$ and $Y_{-(m-1)/2} = Y_{(m+5)/2}$ hold.

Proof. Let m be even. It is easy to see that $g_{\frac{m}{2}} = 0$, $g_{\frac{m}{2}-1} = 1$ and $g_{\frac{m}{2}-2} = a-2$, so that $Y_{\frac{m}{2}} = Y_2^{a-1}x_3^{2-a}$, which agrees with $Y_{-1} = Y_0^{a-1}x_3 = (Y_2x_3^{-1})^{a-1}x_3 = Y_2^{a-1}x_3^{2-a}$. Moreover, it follows that $Y_{\frac{m}{2}-1} = Y_{\frac{m}{2}}^{a-2}Y_2^{-1}x_3 = Y_{-1}^{a-2}Y_2^{-1}x_3$, which agrees with $Y_{-2} = Y_{-1}^{a-2}Y_0^{-1}$.

Now let m be odd. Let us put $g = g_{(m-1)/2}$. Due to Proposition 3.3 we have $g_{(m+1)/2} = -g$, from which we conclude $g_{(m-3)/2} = (a-1)g$ and $g_{(m+3)/2} = -(a-1)g$. Furthermore, the recursion implies $g_{(m-5)/2} = (a^2 - 3a + 1)g$. Proposition 3.2 yields

$$\begin{aligned} Y_{-(m-3)/2} &= Y_0^{(a-1)g+g} x_3^{(a-1)g} = Y_0^{ag} x_3^{(a-1)g}, \\ Y_{-(m-1)/2} &= Y_0^{g-g} x_3^{g} = x_3^{g}. \end{aligned}$$

Using these expressions, we can write the next element of the sequence as

$$Y_{-(m+1)/2} = \frac{x_3^{(a-1)g} + Y_0^{ag} x_3^{(a-1)g}}{Y_0^{ag} x_3^{ag}} = x_3^{-g}(1 + Y_0^{-ag}).$$

On the other hand, Proposition 3.2 yields

$$\begin{aligned} Y_{(m-1)/2} &= Y_0^{(a^2-3a+1)g+(a-1)g} x_3^{(a-1)g} = Y_0^{a(a-2)g} x_3^{(a-1)g}, \\ Y_{(m+1)/2} &= Y_0^{(a-1)g+g} x_3^{g} = Y_0^{ag} x_3^{g}. \end{aligned}$$

Using these expressions, we can write the next elements of the sequence as

$$\begin{aligned} Y_{(m+3)/2} &= \frac{Y_0^{a(a-1)g} x_3^{(a-1)g} + Y_0^{a(a-2)g} x_3^{(a-1)g}}{Y_0^{a(a-1)g} x_3^{ag}} = x_3^{-g}(1 + Y_0^{-ag}), \qquad (3) \\ Y_{(m+5)/2} &= \frac{Y_0^{ag} x_3^{g}}{Y_0^{ag}(1 + Y_0^{-ag}) - 1} = x_3^{g}. \end{aligned}$$

The expressions agree with the expressions that we obtain for $Y_{-(m+1)/2}$ and $Y_{-(m-1)/2}$, and so the statement follows. □

3.4 Numerical comparison of the two sequences

Theorem 3.5. Let $i \in I$. The element $Y_i = Y_i(x_1, x_2)$ is an approximation of $y_i = y_i(x_1, x_2)$ with relative error $\left|\frac{Y_i - y_i}{y_i}\right| = \left|\frac{Y_i(x_1,x_2) - y_i(x_1,x_2)}{y_i(x_1,x_2)}\right| = O(x_1 x_2)$ for $x_1, x_2 \to 0$.

Before we prove the theorem, we state a lemma. For proofs of the lemma and the theorem recall that our assumption $m > 4$ implies $a > 2$. Moreover, note that for $x_1, x_2 \in (0,1)$ we have

$$x_3 = \tfrac{1+x_2^a}{x_1} > x_1^{-1} > 1, \qquad Y_2 = \tfrac{1+x_1+x_2^a}{x_1 x_2} > x_1^{-1} x_2^{-1} > 1,$$
$$Y_2 / x_3 = \tfrac{1+x_1+x_2^a}{x_2(1+x_2^a)} > x_2^{-1} > 1.$$

Lemma 3.6. Suppose that $i \in I$ indexes some element of the sequence $(Y_i)_{i \in I}$.

(a) Suppose that m is even and $i \in \{0\} \cup \{3, 4, \ldots, \frac{m-2}{2}\}$ or m is odd and $i \in \{0, -1, \ldots, -\frac{m-3}{2}\} \cup \{3, 4, \ldots, \frac{m+3}{2}\}$. Then we may write the quotient $(Y_i Y_{i-1} - 1)/(Y_i Y_{i-1})$ as $1 - \epsilon_i$ for some the real-valued function $\epsilon_i = \epsilon_i(x_1, x_2)$ with $|\epsilon_i(x_1, x_2)| = O(x_1 x_2)$.

(b) If $3 \le i \le \frac{m-1}{3}$, then we may write the quotient $(Y_i^{a-1} + Y_{i-1})/(Y_i^{a-1})$ as $1 + \epsilon_i'$ for some real-valued function $\epsilon_i' = \epsilon_i'(x_1, x_2)$ with $|\epsilon_i'(x_1, x_2)| = O(x_1 x_2)$. If $-\frac{m-3}{2} \le i \le 0$, then we may write the quotient $(Y_i^{a-1} + Y_{i+1})/(Y_i^{a-1})$ as $1 + \epsilon_i''$ for some real-valued function $\epsilon_i'' = \epsilon_i''(x_1, x_2)$ with $|\epsilon_i''(x_1, x_2)| = O(x_1 x_2)$. Finally, if m is odd, then we may write the quotient $(Y_{(m+3)/2}^{a-1} + Y_{(m+1)/2})/Y_{(m+1)/2}$ as $1 + \epsilon'''$ for some real-valued function $\epsilon''' = \epsilon'''(x_1, x_2)$ with $|\epsilon'''(x_1, x_2)| = O(x_1 x_2)$.

Proof of the lemma. (a) By definition we have $\epsilon_i(x_1, x_2) = (Y_i Y_{i-1})^{-1}$. If $3 \le i \le \frac{m+1}{2}$, then Proposition 3.2 implies

$$\epsilon_i(x_1, x_2) = (Y_i Y_{i-1})^{-1} = (Y_2/x_3)^{-g_{i-2}-g_{i-3}} Y_2^{-g_{i-1}-g_{i-2}}.$$

Thus, it is enough to show that $g_{i-2} + g_{i-3}$ is nonnegative and $g_{i-1} + g_{i-2}$ is at least 1 for $3 \le i \le \frac{m+1}{2}$, which follows from the explicit formula in Proposition 3.3. If m is odd and $i = \frac{m+3}{2}$, then Proposition 3.2 and formula (3) imply $Y_i Y_{i-1} = 1 + (Y_2/x_3)^{ag} > (Y_2/x_3)^{ag}$ from which we conclude with the statement of the lemma by similar arguments as above. If $-\frac{m-3}{2} \le i \le 0$, then Proposition 3.2 implies

$$\epsilon_i(x_1, x_2) = (Y_i Y_{i-1})^{-1} = (Y_2/x_3)^{-g_{-i+1}-g_{-i+2}} Y_2^{-g_{-i}-g_{-i+1}}.$$

Thus, it is enough to show that $g_{i+1} + g_{i+2}$ is nonnegative and $g_i + g_{i+1}$ is at least 1 for $0 \le i \le \frac{m-3}{2}$, which follows from the explicit formula in Proposition 3.3.

(b) By definition we have $\epsilon_i'(x_1, x_2) = Y_{i-1}/Y_i^{a-1}$ for all i for which the function is defined. Proposition 3.2 yields $\epsilon_i'(x_1, x_2) = (Y_2/x_3)^{g_{i-3}-(a-1)g_{i-2}} Y_2^{g_{i-2}-(a-1)g_{i-1}} = (Y_2/x_3)^{-g_{i-1}-g_{i-2}} Y_2^{-g_i-g_{i-1}}$. Thus, it is enough to show that $g_{i-1} + g_{i-2}$ and $g_i + g_{i-1}$ is greater than 1 for $3 \le i \le \frac{m-1}{2}$, which follows from the explicit formula in Proposition 3.3. The cases ϵ_i'' and ϵ''' are proved similarly. □

With the preparation we are ready to prove the theorem:

Proof of the theorem. The statement is true for $i \in \{1, 2\}$ because $Y_1 = y_1$ and $Y_2 = y_2$ by definition. For $i \in \{3, 0\}$ we can estimate the relative errors as $x_1, x_2 \to 0$:

$$\left|\frac{y_3 - Y_3}{y_3}\right| = \left|\frac{y_1}{y_1 + y_2^{a-1}}\right| = \left|\frac{1}{1 + y_1^{-1} y_2^{a-1}}\right| < \left|\frac{1}{1 + y_1^{-1} y_2}\right| < \left|\frac{1}{1 + x_1^{-1} x_2^{-2}}\right| < x_1 x_2^2 = O(x_1 x_2),$$
$$\left|\frac{y_0 - Y_0}{y_0}\right| = \left|\frac{y_1^{a-1}}{y_1^{a-1} + y_2}\right| = \left|\frac{1}{1 + y_1^{1-a} y_2}\right| < \left|\frac{1}{1 + y_1^{-1} y_2}\right| < \left|\frac{1}{1 + x_1^{-1} x_2^{-2}}\right| < x_1 x_2^2 = O(x_1 x_2).$$

For the other values of i we prove the theorem by induction on the absolute value of i. Assume that $i \ge 3$ and that the statement is true for i and $i - 1$. We put $Y_i/y_i = 1 + \delta_i$ and $Y_{i-1}/y_{i-1} = 1 + \delta_{i-1}$. By induction hypothesis $|\delta_i| = |\delta_i(x_1, x_2)|$ and $|\delta_{i-1}| = |\delta_{i-1}(x_1, x_2)|$ are real-valued functions in the class $O(x_1 x_2)$. Define an auxiliary function

$$\widetilde{Y}_{i+1} = Y_{i+1}(x_1, x_2) = \frac{Y_i^{a-1} + Y_{i-1}}{Y_i Y_{i-1} - 1}.$$

Taylor's theorem for the function $f(x) = (1 + x)^{a-1}$ evaluated at $x = \delta_i$ implies that we may write $(1 + \delta_i)^{a-1} = 1 + \delta_i(a - 1)(1 + \xi)^{a-2}$ for some ξ between 0 and δ_i. We put $Y_i^{a-1}/y_i^{a-1} = 1 + \widetilde{\delta}_i$. The previous discussion implies that the function $|\widetilde{\delta}_i| = |\widetilde{\delta}_i(x_1, x_2)|$ is a real-valued function in the class $O(x_1 x_2)$. The quotient $(Y_i^{a-1} + Y_{i-1})/(y_i^{a-1} + y_{i-1})$ lies between $1 + \widetilde{\delta}_i$ and $1 + \delta_{i-1}$. Hence we may write the quotient as $1 + \delta_i'$ for some the real-valued function $\delta_i' = \delta_i'(x_1, x_2)$ with $|\delta_i'(x_1, x_2)| = O(x_1 x_2)$. Similarly, by the induction hypothesis and the previous lemma we may write the quotient $\frac{y_i y_{i-1} - 1}{Y_i Y_{i-1} - 1} = \frac{y_i y_{i-1} - 1}{Y_i Y_{i-1}} \cdot \frac{Y_i Y_{i-1}}{Y_i Y_{i-1} - 1}$ as $1 + \delta_i''$ for some the real-valued function $\delta_i'' = \delta_i''(x_1, x_2)$ with $|\delta_i''(x_1, x_2)| = O(x_1 x_2)$. We conclude that the quotient $\widetilde{Y}_{i+1}/y_{i+1} = 1 + \delta_i'''$ for some real-valued function $\delta_i''' = \delta_i'''(x_1, x_2)$ with $|\delta_i'''(x_1, x_2)| = O(x_1 x_2)$.

From the previous lemma we can conclude that we may write the quotient $Y_{i+1}/\widetilde{Y}_{i+1}$ as $1 + \widetilde{\epsilon}_i$ for some real-valued function $\widetilde{\epsilon}_i = \widetilde{\epsilon}_i(x_1, x_2)$ with $|\widetilde{\epsilon}_i(x_1, x_2)| = O(x_1 x_2)$. The theorem follows because $Y_i/y_i = (Y_i/\widetilde{Y}_i) \cdot (\widetilde{Y}_i/y_i) = (1 + \widetilde{\epsilon}_i)(1 + \delta_i''')$ with $\widetilde{\epsilon}_i + \delta_i''' + \widetilde{\epsilon}_i \delta_i''' = O(x_1 x_2)$.

The case $i \le 0$ is proved similarly. □

Thus, the sequence $(y_i)_{i\in I}$ is approximately equal to the sequence $(Y_i)_{i\in I}$ which can be extended to a period sequence $(Y_i)_{i\in\mathbb{Z}}$ in view of Theorem 3.4. Hence, the original sequence $(x_i)_{i\in\mathbb{Z}}$ is approximately equal to a periodic sequence.

4 Matrix mutation with real entries

The matrix mutation rule generalizes to real entries. As before, an $n \times n$ matrix $B = (b_{ij})$ with *real* entries is called *skew-symmetrizable* if there exists a diagonal $n \times n$ matrix $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$ with positive real diagonal entries such that the matrix DB is skew-symmetric, i. e. $d_i b_{ij} = -d_j b_{ji}$ for all $1 \le i, j \le n$. Let B be a real skew-symmetrizable $n \times n$ matrix and $k \in \{1, 2, \dots, n\}$ an index. We define the *mutation* of B at k to be the $n \times n$ matrix $B' = (b'_{ij})$ with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}; \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

As before, we denote the mutation also by $B' = \mu_k(B)$. The following proposition is immediate.

Proposition 4.1. Let B be a real skew-symmetrizable $n \times n$ matrix with skew-symmetrizer D and let $k \in \{1, 2, \dots, n\}$. The matrix $\mu_k(B)$ is again skew-symmetrizable with skew-symmetrizer D. Moreover, $\mu_k^2(B) = B$.

As before, we define *mutation equivalence* to be the smallest equivalence relation on the set of skew-symmetrizable real $n \times n$ matrices such that $\mu_k(B) \simeq B$ for all B and all k.

Remark 4.2. For mutation classes of integer matrices we have several structural results. There is classification of mutation-finite, skew-symmetrizable, integer matrices by Felikson-Shapiro-Tumarkin [2]. Effective criteria to test whether a given skew-symmetric matrix is mutation-finite are due to Lawson [5] (via minimal mutation-infinite subquivers) and Warkentin [6] (via forks). On the other hand, structural results for classification of mutation-finite, skew-symmetrizable, real matrices seem to be harder, because it is easy to construct finite mutation classes, as the following example shows.

Example 4.3. Let a, b, c, a', b', c' be positive real numbers. Consider the matrix

$$B = \begin{pmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{pmatrix}.$$

Then B is skew symmetrizable if and only if $abc = a'b'c'$. In particular, let $a, b, c \in \mathbb{R}^+$ such that $abc = 8$. Put $a' = \frac{bc}{2}$, $b' = \frac{ca}{2}$ and $c' = \frac{ab}{2}$. Then B is skew symmetrizable and $\mu_1(B) = \mu_2(B) = \mu_3(B) = -B$. Hence, B is mutation finite.

The Coxeter group of type H_3 is the symmetry group of the regular icosahedron. Besides the Coxeter group of type H_4, it is the only noncrystallographic Coxeter group whose rank is greater than 2. The next lemma shows that the corresponding mutation matrices are mutation finite.

Lemma 4.4. Let $a = 4\cos^2(\frac{\pi}{5})$. The matrices

$$B' = \begin{pmatrix} 0 & a & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad B'' = \begin{pmatrix} 0 & a & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

of type H_3 and H_4 are mutation finite. If we identify matrices that are obtained from each other by a simultaneous row and column permutation, then the mutation classes have sizes 16 and 82.

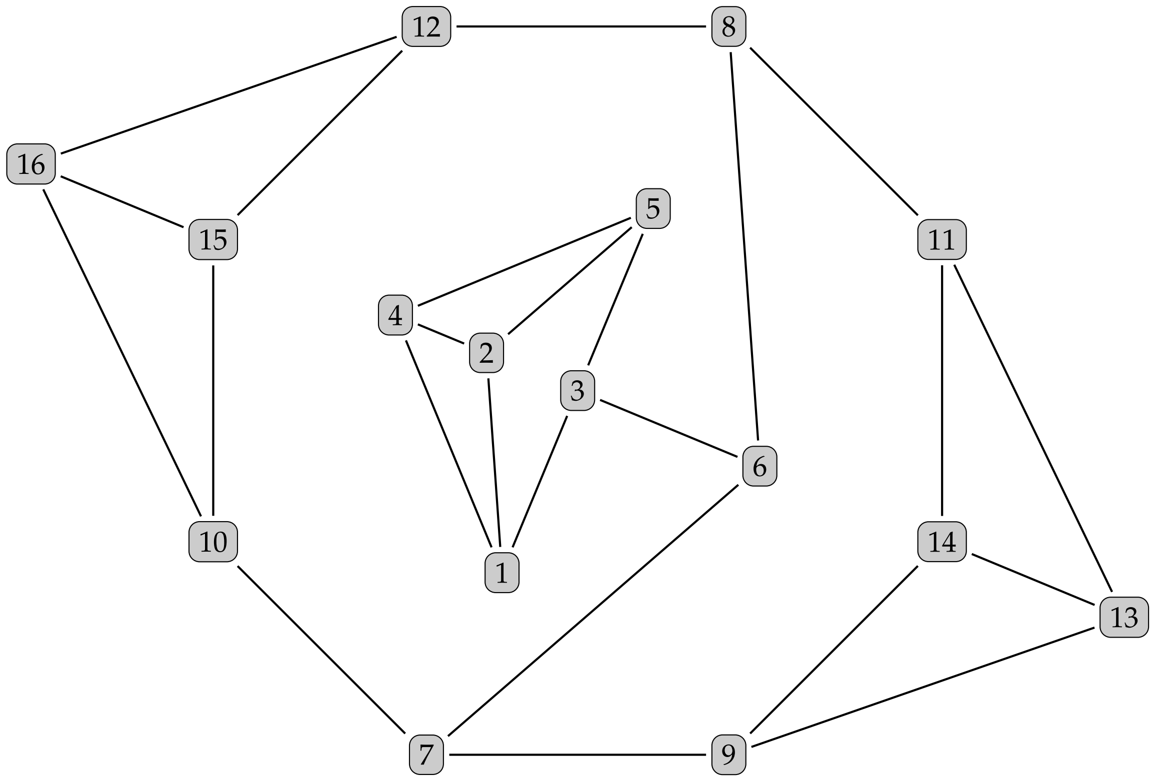

Figure 2: The mutation class for H_3

Proof. A calculation shows that the following set of 16 matrices is closed under mutation. Mutations are visualized in the picture. The set contains the matrix B'.

$$1: \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & a \\ 0 & -1 & 0 \end{pmatrix}, \quad 7: \begin{pmatrix} 0 & 1-a & a \\ a-1 & 0 & 1-a \\ -1 & a-2 & 0 \end{pmatrix}, \quad 12: \begin{pmatrix} 0 & 0 & 1-a \\ 0 & 0 & a \\ a-2 & -1 & 0 \end{pmatrix},$$

$$2: \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 0 \end{pmatrix}, \quad 8: \begin{pmatrix} 0 & 1-a & a-1 \\ a-1 & 0 & -a \\ 2-a & 1 & 0 \end{pmatrix}, \quad 13: \begin{pmatrix} 0 & 1-a & 0 \\ a-1 & 0 & a-1 \\ 0 & 2-a & 0 \end{pmatrix},$$

$$3: \begin{pmatrix} 0 & -1 & a \\ 1 & 0 & -a \\ -1 & 1 & 0 \end{pmatrix}, \quad 9: \begin{pmatrix} 0 & a-1 & 0 \\ 1-a & 0 & a-1 \\ 0 & 2-a & 0 \end{pmatrix}, \quad 14: \begin{pmatrix} 0 & a-1 & 0 \\ 1-a & 0 & 1-a \\ 0 & a-2 & 0 \end{pmatrix},$$

$$4: \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}, \quad 10: \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & a-1 \\ 1 & 2-a & 0 \end{pmatrix}, \quad 15: \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & a-1 \\ -1 & 2-a & 0 \end{pmatrix},$$

$$5: \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}, \quad 11: \begin{pmatrix} 0 & a-1 & 1-a \\ 1-a & 0 & 0 \\ a-2 & 0 & 0 \end{pmatrix}, \quad 16: \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 1-a \\ 1 & a-2 & 0 \end{pmatrix}.$$

$$6: \begin{pmatrix} 0 & a-1 & -a \\ 1-a & 0 & a \\ 1 & -1 & 0 \end{pmatrix},$$

A similar argument works in the case H_4. □

Remark 4.5. The following questions which might interesting to investigate in the future: What is a good notion of *cluster algebra* in this context? For example, what is a good choice of an ambient field? Does a sophisticated version of the *Laurent phenomenon* hold? Can we also define approximately periodic sequences for the noncrystallographic cluster algebras of type H_3 or H_4?

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