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On the approximate periodicity of sequences attached to noncrystallographic root systems

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Abstract

We study Fomin-Zelevinsky’s mutation rule in the context of noncrystallographic root systems. In particular, we construct approximately periodic sequences of real numbers for the noncrystallographic root systems of rank 2 by adjusting the exchange relation for cluster algebras. Moreover, we describe matrix mutation classes for type $H_3$ and $H_4$.

1 Introduction

Fomin and Zelevinsky have introduced cluster algebras in an impactful article [3, Definition 2.3]. In the last ten years diverse authors have found cluster algebra structures in various branches of mathematics such as representation theory, algebra and combinatorics. To define a cluster algebra, Fomin-Zelevinsky have defined seeds and mutations of seeds. Here, a seed (without frozen variables) is a pair $(\mathbf{x}, B)$ which consists of a cluster and a mutation matrix. The cluster $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a sequence of cluster variables and the mutation matrix $B$ is a skew-symmetrizable integer $n \times n$ matrix. Given an initial seed, the cluster algebra is now defined to be generated by all cluster variables in all seeds that are obtained from the initial seed by a sequence of mutations. The natural number $n$ is called the rank of the cluster algebra.

Some cluster algebras are of finite type and some cluster algebras are of infinite type. Here, we say a cluster algebra is of finite type if the mutation process yields only finitely many cluster variables. In another impactful article, Fomin-Zelevinsky [4, Theorem 1.4] have classified the cluster algebras of finite type via finite type root systems. The theorem implies that finite type cluster algebras (without frozen variables) are in bijection with Dynkin diagrams of type $A_n(n \geq 1)$, $B_n(n \geq 2)$, $C_n(n \geq 3)$, $D_n(n \geq 4)$, $E_n(n = 6,7,8)$, $F_4$, and $G_2$.

These Dynkin diagrams classify crystallographic root systems. In particular, such a diagram visualizes the Coxeter structure of the Weyl group of the corresponding root system. In the setup of cluster algebras, the crystallographic condition yields integer entries in the mutation matrix $B$. On the other hand, finite Coxeter groups are in bijection with Coxeter-Dynkin diagrams. Coxeter-Dynkin diagrams do not necessarily satisfy the crystallographic condition. Examples of noncrystallographic Coxeter groups are dihedral groups (with Coxeter-Dynkin diagram $I_2(m)$ with $m = 5$ or $m \geq 7$) and the symmetry group of the icosahedron (with Coxeter-Dynkin diagram $H_3$).

The aim of this note is to generalize Fomin-Zelevinsky’s matrix and seed mutation to noncrystallographic root systems. For every noncrystallographic root system of rank 2 the mutation class of the $B$-matrix contains two elements. Given two initial positive real numbers we define a sequence of real numbers by adjusting the exchange relations for cluster algebras to our setup. It turns out that the sequence is no longer a periodic sequence, but it is an almost periodic sequence meaning that it is approximately equal to a periodic sequence. For the noncrystallographic root
system of type $H_3$ the mutation class of the $B$-matrix is finite, but we do not observe the phenomenon of almost periodicity. The question of approximate periodicity in this setup has also been touched by Reading-Speyer, see Armstrong \[1, \text{Problem 6.4}].

2 Background

2.1 Fomin-Zelevinsky’s cluster algebras

In this section we wish to recall the definition of Fomin-Zelevinsky’s cluster algebras. We only consider coefficient-free cluster algebras without frozen variables over the field of rational numbers.

Let $n \geq 1$ be an integer and let $u_1, u_2, \ldots, u_n$ be algebraically independent variables over the field $Q$ of rational numbers. The field $\mathcal{F} = Q(u_1, u_2, \ldots, u_n)$ of rational functions is also called the ambient field. A cluster is a sequence $x = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ of algebraically independent elements. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called skew-symmetrizable if there exists a diagonal $n \times n$ matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with positive integer diagonal entries such that the matrix $DB$ is skew-symmetric, i.e. $d_i b_{ij} = -d_j b_{ij}$ for all $1 \leq i, j \leq n$. In this case, the diagonal matrix $D$ is called a skew-symmetrizer of $B$. A seed is a pair $(x, B)$ formed by a cluster $x$ and a skew-symmetrizable integer $n \times n$ matrix $B$. We denote the set of seeds by $S$.

Let $k \in \{1, 2, \ldots, n\}$ be a natural number. A mutation in direction $k$ is a map $\mu_k : S \to S, (x, B) \mapsto \mu_k(x, B) = (x', B')$, where $x' = (x'_1, x'_2, \ldots, x'_n)$ is the sequence that we obtain from the cluster $x$ by replacing the variable $x_k$ with

$$x'_k = \frac{1}{x_k} \left( \prod_{b_{ik} > 0} x_k^{b_{ik}} + \prod_{b_{ik} < 0} x_k^{-b_{ik}} \right) \in \mathcal{F},$$

and keeping all other cluster variables $x'_i = x_i$ with $i \neq k$, and $B' = (b'_{ij})$ is the $n \times n$ matrix with

$$b'_{ij} = \begin{cases} b_{ij}, & \text{if } k \in \{i, j\}; \\ b_{ij} + \frac{|b_{ik} b_{kj}|}{2}, & \text{otherwise.} \end{cases}$$

For every seed $(x, B) \in S$ the pair $\mu_k(x, B) = (x', B')$ is again seed, i.e. the elements of the sequence $x'$ are also algebraically independent over the field of rational numbers and the matrix $B'$ has integer entries and is skew-symmetrizable (with the same skew-symmetrizer $D$). Thus the map $\mu_k : S \to S$ is well-defined.

Fomin-Zelevinsky’s mutation of seeds has many remarkable properties. Firstly, for every index $k$ we have $\mu_k^2 = \text{id}_S$ so that the map $\mu_k$ is an involution. We declare two seeds $(x, B), (x', B') \in S$ to be mutation equivalent if there exists a sequence $(k_1, k_2, \ldots, k_r)$ of indices such that $(x, B) = (\mu_{k_1} \circ \mu_{k_2} \circ \ldots \circ \mu_{k_r})(x', B')$. In this case we write $(x, B) \simeq (x', B')$. It follows that $\simeq$ is an equivalence relation on the set of all seeds.

Suppose that $(x, B)$ is an initial seed. The cluster algebra $\mathcal{A}(x, B) \subseteq \mathcal{F}$ is the $Q$-subalgebra generated by all cluster variables $x^\pm_k$ in all seeds $(x', B')$ that are mutation equivalent to $(x, B)$. By construction we have $\mathcal{A}(x, B) \subseteq \mathcal{F}$. More generally, Fomin-Zelevinsky’s Laurent phenomenon \[3, \text{Theorem 3.1}] asserts that $\mathcal{A}(x, B) \subseteq Q[x_1^\pm, x_2^\pm, \ldots, x_n^\pm]$. More generally, every cluster variable is an element in the ring $\mathbb{Z}[x_1^\pm, x_2^\pm, \ldots, x_n^\pm]$.

2.2 Cluster algebras of rank 2

A skew-symmetrizable integer $2 \times 2$ matrix has the form $B = \pm \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}$ for some natural numbers $a, b \geq 1$. Note that the two possible choices of the sign yield isomorphic cluster algebras which we
that in these cases the constant term in the polynomial \( f(x) \) describes the mutation from the cluster. Fomin-Zelevinsky’s classification theorem implies that the cluster algebra \( A(a, b) \) is of finite type if and only if \( ab < 4 \). In these cases, the sequence \( (x_i)_{i \in \mathbb{Z}} \) is a periodic sequence. The period of the sequence is equal to 5, 6, or 8 when \((a, b)\) is equal to \((1, 1), (2, 1), \) or \((3, 1)\).

Assume that \( A(a, b) \) is of finite type and let \( i \in \mathbb{Z} \) be an integer. Due to the Laurent phenomenon we can write \( x_i = f(x_1, x_2)/(x_1^{a_1}x_2^{a_2}) \) for some polynomial \( f \in \mathbb{Z}[x_1, x_2] \). It is easy to see that in these cases the constant term in the polynomial \( f(x_1, x_2) \) is always equal to 1. Evaluation of the Laurent polynomial at the pair \((x_1, x_2)\) yields a function \( x_i: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \). Motivated from tropical geometry, we could approximate \( x_i \approx 1/(x_1^{a_1}x_2^{a_2}) \). In this paper we wish to introduce an approximation, which is more accurate than the tropical approximation and also works in a more general (noncrystallographic) setup where we do not have a Laurent phenomenon.

## 3 Approximately periodic sequences attached to noncrystallographic root systems of rank 2

### 3.1 The definition of the almost periodic sequences for type I

The Dynkin diagrams attached to the cluster algebras \( A(1, 1), A(2, 1) \) and \( A(3, 1) \) are \( A_2, B_2 \) and \( G_2 \), respectively. The corresponding Coxeter groups are the dihedral symmetry groups of the equilateral triangle, the square and the regular hexagon. More generally, the Coxeter-Dynkin diagram associated with the dihedral group of symmetries of the regular \( m \)-gon, for some \( m \geq 3 \), consists of two vertices that are joined by an edge of weight \( a = 4 \cos^2(\frac{\pi}{m}) \). Note that \( a \geq 1 \). Generalizing the classical construction, the two possible orientations of the diagram yield two possible mutation matrices \( B = \pm \left( \begin{array}{cc} 0 & a \\ -1 & 0 \end{array} \right) \). With this data we associate the following recursion. Let \( x = (x_1, x_2) \) be an initial cluster consisting of positive real numbers \( x_1 \) and \( x_2 \). Define a sequence \( (x_i)_{i \in \mathbb{Z}} \) of positive real numbers by

\[
x_{i-1}x_{i+1} = \begin{cases} 
  x_i^a + 1, & \text{if } i \text{ is even;} \\
  x_i^a + 1, & \text{if } i \text{ is odd.}
\end{cases}
\]

In contrast to the cases \( m = 3, 4, 6 \) the sequences are neither periodic nor do we notice the Laurent phenomenon. But we observe some approximate periodicity: in the case \( m = 5 \) (where we have \( a = 4 \cos^2(\frac{\pi}{5}) = \frac{1}{2}(3 + \sqrt{5}) \approx 2.618033988 \)) we have randomly chosen starting values \( x_1 = 0.829497 \) and \( x_2 = 0.363532 \) from the open interval \((0, 1)\), and computed the first few terms numerically, as the first two columns in Figure I illustrate. After 14 steps, we always get close to our starting values, e.g. \( x_{-5} \approx x_9 \) and \( x_{-4} \approx x_{10} \). The same phenomenon also occurs for other values of \( m \), and the number of steps is either \( m + 2 \) or \( 2(m + 2) \) depending on the parity of \( m \). The aim of this section is to explain this phenomenon. From now on we assume that \( m > 4 \), because in the other cases we have exact periodicity. Note that \( m > 4 \) implies \( a > 2 \).

### 3.2 A recursion formula for a subsequence

It is enough to look at every other term of the sequence \( (x_i)_{i \in \mathbb{Z}} \), because we can recover every term from the exchange relation (I) once we know its neighbors. To this end, let us define a sequence...
We define another sequence $(Y_i)_{i \in I}$. Moreover, we view the sequence $(Y_i)_{i \in I}$ as a numerical approximation of the sequence $(y_i)_{i \in \mathbb{Z}}$. The index set $I$ is equal to $I = \{-m, -m+1, \ldots, m, m+1\}$ if $m$ is even and to $I = \{-m+1, -m+2, \ldots, m-2, m-1\}$ if $m$ is odd. In both cases, we put $Y_1 = y_1$ and $Y_2 = y_2$, and define the other elements in the sequence recursively. We put:

\[
Y_0 = \frac{Y_2}{Y_1 Y_2 - 1}; \quad Y_{-1} = \frac{Y_0^{a-1}}{Y_0 Y_1 - 1}; \quad Y_2 = \frac{Y_0^{a-1}}{Y_0 Y_{-1} - 1};
\]

\[
Y_3 = \frac{Y_2^{a-1}}{Y_1 Y_2 - 1}; \quad Y_{i+1} = \frac{Y_i^{a-1}}{Y_{i-1} Y_i} = \frac{Y_i^{a-2}}{Y_{i-1}} \text{ for } 3 \leq i \leq \frac{m-1}{2}.
\]

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</table>

Figure 1: An example of an approximately periodic sequence $(x_n)$ with $m = 5$.

$(y_i)_{i \in \mathbb{Z}}$ of positive real numbers by putting $y_i = x_{2i}$ for all integers $i \in \mathbb{Z}$. As above, we can view every element $y_i = y_i(x_1, x_2)$ as a function $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ in the initial values $x_1, x_2$. We will see that the sequence $(y_i)_{i \in \mathbb{Z}}$ is almost periodic; this will imply that the original sequence $(x_i)_{i \in \mathbb{Z}}$ is also almost periodic. The following proposition shows that there is a self-contained recursion formula for the elements of the sequence $(y_i)_{i \in \mathbb{Z}}$.

**Proposition 3.1.** Let $i \in \mathbb{Z}$ be an integer. Then the elements $y_{i-1}$ $y_i$ and $y_{i+1}$ satisfy the equation $y_{i-1}y_i y_{i+1} = y_{i-1} + y_{i+1} + y_i^{-1}$.

**Proof.** Let $i \in \mathbb{Z}$ be an integer. By construction we have $y_{i-1} = x_{2i-2}$, $y_i = x_{2i}$ and $y_{i+1} = x_{2i+2}$. Note that $y_{i-1}y_i - 1 = x_{2i-1}$ and $y_i y_{i+1} - 1 = x_{2i+1}$. The relation $(y_{i-1}y_i - 1)(y_i y_{i+1} - 1) = y_i^2 + 1$ yields $y_{i-1}y_i y_{i+1} = y_{i-1} + y_{i+1} + y_i^{-1}$. □

### 3.3 An approximation of the sequence

We define another sequence $(Y_i)_{i \in I}$. Moreover, we view the sequence $(Y_i)_{i \in I}$ as a numerical approximation of the sequence $(y_i)_{i \in \mathbb{Z}}$. The index set $I$ is equal to $I = \{-2, -1, 0, 1, \ldots, \frac{m}{2}\}$ if $m$ is even and to $I = \{-\frac{m+1}{2}, -\frac{m-1}{2}, \ldots, \frac{m-3}{2}, \frac{m-1}{2}\}$ if $m$ is odd. In both cases, we put $Y_1 = y_1$ and $Y_2 = y_2$, and define the other elements in the sequence recursively. We put:

\[
Y_0 = \frac{Y_2}{Y_1 Y_2 - 1}; \quad Y_{-1} = \frac{Y_0^{a-1}}{Y_0 Y_1 - 1}; \quad Y_2 = \frac{Y_0^{a-1}}{Y_0 Y_{-1} - 1};
\]

\[
Y_3 = \frac{Y_2^{a-1}}{Y_1 Y_2 - 1}; \quad Y_{i+1} = \frac{Y_i^{a-1}}{Y_{i-1} Y_i} = \frac{Y_i^{a-2}}{Y_{i-1}} \text{ for } 3 \leq i \leq \frac{m-1}{2}.
\]
for all \( m \). If \( m \) is odd, then we define the missing elements in the sequence as follows:

\[
Y_{(m+3)}/2 = \frac{Y_{i}^{m+1}/2 + Y_{(m+1)/2}}{Y_{(m+1)/2}Y_{(m+1)/2}} ,
\]

\[
Y_{i} = \frac{Y_{i}^{m+1}/2}{Y_{i+1}/Y_{i}} = \frac{Y_{i}^{m+2}/2}{Y_{i+1}/Y_{i}} \quad \text{for} \quad \frac{m-3}{2} \leq i \leq -2 ,
\]

\[
Y_{(m+5)/2} = \frac{Y_{(m+1)/2} + Y_{(m+1)/2}}{Y_{(m+3)/2}Y_{(m+1)/2} - 1} ,
\]

\[
Y_{(m+1)/2} = \frac{Y_{(m+1)/2}}{Y_{(m+3)/2}Y_{(m+1)/2} - 1} .
\]

Using the relations \( Y_{1} = x_{2} \) and \( Y_{2} = x_{4} = \frac{1+\sqrt{4x_{1}^{2}}}{2} \), we can view every element \( Y_{i} = Y_{i}(x_{1}, x_{2}) \) as a real-valued function in the initial variables \( x_{1}, x_{2} \) as above. Note that \( x_{3} = Y_{1}x_{2} - 1 \). Moreover, \( (Y_{0}Y_{1} - 1)(Y_{1}Y_{2} - 1) = 1 \) implies \( Y_{0}Y_{1} - 1 = x_{3}^{1} \). Thus we have \( Y_{2} = Y_{0}x_{3} \).

Next, we will give an explicit formula for the elements of the sequence \((Y_{i})_{i \in \mathbb{Z}}\). To do so, we define a sequence \((g_{i})_{i \geq 0}\) of integers by \( g_{0} = 0, g_{1} = 1 \) and \( g_{i+1} = (a-2)g_{i} - g_{i-1} \) for \( i \geq 1 \). The following proposition relates the two sequences.

**Proposition 3.2.** Let \( i \in \mathbb{I} \). If \( 0 \leq i \leq \frac{m-3}{2} \), then we have \( Y_{i+2} = Y_{2}^{i+1/\sqrt{X_{3}}}, x_{3}^{-i} = Y_{0}^{i+1/\sqrt{X_{3}}} \). Moreover, if \( 0 \leq i \leq \frac{m-1}{2} \), then we have \( Y_{i} - Y_{2}^{i+1/\sqrt{X_{3}}}x_{3}^{-i} = Y_{0}^{i+1/\sqrt{X_{3}}} \).

**Proof.** The relation \( Y_{2} = Y_{0}x_{3} \) implies \( Y_{2}^{i+1/\sqrt{X_{3}}}x_{3}^{-i} = Y_{0}^{i+1/\sqrt{X_{3}}}x_{3}^{-i} \) and \( Y_{2}^{i+1/\sqrt{X_{3}}}x_{3}^{-i} = Y_{0}^{i+1/\sqrt{X_{3}}}x_{3}^{-i} \) for all \( i \). Trivially, we have \( Y_{2} = Y_{2}(x_{3}) \) and \( Y_{0} = Y_{0}(x_{3}) \) so that the formulae hold true for \( i = 0 \). By definition, we have \( Y_{3} = Y_{2}^{1/\sqrt{X_{3}}}x_{3}^{2} \) and \( Y_{3} - Y_{2}^{1/\sqrt{X_{3}}}x_{3}^{2} = Y_{0}^{1/\sqrt{X_{3}}}x_{3}^{2} \) so that the formulae hold true for \( i = 1 \). The general case follows from the definition of the sequence \((Y_{i})_{i \in \mathbb{Z}}\) by mathematical induction.

Proposition 3.2 asserts that the sequence \((g_{i})_{i \in \mathbb{Z}}\) controls the sequence \((Y_{i})_{i \in \mathbb{Z}}\). The following Proposition states the main features of this sequence. We denote by \( \omega = \exp\left(\frac{2\pi i}{m}\right) \subset \mathbb{C} \) the root of unity and by \( \overline{\omega} \subset \mathbb{C} \) its complex conjugate.

**Proposition 3.3.** The sequence \((g_{i})_{i \geq 0}\) is periodic. The period is equal to \( m \), if \( m \) is odd, and equal to \( \frac{m}{2} \), if \( m \) is even. Moreover, the following formula holds true for all natural numbers \( i \geq 0 \):

\[
g_{i} = \frac{\omega^{i} - \overline{\omega}^{i}}{\omega - \overline{\omega}} .
\]

**Proof.** Note that the sequence \((g_{i})_{i \in \mathbb{Z}}\) is a homogeneous linear recurrence relation with characteristic polynomial \( X^{2} - (a-2)X + 1 \). By elementary trigonometry we have \( a - 2 = 4\cos^{2}\left(\frac{\pi}{m}\right) - 2 = 2\cos\left(\frac{\pi}{m}\right) \) so that the characteristic polynomial splits as \( (X - \omega)(X - \overline{\omega}) \). Therefore, the sequence \((g_{i})_{i \in \mathbb{Z}}\) is a \( \mathbb{C} \)-linear combination of the sequences \((\omega^{i})_{i \in \mathbb{N}}\) and \((\overline{\omega}^{i})_{i \in \mathbb{N}}\). A comparison of coefficients for the initial values \( g_{0} \) and \( g_{1} \) yields equation (2).

Note that the previous proposition implies that \( g_{i} \) is positive if \( 1 \leq i \leq \frac{m-1}{2} \). Similarly, \( g_{i} \) is negative if \( -\frac{m-1}{2} \leq i \leq -1 \). Moreover \( |g_{i}| \geq 1 \) unless \( i \in \left\{ \frac{m}{2}, \frac{0, m}{2}\right\} \). The explicit formula implies that the following terms in the sequence \((Y_{i})_{i \in \mathbb{Z}}\) are equal.

**Theorem 3.4.** If \( m \) is even, then the equations \( Y_{-2} = Y_{(m-2)/2} \) and \( Y_{-1} = Y_{m/2} \) hold. If \( m \) is odd, then the equations \( Y_{-(m+1)/2} = Y_{(m+3)/2} \) and \( Y_{-(m-1)/2} = Y_{(m+5)/2} \) hold.

**Proof.** Let \( m \) be even. It is easy to see that \( g_{\frac{m}{2}} = 0, g_{\frac{m}{2} - 1} = 1 \) and \( g_{\frac{m}{2} - 2} = a - 2 \), so that \( Y_{\frac{m}{2}} = Y_{\frac{m}{2}}^{a-1}x_{3}^{d-4} \), which agrees with \( Y_{-1} = Y_{0}^{a-1}x_{3} = (Y_{2}x_{3}^{1})^{a-1}x_{3} = Y_{2}x_{3}^{d-4} \). Moreover, it follows that \( Y_{\frac{m}{2} - 1} = Y_{2}^{a-2}Y_{2}^{a-1}x_{3} = Y_{-2}Y_{2}^{a-2}x_{3} \), which agrees with \( Y_{-2} = Y_{2}^{a-2}Y_{0}^{a-1} \).
Now let \( m \) be odd. Let us put \( g = g_{(m-1)/2} \). Due to Proposition 3.2 we have \( g_{(m+1)/2} = -g \), from which we conclude \( g_{(m-3)/2} = (a - 1)g \) and \( g_{(m+3)/2} = -(a - 1)g \). Furthermore, the recursion implies \( g_{(m-5)/2} = (a^2 - 3a + 1)g \). Proposition 3.2 yields

\[
Y_{-(m-3)/2} = \frac{(a-1)g + g_{x_3}^a (a-1)g}{Y_0 x_3^a}, \\
Y_{-(m-1)/2} = \frac{g_{x_3}^a}{Y_0 x_3^a} = x_3^g.
\]

Using these expressions, we can write the next element of the sequence as

\[
Y_{-(m+1)/2} = \frac{(a-1)g + Y_0 x_3^a (a-1)g}{Y_0 x_3^a} = x_3^{-g} (1 + Y_0^{-ag}).
\]

On the other hand, Proposition 3.2 yields

\[
Y_{(m-1)/2} = \frac{(a^2 - 3a + 1)g + (a-1)g_{x_3}^a (a-1)g}{Y_0 x_3^a}, \\
Y_{(m+1)/2} = \frac{g_{x_3}^a}{Y_0 x_3^a} = Y_0 x_3^a.
\]

Using these expressions, we can write the next elements of the sequence as

\[
Y_{(m+3)/2} = \frac{\frac{Y_0 x_3^a}{(1 + Y_0^{-ag})}}{Y_0 x_3^a} = x_3^{-g} (1 + Y_0^{-ag}),
\]

\[
Y_{(m+5)/2} = \frac{Y_0 x_3^a}{(1 + Y_0^{-ag}) - 1} = x_3^g.
\]

The expressions agree with the expressions that we obtain for \( Y_{-(m+1)/2} \) and \( Y_{-(m-1)/2} \), and so the statement follows.

3.4 Numerical comparison of the two sequences

Theorem 3.5. Let \( i \in I \). The element \( Y_i = Y_i(x_1, x_2) \) is an approximation of \( y_i = y_i(x_1, x_2) \) with relative error

\[
\left| \frac{Y_i - y_i}{y_i} \right| = \left| \frac{Y_i(x_1, x_2) - y_i(x_1, x_2)}{y_i(x_1, x_2)} \right| = O(x_1 x_2) \text{ for } x_1, x_2 \to 0.
\]

Before we prove the theorem, we state a lemma. For proofs of the lemma and the theorem recall that our assumption \( m > 4 \) implies \( a > 2 \). Moreover, note that for \( x_1, x_2 \in (0, 1) \) we have

\[
x_3 = \frac{1 + x_1^2}{x_1} > x_1^{-1} > 1, \quad Y_2 = \frac{1 + x_1 + x_2}{x_2(1 + x_2)} > x_1^{-1} x_2^{-1} > 1, \\
Y_2/x_3 = \frac{1 + x_1 + x_2}{x_2(1 + x_2)} > x_1^{-1} x_2^{-1} > 1.
\]

Lemma 3.6. Suppose that \( i \in I \) indexes some element of the sequence \( (Y_i)_{i \in I} \).

(a) Suppose that \( m \) is even and \( i \in \{0\} \cup \{3, 4, \ldots, \frac{m-2}{2} \} \) or \( m \) is odd and \( i \in \{0, -1, \ldots, -\frac{m-3}{2} \} \cup \{3, 4, \ldots, \frac{m+3}{2} \} \). Then we may write the quotient \( (Y_i Y_{i-1})/(Y_i Y_{i-1}) \) as \( 1 - e_i \) for some the real-valued function \( e_i = e_i(x_1, x_2) \) with \( |e_i(x_1, x_2)| = O(x_1 x_2) \).

(b) If \( 3 \leq i \leq \frac{m+1}{3} \), then we may write the quotient \( (Y_{i-1} + Y_{i-1})/(Y_{i-1} Y_{i-1}) \) as \( 1 + e_i' \) for some real-valued function \( e_i' = e_i'(x_1, x_2) \) with \( |e_i'(x_1, x_2)| = O(x_1 x_2) \). If \( -\frac{m-3}{2} \leq i \leq 0 \), then we may write the quotient \( (Y_{i-1} + Y_{i+1})/(Y_{i-1} Y_{i-1}) \) as \( 1 + e_i'' \) for some real-valued function \( e_i'' = e_i''(x_1, x_2) \) with \( |e_i''(x_1, x_2)| = O(x_1 x_2) \). Finally, if \( m \) is odd, then we may write the quotient \( (Y_{(m-3)/2} + Y_{(m+1)/2})/Y_{(m+1)/2} \) as \( 1 + e_i''' \) for some real-valued function \( e_i''' = e_i'''(x_1, x_2) \) with \( |e_i'''(x_1, x_2)| = O(x_1 x_2) \).
Proof of the lemma. (a) By definition we have $e_i(x_1, x_2) = (Y_i Y_{i-1})^{-1}$. If $3 \leq i \leq \frac{m+1}{2}$, then Proposition 3.2 implies

$$e_i(x_1, x_2) = (Y_i Y_{i-1})^{-1} = (Y_2 / x_3)^{-g_{i-2} - g_{i-3}} Y_2^{-g_{i-1} - g_{i-2}}.$$ 

Thus, it is enough to show that $g_{i-2} + g_{i-3}$ is nonnegative and $g_{i-1} + g_{i-2}$ is at least 1 for $3 \leq i \leq \frac{m+1}{2}$, which follows from the explicit formula in Proposition 3.3. If $m$ is odd and $i = \frac{m+3}{2}$, then Proposition 3.2 and formula 5 imply $Y_i Y_{i-1} = 1 + (Y_2 / x_3)^{g_i} > (Y_2 / x_3)^{g_i}$ from which we conclude with the statement of the lemma by similar arguments as above. If $i = \frac{m-3}{2} \leq 0$, then Proposition 3.2 implies

$$e_i(x_1, x_2) = (Y_i Y_{i-1})^{-1} = (Y_2 / x_3)^{-g_{i+1} - g_{i+2}} Y_2^{-g_{i+1} - g_{i+2}}.$$ 

Thus, it is enough to show that $g_{i+1} + g_{i+2}$ is nonnegative and $g_i + g_{i+1}$ is at least 1 for $0 \leq i \leq \frac{m-3}{2}$, which follows from the explicit formula in Proposition 3.3. The cases $e''_i$ and $e'''_i$ are proved similarly.

(b) By definition we have $e'_i(x_1, x_2) = Y_{i-1} / Y_i^{a-1}$ for all $i$ for which the function is defined. Proposition 3.2 yields $e'_i(x_1, x_2) = (Y_2 / x_3)^{g_{i-3} - (a-1)g_{i-2} Y_2^{-g_{i-2} - (a-1)g_{i-1}}} = (Y_2 / x_3)^{-g_{i-2} - g_{i-1} Y_2^{-g_{i-1}}}$. Thus, it is enough to show that $g_{i-2} + g_{i-1}$ is greater than 1 for $3 \leq i \leq \frac{m-1}{2}$, which follows from the explicit formula in Proposition 3.3. The cases $e''_i$ and $e'''_i$ are proved similarly.

With the preparation we are ready to prove the theorem:

Proof of the theorem. The statement is true for $i \in \{1, 2\}$ because $Y_1 = y_1$ and $Y_2 = y_2$ by definition. For $i \in \{3, 0\}$ we can estimate the relative errors as $x_1, x_2 \to 0$:

$$\left| \frac{y_3 - Y_3}{y_3} \right| = \left| \frac{y_1}{y_1 + y_2^{a-1}} \right| < \left| \frac{1}{1 + y_1^{a-1} y_2} \right| < \left| \frac{1}{1 + x_1^{a-1} x_2} \right| \leq O(x_1 x_2),$$

$$\left| \frac{y_0 - Y_0}{y_0} \right| = \left| \frac{y_1^{a-1}}{y_1^{a-1} + x_2} \right| < \left| \frac{1}{1 + y_1^{a-1} y_2} \right| < \left| \frac{1}{1 + x_1^{a-1} x_2} \right| \leq O(x_1 x_2).$$

For the other values of $i$ we prove the theorem by induction on the absolute value of $i$. Assume that $i \geq 3$ and that the statement is true for $i$ and $i - 1$. We put $Y_i / y_i = 1 + \delta_i$ and $Y_{i-1} / y_{i-1} = 1 + \delta_{i-1}$. By induction hypothesis $|\delta_i| = |\delta_i(x_1, x_2)|$. Hence we may write the quotient $Y_i / y_i = 1 + \delta_i$ for some real-valued function $\delta_i(x_1, x_2)$ with $|\delta_i(x_1, x_2)| = O(x_1 x_2)$. Similarly, by the induction hypothesis and the previous lemma we may write the quotient $Y_{i-1} / y_{i-1} = 1 + \delta_{i-1}$ for some real-valued function $\delta_{i-1}(x_1, x_2)$ with $|\delta_{i-1}(x_1, x_2)| = O(x_1 x_2)$. We conclude that the quotient $Y_{i+1} / y_{i+1} = 1 + \delta_i$ for some real-valued function $\delta_i(x_1, x_2)$ with $|\delta_i(x_1, x_2)| = O(x_1 x_2)$.

From the previous lemma we can conclude that we may write the quotient $Y_{i+1} / y_{i+1} = 1 + \tilde{e}_i$ for some real-valued function $\tilde{e}_i(x_1, x_2)$ with $|\tilde{e}_i(x_1, x_2)| = O(x_1 x_2)$. The theorem follows because $Y_i / y_i = (Y_i / \tilde{Y}_i) \cdot (\tilde{Y}_i / y_i) = (1 + \tilde{e}_i)(1 + \delta''')$ with $\tilde{e}_i + \delta''' + \tilde{e}_i \delta''' = O(x_1 x_2)$.

The case $i \leq 0$ is proved similarly.
Thus, the sequence $(y_i)_{i \in I}$ is approximately equal to the sequence $(Y_i)_{i \in I}$ which can be extended to a period sequence $(Y_i)_{i \in \mathbb{Z}}$ in view of Theorem 3.4. Hence, the original sequence $(x_i)_{i \in \mathbb{Z}}$ is approximately equal to a periodic sequence.

4 Matrix mutation with real entries

The matrix mutation rule generalizes to real entries. As before, an $n \times n$ matrix $B = (b_{ij})$ with real entries is called skew-symmetrizable if there exists a diagonal $n \times n$ matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with positive real diagonal entries such that the matrix $DB$ is skew-symmetric, i.e. $d_ib_{ij} = -d_ib_{ji}$ for all $1 \leq i, j \leq n$. Let $B$ be a real skew-symmetrizable $n \times n$ matrix and $k \in \{1, 2, \ldots, n\}$ an index. We define the mutation of $B$ at $k$ to be the $n \times n$ matrix $B' = (b'_{ij})$ with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } k \in \{i, j\}; \\ b_{ij} + \frac{b_{ik}b_{kj} + b_{jk}b_{ki}}{2}, & \text{otherwise.} \end{cases}$$

As before, we denote the mutation also by $B' = \mu_k(B)$. The following proposition is immediate.

**Proposition 4.1.** Let $B$ be a real skew-symmetrizable $n \times n$ matrix with skew-symmetrizer $D$ and let $k \in \{1, 2, \ldots, n\}$. The matrix $\mu_k(B)$ is again skew-symmetrizable with skew-symmetrizer $D$. Moreover, $\mu_k^2(B) = B$.

As before, we define mutation equivalence to be the smallest equivalence relation on the set of skew-symmetrizable real $n \times n$ matrices such that $\mu_k(B) \simeq B$ for all $B$ and all $k$.

**Remark 4.2.** For mutation classes of integer matrices we have several structural results. There is classification of mutation-finite, skew-symmetrizable, integer matrices by Felikson-Shapiro-Tumarkin [2]. Effective criteria to test whether a given skew-symmetric matrix is mutation-finite are due to Lawson [5] (via minimal mutation-infinite subquivers) and Warkentin [6] (via forks). On the other hand, structural results for classification of mutation-finite, skew-symmetrizable, real matrices seem to be harder, because it is easy to construct finite mutation classes, as the following example shows.

**Example 4.3.** Let $a, b, c, a', b', c'$ be positive real numbers. Consider the matrix

$$B = \begin{pmatrix} 0 & a & -c' \\ -a' & 0 & b \\ c & -b' & 0 \end{pmatrix}.$$  

Then $B$ is skew symmetric if and only if $abc = a'b'c'$. In particular, let $a, b, c \in \mathbb{R}^+$ such that $abc = 8$. Put $a' = \frac{bc}{a}, b' = \frac{ca}{b}$ and $c' = \frac{ab}{c}$. Then $B$ is skew symmetric and $\mu_1(B) = \mu_2(B) = \mu_3(B) = B$. Hence, $B$ is mutation finite.

The Coxeter group of type $H_3$ is the symmetry group of the regular icosahedron. Besides the Coxeter group of type $H_4$, it is the only noncrystallographic Coxeter group whose rank is greater than 2. The next lemma shows that the corresponding mutation matrices are mutation finite.

**Lemma 4.4.** Let $a = 4 \cos^2(\frac{\pi}{3})$. The matrices

$$B' = \begin{pmatrix} 0 & a & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & a & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

of type $H_3$ and $H_4$ are mutation finite. If we identify matrices that are obtained from each other by a simultaneous row and column permutation, then the mutation classes have sizes 16 and 82.
Figure 2: The mutation class for $H_3$

**Proof.** A calculation shows that the following set of 16 matrices is closed under mutation. Mutations are visualized in the picture. The set contains the matrix $B'$.

$\begin{align*}
1: \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & a \\ 0 & -1 & 0 \end{pmatrix}, & \quad 7: \begin{pmatrix} 0 & 1-a & a \\ a-1 & 0 & 1-a \\ -1 & a-2 & 0 \end{pmatrix}, & \quad 12: \begin{pmatrix} 0 & 0 & 1-a \\ 0 & 0 & a \\ a-2 & -1 & 0 \end{pmatrix}, \\
2: \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 0 \end{pmatrix}, & \quad 8: \begin{pmatrix} 0 & 1-a & a-1 \\ a-1 & 0 & -a \\ 2-a & 1 & 0 \end{pmatrix}, & \quad 13: \begin{pmatrix} 0 & 1-a & 0 \\ a-1 & 0 & a-1 \\ 0 & 2-a & 0 \end{pmatrix}, \\
3: \begin{pmatrix} 0 & -1 & a \\ 1 & 0 & -a \\ -1 & 1 & 0 \end{pmatrix}, & \quad 9: \begin{pmatrix} 0 & a-1 & 0 \\ 1-a & 0 & a-1 \\ 0 & 2-a & 0 \end{pmatrix}, & \quad 14: \begin{pmatrix} 0 & a-1 & 0 \\ 1-a & 0 & 1-a \\ 0 & a-2 & 0 \end{pmatrix}, \\
4: \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}, & \quad 10: \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & a-1 \\ 1-2-a & 0 \end{pmatrix}, & \quad 15: \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a-1 \\ -1 & 2-a & 0 \end{pmatrix}, \\
5: \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -a \\ 0 & 1 & 0 \end{pmatrix}, & \quad 11: \begin{pmatrix} 0 & a-1 & 1-a \\ 1-a & 0 & 0 \\ a-2 & 0 & 0 \end{pmatrix}, & \quad 16: \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 1-a \\ 1 & a-2 & 0 \end{pmatrix}.
\end{align*}$

A similar argument works in the case $H_4$.

**Remark 4.5.** The following questions which might interesting to investigate in the future: What is a good notion of *cluster algebra* in this context? For example, what is a good choice of an am-
bient field? Does a sophisticated version of the Laurent phenomenon hold? Can we also define approximately periodic sequences for the noncrystallographic cluster algebras of type $H_3$ or $H_4$?

References


