Remarks on certain two-component systems with peakon solutions

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June 11, 2019

Abstract

We consider a Lax pair found by Xia, Qiao and Zhou for a family of two-component analogues of the Camassa-Holm equation, including an arbitrary function H, and show that this apparent freedom can be removed via a combination of a reciprocal transformation and a gauge transformation, which reduces the system to triangular form. The resulting triangular system may or may not be integrable, depending on the choice of H. In addition, we apply the formal series approach of Dubrovin and Zhang to show that scalar equations of Camassa-Holm type with homogeneous nonlinear terms of degree greater than three are not integrable.

This article is dedicated to Darryl Holm on his 70th birthday.

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1 Introduction

The partial differential equation

$$m_t + um_x + 2u_x m = 0, \qquad m = u - u_{xx} \tag{1}$$

was derived from asymptotic expansions in shallow water theory by Camassa and Holm [6], who also obtained a bi-Hamiltonian structure and found remarkable weak solutions in the form of a superposition of peaks (peakons),

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\, e^{-|x-q_j(t)|}, \tag{2}$$

where $(q_j, p_j)_{j=1,\ldots,N}$ form a set of canonical coordinates and momenta in a finite-dimensional Hamiltonian system with time t that is completely integrable in the Liouville sense. Although it is a nonlocal partial differential equation, either in the form (1) in terms of m with $u = (1 - D_x^2)^{-1}\, m$, or rewritten as an evolution equation for u, i.e. $u_t = \ldots$ (cf. equation (5) below), the Camassa-Holm equation has an infinite hierarchy of commuting symmetries which are given by local evolution equations in m.

The integrability of the equation (1) itself was already included in earlier results of Fokas and Fuchssteiner on hereditary symmetries and recursion operators [16], but the work of Camassa and Holm led to new analytical and geometrical insights: in addition to the peakons given by (2), and smooth solitons [22, 25, 33] that appear when linear dispersion is added to (1), other classes of initial data produce wave breaking [26]; and the equation has a variational formulation as a geodesic flow on (an extension of) a diffeomorphism group [30]. The geometrical interpretation of (1) as an Euler-Poincaré equation naturally generalizes to diffeomorphisms in two or more dimensions, and the analogues of the weak solutions (2) can be applied to the problem of template matching in computational anatomy [20]; but in general such higher-dimensional extensions do not to preserve integrability. Further research on the one-dimensional case has been concerned with the derivation [17, 32] and classification [31] of integrable scalar equations analogous to (1), as well as the search for suitable two-component or multi-component analogues [9, 15, 21, 24, 37, 36, 38, 39]. From the analytical point of view, there is also considerable interest in finding dispersive equations with higher order nonlinearity, which (despite not being integrable) display similar features in the form of peakons, wave breaking and one or more higher conservation laws [1, 19].

Recently Xia, Qiao and Zhou introduced the following two-component system of partial differential equations:

$$\begin{aligned} m_t &= (mH)_x + mH - \tfrac{1}{2}m(u-u_x)(v+v_x), \\ n_t &= (nH)_x - nH + \tfrac{1}{2}n(u-u_x)(v+v_x), \end{aligned} \tag{3}$$

with

$$m = u - u_{xx}, \qquad n = v - v_{xx}. \tag{4}$$

In the above, H is an arbitrary function of x and t, which (in particular) can be fixed by choosing it to be a specific function of the fields u, v and their derivatives. The authors of [39] refer to this system in the title of their paper as "synthetical" because it provides a synthesis of several different systems admitting peakon solutions, by choosing H to have a specific dependence on u, v; reductions to integrable scalar partial differential equations can be achieved by imposing further conditions on u and v. For instance, setting $H = u$ and $v = 2$ reduces (3) to the Camassa–Holm equation [6], which can be rewritten as

$$(1 - D_x^2)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}; \tag{5}$$

while setting $H = u^2 - u_x^2$ and $v = 2u$ produces the equation

$$(1 - D_x^2)u_t = D_x\left(u_x^2u_{xx} - u^2u_{xx} - uu_x^2 + u^3\right), \tag{6}$$

which was first derived by Fokas [17], then by Olver and Rosenau [32], and has been studied more recently by Qiao [34].

Coupled systems of Camassa-Holm type (both integrable and non-integrable equations) are very interesting because they exhibit new behaviour: for instance, there are waltzing peakons [10], and the scattering of two peakons need not give rise to a simple phase shift as in the scalar case [4]. Other examples include the two-component system

$$\begin{aligned} m_t &= \tfrac{1}{2}D_x\Big(m(u-u_x)(v+v_x)\Big), \\ n_t &= \tfrac{1}{2}D_x\Big(n(u-u_x)(v+v_x)\Big) \end{aligned} \tag{7}$$

introduced in [36], whose multipeakon solutions were recently analyzed in [8], which arises from (3) by taking

$$H = \frac{1}{2}(u-u_x)(v+v_x); \tag{8}$$

or the system obtained from the choice

$$H = \frac{1}{2}(uv - u_x v_x), \tag{9}$$

that is

$$\begin{aligned} m_t &= \tfrac{1}{2} D_x\Big(m(uv - u_x v_x)\Big) - \tfrac{1}{2}\Big(m(uv_x - u_x v)\Big), \\ n_t &= \tfrac{1}{2} D_x\Big(n(uv - u_x v_x)\Big) + \tfrac{1}{2}\Big(n(uv_x - u_x v)\Big), \end{aligned} \tag{10}$$

which was studied in [38].

The adjective "synthetical" in the title of the paper [39] is rarely used, but the English word "synthetic" is much more common, and it is synonymous with "artificial" or "fake" in everyday language. Although the system (3) arises as the compatibility condition of a linear system (Lax pair), which yields an infinite sequence of conservation laws, we will show that this is not sufficient for this two-component system (or all its reductions) to be integrable. In fact, the linear system obtained in [39] should be considered as an example of a fake Lax pair (see [7] or [5, 35]): by a combination of a change of independent variables (reciprocal transformation) and a gauge transformation, the function H can be removed, and the system decouples into an integrable scalar equation together with an arbitrary evolution equation (which is generically non-integrable). As the consequence, the infinite sequence of conservation laws only depend on a single dependent variable (the variable ϑ below). Thus it turns out that it is appropriate to apply the word "synthetic" in this context.

Our main result can be summarized as follows.

Theorem 1. *Let (3) be specified as an autonomous system of partial differential equations for $u = u(x,t)$ and $v = v(x,t)$, by making a particular choice of function $H = h(u, v, u_x, v_x, \ldots)$ of u, v and their x-derivatives. Then there is a reciprocal transformation to a triangular system for $\vartheta = \vartheta(X,T)$ and $\kappa = \kappa(X,T)$, given by*

$$\begin{aligned} \vartheta_T + \tfrac{1}{4}\left(\tfrac{(\vartheta_{XT}-2)^2}{4\vartheta^2} - \vartheta_T^2 \right)_X &= 0, \\ \kappa_T + \kappa\, \mathcal{F}[\vartheta, \kappa] &= 0, \end{aligned} \tag{11}$$

where $\mathcal{F}[\vartheta, \kappa]$ denotes a (possibly nonlocal) function of ϑ, κ and their X-derivatives.

As we shall see, on its own the first equation for ϑ in the system (11) is integrable: it corresponds to a negative flow in the modified KdV hierarchy; but in general the second equation is not integrable, for an arbitrary choice of $\mathcal{F}$ (which corresponds to the arbitrariness of H). In the next section we will derive the above result, explaining how the dependence of the system (3) on H can be removed from the Lax pair.

Section 3 is concerned with a different question, namely the degree of nonlinearity that appears in integrable peakon equations. Using the approach of Dubrovin and Zhang, which is based on writing equations as series that are perturbations of the dispersionless limit [11, 12, 13], we present a theorem to the effect that there are no integrable homogeneous scalar peakon equations with nonlinearity of degree greater than three. This result should be sufficient to infer that all integrable multi-component analogues of the Camassa-Holm can have only quadratic or cubic nonlinear terms, since such systems reduce to the scalar case by identifying fields or by setting all but one of the fields to zero. The paper ends with a brief discussion of the results.

2 Lax pair and reciprocal transformation

For what follows it will be convenient to rescale the dependent variables u, v in (3) so that

$$u \to 2u, \qquad v \to -2v,$$

which implies that $m \to 2m$, $n \to -2n$, and introduce the quantities

$$A = u - u_x, \qquad B = v + v_x, \tag{12}$$

so that the system takes the form

$$\begin{aligned} m_t &= (mH)_x + mH + 2mAB, \\ n_t &= (nH)_x - nH - 2nAB, \end{aligned} \tag{13}$$

with

$$m = A + A_x, \qquad n = B - B_x. \tag{14}$$

With this choice of scaling, the Lax pair presented in (10) can be rewritten as

$$\Psi_x = \mathbf{U}\Psi, \qquad \Psi_t = \mathbf{V}\Psi, \tag{15}$$

where

$$\mathbf{U} = \begin{pmatrix} -\frac{1}{2} & m\lambda \\ n\lambda & \frac{1}{2} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} -\frac{1}{2}\lambda^{-2} + AB & A\lambda^{-1} + mH\lambda \\ B\lambda^{-1} + nH\lambda & \frac{1}{2}\lambda^{-2} - AB \end{pmatrix}.$$

By a standard method, transforming the x part of (15) into a Riccati equation and making an asymptotic expansion of the Riccati potential in powers of λ, it was shown in [39] that the system (3) has infinitely many conservation laws. With the choice of scaling as in (13), the first of these is

$$q_t = (qH)_x, \qquad q = \sqrt{mn}. \tag{16}$$

Using the latter, we transform the independent variables via the reciprocal transformation

$$\mathrm{d}X = q\,\mathrm{d}x + qH\,\mathrm{d}t, \qquad \mathrm{d}T = \mathrm{d}t, \tag{17}$$

so that partial derivatives transform as $D_x = q\,D_X$, $D_t = D_T + qH\,D_X$. It is then helpful to replace m, n throughout by q, κ, where

$$\kappa = \sqrt{\frac{n}{m}},$$

so that the system (13) becomes

$$\begin{array}{rcl} (q^{-1})_T + H_X & = & 0, \\ (\log\kappa)_T + H + 2AB & = & 0, \end{array} \tag{18}$$

and (14) produces

$$A_X = -Aq^{-1} + \kappa^{-1}, \qquad B_X = Bq^{-1} - \kappa. \tag{19}$$

The reciprocal transformation can also be applied to the Lax pair (15), to yield

$$\Psi_X = \hat{\mathbf{U}}\Psi, \qquad \Psi_T = \hat{\mathbf{V}}\Psi, \tag{20}$$

where

$$\hat{\mathbf{U}} = \begin{pmatrix} -\frac{1}{2}q^{-1} & \kappa^{-1}\lambda \\ \kappa\lambda & \frac{1}{2}q^{-1} \end{pmatrix}, \; \hat{\mathbf{V}} = \begin{pmatrix} -\frac{1}{2}\lambda^{-2} + AB + \frac{1}{2}H & A\lambda^{-1} \\ B\lambda^{-1} & \frac{1}{2}\lambda^{-2} - AB - \frac{1}{2}H \end{pmatrix}.$$

The compatibility conditions for (20), coming from the zero curvature equation $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$, are precisely the equations (18) and (19).

We now explain how these equations can be decoupled into a triangular system consisting of an integrable scalar equation together with an arbitrary evolution equation (which is generically non-integrable). To see this, we introduce the gauge transformation

$$\Psi = \mathbf{g}\Phi, \qquad \mathbf{g} = \begin{pmatrix} \kappa^{-1/2} & 0 \\ 0 & \kappa^{1/2} \end{pmatrix},$$

which transforms the Lax pair (20) to

$$\Phi_X = \mathcal{U}\Phi, \qquad \Phi_T = \mathcal{V}\Phi, \tag{21}$$

where

$$\mathcal{U} = \begin{pmatrix} \vartheta & \lambda \\ \lambda & -\vartheta \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} -\frac{1}{2}\lambda^{-2} & F\lambda^{-1} \\ G\lambda^{-1} & \frac{1}{2}\lambda^{-2} \end{pmatrix},$$

with

$$\vartheta = -\frac{1}{2q} + \frac{1}{2}(\log \kappa)_X, \qquad F = \kappa A, \qquad G = \kappa^{-1}B. \tag{22}$$

The compatibility conditions for (21) are

$$\begin{aligned} \vartheta_T &= F - G, \\ F_X &= 2\vartheta F + 1, \\ G_X &= -2\vartheta G - 1. \end{aligned} \tag{23}$$

The latter system can be written as a single scalar equation for ϑ, by eliminating F and G. This is best achieved by noting that the second and third equations in (23) imply $(FG)_X = G - F$, and so the first equation yields the conservation law

$$\vartheta_T + (FG)_X = 0. \tag{24}$$

The difference of the last two equations in (23) also gives $2\vartheta(F+G) = (F-G)_X - 2 = \vartheta_{XT} - 2$ (using the first equation once again), so that overall we have F and G given in terms of ϑ as

$$F = \frac{\vartheta_{XT} - 2}{4\vartheta} + \frac{1}{2}\vartheta_T, \qquad G = \frac{\vartheta_{XT} - 2}{4\vartheta} - \frac{1}{2}\vartheta_T. \tag{25}$$

Finally, if we substitute these expressions into the conservation law (24), we obtain the equation

$$\vartheta_T + \frac{1}{4}\left(\frac{(\vartheta_{XT} - 2)^2}{4\vartheta^2} - \vartheta_T^2\right)_X = 0, \tag{26}$$

which is an integrable partial differential equation for $\vartheta(X,T)$. In fact, (26) corresponds to the first negative flow in the modified KdV hierarchy, which is best seen by rewriting (21) as a scalar Lax pair for the first component $\phi = \phi_1$ of the vector $\Phi = (\phi_1, \phi_2)^T$, to find

$$\begin{aligned} (D_X + \vartheta)(D_X - \vartheta)\phi &\equiv (D_X^2 + W)\phi = \lambda^2 \phi, \\ \phi_T &= \lambda^{-2}\Big(F\phi_X - \tfrac{1}{2} F_X \phi \Big), \end{aligned}$$

with

$$W = -\vartheta_X - \vartheta^2;$$

the X part is just the KdV spectral problem, and the potential W in the Schrödinger operator is given by the standard Miura expression in terms of ϑ. However, observe that the time evolution of the field $\kappa(X,T)$ is not determined by the Lax pair (21); we shall return to this shortly.

Since (26) takes the form of a conservation law, it is convenient to introduce a potential $f(X,T)$ such that $\vartheta = f_X$, and then the equation

$$f_T + \frac{(f_{XXT} - 2)^2}{16 f_X^2} - \frac{f_{XT}^2}{4} = 0 \tag{27}$$

is obtained, by integrating and absorbing an arbitrary function of T into f. Thus we can describe solutions of the system (13) in the following way.

Theorem 2. *Let $f(X,T)$ be a solution of (27), let $\kappa(X,T)$ be an arbitrary function, and let $\vartheta(X,T) = f_X(X,T)$. Then a solution $(A(x,t), B(x,t))$ of the system (13) with non-autonomous coefficient $H(x,t)$ is given in parametric form by setting*

$$x = \log \kappa(X,t) - 2f(X,t)$$

and $T = t$ in the expressions

$$A = \kappa^{-1} F, \qquad B = \kappa G, \qquad H = -(\log \kappa)_T - 2FG,$$

where F and G are given in terms of ϑ by (25).

In the above formulation, the function κ is arbitrary, and together with ϑ it completely determines the quantity H, viewed as a non-autonomous coefficient appearing in the system (13). However, in [39] the role of H was envisaged somewhat differently: in that setting, one must consider the inverse problem of determining $\kappa(X,T)$, when H is some given function of the original fields and their derivatives that is specified a priori, i.e. $H = h(u, v, u_x, v_x, u_{xx}, v_{xx}, \ldots)$ in (3), or $H = \hat{h}(A, B, A_x, B_x, \ldots)$ in (13). With this alternative perspective, the original coupled system for u, v (or A, B) with independent variables x, t is equivalent under the reciprocal transformation (17) to a system consisting of the integrable equation (26) together with the evolution equation

$$\kappa_T = -\kappa(H + 2FG) \tag{28}$$

for κ, with F and G given by (25). The latter system is triangular, since (26) is an autonomous equation for ϑ alone, while the terms on the right-hand side of (28) generally depend on both κ, ϑ and their derivatives in a complicated way; for instance, given $H = \hat{h}(A, B, A_x, B_x, \ldots)$ we should replace A, B by $A = \kappa^{-1}F$, $B = \kappa G$ and use (25), while A_x should be replaced by $qA_X = q(\kappa^{-1}F)_X$ where

$$q = \Big((\log \kappa)_X - 2\vartheta\Big)^{-1},$$

and so on; alternatively, given $H = h(u, v, u_x, v_x, \ldots)$ one must consider potentially nonlocal expressions, since (12) gives $u - u_x = A$, so $u - qu_X = \kappa^{-1}F$, etc. Moreover, the equation (28) is completely independent of the Lax pair (21), so there is no reason for it to be integrable.

Thus, by identifying $\mathcal{F} = H + 2FG$, we have arrived at Theorem 1, and our main conclusion: in general, for a given choice of H, the Lax pair (15) is insufficient to infer that the system (3) is integrable. An integrable coupled system only arises for certain exceptional choices of H.

One particular exception is the case corresponding to (8) above, namely

$$H = -2AB,$$

which causes the right-hand side of (28) to vanish, since $FG = AB$. In that case, in terms of A, B with m, n given by (14), the system (13) takes the form

$$m_t = -2(ABm)_x, \qquad n_t = -2(ABn)_x, \tag{29}$$

which is one of the coupled cubic integrable systems derived recently in [24]. After rescaling the dependent variables, this corresponds to the system (7) obtained in [36], via the Miura map (12); the equation (6) is a reduction of this system to a scalar equation. This is a situation where the equation (28) trivially decouples from (26), with $\kappa_T = 0$ implying that κ is an arbitrary function of X. For some exact solutions of the system (29), see [24].

Another exceptional situation arises by taking a reduction to a scalar equation with

$$H = ku, \qquad v = \ell, \quad \text{for } k, \ell \text{ constant},$$

so that $B = n = \ell$. From the second equation in (13) it follows that $\ell = -k/2$ must hold, and the first equation becomes the Camassa-Holm equation (5), up to rescaling. To fix the choice of scale we set $k = -2$, $\ell = 1$, to find

$$q = \sqrt{m} = \kappa^{-1},$$

while using $FG = AB$ gives

$$H + 2FG = -2u + 2(u - u_x) = -2qu_X = qH_X,$$

from which it follows that the two equations in (18) are equivalent to each other, and by (22) we see that ϑ is given in terms of q as

$$\vartheta = \frac{q_X - 1}{2q}.$$

The field q satisfies the equation

$$(q^{-1})_T + \Big(q(\log q)_{XT} - 2q^2\Big)_X = 0,$$

which can be rewritten in the form

$$W_T = -2q_X, \qquad W = -\frac{q_{XX}}{2q} + \frac{q_X^2}{4q^2} - \frac{1}{4q^2},$$

identifying it as the first negative flow in the KdV hierarchy (see [18, 23] for more details).

However, for other choices of H we expect that an integrable system does not arise; in particular, it appears that the system (10) and other examples considered in [39] are not integrable.

3 Homogeneous Camassa-Holm type equations

In this section we consider integrable Camassa-Holm type equations with homogeneous nonlinear terms, of the form

$$(1 - D_x^2)u_t = \alpha u^k u_x + \beta u^k u_{xxx} + \gamma u^{k-1} u_x u_{xx} + \delta u^{k-2} u_x^3. \tag{30}$$

Nonlinearities of this type have been considered in [1, 19]. Here $\alpha, \beta, \gamma, \delta, k$ are arbitrary complex constants, and we assume that $\alpha k \neq 0$.

Known integrable examples of Camassa-Holm type equations of the form (30) correspond to $k = 1$ and $k = 2$. We show that under the above assumptions these are the only possible degrees of nonlinearity.

To prove this we adopt the viewpoint of Dubrovin-Zhang, which takes a perturbative approach to integrability, with quasilinear hyperbolic systems as the starting point; the reader is referred to [11, 12, 13] for the origin of these ideas, and to [14] for a more recent review. Consider a formal series

$$\begin{aligned} u_t &= \lambda(u)u_x + \epsilon\,(a_1(u)u_{xx} + a_2(u)u_x^2) \\ &\quad + \epsilon^2\,(b_1(u)u_{xxx} + b_2(u)u_x u_{xx} + b_3(u)u_x^3) + \cdots =: F, \end{aligned} \tag{31}$$

where ϵ is an arbitrary parameter, and we assume that $\lambda'(u) \neq 0$. Expressions at each power ϵ^n are homogeneous differential polynomials in x-derivatives of u of weight $n+1$ if we adopt the convention that the weight of u is 0 and the weight of the jth derivative of u is j, for $j \geq 0$. The series (31) may or may not truncate. In the former case the expression (31) is an evolutionary partial differential equation. Following [14] we adopt the following definition of *integrability* for the formal series (31):

Definition 1. *The series (31) is integrable if there exists another formal series (formal symmetry)*

$$\begin{aligned} u_\tau &= \mu(u)u_x + \epsilon\,(A_1(u)u_{xx} + A_2(u)u_x^2) \\ &\quad + \epsilon^2\,(B_1(u)u_{xxx} + B_2(u)u_x u_{xx} + B_3(u)u_x^3) + \cdots =: G \end{aligned} \tag{32}$$

that commutes with (31) for an arbitrary choice of the function $\mu(u)$.

Definition 1 can be viewed as a reformulation and extension of Definition 3.1 on p. 7 of [14], with the main difference being that we do not require the formal series (31) and (32) to be Hamiltonian. The above definition was also adopted by Arsie, Lorenzoni and Moro in their study of integrable viscous conservation laws [2, 3]. Taking the dispersionless limit $\epsilon \to 0$ in (31) yields an equation in the (dispersionless) Burgers hierarchy, which has $u_\tau = \mu(u)u_x$ as a symmetry for any $\mu(u)$. Since μ is arbitrary, one can produce an infinite sequence of formal symmetries by taking $\mu(u) = u^j$ for $j = 0, 1, 2, \ldots$, corresponding to an infinite number of symmetries for (31), which is the usual requirement of integrability in the symmetry approach [27, 28]. The main difference is that from the latter point of view a symmetry is usually defined by starting from its leading linear dispersion term, whereas in [14] and [2, 3] one starts with the leading nonlinear term, defined by $\mu(u)$.

The commutator of two formal series (31) and (32), is again a series in positive powers of ϵ:

$$[F, G] = \epsilon K_1 + \epsilon^2 K_2 + \epsilon^3 K_3 + \cdots .$$

Each term K_m, $m > 2$ is a homogeneous differential polynomial in derivatives of u of weight $m + 1$ with coefficients expressed in terms of $\lambda, \mu, A_i, B_i, \ldots, a_i, b_i, \ldots$ and their derivatives. Vanishing of K_m leads to two different sets of relations on $\lambda, \mu, A_i, B_i, \ldots, a_i, b_i, \ldots$:

- Vanishing of coefficients of K_m at monomials of the form $D_x^{n_1}(u)D_x^{n_2}(u) \cdots D_x^{n_j}(u)u_x^s$, with $s > 0$ and $n_1, n_2, \ldots, n_j > 1$ leads to *a triangular linear* system of equations on coefficients of (32). From this system one explicitly finds $A_i, B_i, \ldots$ in terms of $\lambda, \mu, a_i, b_i, \ldots$ and their derivatives.

- Vanishing of coefficients of K_m at monomials of the form $D_x^{n_1}(u)D_x^{n_2}(u) \cdots D_x^{n_j}(u)$ with $n_1, n_2, \ldots, n_j > 1$ leads to a system of differential constraints on $\lambda, \mu_i, a_i, b_i, \ldots$. The requirement of vanishing of the commutator $[F, G]$ for *any* function $\mu(u)$ leads to a condition that coefficients at every monomial $\mu(u)^{k_1}\, \mu'(u)^{k_2}\, \mu''(u)^{k_3} \cdots$ should vanish. These form a system of ordinary differential equations for the functions $\lambda(u), a_1(u), a_2(u), b_1(u), \ldots$, which are the integrability conditions for (31). The first integrability condition occurs at order ϵ^3:

$$a_1(-4a_1a_2\lambda' - 2a_1a_1'\lambda' + 3b_1\lambda'^2 + 4a_1^2\lambda'') = 0,$$

Thus, at each order in ϵ, the vanishing of the commutator leads to necessary conditions for (31) to be integrable in the sense of Definition 1, and so checking these conditions provides a test for integrability. Having verified a finite number of conditions, one then requires a different argument to verify whether such conditions are also sufficient for integrability. Let us illustrate the approach by the following examples:

Example 1. Consider the following extension of the Burgers equation:

$$u_t = \lambda(u)u_x + \epsilon u_{xx} = F, \qquad \lambda'(u) \neq 0. \tag{33}$$

Applying the above procedure requires that there exists a formal series G (32) that commutes with F for an arbitrary choice of $\mu(u)$. Then

$$\begin{aligned} G &= \mu(u)u_x + \epsilon\tfrac{1}{\lambda'^2}\left(\mu'\lambda' u_{xx} - (\mu'\lambda'' - \mu''\lambda')u_x^2\right) \\ &\quad +\epsilon^2\left(-\tfrac{2}{3\lambda'^3}(\mu'\lambda'' - \mu''\lambda')u_{xxx} + \cdots\right) + \epsilon^3(\cdots) \end{aligned}$$

The integrability condition at order ϵ^3 then yields $\lambda'' = 0$, which implies that equation (33) is integrable if and only if it is equivalent to the Burgers equation.

Example 2. Consider now another equation of Burgers type, namely

$$u_t = uu_x + \epsilon f(u)u_{xx} = F, \qquad f(u) \neq 0. \tag{34}$$

In this case, with $\mu = \mu(u)$,

$$\begin{aligned} G &= \mu u_x + \epsilon f(\mu' u_{xx} + \mu'' u_x^2) \\ &\quad +\epsilon^2\left(\tfrac{2}{3}f^2\mu'' u_{xx} + \tfrac{5}{3}f(f'\mu'' + f\mu''')u_x u_{xx} + \cdots\right) + \cdots . \end{aligned}$$

The integrability condition at order ϵ^3 gives $f' = 0$ and hence this yields the standard Burgers equation once again.

Example 3. Consider now an equation of KdV type, namely

$$u_t = \lambda(u)u_x + \epsilon^2 u_{xxx} = F, \qquad \lambda'(u) \neq 0. \tag{35}$$

In this case all terms in (32) at odd powers of ϵ vanish. We have

$$G = \mu u_x + \epsilon^2\left(\frac{\mu'}{\lambda'}u_{xx} - 2\frac{\mu'\lambda'' - \mu''\lambda'}{\lambda'^2}u_x u_{xx} + \cdots\right) + \cdots .$$

The first non-identically vanishing integrability conditions occur at order ϵ^8. These are equivalent to $\lambda''' = 0$, yielding the KdV equation and the mKdV equation.

Example 4. Similarly, in the case of

$$u_t = uu_x + \epsilon^2 f(u)u_{xxx} = F, \qquad f(u) \neq 0, \tag{36}$$

we have

$$G = \mu u_x + \epsilon^2 \left(f\mu' u_{xxx} + 2f\mu'' u_x u_{xx} + \frac{1}{2} f\mu''' u_x^3 \right) + \cdots .$$

The first non-identically vanishing integrability conditions occur at order ϵ^8. These are equivalent to

$$3ff'' - f'^2 = 0.$$

Modulo rescaling and shifting $u \to u+\text{const}$, all solutions are equivalent to either $f = 1$ or $f = u^{\frac{3}{2}}$. This leads to the KdV equation and another integrable equation of Harry Dym type, namely

$$u_t = u^{\frac{3}{2}} u_{xxx} + u u_x.$$

The application of the above test to the Camassa-Holm type equations (30) with homogeneous nonlinearity is the following. First of all we rewrite (30) as an evolutionary formal series by inverting the operator $1 - D_x^2$:

$$\begin{aligned} u_t &= (1-D_x^2)^{-1}(\alpha u^k u_x + \beta u^k u_{xxx} + \gamma u^{k-1} u_x u_{xx} + \delta u^{k-2} u_x^3) \\ &= \alpha u^k u_x + (\alpha+\beta) u^k u_{xxx} + (3k\alpha + \gamma) u^{k-1} u_x u_{xx} \\ &\quad + (\delta + k(k-1)\alpha) u^{k-2} u_x^3 + \cdots . \end{aligned} \tag{37}$$

By rescaling x and t it is convenient to introduce the parameter ϵ, which counts the weight of every monomial:

$$u_t = \alpha u^k u_x + \epsilon^2 F_2[u] + \epsilon^4 F_4[u] + \cdots , \tag{38}$$

where

$$F_2[u] = (\alpha+\beta) u^k u_{xxx} + (3k\alpha+\gamma) u^{k-1} u_x u_{xx} + (\delta + k(k-1)\alpha) u^{k-2} u_x^3$$

and the omitted terms are $O(\epsilon^6)$. (It is easy to see that all odd orders of ϵ are absent from the above expression.)

The procedure leads to the following theorem:

Theorem 3. *If equation (30) with $\alpha \neq 0$ and $k \neq 0$ is integrable then $k = 1$ or $k = 2$.*

Sketch of the proof: The proof consists of four main steps which are outlined below; most of the resulting algebraic conditions are omitted, as they are too lengthy to include here.

1. The condition that $\alpha k \neq 0$ in the theorem guarantees the applicability of the test. Since (38) contains only terms with even powers of ϵ, we can seek a formal symmetry without odd order powers of ϵ, that is

$$u_\tau = \mu(u)u_x + \epsilon^2 \left(B_1(u)u_{xxx} + B_2(u)u_x u_{xx} + B_3(u)u_x^3\right) + \cdots .$$

2. Compatibility conditions up to order ϵ^6 do not impose any constraints on the equation (38).

3. At order ϵ^8 one obtains 17 algebraic equations on $\alpha, \beta, \gamma, \delta$ and k. The first of them reads as

$$(-27 + 18k + 4k^2)\alpha^2 + (-54 + 45k - 26k^2)\alpha\beta + 54(k-1)\alpha\gamma - 108\alpha\delta$$
$$-3(k-3)(2k-3)\beta^2 + 9(k-6)\beta\gamma - 108\beta\delta + 9\gamma^2 = 0,$$

   and the remaining ones are increasingly more complicated.

4. The final step of the proof requires computations up to order ϵ^{10}. The resulting algebraic system of equations for $\alpha, \beta, \gamma, \delta$ and k possesses non-trivial (non-zero) solutions only if $k = 1$ or $k = 2$.

4 Discussion

Lax pairs and an infinite number of conservation laws are considered to be hallmarks of integrability for systems of partial differential equations. However, a rigorous definition of these concepts is required, in particular for multi-component systems. Without it, a Lax pair alone is not sufficient to infer integrability [5, 7, 35], and even an infinite number of conservation laws may not be enough. No matter what choice of H is made, the system (3) formally arises from a Lax pair, and this Lax pair yields an infinite number of conservation laws, but our calculations show that this Lax pair is "fake" in the sense that the dependence on H can be removed, and the system can be reduced to an integrable scalar equation coupled with another equation which is not integrable in general. Nevertheless, there are certain specific choices of H for which the second equation is either trivial or equivalent to a copy of the first equation, corresponding to the Camassa-Holm equation, or to the system (7) found in [36] (which includes (6) as a scalar reduction).

There are other systems found in [39] for which compatible bi-Hamiltonian operators are presented, including the system (10) from [38]. As bi-Hamiltonian structures are considered to be another hallmark of integrability, this would seem to contradict our claim that these other choices of H should not give integrable systems. However, it appears that the pairs of compatible Hamiltonian operators J, K presented for these other cases in [39] do not give rise to an infinite hierarchy of *local* symmetries in terms of the fields m, n. Within the symmetry approach to integrable systems [27, 28, 29], there is a requirement of infinitely many local symmetries, yet most of the recursion operators JK^{-1} or KJ^{-1} found in [39] produce only nonlocal equations, so there is no contradiction.

Furthermore, we expect that in the peakon sector it may not possible be obtain a consistent spectral problem from the Lax pair for (10) and the other systems presented in [39], apart from the exceptional system (7), for which the spectral theory for the peakons was derived in [8].

All of the known integrable scalar equations or coupled systems of Camassa-Holm type contain nonlinear terms of degree at most three. The result in the third section above shows that this condition on the degree is necessary for integrability.

Acknowledgments: We are grateful to the referees for useful comments which have helped us to clarify the paper. MH and ANWH both benefited from the hospitality provided by the Dipartimento di Matematica e Fisica di Roma Tre in 2013, when we began discussions on fake Lax pairs. ANWH is supported by Fellowship EP/M004333/1, and JPW by the grant EP/P012698/1, both from the Engineering and Physical Sciences Research Council (EPSRC). ANWH would like to thank the organizers of New Trends in Applied Geometric Mechanics at ICMAT (Madrid, Spain) in July 2017 for such a wonderful celebration of Darryl Holm's 70th birthday. We are all grateful to Darryl for so much warm friendship and inspiration over the years.

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