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QRT maps and related Laurent systems

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Abstract

In recent work it was shown how recursive factorisation of certain QRT maps leads to Somos-4 and Somos-5 recurrences with periodic coefficients, and to a fifth-order recurrence with the Laurent property. Here we recursively factorise the 12-parameter symmetric QRT map, given by a second-order recurrence, to obtain a system of three coupled recurrences which possesses the Laurent property. As degenerate special cases, we first derive systems of two coupled recurrences corresponding to the 5-parameter multiplicative and additive symmetric QRT maps. In all cases, the Laurent property is established using a generalisation of a result due to Hickerson, and exact formulae for degree growth are found from ultradiscrete (tropical) analogues of the recurrences. For the general 18-parameter QRT map it is shown that the components of the iterates can be written as a ratio of quantities that satisfy the same Somos-7 recurrence.

1   Introduction

A rational recurrence relation is said to have the *Laurent property* if all of the iterates are Laurent polynomials in the initial values, with coefficients belonging to some ring (typically $\mathbb{Z}$). We call such a recurrence a *Laurent recurrence*. The first examples of such recurrences were discovered by Michael Somos in the 1980s [14]. Since then many more have been found [1, 10, 13, 23, 24] (also cf. [9]). The Laurent property is a central feature of cluster algebras (see [11, 12] and references).

This paper is concerned with systems of Laurent recurrences related to QRT maps. The QRT maps are an 18-parameter family of birational transformations of the plane, which were introduced in [27, 28], encompassing various examples that appeared previously in a wide variety of contexts, including statistical mechanics, discrete soliton theory and dynamical systems. QRT maps are measure-preserving (symplectic) and have an invariant function (first integral), hence they provide a prototype of a discrete integrable system in finite dimensions. The generic level set of the first integral is a curve of genus one, so there is an associated elliptic fibration of the plane [30]. The rich geometry of QRT maps is described extensively in the monograph by Duistermaat [8]; for a terse overview, see subsection 6.3 below.

It is an open question as to what conditions are necessary for "Laurentification" of a general birational transformation, i.e. to determine whether such a transformation admits a lift to a Laurent recurrence or a system of such recurrences. In [17] two of the authors used ultradiscretization and recursive factorisation (which was employed in [29], but can in fact be found in earlier work by Boukraa and Maillard [4]) to derive recurrence relations for the divisors of iterates of homogenised discrete integrable systems. As the divisors are polynomials, these recurrences should possess the Laurent property, as indeed they do, in all cases considered. A different approach using projective coordinates has been taken in [32], leading to similar results.

Specifically, it was shown in [17] that two particular multiplicative symmetric QRT maps,

namely

$$u_{n+1}u_{n-1} = \frac{\alpha u_n + \beta}{u_n^2}, \tag{1a}$$

$$u_{n+1}u_{n-1} = \frac{\gamma u_n + \delta}{u_n}, \tag{1b}$$

give rise via recursive factorisation to Somos-4 and Somos-5 recurrences, that is

$$c_{n-2}c_{n+2} = \alpha_n c_{n-1}c_{n+1} + \beta_n c_n^2, \tag{2a}$$

$$d_{n-3}d_{n+2} = \gamma_n d_{n-2}d_{n+1} + \delta_n d_{n-1}d_n, \tag{2b}$$

respectively, where the coefficients α_n, β_n and γ_n, δ_n are periodic functions of n, with period 8 in the first case and period 7 in the second. The connection between the QRT maps (1) and the autonomous versions of these Somos recurrences is well known (see e.g. [18, 19]). Both equations (2) are special cases of a non-autonomous Gale-Robinson recurrence [14], which arise as reductions of Hirota's bilinear (discrete KP) equation [16, 24, 25, 34, 36]. Furthermore, it was shown in [17] that the additive QRT map

$$u_{n+1} + u_{n-1} = \frac{\alpha - u_n^2}{u_n}, \tag{3}$$

known as DTKQ-2 [6], is related by recursive factorisation to a fifth-order Laurent recurrence, that is

$$e_{n-1}e_{n-2}^2 e_{n-5} + e_{n-1}^2 e_{n-4}^2 + e_n e_{n-3}^2 e_{n-4} = \alpha e_{n-2}^2 e_{n-3}^2. \tag{4}$$

It is worth pointing out that the Laurent property is neither necessary nor sufficient for integrability. To see why it is not necessary, note that a discrete integrable system, in the form of a birational map satisfying the conditions of Liouville's theorem, need not have the Laurent property: this property is associated with a particular choice of coordinate system, and is easily destroyed by a birational change of coordinates, whereas integrability is not. As for sufficiency, it is known that large families of birational recurrences with the Laurent property arise from certain sequences of mutations in a cluster algebra [10, 13] or an LP algebra [1, 23], yet integrability is a rare property, and only a small minority of such recurrences are discrete integrable systems.

Nevertheless, in an algebraic setting, based on the evidence of a large number of examples, it appears that discrete integrable systems should always admit Laurentification. The advantage of having a system with the Laurent property is that it leads to a very direct way of calculating the sequence of degrees, so that the algebraic entropy of the system can be calculated as the limit $\lim_{n\to\infty} n^{-1} \log d_n$ (where d_n is the degree of the nth iterate). In the approach of Bellon and Viallet [2], discrete integrable systems are characterised by having zero algebraic entropy. For the case of the QRT maps considered here, which can be regularised by a finite number of blowups of the plane [8], and preserve a pencil of invariant curves, general arguments indicate that the degrees grow quadratically with n [3], and thus the entropy is zero. While a geometrical approach via blowups is effective for counting degrees in dimension two, it becomes increasingly difficult in higher dimensions, and this is our motivation for considering Laurent systems here, in a test case where we know the degree growth in advance.

Laurentification is not a unique procedure, and for convenience one should aim to find the simplest system which has the Laurent property. Recursive factorisation can provide a Laurent system, but not always the simplest one: in particular, as shown below (see also [25]), solutions to the non-autonomous Somos recurrences (2) from [17] are related to those of their autonomous versions, which are simpler. In section 3 we obtain a two-component autonomous system directly from the 5-parameter multiplicative symmetric QRT map

$$u_{n+1}u_{n-1} = \frac{a_3 u_n^2 + a_5 u_n + a_6}{a_1 u_n^2 + a_2 u_n + a_3}, \tag{5}$$

by writing the iterates as a ratio $u_n = k_n/l_n$. We prove that this is a Laurent system, and use the Laurent property together with ultradiscretization to derive a polynomial formula for the

growth of degrees (quadratic in n). We also show how our autonomous system degenerates to the non-autonomous Somos-4 (2a) and Somos-5 (2b) in the special cases considered in [17].

In section 4, we Laurentify the 5-parameter additive symmetric QRT map

$$u_{n+1} + u_{n-1} = -\frac{a_2 u_n^2 + a_4 u_n + a_5}{a_1 u_n^2 + a_2 u_n + a_3}, \tag{6}$$

which generalises (3). This gives another system of two recurrences, which degenerates to (4) as a special case. We show that the same quadratic formula as found for (5) describes the degree growth of (6).

In section 5, we recursively factorise the 12-parameter symmetric QRT map (see equation (35) below), and obtain a three-component system, whose Laurentness follows directly from factorisation properties. We describe how the additive and multiplicative Laurent systems obtained from (5) and (6) appear as degenerate cases, and use ultradiscretization to show that the degree growth of the symmetric QRT map is quadratic.

In section 6, we present Somos-7 recurrences that are satisfied by the variables in the Laurent systems introduced in the preceding sections. We prove that the components of iterates of the general 18-parameter QRT map can also be written as a ratio of quantities that satisfy a Somos-7 relation.

Because deriving systems that are likely to possess the Laurent property can now be done routinely, there is a need for verifying the Laurent property routinely. An account of such a procedure for autonomous recurrences, found by Hickerson, was given in [14, 26]. Another approach, built into the axiomatic framework of cluster algebras or LP algebras [23], is to use the Caterpillar Lemma as in [10], but this only applies to relations in multiplicative form (i.e. exchange relations with a product of two terms on the left-hand side). A straightforward generalisation of Hickerson's method to systems of equations, with more general denominators, is given in Theorem 2 in the next section. For the multiplicative and additive Laurent systems (equations (12) and (29) below) it is easy to verify the conditions in the theorem, and hence to establish their Laurentness.

2   Proving the Laurent property

Sufficient conditions for equations of the form

$$\tau_n \tau_{n-k} = P(\tau_{n-k+1}, \ldots, \tau_{n-1}), \qquad k \in \mathbb{N}, \quad P \text{ polynomial over } \mathcal{R} \tag{7}$$

(where $\mathcal{R}$ is a ring of coefficients) to possess the Laurent property were found by Hickerson. Taking $\{\tau_i\}_{i=0}^{k-1}$ as the initial values, the iterates are written as a ratio

$$\tau_n = \frac{p_n(\tau_0, \ldots, \tau_{k-1})}{q_n(\tau_0, \ldots, \tau_{k-1})}$$

of coprime polynomials, so that the greatest common divisor $(p_n, q_n) = 1$. The Laurent property means that all q_n are monomials. The following is Hickerson's result, as mentioned by Gale in [14] and proved by Robinson in [26].

Theorem 1. *Equation (7) has the Laurent property if $(p_k, p_{k+l}) = 1$ for $l = 1, \ldots, k$ and q_{2k} is a monomial.*

Below we provide sufficient conditions for systems of equations to possess the Laurent property. At the same time, we generalise the form of the right-hand side of (7), by allowing a monomial denominator, and consider the case where the iterates are Laurent polynomials in a subset of the initial variables and polynomial in the rest.

Consider a system of d ordinary difference equations of order k,

$$\tau_n^i = \frac{P^i(\tau_{n-k}^1, \ldots, \tau_{n-1}^d)}{Q^i(\tau_{n-k}^1, \ldots, \tau_{n-1}^d)}, \quad P^i \text{ polynomial}, \quad Q^i \text{ monomial}, \qquad i = 1, \ldots, d. \tag{8}$$

From a set of kd initial values $U = \{\tau_l^j\}_{1 \leq j \leq d, 0 \leq l \leq k-1}$, where the superscripts denote components (not exponents), one finds τ_n^i as rational functions of the initial values, given by

$$\tau_n^i = \frac{p_n^i(\tau_0^1, \ldots, \tau_{k-1}^d)}{q_n^i(\tau_0^1, \ldots, \tau_{k-1}^d)}, \tag{9}$$

with $(p_n^i, q_n^i) = 1$. By definition, if $q_n^i \in \mathcal{R}[U]$ is a monomial for all i and $n \geq 0$, then (8) has the Laurent property, meaning that each τ_n^i belongs to the ring $\mathcal{R}[U^{\pm 1}] := \mathcal{R}[(\tau_0^1)^{\pm 1}, \ldots, (\tau_{k-1}^d)^{\pm 1}]$. The form of (8) guarantees that all components q_n^i are monomials for $0 \leq n \leq k$. Suppose these monomials depend on a subset of the initial values $V \subset U$, specified by a set of superscripts $I \subset \{1, \ldots, d\}$. The following conditions guarantee that q_n^i are monomials for all i and $n \geq 0$.

Theorem 2. *Suppose that q_k^i is a monomial in $\mathcal{R}[V]$ for $1 \leq i \leq d$. If p_k^i is coprime to p_{k+l}^j for all $i, j \in I \subset \{1, \ldots, d\}$, $l = 1, \ldots, k$, and $q_m^i \in \mathcal{R}[V]$ is a monomial for $1 \leq i \leq d$, $k+1 \leq m \leq 2k$, then (8) has the Laurent property: all iterates are Laurent polynomials in the variables from V and they are polynomial in the remaining variables from $W = U \setminus V$.*

Proof. The proof is by induction in n. If we regard $\{\tau_l^j\}_{1 \leq j \leq d, 1 \leq l \leq k}$ as initial data, then from (9) we may write

$$\tau_n^i = \frac{p_{n-1}^i(\tau_1^1, \ldots, \tau_k^d)}{q_{n-1}^i(\tau_1^1, \ldots, \tau_k^d)}, \tag{10}$$

while on the other hand, by taking $\{\tau_l^j\}_{1 \leq j \leq d, k+1 \leq l \leq 2k}$ as initial values, we find

$$\tau_n^i = \frac{p_{n-k-1}^i(\tau_{k+1}^1, \ldots, \tau_{2k}^d)}{q_{n-k-1}^i(\tau_{k+1}^1, \ldots, \tau_{2k}^d)}. \tag{11}$$

Then by using (9) again, the arguments τ_m^j for $m = k, \ldots, 2k$ can be expressed as Laurent polynomials in the variables from V with coefficients in $\mathcal{R}[W]$. Thus for each i the denominator of (10) becomes a monomial in the variables from V, multiplied by powers of the polynomials p_k^j for $j \in I$. On the other hand, the denominator of (11) becomes a monomial in variables from V only, multiplied by powers of p_{k+l}^j for $j \in I$ and $l = 1, \ldots, k$. By the coprimality assumption, the only way that these two expressions can be equal is if all the powers of polynomials p_m^j for $j \in I$ appearing in a denominator cancel with the numerator in each case, to leave a reduced expression for τ_n^i as a Laurent polynomial in the ring $\mathcal{R}[W][V^{\pm 1}]$. $\qquad\square$

The preceding result can be modified to include the case where the coefficients in system (8) are periodic functions, e.g. as in (2), but we will not need this in the sequel. However, when discussing ultradiscretization it will be convenient to describe periodic sequences using the following notation.

Notation 3. *A periodic function f_n such that $f_{n+m} = f_n$ is defined by m values: we write $f \bmod m = [v_1, \ldots, v_m]$ to mean $f_n = v_{n \bmod m}$.*

3 The multiplicative symmetric QRT map

In this section, we show how to "Laurentify" the multiplicative symmetric QRT map, i.e. produce a corresponding Laurent system of recurrences, by applying homogenisation. We then use ultradiscretization to derive the degree growth of the map. We also show how the Laurent system reduces to the Somos-4 and Somos-5 equations with periodic coefficients that were found in [17].

3.1 Laurentification of the multiplicative symmetric QRT map

By taking $u_n = \frac{k_n}{l_n}$ in (5) and identifying the numerators and denominators on both sides, we obtain a system that generates sequences (k_n) and (l_n), that is

$$k_{n+1}k_{n-1} = a_3 k_n^2 + a_5 k_n l_n + a_6 l_n^2, \tag{12a}$$

$$l_{n+1}l_{n-1} = a_1 k_n^2 + a_2 k_n l_n + a_3 l_n^2. \tag{12b}$$

Without loss of generality one can choose $(k_0, k_1, l_0, l_1) = (u_0, u_1, 1, 1)$ as initial values for (12). Observe that the system (12) is homogeneous of degree 2: it is a Hirota bilinear form for (5).

Proposition 4. *The system (12) has the Laurent property. Any four adjacent iterates k_n, l_n, k_{n+1}, l_{n+1} are pairwise coprime Laurent polynomials in the ring $\mathcal{R}[k_0^{\pm 1}, k_1^{\pm 1}, l_0^{\pm 1}, l_1^{\pm 1}]$, where $\mathcal{R} = \mathbb{Z}[a_1, a_2, a_3, a_5, a_6]$ is the ring of coefficients.*

Proof. The Laurent property can be verified directly by applying Theorem 2 in the case that the dimension $d = 2$ and the order $k = 2$. For the coprimality, observe that when $n = 0$ this is trivially true, and proceed by induction in n. If a non-constant Laurent polynomial $P \in \mathcal{R}[k_0^{\pm 1}, k_1^{\pm 1}, l_0^{\pm 1}, l_1^{\pm 1}]$ is a common factor of k_{n+1} and k_n, then it divides the right-hand side of (12a), hence $P | l_n$, which contradicts $(k_n, l_n) = 1$. Thus $(k_{n+1}, k_n) = (k_{n+1}, l_n) = 1$, and similarly from (12b) we have $(l_{n+1}, k_n) = (l_{n+1}, l_n) = 1$. Now let P be a common factor of k_{n+1} and l_{n+1},

$$\implies \mathrm{S}\,\mathbf{h}_n = P\,\mathbf{v}_n, \text{ where } \mathrm{S} = \begin{pmatrix} a_3 & a_5 & a_6 & 0 \\ 0 & a_3 & a_5 & a_6 \\ a_1 & a_2 & a_3 & 0 \\ 0 & a_1 & a_2 & a_3 \end{pmatrix}, \quad \mathbf{h}_n = \begin{pmatrix} k_n^3 \\ k_n^2 l_n \\ k_n l_n^2 \\ l_n^3 \end{pmatrix}, \quad P\,\mathbf{v}_n = \begin{pmatrix} k_n k_{n+1} k_{n-1} \\ k_{n+1} k_{n-1} l_n \\ k_n l_{n+1} l_{n-1} \\ l_{n+1} l_{n-1} l_n \end{pmatrix}.$$

Multiplying the above equation by the adjugate of the Sylvester matrix S yields

$$\mathrm{R}\,\mathbf{h}_n = P\,\mathrm{S}^{\mathrm{adj}}\mathbf{v}_n,$$

where the resultant $\mathrm{R} = \det \mathrm{S}$ is a non-zero element of the coefficient ring $\mathcal{R}$, namely

$$\mathrm{R} = a_1^2 a_6^2 - a_1 a_2 a_5 a_6 - 2 a_1 a_3^2 a_6 + a_1 a_3 a_5^2 + a_2^2 a_3 a_6 - a_2 a_3^2 a_5 + a_3^4 \neq 0. \tag{13}$$

Hence P divides each component of the vector $\mathbf{h}_n$, contradicting $(k_n, l_n) = 1$. $\qquad \square$

Remark 5. *The latter result remains true for numerical values a_i such that $a_1 a_3 a_6 \mathrm{R} \neq 0$.*

The Laurent property implies that, in general, the iterates of (12) can be written in the form

$$k_n = \frac{\mathrm{N}_n(\mathbf{k})}{\mathbf{k}^{\mathbf{d}_n}}, \qquad l_n = \frac{\hat{\mathrm{N}}_n(\mathbf{k})}{\mathbf{k}^{\mathbf{e}_n}}, \tag{14}$$

where N_n, $\hat{\mathrm{N}}_n$ are polynomials in $\mathbf{k} = (k_0, k_1, l_0, l_1)$ that are not divisible by any of these four variables, while the denominators are Laurent monomials, i.e.

$$\mathbf{k}^{\mathbf{d}_n} = k_0^{d_n^{(1)}} k_1^{d_n^{(2)}} l_0^{d_n^{(3)}} l_1^{d_n^{(4)}}, \qquad \mathbf{d}_n = (d_n^{(1)}, d_n^{(2)}, d_n^{(3)}, d_n^{(4)})^T$$

and similarly for $\mathbf{k}^{\mathbf{e}_n}$, where the exponents appearing in the denominator vectors $\mathbf{d}_n$ and $\mathbf{e}_n$ are integers. The initial vectors are

$$\mathbf{d}_0 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{d}_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_0 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}. \tag{15}$$

3.2 Growth of degrees of the multiplicative symmetric QRT map

In order to measure the growth of degrees of the map (5), we consider the growth of the degrees of the Laurent polynomials that are generated by the system (12). From the form of this system, it is clear that k_n, l_n are homogeneous rational functions of degree 1 in the initial values $\mathbf{k} = (k_0, k_1, l_0, l_1)$, which implies that

$$\deg_{\mathbf{k}}(\mathrm{N}_n(\mathbf{k})) = 1 + \deg_{\mathbf{k}}(\mathbf{k}^{\mathbf{d}_n}) = 1 + \sum_{j=1}^{4} d_n^{(j)}, \qquad \deg_{\mathbf{k}}(\hat{\mathrm{N}}_n(\mathbf{k})) = 1 + \deg_{\mathbf{k}}(\mathbf{k}^{\mathbf{e}_n}) = 1 + \sum_{j=1}^{4} e_n^{(j)}, \quad (16)$$

where $\deg_{\mathbf{k}}$ denotes the total degree in these variables. Furthermore, from the form of (12), k_n, l_n are also subtraction-free rational expressions in the initial values, meaning that a standard argument which is used in the theory of cluster algebras can be applied (cf. equation (7.7) in [11], or Lemma 8.3 in [12]), and hence the denominator vectors $\mathbf{d}_n, \mathbf{e}_n$ satisfy a tropical version of the Laurent system, given by the max-plus ultradiscretization[1]

$$\begin{aligned} \mathbf{d}_{n+1} + \mathbf{d}_{n-1} &= \max(2\mathbf{d}_n, \mathbf{d}_n + \mathbf{e}_n, 2\mathbf{e}_n), \\ \mathbf{e}_{n+1} + \mathbf{e}_{n-1} &= \max(2\mathbf{d}_n, \mathbf{d}_n + \mathbf{e}_n, 2\mathbf{e}_n), \end{aligned} \qquad (17)$$

where the max applies componentwise on the right-hand side.

Due to the symmetrical form of the tropical system (17) and the initial vectors (15), the solution of this ultradiscrete vector system can be written as

$$\mathbf{d_n} = (d_n, d_{n-1}, e_n, e_{n-1})^T, \qquad \mathbf{e_n} = (e_n, e_{n-1}, d_n, d_{n-1})^T,$$

in terms of a pair of sequences (d_n), (e_n) which satisfy the scalar version of (17), that is

$$\begin{aligned} d_{n+1} + d_{n-1} &= \max(2d_n, d_n + e_n, 2e_n), \\ e_{n+1} + e_{n-1} &= \max(2d_n, d_n + e_n, 2e_n), \end{aligned} \qquad (18)$$

with the initial values $d_0 = -1$, $d_1 = e_0 = e_1 = 0$. If we introduce the sums and differences

$$\Sigma_n = d_n + e_n, \qquad \Delta_n = d_n - e_n,$$

and note the fact that $\max(\Delta, 0, -\Delta) = |\Delta|$, then the scalar system (18) becomes

$$\begin{aligned} \Sigma_{n+2} - 2\Sigma_{n+1} + \Sigma_n &= 2|\Delta_{n+1}|, \\ \Delta_{n+2} + \Delta_n &= 0, \end{aligned} \qquad (19)$$

with initial values

$$\Sigma_0 = \Delta_0 = -1, \qquad \Sigma_1 = \Delta_1 = 0. \qquad (20)$$

The decoupled equation for Δ_n implies that this quantity has period 4, and from the initial values it is clear that

$$\Delta_{\bmod 4} = [0, 1, 0, -1], \qquad (21)$$

so the right-hand side of the first equation in (19) has period 2, which gives the homogeneous linear equation

$$(\mathcal{S}^2 - 1)(\mathcal{S} - 1)^2 \Sigma_n = 0,$$

where $\mathcal{S}$ denotes the shift operator such that $\mathcal{S}\Sigma_n = \Sigma_{n+1}$. Using the fact that Σ_n takes the sequence of values $-1, 0, 1, 4$ for $n = 0, 1, 2, 3$, this fourth-order recurrence is readily solved.

Lemma 6. *The solution of the system (19) with initial values (20) is given by*

$$\Sigma_n = \frac{1}{2}n^2 - \frac{3}{4} - \frac{(-1)^n}{4}$$

together with (21).

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[1]Here, and in the sequel, when going from a discrete equation (recurrence) to an ultradiscrete one, we assume the parameters a_i are generic; in particular, for (12) we assume non-zero a_i such that (13) holds.

Now if we substitute in the initial values $(k_0, k_1, l_0, l_1) = (u_0, u_1, 1, 1)$ then the numerators in (16) become a pair of polynomials in u_0, u_1, denoted $N_n(\mathbf{u})$, $\hat{N}_n(\mathbf{u})$, and we find

$$u_n = \frac{k_n}{l_n} = \frac{N_n(\mathbf{u})}{u_0^{\Delta_n} u_1^{\Delta_{n-1}} \hat{N}_n(\mathbf{u})}. \tag{22}$$

For generic non-zero coefficients such that (13) holds, the polynomials $N_n(\mathbf{u})$ and $\hat{N}_n(\mathbf{u})$ are coprime, and from the form of the recurrence they always contain a term of highest possible total degree in u_0, u_1. Thus, by (16) and the result of Lemma 6, we have

$$\deg_{\mathbf{u}}(N_n(\mathbf{u})) = \deg_{\mathbf{k}}(N_n(\mathbf{k})) = \Sigma_{n-1} + \Sigma_n + 1 = n^2 - n,$$

and the same formula holds for $\deg_{\mathbf{u}}(\hat{N}_n(\mathbf{u}))$. From the periodic sequence (21) it is clear that the Laurent monomial factor in (22) cycles in the pattern $u_0, u_1, u_0^{-1}, u_1^{-1}$, so the degree of the numerator is one more than the degree of the denominator, or vice versa. Hence $\deg_{\mathbf{u}}(u_n) = 1 + \deg_{\mathbf{u}}(N_n(\mathbf{u})) = 1 + \deg_{\mathbf{u}}(\hat{N}_n(\mathbf{u}))$, which yields an exact formula for this degree.

Theorem 7. *As a rational function of the initial values u_0, u_1, the nth iterate u_n of the multiplicative QRT map (5) has degree $n^2 - n + 1$.*

3.3 Degeneration to Somos recurrences

In this subsection, we demonstrate that the Somos-4 and Somos-5 recurrences (2), with periodic coefficients of periods 8 and 7 respectively, arise as special cases of the Laurent system (12).

In [17] the initial data for (1a) were taken as u_1, u_2, while here we use u_0, u_1 instead; mutatis mutandis, the periodic coefficients found in [17] are defined as follows.

Definition 8. *The Somos-4 equation with periodic coefficients which generates the divisors of the QRT map (1a) is given by (2a) with $\alpha_n = \alpha u_0^{\mathfrak{p}_{n+2}} u_1^{\mathfrak{p}_{n-1}}$, $\beta_n = \beta u_0^{\mathfrak{q}_{n+2}} u_1^{\mathfrak{q}_{n-1}}$, and*

$$\mathfrak{p}_{\mod 8} = [0, 1, 0, 0, 1, 0, 0, 1] \text{ and } \mathfrak{q}_{\mod 8} = [2, 0, 0, 1, 0, 1, 0, 0]. \tag{23}$$

Theorem 9. *In the degenerate case $a_1 = 1$, $a_2 = a_3 = 0$, $a_5 = \alpha$ and $a_6 = \beta$ the quantities k_n and l_n can be expressed in terms of solutions to (2a) as*

$$k_n = u_0^{2\zeta_{n+2} + \mathfrak{q}_{n+1} - \mathfrak{p}_{n+1}} u_1^{2\zeta_{n-1} + \mathfrak{q}_{n-2} - \mathfrak{p}_{n-2}} c_{n-2} c_n, \quad l_n = u_0^{2\zeta_{n+2}} u_1^{2\zeta_{n-1}} c_{n-1}^2, \tag{24}$$

where ζ_n satisfies

$$(\mathcal{S} - 1)^2 \zeta_n = \mathfrak{p}_n - \mathfrak{q}_n. \tag{25}$$

Proof. Taking $a_1 = 1$ and $a_2 = a_3 = 0$, $a_5 = \alpha$ and $a_6 = \beta$ in equation (12), it is easy to see that $k_n = \tau_{n-2} \tau_n$, $l_n = \tau_{n-1}^2$ is a solution of (12) whenever τ_n satisfies the autonomous Somos-4 recurrence

$$\tau_{n+2} \tau_{n-2} = \alpha \tau_{n+1} \tau_{n-1} + \beta \tau_n^2. \tag{26}$$

Moreover, for this degenerate choice of coefficients, every solution of (12) can be written in this way, e.g. by taking $\tau_{-2} = k_0/\sqrt{l_1}$, $\tau_{-1} = \sqrt{l_0}$, $\tau_0 = \sqrt{l_1}$, $\tau_1 = k_1/\sqrt{l_0}$ as initial values for (26). Now one can make a gauge transformation between τ_n and c_n, which is required to satisfy (2a) :

$$\tau_n = u_0^{\zeta_{n+3}} u_1^{\zeta_n} c_n \implies (\mathcal{S}^2 - 1)^2 \zeta_n = -\mathfrak{q}_{n+1},$$

a fourth-order recurrence for ζ_n, and also (25) must hold. But the form of the 8-periodic coefficients (23) implies that $(\mathcal{S} + 1)^2 (\mathfrak{p}_n - \mathfrak{q}_n) = -\mathfrak{q}_{n+1}$, so the fourth-order relation is a consequence of (25), which completely determines the sequence of exponents (ζ_n), up to a suitable choice of ζ_0, ζ_1. $\square$

Definition 10. *The Somos-5 equation with periodic coefficients which generates the divisors of the QRT map (1b) is given by (2b), where*

$$\gamma_n = \gamma u_0^{\mathfrak{p}_{n+1}} u_1^{\mathfrak{p}_{n-3}} \quad and \quad \delta_n = \delta u_0^{\mathfrak{q}_{n+1}} u_1^{\mathfrak{q}_{n-3}},$$

with $\mathfrak{p}_{\mod 7} = [1, 0, 0, 0, 1, 0, 0]$ and $\mathfrak{q}_{\mod 7} = [0, 0, 1, 0, 0, 1, 1]$.

Theorem 11. *In the degenerate case $a_1 = a_3 = 0$, $a_2 = 1$, $a_5 = \gamma$ and $a_6 = \delta$, k_n and l_n are expressed in terms of solutions to (2b) as*

$$k_n = u_0^{\eta_{n+3}+\eta_n} u_1^{\eta_{n-1}+\eta_{n-4}} d_n d_{n-3}, l_n = u_0^{\eta_{n+2}+\eta_{n+1}} u_1^{\eta_{n-2}+\eta_{n-3}} d_{n-1} d_{n-2}, \tag{27}$$

where

$$(\mathcal{S}^3 - \mathcal{S}^2 - \mathcal{S} + 1)\eta_n = \mathfrak{p}_n - \mathfrak{q}_n. \tag{28}$$

Proof. This is similar to the proof of Theorem 9. The autonomous Somos-5 relation

$$\tau_{n-3}\tau_{n+2} = \gamma\tau_{n-2}\tau_{n+1} + \delta\tau_{n-1}\tau_n$$

solves this degenerate case of (12) by taking $k_n = \tau_{n-3}\tau_n$, $l_n = \tau_{n-2}\tau_{n-1}$, and then a gauge transformation $\tau_n = u_0^{\eta_{n+3}} u_1^{\eta_{n-1}} d_n$ gives a corresponding solution of (2b), provided η_n satisfies the third-order linear relation (28) above. □

4 The additive symmetric QRT map

The aim of this section is to Laurentify the additive QRT map (also called the McMillan map), and to establish a formula for the growth of degrees.

4.1 Laurentification of the additive symmetric QRT map

Substituting $u_n = \frac{k_n}{l_n}$ into (6), and identifying quadratic numerators and denominators on each side leads to the following associated Laurent system.

Proposition 12. *The system*

$$\begin{aligned}
k_{n+1}l_{n-1} &= -(k_{n-1}l_{n+1} + a_2\,k_n^{\,2} + a_4\,k_n l_n + a_5\,l_n^{\,2}), \\
l_{n+1}l_{n-1} &= a_1\,k_n^{\,2} + a_2\,k_n l_n + a_3\,l_n^{\,2}
\end{aligned} \tag{29}$$

has the Laurent property. Any four adjacent iterates k_n, l_n, k_{n+1}, l_{n+1} are pairwise coprime Laurent polynomials in the ring $\mathcal{R}[k_0, k_1, l_0^{\pm 1}, l_1^{\pm 1}]$, where $\mathcal{R} = \mathbb{Z}[a_1, a_2, a_3, a_4, a_5]$.

Proof. This follows from Theorem 2 and a coprimality argument analogous to the proof of Proposition 4. □

It is straightforward to check that the equation (4), obtained from (3) in [17], corresponds to a degenerate case of (29).

Proposition 13. *In the degenerate case $a_1 = a_3 = a_4 = 0$, $a_2 = 1$ and $a_5 = -\alpha$, the solution of (29) is given by $k_n = e_{n+1}e_{n-2}$ and $l_n = e_n e_{n-1}$, where e_n satisfies equation (4).*

4.2 Growth of degrees of the additive QRT map

Just as in the multiplicative case, the Laurent property means that the iterates of (29) can be factored as in (14), in terms of a general set of initial values $\mathbf{k} = (k_0, k_1, l_0, l_1)$, with initial denominator vectors (15), where (for generic parameter values) N_n and $\hat{N}_n$ are coprime. Observe that the system is also bilinear (homogeneous of degree 2), so the relations (16) hold, and the second equations in (12) and (29) are identical. Note also that, due to the minus sign in the first equation, the system (29) does not generate subtraction-free rational expressions in $\mathbf{k}$. Nevertheless, by considering the dependence on k_0, k_1, it is not hard to show that cancellations cannot occur in the highest degree terms appearing in the numerator on the right-hand side, and arrive at the following

Lemma 14. *The denominator vectors satisfy the max-plus tropical version of (29), namely*

$$\begin{aligned}
\mathbf{d}_{n+1} + \mathbf{e}_{n-1} &= \max(\mathbf{d}_{n-1} + \mathbf{e}_{n+1}, 2\mathbf{d}_n, \mathbf{d}_n + \mathbf{e}_n, 2\mathbf{e}_n), \\
\mathbf{e}_{n+1} + \mathbf{e}_{n-1} &= \max(2\mathbf{d}_n, \mathbf{d}_n + \mathbf{e}_n, 2\mathbf{e}_n).
\end{aligned} \tag{30}$$

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Proof. By Proposition 12, the iterates of (29) can be written in the form (14), where now the (Laurent) monomial denominators are just monomials in l_0, l_1 for all $n \geq 2$. Then, by explicitly considering how the numerators depend on k_0, k_1, $N_2(\mathbf{k}) = -a_1 k_0 k_1^2 + \ldots$, $\hat{N}_2(\mathbf{k}) = a_1 k_1^2 + \ldots$, and it can be shown by induction that for all $n \geq 2$, $N_n(\mathbf{k}) =$ monomial in k_0, k_1 of degree δ_n + lower order terms, and $\hat{N}_n(\mathbf{k}) =$ monomial in k_0, k_1 of degree $\hat{\delta}_n$ + lower order terms, where $\delta_n, \hat{\delta}_n$ denote the respective total degrees of these numerators. To perform the induction, it should be assumed also that $\delta_n > \hat{\delta}_n$ as part of the inductive hypothesis; clearly this holds for $n = 2$, and $n = 3$ is easily checked as well. Now suppose the hypothesis holds up to n, and consider the right-hand side of the second equation in (29): in terms of k_0, k_1, the term of highest degree in the numerator comes from k_n^2, and all other terms are of lower degree, so there can be no cancellation between the numerator and denominator (which only depends on l_0, l_1); thus, comparing with the numerator of the left-hand side, this implies

$$\hat{\delta}_{n+1} + \hat{\delta}_{n-1} = 2\delta_n, \tag{31}$$

and the numerator $\hat{N}_{n+1}(\mathbf{k})$ is of the required form. Similarly, on the right-hand side of the first equation in (29), the term with numerator of largest possible degree can only be $k_{n-1}l_{n+1}$ or k_n^2, but the first term has degree $\delta_{n-1} + \hat{\delta}_{n+1} = 2\delta_n + \delta_{n-1} - \hat{\delta}_{n-1} > 2\delta_n$, using (31) and the inductive hypothesis, which implies that the numerator of $k_{n-1}l_{n+1}$ is of largest degree; so again there can be no cancellation, the numerator $N_{n+1}(\mathbf{k})$ is of the required form, and comparing with the left-hand side yields

$$\delta_{n+1} + \hat{\delta}_{n-1} = \hat{\delta}_{n+1} + \delta_{n-1}, \tag{32}$$

and hence $\delta_{n+1} - \hat{\delta}_{n+1} = \delta_{n-1} - \hat{\delta}_{n-1} > 0$, which gives the other part of the hypothesis. $\square$

From the form of the initial data (15), the solution of the ultradiscrete system (30) is written as

$$\mathbf{d_n} = (d_n, d_{n-1}, \tilde{d}_n, \tilde{d}_{n-1})^T, \qquad \mathbf{e_n} = (e_n, e_{n-1}, \tilde{e}_n, \tilde{e}_{n-1})^T,$$

in terms of two pairs of sequences (d_n, e_n), $(\tilde{d}_n, \tilde{e}_n)$, each of which satisfies the scalar version of (30). Upon introducing the difference vector

$$\mathbf{\Delta}_n = \mathbf{d}_n - \mathbf{e}_n = (\Delta_n, \Delta_{n-1}, \tilde{\Delta}_n, \tilde{\Delta}_{n-1})^T,$$

the system is equivalent to

$$\begin{aligned} \mathbf{\Delta}_{n+1} &= \max(\mathbf{\Delta}_{n-1}, 0), \\ \mathbf{e}_{n+1} - 2\mathbf{e}_n + \mathbf{e}_{n-1} &= \max(2\mathbf{\Delta}_n, 0). \end{aligned} \tag{33}$$

For the first sequence pair (d_n, e_n), the initial data give $\Delta_0 = -1$, $\Delta_1 = 0$, which implies $\Delta_n = 0$ for $n \geq 1$, while for the pair $(\tilde{d}_n, \tilde{e}_n)$ with $\tilde{\Delta}_0 = 1$, $\tilde{\Delta}_1 = 0$, the first equation above gives $\tilde{\Delta}_{\bmod 2} = [0, 1]$. For the second equation in (33), the solution $\mathbf{e}_n$ is then found to be specified by

$$\tilde{e}_n = \frac{1}{2}n^2 - \frac{3}{4} - \frac{(-1)^n}{4} \quad \text{and} \quad e_n = 0 \quad \forall n \geq 0 \implies d_n = 0 \quad \forall n \geq 1,$$

so k_0, k_1 never appear in the denominator of any Laurent polynomials, as is obvious from (29).

Finally, from (16) we see that, consistent with (31) and (32),

$$\deg_{\mathbf{k}}(N_n(\mathbf{k})) = \Delta_n + \Delta_{n-1} + \tilde{\Delta}_n + \tilde{\Delta}_{n-1} + \deg_{\mathbf{k}}(\hat{N}_n(\mathbf{k})) = 1 + \deg_{\mathbf{k}}(\hat{N}_n(\mathbf{k})) \quad \text{for } n \geq 2$$

and $\deg_{\mathbf{k}}(\hat{N}_n(\mathbf{k})) = 1 + e_n + e_{n-1} + \tilde{e}_n + \tilde{e}_{n-1} = n^2 - n$. By setting $\mathbf{k} = (u_0, u_1, 1, 1)$ to find $u_n = k_n/l_n = N_n(\mathbf{u})/\hat{N}_n(\mathbf{u})$ for $n \geq 2$, and noting that the total degrees of N_n and $\hat{N}_n$ remain the same after substitution, this yields the same quadratic expression for the degree growth as for (5).

Theorem 15. *As a rational function of the initial values u_0, u_1, the nth iterate u_n of the additive QRT map (6) has degree $n^2 - n + 1$.*

5 The 12-parameter symmetric QRT map

The symmetric QRT map [27, 28] is constructed as follows. We start with two symmetric 3×3 matrices,

$$\mathcal{A} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{01} & b_{11} & b_{12} \\ b_{02} & b_{12} & b_{22} \end{pmatrix}, \tag{34}$$

and introduce the vectors

$$\mathbf{V}(u,v) = \begin{pmatrix} u^2 \\ uv \\ v^2 \end{pmatrix}, \qquad \mathbf{f}(u) = \mathcal{A}\mathbf{V}(u,1) \times \mathcal{B}\mathbf{V}(u,1),$$

with the components of $\mathbf{f}$ denoted by $f^{(i)}$, $i = 1, 2, 3$. The map of the plane defined by

$$\varphi_{sym} : (u_{n-1}, u_n) \mapsto (u_n, u_{n+1}), \quad \text{with} \quad u_{n+1} = \frac{f^{(1)}(u_n) - u_{n-1}f^{(2)}(u_n)}{f^{(2)}(u_n) - u_{n-1}f^{(3)}(u_n)}, \tag{35}$$

is the general form of the symmetric QRT map, which admits the invariant

$$J = \frac{\mathbf{V}(u_{n-1}, 1)^T \mathcal{A}\mathbf{V}(u_n, 1)}{\mathbf{V}(u_{n-1}, 1)^T \mathcal{B}\mathbf{V}(u_n, 1)}. \tag{36}$$

5.1 Recursive factorisation of the symmetric QRT map

Let us homogenise the map φ_{sym}. Taking $u_n = p_n/q_n$ gives

$$\frac{p_{n+1}}{q_{n+1}} = \frac{q_{n-1}f^{(1)}(\frac{p_n}{q_n}) - p_{n-1}f^{(2)}(\frac{p_n}{q_n})}{q_{n-1}f^{(2)}(\frac{p_n}{q_n}) - p_{n-1}f^{(3)}(\frac{p_n}{q_n})},$$

from which we obtain the polynomial system

$$p_{n+1} = q_{n-1}F_n^{(1)} - p_{n-1}F_n^{(2)}, \quad q_{n+1} = q_{n-1}F_n^{(2)} - p_{n-1}F_n^{(3)}, \tag{37}$$

with $F_n^{(i)} = q_n^4 f^{(i)}(\frac{p_n}{q_n})$. Slightly more explicitly, in terms of the vectors $\mathbf{A}_n := \mathcal{A}\mathbf{V}(p_n, q_n)$, $\mathbf{B}_n := \mathcal{B}\mathbf{V}(p_n, q_n)$, with components denoted $A_n^{(i)}$, $B_n^{(i)}$ respectively, we set $\mathbf{F}_n = \mathbf{A}_n \times \mathbf{B}_n$, and then the system (37) can be written as

$$\begin{aligned} p_{n+1} &= B_n^{(3)}(p_{n-1}A_n^{(1)} + q_{n-1}A_n^{(2)}) - A_n^{(3)}(p_{n-1}B_n^{(1)} + q_{n-1}B_n^{(2)}), \\ q_{n+1} &= B_n^{(1)}(p_{n-1}A_n^{(2)} + q_{n-1}A_n^{(3)}) - A_n^{(1)}(p_{n-1}B_n^{(2)} + q_{n-1}B_n^{(3)}), \end{aligned} \tag{38}$$

which is equivalent to the vector equation

$$(q_{n-1}q_{n+1}, -p_{n-1}q_{n+1} - q_{n-1}p_{n+1}, p_{n-1}p_{n+1})^T = \mathbf{W}_n, \tag{39}$$

where $\mathbf{W}_n = \mathbf{F}_n \times \mathbf{V}_{n-1}$, $\mathbf{V}_n = \mathbf{V}(p_n, q_n)$. The map (38) generates polynomials when iterated forwards, but not backwards, as the inverse does not have the Laurent property (this is analogous to the fact that a generic polynomial map does not have a polynomial inverse); it leaves the ratio of polynomials N_n/D_n invariant, where

$$N_n = \mathbf{V}_n \cdot \mathbf{A}_{n-1} = \mathbf{V}_{n-1} \cdot \mathbf{A}_n, \quad D_n = \mathbf{V}_n \cdot \mathbf{B}_{n-1} = \mathbf{V}_{n-1} \cdot \mathbf{B}_n,$$

since the matrices $\mathcal{A}$, $\mathcal{B}$ are symmetric. The invariance implies that N_n divides N_{n+1} and D_n divides D_{n+1}. We can describe the factorisations as follows.

Lemma 16. *We have $N_{n+1} = N_n Q_n$ and $D_{n+1} = D_n Q_n$ where $Q_n = (F_n^{(2)})^2 - F_n^{(1)} F_n^{(3)}$.*

Proof. Using $\mathbf{F} = \mathbf{A} \times \mathbf{B}$, hence $\mathbf{F} \cdot \mathbf{A} = 0$, $\mathbf{W} = \mathbf{F} \times \mathbf{V} = (\mathbf{A} \cdot \mathbf{V})\mathbf{B} - (\mathbf{B} \cdot \mathbf{V})\mathbf{A}$, we get

$$
\begin{aligned}
N_{n+1} &= \mathbf{V}_{n+1} \cdot \mathbf{A}_n \\
&= p_{n+1}(q_{n-1}F_n^{(1)} - p_{n-1}F_n^{(2)})A_n^{(1)} + (q_{n-1}F_n^{(2)} - p_{n-1}F_n^{(3)})(p_{n+1}A_n^{(2)} + q_{n+1}A_n^{(3)}) \\
&= W^{(3)}(-F_n^{(2)}A_n^{(1)} - F_n^{(3)}A_n^{(2)}) + W^{(2)}F_n^{(3)}A_n^{(3)} + W^{(1)}F_n^{(2)}A_n^{(3)} \\
&= \mathbf{A}_n \cdot \mathbf{V}_{n-1}\left(F_2^n(A_n^{(3)}B_n^{(1)} - A_n^{(1)}B_n^{(3)}) - F_n^{(3)}(A_n^{(2)}B_n^{(3)} - A_n^{(3)}B_n^{(2)})\right) = N_n Q_n.
\end{aligned}
$$

□

Iterating the map one more time, the quotient Q_n arises as a common divisor of p_{n+2} and q_{n+2}. This can be seen from

$$
\begin{aligned}
p_{n+2} &= B_{n+1}^{(3)}(p_n A_{n+1}^{(1)} + q_n A_{n+1}^{(2)}) - A_{n+1}^{(3)}(p_n B_{n+1}^{(1)} + q_n B_{n+1}^{(2)}) \\
&= \frac{B_{n+1}^{(3)}(p_n^2 A_{n+1}^{(1)} + p_n q_n A_{n+1}^{(2)} + p_n^2 A_{n+1}^{(3)})}{p_n} - \frac{A_{n+1}^{(3)}(p_n^2 B_{n+1}^{(1)} + p_n q_n B_{n+1}^{(2)} + p_n^2 B_{n+1}^{(3)})}{p_n} \\
&= \frac{B_{n+1}^{(3)} N_{n+1} - A_{n+1}^{(3)} D_{n+1}}{p_n} = Q_n \left(\frac{B_{n+1}^{(3)} N_n - A_{n+1}^{(3)} D_n}{p_n} \right).
\end{aligned}
$$
(40)

The second term in (40) is polynomial, which can be seen directly from the factorisation

$$
B_{n+1}^{(3)} A_{n-1}^{(3)} - A_{n+1}^{(3)} B_{n-1}^{(3)} = (p_{n-1}q_{n+1} - p_{n+1}q_{n-1})(\mathbf{F}_n \times \mathbf{V}_{n-1}) \cdot \mathbf{P},
$$

where $\mathbf{P} = (a_{02}, a_{12}, a_{22})^T \times (b_{02}, b_{12}, b_{22})^T$, and the observation that $\mathbf{F}_n \equiv \mathbf{P}q_n^4 \bmod p_n$. For q_{n+2} we find the similar expression

$$
q_{n+2} = Q_n \left(\frac{B_{n+1}^{(1)} N_n - A_{n+1}^{(1)} D_n}{q_n} \right).
$$
(41)

Theorem 17. *The polynomial Q_n is a divisor of Q_{n+1}.*

Proof. The proof is by direct computation. We write

$$
Q_n = \sum_{i=0}^{8} d_i p_n^{8-i} q_n^i = Q(p_n, q_n).
$$
(42)

Considering the three components of $\mathbf{F}_n$ as variables, we substitute equation (37) into Q_{n+1} and reduce the result modulo $Q_n = (F_n^{(2)})^2 - F_n^{(1)}F_n^{(3)}$ using a total degree ordering. We denote the coefficients of the resulting polynomial in p_{n-1}, q_{n-1} by e_i, so that $Q_{n+1}\mid_{\text{Eq.}(37)} \equiv \sum_{i=0}^{8} e_i p_{n-1}^{8-i} q_{n-1}^i \bmod Q_n$. There appear to be two non-trivial common factors, namely $X = \gcd(e_0, e_2, e_4, e_6, e_8)$, $Y = \gcd(e_1, e_3, e_5, e_7)$, and we have $Q_n \mid Y - X F_2^n$. Thus it suffices to establish that Q_n is a divisor of X. Curiously, we have the following expression, modulo Q_n: $X(F_n^{(1)})^4 \equiv \sum_{i=0}^{8} d_i (F_n^{(1)})^{8-i}(F_n^{(2)})^i = Q(F_n^{(1)}, F_n^{(2)})$. Finally, with computer algebra (e.g. Maple) it can be verified that Q_n divides $Q(F_n^{(1)}, F_n^{(2)})$. □

From (38), an ultradiscrete system of recurrences for a lower bound on the multiplicities is given as follows:

$$
\mathfrak{m}_{n+1}^p = \mathfrak{m}_{n+1}^q = \min_{r \in \{p,q\}, i+j=4, i,j \geq 0} \left(\mathfrak{m}_{n-1}^r + i\mathfrak{m}_n^p + j\mathfrak{m}_n^q \right).
$$
(43)

So a lower bound on the growth of the multiplicity of divisors is $\mathfrak{m}_{n+1}^p = \mathfrak{m}_{n+1}^q = \mathfrak{m}_{n+1}$ with

$$
\mathfrak{m}_{n+1} = 4\mathfrak{m}_n + \mathfrak{m}_{n-1},
$$
(44)

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where $\mathfrak{m}_0 = 0$ and $\mathfrak{m}_1 = 1$. We are interested in primitive divisors. Therefore, according to Theorem 17 we want to divide Q_n by Q_{n-1}. But there is much more to divide out. As Q_{n-2} divides $\gcd(p_n, q_n)$ and Q_n is homogeneous in p_n, q_n of degree 8, Q_n is also divisible by $(Q_{n-2})^8$, and by $(Q_{n-3})^{32}$, and so on. We can recursively define a polynomial r_n by $r_1 = 1$ and

$$Q_n = \prod_{k=2}^{n-2} r_{n-k}^{8\mathfrak{m}_{k-1}} r_{n-1} r_n \implies r_n = \prod_{j=0}^{n-2} (Q_{n-j})^{1-2j^2}.$$

In terms of r_n, a common divisor of p_n and q_n is given by

$$g_n = \prod_{k=2}^{n-2} r_k^{\mathfrak{m}_{n-k-1}} \implies g_{n+1} = r_{n-1} g_{n-1} g_n^4. \tag{45}$$

As every divisor is a common divisor, we define the quotients s_n, t_n by

$$p_n = g_n s_n \quad \text{and} \quad q_n = g_n t_n. \tag{46}$$

To find a closed system of equations we need to involve r_n and find out how it relates to s_n, t_n. Using $Q_n = Q(p_n, q_n) = g_n^8 Q(s_n, t_n)$ it can be verified that $r_n r_{n-1} = Q(s_n, t_n)$. Substituting (46) into (37), and using (45), we arrive at the system

$$\begin{aligned}
r_n r_{n-1} &= Q(s_n, t_n), \\
s_{n+1} r_{n-1} &= t_{n-1} F^{(1)}(s_n, t_n) - s_{n-1} F^{(2)}(s_n, t_n), \\
t_{n+1} r_{n-1} &= t_{n-1} F^{(2)}(s_n, t_n) - s_{n-1} F^{(3)}(s_n, t_n),
\end{aligned} \tag{47}$$

with $\mathbf{F}(s_n, t_n) = g_n^{-4} \mathbf{F}_n$. Up to shifting indices, initial values can be chosen as $s_0 = u_0$, $s_1 = u_1$, $t_0 = t_1 = r_0 = 1$. The ratio $u_n = \frac{s_n}{t_n}$ is an iterate of the symmetric QRT map (35).

Theorem 18. *The system (47) has the Laurent property: $r_n, s_n, t_n \in \hat{\mathcal{R}} := \mathcal{R}[r_0^{\pm 1}, s_0, s_1, t_0, t_1]$, where $\mathcal{R}$ is the ring of polynomials in the parameters a_{ij}, b_{ij} over $\mathbb{Z}$.*

Proof. Upon noting that $Q(s,t)$ and $F^{(j)}(s,t)$ are homogeneous with weights 8 and 4 respectively, we see that the system (47) is weighted homogeneous, where r_n has degree 4 and s_n, t_n have degree 1. This allows us to show by induction that the iterates can be written in terms of $\mathbf{r} = (r_0, s_0, s_1, t_0, t_1)$ in a similar fashion to (14), as

$$r_n = \frac{\mathrm{N}_n^*(s_0, s_1, t_0, t_1)}{r_0^{c_n}}, \qquad s_n = \frac{\mathrm{N}_n(s_0, s_1, t_0, t_1)}{r_0^{d_n}}, \qquad t_n = \frac{\hat{\mathrm{N}}_n(s_0, s_1, t_0, t_1)}{r_0^{e_n}}, \tag{48}$$

where N_n^*, N_n, $\hat{\mathrm{N}}_n$ are homogeneous polynomials in s_0, s_1, t_0, t_1. The system (47) only involves division by r_{n-1} at each iteration, and since it is of second order in s_n, t_n and only first order in r_n, it is slightly more general than the conditions in Theorem 2, but Hickerson's method still extends to this situation. Clearly the 5 initial values as well as r_1, s_2, t_2 belong to $\hat{\mathcal{R}}$, while from Theorem 17 it follows that $Q(s_1, t_1) | Q(s_2, t_2)$, which implies $r_2 \in \hat{\mathcal{R}}$, and the same computations that yield (40) and (41) also give $s_3, t_3 \in \hat{\mathcal{R}}$. It can also be verified with computer algebra that r_1 and r_2 are coprime; it is sufficient to check that this is so for some particular numerical choice of coefficients a_{ij}, b_{ij}, in which case it must hold when the coefficients are variables. Then from the inductive hypothesis we can write the new iterate produced by the first equation in (47) as

$$r_n = \frac{\mathrm{N}_{n-1}^*(s_1, s_2, t_1, t_2)}{r_1^{c_{n-1}}} = \frac{\mathrm{N}_{n-2}^*(s_2, s_3, t_2, t_3)}{r_2^{c_{n-2}}}.$$

By substituting for $r_1, r_2, s_2, s_3, t_2, t_3$ as Laurent polynomials in $\hat{\mathcal{R}}$ and using $(r_1, r_2) = 1$, it follows from the equality of the latter two expressions above that $r_n \in \hat{\mathcal{R}}$, and the same argument shows that $s_{n+1}, t_{n+1} \in \hat{\mathcal{R}}$. $\qquad \square$

In an appendix, we also prove the following result.

Lemma 19. *The Laurent polynomials s_n and t_n generated by (47) are coprime in $\hat{\mathcal{R}}$ for all $n \geq 0$.*

As before, no cancellations occur in the numerators when the Laurent polynomials are substituted into the right-hand sides of the system (47), so we can immediately write down the ultradiscrete system for the denominator exponents c_n, d_n, e_n in (48), that is

$$
\begin{aligned}
c_n + c_{n-1} &= \max\Big((8-i)d_n + ie_n\Big)_{i=0,\dots,8}, \\
d_{n+1} + c_{n-1} &= \max(d_{n-1}, e_{n-1}) + M_n, \\
e_{n+1} + c_{n-1} &= \max(d_{n-1}, e_{n-1}) + M_n, \quad M_n := \max\Big((4-i)d_n + ie_n\Big)_{i=0,\dots,4}
\end{aligned}
$$

Subtracting the last two equations above implies that $d_n = e_n$ for all $n \geq 2$, and hence

$$
c_n + c_{n-1} = 8d_n, \qquad d_{n+1} + c_{n-1} = 4d_n + d_{n-1} \implies (\mathcal{S} - 1)^3 d_n = 0.
$$

Therefore d_n and e_n both grow quadratically with n, as do the degrees of N_n, $\hat{N}_n$, and (by Lemma 19 and its proof - see appendix) these are coprime and their degrees remain the same after substituting $\mathbf{r} = (1, u_0, u_1, 1, 1)$ into $u_n = N_n/\hat{N}_n$. This yields an exact formula for the degree growth of (35). To be precise, for $n \geq 2$ we find $d_n = e_n = (n^2 - n)/2 \implies \deg_{\mathbf{u}}(N_n) = \deg_{\mathbf{u}}(\hat{N}_n) = \deg_{\mathbf{u}}(u_n) = 2n^2 - 2n + 1$, using the fact that $\deg_{\mathbf{r}}(N_n) = 4d_n + 1$ by the weighted homogeneity of (47).

Theorem 20. *As a rational function of the initial values u_0, u_1, the nth iterate u_n of the symmetric QRT map (35) has degree $2n^2 - 2n + 1$.*

5.2 Degeneration to the Laurentified multiplicative/additive QRT maps

In this subsection, we show how the Laurentified multiplicative and additive QRT maps arise as special cases of the three-component system (47).

Theorem 21. *In the degenerate case $b_{11} = 1$ and all other $b_{ij} = 0$, the polynomials r_n, s_n and t_n can be expressed in terms of the solution to (12) as*

$$
r_n = k_{n+1} l_{n+1} k_n l_n, \qquad s_n = k_n, \qquad t_n = l_n. \tag{49}
$$

Proof. By substituting $b_{11} = 1$ and no other non-zero b_{ij} in (47), we find

$$
r_n r_{n-1} = s_n^2 t_n^2 R_n S_n, \quad s_{n+1} r_{n-1} = t_{n-1} t_n s_n S_n, \quad t_{n+1} r_{n-1} = s_{n-1} s_n t_n R_n, \tag{50}
$$

where $R_n = a_1 s_n{}^2 + a_2 s_n t_n + a_3 t_n{}^2$ and $S_n = a_3 s_n{}^2 + a_5 s_n t_n + a_6 t_n{}^2$. With the substitutions (49), the second and third equations n (50) correspond to the two equations (12), while the first one is a consequence of them. $\qquad\square$

Theorem 22. *In the degenerate case $b_{22} = 1$ and all other $b_{ij} = 0$, r_n, s_n and t_n are expressed in terms of solutions to (29) as*

$$
r_n = -l_n^2 l_{n+1}^2, \qquad s_n = k_n, \qquad t_n = l_n. \tag{51}
$$

Proof. Setting all $b_{ij} = 0$ apart from $b_{22} = 1$ in (47) gives

$$
r_n r_{n-1} = t_n^4 R_n^2, \qquad s_{n+1} r_{n-1} = t_n^2 (s_{n-1} R_n + t_{n-1} S_n), \qquad t_{n+1} r_{n-1} = -t_{n-1} t_n^2 R_n, \tag{52}
$$

where $R_n = a_1 s_n^2 + a_2 t_n s_n + a_3 t_n^2$, $S_n = a_2 s_n^2 + a_4 s_n t_n + a_5 t_n^2$. The first and third equations above follow from the second equation in (29). The second equation in (52) is a consequence of the first equation of the system (29). $\qquad\square$

6 Somos recurrences

In this section we show that the components of the Laurent systems we have derived satisfy Somos-7 recurrence relations. We start with the multiplicative case, and from this we will obtain the additive case, before proceeding to obtain the Somos-7 relations corresponding to the general symmetric QRT map as a corollary of a broader result for asymmetric QRT.

6.1 The multiplicative case

Let

$$I = \frac{a_1 u_0^2 u_1^2 + a_2 u_0 u_1 (u_0 + u_1) + a_3 (u_0^2 + u_1^2) + a_5 (u_0 + u_1) + a_6}{u_0 u_1}, \tag{53}$$

be the invariant of (5) considered as a map on the plane of initial values (u_0, u_1). We will prove the following theorem.

Theorem 23. *The variables k_n, l_n in (12) each satisfy the same Somos-7 recurrence,*

$$x_{n+7} x_n = A x_{n+6} x_{n+1} + B x_{n+5} x_{n+2} + C x_{n+4} x_{n+3}, \tag{54}$$

with coefficients given by

$$A = -(\alpha^3 + \alpha\beta\gamma + \beta^2), \qquad B = (\beta^2 - A)\alpha^2, \qquad C = A(A - \beta^2), \tag{55}$$

with

$$\alpha = a_2 a_5 + a_3^2 - a_1 a_6 + a_3 I, \quad \beta = (a_1 a_5 - 2 a_2 a_3) a_5 + a_2^2 a_6 + (a_1 a_6 - a_3^2) I, \quad \gamma = 4 a_3 + I. \tag{56}$$

The coefficients may be found using computer algebra as follows. If x_n, y_n satisfy the same Somos-7 recurrence, then

$$\det \begin{pmatrix} x_5 x_{-2} & x_4 x_{-1} & x_3 x_0 & x_2 x_1 \\ x_6 x_{-1} & x_5 x_0 & x_4 x_1 & x_3 x_2 \\ y_5 y_{-2} & y_4 y_{-1} & y_3 y_0 & y_2 y_1 \\ y_6 y_{-1} & y_5 y_0 & y_4 y_1 & y_3 y_2 \end{pmatrix} = 0$$

for all initial values (where, in each row of the matrix, the indices can be shifted by an arbitrary amount). The coefficients of the Somos-7 recurrence are then found by calculating a constant vector (independent of n) that spans the one-dimensional kernel of the above matrix - see [21] for an introduction to this method.

However, it is more instructive to prove the result by considering the form of the solution. As an added benefit, this provides more succinct expressions for the coefficients. We first show that k_n, l_n can be written in the form of products

$$k_n = \tau_n \tau_n^*, \qquad l_n = \tilde{\tau}_n \tilde{\tau}_n^*, \tag{57}$$

where each of $\tau_n, \tau_n^*, \tilde{\tau}_n, \tilde{\tau}_n^*$, satisfy a Somos-5 recurrence (in fact, the *same* Somos-5 recurrence, with identical coefficients and a first integral taking the same value); then we use the fact that products of this kind provide special solutions of Somos-7 recurrences.

The general Somos-7 recurrence, of the form (54), arises as a reduction of the bilinear discrete BKP equation, a partial difference equation which is also known as the cube recurrence [10]. It is possible to obtain a general analytic solution in terms of genus two sigma functions, corresponding to a translation on a two-dimensional Jacobian variety, which we intend to present elsewhere. Nevertheless, Somos-7 also admits special solutions given by products of elliptic sigma functions, which are described as follows.

Proposition 24. *Let τ_n satisfy the Somos-5 recurrence*

$$\tau_{n+5}\tau_n = \alpha\tau_{n+4}\tau_{n+1} + \beta\tau_{n+3}\tau_{n+2}, \tag{58}$$

with coefficients α, β, and let τ_n^* satisfy the same recurrence but with coefficients α^*, β^*. Then whenever the constraint

$$A = \frac{\psi_4\psi_2^*\psi_5^*}{\psi_2\psi_4^*} = \frac{\psi_4^*\psi_2\psi_5}{\psi_2^*\psi_4} \tag{59}$$

holds, the product $k_n = \tau_n\tau_n^*$ satisfies a Somos-7 recurrence of the form (54) with A given by (59),

$$B = \frac{\psi_3\psi_4\psi_3^*\psi_4^*}{\psi_2\psi_2^*} - A\psi_3\psi_3^*, \qquad C = \psi_5\psi_5^* - A\frac{\psi_4\psi_4^*}{\psi_2\psi_2^*}, \tag{60}$$

where ψ_n, ψ_n^* denotes the companion elliptic divisibility sequence associated with τ_n, τ_n^* respectively.

Proof. For later use, we record analytic formulae from [19, 20] concerning the solution of (58), and its companion elliptic divisibility sequence (EDS). From the proof of Corollary 2.12 in [19], the general analytic solution of (58) can be written in the form

$$\tau_n = A_\pm B_\pm^{[\frac{n}{2}]}\mu^{[\frac{n}{2}]^2}\sigma(z_0 + n\kappa), \qquad \frac{B_+}{B_-} = -\mu^{-1} = \sigma(\kappa)^4, \tag{61}$$

with the $\pm$ signs selected according to the parity of n, where $\sigma(z) = \sigma(z; g_2, g_3)$ is the Weierstrass sigma function corresponding to the elliptic curve $\mathcal{E}$ in the (x, y) plane given by

$$\mathcal{E}: \qquad y^2 = 4x^3 - g_2 x - g_3, \tag{62}$$

and its companion EDS is given by $\psi_n = \sigma(n\kappa)/\sigma(\kappa)^{n^2-1}$. The coefficients in (58) can be expressed in terms of the companion EDS, as

$$\alpha = \psi_3, \qquad \beta = -\frac{\psi_4}{\psi_2}, \tag{63}$$

and the terms of the sequence τ_n satisfy a particular Somos-7 recurrence as well, namely

$$\tau_{n+7}\tau_n = \frac{\psi_3\psi_4}{\psi_2}\tau_{n+5}\tau_{n+2} - \psi_5\,\tau_{n+4}\tau_{n+3}, \qquad \psi_5 = \psi_2^3\psi_4 - \psi_3^3 \tag{64}$$

(see also [31]). Upon substituting $\bar{k}_n = \tau_n\tau_n^*$ into (54) and using (58), (64) and the corresponding equations (with asterisks) for τ_n^* to eliminate products with shifts of width 5 and 7, one obtains a linear equation in the products XX^*, YY^*, XY^*, YX^*, where $X = \tau_{n+5}\tau_{n+2}$, $Y = \tau_{n+4}\tau_{n+3}$, and similarly for X^*, Y^*. Since this expression must vanish for all n, it is required that the coefficients of each of these four products should be zero, leading to the two different equations for A in (59), which have to be consistent, as well as the equations (60) for B, C. $\qquad\square$

Corollary 25. *If the sequences τ_n, τ_n^* satisfy the Somos-5 recurrence (58) with the same coefficients α, β and have the same value of the invariant*

$$\gamma = \frac{\tau_{n+1}\tau_{n+4}}{\tau_{n+2}\tau_{n+3}} + \frac{\tau_{n+1}\tau_{n+2}}{\tau_n\tau_{n+3}} + \alpha\left(\frac{\tau_{n+1}\tau_{n+4}}{\tau_{n+2}\tau_{n+3}} + \frac{\tau_n\tau_{n+3}}{\tau_{n+1}\tau_{n+2}}\right) + \beta\frac{\tau_{n+2}^2}{\tau_n\tau_{n+4}}, \tag{65}$$

then $k_n = \tau_n\tau_n^*$ satisfies the Somos-7 recurrence (54) with coefficients

$$A = \psi_5, \qquad B = (D - A)\psi_3^2, \qquad C = A(A - D), \qquad \text{where} \quad D = \left(\frac{\psi_4}{\psi_2}\right)^2. \tag{66}$$

Proof of Corollary: The fact that α, β, γ are the same for both sequences implies that they have the same companion EDS ψ_n. This means that the constraint (59) is trivially satisfied, and the expressions for the coefficients A, B, C simplify dramatically. $\qquad\square$

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Theorem 26. *The solution of the bilinear system (12) can be written in the form (57), where* $\tau_n, \tau_n^*, \tilde{\tau}_n, \tilde{\tau}_n^*$ *all satisfy a Somos-5 recurrence with the same coefficients* α, β *and the same value of the invariant* γ *given by (65).*

Proof. By Proposition 2.5 in [19], the solution of the multiplicative QRT map (5) can be written as $u_n = f(z_0 + n\kappa)$ where $f = f(z)$ is an elliptic function of its argument i.e. periodic with respect to the period lattice Λ defined by an elliptic curve $\mathcal{E}$ as in (62). The function $f(z)$ provides a uniformization of the curve $\mathcal{C}$ in the (u_0, u_1) plane defined by fixing the value of the invariant I for the map, and this curve is isomorphic to $\mathcal{E}$. Also, projecting onto the first coordinate, $(u_0, u_1) \mapsto u_0$, defines a $2 : 1$ map $\mathcal{C} \to \mathbb{P}^1$, so the function f is an elliptic function of order 2. Hence f has two zeros and two poles in any period parallelogram for Λ, and so, by a standard result in the theory of elliptic functions (see e.g. §20·53 in [35]), up to a shift in z,

$$f(z) = K\frac{\sigma(z - Z)\sigma(z + Z)}{\sigma(z - P)\sigma(z + P)}$$

for some constants Z, P, K. Thus

$$u_n = \frac{k_n}{l_n} = K\frac{\sigma(z_n - Z)\sigma(z_n + Z)}{\sigma(z_n - P)\sigma(z_n + P)}, \qquad z_n = z_0 + n\kappa, \tag{67}$$

and it remains to check that by inserting suitable n-dependent prefactors as in (61) the numerator k_n and denominator l_n can each be written as a product of two Somos-5 sequences. To be more precise, the general solution of the bilinear system (12) can be written as

$$k_n = \mathrm{a}_\pm \mathrm{b}_\pm^{[\frac{n}{2}]} \mu^{2[\frac{n}{2}]^2} \sigma(z_n - Z)\sigma(z_n + Z), \qquad l_n = \tilde{\mathrm{a}}_\pm \mathrm{b}_\pm^{[\frac{n}{2}]} \mu^{2[\frac{n}{2}]^2} \sigma(z_n - P)\sigma(z_n + P), \tag{68}$$

where the $\pm$ signs are chosen with the parity of n, and the prefactors are taken to satisfy the relations

$$\mu = -\sigma(\kappa)^{-4}, \qquad \frac{\mathrm{a}_+}{\tilde{\mathrm{a}}_+} = \frac{\mathrm{a}_-}{\tilde{\mathrm{a}}_-} = K, \qquad \frac{\mathrm{b}_-}{\mathrm{b}_+} = \mu^2, \qquad \mathrm{b}_+\mathrm{b}_- = \left(\frac{\mathrm{a}_-}{\mathrm{a}_+}\right)^4.$$

Subject to the above, the solution (68) is completely determined by the 9 parameters a_+, a_-, $\tilde{\mathrm{a}}_+$, z_0, κ, Z, P, g_2, g_3, which fix the 4 initial values and 5 coefficients a_1, a_2, a_3, a_5, a_6 in (12). Then from the formula for k_n in (68) there is a factorisation $k_n = \tau_n \tau_n^*$, with τ_n given by (61) with z_0 replaced by $z_0 - Z$, and

$$\tau_n^* = A_\pm^* B_\pm^{*[\frac{n}{2}]} \mu^{[\frac{n}{2}]^2} \sigma(z_0 + Z + n\kappa), \qquad \frac{B_+^*}{B_-^*} = -\mu^{-1} = \sigma(\kappa)^4, \quad \text{where } A_\pm A_\pm^* = \mathrm{a}_\pm, \ B_\pm B_\pm^* = \mathrm{b}_\pm,$$

so that τ_n and τ_n^* satisfy the same Somos-5 recurrence with the same value of the invariant γ. By making an analogous factorisation, the right-hand side of the formula for l_n in (68) can be written as a product $\tilde{\tau}_n \tilde{\tau}_n^*$ for another pair of such Somos-5 sequences. □

It is worth explaining the nature of the factorisation (57) in more detail. First of all, by Proposition 4, each of the terms k_n, l_n is a Laurent polynomial in the ring $\mathcal{R}[k_0^{\pm 1}, k_1^{\pm 1}, l_0^{\pm 1}, l_1^{\pm 1}]$, and can be factored as a product of a polynomial with a Laurent monomial, but the factorisation (57) is not of this kind: it leads to τ_n, τ_n^*, etc. which are algebraic functions of the coefficients and initial data for (12). Secondly, the factorisation (57) is not unique: there is a 6-parameter family of gauge transformations that preserve it, given by

$$\tau_n \to \overline{A}_\pm \overline{B}^n \tau_n, \quad \tau_n^* \to (\overline{A}_\pm)^{-1}\overline{B}^{-n}\tau_n^*, \quad \tilde{\tau}_n \to \hat{A}_\pm \hat{B}^n \tilde{\tau}_n, \quad \tilde{\tau}_n^* \to (\hat{A}_\pm)^{-1}\hat{B}^{-n}\tilde{\tau}_n^*, \tag{69}$$

where $\overline{A}_\pm, \hat{A}_\pm$ vary with the parity of n, for arbitrary non-zero $\overline{A}_+, \overline{A}_-, \hat{A}_+, \hat{A}_-, \overline{B}, \hat{B}$. The above theorem guarantees that Somos-5 sequences $\tau_n, \tau_n^*, \tilde{\tau}_n, \tilde{\tau}_n^*$ exist such that (57) holds. However, given the initial data k_0, l_0, k_1, l_1 and coefficients $a_1, a_3, \ldots, a_6$ for (12), although the coefficients α, β are given by the polynomial expressions (56), it is necessary to solve algebraic equations in

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order to find the four sets of initial data for Somos-5, i.e. $\tau_0, \tau_1, \ldots, \tau_4$ for the first sequence, and so on. Without loss of generality, by exploiting the gauge symmetry (69), the 6 values $\tau_0^*, \tau_1^*, \tau_2^*, \tilde{\tau}_0^*, \tilde{\tau}_1^*, \tilde{\tau}_2^*$ can all be fixed to 1, so that $\tau_0 = k_0$, $\tau_1 = k_1$, $\tau_2 = k_2$ and $\tilde{\tau}_0 = l_0$, $\tilde{\tau}_1 = l_1$, $\tilde{\tau}_2 = l_2$. By fixing γ as in (56), the formula (65) determines τ_4 as an algebraic function of $\tau_0, \tau_1, \tau_2, \tau_3$ and the coefficients and initial data for (12), by solving a quadratic equation, and similarly for $\tau_4^*, \tilde{\tau}_4, \tilde{\tau}_4^*$. Thus, in order to have four complete sets of initial data for Somos-5, it remains to determine the four quantities $\tau_3, \tau_3^*, \tilde{\tau}_3, \tilde{\tau}_3^*$. The product $\tau_3 \tau_3^* = k_3$ is a known (Laurent polynomial) function of k_0, l_0, k_1, l_1 and $a_1, a_3, \ldots, a_6$, and similarly for $\tilde{\tau}_3 \tilde{\tau}_3^* = l_3$, but two more relations in are needed to obtain the remaining four quantities. Then, by using (58) to write

$$k_5 = \tau_5 \tau_5^* = \frac{(\alpha \tau_1 \tau_4 + \beta \tau_2 \tau_3)(\alpha \tau_1^* \tau_4^* + \beta \tau_2^* \tau_3^*)}{\tau_0 \tau_0^*},$$

the left-hand side is determined by the coefficients/initial data for (12), while the ratio on the right-hand side is a rational function of these together with the Somos-5 initial data, and similarly for $l_5 = \tilde{\tau}_5 \tilde{\tau}_5^*$, thus providing the necessary two additional relations, which are best solved using resultants.

Proof of Theorem 23: By Theorem 26, both k_n and l_n are products of Somos-5 sequences, so by Corollary 25 they satisfy the same Somos-7 recurrence with coefficients given by (66). It remains to relate the quantities ψ_2, ψ_3, ψ_4 that define the companion EDS to the parameters a_j, $j = 1, 2, 3, 5, 6$ and the value of the integral I for the QRT map (5). This is achieved by using the formulae for the EDS in [20], which give the identity $\psi_2^4 = \beta + \alpha \gamma$ in addition to (63), and noting that fixing the value of the invariant γ defines a curve of genus one in the plane with coordinates $(x, y) = (\tau_n \tau_{n+3}/(\tau_{n+1} \tau_{n+2}), (\tau_{n+1} \tau_{n+4}/(\tau_{n+2} \tau_{n+3}))$, that is

$$(x + y)xy + \alpha(x + y) - \gamma xy + \beta = 0. \tag{70}$$

The latter curve is isomorphic to the curve $\mathcal{C}$, and by the use of an algebra package such as `algcurves` in Maple one can relate α, β, γ to the coefficients appearing in $\mathcal{C}$; this leads to the relations (56). □

6.2 The additive case

The quantity

$$J = a_1 u_0^2 u_1^2 + a_2 u_0 u_1(u_0 + u_1) + a_3(u_0^2 + u_1^2) + a_4 u_0 u_1 + a_5(u_0 + u_1), \tag{71}$$

is an invariant of (6) considered as a map on the plane of initial values (u_0, u_1). There is a well known link between the additive form (6) of the QRT map and the multiplicative one (5).

Lemma 27. *Suppose that an orbit of (5) starting from the initial point (u_0, u_1) is such that the value of the invariant is $I = -a_4$. Then this coincides with the orbit of (6) starting from the same initial point with the invariant taking the value $J = -a_6$.*

Proof. This follows from (53), expressing a_6 in terms of I, substituting into the formula for the corresponding map, then comparing with a_4 in terms of J obtained from (71). □

Theorem 28. *The variables k_n, l_n in (29) each satisfy the same Somos-7 recurrence, of the form*

$$x_{n+7} x_n = A x_{n+6} x_{n+1} + B x_{n+5} x_{n+2} + C x_{n+4} x_{n+3} \tag{72}$$

with coefficients given by (55), but with

$$\alpha = a_1 J + a_2 a_5 + a_3^2 - a_3 a_4, \quad \beta = a_1 a_5^2 - 2a_2 a_3 a_5 + a_3^2 a_4 + (a_1 a_4 - a_2^2)J, \quad \gamma = 4a_3 - a_4.$$

Proof. It is enough to observe that, by Lemma 27, any orbit of the multiplicative map corresponds to an orbit of the additive version, and by identifying k_n, l_n in (12) with k_n, l_n in (29) the result follows. □

As already noted, the DTKQ-2 equation (3) is a special case of the additive QRT map, with

$$J = (u_n u_{n-1} - \alpha)(u_n + u_{n-1})$$

being a first integral in this case. Moreover, (4) arises from (3) by setting

$$u_n = \frac{e_n e_{n+3}}{e_{n+1} e_{n+2}}.$$

To see that how this is a degenerate case of the preceding result on Somos-7 recurrences, observe that, by Lemma 27, the iterates of (3) also satisfy the multiplicative QRT map

$$u_{n+1} u_{n-1} = -\frac{(\alpha u_n + J)}{u_n},$$

which leads to the following result.

Proposition 29. *Any sequence (e_n) generated by (4) also satisfies the Somos-5 relation*

$$e_{n+5} e_n + \alpha\, e_{n+4} e_{n+1} + J\, e_{n+3} e_{n+2} = 0. \tag{73}$$

Corollary 30. *For all $m, n \in \mathbb{Z}$ the following relation holds:*

$$\begin{vmatrix} e_n e_{n+5} & e_m e_{m+5} \\ e_{n+2} e_{n+3} & e_{m+2} e_{m+3} \end{vmatrix} + \alpha \begin{vmatrix} e_{n+1} e_{n+4} & e_{m+1} e_{m+4} \\ e_{n+2} e_{n+3} & e_{m+2} e_{m+3} \end{vmatrix} = 0. \tag{74}$$

Proof of Corollary: This follows immediately from (73). By eliminating e_{m+5} from the top right entry and expanding the two determinants, the left-hand side of (74) becomes

$$\begin{vmatrix} e_n e_{n+5} & -\alpha\, e_{m+4} e_{m+1} - J\, e_{m+3} e_{m+2} \\ e_{n+2} e_{n+3} & e_{m+2} e_{m+3} \end{vmatrix} + \alpha \begin{vmatrix} e_{n+1} e_{n+4} & e_{m+1} e_{m+4} \\ e_{n+2} e_{n+3} & e_{m+2} e_{m+3} \end{vmatrix}$$
$$= e_{m+2} e_{m+3} (e_{n+5} e_n + \alpha\, e_{n+4} e_{n+1} + J\, e_{n+3} e_{n+2}) = 0$$

Thus apart from an overall m-dependent prefactor, which can be removed, for each n the formula (74) just corresponds to the same Somos-5 relation (73). $\qquad \square$

Remark 31. *The equation (74) is of degree four and depends on two indices m, n. This is reminiscent of Ward's defining relation for an elliptic divisibility sequence [33].*

6.3 The asymmetric QRT map

It turns out that Theorem 23 admits a straightforward generalization to both the general 12-parameter symmetric QRT map and the full 18-parameter asymmetric QRT map, with the result for the former being a special case of that for the latter. For convenience, we briefly outline the geometrical description of the general QRT map due to Tsuda [30]; for more details see Duistermaat's book [8]. Starting from a pencil of biquadratic curves

$$\mathcal{P}(u, v; \lambda) \equiv \sum_{i,j=0,1,2} (a_{ij} + \lambda b_{ij}) u^{2-i} v^{2-j} = 0 \tag{75}$$

in the (u, v) plane, labelled by $\lambda \in \mathbb{P}^1$, there are two natural birational involutions on each curve (for fixed λ), called the horizontal/vertical switch, denoted ι_H and ι_V respectively, given by

$$\iota_H : \quad (u, v) \mapsto (u^\dagger, v), \qquad \iota_V : \quad (u, v) \mapsto (u, v^\dagger), \tag{76}$$

where $u^\dagger$ denotes the conjugate root of (75) considered as a quadratic in u (for fixed v), and similarly for $v^\dagger$. Note that the birationality of ι_H (and similarly that of ι_V) can be seen directly by writing (75) as

$$\mathcal{P} \equiv \mathrm{A}(v, \lambda) u^2 + \mathrm{B}(v, \lambda) u + \mathrm{C}(v, \lambda) = 0,$$

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and using either the formula for the sum or the product of the roots, i.e.

$$u^\dagger + u = -\frac{\mathrm{B}(v, \lambda)}{\mathrm{A}(v, \lambda)}, \qquad u^\dagger u = \frac{\mathrm{C}(v, \lambda)}{\mathrm{A}(v, \lambda)}. \tag{77}$$

Then the QRT map φ is defined on each curve in the pencil to be the product of these two involutions,

$$\varphi = \iota_V \cdot \iota_H, \tag{78}$$

and this lifts to a birational map on the (u, v) plane: to apply ι_H one can eliminate λ from the pair of equations (77) to yield a formula for $u^\dagger$ as a rational function of u and v alone, and similarly for the application of ι_V. Moreover, solving (75) for λ gives

$$\lambda = -\frac{\sum_{i,j} a_{ij} u^j v^k}{\sum_{i,j} b_{ij} u^j v^k}, \tag{79}$$

and a generic point (u, v), as well as its orbit under the QRT map φ, belongs to a unique curve $\mathcal{C} = \mathcal{C}(\lambda)$ with the corresponding value of λ being determined by the above ratio (the exception being the base points, for which both the numerator and denominator vanish).

In the special case where the coefficient matrices $\mathcal{A} = (a_{ij})$ and $\mathcal{B} = (b_{ij})$ are symmetric, each curve in the pencil admits the additional involution $\sigma : (u, v) \mapsto (v, u)$, and the symmetric QRT map φ_{sym} is defined as $\varphi_{sym} = \sigma \cdot \iota_H$; the vertical switch is conjugate to the horizontal, so $\iota_V = \sigma \cdot \iota_H \cdot \sigma$, and $\varphi = (\varphi_{sym})^2$. With symmetric matrices as in (34), φ_{sym} coincides with the map (35) defined previously, and comparing (36) with (79) we see that $J = -\lambda$.

Considering the general asymmetric QRT map, we can start from an initial point (u_0, v_0) and apply φ repeatedly to obtain the sequence

$$(u_n, v_n) = \varphi^n(u_0, v_0). \tag{80}$$

We have the following result.

Theorem 32. *Any orbit (80) produced by the general QRT map can be written as $u_n = k_n/l_n$, $v_n = p_n/q_n$, where k_n, l_n, p_n, q_n all satisfy the same Somos-7 recurrence of the form (54), with the coefficients A, B, C given by (55) in terms of quantities α, β, γ which are determined precisely by the 18 parameters a_{ij}, b_{ij} and the value of the invariant λ as in (79).*

Proof. This is similar to the proof of Theorem 26, so rather than repeating this it is sufficient to explain the main things that are different. For generic values of a_{ij}, b_{ij} and λ the curve $\mathcal{C}$ defined by (75) is smooth for $(u, v) \in \mathbb{P}^1 \times \mathbb{P}^1$ and has genus one, so it is isomorphic to $\mathbb{C}/\Lambda$ where Λ is the period lattice of an elliptic curve $\mathcal{E}$ given in the Weierstrass form (62). Each of the maps $(u, v) \mapsto u$ and $(u, v) \mapsto v$ defines a double cover of $\mathbb{P}^1$, with associated elliptic functions of order two, denoted f, g respectively, such that $(u, v) = (f(z), g(z))$ provides the uniformization of $\mathcal{C}$. Now, up to an overall shift in z, we may write

$$f(z) = K \frac{\sigma(z - Z)\sigma(z + Z)}{\sigma(z - P)\sigma(z + P)}, \qquad g(z) = K' \frac{\sigma(z + \delta - Z')\sigma(z + \delta + Z')}{\sigma(z + \delta - P')\sigma(z + \delta + P')}$$

for some constants $\delta, Z, P, K, Z', P', K'$ which specify the poles and zeros of f, g. Then by a standard argument (see e.g. Theorem 2.4 in [30] or the proof of Proposition 2.5 in [19]), the composition of two involutions $\varphi = \iota_V \cdot \iota_H$ corresponds to a translation $z \mapsto z + \kappa$ on the torus $\mathbb{C}/\Lambda$, so the iterates (80) of the QRT map have an analytic expression given by (67) and

$$v_n = \frac{p_n}{q_n} = K' \frac{\sigma(z_n + \delta - Z')\sigma(z_n + \delta + Z')}{\sigma(z_n + \delta - P')\sigma(z_n + \delta + P')}, \qquad z_n = z_0 + n\kappa. \tag{81}$$

Thus once again we can write down a factorisation $k_n = \tau_n \tau_n^*$ into a product of Somos-5 sequences, and similarly for l_n, p_n, q_n. Therefore all of k_n, l_n, p_n, q_n satisfy the same Somos-7 recurrence, whose coefficients can be determined in terms of the parameters α, β, γ for a cubic curve (70) that is isomorphic to $\mathcal{C}$. (See e.g. Remark 2.2 in [30] for details.) $\qquad\square$

Remark 33. *The formulae for k_n, l_n, p_n, q_n as a product of two tau functions/sigma functions, as in (57), (68), and in the proof above, have a counterpart in the non-autonomous setting, in the form of the bilinearization of the q-P_{III} and q-P_{VI} equations by Jimbo et al. [22].*

The analogue of the above result for the map $\varphi_{sym} = \sigma \cdot \iota_H$ is simpler because for a symmetric biquadratic curve the shift κ for the original QRT map φ can be chosen so that $g(z) = f(z + \kappa/2)$, and then we can identify $v_n = u_{n+1/2}$ in (80). Lemma 27 generalizes to the full 12-parameter symmetric case, such that on every orbit of (35) one may write u_n as a ratio of quantities which satisfy both bilinear systems (12) and (29), with parameters that are linear functions of λ. By combining the above results on the multiplicative and additive cases, we arrive at the following.

Corollary 34. *Consider the general symmetric QRT map (35) with $u_n = p_n/q_n$, where the variables p_n, q_n satisfy a rational, non-Laurent system which is a variant of (37), namely*

$$p_{n+1} = \frac{q_{n-1}F_n^{(1)} - p_{n-1}F_n^{(2)}}{D_n}, \quad q_{n+1} = \frac{q_{n-1}F_n^{(2)} - p_{n-1}F_n^{(3)}}{D_n}, \tag{82}$$

with $F_n^{(i)} = q_n^4 f^{(i)}(\frac{p_n}{q_n})$ and $D_n = \mathbf{V}(p_{n-1}, q_{n-1})^T \mathcal{B}\mathbf{V}(p_n, q_n)$ as above. Then p_n and q_n are both solutions of the same Somos-7 relation, namely

$$p_{n+7}p_n = A p_{n+6}p_{n+1} + B p_{n+5}p_{n+2} + C p_{n+4}p_{n+3}, \tag{83}$$

where the formulae (55) and (56) for the coefficients A, B, C are still valid if we set

$$a_1 = a_{00} - Jb_{00}, \; a_2 = a_{01} - Jb_{01}, \; a_3 = a_{02} - Jb_{02}, \; a_5 = a_{12} - Jb_{12}, \; a_6 = a_{22} - Jb_{22}, \; I = -a_{11} + Jb_{11}$$

in terms of the 12 parameters a_{ij}, b_{ij}, $0 \le i \le j \le 2$ and first integral $J = -\lambda$, as in (36), for φ_{sym}.

Corollary 35. *The variables s_n, t_n in the system (47) both satisfy the same Somos-7 relation, given by (83) but with the coefficients rescaled according to*

$$A \to K^{12}A, \qquad B \to K^{20}B, \qquad C \to K^{24}C,$$

where on each orbit the constant K is given by

$$K^2 = \frac{\rho_{n+1}\rho_{n-1}}{\rho_n^2}, \qquad \rho_n = \frac{s_n}{p_n} = \frac{t_n}{q_n}, \tag{84}$$

with p_n, q_n being the corresponding solution to (82).

Proof of Corollary 35: Since the solutions of (47) and (82) are both related to the same solution of (35) via $u_n = s_n/t_n = p_n/q_n$, we can introduce $\rho_n = s_n/p_n = t_n/q_n$. Taking the quotient of each side of the second equation (47) with each side of the first equation (82), and doing the same for the third equation (47) with the second equation (82), leads in both cases to

$$\rho_{n+1}r_{n-1} = D_n \rho_n^4 \rho_{n-1}, \tag{85}$$

while eliminating s_n, t_n from the first equation (47) gives

$$r_n r_{n-1} = \rho_n^8 Q(p_n, q_n). \tag{86}$$

Then using (85) in (86) and further substituting for p_{n+1}, q_{n+1} from (82) gives

$$\left(\frac{\rho_{n+1}\rho_{n-1}}{\rho_n^2}\right)\left(\frac{\rho_{n+2}\rho_n}{\rho_{n+1}^2}\right)^{-1} = \frac{Q(p_n, q_n)}{D_n D_{n+1}} = 1 \implies \frac{\rho_{n+1}\rho_{n-1}}{\rho_n^2} = \text{const} \implies \rho_n = \rho_0 \left(\frac{\rho_1}{K\rho_0}\right)^n K^{n^2}$$

for some K, which verifies (84). The quadratic exponential form of the gauge transformation from p_n to s_n means that s_n satisfies (83) but with coefficients rescaled as stated, and likewise for t_n. If the initial values are chosen so that $r_0 = 1$, $s_0 = p_0 = u_0$, $s_1 = p_1 = u_1$ and $t_0 = q_0 = t_1 = q_1 = 1$ then $\rho_0 = \rho_1 = 1$ and hence $K^2 = \rho_2 = D_1 = \mathbf{V}(u_0, 1)^T \mathcal{B}\mathbf{V}(u_1, 1)$. $\qquad\square$

7 Concluding remarks

We have applied homogenisation and/or recursive factorisation to symmetric QRT maps, and have shown directly that this produces systems with the Laurent property. In the multiplicative case, the resulting system (12), appears to correspond to a pair of mutations in an LP algebra [23], which would give an alternative way to verify the Laurent property. However, neither (29) nor the system (47) obtained from the general symmetric QRT are of the right form for successive LP algebra mutations.

Our results show that obtaining Laurent systems and using their ultradiscrete (or tropical) versions is a very efficient method for calculating the degree growth of maps. For the QRT maps, which preserve an elliptic fibration, the fact that the degree growth is quadratic is well known in the context of algebraic entropy of maps with invariant curves [3]. Earlier results on automorphisms of rational surfaces admitting an elliptic fibration were presented by Gitzatullin [15]. The quadratic growth observed in QRT maps also fits into Diller and Favre's classification of bimeromorphic maps of surfaces in [7], and can be understood by making a sequence of blowing-up transformations which lifts the QRT map to an automorphism of a complex analytic surface, then considering the action on homology (see e.g. the discussion of geometric singularity confinement in chapter 3 of [8]). However, this approach may become intractable for maps in dimension 3 and above, whereas the combination of Laurentification and ultradiscretization extends to higher dimensions in a straightforward manner.

In future work we propose to consider the analogue of (47) for the general asymmetric QRT map, as well as Laurent systems for higher-dimensional maps. As a starting point, it would be worth considering the bilinearization of the discrete Painlevé VI (q-P_{VI}) equation presented in [22], for which the autonomous limit is just the multiplicative version of the asymmetric QRT map.

Appendix

Here we provide an inductive proof of Lemma 19, by making repeated use of two facts: (i) for a pair of polynomials $f, g \in \mathbb{K}[x_1, \ldots, x_m]$ in m variables over a field $\mathbb{K}$ with positive degree in x_1, there are polynomials A, B such that $Af + Bg = \mathrm{Res}(f, g, x_1)$; and (ii) f, g have a common factor with positive degree in x_1 if and only if this resultant vanishes (see e.g. Proposition 1, section §6 of chapter 3 in [5]). This extends directly to the case at hand, where the ring of coefficients $\mathcal{R}$ is a unique factorisation domain, by working in the corresponding field of fractions.

As our inductive hypothesis, we assume that $(s_k, t_k) = 1$ for $0 \le k \le n$. The base cases $k = 0, 1$ are trivial, while for $k = 2$ we can write

$$\begin{pmatrix} r_0 s_2 \\ r_0 t_2 \end{pmatrix} = \begin{pmatrix} -F^{(2)}(s_1, t_1) & F^{(1)}(s_1, t_1) \\ -F^{(3)}(s_1, t_1) & F^{(2)}(s_1, t_1) \end{pmatrix} \begin{pmatrix} s_0 \\ t_0 \end{pmatrix}, \tag{87}$$

and verify directly that $N_2 = r_0 s_2$ and $\hat{N}_2 = r_0 t_2$ are coprime polynomials in $\mathcal{R}[s_0, t_0, s_1, t_1]$, and homogeneous in s_0, t_0 and s_1, t_1 separately (of degree 1 and 4, respectively). Moreover, if we regard $N_2, \hat{N}_2$ as (linear) polynomials in $u_0 = s_0/t_0$, then from the aforementioned facts their coprimality means that there are polynomials A, B (also linear in u_0), whose coefficients are homogeneous polynomials in $\mathcal{R}[s_1, t_1]$, such that

$$t_0^{-1}(Ar_0 s_2 + Br_0 t_2) = \mathrm{Res}(t_0^{-1} r_0 s_2, t_0^{-1} r_0 t_2, u_0) \ne 0.$$

As it happens, from the linearity of the system (87) we see that the resultant in this case is just the determinant of the 2×2 matrix on the right, and turns out to be equal to the polynomial

$-Q(s_1, t_1)$, homogeneous of degree 8, so after rescaling by t_0^2 we have

$$A' r_0 s_2 + B' r_0 t_2 = -t_0^2 Q(s_1, t_1),$$

where A', B' are homogeneous in s_0, t_0 and s_1, t_1 (separately). Upon shifting indices, by applying the pullback of the map defined by the Laurent system (47), this gives an identity of Laurent polynomials for all n, namely

$$A'(s_{n-1}, t_{n-1}, s_n, t_n) r_{n-1} s_{n+1} + B'(s_{n-1}, t_{n-1}, s_n, t_n) r_{n-1} t_{n+1} = -t_{n-1}^2 Q(s_n, t_n). \qquad (88)$$

Now we suppose that s_{n+1} and t_{n+1} have a non-trivial common factor P, and show that under the inductive hypothesis this leads to a contradiction. Indeed, from the right-hand side of (88) there are two possibilities: (a) $P|t_{n-1}$, or (b) $P|Q(s_n, t_n)$. In case (a), (47) directly yields

$$r_{n-1} s_{n+1} - t_{n-1} F^{(1)}(s_n, t_n) = -s_{n-1} F^{(2)}(s_n, t_n), \ \ r_{n-1} t_{n+1} - t_{n-1} F^{(2)}(s_n, t_n) = -s_{n-1} F^{(3)}(s_n, t_n),$$

therefore P must divide both right-hand sides above, and, since $(s_{n-1}, t_{n-1}) = 1$ by the inductive hypothesis, this yields $P|F^{(2)}(s_n, t_n)$ and $F^{(3)}(s_n, t_n)$. By applying the same argument as in the proof of Proposition 4, we form the Sylvester matrix corresponding to the coefficients of $F^{(2)}$ and $F^{(3)}$, whose determinant is a non-zero element of $\mathcal{R}$, and since also $(s_n, t_n) = 1$ by hypothesis, this gives a contradiction. Thus we are left with case (b). The fact that s_2, t_2 are both coprime to $Q(s_1, t_1)$ can be verified directly, so by taking resultants with respect to u_1 and shifting indices this leads to a pair of identities

$$A_n^* r_{n-1} s_{n+1} + B_n^* Q(s_n, t_n) = t_n^M R^*(s_{n-1}, t_{n-1}), \quad A_n^\dagger r_{n-1} s_{n+1} + B_n^\dagger Q(s_n, t_n) = t_n^M R^\dagger(s_{n-1}, t_{n-1}),$$

where $A_n^*, B_n^*, A_n^\dagger, B_n^\dagger$ are homogeneous polynomials in $s_{n-1}, t_{n-1}, s_n, t_n$, M is a positive integer, and the resultants $R^*, R^\dagger$ are a pair of coprime homogeneous polynomials in their arguments. Using the same method as for Proposition 4 once again, the latter coprimality means that, since $(s_{n-1}, t_{n-1}) = 1$ by hypothesis, P cannot divide both $R^*(s_{n-1}, t_{n-1})$ and $R^\dagger(s_{n-1}, t_{n-1})$, hence $P|t_n$. To finish off the argument, it is enough to use $(s_2, t_1) = 1 = (t_2, t_1)$ and take resultants with respect to $v_1 = t_1/s_1$, then shift to obtain further relations of the form

$$\tilde{A}_n r_{n-1} s_{n+1} + \tilde{B}_n t_n = s_n^L \tilde{R}(s_{n-1}, t_{n-1}), \quad \hat{A}_n r_{n-1} s_{n+1} + \hat{B}_n t_n = s_n^L \hat{R}(s_{n-1}, t_{n-1})$$

for all n, where L is a positive integer and the resultants $\tilde{R}, \hat{R}$ are coprime homogeneous polynomials in their arguments. From $(s_n, t_n) = 1$ it follows from the two right-hand sides above that $P|\tilde{R}(s_{n-1}, t_{n-1}), \hat{R}(s_{n-1}, t_{n-1})$, which is seen to be a contradiction by applying the argument used for Proposition 4 once more. This completes the proof of Lemma 19. □

Analogous arguments can be used to obtain similar coprimality conditions for all Laurent polynomials r_n, s_n, t_n generated by (47), and it appears that more is true: these iterates should be distinct irreducible elements of $\hat{\mathcal{R}} = \mathcal{R}[r_0^{\pm 1}, s_0, s_1, t_0, t_1]$ for all n.

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