

# AN EFFICIENT GRADIENT PROJECTION METHOD FOR STOCHASTIC OPTIMAL CONTROL PROBLEMS \*

BO GONG\*, WENBIN LIU†, TAO TANG‡, WEIDONG ZHAO§, AND TAO ZHOU¶

**Abstract.** In this work, we propose a simple yet effective gradient projection algorithm for a class of stochastic optimal control problems. We first reduce the optimal control problem into an optimization problem for a convex functional by means of a projection operator. Then we propose an convergent iterative scheme for the optimization problem. The key issue in our iterative scheme is to compute the gradient of the objective functional by solving the adjoint equations that are given by backward stochastic differential equations (BSDEs). The Euler method is used to solve the resulting BSDEs. Rigorous convergence analysis is presented and it is shown that the entire numerical algorithm admits a first order rate of convergence. Several numerical examples are carried out to support the theoretical finding.

**Key words.** Stochastic optimal control, gradient projection methods, backward stochastic differential equations, conditional expectations

**AMS subject classifications.** 60H35, 65C20, 35Q99, 35R35

**1. Introduction.** In recent years, stochastic optimal control has been extensively studied and has become an essential tool in various fields, such as financial mathematics and engineering. There exists a very extensive body of literature in both theoretical and practical studies of stochastic optimal control problems, see e.g., [4, 5, 17, 25, 8, 11] and references therein.

In this work, we are concerned with the following stochastic optimal control problem

$$\min_{u \in K} J(u) = \mathbb{E} \left[ \int_0^T \left( h(x_t^u) + j(u(t)) \right) dt + k(x_T^u) \right], \quad (1.1)$$

where  $u$  is the control policy, and  $x^u$  is the corresponding state process that satisfies the following stochastic differential equation

$$dx_t^u = b(x_t^u, u(t))dt + \sigma(x_t^u, u(t))dW_t, \quad t \in (0, T], \quad x|_{t=0} = x_0. \quad (1.2)$$

Theoretical investigations for the above model can be found in [4, 13, 17, 24, 3, 7, 15, 26, 32, 36]. For practical applications of (1.1)-(1.2), one can refer to [7, 23, 26, 36, 38] for engineering applications, to [21, 22, 30, 40, 42] for applications in option pricing and portfolio optimization, to [1] for analysis of climate changes, and to [16] for biological and medical problems, to name a few.

In general the above model does not admit explicitly closed form solutions and thus efficient numerical algorithms have been widely studied in recent years. Roughly

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†Department of Mathematics, Hongkong Baptist University, Hongkong, China (13479245@life.hkbu.edu.hk)

‡Kent Business School, University of Kent, UK (W.B.Liu@kent.ac.uk)

§Department of Mathematics, Southern University of Sciences and Technology, Shenzhen, China (tangt@sustc.edu.cn)

¶School of Mathematics & Finance Institute, Shandong University, Jinan, China (wdzhao@sdu.edu.cn)

¶LSEC, Institute of Computational Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China (tzhou@lsec.cc.ac.cn)

speaking, we can characterize numerical algorithms into four categories: (i) transferring the control problem into finite dimensional stochastic programming, see e.g., [9, 15, 18, 19, 26, 36, 39, 41]; (ii) dynamic programming principle (DPP) based approach [6, 25]. In this framework, one usually needs to solve the corresponding Hamilton-Bellman-Jacobi (HJB) equations, and this is one of the most widely used numerical methods [2, 4, 5, 10, 20]; (iii) martingale based methods [21, 22, 37]; and (iv) stochastic maximum principle (SMP) based methods, see e.g., [17] and references therein.

Basically, the SMP procedure is to directly compute directional derivative for the objective functional  $J(\cdot)$  by introducing an adjoint process. Then by introducing an optimality condition for the control problem, a variational inequality coupled with the state and adjoint equations forms an optimality condition system (we call it SMP system) that can be used to solve the optimal control problem. While SMP is a popular tool for theoretical studies of stochastic optimal control, see, e.g., [40, 42], it has not been widely used in the numerical setting.

In this work, we propose a simple yet effective gradient projection algorithm for the stochastic optimal control problem (1.1)-(1.2). We first reduce the optimal control problem to an optimization problem for a convex functional by means of a projection operator. Then we propose a convergent iterative scheme for the optimization problem. The key idea in our iterative scheme is to compute the gradient of the objective functional in an efficient way, and this is done by solving the adjoint equations that is given by backward stochastic differential equations (BSDEs). Our approach belongs to SMP based approach, and it relies on solving the SMP system in an efficient way. To this end, we propose a simple yet effective Euler-type method for solving the resulting BSDEs. Furthermore, we perform a sharp convergence analysis and we show that our numerical method admits a first order rate of convergence. Several numerical examples are presented to support the theoretical founding.

The rest of the paper is organized as follows. In Section 2 we set up the stochastic optimal control problem and provide with some assumptions. The gradient projection method is presented in Section 3. Section 4 is devoted to convergence analysis of the proposed numerical approach. Several numerical experiments are presented to show the effective of the proposed numerical method in Section 5. We finally give some conclusions in Section 6.

**2. Problem Setup.** For notational simplicity, we shall narrow our discussion to the one dimensional case, however, the whole framework applies easily to multi-dimensional cases. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$  be a complete probability space with filtration  $\mathcal{F}_t$  generated by the Brownian motion  $\{W_s\}_{0 \leq s \leq t}$ . Here  $T$  is the terminal time. We denote by  $U = L^2([0, T]; \mathbb{R})$  the space of all square integrable functions  $x : [0, T] \rightarrow \mathbb{R}$ , and denote by  $U_{\mathcal{F}} = \mathbb{L}_{\mathcal{F}}^2([0, T] \times \Omega; \mathbb{R})$  the space of all adapted stochastic processes  $x : [0, T] \times \Omega \rightarrow \mathbb{R}$  that satisfy

$$\mathbb{E} \left[ \int_0^T (x_t)^2 dt \right] < +\infty.$$

Let  $\mathcal{C} \subset \mathbb{R}$  be a nonempty, convex and closed subset, and we define the following control set

$$K = \{u \in U \mid u(t) \in \mathcal{C} \text{ a.e.}\}.$$

Note that we have assumed that the control  $u$  is deterministic. We remark that a deterministic control can still be useful for future planning as discussed e.g. in

[7, 26] for engineering applications, in [9] for financial applications and in [36] for an application in stochastic hybrid systems. Moreover, stochastic control (i.e.,  $u \in U_{\mathcal{F}}$ ) can also be included in our approach and this will be discussed in Section 5 via numerical examples.

Given  $u \in K$ , the controlled state process  $x_t^u$  is governed by the SDE

$$dx_t^u = b(x_t^u, u(t))dt + \sigma(x_t^u, u(t))dW_t, \quad t \in (0, T], \quad x|_{t=0} = x_0 \in \mathbb{R}. \quad (2.1)$$

The considered cost functional is given by

$$J(u) = \mathbb{E} \left[ \int_0^T \left( h(x_t^u) + j(u(t)) \right) dt + k(x_T^u) \right], \quad (2.2)$$

where  $h(\cdot), j(\cdot), k(\cdot)$  are given functions and  $x_t^u$  is the solution of (2.1). We now state our stochastic optimal control problem as follows:

$$\text{Find } u^* \in K \text{ such that } J(u^*) = \min_{u \in K} J(u). \quad (2.3)$$

Throughout the paper, we shall make the following assumption.

ASSUMPTION 1.

- The functions  $b = b(x, u)$  and  $\sigma = \sigma(x, u)$  are continuously differentiable with respect to  $x$  and  $u$ , and have bounded derivatives.
- The functions  $h, j$  and  $k$  are continuously differentiable, and their derivatives have at most a linear growth with respect to the underlying variables.

Notice that under Assumption 1, the solution  $x_t^u$  of (2.1) and the cost functional  $J(u)$  are all well defined for  $u \in K$ .

**3. The gradient projection method.** In this section, we will present details of our gradient projection method. For the stochastic optimal control problem (2.1)-(2.2), it is well known that for the optimal control  $u^*$  it holds

$$(J'(u^*), v - u^*) \geq 0, \quad \forall v \in K, \quad (3.1)$$

where  $(J'(u), v)$  is the variation of  $J(u)$  along the direction  $v$ , i.e., for  $v \in U$  such that  $u + v \in K$ , we have

$$(J'(u), v) = \lim_{\rho \downarrow 0} \frac{J(u + \rho v) - J(u)}{\rho}. \quad (3.2)$$

The existence of such derivatives has been discussed in [13, 32, 40]. Here we slightly abuse the notation by referring  $J'(u)$ , the functional, to the corresponding element in  $U$ , as  $U$  is a Hilbert space.

Next, we propose a gradient projection method for solving the optimal condition (3.1). To this end, let  $\|\cdot\|$  be the norm of  $U$ . We define the projection operator  $P_K$ :  $\omega \mapsto P_K \omega$  as

$$\|P_K \omega - \omega\| = \min_{u \in K} \|u - \omega\|. \quad (3.3)$$

Notice that the projection problem (3.3) is equivalent to the inequality

$$(P_K \omega - \omega, v - P_K \omega) \geq 0, \quad \forall v \in K. \quad (3.4)$$

For any positive constant  $\rho$ , the variational inequality (3.1) is equivalent to the following inequality

$$(u^* - (u^* - \rho J'(u^*)), v - u^*) \geq 0, \quad \forall v \in K. \quad (3.5)$$

By the fact of wellposedness of convex optimizations and by comparing the above inequality with the inequality (3.4), we conclude that for the optimal control  $u^*$ , it holds

$$u^* = P_K(u^* - \rho J'(u^*)). \quad (3.6)$$

That is, the optimal control  $u^*$  is the fixed point of  $P_K(u - \rho J'(u))$  on  $K$ .

We shall approximate the control  $u^*$  numerically by step functions. To this end, we introduce the following uniform time partition:

$$0 = t_0^N < t_1^N < \dots < t_N^N = T, \quad t_{n+1}^N - t_n^N = T/N =: \Delta t. \quad (3.7)$$

We will denote by  $I_n^N$  the intervals  $[t_{n-1}^N, t_n^N]$  for  $1 \leq n \leq N-1$ , and by  $I_N^N$  the interval  $[t_{N-1}^N, t_N^N]$ . In the context where  $N$  is fixed, we shall omit the superscript  $N$  of  $t_n^N$ . We also define the associated space of piecewise constant functions by

$$U_N = \left\{ u \in U \mid u = \sum_{n=1}^N \alpha_n \mathcal{X}_{I_n^N} \text{ a.e., } \alpha_n \in \mathbb{R} \right\}.$$

Let  $K_N = K \cap U_N$ , then it is clear that  $K_N$  is also convex and closed. Now, we define the approximated problem of (2.3) by

$$J(u^{*,N}) = \min_{u \in K_N} J(u).$$

Using similar arguments, one can show that

$$u^{*,N} = P_{K_N}(u^{*,N} - \rho J'(u^{*,N})). \quad (3.8)$$

Based on the above optimal condition, we propose the following fixed-point iteration scheme to get the approximated optimal control

$$u^{i+1,N} = P_{K_N}(u^{i,N} - \rho_i J'_N(u^{i,N})), \quad i = 1, 2, \dots, \quad (3.9)$$

where  $\rho_i$  is a positive constant. Notice that in the above equation we have changed  $J'(\cdot)$  into  $J'_N(\cdot)$ , as one cannot compute  $J'(\cdot)$  exactly in general, and thus it is obtained by numerical approaches. It is clear that  $J'_N(\cdot)$  depends on particular numerical schemes, and we shall discuss the numerical approximation of  $J'_N(\cdot)$  in later sections. We will denote the error between  $J'(\cdot)$  and  $J'_N(\cdot)$  by

$$\epsilon_N = \sup_i \|J'(u^{N,i}) - J'_N(u^{N,i})\|. \quad (3.10)$$

Next, we show in Theorem 1 the convergence property of the iteration scheme (3.9).

**THEOREM 1.** *Assume that  $J'(\cdot)$  is Lipschitz and uniformly monotone around  $u^*$  and  $u^{*,N}$  in the sense that there exist positive constants  $c$  and  $C$  such that*

$$\begin{aligned} \|J'(u^*) - J'(v)\| &\leq C\|u^* - v\|, \quad \forall v \in K, \\ (J'(u^*) - J'(v), u^* - v) &\geq c\|u^* - v\|^2, \quad \forall v \in K, \\ \|J'(u^{*,N}) - J'(v)\| &\leq C\|u^{*,N} - v\|, \quad \forall v \in K_N, \\ (J'(u^{*,N}) - J'(v), u^{*,N} - v) &\geq c\|u^{*,N} - v\|^2, \quad \forall v \in K_N. \end{aligned}$$

Moreover, we assume that

$$\epsilon_N = \sup_i \|J'(u^{i,N}) - J'_N(u^{i,N})\| \rightarrow 0, \quad N \rightarrow \infty.$$

Suppose that  $\rho_i$  is chosen such that  $0 < 1 - 2c\rho_i + (1 + 2C)\rho_i^2 \leq \delta^2$  for some constant  $0 < \delta < 1$ . Then, the iteration scheme (3.9) is convergent, more precisely, we have

$$\|u^* - u^{i,N}\| \rightarrow 0, \quad i, N \rightarrow \infty.$$

*Proof.* By (3.8) and (3.9), we have

$$\begin{aligned} \|u^{*,N} - u^{i+1,N}\|^2 &\leq \|u^{*,N} - u^{i,N} - \rho_i(J'(u^{*,N}) - J'_N(u^{i,N}))\|^2 \\ &= \|u^{*,N} - u^{i,N}\|^2 - 2\rho_i(u^{*,N} - u^{i,N}, J'(u^{*,N}) - J'_N(u^{i,N})) \\ &\quad + \rho_i^2 \|J'(u^{*,N}) - J'_N(u^{i,N})\|^2. \end{aligned}$$

By the Lipschitz condition and monotonicity property of  $J(\cdot)$ , we have

$$\begin{aligned} &-2\rho_i(u^{*,N} - u^{i,N}, J'(u^{*,N}) - J'_N(u^{i,N})) \\ &= -2\rho_i(u^{*,N} - u^{i,N}, J'(u^{*,N}) - J'(u^{i,N}) + J'(u^{i,N}) - J'_N(u^{i,N})) \\ &\leq -2c\rho_i \|u^{*,N} - u^{i,N}\|^2 + \rho_i^2 \|u^{*,N} - u^{i,N}\|^2 + \epsilon_N^2. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \rho_i^2 \|J'(u^{*,N}) - J'_N(u^{i,N})\|^2 &= \rho_i^2 \|J'(u^{*,N}) - J'(u^{i,N}) + J'(u^{i,N}) - J'_N(u^{i,N})\|^2 \\ &\leq 2C\rho_i^2 \|u^{*,N} - u^{i,N}\|^2 + 2\rho_i^2 \epsilon_N^2. \end{aligned}$$

It is easy to show that for sufficiently small  $\rho_i$ , there is a constant  $0 < \delta < 1$  such that  $0 < 1 - 2c\rho_i + (1 + 2C)\rho_i^2 \leq \delta^2$ , then we get

$$\|u^{*,N} - u^{i+1,N}\|^2 \leq \delta^2 \|u^{*,N} - u^{i,N}\|^2 + (1 + 2\rho_i^2)\epsilon_N^2, \quad (3.11)$$

Then, there exists a constant  $C_1$  that independent of  $N$  and  $i$  such that

$$\|u^{*,N} - u^{i,N}\| \leq \delta^i \|u^{*,N} - u^{0,N}\| + C_1 \epsilon_N.$$

Under the assumption  $\epsilon_N \rightarrow 0$ , we get

$$\|u^{*,N} - u^{i,N}\| \rightarrow 0, \quad (N, i \rightarrow \infty). \quad (3.12)$$

On the other hand, using similar arguments as for deriving (3.11), we obtain

$$\begin{aligned} \|u^* - u^{*,N}\| &= \|u^* - P_{K_N}(u^* - \rho J'(u^*)) + P_{K_N}(u^* - \rho J'(u^*)) - u^{*,N}\| \\ &\leq \|u^* - P_{K_N}(u^* - \rho J'(u^*))\| + \sqrt{1 - 2c\rho + C\rho^2} \|u^* - u^{*,N}\|. \end{aligned}$$

Let  $\rho = c/C$ ,  $C_2 = (1 - \sqrt{1 - 2c\rho + C\rho^2})^{-1}$ , we have

$$\|u^* - u^{*,N}\| \leq C_2 \|u^* - P_{K_N}(u^* - \rho J'(u^*))\|.$$

Since  $\mathcal{C}$  is invariant in time, for  $v \in U_N$ , it holds that  $P_K v \in U_N$ . Thus we have  $P_K v \in K_N$ , and then we have  $P_K v = P_{K_N} v$ . Now, denoting  $\omega := u^* - \rho J'(u^*)$ , we have

$$\begin{aligned} \|u^* - u^{*,N}\| &\leq C_2 \|u^* - P_{K_N}(u^* - \rho J'(u^*))\| = C_2 \|P_K \omega - P_{K_N} \omega\| \\ &\leq C_2 (\|P_K \omega - P_K P_{U_N} \omega\| + \|P_K P_{U_N} \omega - P_{K_N} \omega\|) \\ &= C_2 (\|P_K \omega - P_K P_{U_N} \omega\| + \|P_{K_N} P_{U_N} \omega - P_{K_N} \omega\|) \\ &\leq 2C_2 \|\omega - P_{U_N} \omega\|. \end{aligned}$$

As  $U_N$  is dense in  $U$ , we have  $\|\omega - P_{U_N} \omega\| \rightarrow 0$ , and thus  $\|u^* - u^{*,N}\| \rightarrow 0$ . Then, the conclusion follows from this argument and (3.12).  $\square$

In Theorem 1 we have shown the convergence of  $\|u^{*,N} - u^{i,N}\|$  under the assumption  $\epsilon_N \rightarrow 0$ . Note that this is a reasonable assumption. In fact, under certain regularity requirements, and by designing suitable numerical approaches for  $J'_N(\cdot)$ , one could further expect that  $\epsilon_N \sim \mathcal{O}(\Delta t)$ . In such a case, we could expect a first order rate of convergence of our iteration scheme (3.9), as illustrated in the following corollary.

**COROLLARY 1.** *Suppose that the conditions in Theorem 1 holds, and furthermore, we assume that  $u^*$  and  $J'(u^*)$  are both Lipschitz continuous functions in  $U$ , then under the condition  $\epsilon_N \sim \mathcal{O}(\Delta t)$  we have*

$$\|u^* - u^{i,N}\| \sim \mathcal{O}(\Delta t), \quad i \rightarrow \infty.$$

The iteration scheme (3.9) is the starting point of our numerical approach for stochastic optimal control problems. In the following sections, we shall show how to get the numerical approximation  $J'_N(u)$  of  $J'(u)$  in each iteration.

**3.1. The representation of  $J'(u)$ .** It is noticed that the iteration scheme (3.9) involves the computation of  $J'(u)$ . In this section, we will derive a new formula of  $J'(u)$  for fixed  $u \in K$  by introducing a pair of adjoint processes. Again in all our arguments,  $J'(u)$  is referred to the element of  $U$ .

By the definition (3.2), we have

$$\begin{aligned} (J'(u), v) &= \lim_{\rho \downarrow 0} \frac{J(u + \rho v) - J(u)}{\rho} \\ &= \mathbb{E} \left[ \int_0^T h'(x_t^u) D x_t^u(v) dt + \int_0^T j'(u(t)) v(t) dt + k'(x_T^u) D x_T^u(v) \right], \end{aligned} \quad (3.13)$$

where  $x_t^u$  is the solution of the SDE (2.1), and

$$D x_t^u(v) := \lim_{\rho \downarrow 0} \frac{x_t^{u+\rho v} - x_t^u}{\rho}.$$

Under Assumption 1, the process  $D x_t^u(v)$  satisfies the SDE

$$\begin{aligned} dD x_t^u(v) &= \left( b'_x(x_t^u, u(t)) D x_t^u(v) + b'_u(x_t^u, u(t)) v(t) \right) dt \\ &\quad + \left( \sigma'_x(x_t^u, u(t)) D x_t^u(v) + \sigma'_u(x_t^u, u(t)) v(t) \right) dW_t, \quad D x_0^u(v) = 0. \end{aligned} \quad (3.14)$$

Notice that one can resort to the above equation to get  $J'(u)$ , however, this would involve very complicate numerical schemes for solving (3.14) (see e.g. [7]). To overcome

this, we shall introduce a pair of adjoint processes  $(p^u, q^u)$  that solves the following backward stochastic differential equation (BSDE):

$$-dp_t^u = f(x_t^u, p_t^u, q_t^u, u(t))dt - q_t^u dW_t, \quad p_T^u = g(x_T^u) = k'(x_T^u), \quad (3.15)$$

where  $f$  is defined as

$$f(x, p, q, u) = h'(x) + p b'_x(x, u) + q \sigma'_x(x, u).$$

Notice that by the standard BSDE theory, under Assumption 1, the BSDE (3.15) admits a unique solution  $(p_t^u, q_t^u)$  for  $u \in K$ . We remark that theoretical studies of BSDEs has been a hot topic recently. In particular, the wellposedness of our adjoint equation, i.e., the BSDEs (3.15), has been well discussed under mild assumptions. One can refer to [34, 33, 29] for more details on the BSDEs theory.

We shall show in the following that by introducing the pair  $(p_t^u, q_t^u)$ , the involving terms  $Dx_t^u(v)$  in (3.13) will be canceled. More precisely, by Itô's formula, we have

$$\begin{aligned} & h'(x_t^u)Dx_t^u(v)dt \\ &= -Dx_t^u(v)dp_t^u - \left( p_t^u b'_x(x_t^u, u(t)) + q_t^u \sigma'_x(x_t^u, u(t)) \right) Dx_t^u(v)dt + q_t^u Dx_t^u(v)dW_t \\ &= -d(p_t^u Dx_t^u(v)) + p_t^u dDx_t^u(v) + q_t^u \left( \sigma'_x(x_t^u, u(t))Dx_t^u(v) + \sigma'_u(x_t^u, u(t))v(t) \right) dt \\ & \quad - \left( p_t^u b'_x(x_t^u, u(t)) + q_t^u \sigma'_x(x_t^u, u(t)) \right) Dx_t^u(v)dt + q_t^u Dx_t^u(v)dW_t \\ &= -d(p_t^u Dx_t^u(v)) + \left( p_t^u b'_u(x_t^u, u(t)) + q_t^u \sigma'_u(x_t^u, u(t)) \right) v(t)dt \\ & \quad + \left( p_t^u \sigma'_x(x_t^u, u(t))Dx_t^u(v) + p_t^u \sigma'_u(x_t^u, u(t))v(t) + q_t^u Dx_t^u(v) \right) dW_t. \end{aligned} \quad (3.16)$$

Then, by inserting (3.16) into (3.13), we obtain

$$(J'(u), v) = \int_0^T \left( \mathbb{E}[p_t^u b'_u(x_t^u, u(t)) + q_t^u \sigma'_u(x_t^u, u(t))] + j'(u(t)) \right) v(t)dt.$$

Then, we can re-define  $J'(u)$  by

$$J'(u)|_t = \mathbb{E}[p_t^u b'_u(x_t^u, u(t)) + q_t^u \sigma'_u(x_t^u, u(t))] + j'(u(t)). \quad (3.17)$$

Here  $J'(u)|_t$  represents  $J'(u)$ , as an element of  $U$ , valued at  $t$ .

To simplify the expression of  $J'(u)$ , we have introduced a pair of adjoint processes  $(p^u, q^u)$  that satisfies the BSDE (3.15), to get rid of the term  $Dx_t^u(\cdot)$ . Then, by solving the BSDE (3.15), we can get the solution pair  $(p^u, q^u)$  numerically, and then further get an approximated  $J'_N(u)$  of  $J'(u)$  by using (3.17). In the next section, we shall propose an Euler type method for solving the adjoint BSDE (3.15).

**REMARK 1.** *We remark that the authors in [12] also introduced an adjoint equation to cancel the term  $Dx_t^u(\cdot)$ . The adjoint equation therein is an anticipating integrand stochastic differential equation, where the solution is required to be backward-adapted instead of the classic forward adapted. However, such a requirement is not true in general. In other words, the wellposedness of the adjoint equation in [12] is unclear for general situations.*

**3.2. Numerical approximations for adjoint equations.** By (3.15), it is noticed that the solution pair  $(p_t^u, q_t^u)$  depends on the forward process  $x_t^u$ . Hence, we need to solve (for  $t \in [0, T]$ ) the following forward-backward stochastic differential equations (FBSDEs)

$$\begin{cases} dx_t^u = b(x_t^u, u(t))dt + \sigma(x_t^u, u(t))dW_t, & x_{t=0} = x_0, \\ -dp_t^u = f(x_t^u, p_t^u, q_t^u, u(t))dt - q_t^u dW_t, & p_T^u = g(x_T^u). \end{cases} \quad (3.18)$$

Next, we shall discuss how to solve the above FBSDEs numerically with a given  $u \in K$ . For notation simplicity, we shall omit the superscript  $u$  in this section, such as  $x_t = x_t^u$ ,  $p_t = p_t^u$ ,  $q_t = q_t^u$ .

Under mild assumptions, it is well known that the above backward equation is wellposed [35]. Moreover, the solutions  $p_t$  and  $q_t$  have the representations

$$p_t = \eta(t, x_t), \quad q_t = \sigma(x_t, u(t))\partial_x \eta(t, x_t), \quad (3.19)$$

where  $\eta(t, x) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is the solution of the following parabolic PDE

$$\mathcal{L}^0 \eta(t, x) = -f(x, \eta(t, x), \sigma(x, u(t))\partial_x \eta(t, x), u(t)), \quad \eta(T, x) = g(x), \quad (3.20)$$

with

$$\mathcal{L}^0 \eta(t, x) = \partial_t \eta(t, x) + b(x, u(t))\partial_x \eta(t, x) + \frac{1}{2}\sigma(x, u(t))^2 \partial_{xx} \eta(t, x),$$

The representations in (3.19) is the so called nonlinear Feynman-Kac formula [35].

We remark that numerical methods for FBSDEs is a hot topic recently, and one can refer to [27, 28, 44, 45, 46] and references therein for variable numerical approaches. Here in this paper, we shall introduce a simple scheme, namely the Euler scheme, for solving the FBSDEs (3.18).

**3.3. The Euler scheme for FBSDEs.** We now follow closely the work [45] and [46] to introduce the Euler method for solving the FBSDEs (3.18). The time partition was defined in (3.7). By integrating both sides of the backward equation on  $[t_n, t_{n+1}]$  we obtain

$$p_{t_n} = p_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f(x_t, p_t, q_t, u(t)) dt - \int_{t_n}^{t_{n+1}} q_t dW_t. \quad (3.21)$$

Then by taking conditional expectation  $\mathbb{E}_{t_n}^x[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_{t_n}, x_{t_n} = x]$  on both sides of (3.21) and applying the left-point rectangular rule, we have

$$p_{t_n}^x = \mathbb{E}_{t_n}^x [p_{t_{n+1}}] + \Delta t f(x, p_{t_n}^x, q_{t_n}^x, u(t_n)) + \bar{R}_{p,n}^x, \quad (3.22)$$

where

$$\bar{R}_{p,n}^x = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x [f(x_t, p_t, q_t, u(t))] dt - \Delta t f(x, p_{t_n}^x, q_{t_n}^x, u(t_n))$$

is the truncation error due to the left-point rectangular rule. Equation (3.22) is our reference equation for solving  $p$ .

Next, we aim to deriving another reference equation for solving  $q$ . To this end, by multiplying (3.21) by  $\Delta W_{n+1} := W_{t_{n+1}} - W_{t_n}$  and taking conditional expectation



$\mathbb{E}_{t_n}^x[\cdot]$  on both sides of the derived equation and then applying again the left-point rectangular rule, we obtain

$$q_{t_n}^x = \frac{1}{\Delta t} \left( \mathbb{E}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] + \bar{R}_{q,n}^x \right), \quad (3.23)$$

where

$$\bar{R}_{q,n}^x = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x [f(x_t, p_t, q_t, u(t)) \Delta W_{n+1}] dt - \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x [q_t] dt + \Delta t q_{t_n}^x$$

is again the corresponding truncation error.

By removing the error terms  $\bar{R}_{p,n}^x$  and  $\bar{R}_{q,n}^x$  in (3.22) and (3.23), we get the following semi-discretization scheme for the BSDE in (3.18): impose the initial value of  $p_N^x = g(x)$  on  $x \in \mathbb{R}$ , and then for  $n = N-1, \dots, 1, 0$ , compute  $p_n^x = p_n(x)$  and  $q_n^x = q_n(x)$  with  $x \in \mathbb{R}$  in a backward way by

$$p_n^x = \mathbb{E}_{t_n}^x [p_{n+1}] + \Delta t f(x, p_n^x, q_n^x, u(t_n)), \quad (3.24)$$

$$q_n^x = \frac{1}{\Delta t} \mathbb{E}_{t_n}^x [p_{n+1} \Delta W_{n+1}]. \quad (3.25)$$

Notice that in the above semi-discretization schemes (3.24) and (3.25), solving  $p_n^x$  and  $q_n^x$  for each  $x \in \mathbb{R}$  may involve the knowledge of  $p_{n+1}$  on the whole space region  $\mathbb{R}$ . To apply this scheme into practice, the spacial discretization of  $\mathbb{R}$  and the approximations of the conditional expectation  $\mathbb{E}_{t_n}^x[\cdot]$  are required.

To do this, we introduce a uniform partition  $\mathbb{R}_h$  of the  $\mathbb{R}$  as

$$\mathbb{R}_h = \{x_k \mid k = 0, \pm 1, \pm 2, \dots\}, \quad \text{with} \quad \Delta x = x_{k+1} - x_k.$$

We shall denote  $I_k =: [x_k, x_{k+1}]$ . Notice that the above partition involves infinite grid points, however, this is unnecessary in practical applications, as we are always interested in the final information ( $t = 0$ ) in a finite interval. This means that we can consider a finite partition with  $|k| \leq P$  with  $P$  being a positive integer (which can be very large and problem dependent). In what follows, we shall consider a finite partition with the parameter  $P$ . We remark that how to choose a reasonable  $P$  is not a trivial work, and we refer to [46] for further discussions.

On the partition  $\mathbb{R}_h$ , we introduce a continuous piece-wise linear function space  $V_h$ , the element of which  $v \in V_h$  can be represented as follows

$$v(x) = \sum_{|k| \leq P} v_k(x_k) \phi_k(x), \quad \text{with} \quad \phi_k(x) = \begin{cases} \frac{x - x_{k-1}}{x_k - x_{k-1}}, & x \in I_{k-1}, \\ \frac{x_{k+1} - x}{x_{k+1} - x_k}, & x \in I_k, \\ 0, & \text{otherwise.} \end{cases}$$

For a continuous function  $f(x)$ , we now introduce the associated interpolation operator  $\mathcal{I}_h$  by

$$\mathcal{I}_h f(x) = \sum_{|k| \leq P} f(x_k) \phi_k(x),$$

i.e., a function in  $V_h$  is determined by its values at the grid points in  $\mathbb{R}_h$ .

**3.3.1. The approximation of conditional expectations.** We now discuss the approximation of conditional expectations. Let  $\tilde{x}_{t_{n+1}}^{t_n, x}$  be the Euler approximation of the state  $x_{t_{n+1}}^{t_n, x}$ , namely,

$$\begin{aligned}\tilde{x}_{t_{n+1}}^{t_n, x} &= x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\Delta W_{n+1} \\ &= x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\zeta,\end{aligned}\quad (3.26)$$

where  $\zeta \sim \mathcal{N}(0, 1)$  is a normal random variable.

We choose  $\tilde{p}_{t_{n+1}}^{t_n, x} = p_{t_{n+1}}(\tilde{x}_{t_{n+1}}^{t_n, x}) = \eta(t_{n+1}, \tilde{x}_{t_{n+1}}^{t_n, x})$  to approximate  $p_{t_{n+1}}^{t_n, x}$ , where  $\eta(t, x)$  is the solution of the problem (3.19). As a result,  $\tilde{p}_{t_{n+1}}^{t_n, x}$  is a function of  $\tilde{x}_{t_{n+1}}^{t_n, x}$ , thus a function of the increment  $\Delta W_{n+1}$ . Therefore, the conditional expectation  $\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}]$  (as well as  $\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}\Delta W_{n+1}]$ ) can be written into an integral on  $\mathbb{R}$  with the Gaussian probability density function  $\rho(\xi) = \frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}$ . Hence we propose the Gauss-Hermite quadrature rule to approximate these conditional expectations. The  $L$ -point Gauss-Hermite quadrature rule for a function  $f$  writes

$$\mathbb{E}[f(\zeta)] = \int_{\mathbb{R}} f(\xi)\rho(\xi)d\xi \approx \sum_{\ell=1}^L f(\xi_\ell)\omega_\ell, \quad (3.27)$$

where  $\{\xi_\ell\}$  and  $\{\omega_\ell\}$  are the Gaussian-Hermite quadrature points and the associated weights, respectively.

Consider for example the approximation of the conditional expectation  $\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}]$ , we have

$$\begin{aligned}\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}] &= \mathbb{E}[p_{t_{n+1}}(\tilde{x}_{t_{n+1}}^{t_n, x})] \\ &= \mathbb{E}[p_{t_{n+1}}(x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\zeta)] \\ &\approx \sum_{\ell=1}^L p_{t_{n+1}}(x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\xi_\ell)\omega_\ell.\end{aligned}\quad (3.28)$$

We shall denote by  $\tilde{\mathbb{E}}_{t_n}^x[p_{t_{n+1}}]$  the approximation of  $\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}]$ , more precisely,

$$\tilde{\mathbb{E}}_{t_n}^x[p_{t_{n+1}}] = \sum_{\ell=1}^L p_{t_{n+1}}(x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\xi_\ell)\omega_\ell. \quad (3.29)$$

Similarly we denote by  $\tilde{\mathbb{E}}_{t_n}^x[p_{t_{n+1}}\Delta W_{n+1}]$  the approximation of  $\mathbb{E}_{t_n}^x[\tilde{p}_{t_{n+1}}\Delta W_{n+1}]$ :

$$\tilde{\mathbb{E}}_{t_n}^x[p_{t_{n+1}}\Delta W_{n+1}] = \sum_{\ell=1}^L p_{t_{n+1}}(x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\xi_\ell)\sqrt{\Delta t}\xi_\ell\omega_\ell. \quad (3.30)$$

In the quadrature rule (3.29), it is noticed that  $\hat{x} = x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\xi_\ell$  may not be on the partition  $\mathbb{R}_h$ . Therefore, we shall resort to the linear interpolation  $\mathcal{I}_h$  to get the desired information. To this end, we define

$$\hat{\mathbb{E}}_{t_n}^x[p_{t_{n+1}}] = \sum_{\ell=1}^L \mathcal{I}_h p_{t_{n+1}}(x + b(x, u(t_n))\Delta t + \sigma(x, u(t_n))\sqrt{\Delta t}\xi_\ell)\omega_\ell. \quad (3.31)$$

Similarly, we define  $\hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}]$  as

$$\begin{aligned} \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] &= \sum_{\ell=1}^L \mathcal{I}_h p_{t_{n+1}}(x + b(x, u(t_n))) \Delta t \\ &\quad + \sigma(x, u(t_n)) \sqrt{\Delta t} \xi_\ell \sqrt{\Delta t} \xi_\ell \omega_\ell. \end{aligned} \quad (3.32)$$

Notice that the approximated expectation  $\hat{\mathbb{E}}_{t_n}^x [\cdot]$  is a function of  $x$ . In the partition space  $\mathbb{R}_h$ , we denote  $\hat{\mathbb{E}}_{t_n}^x [\cdot] := \hat{\mathbb{E}}_{t_n}^{x_{t_n}} [\cdot]$ . In addition, for functions  $f \in V_h$ , we have

$$\begin{aligned} \hat{\mathbb{E}}_{t_n}^x [f(\tilde{x}_{t_{n+1}}^{t_n, x})] &= \tilde{\mathbb{E}}_{t_n}^x [f(\tilde{x}_{t_{n+1}}^{t_n, x})], \\ \hat{\mathbb{E}}_{t_n}^x [f(\tilde{x}_{t_{n+1}}^{t_n, x}) \Delta W_{n+1}] &= \tilde{\mathbb{E}}_{t_n}^x [f(\tilde{x}_{t_{n+1}}^{t_n, x}) \Delta W_{n+1}]. \end{aligned}$$

Based on the above observations, we finally get the following approximations  $\hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}]$  and  $\hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}]$  of  $\mathbb{E}_{t_n}^x [p_{t_{n+1}}]$  and  $\mathbb{E}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}]$ :

$$\begin{aligned} \mathbb{E}_{t_n}^x [p_{t_{n+1}}] &= \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}] + \hat{R}_{p,n}^x, \\ \mathbb{E}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] &= \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] + \hat{R}_{q,n}^x, \end{aligned} \quad (3.33)$$

where  $\hat{R}_{p,n}^k$  and  $\hat{R}_{q,n}^k$  are the truncation errors:

$$\begin{aligned} \hat{R}_{p,n}^x &= \tilde{R}_{p,n}^x + R_{E,p,n}^x + R_{I_h,p,n}^x, \\ \hat{R}_{q,n}^x &= \tilde{R}_{q,n}^x + R_{E,q,n}^x + R_{I_h,q,n}^x, \end{aligned} \quad (3.34)$$

with

$$\begin{aligned} \tilde{R}_{p,n}^x &= \mathbb{E}_{t_n}^x [p_{t_{n+1}}] - \mathbb{E}_{t_n}^x [\tilde{p}_{t_{n+1}}], \quad \tilde{R}_{q,n}^x = \mathbb{E}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] - \mathbb{E}_{t_n}^x [\tilde{p}_{t_{n+1}} \Delta W_{n+1}], \\ R_{E,p,n}^x &= \mathbb{E}_{t_n}^x [\tilde{p}_{t_{n+1}}] - \tilde{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}], \quad R_{E,q,n}^x = \mathbb{E}_{t_n}^x [\tilde{p}_{t_{n+1}} \Delta W_{n+1}] - \tilde{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}], \\ R_{I_h,p,n}^x &= \tilde{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}] - \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}], \quad R_{I_h,q,n}^x = \tilde{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] - \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}]. \end{aligned}$$

**3.3.2. The fully discrete scheme.** By the reference equations (3.22) and (3.23) and the approximations of the conditional expectations in (3.31), we get the following two equations:

$$p_{t_n}^x = \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}}] + \Delta t f(x, p_{t_n}^x, q_{t_n}^x, u(t_n)) + R_{p,n}^x, \quad p_{t_N}^x = g(x), \quad (3.35)$$

$$q_{t_n}^x = \frac{1}{\Delta t} \left( \hat{\mathbb{E}}_{t_n}^x [p_{t_{n+1}} \Delta W_{n+1}] + R_{q,n}^x \right), \quad (3.36)$$

where  $R_{p,n}^x$  and  $R_{q,n}^x$  are the total truncation errors defined by

$$R_{p,n}^x = \bar{R}_{p,n}^x + \hat{R}_{p,n}^x, \quad R_{q,n}^x = \bar{R}_{q,n}^x + \hat{R}_{q,n}^x \quad (3.37)$$

with  $\bar{R}_{p,n}^x$  and  $\bar{R}_{q,n}^x$  defined in (3.22) and (3.23), and  $\hat{R}_{p,n}^x$  and  $\hat{R}_{q,n}^x$  defined in (3.34).

Based on the two reference equations (3.35) and (3.36), we propose a fully discrete numerical scheme for solving the FBSDEs (3.18) as follows: given the terminal condition  $p_N = \sum_{|k| \leq P} g(x_k) \phi_k(\cdot) \in V_h$ , for  $n = N - 1, \dots, 1, 0$ , and each  $x_k \in \mathbb{R}_h$ , solve  $p_n \in V_h$  and  $q_n \in V_h$  by

$$p_n^k = \hat{\mathbb{E}}_{t_n}^{x_k} [p_{n+1}] + \Delta t f(x_k, p_n^k, q_n^k, u(t_n)), \quad (3.38)$$

$$q_n^k = \frac{1}{\Delta t} \hat{\mathbb{E}}_{t_n}^{x_k} [p_{n+1} \Delta W_{n+1}]. \quad (3.39)$$

**3.4. Summary of the numerical approach.** We now summarize the entire algorithm of our gradient projection method. In the fixed-point iteration (3.9), we have introduced  $J'_N(\cdot)$  as the approximation of  $J'(\cdot)$ . As the relation between  $J'(\cdot)$  and the adjoint processes  $(p, q)$  has been revealed in (3.17), it is natural to define the approximation  $J'_N(\cdot)$  by replacing  $p, q$  and  $\mathbb{E}[\cdot]$  in (3.17) with the numerical ones. More precisely, we define

$$J'_N(u)|_{t_n} = \hat{\mathbb{E}}[p_n b'_u(\cdot, u(t_n)) + q_n \sigma'_u(\cdot, u(t_n))] + j'(u(t_n)), \quad (3.40)$$

where  $\hat{\mathbb{E}}[\cdot]$  is defined by

$$\hat{\mathbb{E}}[\phi_{t_0}] = \phi_{t_0}, \quad \hat{\mathbb{E}}[\phi_{t_n}] = \hat{\mathbb{E}}_{t_0}^{x_0} [\hat{\mathbb{E}}_{t_1} [\cdots \hat{\mathbb{E}}_{t_{n-1}} [\phi_{t_n}]]], \quad n \geq 1. \quad (3.41)$$

To make sure that  $J'_N(\cdot) \in U_N$ , we define

$$J'_N(u)|_t = \sum_{n=0}^{N-1} J'_N(u)|_{t_n} \mathcal{X}_{I_n^N}(t). \quad (3.42)$$

Then, the gradient projection method is summarized in Algorithm 1.

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**Algorithm 1** Gradient projection method

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- Set the initial guess of the control  $u_0 \in U_N$  and the error tolerance  $\epsilon_0$ ;
1. Set the terminal condition:  $p_N^k = g(x_k)$ ,  $x_k \in \mathbb{R}_h$ ;
  2. For  $n = N - 1, \dots, 1, 0$ , solve  $(p_n, q_n)$  by (3.38)-(3.39);
  3. Compute  $J'_N(u)|_{t_n}$  by (3.40);
  4. Update  $u$  by (3.9);
- Repeat the above steps until the error  $\|u^{i+1,N} - u^{i,N}\|$  reaches the tolerance  $\epsilon_0$ .
- 

**4. Error estimates.** In this section, we shall perform a rigorous error analysis for our gradient projection method. As concluded in Corollary 1, the first order rate of convergence relies on the estimate  $\epsilon_N = \mathcal{O}(\Delta t)$ . By observing the definition of (3.17) and (3.40), we see that the error  $\epsilon_N$  contains two parts: the numerical error of  $(p_n^k, q_n^k)$  and the approximation error of  $\hat{\mathbb{E}}[\cdot]$ . In the following sections, we shall estimate the two parts one by one.

**4.1. Preliminary results of the discrete operator  $\hat{\mathbb{E}}[\cdot]$ .** In this subsection, we first show some basic properties of the approximated conditional expectations  $\hat{\mathbb{E}}_{t_n}^{x_k}[\cdot]$  and  $\hat{\mathbb{E}}[\cdot]$  which are defined in (3.31) and (3.41) respectively.

Notice that the weights of the quadrature rule  $\{\omega_\ell\}$  are all positive and there holds

$$\sum_{\ell} \omega_\ell = 1.$$

Moreover, the  $L$ -point Gauss-Hermite quadrature rule is exact for polynomials with degree less than or equal to  $2L - 1$ . We now state some basic properties of  $\hat{\mathbb{E}}_{t_n}^{x_k}[\cdot]$  in the following:

**PROPOSITION 1.** *Given variables  $\phi_{t_{n+1}} = \bar{\phi}(t_{n+1}, x_{t_{n+1}})$ , for  $L \geq 2$ , we have*

- $\hat{\mathbb{E}}[\hat{\mathbb{E}}_{t_n}[\phi_{t_{n+1}}]] = \hat{\mathbb{E}}[\phi_{t_{n+1}}]$ .
- *If for any  $x$ , it holds  $\phi_{t_{n+1}}^x \geq 0$ , then we have  $\hat{\mathbb{E}}_{t_n}^{x_k}[\phi_{t_{n+1}}] \geq 0$ ,  $\hat{\mathbb{E}}[\phi_{t_{n+1}}] \geq 0$ .*

$$\begin{aligned} \bullet \quad & \left( \hat{\mathbb{E}}_{t_n}^{x_k}[\phi_{t_{n+1}}] \right)^2 \leq \hat{\mathbb{E}}_{t_n}^{x_k}[(\phi_{t_{n+1}})^2], \quad (\hat{\mathbb{E}}[\phi_{t_n}])^2 \leq \hat{\mathbb{E}}[(\phi_{t_n})^2], \\ & \left( \hat{\mathbb{E}}_{t_n}^{x_k}[\phi_{t_{n+1}} \Delta W_{t_{n+1}}] \right)^2 \leq \left( \hat{\mathbb{E}}_{t_n}^{x_k}[(\phi_{t_{n+1}})^2] - (\hat{\mathbb{E}}_{t_n}^{x_k}[\phi_{t_{n+1}}])^2 \right) \Delta t. \end{aligned}$$

The above propositions are all well known and easy to prove. It is also known that under Assumption 1, for  $m \geq 1$  it holds that

$$\mathbb{E}[|x_t|^m] \leq C(|x_0|^m + 1).$$

In the following, we shall provide a similar result for the approximated expectation  $\hat{\mathbb{E}}[\cdot]$ . Notice that in what follows,  $C$  shall stand for a constant that is independent of  $\Delta t$ ,  $\Delta x$ ,  $n$  and  $k$ , while its value may vary from place to place.

**PROPOSITION 2.** *Under Assumption 1, for  $m \geq 2$ ,  $L \geq 2$ , and  $\Delta x = \mathcal{O}(\sqrt{\Delta t})$ , it holds*

$$\hat{\mathbb{E}}[|x_{t_n}|^m] \leq C(|x_0|^m + 1).$$

*Proof.* We denote by  $\mathcal{I}_h|x|^m$  the linear interpolation of the function  $|\cdot|^m$  at  $x$ . By the interpolation theory, there exists  $\theta \in [x^-, x^+]$  (where  $x^- < x^+$  are two grid points around  $x$ ) such that for  $\Delta x$  sufficiently small, it holds

$$\begin{aligned} \mathcal{I}_h|x|^m & \leq |x|^m + \frac{1}{8}m(m-1)|\theta|^{m-2}(\Delta x)^2 \\ & \leq |x|^m + \frac{1}{8}m(m-1)(|x| + \Delta x)^m + 1)(\Delta x)^2 \\ & \leq |x|^m + C(|x|^m + C\Delta x(|x|^m + 1) + 1)(\Delta x)^2 \\ & \leq (1 + C(\Delta x)^2)|x|^m + C(\Delta x)^2. \end{aligned} \tag{4.1}$$

For fixed  $k$  and  $\ell$ , let  $a_1 = x_k + b(x_k, u(t_n))\Delta t$  and  $a_2 = \sigma(x_k, u(t_n))\sqrt{\Delta t}\xi_\ell$ , then there exists  $\theta \in [a_1, a_1 + a_2]$  such that

$$\begin{aligned} |x_{k,\ell}|^m & = |a_1 + a_2|^m = |a_1|^m + m|a_1|^{m-1}\text{sgn}(a_1)a_2 + m(m-1)|\theta|^{m-2}(a_2)^2 \\ & \leq |a_1|^m + m|a_1|^{m-1}\text{sgn}(a_1)a_2 + m(m-1)(|a_1| + |a_2|)^{m-2}(a_2)^2. \end{aligned} \tag{4.2}$$

By the assumptions on  $b$  and  $\sigma$ , for sufficiently small  $\Delta t$  we have

$$\begin{aligned} |a_1|^m & \leq ((1 + C\Delta t)|x_k| + C\Delta t)^m \\ & \leq (1 + C\Delta t)^m|x_k|^m + C\Delta t(|x_k|^m + 1) \\ & \leq (1 + C\Delta t)|x_k|^m + C\Delta t, \\ |a_1| + |a_2| & \leq C(|x_k| + 1). \end{aligned}$$

Using together (4.1)-(4.2) and the definition

$$x_{k,\ell} = x_k + b(x_k, u(t_n))\Delta t + \sigma(x_k, u(t_n))\sqrt{\Delta t}\xi_\ell,$$

we have

$$\begin{aligned}
\hat{\mathbb{E}}_{t_n}^{x_k}[|x_{t_{n+1}}|^m] &= \sum_{\ell=1}^L \mathcal{I}_h |x_{k,\ell}|^m \omega_\ell \\
&\leq (1 + C(\Delta x)^2) \sum_{\ell=1}^L |x_{k,\ell}|^m \omega_\ell + C(\Delta x)^2 \\
&\leq (1 + C(\Delta x)^2) \left( (1 + C\Delta t) |x_k|^m + C\Delta t + C(|x_k|^m + 1)\Delta t \right) + C(\Delta x)^2 \\
&\leq (1 + C(\Delta x)^2) \left( (1 + C\Delta t) |x_k|^m + C\Delta t \right) + C(\Delta x)^2.
\end{aligned}$$

Consequently, by the definition (3.41) and the assumption  $\Delta x = \mathcal{O}(\sqrt{\Delta t})$ , we have

$$\begin{aligned}
\hat{\mathbb{E}}[|x_{t_{n+1}}|^m] &= \hat{\mathbb{E}}[\hat{\mathbb{E}}_{t_n}[|x_{t_{n+1}}|^m]] \\
&\leq (1 + C(\Delta x)^2) \left( (1 + C\Delta t) \hat{\mathbb{E}}[|x_{t_n}|^m] + C\Delta t \right) + C(\Delta x)^2 \\
&\leq (1 + C(\Delta x)^2)^{n+1} (1 + C\Delta t)^{n+1} \left( |x_0|^m + (n+1)C\Delta t + (n+1)C(\Delta x)^2 \right) \\
&\leq C(|x_0|^m + 1).
\end{aligned}$$

This completes the proof.  $\square$

Next, by the variational arguments, we can easily present an approximation property for the expectation  $\hat{\mathbb{E}}[\cdot]$ .

LEMMA 1. Assume that  $b, \sigma \in C_b^{0,4}$ . For  $\phi_t = \bar{\phi}(t, x_t)$  with  $\bar{\phi} \in C_b^{0,4}$ , we define  $\Phi_{t_i}(x) = \mathbb{E}_{t_i}^x[\phi_{t_n}]$ , then it holds  $\Phi_{t_i} \in C_b^{0,4}$ , and furthermore, we have

$$\mathbb{E}[\phi_{t_n}] = \hat{\mathbb{E}}[\phi_{t_n}] + \sum_{i=0}^{n-1} \hat{\mathbb{E}}[\hat{R}_{\Phi,i}],$$

$$\text{with } \hat{R}_{\Phi,i} = \mathbb{E}_{t_i}^{x_{t_i}}[\Phi_{t_{i+1}}] - \hat{\mathbb{E}}_{t_i}^{x_{t_i}}[\Phi_{t_{i+1}}], \quad 1 \leq i \leq n.$$

#### 4.2. The error estimates of $(p_n^k, q_n^k)$ . We let

$$\mu_n = p_{t_n} - p_n, \quad \nu_n = q_{t_n} - q_n,$$

where  $(p_t, q_t)$  and  $(p_n, q_n)$  are the exact solutions of the FBSDEs (3.18) and numerical solutions of the scheme (3.38)-(3.39), respectively. Notice that

$$p_n(x) = \sum_{k=-\infty}^{\infty} p_n^k \phi_k(x), \quad q_n(x) = \sum_{k=-\infty}^{\infty} q_n^k \phi_k(x),$$

where  $(p_n^k, q_n^k)$  are numerical solutions by scheme (3.38)-(3.39), and for  $x_k \in \mathbb{R}_h$  we have  $(p_n^k, q_n^k) = (p_n(x_k), q_n(x_k))$ . We now define

$$\mu_n^k = p_{t_n}^{x_k} - p_n^k, \quad \nu_n^k = q_{t_n}^{x_k} - q_n^k.$$

Then, by subtracting the equation (3.38) from the equation (3.35), and the equation (3.39) from the equation (3.36), respectively, we deduce that

$$\mu_n^k = \hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{n+1}] + \Delta t \delta f_n^k + R_{p,n}^k, \quad \mu_N^k = p_{t_N}^{x_k} - p_N^k, \quad (4.3)$$

$$\nu_n^k = \frac{1}{\Delta t} \left( \hat{\mathbb{E}}_{t_n}^{x_k}[\mu_{n+1} \Delta W_{n+1}] + R_{q,n}^k \right), \quad (4.4)$$

where

$$\delta f_n^k = f(x_k, p_{t_n}^{x_k}, q_{t_n}^{x_k}, u(t_n)) - f(x_k, p_n^k, q_n^k, u(t_n)), \quad R_{p,n}^k = R_{p,n}^{x_k}, \quad R_{q,n}^k = R_{q,n}^{x_k}.$$

Now, we are ready to give the estimates of  $\mu_n^k$  and  $\nu_n^k$  in the following Lemma. The estimates also imply the stability of the scheme (3.38)-(3.39) and will be used in our final error estimates.

LEMMA 2. *Under proposition 1, namely, assume that  $f(x, p, q, u)$  is Lipschitz continuous with respect to  $p$  and  $q$ , uniformly in  $x$  and  $u$ , then there holds*

$$\hat{\mathbb{E}}[(\mu_n)^2] + \Delta t \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(\nu_n)^2] \leq C \hat{\mathbb{E}}[(\mu_N)^2] + \frac{C}{\Delta t} \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2].$$

*Proof.* By taking square of the equations (4.3)-(4.4), and using together Proposition 1, and the inequality  $(a+b)^2 \leq (1+\epsilon)a^2 + (1+1/\epsilon)b^2$  we get

$$\begin{aligned} (\mu_n^k)^2 &\leq (1 + \gamma \Delta t) (\hat{\mathbb{E}}_{t_n}^{x_k} [\mu_{n+1}])^2, \\ &\quad + C \left(1 + \frac{1}{\gamma \Delta t}\right) \left(\Delta t^2 ((\mu_n^k)^2 + (\nu_n^k)^2) + (R_{p,n}^k)^2\right) \\ (\nu_n^k)^2 &\leq \frac{C}{\Delta t} \left(\hat{\mathbb{E}}_{t_n}^{x_k} [(\mu_{n+1})^2] - (\hat{\mathbb{E}}_{t_n}^{x_k} [\mu_{n+1}])^2\right) + \frac{C}{(\Delta t)^2} (R_{q,n}^k)^2. \end{aligned}$$

Let  $\Delta t \leq 1/\gamma$ ,  $\gamma = 4C^2$ , and add up the above inequalities to get

$$\begin{aligned} (\mu_n^k)^2 + \frac{\Delta t}{2C} (\nu_n^k)^2 &\leq (1 + \gamma \Delta t) \hat{\mathbb{E}}_{t_n}^{x_k} [(\mu_{n+1})^2] + \frac{\Delta t}{2C} (\mu_n^k)^2 \\ &\quad + \frac{1}{2\Delta t} \left((R_{p,n}^k)^2 + (R_{q,n}^k)^2\right), \end{aligned}$$

which yields

$$(\mu_n^k)^2 + C \Delta t (\nu_n^k)^2 \leq (1 + C \Delta t) \hat{\mathbb{E}}_{t_n}^{x_k} [(\mu_{n+1})^2] + \frac{1}{\Delta t} \left((R_{p,n}^k)^2 + (R_{q,n}^k)^2\right).$$

By taking discrete expectation on the above inequality, using together property i, we have

$$\hat{\mathbb{E}}[(\mu_n)^2] + C \Delta t \hat{\mathbb{E}}[(\nu_n)^2] \leq (1 + C \Delta t) \hat{\mathbb{E}}[(\mu_{n+1})^2] + \frac{1}{\Delta t} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2]. \quad (4.5)$$

Then, we get

$$\hat{\mathbb{E}}[(\mu_n)^2] \leq C \hat{\mathbb{E}}[(\mu_N)^2] + \frac{C}{\Delta t} \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2]. \quad (4.6)$$

Taking the summation of (4.5) from  $n = 0$  to  $N - 1$  we get

$$\begin{aligned} C \Delta t \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(\nu_n)^2] &\leq \sum_{n=0}^{N-1} \left(C \Delta t \hat{\mathbb{E}}[(\mu_{n+1})^2] + \frac{1}{\Delta t} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2]\right) \\ &\leq C \hat{\mathbb{E}}[(\mu_N)^2] + \frac{C}{\Delta t} \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2]. \end{aligned} \quad (4.7)$$

Then, the proof is completed.  $\square$

We now provide with the following lemma for estimating the truncation errors, and the proof is somehow standard using the arguments of approximation theory.

LEMMA 3. *Suppose that Assumptions 1 holds, and moreover, we assume that  $b(\cdot, w), \sigma(\cdot, w) \in C_b^4$ , and  $f(\cdot, \cdot, \cdot, w) \in C_b^{2,2,2}$  hold uniformly in  $w \in \mathcal{C}$ ,  $\eta \in C_b^{1,4}$ . Then, we have*

$$\frac{1}{\Delta t} \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(R_{p,n})^2 + (R_{q,n})^2] = \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^4/(\Delta t)^2).$$

*Proof.* As shown in (3.34), (3.35) and (3.36),  $R_{p,n}^k$  and  $R_{q,n}^k$  consist of the following parts of errors.

$$\begin{aligned} R_{p,n}^k &= \bar{R}_{p,n}^k + \tilde{R}_{p,n}^k + R_{E,p,n}^k + R_{I_h,p,n}^k, \\ R_{q,n}^k &= \bar{R}_{q,n}^k + \tilde{R}_{q,n}^k + R_{E,q,n}^k + R_{I_h,q,n}^k. \end{aligned} \quad (4.8)$$

By the interpolation theory, we have the following estimate

$$R_{I_h,p,n}^k = \mathcal{O}((\Delta x)^2), \quad R_{I_h,q,n}^k = \mathcal{O}((\Delta x)^2).$$

Also, by the error estimate of the Gauss quadrature rule [31], for  $f \in C^r$  and  $0 < \epsilon < 1$  we have

$$\left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\xi) e^{-\xi^2/2} d\xi - \sum_{\ell=1}^L f(\xi_\ell) \omega_\ell \right| \leq \frac{CL^{-r/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} |f^{(r)}(\xi) e^{-(1-\epsilon)\xi^2/2}| d\xi,$$

where the constant  $C$  is depends on  $r$  while independent of  $L$  and  $f$ . Therefore, we have that

$$\begin{aligned} |R_{E,p,n}^k| &\leq C\sigma(x_k, u(t_n))^4 (\Delta t)^2, \\ |R_{E,q,n}^k| &\leq C\sigma(x_k, u(t_n))^4 (\Delta t)^{5/2} + C\sigma(x_k, u(t_n))^3 (\Delta t)^2. \end{aligned}$$

Then, by Proposition 2, we have

$$\hat{\mathbb{E}}[(R_{E,p,n})^2] = \mathcal{O}((\Delta t)^4), \quad \hat{\mathbb{E}}[(R_{E,q,n})^2] = \mathcal{O}((\Delta t)^4).$$

For  $\tilde{R}_{p,n}^k$  and  $\tilde{R}_{q,n}^k$ , a rough estimation follows by the Taylor expansion: by subtracting (4.10) from (4.9) we obtain

$$\mathbb{E}_{t_n}^{x_k}[p_{t_{n+1}}] = p_{t_{n+1}}^{x_k} + \Delta t \mathcal{L} \eta(t_{n+1}, x_k) + \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathbb{E}_{t_n}^{x_k}[\mathcal{L} \mathcal{L} \eta(t_{n+1}, y_r)] dr ds, \quad (4.9)$$

$$\mathbb{E}_{t_n}^{x_k}[\tilde{p}_{t_{n+1}}] = p_{t_{n+1}}^{x_k} + \Delta t \mathcal{L} \eta(t_{n+1}, x_k) + \int_{t_n}^{t_{n+1}} \int_{t_n}^s \mathbb{E}_{t_n}^{x_k}[\mathcal{L} \mathcal{L} \eta(t_{n+1}, \tilde{y}_r)] dr ds, \quad (4.10)$$

then we can deduce that  $\tilde{R}_{p,n}^k = \mathcal{O}((\Delta t)^2)$ , where

$$\mathcal{L} \eta(t, x) = b(x, u(t)) \partial_x \eta(t, x) + \frac{1}{2} \sigma(x, u(t))^2 \partial_{xx} \eta(t, x).$$



Similarly, for  $\tilde{R}_{q,n}^k$  we can derive that  $\tilde{R}_{q,n}^k = \mathcal{O}((\Delta t)^2)$ . Finally, for the semi-discretization error, we have

$$\begin{aligned}\bar{R}_{p,n}^{x_k} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^t \mathbb{E}_{t_n}^{x_k} [\mathcal{L}^0 \bar{f}(s, y_s)] ds dt, \\ \bar{R}_{q,n}^{x_k} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^t \mathbb{E}_{t_n}^{x_k} [\mathcal{L}^0 \bar{f}(s, y_s) \Delta W_{n+1} + \mathcal{L}^1 \bar{f}(s, y_s) - \mathcal{L}^0 \zeta(s, y_s)] ds dt,\end{aligned}$$

where  $\bar{f}(t, x) = f(x, \eta(t, x), \zeta(t, x), u(t))$ . Thus, by recalling that  $u \in U_N$ , we have

$$\bar{R}_{p,n}^{x_k} = \mathcal{O}((\Delta t)^2), \quad \bar{R}_{q,n}^{x_k} = \mathcal{O}((\Delta t)^2).$$

Then, the desired result follows by combining all the estimates above.  $\square$

Using together the above arguments (Lemma 1-3), we can finally get the following error estimates for our numerical schemes:

**THEOREM 2.** *Suppose that assumption 1 holds, and under similar assumptions as in the above lemmas. Then, the conclusions of Lemma 2 and Lemma 3 imply*

$$\hat{\mathbb{E}}[(\mu_n)^2] + \Delta t \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(\nu_n)^2] = \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^4/(\Delta t)^2).$$

Then, we can further get

$$\epsilon_N = \sup_i \|J'(u^{N,i}) - J'_N(u^{N,i})\| = \mathcal{O}(\Delta t) + \mathcal{O}((\Delta x)^2/\Delta t).$$

In particular, by taking  $\Delta x = \Delta t$ , we have that  $\epsilon_N = \mathcal{O}(\Delta t)$ . Then, using together Corollary 1, we have

$$\|u^* - u^{N,i}\| = \mathcal{O}(\Delta t), \quad i \rightarrow \infty.$$

*Proof.* Given  $u \in U_N$ , we define

$$\begin{aligned}\phi_t &= p_t b'_u(x_t, u(t)) + q_t \sigma'_u(x_t, u(t)) + j'(u(t)), \\ \phi_n^k &= p_n^k b'_u(x_k, u(t_n)) + q_n^k \sigma'_u(x_k, u(t_n)) + j'(u(t_n)).\end{aligned}$$

Then by the assumptions, we have  $\bar{\phi} \in C_b^{1,4}$  in  $[t_n, t_{n+1}) \times \mathbb{R}$ , where  $\bar{\phi}$  is such that

$\phi_t = \bar{\phi}(t, x_t)$ . Moreover,  $J'(u)|_t = \mathbb{E}[\phi_t]$ ,  $J'_N(u)|_{t_n} = \hat{\mathbb{E}}[\phi_n]$ . Then, we have

$$\begin{aligned}
& \|J'(u) - J'_N(u)\|^2 \\
& \leq C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} (J'(u)|_t - J'(u)|_{t_n})^2 + (J'(u)|_{t_n} - J'_N(u)|_{t_n})^2 dt \\
& = C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left( \int_{t_n}^t \frac{d}{dr} \mathbb{E}[\phi_r] \Big|_{r=s} ds \right)^2 dt + C\Delta t \sum_{n=0}^{N-1} (\mathbb{E}[\phi_{t_n}] - \hat{\mathbb{E}}[\phi_n])^2 \\
& \leq C\Delta t \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{t_n}^t \left( \mathbb{E}[\mathcal{L}^0 \bar{\phi}(s, y_s)] \right)^2 ds dt \\
& \quad + C\Delta t \sum_{n=0}^{N-1} \left( (\mathbb{E}[\phi_{t_n}] - \hat{\mathbb{E}}[\phi_{t_n}])^2 + (\hat{\mathbb{E}}[\phi_{t_n}] - \hat{\mathbb{E}}[\phi_n])^2 \right) \\
& \leq C(\Delta t)^2 + C(\Delta x)^4/(\Delta t)^2 + C\Delta t \sum_{n=0}^{N-1} \hat{\mathbb{E}}[(\mu_n)^2 + (\nu_n)^2] \\
& = \mathcal{O}((\Delta t)^2) + \mathcal{O}((\Delta x)^4/(\Delta t)^2).
\end{aligned}$$

Notice that the above estimation of  $\|J'(u) - J'_N(u)\|^2$  does not depend on the choice of  $u$ , and then, we complete the proof.  $\square$

**5. Numerical experiments.** In this section, we present several numerical examples to verify the efficiency of our numerical approach. In all our computations, we need to choose a reasonable parameter  $\rho$ . Motivated by the error estimates in the last section, it is noticed that the scheme admits good convergence property with sufficiently small  $\rho$ . However, extremely small  $\rho$  would decrease the convergence rate of the iteration. In our examples, we shall simply choose  $\rho_i = 1/\sqrt{i}$ . And in what follows, we shall denote by "CT" the convergence rate.

*Example 1.* Our first example has been used in [12]. The optimal control problem is stated as

$$J(u^*) = \min_{u \in K} J(u)$$

with the cost functional

$$J(u) = \frac{1}{2} \int_0^T \mathbb{E}[(x_t - x^*(t))^2] dt + \frac{1}{2} \int_0^T u^2(t) dt, \quad K = U,$$

and the controlled state equation

$$dx_t = u(t)x_t dt + \sigma x_t dW_t.$$

Here  $\sigma$  is a constant. The deterministic function  $x^*$  and the corresponding exact solution  $u^*$  are given by

$$u^*(t) = \frac{T-t}{\frac{1}{x_0} - Tt + \frac{t^2}{2}}, \quad x^*(t) = \frac{e^{\sigma^2 t} - (T-t)^2}{\frac{1}{x_0} - Tt + \frac{t^2}{2}} + 1. \quad (5.1)$$

We set  $x_0 = 1$ ,  $T = 1$  and  $\sigma = 0.1$ , and the number of samples for approximating the expectation is chosen as  $M = 10^5$ , and we set the tolerance as  $\epsilon_0 = 10^{-5}$ . Numerical results by our gradient projection method are presented in Figure 1.

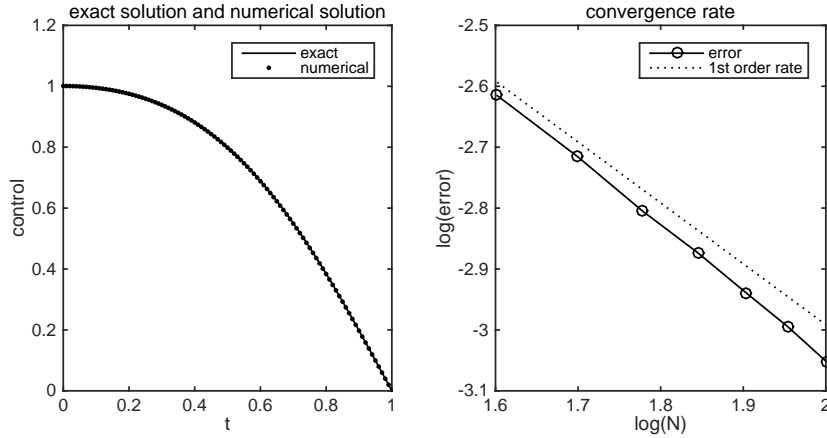


FIG. 1. Numerical results for example 1 with solution (5.1).

The left plot shows that the numerical solution matches the exact solution very well when  $N = 100$ . In the right plot, we have tested the error decays with  $N = 40, 50, \dots, 100$ , and it is clearly shown that the method admits a first order rate of convergence.

Next, we test a different pair  $(x^*, u^*)$  which is given by

$$u^*(t) = \frac{e^{-T} - e^{-t}}{\frac{1}{x_0} + 1 - e^{-t} - te^{-T}}, \quad x^*(t) = \frac{e^{\sigma^2 t} - (e^{-T} - e^{-t})^2}{\frac{1}{x_0} + 1 - e^{-t} - te^{-T}} - e^{-t}. \quad (5.2)$$

We set  $\sigma = 0.1$ ,  $M = 10^5$ ,  $\epsilon_0 = 10^{-5}$  and  $N = 40, 50, \dots, 100$ . The numerical results are given in Figure 2. Again, the numerical solution matches the exact solution very well and first order convergence rate is observed.

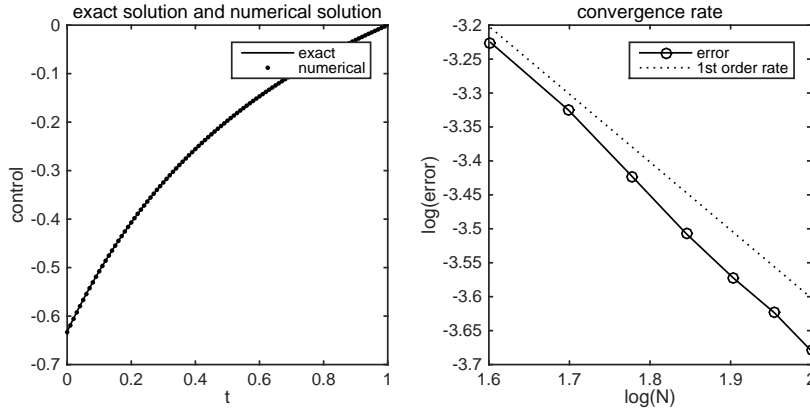


FIG. 2. Numerical results for example 1 with solution (5.2).

*Example 2.* Our second example is also from [12]. More precisely, we consider

$$J(u^*) = \min_{u \in K} J(u)$$

with

$$J(u) = \frac{1}{2} \int_0^T \mathbb{E}[(x_t - x^*(t))^2] dt + \frac{1}{2} \int_0^T u^2(t) dt, \quad K = U,$$

$$dx_t = (u(t) - r(t))dt + \sigma u(t)dW_t.$$

Here we set  $r(t) = u^*(t)/2$ ,  $x_0 = 0$ ,  $T = 1$ , and  $\sigma$  is a constant. The deterministic function  $x^*$  and the corresponding exact solution  $u^*$  are chosen as

$$u^*(t) = \frac{T-t}{\sigma^2(T-t)+1}, \quad x^*(t) = \frac{t}{2\sigma^2} - \frac{1}{2\sigma^4} \ln \frac{\sigma^2 T + 1}{\sigma^2(T-t) + 1} + 1.$$

In our computations, we choose  $\sigma = 0.1$ ,  $M = 10^5$ ,  $\epsilon_0 = 10^{-5}$ , and  $N = 40, 50, \dots, 100$ . The numerical results are shown in Figure 3. Similar conclusions can be made as for example 1. The method converge with the first order accuracy.

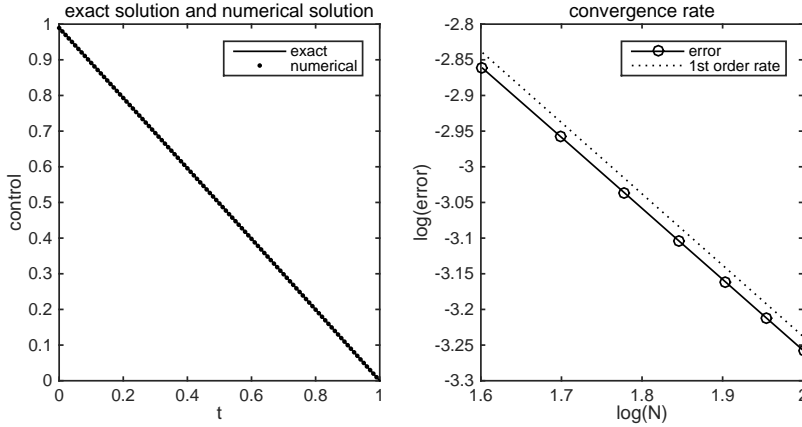


FIG. 3. Numerical results for example 2.

*Example 3.* The previous discussions have been focused on the deterministic control, that is,  $u \in U$ . In this example, we will show that our method can also be used to solve stochastic optimal control problems with feedback control.

This example is set to be the same as in (2.1)-(2.2), except that the control constraint set is now a set of stochastic controls:

$$K_{\mathcal{F}} = \{u \in U_{\mathcal{F}} \mid u_t(\omega) \in \mathcal{C} \text{ a.e. a.s.}\}. \quad (5.3)$$

It follows from the stochastic optimal control theory that the optimal control is actually a feedback control, more precisely, there exists a function  $\bar{u}^*$  such that  $u_t^* = \bar{u}^*(t, x_t)$ , see e.g. [43, 14]. Given a feedback control  $u$  with  $u_t = \bar{u}(t, x_t)$ , by introducing the adjoint processes  $(p, q)$  in the same way as in the deterministic case, and by applying the Itô's formula, we can show that

$$J'(u)_t = p_t b'_u(x_t, u_t) + q_t \sigma'_u(x_t, u_t) + j'(u_t). \quad (5.4)$$

Notice that  $u_t$  is a function of  $t$  and  $x_t$ , then by (5.4) we know that  $J'(u)_t$  is also a function of  $t$  and  $x_t$ . Therefore, due to the feedback property of the control, we can write  $J'(u)$  pointwisely in time-space grids, namely,

$$J'(u)_{t_n}^x = p_{t_n}^x b'_u(x, \bar{u}(t_n, x)) + q_{t_n}^x \sigma'_u(x, \bar{u}(t_n, x)) + j'(\bar{u}(t_n, x)), \quad (5.5)$$

where  $x \in D_h$  and  $J'(u)_t^x$  denotes  $J'(u)_t$  valued at  $x_t = x$ . In the above equation, by introducing our numerical solutions  $p_n$  and  $q_n$ , we get the approximated  $J'_N(\cdot)$  of  $J'(\cdot)$ :

$$J'_N(u)_n^k = p_n^k b'_u(x_k, \bar{u}(t_n, x_k)) + q_n^k \sigma'_u(x_k, \bar{u}(t_n, x_k)) + j'(\bar{u}(t_n, x_k)). \quad (5.6)$$

Since the constraint  $K$  (5.3) is also pointwise in time and space, the projection problem at the grid point  $(t_n, x)$ ,  $x \in D_h$  can be written as

$$\bar{u}^*(t_n, x) = P_C(\bar{u}^*(t_n, x) - \rho J'(u^*)_{t_n}^x).$$

Here we shall not compute the feedback law explicitly, however, we do compute the values of the control at the grid point. Then  $u^*$  is updated in the following way:

$$\bar{u}^{i+1}(t_n, x_k) = P_C(\bar{u}^i(t_n, x_k) - \rho_i J'_N(u^i)_n^k). \quad (5.7)$$

Notice that due to the change of the space of control, we get rid of the expectation in the computation of  $J'(u)$ , meaning that we no longer need the history information before time  $t$  to compute  $J'(u)_t$ , but only the information at time instance  $t$ . Consequently, if a proper space partition  $\{x_k\}_k$  is obtained, and the constraint  $K$  is pointwise in time, then we can run the algorithm in a backward manner as described in Algorithm 2. Compared to Algorithm 1, we notice that under the same spacial

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**Algorithm 2** Gradient projection method

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Set the initial guess of the control  $\{\bar{u}(t_n, x_k)\}_{n,k}$  and the error tolerance  $\epsilon_0$ ;

1. Set the terminal condition:  $p_N^k = g(x_k)$ ,  $x_k \in \mathbb{R}_h$ ;
  2. For  $n = N - 1, \dots, 1, 0$ , do
    - a. solve  $(p_n, q_n)$  by (3.38)-(3.39),
    - b. compute  $J'_N(u)_n^k$  by (5.6),
    - c. update  $u$  by (5.7),
- repeat a-c until  $\sup_k |\bar{u}^{i+1}(t_n, x_k) - \bar{u}^i(t_n, x_k)| \leq \epsilon_0$ .
- 

partition, Algorithm 2 can save a lot restoration.

We now test Algorithm 2 for Example 3 with  $K$  defined in (5.3), and compare the results using feedback control with the results obtained by using the deterministic control. For the feedback control, we shall use the rectangular rule and Monte Carlo method to compute the integral and the expectation of objective functional, respectively. The numerical results are listed in the Table 1. It is shown that the use of feedback control can indeed improve the results (produces a smaller value of objective functional), and this is reasonable as we are minimizing the objective functional within a larger control set.

$N$	$J(u)$ with Algorithm 1	$J(u)$ with Alogrithm 2
100	0.84833	0.62535
200	0.84797	0.64507
400	0.84777	0.65509
800	0.84770	0.66013

TABLE 1  
Numerical results for example 5.3.

*Example 4.* Our last example is a portfolio problem. We consider the following example which has been used in [9].

$$J(u^*) = \min_{u \in K} J(u)$$

with

$$J(u) = \frac{1}{2} \mathbb{E} [(x_T - \kappa)^2], \quad K = \{u \in U_{\mathcal{F}}; -1 \leq u_t \leq 1, \text{ a.e. a.s.}\},$$

$$dx_t = (\zeta \sigma u_t + r)x_t dt + \sigma u_t x_t dW_t.$$

The parameters are chosen as

$$T = 50, \quad \kappa = 1000, \quad x_0 = 300, \quad r = 0.02, \quad \sigma = 0.1, \quad \zeta = 0.05.$$

We set  $\epsilon_0 = 10^{-4}$ ,  $L = 4$ ,  $\rho_i = 0.01/i$ , and the space region is given by  $[-100, 900]$ . The optimal value of  $J(u)$  given in [9] is 15023. To show the convergence rate, we perform experiments with  $N = 1000, 2000, 4000, 8000$ , and choose  $M = N^2/10$ . The corresponding numerical solutions for  $J(u)$  are listed in the Table 2. It is clear that the method admits a first order rate of convergence. This example shows that the Algorithm 2 is capable of solving some optimal control problems involving feedback control.

$N$	1000	2000	4000	8000	optimal
$J(u)$	15196	15107	15069	15045	15023
CR	-	1.0423	0.8688	1.0641	

TABLE 2  
Numerical results for example 5.4.

**6. Conclusion.** In this work, we propose a gradient projection method for solving stochastic optimal control problems. The scheme contains a fixed-point iteration of the control, and an Euler scheme for solving the adjoint equation that is given by BSDEs. The Euler method is used to solve the adjoint BSDEs. We rigorously prove that our numerical method admits a first order rate of convergence. Several numerical tests are presented to support our theoretical finding.

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