MULTIPLICITY-FREE KRONECKER PRODUCTS OF CHARACTERS OF THE SYMMETRIC GROUPS

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ABSTRACT. We provide a classification of multiplicity-free inner tensor products of irreducible characters of symmetric groups, thus confirming a conjecture of Bessenrodt. Concurrently, we classify all multiplicity-free inner tensor products of skew characters of the symmetric groups. We also provide formulae for calculating the decomposition of these tensor products.

1. INTRODUCTION

The inner and outer tensor products of irreducible characters of the symmetric groups (or equivalently of Schur functions) have been of central interest in representation theory and algebraic combinatorics since the landmark papers of Littlewood and Richardson [LR34] and Murnaghan [Mur38]. More recently, these coefficients have provided the centrepiece of geometric complexity theory in an approach that seeks to settle the P versus NP problem [Mul07]; it was recently shown to require not only positivity but precise information on the coefficients [BIP]. The Kronecker coefficients have also been found to have deep connections with quantum information theory [CHM07].

The coefficients arising in the outer tensor product are the most well-understood. The Littlewood–Richardson rule provides an efficient positive combinatorial description for their computation. Using this algorithm, a classification of multiplicity-free outer tensor products was obtained by Stembridge [Ste01]. This was extended to a classification of multiplicity-free skew characters by Gutschwager [Gut10b], a result equivalent to the classification of multiplicity-free products of Schubert classes obtained around the same time by Thomas and Yong [TY10].

By contrast, the coefficients arising in the inner tensor product are much less well-understood; indeed, they have been described as 'perhaps the most challenging, deep and mysterious objects in algebraic combinatorics' [PP]. The determination of these coefficients has been described by Richard Stanley as 'one of the main problems in the combinatorial representation theory of the symmetric group' [Sta99]. While 'no satisfactory answer to this question is known' [JK81] there have, over many decades, been a number of contributions made towards computing special products (such as those labelled by 2-line or hook partitions [Bla17, BWZ10, Ro01, Rem92, RW94] or certain powers [GC]) or the multiplicity of special constituents (for example those with few homogenous components [BK99, BW14]).

In 1999, Bessenrodt conjectured a classification of multiplicity-free Kronecker products of irreducible characters of the symmetric groups. Mainly using results of Remmel, Saxl and Vallejo, it was shown at that time that the products on the conjectured list were indeed multiplicity-free and the conjecture was verified by computer calculations for all $n \leqslant 40$. Since then, multiplicity-free Kronecker products have been studied in [BO06, BWZ10, Gut10a, Man10]. In this paper we prove that the classification list is indeed complete for all $n \in \mathbb{N}$ and hence confirm the conjecture, that is, we have the following result:

Theorem 1.1. *Let λ, μ be partitions of $n \in \mathbb{N}$. Then the Kronecker product $[\lambda] \cdot [\mu]$ of the irreducible characters $[\lambda], [\mu]$ of $\mathfrak{S}_n$ is multiplicity-free if and only if the partitions λ, μ satisfy one of the following conditions (up to conjugation of one or both of the partitions):*

(1) One of the partitions is (n), and the other one is arbitrary;

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(2) one of the partitions is $(n-1,1)$, and the other one is a fat hook (here, a fat hook is a partition with at most two different parts, i.e. it is of the form (a^b, c^d), $a \geqslant c$);
(3) $n = 2k+1$ and $\lambda = (k+1,k) = \mu$, or $n = 2k$ and $\lambda = (k,k) = \mu$;
(4) $n = 2k$, one of the partitions is (k,k), and the other one is one of $(k+1,k-1), (n-3,3)$ or a hook;
(5) one of the partitions is a rectangle, and the other one is one of $(n-2,2), (n-2,1^2)$;
(6) the partition pair is one of the pairs $((3^3),(6,3))$, $((3^3),(5,4))$, and $((4^3),(6^2))$.

We also provide the explicit combinatorial formulae for calculating any multiplicity-free Kronecker product in Section 3. Using this we can then easily prove the following consequence of Theorem 1.1:

Theorem 1.2. *Let λ, μ, ν be partitions of $n \in \mathbb{N}$, all different from (n) and (1^n). Then the Kronecker product $[\lambda] \cdot [\mu] \cdot [\nu]$ of the irreducible characters $[\lambda], [\mu], [\nu]$ of $\mathfrak{S}_n$ is not multiplicity-free.*

Assuming the classification of multiplicity-free Kronecker products for a symmetric group $\mathfrak{S}_n$, with some further work the complete list of multiplicity-free products involving *skew characters* of $\mathfrak{S}_n$ is obtained; we state this below. Indeed, this will be an important tool in the inductive proof of Theorem 1.1. A proper skew diagram is one that is not the diagram of a partition up to rotation, the corresponding skew character has two distinct irreducible constituents by [BK99, Lemma 4.4]; we shall refer to such a character as a *proper skew character*.

Theorem 1.3. *No product of two proper skew characters is multiplicity-free. Now, let α be a partition of n and let χ denote a proper skew character of $\mathfrak{S}_n$. The product $\chi \cdot [\alpha]$ is multiplicity-free if and only if one of the following holds (up to conjugation of one or both of the diagrams):*

(1) $\alpha = (n)$, and χ is a multiplicity-free skew character;
(2) $n = ab$, $a, b \geqslant 2$: $\alpha = (a^b)$, $\chi = [(n,1)/(1)] = [n] + [n-1,1]$;
(3) $n = 2k$, $k \geqslant 2$, $\alpha = (k,k)$, $\chi = [(k+1,k)/(1)] = [k+1,k-1] + [k,k]$.

We remark that there are classification conjectures also on multiplicity-free Kronecker products for groups related to the symmetric groups. In these cases, even less is known about the Kronecker coefficients and crucial reduction tools available for the symmetric groups are still missing.

The layout of the paper is as follows. In Section 2, we recall the results concerning Kronecker and Littlewood–Richardson coefficients which will be useful for the remainder of the paper; chief among these are Dvir's recursion formula and Manivel's semigroup property. We also explain our methodology and the intersection diagrams which will be essential in the bulk of the paper. In Section 3, we verify that the products on our list are indeed multiplicity-free and provide formulae for decomposing these inner tensor products; using some of these, we also show how to deduce Theorem 1.2 from Theorem 1.1. Sections 4 to 8 are dedicated to proving the converse, namely that any product $[\lambda] \cdot [\mu]$ such that the pair (λ, μ) is not on the list in Theorem 1.1, contains multiplicities strictly greater than 1. Section 4 serves as a gentle introduction to the techniques which will be used in Sections 6, 7, and 8; here we consider tensor squares, products involving a hook, and products involving a 2-line partition. In Section 5, we show that if Theorem 1.1 is true for all partitions of degree less than or equal to n, then Theorem 1.3 is also true for all skew-partitions of degree less than or equal to n. We then begin our inductive proof of Theorem 1.1 in earnest. In Sections 6 and 7 we consider products involving either a character labelled by a rectangle or fat hook partition; such products are the most difficult to tackle using Dvir's recursion formula and the semigroup property as one is more likely to reduce to a multiplicity-free product. Finally, in Section 8 we prove that if Theorem 1.1 and thus also Theorem 1.3 are true for all partitions of degree less than or equal to $n-1$, then they also hold true for any product involving partitions of degree n. The hard work in earlier sections has a surprising pay-off: the large reduction from arbitrary tensor products to those involving a fat hook is much simpler than one would expect. The main technique in the final section is to reduce to a product involving a fat hook or a rectangle and to appeal to the earlier sections. The cases for small n are handled by computer calculations, using either Maple together with Stembridge's SF package, or GAP [GAP4]; in both contexts, the computation of Kronecker products for symmetric group characters and their decomposition is already provided.

2. Background and useful results

2.1. **Symmetric group combinatorics.** We let $\mathfrak{S}_n$ denote the symmetric group on n letters. The combinatorics underlying the representation theory of the symmetric group is based on partitions. A *partition* λ of n, denoted $\lambda \vdash n$, is defined to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ equals n. The *length* of a partition $\lambda \vdash n$ is the number of nonzero parts, we denote this by $\ell(\lambda)$. The *width* of a partition $\lambda \vdash n$ is the size of the first part and is denoted $w(\lambda) = \lambda_1$. The *depth* of a partition $\lambda \vdash n$ is $n - \lambda_1$.

We identify a partition, λ, with its associated *Young diagram*, that is the set of *nodes*

$$\left\{(i,j) \in \mathbb{Z}^2_{>0} \ | \ j \leqslant \lambda_i \right\}.$$

A node (i, λ_i) of λ is *removable* if it can be removed from the diagram of λ to leave the diagram of a partition, while a node not in the diagram of λ is an *addable* node of λ if it can be added to the diagram of λ to give the diagram of a partition. The set of removable (respectively addable) nodes of a partition, λ, is denoted by $\mathrm{rem}(\lambda)$ (respectively $\mathrm{add}(\lambda)$). Given $A \in \mathrm{rem}(\lambda)$ (respectively $A \in \mathrm{add}(\lambda)$) we let λ_A (respectively λ^A) denote the partition obtained by removing the node A from (respectively adding the node A to) the partition λ.

Given $\lambda \vdash n$, we define the *conjugate* or *transpose* partition, λ^t, to be equal to the partition obtained from λ by reflecting its Young diagram through the 45° diagonal. The Durfee length of λ is the diagonal length of the Young diagram of λ, and thus gives the side lengths of the largest square which fits into the Young diagram of λ.

Given μ and λ partitions such that $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$, we write $\mu \subseteq \lambda$. If $\mu \subseteq \lambda$, then the *skew partition* or *skew Young diagram* (denoted λ/μ) is simply the set difference between the Young diagrams of λ and μ. If $n = |\lambda| - |\mu|$ then we say that λ/μ is a skew partition of n. We let γ^{rot} denote the diagram obtained by rotating the Young diagram of γ through 180°. We say that a skew diagram γ is a *proper skew diagram* if neither γ nor γ^{rot} is the diagram of a partition. We say that a skew diagram λ/μ is *basic* if it does not contain empty rows or columns, in other words $\mu_i < \lambda_i$, $\mu_i \leqslant \lambda_{i+1}$ for each $1 \leqslant i \leqslant \ell(\lambda)$.

Over the complex numbers, the irreducible characters, $[\lambda]$, of $\mathfrak{S}_n$ are indexed by the partitions, $\lambda \vdash n$. Given a skew partition λ/μ of n, we have an associated *skew character* $[\lambda/\mu]$ of $\mathfrak{S}_n$, see [JK81, Section 2.4] for more details. For the corresponding definitions of Schur and skew Schur functions, see [Sta99].

Theorem 2.1. *[The Littlewood–Richardson Rule] For $\lambda \vdash r_1$, $\mu \vdash r_2$ and $\nu \vdash r_1 + r_2$,*

$$[\lambda] \boxtimes [\mu] = \sum_\nu c(\lambda, \mu, \nu)[\nu]$$

where the $c(\lambda,\mu,\nu)$ are the Littlewood–Richardson coefficients *computed as follows.*

The coefficient $c(\lambda,\mu,\nu)$ is zero, unless $\lambda \subseteq \nu$. Otherwise it is the number of fillings of ν constructed as follows.
For each node (i,j) of μ, take a symbol $u_{i,j}$. Begin with the diagram λ and:

(1) *Add to it all symbols $u_{1,j}$ (corresponding to the first row of nodes of μ) in such a way as to produce the diagram of a partition and to satisfy (3).*
(2) *Next add all symbols $u_{2,j}$ (corresponding to the second row of nodes of μ) following the same rules. Continue this process with all rows of μ.*
(3) *The added symbols must satisfy: (a) for all i, if $y < j$, then $u_{i,y}$ appears in a later column than $u_{i,j}$; and (b) for all j, if $x < i$, then $u_{x,j}$ is in an earlier row than $u_{i,j}$.*

2.2. **Multiplicity-free skew characters.** We recall the classification of multiplicity-free outer products of irreducible characters and multiplicity-free skew characters of symmetric groups as in [Ste01] and [Gut10b, TY10], respectively.

Theorem 2.2 (Multiplicity-free outer products of irreducible characters [Ste01])**.** *A complete list of multiplicity-free outer products of two irreducible characters of symmetric groups is given as follows:*

- *[rectangle]* $\boxtimes$ *[rectangle];*
- *[rectangle]* $\boxtimes$ *[near-rectangle];*
- *[2-line rectangle]* $\boxtimes$ *[fat hook];*
- *[linear]* $\boxtimes$ *[anything].*

Here, a linear partition (2-line rectangle) means a partition with one row or one column (two rows or two columns). A near-rectangle is obtained from a rectangle by adding a single row or column to a rectangle, so a near-rectangle is a special fat hook.

Generalising this result, Gutschwager [Gut10b] classified the basic skew partitions giving multiplicity-free skew characters; this is closely connected to the classification of multiplicity-free products of Schubert classes given by Thomas and Yong [TY10].

Let ρ/σ be a basic skew diagram; it may be connected or decompose into two or more pieces (where two adjacent pieces only meet in a point). We define two paths along the rim of ρ/σ. The *inner path* starts in the lower left corner with an upward segment, follows the shape of σ and ends with a segment to the right in the upper right corner; here, by a *segment* we mean the maximal pieces of the path where the direction does not change. The *outer path* starts in the lower left corner with a segment to the right, follows the shape of ρ and ends with an upward segment in the upper right corner.

We let s_{in} and s_{out} denote the length of the shortest straight segment of the inner path and of the outer path, respectively. Figure 1 depicts several basic skew diagrams, where the partition ρ is shown embedded in a rectangle, with complementary partition τ. In the middle picture, the skew diagram ρ/σ decomposes into two pieces δ' and δ''.

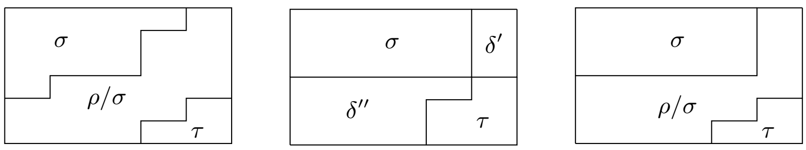

FIGURE 1. Basic skew diagrams

Before we state the classification of the basic skew diagrams labelling multiplicity-free skew characters, we recall that the character associated to a skew diagram is homogeneous if and only if the diagram is a partition diagram up to a possible rotation by 180°; in which case it is already irreducible (see [BK99]). Thus, the skew diagram is proper if and only if the corresponding skew character is proper, i.e., it has at least two different constituents.

Theorem 2.3 (Multiplicity-free skew characters)**.** [Ste01, Gut10b, TY10] *Let D be a basic proper skew diagram. Then the skew character $[D]$ is multiplicity free if and only if up to rotation of D by* 180°*, we have $D = \rho/\sigma$ with σ a rectangle, and additionally one of the following conditions holds:*

(1) $s_{in} = 1$*;*
(2) $s_{in} = 2$, $|\mathrm{rem}(\rho)| = 3$*;*
(3) $s_{out} = 1$, $|\mathrm{rem}(\rho)| = 3$*;*
(4) $|\mathrm{rem}(\rho)| = 2$.

Remark. *We emphasise that Theorem 2.3 covers all cases of multiplicity-free proper skew characters; in particular, the skew character $[\rho/\sigma]$ is not multiplicity-free when the diagram ρ/σ decomposes into more than two connected components, or if it decomposes into two components and one of them is a proper skew partition.*

Furthermore, note that in the cases (2)-(4) described above, the complementary partition τ to ρ/σ (in the pictures above) is a (rotated) fat hook, as in Figure 1.

Assuming that the two pictures to the right in Figure 1 are scaled such that the short segments on the outer path are of length 1, the theorem tells us that these skew diagrams correspond to multiplicity-free characters, whereas the skew diagram in the left picture certainly does not as both σ and τ are not rectangular.

2.3. **The semigroup property for Kronecker coefficients.** We now recall Manivel's semigroup property for Kronecker coefficients [Man11]. This will be one of the two main tools used in proving the classification theorem.

Let λ, μ, ν be partitions of n. We define the Kronecker coefficients $g(\lambda, \mu, \nu)$ to be the coefficients in the expansion

$$[\lambda] \cdot [\mu] = \sum_{\nu \vdash n} g(\lambda, \mu, \nu)\, [\nu]\,.$$

In principle, they may be computed via the scalar product, in other words,

$$g(\lambda, \mu, \nu) = \langle [\lambda] \cdot [\mu], [\nu] \rangle = \frac{1}{n!} \sum_{g \in \mathfrak{S}_n} [\lambda](g)[\mu](g)[\nu](g),$$

from which it also shows that the Kronecker coefficients are symmetric in λ, μ, ν. For $\lambda, \mu \vdash n$ we also define

$$g(\lambda, \mu) = \max\{g(\lambda, \mu, \nu), \nu \vdash n\},$$

so that the Kronecker product $[\lambda] \cdot [\mu]$ is multiplicity-free if and only if $g(\lambda, \mu) = 1$.

Proposition 2.4. *Let $\alpha, \beta, \gamma \vdash n_1$ and $\lambda, \mu, \nu \vdash n_2$. If both $g(\alpha, \beta, \gamma) > 0$ and $g(\lambda, \mu, \nu) > 0$ then*

$$g(\lambda + \alpha, \mu + \beta, \nu + \gamma) \geqslant \max\{g(\lambda, \mu, \nu), g(\alpha, \beta, \gamma)\}\,.$$

In particular,

$$g(\lambda + \alpha, \mu + \beta) \geqslant \max\{g(\lambda, \mu), g(\alpha, \beta)\}\,.$$

Remark 2.5. *We will often use this as a reduction procedure, in particular by removing rows and columns from two partitions under consideration.*

As $g(\lambda, \mu) = g(\lambda^t, \mu) = g(\lambda^t, \mu^t)$, we can conjugate one or both of the partitions in the result above. This means that for the inequality, we do not have to take both partitions away from rows at the top but may take off one (or both) from columns at the bottom.

For a given partition ν and $I \subset \{1, \ldots, \ell(\nu)\}$, we let $\nu_I = (\nu_{i_1}, \nu_{i_2}, \ldots)_{i_k \in I}$ and $\nu^I = (\nu_{j_1}, \nu_{j_2}, \ldots)_{j_k \notin I}$.

Corollary 2.6. *Let λ, μ be partitions of n, and suppose there exist some I and J such that $|\lambda_I| = |\mu_J|$. Then*

$$g(\lambda, \mu) \geqslant \max\{g(\lambda_I, \mu_J), g(\lambda^I, \mu^J)\}\,.$$

In particular, if either $g(\lambda_I, \mu_J) > 1$ or $g(\lambda^I, \mu^J) > 1$, then it follows that $g(\lambda, \mu) > 1$ also.

Proof. Suppose that $|\lambda_I| = |\mu_J| = k$. Let $\sigma \vdash k$ and $\tau \vdash n - k$ be partitions such that $g(\lambda_I, \mu_J, \sigma) = g(\lambda_I, \mu_J), g(\lambda^I, \mu^J, \tau) = g(\lambda^I, \mu^J)$. We have that

$$g(\lambda, \mu) \geqslant g(\lambda, \mu, \sigma + \tau) \geqslant \max\{g(\lambda_I, \mu_J, \sigma), g(\lambda^I, \mu^J, \tau)\},$$

and so the result follows. □

Notation. If $\lambda = \mu + \nu$, we say that λ/ν is an *(SG)-removable* (or *semigroup removable*) skew partition. See Example 2.10 for an example of how one can use this procedure to prove that a product contains multiplicities.

2.4. **The Dvir Recursion.** We now recall Dvir's recursive approach to calculating the value of a given Kronecker coefficient. This is the second main tool which we shall use in our proof of the classification theorem.

In the following, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is a partition, we set $\hat{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_\ell)$.

Theorem 2.7. *(*[Dvi93, 1.6 and 2.4], [CM93, 1.1 and 2.1(d)]*) Let λ, μ be partitions of n. Then*

$$\max\{\nu_1 \mid g(\lambda,\mu,\nu) > 0\} = |\lambda \cap \mu|$$

Let $\mu, \nu, \lambda \vdash n$ and set $\beta = \lambda \cap \mu$. If $\nu_1 = |\lambda \cap \mu|$, then

$$g(\lambda,\mu,\nu) = \langle [\lambda/\beta] \cdot [\mu/\beta], [\hat{\nu}] \rangle\ .$$

Remark. *In the situation above, note that by the Littlewood–Richardson rule and the bound on the width given above, any constituent $[\alpha]$ of $[\lambda/\beta] \cdot [\mu/\beta]$ has width at most m, so $\alpha = (\alpha_1, \alpha_2, \dots)$ can always be extended to a partition $(m, \alpha) = (m, \alpha_1, \alpha_2, \dots)$, giving a constituent in $[\lambda] \cdot [\mu]$.*

Since skew characters of $\mathfrak{S}_n$ can be decomposed into irreducible characters using the Littlewood–Richardson rule, the following theorem provides a recursive formula for the coefficients $g(\lambda,\mu,\nu)$.

Theorem 2.8. [Dvi93, 2.3]. *Let λ, μ and $\nu = (\nu_1, \nu_2, \dots)$ be partitions of n. Define $Y(\nu)$ to be the set of all partitions obtained by adding a horizontal strip of size ν_1 to $\hat{\nu}$; explicitly, the set is described as*

$$Y(\nu) = \{(\eta_1, \eta_2, \dots) \vdash n \mid \eta_i \geqslant \nu_{i+1} \geqslant \eta_{i+1} \text{ for all } i \geqslant 1\}\ .$$

Then

$$g(\lambda,\mu,\nu) = \sum_{\substack{\alpha \vdash \nu_1 \\ \alpha \subseteq \lambda\cap\mu}} \langle [\lambda/\alpha] \cdot [\mu/\alpha], [\hat{\nu}] \rangle - \sum_{\substack{\eta \in Y(\nu) \\ \eta \neq \nu \\ \eta_1 \leqslant |\lambda\cap\mu|}} g(\lambda,\mu,\eta)\ .$$

This is crucial for the following result that will be useful later.

Lemma 2.9. [BK99, Lemma 4.6], [BW14, Lemma 2.6] *Let $\lambda, \mu \vdash n$ be partitions not of the form (n) or $(n-1,1)$ up to conjugation. Set $\beta = \lambda \cap \mu \vdash m$. Assume that λ/β is a single row and that $[\mu/\beta]$ is an irreducible character $[\alpha]$, with a partition α. Then we have $g(\lambda,\mu,(m,\alpha)) > 0$. Furthermore, we define the virtual character*

$$\chi = \sum_{A \in \mathrm{rem}(\beta)} [\lambda/\beta_A] \cdot [\mu/\beta_A] - \sum_{B \in \mathrm{add}(\alpha)} \alpha^B\ . \tag{2.1}$$

Then if $\langle \chi, [\kappa] \rangle > 0$, for $\kappa \vdash n-m+1$, then $\nu = (m-1,\kappa)$ is a partition of n, and $g(\lambda,\mu,\nu) = \langle \chi, [\kappa] \rangle$.

2.5. **Terminology, notation, and methods.** We shall frequently use the following terms:

- linear partition (or linear character) to mean a partition of the form (k) or (1^k) (or the corresponding character $[k]$ or $[1^k]$) for some $k \geqslant 1$;
- the natural character to mean the character $[k-1,1]$ for some $k \geqslant 3$;
- 2-line partition to mean a partition, λ, such that $\ell(\lambda) = 2$ or $w(\lambda) = 2$;
- proper hook to mean a partition of the form $(n-a, 1^a)$ for $1 \leqslant a < n-1$;
- fat rectangle to mean a rectangle which is not linear or a 2-line rectangle;
- proper fat hook to mean a fat hook which is not equal to a rectangle, hook, or 2-line partition;
- proper skew partition to mean a skew partition, λ, such that neither λ nor λ^{rot} is a proper partition.

Given $\lambda, \mu \vdash n$, we shall refer to the diagram for this pair of partitions to be the diagram obtained by placing the partitions λ and μ on top of one another so that one can see the intersection of these partitions (usually denoted $\beta = \lambda \cap \mu$) and the set differences $\mu/(\lambda \cap \mu)$ and $\lambda/(\lambda \cap \mu)$ explicitly, see for example Figure 2.

Example 2.10. Suppose we wish to show that the tensor square of the character $[a^3]$ contains multiplicities. We do this by considering the possible ways in which we can reduce our problem (using Dvir's recursion formula or the semigroup property) to a problem for a pair of smaller partitions. We have that $\lambda = \mu = (3^3) + ((a-3)^3)$ and

$$g(\lambda, \mu) \geqslant g((3^3), (3^3)) > 1,$$

by the semigroup property, as required (for example, the coefficient $g((3^3), (3^3), (5, 2, 2)) = 2)$.

Alternatively, one can prove that $[\lambda] \cdot [\mu]$ contains multiplicities (for $a \geqslant 3$) as follows. If $a \leqslant 6$ then the result can be verified by direct computation. For $a > 6$, we can conjugate and obtain $g((a^3), (a^3)) = g((3^a), (a^3)) \geqslant g((3^{a-3}), ((a-3)^3))$ by using Dvir's recursion formula. The result then follows by induction.

Example 2.11. Suppose we wish to show that the product $[11, 10^3, 6, 5, 2^4, 1] \cdot [11, 7^3, 6, 5^4, 2, 1]$ contains multiplicities. The diagram is the rightmost depicted in Figure 2. We have that $\gamma = \delta = (3^3)$ and so

$$g(\lambda, \mu) \geqslant g(\gamma, \delta) = g((3^3), (3^3)) > 1,$$

using Dvir's recursion formula, as required. Alternatively, one can use Corollary 2.6 to remove all rows and columns which are common to both λ and μ to obtain the pair of partitions $\tilde{\lambda} = (3^6)$ and $\tilde{\mu} = (6^3)$. The result then follows from the previous example.

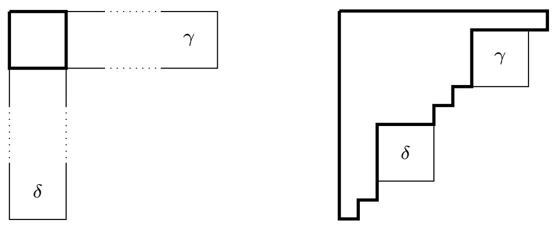

FIGURE 2. The diagrams for the pairs of partitions $(\lambda, \mu) = ((3^a), (a^3))$ and $(\lambda, \mu) = ((11, 7^3, 6, 5^4, 2, 1), (11, 10^3, 6, 5, 2^4, 1))$

In Section 4 we shall first prove Theorem 1.1 in the case of products $[\lambda] \cdot [\mu]$ such that $\lambda = \mu$; one of λ or μ is a hook; and one of λ or μ is a 2-line partition. This allows us to avoid the discussion of small critical cases in the later sections and serves as an introduction to the methods used. In Section 5, we shall then show that if Theorem 1.1 holds by induction on the degree, n, then so does Theorem 1.3.

We shall then begin our inductive proof of Theorem 1.1, assuming the validity of both Theorems 1.1 and 1.3 for partitions of strictly smaller degree. We shall then consider products with a rectangle, followed by products with a fat hook, and then finally arbitrary Kronecker products. At each stage, our strategy will be to prove the result by using the semigroup property and Dvir's recursion formula to reduce the problem to (i) a pair of partitions of strictly smaller degree and then appealing to our inductive proof, or (ii) a pair of partitions of degree n which have already been considered. For example in Section 6 we shall reduce to pairs involving a 2-line or hook partition; in Section 7 we shall reduce to pairs involving a rectangle, or a 2-line, or hook partition.

3. The products on the list are multiplicity-free

Around the time of the classification conjecture, a number of formulae for special products and for constituents of small depth had already been obtained, notably by Remmel and his collaborators, as well as Saxl and Vallejo. This allowed Bessenrodt to check, prior to making the conjecture, that all the products on the list were indeed multiplicity-free. In this section we collect together the non-trivial formulae for the products on our list (up to conjugation). Some of these have appeared in the literature in the past years, and in these cases we refrain from giving proofs and provide references instead.

We start by recalling the products with the character $[n-1, 1]$, which are easy to compute, and then the classification of such multiplicity-free products is not hard to deduce (see [BK99]).

The following result appears in many contexts, for example it underlies the branching graph (of oscillating tableaux) for the partition algebra [HR05].

Lemma 3.1. [BK99, Lemma 4.1] *Let $n \geqslant 3$, and let μ be a partition of n. Let $r = |\mathrm{rem}(\mu)|$. Then*

$$[\mu] \cdot [n-1,1] = \left(\sum_{A \in \mathrm{rem}(\lambda)} \sum_{B \in \mathrm{add}(\mu_A)} [(\mu_A)^B] \right) - [\mu] = (r-1)[\mu] + \text{ other constituents } .$$

Applying the formula above, the multiplicity-free products occurring below can easily be given explicitly in any concrete case. We set $\chi_{(x>y)} = 1$ if $x > y$, and 0 otherwise. Similarly, we set $\chi_{(x>y>z)} = 1$ if $x > y > z$, and 0 otherwise. We extend this notation to other inequalities in the obvious fashion.

Corollary 3.2. *Let $n \geqslant 3$, and let μ be a partition of n. Then*

(i) $[\mu] \cdot [n-1,1]$ is multiplicity-free if and only if μ is a fat hook.
(ii) $[\mu] \cdot [n-1] \uparrow^{\mathfrak{S}_n}$ is multiplicity-free if and only if μ is a rectangle.

In particular, for $n > 2$ we have that

$$[n-1,1]^2 = [n] + [n-1,1] + \chi_{(n>3)}[n-2,2] + [n-2,1^2] .$$

The classification of Kronecker squares was also verified in the course of making the classification conjecture in 1999 using the formulae stated below (which follow as special cases from [RW94, Ro01]) and work of Saxl, Zisser and Vallejo [Sax87, Zis92, Val97]. In the next section we will provide a short proof that the square products in Corollary 3.2 and Proposition 3.3 constitute a complete list of nontrivial multiplicity-free square products (up to conjugation) using the semigroup property.

Proposition 3.3. *Let $k \in \mathbb{N}$.*

(i) For $n = 2k+1$, we have

$$[k+1,k]^2 = \sum_{\substack{\lambda \vdash 2k+1 \\ \ell(\lambda) \leqslant 4}} [\lambda] .$$

(ii) Let $n = 2k$, we let $E(n)$ and $O(n)$ denote the sets of partitions of n into only even parts and only odd parts, respectively, then

$$[k,k]^2 = \sum_{\substack{\lambda \in E(n) \\ \ell(\lambda) \leqslant 4}} [\lambda] + \sum_{\substack{\lambda \in O(n) \\ \ell(\lambda) = 4}} [\lambda] .$$

(iii) Let $n = 2k$, we have that

$$[k,k] \cdot [k+1,k-1] = \sum_{\substack{\lambda \vdash n, \lambda \notin E(n) \\ \ell(\lambda) < 4}} [\lambda] + \sum_{\substack{\lambda \vdash n, \lambda \notin O(n) \cup E(n) \\ \ell(\lambda) = 4}} [\lambda] .$$

Proof. The decompositions (i), (ii), (iii) have since appeared explicitly in [BWZ10, GWXZ, Man10], so we refrain from elaborating on the proof. □

Remark 3.4. *Note that the products in (ii) and (iii) are in the following sense complementary; we have*

$$[k,k] \cdot ([k,k] + [k+1,k-1]) = \sum_{\substack{\lambda \vdash 2k \\ \ell(\lambda) \leqslant 4}} [\lambda] .$$

The decomposition of the products of characters involving a 2-line partition and a hook partition has been determined explicitly by Remmel [Rem92] and Rosas [Ro01]. The formulae there are quite involved, but can be applied in our special case to show

Proposition 3.5. *Let $n = 2k$, and let $\mu \vdash n$ be a hook. Then $[k,k] \cdot [\mu]$ is multiplicity-free.*

Proof. By [Rem92, Theorem 2.2(i)] or [Ro01, Theorem 4], no constituent to a partition of Durfee length 3 can appear, but only hooks and double-hooks.

From the formula in [Rem92, Theorem 2.2(ii)] for the multiplicity of hook constituents in the product, it is immediate that each of these can appear at most once (note that in [Rem92, Theorem 2.2(ii)(c)] the second term can not appear for $(m,n)=(k,k)$).

For a double-hook ν, we might use either [Rem92, Theorem 2.2(iii)] or [Ro01, Theorem 4] to deduce that $g((k,k),\mu,\nu)=0$ or 1. Let $\mu=(n-b,1^b)$ be our hook, and let ν be a double-hook that is not a hook, written as $\nu=(a_1,a_2,2^{b_2},1^{b_1})$ (here $a_1,a_2>0$, $b_1,b_2\geq 0$); we may assume (by conjugation if necessary) that $a_1-a_2\leq b_1$. We recall the formula from [Ro01, Theorem 4]:

$$g((k,k),\mu,\nu)=X_1+X_2+X_3-X_4$$

where

$$\begin{aligned} X_1&=\chi_{(a_2\leq k-b_2-1\leq a_1)}\,\chi_{(b_1+2b_2<b<b_1+2b_2+3)}, & X_2&=\chi_{(a_2\leq k-b_2\leq a_1)}\,\chi_{(b_1+2b_2\leq b\leq b_1+2b_2+3)},\\ X_3&=\chi_{(a_2\leq k-b_2+1\leq a_1)}\,\chi_{(b_1+2b_2<b<b_1+2b_2+3)}, & X_4&=\chi_{(a_2+b_2+b_1=k)}\,\chi_{(b_1+2b_2+1\leq b\leq b_1+2b_2+2)}. \end{aligned} \tag{3.1}$$

First we consider the case where $X_1=1=X_2$ and $X_3=0$. Then $a_1=k-b_2$, so $a_1+b_2=k=a_2+b_1+b_2$, and hence $X_4=1$.

If $X_1=0$ and $X_2=1=X_3$, then $a_2=k-b_2$, hence $a_2+b_2=k=a_1+b_1+b_2\geq a_2+b_1+b_2$, so we must have $b_1=0$ and then $a_1=a_2$. But then $X_3=0$, a contradiction.

If $X_1=1=X_3$, then we also have $X_2=1$. In this case, we must have $a_2\leq k-b_2-1$ and $k-b_2+1\leq a_1$. By our assumption, $a_1-a_2\leq b_1$, hence

$$k\leq a_1+b_2-1\leq a_2+b_1+b_2-1\,.$$

Since $2k=a_1+a_2+b_1+2b_2$, we obtain $a_1+b_2+1\leq k$, and thus we have the contradiction

$$k-b_2+1\leq a_1\leq k-b_2-1\,.$$

Hence the multiplicity $g((k,k),\mu,\nu)=X_1+X_2+X_3-X_4$ is always at most 1. □

We now provide explicit formulae for the remaining Kronecker products with a factor of small depth listed in Theorem 1.1. Also these products were checked in the course of making the classification conjecture in 1999 using [Dvi93, Val97]. We use this opportunity to correct a small mistake in the statement of the formula for the decomposition given in [BO06, Corollary 4.6]; this correction is provided in case (i) below.

Proposition 3.6. *The remaining Kronecker products listed in Theorem 1.1 can be calculated as follows. Here we take the convention that if λ is not a partition, then $[\lambda]$ is zero.*

(i) Let $n=ab\geqslant 6$, $\lambda=(a^b)$, with $a,b>1$. Then the decomposition of the product $[n-2,2]\cdot[a^b]$ is as follows,

$$\begin{aligned}&[a^b]+\chi_{(a>2)}[a^{b-1},a-1,1]+[a^{b-2},(a-1)^2,1^2]+\chi_{(b>3)}[(a+1)^2,a^{b-4},(a-1)^2]\\ &+\chi_{(b>2)}[a+1,a^{b-2},a-1]+\chi_{(b>2)}[a+1,a^{b-3},(a-1)^2,1]+[a+2,a^{b-2},a-2]\\ &+\chi_{(a>2)}[a+1,a^{b-2},a-2,1]+\chi_{(a>3)}[a^{b-1},a-2,2]\,.\end{aligned}$$

(ii) Let $n=ab$ and $\lambda=(a^b)$, with $a\geqslant b>1$. Then the decomposition of the product $[n-2,1^2]\cdot[a^b]$ is as follows,

$$\begin{aligned}&\chi_{(b>2)}[a+2,a^{b-3},(a-1)^2]+[a+1,a^{b-2},a-1]+[a+1,a^{b-2},a-2,1]+[a^{b-2},(a-1)^2,2]\\ &+\chi_{(b>2)}[a+1,a^{b-3},(a-1)^2,1]+\chi_{(b>2)}[(a+1)^2,a^{b-3},a-2]+[a^{b-1},a-2,1^2]+[a^{b-1},a-1,1]\,.\end{aligned}$$

(iii) Let $n=2k>16$. Then the decomposition of the product $[n-3,3]\cdot[k,k]$ is as follows,

$$\begin{aligned}&[k+1,k-1]+[k+1,k-2,1]+[k,k-1,1]+[k,k-2,1^2]+[k,k-2,2]+[k,k-3,3]+\\ &[k-1,k-1,2]+[k-1,k-2,2,1]+[k+3,k-3]+[k+2,k-3,1]+[k+1,k-3,2]\,.\end{aligned}$$

For $6\leqslant n\leqslant 16$ the remaining multiplicity-free products of the form $[n-3,3]\cdot[\lambda]$ are precisely those with $\lambda\in\{(4,2),(4,1^2),(4,3),(3^3)\}$ (up to conjugation), for the corresponding n.

Proof. Case (i) can be proved directly using Dvir's recursion formula and the Littlewood-Richardson rule. Also, Case (ii) may be proved easily by applying Dvir's recursion formula, computing the multiplicity of constituents $[\lambda]$ using $g((k,k),(n-2,1^2),\lambda) = g((k,k),\lambda,(n-2,1^2))$ and the known formula for $g((k,k),\lambda,(n-1,1))$. Case (iii) can be proved easily using [Val97]; as it has since appeared in [BO06, Theorem 4.8] we refrain from elaborating on the proof. □

Finally we show in this section that our main result implies that no product of three non-linear irreducible characters of the symmetric groups is multiplicity-free; hence at the end of this article also Theorem 1.2 is confirmed.

Proposition 3.7. *Assume that Theorem 1.1 is true. Then also Theorem 1.2 holds.*

Proof. Let λ,μ,ν be partitions of n, all different from (n) and (1^n). First we show that a product of the form $[\lambda][\lambda][\mu]$ cannot be multiplicity-free. Since no product of two non-linear irreducible characters of $\mathfrak{S}_n$ is irreducible by [BK99] or [Zis92], we have

$$\langle[\lambda][\lambda][\mu],[\mu]\rangle = \langle[\lambda][\mu],[\lambda][\mu]\rangle > 1,$$

so we are done in this case.

Hence λ,μ,ν have to be three different partitions with pairwise multiplicity-free products. To avoid discussion of small cases, for $n \le 12$ the assertion of Theorem 1.2 is checked by computer, so we assume now $n \ge 13$. By conjugating if necessary, we only have to consider the following triples (λ,μ,ν): $((n-1,1),(k+1,k-1),(k,k))$, $((n-1,1),(n-3,3),(k,k))$, $((n-1,1),(n-a,1^a),(k,k))$ with $2 < a \le \frac{n-1}{2}$, $((n-1,1),(n-2,2),(a^b))$, $((n-1,1),(n-2,1^2),(a^b))$. In all of these cases, $[\lambda][\mu]$ is easily seen to have a constituent $[\alpha]$ where α is not a fat hook by Lemma 3.1. It immediately follows from Theorem 1.1 that $[\alpha][\nu]$, and hence $[\lambda][\mu][\nu]$, cannot be multiplicity-free. □

4. Squares, and products with a hook or with a 2-line partition

As a warm-up to the later sections, we shall now give a self-contained proof of the classification theorem for products $[\lambda]\cdot[\mu]$ involving a hook or 2-line partition or for which $\lambda = \mu$.

4.1. **Squares.** We first consider products of the form $[\lambda]\cdot[\lambda]$. We use the semigroup property to give a simple proof of the converse to Proposition 3.3 (that any product not on the list contains multiplicities).

Proposition 4.1. *Let λ be a partition of n. Then $[\lambda]\cdot[\lambda]$ is multiplicity-free if and only if λ or its conjugate is one of the following*

$$(n),\ (n-1,1),\ (\left\lceil \frac{n}{2}\right\rceil, \left\lfloor \frac{n}{2}\right\rfloor)\,.$$

Proof. By Proposition 3.3, it will suffice to show that any product not of the above form contains multiplicities. Suppose that λ is a 2-line partition not of the above form. Up to conjugation, we can assume that $\lambda = (\lambda_1,\lambda_2)$ such that (i) $\lambda_2 > 1$ and (ii) $\lambda_1 - \lambda_2 \geqslant 2$. The smallest partition satisfying these properties is $(4,2)$; this shall be our *seed* and we shall *grow* all other 2-line cases from this one. Given any λ of the above form, we have that

$$\lambda = (4,2) + (\lambda_1 - 4, \lambda_2 - 2),$$

where the latter term on the right-hand side is a partition because of (ii). By Proposition 2.4, we have that

$$g(\lambda,\lambda) \geqslant g((4,2),(4,2)) = 2$$

(for example, $g((4,2),(4,2),(3,2,1)) = 2$) and so the product $[\lambda]^2$ is not multiplicity-free.

It remains to consider the case in which λ is a partition with $\ell(\lambda), w(\lambda) \geqslant 3$. Set $I = \{1,2,3\}$. Then

$$g(\lambda,\lambda) \geqslant g(\lambda_I,\lambda_I) = g((\lambda_I)^t,(\lambda_I)^t) \geqslant g(((\lambda_I)^t)_I,((\lambda_I)^t)_I)\,.$$

Now $\tilde\lambda = ((\lambda_I)^t)_I = \lambda^t \cap (3^3)$ is a partition with $\ell(\tilde\lambda), w(\tilde\lambda) = 3$. Up to conjugacy we only need to consider $(3,1^3),(3,2,1),(3,3,1),(3,3,2),(3^3)$, with $g(\tilde\lambda,\tilde\lambda)$ equal to 2, 5, 3, 3, 2, respectively. □

We will later use some more detailed information on squares. By work of Saxl [Sax87] , Zisser [Zis92] , Vallejo [Val97, Val14] we have the following result on constituents in squares. We refer to [JK81, Section 2.3.17] for the definition of a hook in a diagram.

Proposition 4.2. *Let $\lambda \vdash n$, $\lambda \neq (n), (1^n)$. Let $h_k = \#\{k\text{-hooks in } \lambda\}$ for $k = 1, 2, 3$ and $h_{21} = \#\{\text{non-linear 3-hooks } H \text{ in } \lambda\}$. Then*

$$\begin{aligned}[\lambda]^2 &= [n] + a_1[n-1,1] + a_2[n-2,2] + b_2[n-2,1^2] \\ &\quad + a_3[n-3,3] + b_3[n-3,1^3] + c_3[n-3,2,1] + \text{constituents of greater depth}\end{aligned}$$

with $a_1 = h_1 - 1$, $b_2 = (h_1-1)^2$, $a_2 = h_2 + h_1(h_1-2)$, for $n \geqslant 4$,
$a_3 = h_1(h_1-1)(h_1-3) + h_2(2h_1-3) + h_3$, for $n \geqslant 6$,
$b_3 = h_1(h_1-1)(h_1-3) + (h_1-1)(h_2+1) + h_{21}$, for $n \geqslant 4$,
$c_3 = 2h_1(h_1-1)(h_1-3) + h_2(3h_1-4) + h_1 + h_{21}$, for $n \geqslant 5$.
In particular, for $n \geq 4$ we always have $a_2 > 0$.

Remark 4.3. *Proposition 4.2 was used to verify the classification of multiplicity-free square products in Proposition 4.1 prior to 1999. Applying this result to a partition λ for which $[\lambda]^2$ is multiplicity-free, immediately yields that λ is a rectangle or $(n-1,1)$ or $(k+1,k)$ (or conjugate). Squares of rectangles can then be dealt with using Dvir's recursion formula.*

4.2. **Hook partitions.** We shall now cover the case of products $[\lambda]\cdot[\mu]$ such that one of λ or μ is a hook, different from (n), $(n-1,1)$ and their conjugates, and the other is an arbitrary partition.

Proposition 4.4. *Let $n \geqslant 5$, and let $\mu = (n-a, 1^a)$ with $2 \leqslant a \leqslant n-3$. Let $\lambda \vdash n$, λ not equal to (n) or $(n-1,1)$ up to conjugation. If $[\mu]\cdot[\lambda]$ is multiplicity-free then (up to conjugation of λ, μ) we have that λ is equal to (k,k) for $n = 2k$, or $a = 2$ and λ is a rectangle.*

Proof. From our computational data, we know that the result holds for all $n \leq 20$, so we may assume that $n \geq 21$. We will proceed by induction, so we assume that the result holds for products with hooks of size smaller than n. Furthermore, by conjugating if necessary, we may (and will) assume that for both $\mu = (n-a, 1^a)$ and λ the length is at most as large as the width, so $a \leq \frac{n-1}{2}$. We have to show $g(\lambda,\mu) > 1$ for any λ different from $(n), (n-1,1)$ and their conjugates, and with (μ,λ) not on the classification list above.

We start with the case in which $\ell(\lambda) = 2$, so $\lambda = (n-b, b)$ where by our assumptions $n - b > b \geqslant 2$. We remove the third column of λ, of height $h \in \{1,2\}$, to obtain a partition $\tilde{\lambda}$, and we set $\tilde{\mu} = (n-a-h, 1^a)$. By our assumptions, $(\tilde{\lambda}, \tilde{\mu})$ is a pair not on our classification list for $n-h$, hence by Corollary 2.6 we conclude

$$g(\lambda,\mu) \geq g(\tilde{\lambda},\tilde{\mu}) > 1$$

and we are done in this case.

We now assume $\ell(\lambda) \geqslant 3$; since (λ,μ) is not on our classification list, λ is not a rectangle when $a = 2$. We remove the third row λ_3 from λ to obtain $\tilde{\lambda}$. As $\lambda_3 \leqslant n/3$ and $a \leq \frac{n-1}{2}$, we have

$$n - a - \lambda_3 \geq n - a - \frac{n}{3} \geq \frac{1}{6}(n+3) \geq 4\,.$$

Hence $\tilde{\mu} = (n-a-\lambda_3, 1^a)$ still satisfies the conditions of the proposition we want to prove. Hence by induction and Corollary 2.6, we have that $g(\lambda,\mu) \geqslant g(\tilde{\mu},\tilde{\lambda}) > 1$ unless $\tilde{\lambda} = (m,m)$ for $m = \frac{n-\lambda_3}{2}$, or $a = 2$ and $\tilde{\lambda}$ is an arbitrary rectangle.

Indeed, both cases can only occur when $\lambda = (m,m,r)$, with $r \geq 1$, and if $a = 2$, we also have $m > r$ (note that $m \geqslant 7$ as $n \geq 21$). In which case, we let $\tilde{\lambda}$ be obtained by removing the second column from λ, of height $h \in \{1,2,3\}$, and we set $\tilde{\mu} = (n-a-h, 1^a)$. Then $\ell(\tilde{\lambda}) = 3$, and λ is not a rectangle in case $a = 2$, so hence again we have by induction and Corollary 2.6 that $g(\lambda,\mu) \geqslant g(\tilde{\mu},\tilde{\lambda}) > 1$. □

4.3. **2-line partitions.** We now consider products in which one factor is labelled by a partition μ with two rows or two columns. Conjugating if necessary, we may assume that μ has two rows.

Lemma 4.5. *Let $n \in \mathbb{N}$ and λ be a partition of n, not equal to (n) or $(n-1,1)$ up to conjugation.*

(1) Let $n \geq 4$. If the product $[n-2,2]\cdot[\lambda]$ is multiplicity-free, then λ is a rectangle or λ is equal to $(3,2)$ (up to conjugation).

(2) Let $n \geq 6$. If the product $[n-3,3]\cdot[\lambda]$ is multiplicity-free, then $\lambda=(k,k)$ or $(4,1^2)$, $(4,2)$, $(4,3)$, (3^3) (up to conjugation).

Proof. As the smaller cases hold by computer calculations, we may assume that $n > 17$. In both cases of the lemma we proceed by induction. By Subsection 4.2, we may assume that λ is not a hook. Conjugating if necessary, we assume that $w(\lambda) \geqslant \ell(\lambda)$; thus $w(\lambda) \geqslant 5$ by our assumption on n.

We have $\mu=(\mu_1,\mu_2)$ with $\mu_2 \in \{2,3\}$, and we take a partition λ (satisfying the assumptions above) that is not a rectangle when $\mu_2=2$, and not (k,k) when $\mu_2=3$. We remove the fourth column of λ, of height h say, to obtain $\tilde{\lambda}$, and we set $\tilde{\mu}=(\mu_1-h,\mu_2)$. By our assumptions, $\tilde{\lambda}$ is not a hook, neither is it a rectangle, nor $(3,2)$ or its conjugate when $\mu_2=2$, and not of the form $(\tilde{k},\tilde{k})$ or one of the exceptional small partitions or one of their conjugates when $\mu_2=3$.. Hence by induction $g(\tilde{\lambda},\tilde{\mu})>1$, and we are done by Proposition 2.4. □

Lemma 4.6. *Let $n=2k \geqslant 2$. Let $\lambda \vdash n$. If the product $[k,k]\cdot[\lambda]$ is multiplicity-free, then λ is a hook, $(n-2,2),(n-3,3),(k,k),(k+1,k-1)$ or (4^3) (up to conjugation).*

Proof. As the smaller cases hold by computer calculations, we may assume that $n \geq 26$. We set $\mu=(k,k)$. We now assume that λ is not one of the partitions listed above that are already known to give a multiplicity-free product with $[k,k]$. We shall again proceed by induction. We assume that $w(\lambda) \geqslant \ell(\lambda)$ and therefore (by our assumption on n) we conclude that $w(\lambda) \geq 6$. If the fifth or sixth column is of even height (and equal to $2h$ say), remove this column from λ to obtain $\tilde{\lambda}$. Otherwise, both the fifth and sixth columns are of odd height (and their sum is equal to $2h$ say); then remove both columns from λ to obtain $\tilde{\lambda}$ In both cases set $\tilde{\mu}=(k-h,k-h)$. We then have a pair of partitions $(\tilde{\mu},\tilde{\lambda})$ such that $g(\tilde{\mu},\tilde{\lambda})>1$ by induction (keeping in mind that $n \geqslant 26$), and hence, by Proposition 2.4, $g(\mu,\lambda)>1$. □

Proposition 4.7. *For μ a 2-line partition, a product $[\mu]\cdot[\lambda]$ is multiplicity-free if and only if the product is on the classification list of Theorem 1.1.*

Proof. We may assume that $n \geqslant 26$ by our computational data, and we proceed again by induction. By Section 3 it is enough to prove that any product not on the list contains multiplicities. By Subsection 4.2 we can assume that λ is not a hook. As before, we may (and will) assume that $w(\lambda) \geqslant \ell(\lambda)$; note that then $w(\lambda) \geq 6$ by our assumption on n. By Lemmas 4.5 and 4.6 we may assume for $\mu=(\mu_1,\mu_2)$ that $\mu_1>\mu_2>3$; we can also assume that λ is not of the form dealt with in these lemmas. By Proposition 4.1, we may also assume that $\lambda \neq \mu$.

We first suppose that $\ell(\lambda)=2$. For $\alpha=(\alpha_1,\alpha_2)$ with $\alpha_2>3$, we define α' by $\alpha'+(1^2)=\alpha$ if $\alpha_2 \geqslant 5$, and by $\alpha'+(2)=\alpha$ otherwise. With this notation in place, $g(\lambda,\mu) \geqslant g(\lambda',\mu')>1$ (given our assumption on n).

We now assume $\ell(\lambda) \geq 3$. Remove the fifth column of λ, of height h say, to obtain $\tilde{\lambda}$. We have three cases to consider: (i) $h<\mu_1-\mu_2$, (ii) $h=\mu_1-\mu_2+2h'$, or (iii) $h=\mu_1-\mu_2+2h'+1$ for $h' \geqslant 0$. Corresponding to these cases we write (i) $\mu=\tilde{\mu}+(h)$, (ii) $\mu=\tilde{\mu}+(\mu_1-\mu_2+h',h')$, or (iii) $\mu=\tilde{\mu}+(\mu_1-\mu_2+h',h'+1)$. We hence obtain a pair of partitions $(\tilde{\lambda},\tilde{\mu})$ such that by induction $g(\tilde{\lambda},\tilde{\mu})>1$ (keeping in mind that $n \geqslant 26$), and hence we are again done by Proposition 2.4. □

5. Multiplicity-free products of skew characters

It is the aim of this section to show that if Theorem 1.1 holds for a fixed $n \in \mathbb{N}$, then so does Theorem 1.3. In the final proof of Theorem 1.1 (and hence also of Theorem 1.3) by induction we may thus always assume that both Theorem 1.1 and Theorem 1.3 hold for all symmetric groups of degree strictly less than n.

First we require some preparatory results on how (multiplicity-free) skew characters decompose into simple constituents. An observation on neighbouring constituents made by Gutschwager will be a very useful tool later on. In [Gut06], he gave a short proof for this based on [Gut10b, Theorem 3.1]; while the proof is constructive, in general the relationship between the neighbours is subtle. The observation is also a special case of the more precise result given in [Gut, Theorem 3.9].

Lemma 5.1. [Gut06, Gut] *Any proper skew character of $\mathfrak{S}_n$ has two neighbouring constituents, i.e., constituents $[\lambda], [\mu]$ such that $|\lambda \cap \mu| = n-1$.*

We now describe multiplicity-free proper skew characters with large maximal constituents (in the lexicographic ordering of the partition labels).

Lemma 5.2. *Let χ be a multiplicity-free proper skew character of $\mathfrak{S}_n$.*

(1) If χ has a constituent $[n]$, then $\chi = [n-k]\boxtimes[k] = \sum_{i=0}^{k}[n-k+i, k-i]$, for some $0 \leqslant k \leqslant n/2$.

(2) If χ has maximal constituent $[n-1,1]$, then we have one of the following:
- *$k \leqslant (n-2)/2$ and*
$$\chi = [n-k-1,1] \boxtimes [k] = \sum_{i=0}^{k}[n-1-i, 1+i] + \sum_{i=0}^{k-1}[n-2-i, 1+i, 1];$$
- *$a > b$, $m = \max(\lfloor (2b-a)/2 \rfloor, 0)$ and*
$$\chi = [(a,b)/(b-1)] = \sum_{i=m}^{b-1}[a-b+1+i, b-i];$$

(3) If χ has maximal constituent $[n-2,2]$, and also has $[n-2,1^2]$ appearing as a constituent, then we have one of the following:
- *$\chi = [n-3,1] \boxtimes [1^2] = [n-2,2] + [n-2,1^2] + [n-3,1^3]$;*
- *$\chi = [(n-2,s,1)/(s-1)] = [((n-2)^2,s)/(n-3,s-1)]$, $1 < s < n-2$;*
- *$\chi = [((n-2)^2,1)/(n-3)]$, $n > 3$.*

(4) If χ has maximal constituent $[n-2,2]$, and contains also $[n-3,3]$, but not $[n-2,1^2]$, then we have one of:
- *$\chi = [n-k-2,2] \boxtimes [k] = [n-2,2] + [n-3,3] + ... + [n-3,2,1] + ... + [n-4,2^2]$;*
- *$\chi = [(n-2,s+2)/(s)] = [n-2,2] + [n-3,3] + ... = \sum_{a=2}^{\lfloor n/2 \rfloor}[n-a,a]$.*

Proof. We may write the skew character χ as $\chi = [\lambda/\mu]$ with a basic skew diagram λ/μ. First we note that the skew diagram λ/μ can have at most two components as no outer product of three characters is multiplicity-free, by Subsection 2.2. Our tactic for the proof will be to examine the maximal constituents of skew characters using the Littlewood–Richardson rule and then considering which characters also satisfy the conditions of Subsection 2.2.

(1) Assuming that χ contains $[n]$, we immediately deduce (from the Littlewood–Richardson rule) that the skew diagram λ/μ consists solely of disconnected single rows. By Theorem 2.2, there are at most two such disconnected rows in χ. Hence the skew character is of the form $\chi = [n-k] \boxtimes [k] = \sum_{i=0}^{k}[n-k+i, k-i]$, for some $k \leqslant n/2$.

(2) Assume now that the maximal constituent of the skew character χ is $[n-1,1]$. Again by the Littlewood–Richardson rule, we deduce that the skew diagram λ/μ then must have one column of length 2, and all other columns are of length 1. As the skew diagram has at most two components, this leaves only a few possibilities for the skew character, and we can only have the two types listed in (2), by Subsection 2.2.

To (3) and (4). We assume that χ has maximal constituent $[n-2,2]$. Note that the skew diagram λ/μ then must have two columns of length 2, and all others are of length 1, and as before, it has at most two components; if it is disconnected, then both components are of partition shape (up to rotation) and are as in Theorem 2.2.

We first consider case (3), in which χ contains $[n-2,1^2]$ as a constituent. Then, if the diagram has two components, both components have a column of length 2 (by the Littlewood–Richardson rule). By Theorem 2.2, the only possibility is then that $\chi = [n-3,1] \boxtimes [1^2] = [n-2,2]+[n-2,1^2]+[n-3,1^3]$.

Now assume that λ/μ is connected and $[n-2,1^2]$ appears as a constituent of χ. In which case the diagram of (2^2) does not appear as a subdiagram of λ/μ (by the Littlewood–Richardson rule). Now, Theorem 2.3 leaves only the possibilities $\lambda/\mu = (r,s,1)/(s-1)$, with $r \geqslant s > 1$, or $\lambda/\mu = (r^2,s)/(r-1,s-1)$, with $r > s > 1$, and for $r > s$ the corresponding skew characters are equal since the diagrams only differ by a rotation. Since $|\lambda/\mu| = n$, we have $r = n-2$.

For case (4), we now assume that χ has maximal constituent $[n-2,2]$ and contains $[n-3,3]$, but not $[n-2,1^2]$. If the diagram is disconnected, the only possibility for χ is

$$\chi = [n-k-2,2] \boxtimes [k] = [n-2,2]+[n-3,3]+...+[n-3,2,1]+\cdots+[n-4,2,2].$$

When λ/μ is connected, the diagram of (2^2) appears as a subdiagram of λ/μ, by our assumption that χ does not contain $[n-2,1^2]$. Therefore

$$\chi = [(s+r,s+2)/(s)] = [n-2,2]+[n-3,3]+... = \sum_{a=2}^{\lfloor n/2 \rfloor} [n-a,a],$$

and $|\lambda/\mu| = n$ implies $s+r = n-2$. □

A first contribution towards classifying multiplicity-free products of skew characters with irreducible characters is contained in the following easy result.

Lemma 5.3. *Let χ be a proper skew character of $\mathfrak{S}_n$. Then $\chi \cdot [n-1,1]$ is multiplicity-free if and only if $n=2$, and the product is then $([2]+[1^2])\cdot[1^2] = [1^2]+[2]$.*

Proof. By Lemma 5.1, χ has two neighbouring constituents, which we may write as $[\alpha^X]$ and $[\alpha^Y]$, for α a partition of $n-1$ and $X \neq Y$ addable nodes for α. If one of the two partitions α^X, α^Y is not a rectangle, say α^X, then $([\alpha^X]+[\alpha^Y])\cdot[n-1,1]$ contains $2[\alpha^X]$, and hence the product is not multiplicity-free. But if both α^X and α^Y are rectangles, then we have $\alpha = (1)$ and α^X, α^Y are the rectangles (2), (1^2). □

For later usage, we now consider products of irreducible characters with characters that will appear as subcharacters in certain skew characters.

Lemma 5.4. *Let $n \geqslant 4$ and let α be a partition of n.*

(1) Let $\chi = [n-2,2]+[n-2,1^2]$, and let α be a rectangle. Then $\chi\cdot[\alpha]$ is multiplicity-free if and only if α is linear, or $n=4$ and $\alpha = (2^2)$. In the latter case, $\chi\cdot[2^2] = [4]+[3,1]+[2^2]+[2,1^2]+[1^4]$.

(2) Let $\chi = [n]+[n-2,2]$. Then $\chi\cdot[\alpha]$ is multiplicity-free if and only if α is linear.

Proof. If α is a linear partition, both products $\chi\cdot[\alpha]$ are clearly multiplicity-free.

(1) The assertion is easily checked for $\alpha = (2^2)$. Summing the formulas in (i) and (ii) of Proposition 3.6 we immediately see that $[a^{b-1}, a-1, 1]$ appears with multiplicity 2 in $[\alpha]\cdot\chi$.

(2) If α is not a linear partition, then $[\alpha]^2$ contains $[n]+[n-2,2]$, by Proposition 4.2. Thus $\langle \chi\cdot[\alpha], [\alpha]\rangle = \langle [n]+[n-2,2], [\alpha]\cdot[\alpha]\rangle \geqslant 2$, and hence $\chi\cdot[\alpha]$ is not multiplicity-free. □

A crucial step towards Theorem 1.3, and thus a contribution towards the induction strategy mentioned previously, is contained in the next proposition.

Proposition 5.5. *Assume that Theorem 1.1 holds for a fixed $n \in \mathbb{N}$. Let χ be a proper skew character of $\mathfrak{S}_n$, and let $[\alpha]$ be an irreducible character of $\mathfrak{S}_n$. Then $\chi\cdot[\alpha]$ is multiplicity-free if and only if χ and α are one of the following pairs (up to multiplication of χ by a linear character or conjugation of the partition α):*

(1) χ is any multiplicity-free proper skew character, and α is a linear partition;
(2) $n = ab$ for $a, b \geqslant 2$, $\alpha = (a^b)$, $\chi = [n-1] \boxtimes [1] = [n] + [n-1, 1]$;
(3) $n = 2k$ for $k \geqslant 2$, $\alpha = (k, k)$, $\chi = [(k+1, k)/(1)] = [k+1, k-1] + [k, k]$.

Proof. We know that the products in the three situations above are indeed multiplicity-free; for cases (2) and (3) this was already covered in Section 3 (and (1) is obvious).

So we now assume that $\chi \cdot [\alpha]$ is multiplicity-free and that $[\alpha]$ is not linear (in other words α is not a linear partition) and hence we may assume $n > 2$. By Lemma 5.3 we already know that $[n-1, 1] \cdot \chi$ is not multiplicity-free. Therefore we need only consider $\alpha \neq (n-1, 1)$ (or its conjugate) and hence we may assume that $n \geqslant 4$.

We have assumed that the classification list in Theorem 1.1 is complete for our fixed $n \in \mathbb{N}$ and that $\chi \cdot [\alpha]$ is multiplicity-free. Every partition on the list is a fat hook and so we deduce that all constituents of χ are labelled by fat hooks. Also, since χ has a non-linear constituent, α must be a fat hook.

Thus α is a fat hook different from (n), $(n-1, 1)$ (and their conjugates, by our assumption and Lemma 5.3 respectively) that has a multiplicity-free product with two neighbouring fat hooks (because of Lemma 5.1).

We shall now consider the possible partitions α from the list in Theorem 1.1 satisfying these conditions. Case-by-case, we consider α on the list $(n-2, 2)$, $(n-3, 3)$, $(n-2, 1^2)$, $(k+1, k)$, (k, k), $(k+1, k-1)$, hooks and rectangles, and the few cases for $n \leqslant 12$.

For $\alpha = (n-2, 2)$, we consider the possible constituents in χ. When $n > 6$, the only non-trivial possible constituents of χ such that $\chi \cdot [\alpha]$ is multiplicity-free are $(n-1, 1)$ (and its conjugate) or a rectangle. Since χ has to have two neighbouring constituents, it must contain $\chi_0 = [n] + [n-1, 1]$ (up to conjugating); but $[n-2, 2] \cdot \chi_0$ is not multiplicity-free (for $n \geqslant 5$).

For $n = 5$, the character $[3, 2]$ has a multiplicity-free product with all $[\beta]$, $\beta \vdash 5$, $\beta \neq (3, 1^2)$. Given the previous arguments, we only have to consider the products with the neighbour pair sums $[4, 1] + [3, 2]$ and $[3, 2] + [2^2, 1]$, and neither of these products are multiplicity-free.

For $n = 4$, the proper skew partitions (up to conjugation) which give multiplicity-free products with α are

$$[2, 2] \cdot ([3] \boxtimes [1]) = [3, 1] + [2, 2] + [2, 1^2]$$
$$[2, 2] \cdot [(3, 2)/(1)] = [4] + [3, 1] + [2, 2] + [2, 1^2] + [1^4]$$

and correspond to cases (2) and (3). The remaining proper skew characters $[2, 1] \boxtimes [1]$ and $[2] \boxtimes [2]$ which are equal to $[(3, 2)/(1)] + [2, 1^2]$ and $[(3, 2)/(1)] + [4]$ respectively and so their products with $[2, 2]$ are not multiplicity-free.

Now we consider $\alpha = (n-3, 3)$, and look again for the possible constituents in χ that have a multiplicity-free product with $[\alpha]$. For $n \geqslant 7$ the only possible constituents of χ whose product with $[\alpha]$ is multiplicity-free are (n), $(n-1, 1)$, or (k, k) and their conjugates; with the exceptions of $(4, 3)$ for $n = 7$ (and their conjugates). Recall χ has a neighbouring pair of constituents, and therefore must contain $\chi_0 = [n] + [n-1, 1]$ for all $n \geqslant 7$ up to conjugation. However, $[n-3, 3] \cdot \chi_0$ is not multiplicity-free for $n \geqslant 7$.

For $n = 6$, the character $[3^2]$ has multiplicity-free products with $[6]$, $[5, 1]$, $[4, 2]$, $[4, 1^2]$ and $[3, 3]$ (and their conjugates). The only neighbour pair sums that have multiplicity-free products with $[\alpha]$ are $[6] + [5, 1] = [5] \boxtimes [1]$ and $[3, 3] + [4, 2] = [(4, 3)/(1)]$ (up to conjugation) which correspond to cases (2) and (3) of the proposition. The first (respectively second) skew character can only be extended to $[6] + [5, 1] + [1^6]$ (respectively cannot be extended) so that the product with $[3^2]$ remains multiplicity-free. The former is not a skew character and so does not provide a counter example. Hence, these considerations for $\alpha = (n-3, 3)$ have only led to the cases for $n = 6$ in (2) and (3).

For $\alpha = (n-2, 1^2)$, we consider the possible constituents in χ. The only possible constituents in χ are then (n), $(n-1, 1)$ (and their conjugates) and rectangles. As before, χ must then contain $\chi_0 = [n] + [n-1, 1]$ (up to conjugating), except when $n = 4$, where there are further possible neighbour pairs. But $[n-2,1^2]\cdot\chi_0$ is not multiplicity-free, and for $n=4$, no neighbour pair sum has a multiplicity-free product with $[2,1^2]$.

For α a hook partition not equal to (n), $(n-1,1)$, $(n-2,1^2)$ up to conjugation (which have already been considered) we consider the possible constituents in χ. The only possible constituents of χ are (n), $(n-1,1)$ or (k,k) (and their conjugates). Again, χ must then contain $\chi_0=[n]+[n-1,1]$ (up to conjugating), but $[\alpha]\cdot\chi_0$ is not multiplicity-free.

For $\alpha=(k+1,k)$ for $k>3$, we consider the possible constituents in χ. Then χ could only have (n), $(n-1,1)$ or $(k+1,k)$ (and their conjugates) as constituents, except for $n=9$, when also (3^3) can also appear. As before, χ must then contain $\chi_0=[n]+[n-1,1]$ (up to conjugating), but $[k+1,k]\cdot\chi_0$ is not multiplicity-free.

For $\alpha=(k+1,k-1)$ for $k>4$, we consider the possible constituents in χ. Then χ could only have (n), $(n-1,1)$ or (k,k) (and their conjugates) as constituents. As before, χ must then contain $\chi_0=[n]+[n-1,1]$ (up to conjugating), but $[k+1,k-1]\cdot\chi_0$ is not multiplicity-free.

Finally we turn to rectangles. First, let $\alpha=(a^b)$ with $a\geq b$, and assume $b>2$. Then χ could only have (n), $(n-1,1)$, $(n-2,2)$ or $(n-2,1^2)$ and their conjugates as constituents, except (i) for $\alpha=(3^3)$ when $n=9$ where $(5,4)$ and $(6,3)$ or their conjugates possibly appear, or (ii) $\alpha=(4^3)$ when $n=12$, where (6^2) or their conjugates possibly appear.

We first exclude the cases (i) and (ii).

If the maximal constituent in χ is $[n]$, by Lemma 5.2 and Lemma 5.4 we must have $\chi=[n-1]\boxtimes[1]=[n]+[n-1,1]$. In this case, $\chi\cdot[\alpha]$ is indeed multiplicity-free, and we are in situation (2) of the proposition.

If the maximal constituent in χ is $[n-1,1]$, by Lemma 5.2 we only have to discuss the cases when χ is one of the skew characters $[n-2,1]\boxtimes[1]=[n-1,1]+[n-2,2]+[n-2,1^2]$, $[1^2]\boxtimes[n-2]=[n-1,1]+[n-2,1^2]$, or $[(n-1,n-2)/(n-3)]=[n-1,1]+[n-2,2]$. By Lemma 5.4(1) we already know that in the first case the product $\chi\cdot[\alpha]$ is not multiplicity-free.

We now consider the second case, where $\chi=[n-1,1]+[n-2,1^2]$. In the computation of the following scalar product we use the information on special constituents in squares given by Proposition 4.2 several times (here, we just write $\uparrow$ for $\uparrow^{\mathfrak{S}_n}$).

$$\begin{aligned}
\chi\cdot[\alpha] \geqslant&\langle[\alpha]\cdot[n-1,1],[\alpha]\cdot[n-2,1^2]\rangle\\
=&\langle[\alpha]\cdot[n-1]\!\uparrow-[\alpha],[\alpha]\cdot[n-2,1^2]\rangle\\
=&\langle[\alpha]^2,[n-1]\!\uparrow\cdot[n-2,1^2]\rangle-\langle[\alpha]^2,[n-2,1^2]\rangle\\
=&\langle[\alpha]^2,[n-1]\!\uparrow\cdot[n-2,1^2]\rangle\\
=&\langle[\alpha]^2,([n-3,1^2]+[n-2,1])\!\uparrow\rangle\\
=&\langle[\alpha]^2,2[n-2,1^2]+[n-3,2,1]+[n-3,1^3]+[n-1,1]+[n-2,2]\rangle\\
=&\langle[\alpha]^2,[n-3,1^3]+[n-2,2]\rangle=2,
\end{aligned}$$

and hence $\chi\cdot[\alpha]$ is not multiplicity-free in this case. In the third case, where $\chi=[n-1,1]+[n-2,2]$, we follow the same strategy as above and compute

$$\begin{aligned}
\chi\cdot[\alpha] \geqslant&\langle[\alpha]\cdot[n-1,1],[\alpha]\cdot[n-2,2]\rangle\\
=&\langle[\alpha]\cdot[n-1]\!\uparrow-[\alpha],[\alpha]\cdot[n-2,2]\rangle\\
=&\langle[\alpha]^2,[n-1]\!\uparrow\cdot[n-2,2]\rangle-\langle[\alpha]^2,[n-2,2]\rangle\\
=&\langle[\alpha]^2,([n-3,2]+[n-2,1])\!\uparrow\rangle-1\\
=&\langle[\alpha]^2,2[n-2,2]+[n-3,3]+[n-3,2,1]+[n-1,1]+[n-2,1^2]\rangle-1\\
=&\langle[\alpha]^2,2[n-2,2]+[n-3,3]\rangle-1=2\,.
\end{aligned}$$

Again, it follows that $\chi\cdot[\alpha]$ is not multiplicity-free.

Now we may assume that χ contains none of $[n],[n-1,1]$ (or their conjugates). Note that χ must contain a neighbour pair sum, so (up to conjugating) we may now assume that χ contains

$[n-2,2]+[n-2,1^2]$. Given our assumption that $a \geqslant b > 2$, it follows from Lemma 5.4 that $\chi \cdot [\alpha]$ is not multiplicity-free.

We now consider the special cases (i) $\alpha = (3^3)$ and $(5,4)$ and $(6,3)$ or their conjugates appear in χ, or (ii) $\alpha = (4^3)$ and (6^2) or its conjugate appears in χ. We remark that these cases can also be checked by computer.

We first consider case (i). We assume that χ has none of the pair sums discussed above; in which case χ must have one of the pair sums $[5,4]+[6,3]$ or $[6,3]+[7,2]$ (up to conjugation). From the formula in Theorem 1.1 we immediately see that the first pair sum does not give a multiplicity-free product. By Proposition 4.2 we see that $[3^3]\cdot[3^3]$ contains both $[6,3]$ and $[7,2]$ (with multiplicity 1), so $[3^3]\cdot([6,3]+[7,2])$ contains $[3^3]$ with multiplicity 2.

We now consider case (ii). We assume that χ has none of the pair sums discussed above; in which case χ must have $[6^2]$ as a constituent (up to conjugation). But χ cannot contain a neighbour of this constituent, so χ must contain one of the pair sums considered above, and so we are done. This finishes the case of rectangles (a^b) such that $a \geqslant b > 2$.

Now let $\alpha = (k,k)$ for $k > 3$. Then the constituents in χ could only be labelled by (n), $(n-1,1)$, (k,k), $(k+1,k-1)$, $(n-2,2)$, $(n-3,3)$ $(n-2,1^2)$ (and their conjugates) and hooks, except for $n=12$ and $k=6$, when also (4^3) or its conjugate can appear.

We follow a similar strategy as before. We assume first that $n > 12$.

If the maximal constituent in χ is $[n]$, then Lemma 5.2 and Lemma 5.4(2) imply that $\chi = [n-1]\boxtimes[1] = [n]+[n-1,1]$. In this case, $\chi\cdot[\alpha]$ is indeed multiplicity-free, and we are in situation (2) of the proposition.

If the maximal constituent in χ is $[n-1,1]$, Lemma 5.2 now implies that χ is one of the following three skew characters: $[1^2]\boxtimes[n-2] = [n-1,1]+[n-2,1^2]$; $[(n-1,n-2)/(n-3)] = [n-1,1]+[n-2,2]$; or $[(n-1,n-3)/(n-4)] = [n-1,1]+[n-2,2]+[n-3,3]$. For each of the first two skew characters, the simple character $[k+1,k-1]$ appears with multiplicity 2 in $\chi\cdot[\alpha]$ using Proposition 3.6(i) and Lemma 3.1. The third character contains the second and so the product $\chi\cdot[\alpha]$ also contains multiplicities.

Hence we may now assume that χ contains none of $[n],[n-1,1]$ (or their conjugates). As χ must contain a neighbour pair sum, we may now assume that χ contains (up to conjugating) one of the skew characters (i) $[n-2,2]+[n-2,1^2]$, (ii) $[n-2,2]+[n-3,3]$, (iii) a sum of two neighbour hooks (not involving $[n],[n-1,1]$ and their conjugates), or (iv) $[k,k]+[k+1,k-1]$.

In case (i), we know by Lemma 5.4 that $\chi\cdot[\alpha]$ is not multiplicity-free. In case (ii) the simple character $[k+1,k-1]$ appears with multiplicity 2 in $\chi\cdot[\alpha]$ using Proposition 3.6(i) and (iii).

In case (iii), the character χ contains a sum of two neighbouring hooks, say $[n-a,1^a]+[n-a-1,1^{a+1}]$, with $1 < a \leqslant k-1$. By equation (3.1), the character $[k+1,k-a,1^{a-1}]$ appears with multiplicity 1 in both $[k,k]\cdot[n-a,1^a]$ and $[k,k]\cdot[n-a-1,1^{a+1}]$. Hence $\chi\cdot[k,k]$ is not multiplicity-free.

Finally, we consider the last possible neighbouring pair (from case (iv)) which can appear in χ. If $\chi = [(k+1,k)/(1)] = [k,k]+[k+1,k-1]$, then by Proposition 3.3 the product

$$([k,k]+[k+1,k-1])\cdot[\alpha] = ([k,k]+[k+1,k-1])\cdot[k,k] = \sum_{\ell(\lambda)\leqslant 4}[\lambda]$$

is indeed multiplicity-free. Now assume the containment $[k,k]+[k+1,k-1] \subseteq \chi$ is strict; in which case χ must contain (in addition to $[k,k]+[k+1,k-1]$) one of the other possible constituents $[n-2,2]$, $[n-2,1^2]$, $[n-3,3]$ or their conjugates, or a hook.

The product of $[k,k]$ with any of $[n-2,2]$, $[n-2,1^2]$, $[n-3,3]$ has a constituent of length 4, and therefore χ cannot contain any of these. Next we want to show that χ cannot contain any of the conjugates of $[n-2,2]$, $[n-2,1^2]$, $[n-3,3]$; note that the first is a neighbour of the other two, so it cannot occur together with one of those.

First assume that $\chi = [k+1,k-1]+[k,k]+[2^2,1^{n-4}]$. Note that this implies that λ/μ has $k-1$ columns of length 2 and two of length 1, and it has two rows of length 2 and $n-4$ of length 1. But since λ/μ is the diagram of a multiplicity-free skew character, it is connected or has two components of shape as described in Theorem 2.2, up to rotation of the pieces, and this is clearly impossible (recall that $n > 12$).

Next assume that $\chi = [k+1, k-1] + [k, k] + [2^3, 1^{n-6}]$ or $\chi = [k+1, k-1] + [k, k] + [3, 1^{n-3}]$. Then, similarly as above, we obtain a contradiction.

It remains to exclude the case of an additional hook appearing in χ. As before, we may assume that χ does not contain $[n]$ or $[n-1, 1]$ (or their conjugates), or pair sums already dealt with. So assume χ has a hook constituent $[n-m, 1^m]$, with $2 < m < n-3$; if there is more than one hook constituent we consider the one with minimal m. If $n - m > k$, then the hook constituent would be maximal, and hence then λ/μ has one column of length $m+1$, and all others are of length 1. But then it is clearly impossible that χ contains $[k, k]$. On the other hand, if $n - m \leq k$, and χ also contains any of the conjugates of $[n-2, 2]$, $[n-2, 1^2]$, $[n-3, 3]$, the previous arguments give again a contradiction, and finally, the case $\chi = [k+1, k-1] + [k, k] + [n-m, 1^m]$ can be handled similarly as above.

Now as the last case for $\alpha = (k, k)$, it only remains to consider the small cases $k \in \{4, 5, 6\}$. Here, the arguments used above supplemented by computer calculations give the claim. Note that for $n = 10$, $k = 5$ we may also have the possible neighbour pair sum $\chi_0 = [6, 4] + [7, 3] = [(7, 4)/(1)]$ in χ, but $\chi_0 \cdot [5, 5]$ is not multiplicity-free. This concludes the case in which α is a rectangle.

Finally, it remains to consider the case where α is a fat hook that is not of one of the special types discussed so far. Excluding the cases considered so far, we may conclude that $n > 4$ and $|\mathrm{rem}(\alpha)| \geqslant 2$. Therefore χ must contain $\chi_0 = [n] + [n-1, 1]$ (up to conjugation), but $[\alpha] \cdot \chi_0$ is not multiplicity-free, as required. $\square$

Proposition 5.6. *Assume that Theorem 1.1 holds for a fixed $n \in \mathbb{N}$. Then no product of two proper skew characters of $\mathfrak{S}_n$ is multiplicity-free.*

Proof. Under the assumption of our proposition, we have already classified in Proposition 5.5 the multiplicity-free products of a proper skew character and an irreducible character.

Let χ be a multiplicity-free proper skew character of $\mathfrak{S}_n$ (and therefore $n > 2$). Now by Proposition 5.5, if $\alpha \vdash n$ is such that $\chi \cdot [\alpha]$ is multiplicity-free, then α is a rectangle. If β is a neighbour of α, then β is not a rectangle (as $n > 2$) and so $\chi \cdot [\beta]$ is not multiplicity-free. But every proper skew character ψ has two neighbouring constituents, by Lemma 5.1, hence $\chi \cdot \psi$ cannot be multiplicity-free. $\square$

Corollary 5.7. *If Theorem 1.1 holds for a fixed $n \in \mathbb{N}$, then Theorem 1.3 also holds for n.*

Remark 5.8. *For the remainder of the paper, we shall assume that Theorem 1.1 (and hence also Theorem 1.3) has been proven by induction for all pairs of partitions of degree strictly less than $n \in \mathbb{N}$. We refer to any pair (ρ, σ) of partitions of degree strictly less than n and satisfying $g(\rho, \sigma) > 1$ as a* seed *(for multiplicity).*

Theorem 2.7 implies that a necessary condition for $g(\lambda, \mu) = 1$ is that the pair $[\lambda/\lambda \cap \mu]$, $[\mu/\lambda \cap \mu]$ belongs to the lists in Theorems 1.1 and 1.3.

6. Products with a rectangle

In this section, we shall assume that $\mu = (a^b)$ is a partition of $n = ab$ with $a, b \geqslant 3$.

Proposition 6.1. *Let $\lambda \vdash n$. The product $[\mu] \cdot [\lambda]$ is multiplicity-free if and only if λ is one of $(n-2, 2)$, $(n-2, 1^2)$, $(n-1, 1)$, (n), or one of the special partitions $(6, 3), (5, 4), (6^2)$ (or conjugate to one of the listed partitions).*

One half of the proposition follows from Section 3. In this section, we prove the other half of this proposition via a series of lemmas.

Important standing assumption: For the remainder of this section, we assume that λ is not one of the listed partitions giving a multiplicity-free product, and we want to deduce that $[\lambda] \cdot [\mu]$ contains multiplicities. We may assume that λ is neither a hook, or 2-line partition, and that $\lambda \neq \mu$, as we have already dealt with these cases in Section 4.

There are two possible intersection diagrams for λ and μ, up to conjugation; these are given in Figure 3. As indicated in the intersection diagram, we may assume (by conjugating if necessary) that $w(\lambda) \geqslant w(\mu)$ for the remainder of this section. We will also use the notation indicated there, in other words we let $\beta = \lambda \cap \mu$, $\delta = \lambda/\beta$ (in the second case $\lambda/\beta = \delta = \delta' \cup \delta''$) and $\gamma = \mu/\beta$.

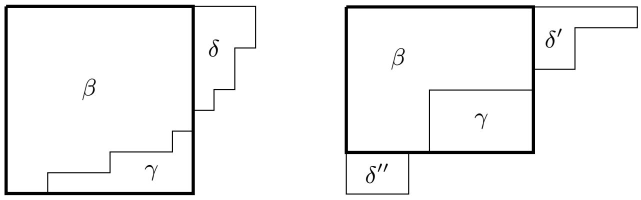

FIGURE 3. The two distinct possible intersection diagrams for a pair (λ, μ) such that μ is a rectangular partition (up to conjugation).

Lemma 6.2. *If $\lambda = (c^d)$ is a rectangular partition of $n = cd$ for $c, d \geq 2$, then $g(\lambda, \mu) > 1$.*

Proof. We may assume that $a \geqslant b$. Without loss of generality we may assume that $a > c$, and thus $b < d$ (as we have assumed $\lambda \neq \mu$). As we have already dealt with 2-line partitions, we may also assume that $c, d \geqslant 3$.

Under these assumptions, $\beta = (c^b) \supseteq (3^3)$, and $\gamma = ((a-c)^b)$ and $\delta = (c^{d-b})$ are (SG)-removable. It then follows that $1 < g(\beta, \beta) \leq g(\lambda, \mu)$ by Lemma 4.1 and Proposition 2.4. $\square$

Lemma 6.3. *If the partition γ^{rot} is (1^k) for $k \geqslant 1$, or $(2, 1^{k-2})$ for $k \geqslant 3$, and δ has one connected component, then $g(\lambda, \mu) > 1$.*

Proof. The structure of this proof (and the idea behind many future proofs) is as follows. Our assumption on γ implies that $w(\gamma) \leqslant 2$. Generically, we can proceed by removing the first $(a-2)$ columns common to both partitions λ and μ to obtain partitions $\tilde{\lambda}$ and $\tilde{\mu}$, such that $\tilde{\mu}$ is a 2-column partition and $g(\lambda, \mu) \geqslant g(\tilde{\lambda}, \tilde{\mu})$ by the semigroup property. As $\tilde{\mu}$ is a 2-line partition, we can then (in most cases) apply Proposition 4.7 to deduce that $g(\tilde{\lambda}, \tilde{\mu}) > 1$. However, if $\ell(\gamma) = b-1$, we shall see that this argument can fail because it is possible that we have reduced to a pair $(\tilde{\lambda}, \tilde{\mu})$ for which $g(\tilde{\lambda}, \tilde{\mu}) = 1$. We therefore refer to the case in which $\ell(\gamma) = b-1$ as an 'exceptional case' and provide a separate argument.

We begin with the generic case. Given γ such that $1 \leqslant \ell(\gamma) \leqslant b-2$, we may remove the first $a-2$ columns from λ and μ and hence obtain partitions $\tilde{\mu} = (2^b)$ and $\tilde{\lambda} = (2^{b-k}, 1^k) + \delta$ (respectively $\tilde{\lambda} = (2^{b-k+1}, 1^{k-2}) + \delta$) for $\gamma^{\text{rot}} = (1^k)$ (respectively $\gamma^{\text{rot}} = (2, 1^{k-2})$). The result then follows from the case for 2-line partitions.

Now assume $\ell(\gamma) = b-1$; we have that λ is equal to either $(a+b-1, (a-1)^{b-1})$ or $(a+b, (a-1)^{b-2}, a-2)$ for γ^{rot} being (1^{b-1}) or $(2, 1^{b-2})$, respectively. We first deal with the case $\gamma^{\text{rot}} = (2, 1^{b-2})$. We set $\tilde{\mu} = (2^b)$ and $\tilde{\lambda} = (3, 2^{b-2}, 1)$ and rewrite our partitions as follows

$$\mu = ((a-2)^b) + \tilde{\mu}\ ,\ \lambda = (a-3+b, (a-3)^{b-1}) + \tilde{\lambda}\ ,$$

and by Proposition 2.4, we have that $g(\mu, \lambda) \geqslant g(\tilde{\mu}, \tilde{\lambda})$. Now, by Proposition 4.7 we have that $g(\tilde{\mu}, \tilde{\lambda}) > 1$, and so the result follows.

We now deal with the case $\gamma^{\text{rot}} = (1^{b-1})$. We set $\tilde{\mu} = (3^b)$ and $\tilde{\lambda} = (b+2, 2^{b-1})$ and rewrite our partitions as follows

$$\mu = ((a-3)^b) + \tilde{\mu}\ ,\ \lambda = ((a-3)^b) + \tilde{\lambda}\ .$$

For $b = 3$ or $b = 4$, a direct computation shows $g(\tilde{\mu}, \tilde{\lambda}) > 1$. When $b \geq 5$, we have that

$$[\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu}^t)] = [2^{b-3}] \boxtimes [2]\ ,\ [\tilde{\mu}^t/(\tilde{\lambda} \cap \tilde{\mu}^t)] = [(b-2)^2]\ .$$

The product of these characters is not multiplicity-free by Proposition 4.7, and our inductive proof. Therefore, by Proposition 2.4 and Corollary 2.7, we have $g(\lambda, \mu) \geqslant g(\tilde{\lambda}, \tilde{\mu}) > 1$. $\square$

Lemma 6.4. *If γ^{rot} is (k) or $(k-1,1)$, and δ has one connected component, then $g(\lambda,\mu)>1$.*

Proof. By Lemma 6.3, we may assume that $\gamma^{\text{rot}} \neq (1)$ or $(2,1)$.

We first consider the case $\gamma^{\text{rot}} = (k)$ for some $2 \leqslant k \leqslant a$. We first deal with the exceptional cases which occur for small values of k; namely $k=2,3$, and $(\gamma,\delta) = ((4),(2^2))$ for $k=4$.

We consider the exceptional cases for $k=2$ in detail. Here δ is equal to (2) or (1^2). If we remove all rows and columns common to λ and μ, we obtain $(\tilde{\lambda},\tilde{\mu}) \in \{((4),(2^2)),((3^2),(2^3))\}$. Unfortunately, $g(\tilde{\lambda},\tilde{\mu})=1$ in these cases, and so we have gone too far. In other words, we have removed too many rows or columns. If $\delta=(2)$, then there are three ways in which we may have removed too many rows or columns,

$$((5,3,1),(3^3))\ ,\ ((4,2,2),(2^4))\ ,\ ((8,4),(6^2)).$$

However, since our original partition μ contained (3^3) (by assumption), we can choose to reduce only to $((5,3,1),(3^3))$. One can deal with $\delta=(1^2)$ in a similar fashion, and here reduce to the exceptional case $((4^2,1),(3^3))$. For all these pairs we have $g(\tilde{\lambda},\tilde{\mu})>1$ by direct computation and the result follows by Proposition 2.4.

For $k=3$, we remove almost all rows and columns common to λ and μ until we reach one of the following pairs $(\tilde{\lambda},\tilde{\mu})$:

$$((6,3^2),(3^4))\ ,\ ((7,4,1),(4^3))\ ,\ ((6,5,1),(4^3))\ ,\ ((5,4,3),(3^4))\ ,\ ((4^3),(3^4)).$$

For all these pairs we have $g(\tilde{\lambda},\tilde{\mu})>1$ by direct computation. If $(\gamma,\delta)=((4),(2^2))$, we remove most rows and columns common to λ and μ and reduce to $(\tilde{\lambda},\tilde{\mu}) = ((6^2,4),(4^4))$ or $((7^2,1),(5^3))$, which satisfy $g(\tilde{\lambda},\tilde{\mu})>1$ by direct computation.

We now assume that we are not in one of the exceptional cases outlined above and so $k \geqslant 4$ and $(\gamma,\delta) \neq ((4),(2^2))$. Remove all columns common to λ and μ to obtain partitions $\tilde{\lambda}$ and $\tilde{\mu}$. In the case $b=3$, we have that $\tilde{\lambda}$ is a 2-line partition and $\tilde{\mu}=(k^3)$ such that $(\tilde{\lambda},\tilde{\mu}) \neq ((6^2),(4^3))$. Therefore $g(\tilde{\lambda},\tilde{\mu})>1$ by Proposition 4.7. In the case $b \geqslant 4$, $\tilde{\lambda}\cap\tilde{\mu} = \tilde{\beta} = (k^{b-1})$ with $k \geqslant 4$ and $b-1 \geqslant 3$ and γ and δ are (SG)-removable; the result follows as $g(\lambda,\mu) \geqslant g(\tilde{\lambda},\tilde{\mu}) \geqslant g(\tilde{\beta},\tilde{\beta})>1$.

We now assume that $\gamma^{\text{rot}}=(k-1,1)$ with $k \geq 4$; by Remark 5.8 we can assume that δ is a fat hook. We first deal with the exceptional cases in which $k=4$ or 5. If $(\gamma^{\text{rot}},\delta)=((3,1),(4))$ then we remove all but one row or column common to both λ and μ to obtain pairs of partitions $(\tilde{\lambda},\tilde{\mu})$. For all other pairs of partitions of 4 or 5, we remove all rows and columns common to both λ and μ to obtain $(\tilde{\lambda},\tilde{\mu})$. The partitions $(\tilde{\lambda},\tilde{\mu})$ obtained in this fashion are all of degree less than or equal to 28, and so can be checked directly (one can reduce this degree even further using the semigroup property, but we do not wish to go into these arguments here).

We now assume that $\gamma^{\text{rot}}=(k-1,1)$ and $k \geqslant 6$. We remove all rows and columns common to both λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\ell(\delta) \leqslant k-4$ then $\tilde{\lambda}\cap\tilde{\mu}^t = ((k-2)^{k-3})$ and so both $\tilde{\lambda}/(\tilde{\lambda}\cap\tilde{\mu}^t)$ and $\tilde{\mu}^t/(\tilde{\lambda}\cap\tilde{\mu}^t)$ are (SG)-removable; therefore $g(\lambda,\mu) \geq g(\tilde{\lambda},\tilde{\mu}^t) \geqslant g(\tilde{\lambda}\cap\tilde{\mu}^t,\tilde{\lambda}\cap\tilde{\mu}^t)>1$.

If $\ell(\delta) \in \{k-3,k-2,k-1,k\}$ it remains to check each of the possible seven such cases. If $\delta=(2,1^{k-2}),(3,1^{k-3}),(4,1^{k-4})$ then we may remove an appropriate hook of length $k-1$ from $\tilde{\lambda}$ (namely $(1^{k-1}),(2,1^{k-3}),(3,1^{k-4})$, respectively) and the final row of length $k-1$ from $\tilde{\mu}$ to obtain a pair of partitions $\hat{\lambda},\hat{\mu}$ which differ only by adding and removing a single node; so the result follows from Lemma 6.3. If $\delta=(1^k)$ then $\tilde{\lambda}\cap\tilde{\mu}^t=(k^{k-1})$ and $\tilde{\lambda}/(\tilde{\lambda}\cap\tilde{\mu}^t)$ and $\tilde{\mu}^t/(\tilde{\lambda}\cap\tilde{\mu}^t)$ are both (SG)-removable; the result follows as $g(\lambda,\mu) \geqslant g(\tilde{\lambda}\cap\tilde{\mu}^t,\tilde{\lambda}\cap\tilde{\mu}^t)>1$ by Subsection 4.1. If $\delta=(2^2,1^{k-4})$ or $(2^3,1^{k-6})$ then $\tilde{\lambda}/(\tilde{\lambda}\cap\tilde{\mu}^t)$ and $\tilde{\mu}^t/(\tilde{\lambda}\cap\tilde{\mu}^t)$ are both linear and the result follows from Lemma 6.3. If $\delta=(3,2,1^{k-5})$ then $g(\delta,\gamma^{\text{rot}})>1$ and so we are done by Theorem 2.7. $\square$

Remark 6.5. *In the proof of Lemma 6.4, we used our assumptions on λ and μ to reduce our list of exceptional cases to the pairs $((5,3,1),(3^3))$ and $((4^2,1),(3^3))$, whereas one naively could have thought we had to check*

$$((5,3,1),(3^3))\ ,\ ((4,2^2),(2^4))\ ,\ ((8,4),(6^2))\ ,\ ((4^2,1),(3^3))\ ,\ ((4,2^2),(2^4)).$$

In future proofs, we shall use this technique (as detailed in the proof above) without going into further detail.

Lemma 6.6. *If $\delta = (1^k)$ or $(2, 1^{k-2})$ for $k \geqslant 4$, then $g(\lambda, \mu) > 1$.*

Proof. Note that by Lemmas 6.3 and 6.4, we may assume that γ is non-linear. If $\gamma = (2^2)$ and $\delta = (1^4)$ then we remove all but possibly one common row or column from λ and μ to obtain a pair of partitions $(\tilde{\lambda}, \tilde{\mu})$ equal to one of the seeds $((4^4, 1^2), (3^6))$ or $((3^4, 2), (2^7))$.

We may now assume that γ is non-linear and $(\gamma, \delta) \neq ((2^2), (1^4))$. Remove all rows and columns common to both λ and μ to obtain pairs of partitions $(\tilde{\mu}, \tilde{\lambda})$ equal to

$$((w(\gamma)^{k+\ell(\gamma)}), ((w(\gamma)+1)^k, \gamma^c)) \ , \ ((w(\gamma)^{k+\ell(\gamma)-1}), (w(\gamma)+2, (w(\gamma)+1)^{k-2}, \gamma^c))$$

for $\delta = (1^k)$ and $\delta = (2, 1^{k-2})$, respectively; here $\gamma^c = (w(\gamma)^{\ell(\gamma)}/\gamma)$ is the rectangular complement of γ.

Let $w(\gamma) = 2$ (and $\delta = (1^k), (2, 1^{k-2})$) for $k \geqslant 4$. Then $(3^3) \subseteq \tilde{\lambda}$ and $\tilde{\mu} = (2^{k+\ell(\gamma)})$. The result follows as $g(\tilde{\lambda}, \tilde{\mu}) > 1$ by Subsection 4.3.

If $w(\gamma) \geqslant 3$ and $\delta = (1^k)$ then $\tilde{\mu}^t/(\tilde{\mu}^t \cap \tilde{\lambda})$ and $\tilde{\lambda}/(\tilde{\mu}^t \cap \tilde{\lambda})$ are (SG)-removable and $\tilde{\mu}^t \cap \tilde{\lambda} = ((w(\gamma)+1)^{w(\gamma)})$. The result follows as $g(\tilde{\mu}^t \cap \tilde{\lambda}, \tilde{\mu}^t \cap \tilde{\lambda}) > 1$ by Subsection 4.1.

If $\delta = (2, 1^{k-2})$, and γ is a rectangle such that $w(\gamma) \geqslant 3$, then the partitions γ and δ are (SG)-removable and $g(\tilde{\beta}, \tilde{\beta}) > 1$ by Lemma 6.2. By Lemma 6.4 and the above, we can now assume that γ^{rot} is a hook not equal to (k) or $(k-1, 1)$. If $\gamma^{\mathrm{rot}} \neq (k-2, 2)$, then $\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu}^t)$ is a proper fat hook and $[\tilde{\mu}^t/(\tilde{\lambda} \cap \tilde{\mu}^t)]$ is not the natural character; therefore $g(\tilde{\lambda}, \tilde{\mu}) > 1$ by Remark 5.8. Finally, if $\gamma^{\mathrm{rot}} = (k-2, 2)$ then

$$\begin{aligned}(\tilde{\mu}, \tilde{\lambda}) &= (((k-2)^{k+1}), (k, (k-1)^{k-2}, k-4)) \\ &= (((k-3)^{k+1}), ((k-2)^{k-2}, k-5)) + ((1^{k+1}), (2, 1^{k-1})).\end{aligned}$$

We have that $g((k-3)^{k+1}), ((k-2)^{k-2}, k-5)) > 1$ by Lemma 6.4. The result follows by Proposition 2.4. $\square$

Lemma 6.7. *If $\delta = (k)$ then $g(\lambda, \mu) > 1$.*

Proof. We first consider the case where γ is a rectangle. If $\gamma = (2^2)$, then we remove almost all rows and columns common to λ and μ to obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to one of the seeds $((6, 2^2), (2^5))$ $((7, 3, 1^2), (3^4))$ $((8, 2^2), (4^3))$. If $\gamma = (2^k)$ for $k \geqslant 3$ then we remove almost all common rows and columns of λ and μ to obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to either of

$$((2k+3, 1^k), (3^{k+1})) \ , \ ((2k+2, 2^2), (2^{k+3})) \ .$$

In the former case, the result follows by Subsection 4.2 as $\tilde{\lambda}$ is a hook. In the latter case the result follows from Subsection 4.3 as $\tilde{\mu}$ is a 2-line partition.

Now assume $\gamma = (k, k)$ for $k \geqslant 3$. We remove almost all common rows and columns to obtain pairs of partitions $(\tilde{\lambda}, \tilde{\mu})$ equal to either of

$$((3k, k), (k^4)) \ . \ ((3k+2, 2^2), ((k+2)^3)) \ .$$

In the former case, $\tilde{\lambda}$ is a 2-line partition and the result follows. In the latter case, remove the final row of $\tilde{\mu}$ and the partition $(k+2)$ from the first row of $\tilde{\lambda}$ to obtain partitions $\hat{\mu} = (k+2, k+2)$ and $\hat{\lambda} = (2k, 2, 2)$. The result again follows from Subsection 4.3.

We now consider the case that $\gamma = (t^u)$ is a fat rectangle for $t, u \geqslant 3$. We may proceed as above by removing all but one common row or column to obtain pairs of partitions $(\tilde{\lambda}, \tilde{\mu})$ equal to either of

$$((tu+t, t), (t^{u+2})) \ , \ ((tu+t+1, 1^u), ((t+1)^{u+1})) \ ,$$

respectively. In the former (respectively latter) case the result follows from Subsection 4.3 (respectively Subsection 4.2).

We now assume that $\gamma^{\mathrm{rot}} = (t^u, v^w)$ is a non-rectangular fat hook, in other words $t \neq v$ and $u, w \neq 0$. We first consider the case where $\ell(\gamma) < b-1$ or $w(\gamma) < a$. By assumption, $\beta = \lambda \cap \mu$ has at least two removable nodes A_1 and A_2 such that A_i and δ are disconnected for $i = 1, 2$. We may assume that $\gamma \cup \{A_1\}$ is not a rectangular partition.

We want to apply Lemma 2.9 and recall the definition of the virtual character χ given there in equation (2.1); note that here $\alpha = \gamma^{\mathrm{rot}}$.

$$\chi = \sum_{A \in \mathrm{rem}(\beta)} [\lambda/\beta_A] \cdot [\mu/\beta_A] - \sum_{B \in \mathrm{add}(\alpha)} \alpha^B \,. \tag{6.1}$$

For the two terms on the right-hand side, we note that the subtracted term is multiplicity-free. By assumption $[\lambda/\beta_{A_i}] = [k+1]+[k,1]$ for $i = 1, 2$. Also note that $[\gamma \cup \{A_1\}] = [\alpha^{A_1}]$. Therefore, we have that

$$\begin{aligned}
\langle \chi, [\alpha^{A_1}] \rangle &\geqslant \sum_{i=1,2} \langle [\mu/\beta_{A_i}] \cdot [\lambda/\beta_{A_i}], [\alpha^{A_1}] \rangle - 1 \\
&\geqslant \langle [\mu/\beta_{A_1}] \cdot ([k+1] + [k,1]), [\alpha^{A_1}] \rangle + \langle [\mu/\beta_{A_2}] \cdot ([k+1] + [k,1]), [\alpha^{A_1}] \rangle - 1 \\
&\geqslant 2 + 1 - 1
\end{aligned}$$

and the result follows by Lemma 2.9.

We now consider the case in which $\ell(\gamma) = b - 1$ and $w(\gamma) = a$ and so $t \geqslant 3, u + w \geqslant 2$. If $w = 1$, then the result follows as λ is a 2-line partition. If $w > 1$, we remove the final u rows from μ and (tu) from the first row of λ to obtain $\tilde{\mu} = (a^{1+w})$ (and so has at least three lines) and $\tilde{\lambda}$ a partition which is neither a hook nor a 2-line partition. In this case, $\tilde{\gamma} = \tilde{\mu}/(\tilde{\mu} \cap \tilde{\lambda})$ is a rectangle, therefore the result follows from the above and Proposition 2.4.

Finally, we consider the case in which γ^{rot} is not a fat hook, i.e., $|\mathrm{rem}(\gamma^{\mathrm{rot}})| > 2$. Then we apply the following iterative procedure to reduce to the situation dealt with before.

(1) If $w(\gamma) \neq w(\mu)$, and $|\mathrm{rem}(\gamma^{\mathrm{rot}})| > 2$, then we remove all columns common to both λ and μ to obtain a pair $(\tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\mu}/(\tilde{\lambda} \cap \tilde{\mu}) = \gamma$ and therefore $w(\tilde{\mu}) \geqslant 3$, $\ell(\tilde{\mu}) \geqslant 4$.
(2) If $w(\gamma) = w(\mu)$, and $|\mathrm{rem}(\gamma^{\mathrm{rot}})| > 2$, then we remove the final $\ell(\mu) - \ell(\lambda)$ rows from μ and the corresponding number of nodes from λ_1 to obtain a pair $(\tilde{\lambda}, \tilde{\mu})$ such that $|\mathrm{rem}((\tilde{\mu}/\tilde{\lambda} \cap \tilde{\mu})^{\mathrm{rot}})| = |\mathrm{rem}(\gamma^{\mathrm{rot}})| - 1$ and $w(\tilde{\mu}/(\tilde{\lambda} \cap \tilde{\mu})) < w(\tilde{\mu})$ and $w(\tilde{\mu}) \geqslant 3$, $\ell(\tilde{\mu}) \geqslant 3$.
(3) Having completed (1) or (2) above, relabel the partitions $(\lambda, \mu) := (\tilde{\lambda}, \tilde{\mu})$ and apply (1) or (2) again, if possible.

The above procedure eventually terminates by producing a pair of partitions (λ, μ) such that $w(\mu) \geqslant 3$, $\ell(\mu) \geqslant 3$, $|\mathrm{rem}(\gamma)| = 2$; therefore the result follows by the semigroup property and the case for fat hooks, covered above. □

Lemma 6.8. *If $\delta = (k-1, 1)$, then $g(\lambda, \mu) > 1$.*

Proof. By Remark 5.8, we may assume that γ^{rot} is a fat hook. If $\gamma = (2^k)$ and $\delta = (2k-1, 1)$, then we remove all but one row or column of λ and μ to obtain partitions $\tilde{\lambda}$ and $\tilde{\mu}$ such that $\tilde{\lambda} \cap \tilde{\mu} = (2^3)$ or $(3^2, 1^k)$ respectively. If $\gamma \neq (2^k)$, then remove all rows and columns common to both λ and μ to obtain partitions $\tilde{\lambda}$ and $\tilde{\mu} = (w(\gamma)^{2+\ell(\gamma)})$.

In either case, we now remove the final row of $\tilde{\mu}$ to obtain $\hat{\mu}$ and we let $\hat{\lambda}$ denote the partition such that $\hat{\lambda} + (w(\hat{\mu}) - 1, 1) = \tilde{\lambda}$. The partition $\hat{\mu}$ is a rectangle and $\hat{\lambda}$ is either a proper fat hook or $|\mathrm{rem}(\hat{\lambda})| = 3$ and such that $\hat{\lambda}/(\hat{\lambda} \cap \hat{\mu}) = (k - w(\tilde{\mu}))$. The result follows from Lemma 6.7. □

Remark 6.9. *For the remainder of this section, we shall assume that $[\delta]$ is not equal to a linear character or the natural character or its conjugate. Similarly if δ has one connected component, then we shall assume that $[\gamma]$ is not equal to a linear character or the natural character or its conjugate.*

Lemma 6.10. *If γ^{rot} and δ are both 2-line partitions, then $g(\lambda, \mu) > 1$.*

Proof. We first suppose that $w(\gamma) = \ell(\delta) = 2$. There are three cases to consider: (i) $\gamma = (2^k)$; (ii) $\gamma \neq (2^k)$ and $\delta = (k^2)$; (iii) $\gamma \neq (2^k)$ and $\delta \neq (k^2)$.

Case (i). Remove all common rows and all but one common column of λ and μ to obtain $(\tilde{\lambda}, \tilde{\mu}) = ((3^2, 1^k) + \delta, (3^{k+2}))$. For $k = 2$ and $k = 3$ it is easily checked that the corresponding pairs are seeds. When $k > 3$, we note that at least one of $(3^2), (4, 2), (5, 1)$ is (SG)-removable from δ (and hence is also (SG)-removable from the first two rows of $\tilde{\lambda}$). In this case, we remove the final two rows of $\tilde{\mu}$ and the relevant partition from $\tilde{\lambda}$ to obtain $(\hat{\mu}, \hat{\lambda})$ such that $\hat{\mu}/(\hat{\lambda} \cap \hat{\mu})$ is a non-linear rectangle and $\hat{\lambda}/(\hat{\lambda} \cap \hat{\mu})$ is a proper skew partition not of one of the forms described in cases (2) and (3) in Theorem 1.3. The result then follows by Remark 5.8.

In case (ii) (respectively (iii)) we remove all rows and columns common to λ and μ to obtain $\tilde{\mu}$ a 2-column rectangle and $\tilde{\lambda}$ a proper fat hook (respectively $\tilde{\lambda}$ such that $|\mathrm{rem}(\tilde{\lambda})| = 3$). The result then follows from Subsection 4.3.

For the remainder of the proof we assume that at least one of $w(\gamma)$ and $\ell(\delta)$ is greater than 2. In the generic case, we remove all common rows and columns from λ and μ to obtain partitions $\tilde{\lambda}$ and $\tilde{\mu}$ and proceed case-by-case. We will deal with the exceptional cases when they appear in that discussion.

If $\gamma = (k^2)$ and $\ell(\delta) = 2$ (respectively $\gamma = (2^k)$ and $\ell(\delta) > 2$) then $\tilde{\lambda}$ (respectively $\tilde{\mu}$) is a 2-line partition and the result follows from Subsection 4.3 as long as we are not in the case $\gamma = (3^2) = \delta$. In the exceptional cases we remove all but one common row or column from λ and μ to obtain $(\tilde{\lambda}, \tilde{\mu})$. For $\gamma = (k^2)$, we have in the exceptional case $(\tilde{\lambda}, \tilde{\mu}) = ((6^2, 3), (3^5)) = ((4^2, 2) + (2^2, 1)), ((2^5) + (1^5)))$ or $((7^2, 1^2), (4^4)) = ((3^2, 1^2) + (4^2), (2^4) + (2^4))$, respectively. Hence we can reduce to $(\hat{\lambda}, \hat{\tilde{\mu}}) = ((4^2, 2), ((2^5))$ or $((3^2, 1^2), (2^4))$, respectively, and $g(\hat{\lambda}, \hat{\tilde{\mu}}) > 1$ by Subsection 4.3. In the exceptional case for $\gamma = (2^k)$ we quickly reduce to a pair involving a 2-column partition where we can again appeal to Subsection 4.3.

If $\gamma = (k^2)$ and $\ell(\delta) > 2$, then δ and γ are (SG)-removable and $(3^3) \subseteq \tilde{\lambda} \cap \tilde{\mu}$ and the result follows from Subsection 4.1.

We may now assume $\gamma \neq (k^2)$ up to conjugation. If $w(\gamma) = 2$, then $\tilde{\mu}$ is a 2-line partition and $\tilde{\lambda}$ is a proper fat hook or $|\mathrm{rem}(\tilde{\lambda})| = 3$. Now assume $\ell(\gamma) = 2$ and $\ell(\delta) = 2$. If $\gamma^{\mathrm{rot}} = (k, k-1)$, remove the two lower rows from $\tilde{\mu}$ to obtain $\hat{\mu} = (k^2)$, and note that $\tilde{\lambda} = (k^2) + \hat{\lambda}$ where $\hat{\lambda}$ is a partition with $|\,\mathrm{rem}(\hat{\lambda})| = 3$; hence the result follows by Subsection 4.3. If $\gamma^{\mathrm{rot}} = (k, k-j)$ for $j > 1$, we have $\tilde{\lambda} = (3^2, 2) + \hat{\lambda}$ for some partition $\hat{\lambda}$, and $\tilde{\mu} = (2^4) + ((k-2)^4)$, so with $g(\lambda, \mu) \geq g(\tilde{\lambda}, \tilde{\mu}) \geq g((3^2, 2), (2^4))$ the claim follows.

It remains to check the cases in which $\ell(\gamma) = 2$ and $\ell(\delta) > 2$; namely (i) $\gamma^{\mathrm{rot}} = (2k-2, 2)$ and $\delta = (2^k)$; (ii) $\gamma^{\mathrm{rot}} = (2k-3, 3)$ and $\delta = (2^k)$; (iii) $\gamma^{\mathrm{rot}} = (k+1, k-1)$ and $\delta = (2^{k-1}, 1^2)$; (iv) $\gamma^{\mathrm{rot}} = (k+1, k-1)$ and $\delta = (2^k)$; (v) $\gamma^{\mathrm{rot}} = (k+1, k)$ and $\delta = (2^k, 1)$.

In case (i), for $k \geqslant 6$ (one can check the seeds for $k = 3, 4, 5$ directly) we have that $\tilde{\lambda}^t/(\tilde{\lambda}^t \cap \tilde{\mu})$ and $\tilde{\mu}/(\tilde{\lambda}^t \cap \tilde{\mu})$ are (SG)-removable and the rectangle $\tilde{\lambda}^t \cap \tilde{\mu}$ contains (3^3), so the result follows from Subsection 4.1. Case (ii) is similar. In cases (iii) to (v), we have that $\tilde{\lambda}^t/(\tilde{\lambda}^t \cap \tilde{\mu})$ and $\tilde{\mu}/(\tilde{\lambda}^t \cap \tilde{\mu})$ are linear partitions and the result follows from Lemma 6.3. □

Lemma 6.11. *If γ or δ is a fat rectangle and δ has one connected component, then $g(\lambda, \mu) > 1$.*

Proof. By Remark 5.8 and Remark 6.9 we may assume that one of δ and γ is a fat rectangle and the other is $(k-2, 2)$, or $(k-2, 1^2)$ or $(5, 4)$ or $(6, 3)$ up to conjugation.

We first suppose that γ is a fat rectangle. Remove all rows and columns common to both partitions λ, μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\ell(\delta) \geqslant 3$, then the result holds as γ and δ are (SG)-removable and $(3^3) \subseteq \tilde{\lambda} \cap \tilde{\mu}$. In the remaining cases, $\delta = (k-2, 2)$, $(5, 4)$ or $(6, 3)$, the partition $\tilde{\lambda}$ is a 2-line partition and $\tilde{\mu}$ is a fat rectangle; the result follows by Subsection 4.3.

We now suppose δ is a fat rectangle. Remove all rows and columns common to both partitions λ, μ to obtain $\tilde{\lambda}, \tilde{\mu}$. For $\gamma^{\mathrm{rot}} = (5, 4)$ or $(6, 3)$, this follows via (SG)-removability from $g((5^3), (3^5)) > 1$, the conjugate case is immediate from Subsection 4.3.

If $\gamma^{\mathrm{rot}} = (k-2, 2), (k-2, 1^2)$ or $(3, 1^{k-3})$ then $\tilde{\lambda}^t \cap \tilde{\mu}$ is a fat rectangle and $\tilde{\lambda}^t/\tilde{\lambda}^t \cap \tilde{\mu}$ and $\tilde{\mu}^t/\tilde{\lambda}^t \cap \tilde{\mu}$ are (SG)-removable; the result follows by Subsection 4.1. If $\gamma^{\mathrm{rot}} = (2^2, 1^{k-4})$, then $\tilde{\mu}$ is a 2-line partition and $\tilde{\lambda}$ is a proper fat hook; the result follows by Subsection 4.3. □

Lemma 6.12. *If one of γ^{rot} or δ is a hook and the other is equal to (k^2) or (2^k), then $g(\lambda, \mu) > 1$.*

Proof. First note that our assumptions in Remark 6.9 imply that $k > 2$. If δ is a hook, remove all columns common to λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\gamma = (2^k)$ then the result follows from the result for 2-line partitions. If $\gamma = (k^2)$, then $\tilde{\lambda}/\tilde{\lambda} \cap \tilde{\mu}$ and $\tilde{\mu}/\tilde{\lambda} \cap \tilde{\mu}$ are (SG)-removable and $\tilde{\lambda} \cap \tilde{\mu}$ is a fat rectangle; the result follows by Lemma 6.2.

Now assume that γ^{rot} is a hook and $\delta = (2^k)$ or (k^2). Remove all rows and columns common to both λ, μ to obtain partitions $\tilde{\mu} = (t^u)$ and $\tilde{\lambda} = ((t+2)^k, (t-1)^{u-k-1})$ or $((t+k)^2, (t-1)^{u-3})$ respectively; in these cases, $t + u = 3k + 1$ and $t + u = 2k + 3$, respectively.

In the case $t = u$, i.e., $\tilde{\mu} = (t^t)$ is a square, we must have $\delta = (2^k)$. Let $\gamma^{\mathrm{rot}} = (2k - m, 1^m)$, where we have $2 \leq m \leq 2k - 3$. Since $t = k + m + 1$ and $t + m = 2k$, we obtain $t = 3m + 2$. Hence the final $m + 1$ rows of $\tilde{\mu} = (t^t)$ form a partition of size $3m^2 + 5m + 2$; removing this gives $\hat{\mu} = ((3m + 2)^{2m+1})$. On the other hand, we can remove a partition of the corresponding size from $\tilde{\lambda}$, as $\tilde{\lambda} = ((m + 2)^{2m+1}, m^m) + \hat{\lambda}$, with $\hat{\lambda} = ((2m + 2)^{2m+1}, (2m + 1)^m)$. Thus $\hat{\lambda}$ and $\hat{\mu}$ are (SG)-removable; since $\hat{\lambda} \cap \hat{\mu} = ((2m + 2)^{2m+1})$ and $g(\hat{\lambda} \cap \hat{\mu}, \hat{\lambda} \cap \hat{\mu}) > 1$, we are done in this case.

Hence we may now assume that $t \neq u$. We conjugate the partition $\tilde{\mu}$ and consider the possible intersection diagrams $D_1 = \tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu}^t)$ and $D_2 = \tilde{\mu}^t/(\tilde{\lambda} \cap \tilde{\mu}^t)$.

By our assumptions we have $k > 2$, $t \geq 3$, $u \geq 5$. Thus if D_1 is disconnected, both components are of size strictly greater than 1. When $\delta = (2^k)$, D_1 could be disconnected only when both $u < t + 2$ and $t < u - 1$, which is impossible. When $\delta = (k^2)$, D_1 is disconnected if and only if $t + 1 < u < t + k$. Then D_2 is a rectangle of width $u - t + 1 \neq 1$ and height $t - 2$. If $t = 3$, then $u = u + t - 3 = 2k$, hence $k < t = 3$, contradiction. Hence D_2 is a non-linear rectangle. But then the character pair $([D_1], [D_2])$ is not on the list in Theorem 1.3; hence by induction $g(\tilde{\lambda}, \tilde{\mu}^t) > 1$, and thus $g(\lambda, \mu) > 1$.

So we now assume that D_1, D_2 are both connected; in fact, then D_1 must be a rectangle and D_2 is a fat hook. If D_2 is a proper fat hook or D_1 a fat rectangle, or if one is a rectangle and the other is not a hook, we are done by the previous results of this section. It remains to consider the case where D_1 is a 2-line rectangle of size $2r > 4$ and D_2 is a hook of the form $(2r - m, 1^m)$ for $2 \leqslant m \leqslant 2r - 3$. When $\delta = (k^2)$, this implies that $t = 3$ and $2k = u = t + k + 1$, hence $k = 4$, $u = 8$. When $\delta = (2^k)$, this implies that $t = k + 1$ and $t + 3 = u = 2k$, so again $k = 4$, $u = 8$. In both cases, we can remove a column of length 8 from $\tilde{\mu}$ and remove $\delta = (4^2)$ from $\tilde{\lambda}$, and the result then follows from Lemma 6.4. ☐

Lemma 6.13. *If δ has two connected components, then $g(\lambda, \mu) > 1$.*

Proof. By Remark 5.8, it suffices to consider (up to conjugation of λ and μ) the cases (i) $\delta'' = (l)$ or (1^l) and $\delta' = (1)$ and $\gamma \vdash l + 1$ is a rectangle; (ii) $\gamma = (k + l)$ and $[\delta] = [\delta'] \boxtimes [\delta'']$ is one of the products from the list in Theorem 2.2 with δ', δ'' of size k, l, respectively, and (δ', δ'') not a pair as in (i). We cover both cases uniformly.

The unique exceptional subcase is $\gamma = (k + l)$ and $\delta' = (k)$, $\delta'' = (1^l)$ (up to conjugation of λ and μ) in which case we remove all rows and columns common to both partitions with the exception of one row (which exists by our assumption that μ is not a 2-line partition) to obtain $(\tilde{\lambda}, \tilde{\mu})$ of the form

$$((2k + l + 1, k + l + 1, 1^{l+1}), ((k + l + 1)^3)).$$

Now suppose that $\gamma = (k + l)$ and $\delta' \vdash k$ and $\delta'' \vdash l$ are not of the above form. Remove all rows and columns common to λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$.

Now, if $\delta' = (1)$ and $\delta'' = (1^l)$ then $\tilde{\lambda}/(\tilde{\mu}^t \cap \tilde{\lambda})$ is disconnected, with two components $(l, l - 1)$ and (1), and $\tilde{\mu}^t/(\tilde{\mu}^t \cap \tilde{\lambda}) = (2^l)$; for $l > 1$ the result follows as this product is not on the list in Theorem 1.3, and for $l = 1$, the pair $((4, 3, 1^2), (3^3))$ is a seed. If $k > 1$ (in either the exceptional or generic cases) then $\tilde{\mu}$ is a rectangle and

$$w(\tilde{\mu}) = k + l + 1 = |\delta'| + |\delta''| + 1 \geqslant \ell(\delta') + \ell(\delta'') + 1 = \ell(\tilde{\lambda})$$

and therefore $\tilde{\mu}^t/(\tilde{\lambda} \cap \tilde{\mu}^t)$ and $\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu}^t)$ each have precisely one component. The result follows by the earlier results in this section.

We now suppose that $\delta' = (1)$ and $\delta'' = (l)$ or (1^l) for some $l \geqslant 3$ and that $\gamma = (t^u)$ for $t, u > 1$. Remove all rows and columns common to λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\delta'' = (1^l)$, then $\tilde{\lambda} \supset (4, 1^3)$ is a hook partition and $\tilde{\mu}$ is a fat rectangle and the result holds by Subsection 4.2. If $\delta'' = (l)$, then we remove the first row $\tilde{\lambda}_1 = (t + tu)$ of $\tilde{\lambda}$ and the final u columns of $\tilde{\mu}$ to obtain the pair $\hat{\lambda} = \hat{\mu} = ((u+2)^l) \supseteq (3^3)$ and the result follows by Subsection 4.1. □

7. Products with a proper fat hook

In this section, we shall consider tensor products in which one of the labelling partitions is a *proper* fat hook (in other words, a fat hook which is not a 2-line, hook, or rectangular partition). We assume throughout this section that $\mu = (a^b, c^d)$ is a proper fat hook partition.

Proposition 7.1. *Let $\lambda \vdash n$. The product $[\mu] \cdot [\lambda]$ is multiplicity-free if and only if $\lambda = (n)$ or $(n-1, 1)$ up to conjugation.*

One half of the proposition follows from Section 3. In this section, we prove the other half of this proposition via a series of lemmas. For the remainder of this section, we assume that $\lambda \neq (n)$ or $(n-1, 1)$ up to conjugation and we want to deduce that $[\lambda] \cdot [\mu]$ contains multiplicities. We may assume that λ is neither a rectangle, a hook, or 2-line partition, and that $\lambda \neq \mu$, as we have already dealt with these cases in Sections 4 and 6.

The possible intersection diagrams for λ and μ, up to conjugation, are given in Figures 4 and 5. We will also use the notation indicated there, in other words we let $\beta = \lambda \cap \mu$, $\delta = \lambda / \beta$ and $\gamma = \mu / \beta$. Informally, we refer to the overlapping rectangles of shape (a^b) and of shape (c^{b+d}) as the arm and the leg of μ, respectively.

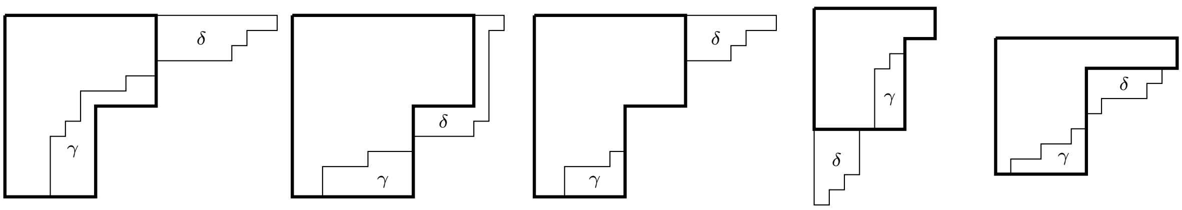

Figure 4. The possible intersection diagrams for which μ is a proper fat hook and γ and δ each have one connected component. We label these diagrams by (1a), (1b), (1c), (1d) and (1e) respectively.

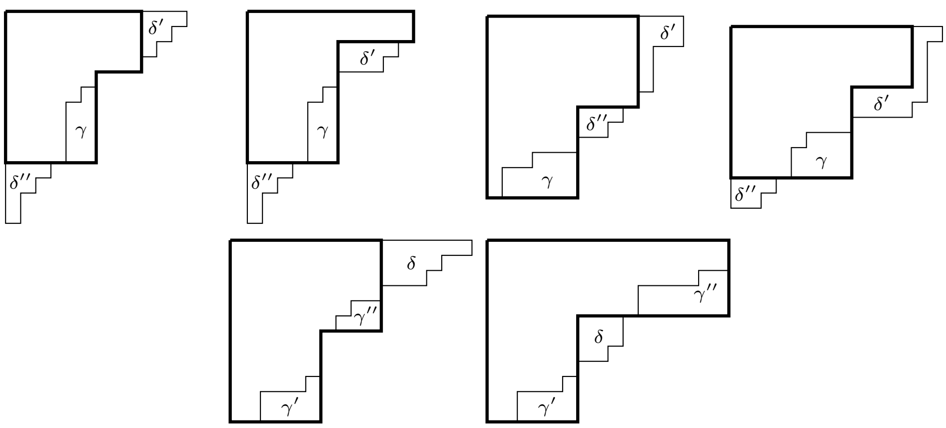

Figure 5. The possible intersection diagrams for which μ is a proper fat hook and one of γ and δ has two connected components. We label these diagrams by (2a), (2b), (2c), (2d) (2e) and (2f) respectively.

Lemma 7.2. *If γ and δ are both rectangles, then $g(\lambda, \mu) > 1$.*

Proof. We invite the reader to check the cases where the size of the partitions γ and δ is at most 2 by hand. These can easily be reduced to small cases (however, listing them is a somewhat tedious exercise). One can easily show that these products contain multiplicities using simplifications of the arguments used here (or these can be checked by computer as the degrees of the partitions are small). We shall assume throughout that γ, δ are of size strictly greater than 2. Assuming γ and δ are both rectangles, cases $(1a)$ and $(1b)$ are empty.

We first consider case $(1c)$. If $\delta = (k)$ and and $\gamma = (k)$ (and by assumption $a - c \geqslant 1$ and either b or d is strictly greater that 1) we remove all but one column in the arm and all but one row common to both partitions and hence arrive at $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the following subcases

$$((2k+1,k),(k+1,k^2))\ ,\ ((2k+1,k+1),((k+1)^2,k)).$$

If $\delta = (k)$ and $\gamma = (1^k)$ (and by assumption $a - c \geqslant 1$ and either b or d is strictly greater than 1) we remove most rows and columns in order to arrive at

$$((3+k,1^k),(3,2^k))\ ,\ ((k+2,2),(2^2,1^k)).$$

In all four of the above subcases, we have that $g(\lambda,\mu) \geqslant g(\tilde{\lambda},\tilde{\mu}) > 1$ by Subsections 4.2 and 4.3.

Now assume that $\delta = (k^2)$ and $\gamma = (1^{2k})$, with $k > 1$. By Remark 6.5 and our assumption that $a > 2$, we can remove all rows and all but two of the columns common to both partitions until we obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to one of the following subcases

$$((3^2,1^{2k}),((k+3)^2))\ ,\ ((3^2,2^{2k}),((k+3)^2,1^{2k}))$$

the latter case follows by Subsection 4.3. In the former case, remove the final two rows of $\tilde{\lambda}$ and the final column of $\tilde{\mu}$ to obtain $(((k+3)^2,1^{2k-2}),(2^{2k+2}))$; the result follows by Subsection 4.3. By Remark 6.5, if $\gamma = \delta = (1^3)$ we can reduce using the semigroup property to the seed $((3^3),(3^3))$.

We now consider the generic case (not of the above form) for $(1c)$. Remove all rows and all columns common to both λ and μ with the exception of one column from the arm. We hence obtain $\tilde{\mu} \cap \tilde{\lambda} = ((w(\gamma)+1)^{\ell(\delta)})$. If $w(\gamma) = 1$ and $\ell(\delta) \geqslant 3$ then the result follows from Subsection 4.3 (the $\ell(\delta) \leqslant 2$ case was covered above). If $w(\gamma) > 1$ and $\ell(\delta) = 2$ then the result follows from Subsection 4.3. If $w(\gamma) > 1$ and $\ell(\delta) > 2$ then γ and δ are (SG)-removable and $(3^3) \subseteq \tilde{\mu} \cap \tilde{\lambda}$ and the result follows by Subsection 4.1.

We now consider case $(1d)$; there are two subcases. If $\delta \neq (1^k)$, then we remove all common columns from the arms of μ and λ until we obtain the partitions $\tilde{\mu} = (c^{b+d})$ and $\tilde{\lambda}$ a proper fat hook. If $\delta = (1^k)$, then we remove common columns and rows until we obtain $\tilde{\mu} = (w(\gamma)+2,(w(\gamma)+1)^{\ell(\gamma)})$ and $\tilde{\lambda} = (w(\gamma)+2,1^{k+\ell(\gamma)})$. The result follows by Subsections 4.2 and 4.3.

We now consider case $(1e)$. By Remark 6.5, if $w(\gamma) = \ell(\delta) = 1$ we can remove successive rows and columns from μ and λ until we obtain $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the following pairs:

$$(((k+1)^2,1^{k+1}),((k+1)^3))\ .\ ((k+2,2^{k+1}),((k+2)^2,1^k)).$$

The first (respectively second) case follows by Section 6 (respectively Subsection 4.1). By Remark 6.5, if $\gamma = (1^k)$ and $\delta = (1^k)$ we can remove all but one row in the arm and all but one common column to obtain $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the pairs:

$$((3,1^{2k}),(3,2^k))\ ,\ ((3,2^{2k}),(3^{k+1},1^k)).$$

The latter case follows by Subsection 4.3. The former can be further reduced to the seed $((3^2,1^2),(3^2,2))$. By Remark 6.5, if $\gamma = (1^{2k})$ and $\delta = (2^k)$ we can successively remove common rows and columns until we obtain $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the following pairs:

$$((3^2,1^{3k}),(3^{k+2}))\ ,\ ((2^{3k}),(4^k,1^{2k}))$$

and the result follows by Section 6 and Subsection 4.3 respectively. We now consider the generic case for $(1e)$. Remove all rows and columns common to both μ and λ with the exception of one row in the arm to obtain $\tilde{\mu} = ((w(\delta)+w(\gamma),(w(\gamma))^{\ell(\gamma)+\ell(\delta)})$ a proper fat hook and $\tilde{\lambda} = ((w(\delta)+w(\gamma))^{\ell(\delta)+1})$ a non-linear rectangle. The result follows by Section 6. ☐

Lemma 7.3. *If either γ or δ is linear and the other is connected, then $g(\lambda,\mu) > 1$.*

Proof. We may assume that one diagram is linear and the other is not a rectangle, as the case of two rectangles has already been addressed in Lemma 7.2.

We first consider cases $(1c, d, e)$ with γ a linear partition.

Suppose we are in case $(1c)$ with $\gamma = (k)$. If $\ell(\delta) = 2$, remove all rows and columns common to λ and μ with the exception of one column in the arm to obtain $\tilde{\mu}$ a proper fat hook and $\tilde{\lambda} \supset (4^2)$ a 2-line partition. Hence the result follows by Subsection 4.3. If $\ell(\delta) > 2$ remove all rows and columns common to both μ and λ to obtain $(\tilde{\mu}, \tilde{\lambda})$ such that $(4^4) \subseteq \tilde{\mu}$ a rectangle and $(4^3) \subset \tilde{\lambda}$. The result follows by Section 6. Now consider case $(1c)$ with $\gamma = (1^k)$. Remove all rows and columns shared by μ and λ with the exception of one column in the arm to obtain $\tilde{\mu}$ and $\tilde{\lambda}$ such that $\tilde{\mu} \supseteq (2^2, 1^3)$ is a 2-line partition and $\tilde{\lambda} \supset (3^2)$ is a non-rectangular partition. The result follows by Subsection 4.3.

For case $(1d)$ with γ linear, we remove all rows and columns common to both λ and μ with the exception of one row from the arm to obtain $\tilde{\mu}$ a non-linear rectangle and $\tilde{\lambda}$ such that $|\mathrm{rem}(\tilde{\lambda})| \geqslant 3$. The result follows by Section 6.

In case $(1e)$ and $\gamma = (k)$ with $k > 3$, remove all rows and columns common to both μ and λ to obtain $\tilde{\mu} \supseteq (4^3)$ a rectangle and $\tilde{\lambda} \supset (3^2)$ a non-rectangular partition; for $k = 3$ we reduce to the seed $((5^2, 4), (5, 3^3))$. In case $(1e)$ and $\gamma = (1^k)$, remove all rows and columns common to both λ and μ with the exception of one row in the arm to obtain $(\tilde{\mu}, \tilde{\lambda})$. We have that $\tilde{\mu} \supseteq (3, 1^5)$ is a hook partition and $\tilde{\lambda} \supseteq (3^2, 2)$; the result follows by Subsection 4.2.

We now consider cases $(1c, d, e)$ for δ a linear partition. Recall that $\gamma^c = ((w(\gamma))^{\ell(\gamma)})/\gamma$.

Assume we are in case $(1c, e)$ with $|\gamma^c| > 2$. If $\delta = (k)$ for case $(1c, e)$ and $w(\gamma) > 2$, remove all rows and columns common to λ and μ to obtain $\tilde{\mu}$ a fat rectangle and $\tilde{\lambda}$ such that $\tilde{\lambda}_1 \geq w(\gamma) + 4$ and $\tilde{\lambda}$ is of depth at least 3. Therefore the result follows by Remark 5.8 for cases $(1c, e)$ with $\delta = (k)$, $w(\gamma) > 2$, and $|\gamma^c| > 2$.

Continuing with case $(1e)$ with $\delta = (k)$, we now assume that either $w(\gamma) = 2$ or $|\gamma^c| \leqslant 2$. In either case, remove all rows and columns common to both λ and μ with the exception of one row in the arm to obtain a pair of proper fat hooks of the form

$$(\tilde{\lambda}, \tilde{\mu}) = \Big(\Big((w(\gamma) + |\gamma|)^2, \gamma^c\Big), \Big(w(\gamma) + |\gamma|, w(\gamma)^{\ell(\gamma)+1}\Big)\Big).$$

We have that $|\gamma^c| < |\gamma|$ by our assumption that $w(\gamma) = 2$ or $|\gamma^c| \leqslant 2$. By the semigroup property, we can reduce to $(\tilde{\lambda}, \tilde{\mu}) = (((w(\gamma) + |\gamma|)^2), (w(\gamma) + |\gamma| - |\gamma^c|, w(\gamma)^{\ell(\gamma)+1}))$ and the result follows by Subsection 4.3.

Continuing with case $(1c)$ with $\delta = (k)$, we now assume that either $w(\gamma) = 2$ or $|\gamma^c| \leqslant 2$. Remove all rows and columns common to λ and μ with the exception of either one arbitrary row, or one column in the leg. We hence obtain $\tilde{\mu}$ a rectangle and $\tilde{\lambda}$ with at least three removable nodes; the result follows by Section 6.

We now consider cases $(1c, e)$ with $\delta = (1^k)$. Remove all rows and columns common to both partition to obtain $\tilde{\mu}$ a non-linear rectangle and $\tilde{\lambda} \supseteq (3^3, 1)$; the result follows by Section 6.

For case $(1d)$ with δ linear, remove all columns and all but one row common to both λ and μ to obtain $\tilde{\mu}$ a fat rectangle and $\tilde{\lambda}$ a partition with at least 3 removable nodes. The result follows by Section 6.

We now consider case $(1b)$; here, only γ can be linear. If $\gamma = (k)$ remove all common rows and columns to obtain $\tilde{\mu}$ and $\tilde{\lambda}$. If $\ell(\delta) = 2$ then the result follows by Subsection 4.3. Suppose that $\ell(\delta) \geqslant 3$. The shortest row of $\tilde{\mu}$ is longer than the longest column in $\tilde{\lambda}$ and so $\tilde{\lambda}^t \cap \tilde{\mu}$ is a rectangle. By assumption, $\ell(\lambda) \geqslant 3$ and so $\tilde{\lambda}^t \cap \tilde{\mu} \supseteq (3^4)$ and the result follows by Subsection 4.1.

We now consider the case $(1b)$ with $\gamma = (1^k)$. The exceptional cases are (i) $(a - c)b \leqslant 2$ and (ii) $\ell(\delta) = 2$. In either case, remove all rows and columns with the exception of one column in the leg (which exists by assumption that μ is neither a hook, nor a 2-line partition) to obtain $(\tilde{\lambda}, \tilde{\mu})$. We have that $(a - c)b < k$ by assumption and so we can remove the final $(a - c)$ columns of $\tilde{\mu}$ and the final $(a - c)b$ rows of $\tilde{\lambda}$ to obtain $\hat{\mu} = (2^{k+\ell(\delta)})$ and $\hat{\lambda} \supset (3, 2, 1)$. The result follows by Subsection 4.3.

Now suppose we are in case $(1b)$ with $\gamma = (1^k)$ and we are not in one of the exceptional cases (i) and (ii) above. Remove all common rows and columns from μ and λ to obtain $\tilde{\mu}$ and $\tilde{\lambda}$. If $\tilde{\mu}$ is a 2-column partition, the result follows. Otherwise, remove all nodes in $\tilde{\lambda}$ to the right of the final column of $\tilde{\mu}$ and remove the corresponding number of nodes from the first column of $\tilde{\mu}_1$ to obtain a pair $(\hat{\lambda}, \hat{\mu})$. We have that $\hat{\delta} = \hat{\lambda}/(\hat{\lambda} \cap \hat{\mu})$ is a proper partition and $\hat{\gamma} = \hat{\lambda}/(\hat{\lambda} \cap \hat{\mu})$ is linear. The result follows from the case $(1e)$ for $\hat{\delta}$ a proper partition, above.

Finally, suppose we are in case $(1a)$; here only δ can be linear. If γ is a proper partition, remove all common rows (or all common columns, respectively) from λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. The partitions $\tilde{\lambda}^t$ and $\tilde{\mu}^t$ are now as in case $(1e)$ (respectively $(1c)$) above and therefore $g(\lambda, \mu) > 1$.

It remains to consider the case when γ is a proper skew partition.

Case (i). If $\gamma = \rho/(1)$ and $\delta = (k)$, remove all but one row (in the arm) or one column (in the leg) to obtain a pair of partitions $(\tilde{\lambda}, \tilde{\mu})$. In the former case, we remove successive rows from $\tilde{\mu}$ (and the corresponding number of nodes from the first row of $\tilde{\lambda}$) until we obtain $\tilde{\mu}$ a fat rectangle and $\tilde{\lambda}$ such that $|\mathrm{rem}(\tilde{\lambda})| = 3$. The result follows by Section 6. In the latter case, remove the final row of $\tilde{\mu}$ and the corresponding number of nodes from the first row of $\tilde{\lambda}$ to obtain a pair $(\hat{\lambda}, \hat{\mu})$. If $\hat{\mu}$ is a rectangle the result follows. Otherwise, $\hat{\mu}/(\hat{\lambda} \cap \hat{\mu})$ is a proper skew partition and $[\hat{\lambda}/(\hat{\lambda} \cap \hat{\mu})] = [k'] \boxtimes [1]$ with $k' < k$, and the result follows from Remark 5.8.

Case (ii). Now assume $\gamma \neq \rho/(1)$ and $\delta = (k)$ and remove all rows and columns common to λ and $\tilde{\mu}$ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. If $\tilde{\lambda}$ is a hook or 2-line partition the result follows. Otherwise, if $\gamma = (\rho/\sigma)^{\mathrm{rot}}$ for $|\mathrm{rem}(\rho)| = 3$ (respectively 2) remove all rows in $\tilde{\mu}$ which occur below the final row of $\tilde{\lambda}$ and remove the corresponding number of nodes from the first row of $\tilde{\lambda}$ to obtain $\hat{\lambda}$ and $\hat{\mu}$. We have that $|\mathrm{rem}(\hat{\lambda})| = 2$ (respectively 3) and $\hat{\mu}$ is either a rectangle or a fat hook such that $\hat{\mu}/\hat{\lambda} \cap \hat{\mu}$ is a proper partition (respectively $\hat{\mu}/\hat{\lambda} \cap \hat{\mu} = (\hat{\rho}/\hat{\sigma})^{\mathrm{rot}}$ for $|\mathrm{rem}(\hat{\rho})| = 2$).

In the former case, the result follows either from Section 6 or from noting that $(\hat{\lambda}, \hat{\mu})$ are as in case $(1a)$ for γ a proper partition. In the latter case, repeat the above argument for case (i) or case (ii) as appropriate.

Finally assume $\delta = (1^k)$ in case $(1a)$. Remove all rows and columns common to both λ and μ to obtain a pair $(\tilde{\lambda}, \tilde{\mu})$. If $w(\gamma) = 2$, then $\tilde{\lambda}$ is a proper fat hook and $\tilde{\mu}$ is a 2-line partition and so the result follows by Subsection 4.3. Otherwise, by our assumptions $k \geqslant 4$ and $3 \leqslant w(\gamma) < k$. The shortest column of $\tilde{\lambda}$ (which is of length equal to k) is longer than the widest row of $\tilde{\mu}$ (equal to $w(\gamma)$) and so $\tilde{\lambda}^t \cap \tilde{\mu} = (w(\gamma)^k) \supseteq (3^4)$ and so the result follows by Subsection 4.1. □

Lemma 7.4. *If either $[\gamma]$ or $[\delta]$ is equal to $[k-1] \boxtimes [1]$ up to conjugation, then $g(\lambda, \mu) > 1$.*

Proof. If $[\gamma]$ or $[\delta]$ is of the form $[1] \boxtimes [k-1]$ up to conjugacy, we may assume that the other is a rectangle by Remark 5.8. It is easy to see that case $(2d)$ is never of this form. We first consider the pairs of partitions $(\tilde{\lambda}, \tilde{\mu})$ which form our exceptional cases, in which it is not possible to remove all rows and columns common to both partitions λ, μ.

In case $(2a)$, suppose that γ is linear and $[\delta] = [1] \boxtimes [k-1]$. By Remark 6.5, we can remove most rows and columns common to μ and λ to obtain $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the following

$$((3, 2^k), (2{+}k, 1^{k+1}))\ ,\ ((3, 2^k), (4, 1^{2k-1}))\ ,\ ((k{+}1)^3)), ((2k, k{+}1, 1^2)\ ,\ ((k{+}1)^3)), ((k{+}2, k{+}1, 1^k).$$

Otherwise, remove all rows and columns common to both λ and μ to obtain $(\tilde{\lambda}, \tilde{\mu})$. The result follows by Subsection 4.2 and Section 6.

We now consider case $(2b)$ (in case $(2f)$ one can use the semigroup property to reduce to the same set of cases, and we therefore do not consider this case explicitly). Suppose that γ is linear and $[\delta] = [1] \boxtimes [k-1]$ up to conjugation. The exceptional cases are precisely those in which $\ell(\delta') = w(\delta'') = 1$ (with notation as in case $(2b)$ of Figure 5) and γ is linear. Remove all rows and columns common to both μ and λ with the exception of one row in the arm to obtain $(\tilde{\mu}, \tilde{\lambda})$ equal to one of the following up to conjugation

$$((3, 2^{k+1}), (3^2, 1^{2k-1}))\ ,\ ((k+1, 2^{k+1}), ((k+1)^2, 1^{k+1}))$$
$$((k+2, (k+1)^2), ((k+2)^2, 1^k))\ ,\ ((2k, (k+1)^2), ((2k)^2, 1^2)).$$

In each case we can remove a single node from the first row of $\tilde{\mu}$ and a single node from the first column of $\tilde{\lambda}$ to obtain a pair $(\hat{\mu}, \hat{\lambda})$. In the first case $\hat{\mu} = (2^{k+2})$ and $\hat{\lambda}$ is a proper fat hook and the result follows by Subsection 4.3. In the third case $\hat{\mu}$ is a fat rectangle and $\hat{\lambda}$ is a proper fat hook and the result follows by Subsection 4.3. In the second and fourth cases with $k > 2$ (the $k = 2$ cases are covered by the first and third cases) $\hat{\mu}, \hat{\lambda}$ are both proper fat hooks and $\hat{\mu}/(\hat{\lambda} \cap \hat{\mu})$ is linear and $[\hat{\lambda}/(\hat{\lambda} \cap \hat{\mu})]$ is the standard character and so the result follows by Lemma 7.3.

The only exceptional case for $(2c)$ is when $\ell(\delta') = w(\delta'') = 1$ and $\gamma = (1^k)$. Then remove all rows and columns common to both λ and μ with the exception of one column in the arm or leg (which must exist as μ is not a 2-line partition) to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. In the former case $\tilde{\lambda}$ is a proper fat hook and $\tilde{\mu} \supset (3, 1^2)$ is a hook partition; the result then follow by Subsection 4.2. In the latter case remove a single node from the first column of $\tilde{\lambda}$ and the first row of $\tilde{\mu}$; the result then follows by Subsection 4.3.

The only exceptional case for $(2e)$ is that in which $w(\gamma') = \ell(\gamma'') = 1$ and $\delta = (k)$. Remove all rows and columns common to both λ and μ with the exception of one row or column in the arm or in the leg to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. For a single row in the leg (respectively column in the arm) the result then follows by Subsection 4.2 (respectively Subsection 4.3). For a single row in the arm, remove a single node from the first column of $\tilde{\lambda}$ and the first column of $\tilde{\mu}$; the result then follows by Subsection 4.3. For a single column in the leg, remove the final row of $\tilde{\mu}$ and the final two columns of $\tilde{\lambda}$ to obtain $(\hat{\lambda}, \hat{\mu})$. For $k > 2$, both $\hat{\lambda}/\hat{\lambda} \cap \hat{\mu}$ and $\hat{\mu}/\hat{\lambda} \cap \hat{\mu}$ have two connected components and the result follows by Remark 5.8; for $k = 2$, we have the seed $((5, 2, 1), (3^2, 2))$.

Now suppose that we are in one of the cases $(2a, b, c, e, f)$ and (γ, δ) is not one of the exceptional cases (all of which were dealt with above). In cases $(2a, b, f)$ we remove all rows and columns common to both μ and λ to obtain a pair $(\tilde{\lambda}, \tilde{\mu})$ where $\tilde{\mu}$ or $\tilde{\lambda}$ is a proper rectangle, and which is not on our list. In case $(2c)$ we remove all common rows and columns from λ, μ and obtain either a 2-line partition $\tilde{\lambda}$ with $(\tilde{\lambda}, \tilde{\mu})$ not on our list, or a pair which can be reduced in one further step to a pair not on our list where at least one is a proper rectangle. In case $(2e)$ again remove all common rows and columns and obtain either a 2-line partition $\tilde{\lambda}$ with $(\tilde{\lambda}, \tilde{\mu})$ not on our list, or a pair $(\tilde{\lambda}, \tilde{\mu})$ where we can remove a shape corresponding to δ from $\tilde{\lambda}$ and the final boxes from the k columns of $\tilde{\mu}$, and $g(\tilde{\lambda} \cap \tilde{\mu}, \tilde{\lambda} \cap \tilde{\mu}) > 1$. So the result follows from Subsection 4.3, Subsection 4.1 and Section 6. ☐

Lemma 7.5. *If either γ or δ is a proper hook partition up to rotation, then $g(\lambda, \mu) > 1$.*

Proof. By Section 4 and Theorem 2.7 we may assume that up to rotation one of γ, δ is a proper hook and the other is a fat hook. By Lemma 7.3 we may assume that neither partition is linear.

Assume (λ, μ) are as in $(1a)$ of Figure 4. We begin with the case $\gamma = (k-1, 1)$ (the case $\delta = (2, 1^{k-2})$ is identical). For $k = 3$ we remove all rows and columns common to both partitions with the exception of one column in the leg to obtain the seed $(\tilde{\mu}, \tilde{\lambda}) = ((3^3, 2), (5, 4, 1^2))$. For $k > 3$, we remove all rows and columns common to both λ, μ to obtain $(\tilde{\mu}, \tilde{\lambda})$. If $\ell(\tilde{\lambda}) = 2$, then the result follows from Subsection 4.3. If $\ell(\tilde{\lambda}) > 2$, then γ and δ are (SG)-removable and $(3^3) \subseteq \tilde{\mu} \cap \tilde{\lambda}$.

Now assume that $\gamma = (2, 1^{k-2})$, with $k > 3$. If k is even and $\delta = (k/2, k/2)$, then we remove all rows and columns common to both partitions with the exception of one column in the leg to obtain $\tilde{\mu}$ and $\tilde{\lambda}$. We then remove $(k/2, k/2)$ from the top of $\tilde{\lambda}$ and $(2^{k/2})$ from the bottom of $\tilde{\mu}$ to obtain $((3^3, 2^{k/2-2}), (3^2, 1^{k-1}))$ and then the result follows by Lemma 7.3. For δ not of the above form, remove all rows and columns common to both partitions to obtain $(\tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\mu}$ is a 2-line partition and $(\tilde{\mu}, \tilde{\lambda})$ is not a pair listed in Theorem 1.1; the result follows by Subsection 4.3.

If $\delta = (k-1, 1)$ and γ is not of the above form, remove all rows and columns common to λ and μ to obtain $\tilde{\lambda} = (w(\gamma) + k - 1, w(\gamma) + 1)$ and $\tilde{\mu}$ a proper fat hook. The result follows from Subsection 4.3.

We now assume that $\gamma, \delta \neq (k-1,1)$ up to conjugation. By Section 4 and Theorem 2.7, being in case $(1a)$ implies that γ is a proper hook and δ is a non-linear rectangle. Remove all rows and columns common to μ and λ to obtain $\tilde{\mu}$ a proper fat hook and $\tilde{\lambda}$ a rectangular partition; the result follows then by Section 6.

Before addressing case $(1b)$ of Figure 4, we first consider cases $(1c,d,e)$. Assume that δ is a proper hook. In case $(1c)$, there is a single exceptional subcase, where $\gamma = (2^k)$ and $\delta = (2k-1,1)$; here we remove all rows and columns common to both partitions with the exception of one column in the arm to obtain $(\tilde{\lambda}, \tilde{\mu}) = ((2k+2,4),(3^2,2^k))$; the result follows by Subsection 4.3. In case $(1d)$, the unique exceptional subcase is $(\gamma, \delta) = ((2^2),(2,1,1))$, which we can reduce to the seed $(\tilde{\lambda}, \tilde{\mu}) = ((4,2^3,1^2),(4^3))$. In case $(1e)$, the single exceptional subcase is given by $\gamma = (2^k)$ and $\delta = (2k-1,1)$; remove all rows and columns common to both partitions with the exception of one row in the arm to obtain $(\tilde{\lambda}, \tilde{\mu})$. In which case $(\tilde{\lambda}^t, \tilde{\mu}^t)$ is equal to a pair of partitions as in the second exceptional case for $(1a)$, above.

Continuing with $(1c,d,e)$ with δ a proper hook, we now argue for the generic case. Remove all rows and columns common to λ and μ to reduce to a pair of partitions $(\tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\mu}$ is a rectangle and $(\tilde{\lambda}, \tilde{\mu})$ does not belong to our list. Thus the result follows by Section 6.

Suppose that we are in cases $(1c,d,e)$ and that γ is a rotated proper hook. Remove all rows and columns common to both partitions to obtain $(\tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\mu}$ is a rectangle and $\tilde{\lambda}$ is a proper fat hook or has three removable nodes (as δ is non-linear). The result follows from Section 6.

Finally, we consider case $(1b)$. If δ^{rot} is a proper hook and γ is a non-linear rectangle, remove all rows and columns common to λ, μ to obtain $(\tilde{\lambda}, \tilde{\mu})$ such that $\tilde{\lambda}$ is a rectangle and $\tilde{\mu}$ is a proper fat hook; the result follows then by Section 6. We may now assume that one of γ^{rot} or δ^{rot} is equal to $(k-1,1)$ up to conjugation and the other is a non-rectangular fat hook. This case is symmetric in swapping γ and δ and therefore we can assume that $\delta^{\mathrm{rot}} = (k-1,1)$ up to conjugation and $\gamma^{\mathrm{rot}} = (t^u, v^w) \vdash k$ is not a rectangle. Remove all rows and columns common to λ, μ to obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to either

$$((t+k-1)^2,(t-v)^w),(t+k-2,t^{u+w+1}))\ ,\ ((t+2)^{k-1},(t-v)^w),((t+1)^{k-2},t^{u+w+1}).$$

The $k=3$ case is the seed $(4^2,1),(3,2^3)$. For $k>3$ in the latter case, if γ^{rot} is of depth at least 4, then $\tilde{\lambda} \cap \tilde{\mu}^t = ((t+2)^{t+1})$ and so $\tilde{\lambda}/(\tilde{\lambda} \cap \tilde{\mu}^t)$ and $\tilde{\mu}^t/(\tilde{\lambda} \cap \tilde{\mu}^t)$ are (SG)-removable and $g(\tilde{\lambda} \cap \tilde{\mu}^t, \tilde{\lambda} \cap \tilde{\mu}^t) > 1$. If the depth of γ^{rot} is smaller than 4, and $\gamma^{\mathrm{rot}} \neq (2^2,1)$ then the sum of the first and final columns in $\tilde{\lambda}$ is equal to the sum of the first and final columns in $\tilde{\mu}$ (equal to $2k-1$ in both cases). Now if γ^{rot} is not one of $(2,1^2),(2,1^3),(2^2,1)$, we remove these columns and obtain $\hat{\mu}$ a non-linear rectangle, and $\hat{\lambda}$ a proper fat hook; the result follows by Corollary 2.6 and Section 6. If $\gamma^{\mathrm{rot}} = (2^2,1)$, we reduce to the seed $((3^4,1),(3^3,2^2))$. If $\gamma^{\mathrm{rot}} = (2,1^2)$, we remove the final two rows from $\tilde{\lambda}$ and the final row from $\tilde{\mu}$, giving a rectangle and a proper fat hook; the result follows by Corollary 2.6 and Section 6. If $\gamma^{\mathrm{rot}} = (2,1^3)$, we remove the final two columns from $\tilde{\lambda}$ and the first column from $\tilde{\mu}$, giving a pair of 2-line partitions not on our list, so the result follows.

In the former case with $k>3$, remove the final column of $\tilde{\lambda}$ and the final two columns from $\tilde{\mu}$ to obtain $\hat{\lambda}$ and $\hat{\mu}$ such that $\hat{\lambda}/\hat{\lambda} \cap \hat{\mu} = (k-4,2)^{\mathrm{rot}}$ and $\hat{\mu}/\hat{\lambda} \cap \hat{\mu} = \gamma$. By Remark 5.8, if γ^{rot} is not equal to $(k-1,1)$ up to conjugation, we are done. If $\gamma^{\mathrm{rot}} = (2,1^{k-2})$ then $g(\tilde{\lambda}, \tilde{\mu}) > 1$ by Subsection 4.1 and if $\gamma^{\mathrm{rot}} = (k-1,1)$, then the result follows by conjugating to the latter case, discussed above. ☐

Lemma 7.6. *If either γ or δ is a 2-line partition, then $g(\lambda, \mu) > 1$.*

Proof. By Remark 5.8 and the previous results in this section, it will suffice to consider γ and δ such that up to conjugation

- one is equal to (k,k) and the other is $(k+1,k)/(1)$;
- the pair is equal to one of the special pairs $((3^3),(6,3))$ or $((3^3),(5,4))$;
- one is equal to $(k-2,2)$ and the other is a rectangle;

- the pair is equal to one of $\{(k+1,k),(k+1,k)\}$, $\{(k^2),(k+1,k-1)\}$, or $\{(k^2),(2k-3,3)\}$.

We consider the proper skew partition case first. We assume without loss of generality that we are in case $(1a)$ (case $(1b)$ is identical and such a pair γ and δ cannot occur in cases $(1c,d,e)$). Remove all rows common to λ and μ to obtain $(\tilde{\lambda},\tilde{\mu})$ equal to one of

$$(((k{+}3)^k,1),((k{+}1)^{k+1},k))\ ,\ (((2k{+}1)^2,1),((k{+}1)^3,k))\ ,\ (((k{+}2)^2,1),(2^{k+2},1))\ ,\ ((4^k,1),(2^{2k},1)).$$

In the first case, we have that $\tilde{\lambda}^t/\tilde{\lambda}^t\cap\tilde{\mu}$ and $\tilde{\mu}/\tilde{\lambda}^t\cap\tilde{\mu}$ are both linear and so the claim follows from Lemma 7.3. In the second case, we have that $\tilde{\lambda}^t/\tilde{\lambda}^t\cap\tilde{\mu}=(2^{(k-1)})$ and $\tilde{\mu}/\tilde{\lambda}^t\cap\tilde{\mu}=((k-1)^3,k-2)/(1)$ and so the result follows for $k>3$ by Remark 5.8; at $k=2$ we have the seed $((5^2,1),(3^3,2))$, to which we also easily reduce in the case $k=3$. The third and fourth cases follow from Subsection 4.3. For the remainder of the proof, we assume that γ and δ are both proper partitions (up to rotation) and proceed case-by-case through $(1a)$ to $(1e)$.

We first consider case $(1a)$ depicted in Figure 4. By the above, γ is a (non-rectangular) 2-line partition. If γ is equal to $(k,k-1)$ up to conjugation, then we can remove rows and columns common to μ and λ until we are in one of the four cases in Figure 6 or at one of $((7^3),(4^4,3))$ or $((5^3),(2^5,1))$. In the first three cases in Figure 6, the result follows by Subsections 4.1 and 4.3. In the fourth case in Figure 6, we remove the final row of $\tilde{\mu}$ and the penultimate column of $\tilde{\lambda}$ to obtain $\hat{\mu}$ a fat rectangle and $\hat{\lambda}$ with $|\mathrm{rem}(\hat{\lambda})|=3$; so this case and the special cases follow by Section 6.

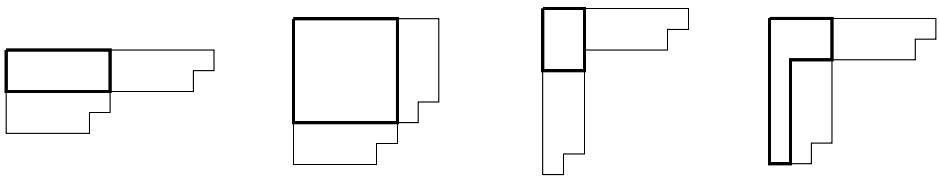

FIGURE 6. The four families, up to conjugation, for $\gamma=(k,k-1)$ and $k\geqslant 3$ in case $(1a)$.

Continuing with case $(1a)$, suppose that γ is equal to one of $(k{+}1,k{-}1)$, $(k{-}3,3)$ or $(k{-}2,2)$ and δ is a non-linear rectangle. Remove all rows and columns common to both μ and λ to obtain $\tilde{\mu}$ a proper fat hook and $\tilde{\lambda}$ a non-linear rectangle; the result follows by Section 6.

It remains to consider the cases where γ is equal to one of $(2^{k-1},1^2)$, $(2^3,1^{k-6})$ or $(2^2,1^{k-4})$, or $(2^4,1)$ and δ is a non-linear rectangle.

Suppose $\gamma=(2^{k-1},1^2)$. If $\ell(\delta)>2$, remove all rows and columns common to λ and μ to obtain $\tilde{\mu}=(2^{k+\ell(\delta)-1},1^2)$ and $\tilde{\lambda}$ a fat rectangle. If $\ell(\delta)=2$, remove all rows and columns common to λ and μ with the exception of one column in the leg to obtain $\tilde{\mu}=(3^{k+1},2^2)$ and $\tilde{\lambda}=((k+3)^2,1^{k+1})$. Remove the final two rows of $\tilde{\mu}$ and the final two columns of $\tilde{\lambda}$ to obtain $\hat{\mu}$ a fat rectangle and $\hat{\lambda}$ a proper fat hook. The result follows by Section 6.

Now suppose that $\gamma=(2^3,1^{k-6})$ or $(2^2,1^{k-4})$, or $(2^4,1)$ and that $\gamma\supseteq(2^2,1^3)$ (as the other cases were handled above). Remove all rows and columns common to both μ and λ to obtain $\tilde{\mu}\supseteq(2^4,1^3)$ a 2-line partition not of the form $(2^{k-1},1^2)$ or (2^k) and $\tilde{\lambda}$ is a non-linear rectangle. The result follows by Subsection 4.3.

Now consider the cases where both γ and δ are equal to $(k+1,k)$ (up to conjugation) for $(1b,c,d,e)$, where $k>1$. Remove all rows and columns common to both μ and λ and arrive at twelve distinct cases (as $(1c)$ and $(1e)$ produce the same set of cases). Eleven of the twelve cases follow by Subsections 4.1 and 4.3 and Section 6. The final case is $((k{+}2,(k{+}1)^3),((2k{+}2)^2,1))$. For $k>2$, remove the final column of $\tilde{\mu}$ and the final row of $\tilde{\lambda}$ to obtain a pair of rectangular partitions. The result follows from Section 6. For $k=2$ we obtain the seed $((6^2,1),(4,3^3))$.

It remains to consider cases $(1b,c,d,e)$ in which precisely one of γ and δ is a rectangle.

In case $(1b)$, where γ has to be a rectangle (respectively in cases $(1c,d,e)$ when δ is a rectangle) remove all rows and columns common to μ and λ to obtain $\tilde{\lambda}$ a non-linear rectangle and $\tilde{\mu}$ a proper fat hook (respectively $\tilde{\mu}$ a rectangle and $\tilde{\lambda}$ a proper fat hook). The result follows from Section 6.

It remains to consider cases $(1c, d, e)$ for γ a rectangle. In cases $(1c)$ (respectively $(1e)$) with $\gamma = (2^k)$ and $\delta = (k+1, k-1)$, remove all rows and columns common to λ and μ with the exception of one column in the leg (respectively row in the arm) to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. In case $(1e)$, remove the final two columns of $\tilde{\lambda}$ and the final two rows of $\tilde{\mu}$ to obtain $\tilde{\lambda}$ a fat rectangle and $\tilde{\mu}$ a proper fat hook. In both cases the result follows by Section 6. For a case not of the above form, remove all rows and columns common to λ and μ to obtain $\tilde{\lambda}$ and $\tilde{\mu}$. The result follows by Section 6. □

Lemma 7.7. *If either of γ or δ is linear and the other has two connected components, then $g(\lambda, \mu) > 1$.*

Proof. By Theorems 2.2, 2.3 and 2.7, the non-linear diagram (of the pair γ and δ) belongs to the list of skew partitions in Theorem 2.2. By Lemma 7.4, we can assume that neither of $[\gamma]$ or $[\delta]$ is $[k-1] \boxtimes [1]$, up to conjugation. We first consider the exceptional cases where we cannot remove all rows and columns common to μ and λ. These only happen in a few cases in which all three external components in the diagram are linear. Up to conjugation of both λ and μ, our exceptional cases are listed below. By aggressive application of Remark 6.5, we can remove all rows and columns common to both partitions with the exception of a single row, R, or column, C, to obtain $\tilde{\lambda}, \tilde{\mu}$. These rows and columns are also listed.

$(2a)$ γ linear, $\delta' = (l)$, $\delta'' = (1^m)$ and C a single column in the arm;
$(2b)$ γ linear, $\delta' = (l)$, $\delta'' = (1^m)$ and R a single row in the arm;
$(2c)$ $\gamma = (1^{l+m})$, $\delta' = (l)$ and $\delta'' = (1^m)$ and C a single column in the leg;
$(2d)$ no exceptions;
$(2e)$ $\gamma' = (1^l), \gamma'' = (m)$, $\delta = (l+m)$ and C a single column in the arm or the leg;
$(2f)$ $\gamma' = (1^l)$ $\gamma'' = (m)$ and $\delta = (l+m)$ or (1^{l+m}) and R a single row in the arm (the resulting partitions are the same as in the $(2b)$ case).

For any γ and δ and any case $(2a$–$e)$ not on the above list, remove all columns and rows common to μ and λ to obtain $\tilde{\mu}$ and $\tilde{\lambda}$. For an example of how Remark 6.5 is used, we compare $(2b)$ and $(2f)$; here we have reduced to the same set of exceptional cases, but using different arguments. For $(2b)$, we know there must exist a row in the arm as μ is non-rectangular. For $(2f)$, we know that λ is not a hook, and so there must be an extra row in the arm or column in the leg. However, case $(2f)$ is symmetric under conjugation (note that case $(2b)$ is not) and so we can assume there is an extra row in the arm.

For the exceptional cases of type $(2a)$, we have that $\tilde{\mu}$ is non-rectangular (up to conjugation, $\tilde{\mu}$ is obtained by adding a single node to the partition $((l+m+1)^2))$ and $\tilde{\lambda} \supset (3, 1^2)$ is a hook partition. Therefore the result follows from Subsection 4.2. The generic case follows from Section 6, as $\tilde{\mu}$ is a fat rectangle and $\tilde{\lambda} \supset (3, 1^2)$.

In case $(2d)$, we know by Remark 5.8 and Theorem 2.2 that $\delta' = (t^u, v^w)$ is a fat hook and therefore $\delta'' = (r^s)$ is a rectangle. If $\gamma = (1^{l+m})$ we remove the final rs rows (each of width $r+1$) from $\tilde{\mu}$ and the final $s(r+1)$ rows (each of width r) from $\tilde{\lambda}$ and hence obtain a pair $(\hat{\lambda}, \hat{\mu})$ as in $(1b)$. If $\gamma = (l+m)$, then the shortest row of $\tilde{\mu}$ (equal to $l+m+r$) is longer than the longest column of $\tilde{\lambda}$ (equal to $s+u+w+1$) and therefore $\tilde{\lambda}/\tilde{\lambda} \cap \tilde{\mu}^t$ and $\tilde{\mu}^t/\tilde{\lambda} \cap \tilde{\mu}^t$ are both connected (in fact $(\tilde{\lambda}, \tilde{\mu}^t)$ are as in case $(1a)$). In both cases, the result follows by earlier results in this section.

We now consider case $(2e)$. In the exceptional case with C a single column in the arm, we have that $\tilde{\lambda} = (2m+l+2, 2)$ and $\tilde{\mu}$ is a proper fat hook; the result holds by Lemma 4.5 and Remark 5.8. In the exceptional case with C a single column in the leg, we remove the final two columns of $\tilde{\lambda}$ and the final row of $\tilde{\mu}$ to obtain $(\hat{\lambda}, \hat{\mu})$ such that $\hat{\lambda}/\hat{\lambda} \cap \hat{\mu}$ and $\hat{\mu}/\hat{\lambda} \cap \hat{\mu}$ are both proper skew partitions with two components each and the result follows by Remark 5.8.

We now consider the generic case of $(2e)$ with $\delta = (l+m)$. We first consider the case where $w(\gamma')$ or $\ell(\gamma'')$ is equal to 1. If $w(\gamma') = 1$ and γ'' is a rectangle, then $\tilde{\lambda}$ is a hook and the result follows from Subsection 4.2. If $w(\gamma') = 1$ and γ'' is not a rectangle, then remove γ' from $\tilde{\mu}$ and

$|\gamma'|$ nodes from $\tilde{\lambda}_1$ to obtain $\hat{\mu}$ a fat rectangle and $\hat{\lambda}$ a partition with at least three removable nodes; the result follows by Section 6. We now assume that $\ell(\gamma'') = 1$ and $w(\gamma'') > 1$. If γ' is a rectangle, then the result follows by Subsection 4.3. If γ' is not a rectangle, remove the final $w(\gamma'')$ columns from $\tilde{\mu}$ and $2w(\gamma'')$ nodes from $\tilde{\lambda}_1$ to obtain $\hat{\mu}$ a non-linear rectangle and $\hat{\lambda}$ such that $|\mathrm{rem}(\hat{\lambda})| \geqslant 3$.

By Remark 6.5 (see Theorem 2.2 in particular) we may assume that at least one of γ' or γ'' is a rectangle and that $w(\gamma'), \ell(\gamma'') > 1$. If γ' is a rectangle, remove γ' from the bottom of $\tilde{\mu}$ and $|\gamma'|$ nodes from the $\tilde{\lambda}_1$ to obtain $\hat{\mu}$ a fat rectangle and $\hat{\lambda} \supset (3, 2^2)$; the result follows by Section 6. We now assume that γ'' is a rectangle. Remove the final $w(\gamma'')$ columns of $\tilde{\mu}$ (each of length $\ell(\gamma'')+1$) and $w(\gamma'')(\ell(\gamma'')+1)$ nodes from $\tilde{\lambda}_1$ to obtain $\hat{\mu}$ a non-linear rectangle and $\hat{\lambda} \supset (3, 2^2)$. The result follows by Section 6.

We now consider the case $\delta = (1^{l+m})$. If $\gamma' = (l)$, $\gamma'' = (m)$ (with $l, m \neq 1$ by our assumptions), remove the first row of $\tilde{\mu}$ and the final column of $\tilde{\lambda}$ to obtain $\hat{\lambda} = \hat{\mu} \supset (4^4)$; the result follows by Subsection 4.1. Now assume that $\delta = (1^{l+m})$ and γ', γ'' are not of the above form. The shortest column of $\tilde{\lambda}$ (of length $l+m$) is longer than the longest row of $\tilde{\mu}$ (of length $w(\gamma') + w(\gamma'')$) and therefore $\tilde{\lambda}^t/\tilde{\lambda}^t \cap \tilde{\mu}$ and $\tilde{\mu}/\tilde{\lambda}^t \cap \tilde{\mu}$ are both connected and the result follows by earlier results in this section.

We now consider the cases $(2f)$ and $(2b)$. We first consider the generic case of $(2f)$. If γ' and γ'' are both rectangles, then $(\tilde{\lambda}, \tilde{\mu})$ are as in case $(2a)$ considered above. If one of γ' and γ'' is a rectangle and the other is a non-rectangular fat hook, then the pair $(\tilde{\lambda}, \tilde{\mu})$ are as in case $(2d)$ considered above. Up to conjugation, it remains to consider the case in which γ' is linear and $(\gamma'')^{\mathrm{rot}}$ is such that $\mathrm{rem}((\gamma'')^{\mathrm{rot}}) \geqslant 3$; in particular $(\gamma'')^{\mathrm{rot}} \supseteq (3, 2, 1)$. If $\delta = (l+m)$, then the shortest row of $\tilde{\lambda}$ is of length $l + m + w(\gamma')$, and the longest column of $\tilde{\mu}$ is less than or equal to $l+m-2$. Therefore $\tilde{\lambda}^t/\tilde{\lambda}^t \cap \tilde{\mu}$ and $\tilde{\mu}/\tilde{\lambda}^t \cap \tilde{\mu}$ are both connected and the result follows by earlier results in this section. If $\delta = (1^{l+m})$ and $\gamma' = (l)$, remove the final $(l+1)$ rows (of width l) from $\tilde{\mu}$ and the final l rows (of width $l+1$) from $\tilde{\lambda}$ to obtain $(\hat{\lambda}, \hat{\mu})$. If $\delta = (1^{l+m})$ and $\gamma'' = (1^l)$, remove the final $2l$ rows (of width 1) from $\tilde{\mu}$ and the final l rows (of width 2) from $\tilde{\lambda}$ to obtain $(\hat{\lambda}, \hat{\mu})$. In either case, $\hat{\lambda}^t/\hat{\lambda}^t \cap \hat{\mu}$ and $\hat{\mu}/\hat{\lambda}^t \cap \hat{\mu}$ are both connected and the result follows by earlier results in this section.

The generic case for $(2b)$ follows from Section 6 as $\tilde{\mu}$ is a rectangle. We now argue for the exceptional case for $(2b)$ (the exceptional case for $(2f)$ is identical but with the roles of γ and δ switched). For $\gamma = (1^{l+m})$ (respectively $(l+m)$) remove the final row of $\tilde{\mu}$ (respectively final two columns of $\tilde{\mu}$) and the final column of $\tilde{\lambda}$ to obtain $(\hat{\mu}, \hat{\lambda})$ such that $\hat{\mu}/\hat{\lambda} \cap \hat{\mu}$ and $\hat{\lambda}/\hat{\lambda} \cap \hat{\mu}$ both having two connected components (respectively $(\tilde{\lambda}, \tilde{\mu})$ are as in the generic case of $(2f)$).

For the exceptional case of $(2c)$, remove the final row of $\tilde{\mu}$ and the final two columns of $\tilde{\lambda}$ to obtain $(\hat{\lambda}, \hat{\mu})$. If $l = 2$, then $(\hat{\lambda}, \hat{\mu})$ are as in the exceptional case for $(2b)$. If $l > 2$, then $\hat{\lambda}/\hat{\lambda} \cap \hat{\mu}$ has three connected components and so the result follows by Remark 5.8. Now assume that we are in the generic case with $\gamma = (1^k)$. If $\tilde{\mu}$ is a hook or 2-line partition, the result follows by Subsections 4.2 and 4.3. Otherwise, we remove δ' from $\tilde{\lambda}$ and $|\delta'|$ nodes from the first column of $\tilde{\mu}$ to obtain a pair as in case $(1e)$ with $\tilde{\mu}$ a proper fat hook. For $\gamma = (k)$, if $\ell(\delta') + \ell(\delta'') = 2$ then the result follows by Subsection 4.3. Otherwise, $\tilde{\lambda} \cap \tilde{\mu}^t = ((\ell(\delta') + \ell(\delta'') + 1)^{\ell(\delta') + \ell(\delta'')}) \supseteq (4^3)$ and the result follows by Subsection 4.1. $\square$

8. The general case

In this section, we continue to assume that Theorem 1.1 has been proven for all Kronecker products labelled by pairs of partitions of degree less than or equal to $n-1$. Armed with the proof of Theorem 1.1 for the case where one partition is a fat hook of degree n, we now embark on proving the general case for arbitrary pairs of partitions of degree n.

We shall assume throughout that $\lambda, \mu \vdash n$ are a pair of partitions such that $\lambda \neq \mu$ and neither λ nor μ is a fat hook. Furthermore, by Theorem 2.7 and Remark 6.5 we may also assume that the pair of characters associated with the skew diagrams $\gamma = \mu/(\lambda \cap \mu)$ and $\delta = \lambda/(\lambda \cap \mu)$ belongs to the lists in Theorem 1.1 and Theorem 1.3. In particular, we may (and will) assume without loss of generality that γ has one connected component and that δ has either one or two connected components.

We shall systematically work through the list of possible pairs of shapes $\lambda/(\lambda\cap\mu)$ and $\mu/(\lambda\cap\mu)$ and reduce the corresponding pairs of partitions λ and μ to pairs of partitions $\tilde{\lambda}$, $\tilde{\mu}$ such that $g(\tilde{\lambda},\tilde{\mu}) > 1$ and the semigroup property implies $g(\lambda,\mu) > 1$. Our typical approach will be to reduce to the case that one of $\tilde{\lambda}$ or $\tilde{\mu}$ is a 2-line, rectangle, or fat hook partition and then appeal to the results of Sections 4, 6 and 7.

Lemma 8.1. *Suppose* $\gamma = \delta = (1)$*, then* $g(\lambda,\mu) > 1$.

Proof. We let $\gamma = (r_1, c_1)$, $\delta = (r_2, c_2)$ and we suppose, without loss of generality, that $r_1 < r_2$ and $c_1 > c_2$. Our general strategy shall be to remove all rows and columns outside of the region labelled by $[r_1,\dots,r_2]\times[c_2,\dots,c_1]$, an example is depicted in Figure 7, below. We first consider the exceptional cases in which

$$\zeta = ([r_1,\dots,r_2]\times[c_2,\dots,c_1]) \cap \lambda \cap \mu$$

is equal to the Young diagram of a partition of the form (k,k), $(k,k-1)$, $(k-1,1)$, or (k) up to conjugation.

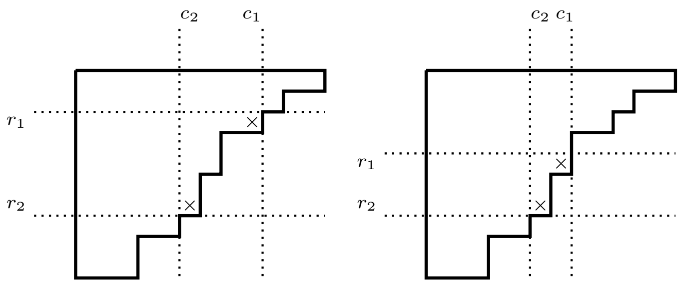

Figure 7. An example of a generic and an exceptional pair of partitions λ and μ such that $\gamma = \delta = (1)$. We have decorated the diagram with the region $[r_1,\dots,r_2]\times[c_2,\dots,c_1]$. In the former case the partition ζ has 3 removable nodes, in the latter case ζ is linear.

We start with the case $\zeta = (k)$ (the case $\zeta = (1^k)$ is similar); we remove most rows common to λ and μ to obtain three distinct cases. If both $r_1, c_2 \neq 1$, we can remove all but one column to the left of the region and all but one row above the region from λ,μ to obtain partitions $\tilde{\lambda},\tilde{\mu}$ such that

$$\tilde{\lambda}\cap\tilde{\mu} = (k+2, k+1, 1)\,.$$

In this case $g(\tilde{\lambda},\tilde{\mu}) = g(((k+2)^2,1),(k+2,k+1,2)) > 1$, by Section 7.

Now suppose that $c_2 = 1$, i.e., $\delta = (r_2, 1)$. By assumption, we have that μ is not a rectangle and so $\mu_1 > \mu_{r_1}$. We remove all but the longest row (of width $\lambda_1 = \mu_1 > \mu_{r_1}$) above the region; we then truncate this row to be of length $k+2$; we hence obtain $\tilde{\lambda}$ and $\tilde{\mu}$ such that

$$\tilde{\lambda}\cap\tilde{\mu} = (k+2, k)\,.$$

In this case $g(\tilde{\lambda},\tilde{\mu}) = g((k+2,k,1),(k+2,k+1)) > 1$, by Subsection 4.3.

Now suppose that $\gamma = (1, c_1)$, in which case we can remove all but the longest column to the left of the region (which is of length greater than or equal to 3, by assumption that neither of λ or μ is a 2-line partition), we then truncate this column to be of length 3 and hence obtain $\tilde{\lambda}$ and $\tilde{\mu}$ such that

$$\tilde{\lambda}\cap\tilde{\mu} = (k+1, 1^2)\,.$$

In this case $g(\tilde{\lambda},\tilde{\mu}) = g((k+1,2,1),(k+2,1,1)) > 1$, by Subsection 4.2.

We now assume that ζ is of the form (k,k), $(k+1,k)$, or $(k-1,1)$ up to conjugation, but not $(1,1)$. We can treat these cases uniformly without any restriction on n and k. In all of these cases, we know that there is at least one extra column or row common to λ and μ which we may consider; this follows from our assumption that neither λ nor μ is a 2-line partition. This leads us to define $\tilde{\lambda}, \tilde{\mu}$ as the intersections of λ, μ with the region $[r_1 - 1, r_1, \dots, r_2] \times [c_2, \dots, c_1]$ or $[r_1, \dots, r_2] \times [c_2 - 1, c_2, \dots, c_1]$, so that $\tilde{\lambda} \cap \tilde{\mu}$ is equal to one of

$$([r_1 - 1, r_1, \dots, r_2] \times [c_2, \dots, c_1]) \cap \lambda \cap \mu \qquad \text{or} \qquad ([r_1, \dots, r_2] \times [c_2 - 1, c_2, \dots, c_1]) \cap \lambda \cap \mu\,.$$

It will then suffice to show that $g(\tilde{\lambda}, \tilde{\mu}) > 1$ in both cases for all three possible partitions, ζ. In the latter case, for $\zeta = (k+1, k)$ we have that $(\tilde{\lambda}, \tilde{\mu}) = ((k+3, k+1, 1), (k+2, k+1, 2))$; removing (2) from $\tilde{\lambda}_1$ and (2) from $\tilde{\mu}_3$, the result follows from Subsection 4.3. In the other five cases, we note that one of the two partitions is a fat hook but $(\tilde{\lambda}, \tilde{\mu})$ is not on the list of Theorem 1.1, so $g(\tilde{\lambda}, \tilde{\mu}) > 1$ follows by Section 7.

We now deal with the generic case (in which $\zeta \neq (k)$, (k, k), $(k+1, k)$, or $(k-1, 1)$, up to conjugation); remove all rows and columns outside of the region labelled by $[r_1, \dots, r_2] \times [c_2, \dots, c_1]$ from λ, μ, to obtain $\tilde{\lambda}$ and $\tilde{\mu}$ such that

$$\tilde{\lambda} \cap \tilde{\mu} = [r_1, \dots, r_2] \times [c_2, \dots, c_1] \cap \lambda \cap \mu.$$

We note that the node γ (respectively δ) is (SG)-removable from $\tilde{\mu}$ (respectively $\tilde{\lambda}$); the result follows as $g(\tilde{\lambda} \cap \tilde{\mu}, \tilde{\lambda} \cap \tilde{\mu}) > 1$ by Proposition 4.1. □

Lemma 8.2. *If γ and δ are both linear, then $g(\lambda, \mu) > 1$.*

Proof. We assume, without loss of generality, that γ appears higher than δ in the diagram and $(\gamma, \delta) = ((1^k), (1^k))$, $((k), (1^k))$ or $((1^k), (k))$. By Lemma 8.1, we may assume that $k \geqslant 2$. The case $((k), (k))$ can be obtained by conjugation.

Case 1: $(\gamma, \delta) = ((1^k), (1^k))$. Assume there is a column, C, to the left of δ (respectively to the right of γ). Remove from the intersection all rows and columns excluding column C (respectively all columns excluding C, and all but one of the rows of width $c > c_1$ above γ) to obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to either of

$$((3^k, 1^k), (2^{2k}))\;,\; ((3, 2^k), (3, 1^{2k}))$$

and the result follows from Section 7. Now assume there is no such column to the left or right and recall our assumption that neither λ nor μ is a 2-line partition. There are two distinct cases to consider, namely

- $k \geqslant 2$ and there is a single column, C, in between δ and γ and a row, R, above γ;
- $k \geqslant 3$ and there are at least two columns, C_1, C_2 in between γ and δ and no rows above γ.

In the former case, we remove from the intersection all rows and columns excluding R and C to obtain $(\tilde{\lambda}, \tilde{\mu}) = ((3, 2^k, 1^k), (3^{k+1}))$. In the latter case, we remove from the intersection all rows and all columns except C_1 and C_2 to obtain $(\tilde{\lambda}, \tilde{\mu}) = ((3^k, 1^k), (4^k))$. In both cases the result follows from Section 6 as $\tilde{\mu}$ is a rectangle.

Case 2: $(\gamma, \delta) = ((k), (1^k))$ for $k \geqslant 2$. If there is both a column and a row between γ and δ, then we reduce to the case $(\tilde{\lambda}, \tilde{\mu}) = ((2^2, 1^k), (k+2, 2))$ and the result follows Subsection 4.3. We may now assume that there is not both a column and a row between γ and δ. Conjugating if necessary, we may assume that there is no column between γ and δ. Suppose that there are no rows above γ. Then by our assumption that $w(\tilde{\lambda}) > 2$, there are two columns C and C' to the left of δ. We remove from the intersection all rows and columns except for C and C' to obtain $(\tilde{\lambda}, \tilde{\mu}) = ((3^{k+1}), (3+k, 2^k)$ and the result follows from Section 6. We may now suppose that there is a row, R, above γ. By assumption, $\tilde{\lambda}$ is not a hook partition and so there is either (i) a single column, C, to the left of δ or (ii) an extra row R' above γ. In the former case, we remove all rows except R and all columns except C and hence obtain $(\tilde{\lambda}, \tilde{\mu}) = ((k+2, 2^{k+1}), ((k+2)^2, 1^k))$ with $k \geqslant 2$ and so the result follows from Section 7. In the latter case, we remove from the intersection all rows and columns with the exception of R and R' to obtain $(\tilde{\lambda}, \tilde{\mu}) = (((k+1)^2, 1^{k+1}), ((k+1)^3))$; the result follows from Section 6.

Case 3: $(\gamma, \delta) = ((1^k), (k))$. For $k = 2$, we can remove all but one column to the left of δ or all but one column between γ and δ (up to conjugation) to obtain $(\tilde{\lambda}, \tilde{\mu})$ equal to either of the small seeds $((3^3),(4^2,1))$, $((3^2,2),(4^2))$. Otherwise, we may remove all rows and columns common to both partitions, and the result follows by Proposition 4.1. ☐

Lemma 8.3. *If one of γ, δ is linear and the other is a proper partition up to rotation, then $g(\lambda,\mu)>1$.*

Proof. We assume, without loss of generality, that γ is linear and that it appears higher than δ in the diagram. By Lemma 8.2, we can assume that δ is non-linear. We start with the discussion of the cases where δ is a proper partition.

Case 1: $\gamma=(k)$ and δ is a proper partition. Suppose there are no rows either above γ or between γ and δ. In which case (by our assumption that μ is neither linear, nor a hook) there exist two columns C and C' to the left of δ. We remove from the intersection all rows and all columns with the exception of C and C'. The result follows as $\tilde\mu$ is a fat hook.

Suppose that there is a row, R, above γ. Remove all rows and columns common to both λ and μ with the exception of R, to obtain $\tilde\mu=((k+w(\delta))^2)$ and $\tilde\lambda=(k+w(\delta),w(\delta),\delta)$; we have that $\tilde\lambda$ is either a proper fat hook or $|\mathrm{rem}(\tilde\lambda)|\geqslant 3$ (our assumptions imply that $w(\delta)\geqslant 2$, $k\geqslant 3$). The result then follows from Subsection 4.3.

Now assume that there is a row, R, between γ and δ and no row above γ. If δ is not a fat hook, then remove all common rows and columns from λ and μ with the exception of R to obtain $\tilde\mu$ a 2-line partition and $\tilde\lambda$ a partition such that $|\mathrm{rem}(\tilde\lambda)|\geqslant 3$. We now assume that δ is a fat hook. Now (by assumption that μ is not a 2-line partition) there is either a second row R' between γ and δ or an extra column, C, to the left of δ. In either case remove all rows and columns from λ and μ with the exception of R and R' or R and C to obtain a pair $(\tilde\lambda,\tilde\mu)$. In the former case, $\tilde\mu$ is a proper fat hook and $[\tilde\lambda]$ is neither a linear character nor the natural character or its dual. In the latter case, $\tilde\lambda$ is a proper fat hook and $\tilde\mu$ has three removable nodes. In either case, the result follows from Section 7.

Case 2: $\gamma=(1^k)$ and δ is a proper partition. We have two exceptional cases to consider, in which $\delta=(2,1)$ or $(2,2)$. In either case, we remove all but a single row or column from λ and μ to obtain 12 seeds $(\tilde\lambda,\tilde\mu)$ of degree less than or equal to 18. Assume $\delta\neq(2,1)$, or $(2,2)$; remove all rows and columns common to λ and μ to obtain $\tilde\lambda$ and $\tilde\mu$. If $w(\delta)=2$, the result follows from Subsection 4.3. Otherwise, $\tilde\lambda\cap\tilde\mu=(w(\gamma)^k)$ with $w(\gamma),k\geqslant 3$ and so $g(\lambda,\mu)\geqslant g(\tilde\lambda\cap\tilde\mu,\tilde\lambda\cap\tilde\mu)>1$ by Subsection 4.1.

Case 3: $\gamma=(k)$ and δ^{rot} is a proper partition. By assumption, λ is not a fat hook and so there exists at least two columns, C and C', of *distinct lengths* belonging to one or two of the regions: to the left of δ, between γ and δ, or to the right of γ. We can assume that the final node in column, C say, does not belong to the same row as the nodes in the partition γ. Remove all rows and columns except for C to obtain $\tilde\lambda$ a proper fat hook and $\tilde\mu$ such that $\ell(\tilde\mu),w(\tilde\mu)>2$; the result follows from Section 7.

Case 4: $\gamma=(1^k)$ and δ^{rot} is a proper partition. We remove all rows and columns common to λ and μ, to obtain $\tilde\lambda$ a non-linear rectangle and $\tilde\mu$ a non-rectangular partition such that $(3^3)\subseteq\tilde\mu$; the result follows from Section 6. ☐

We fix some notation which will be used throughout the remainder of this section. If δ and γ each have exactly one connected component, then we can assume without loss of generality that δ lies below γ on the diagram, as depicted in the leftmost diagram in Figure 8. We shall let R_1 (respectively R_2) denote the longest row in $\lambda\cap\mu$ which appears above γ (respectively between δ and γ) if such a row exists, and let R_1 (respectively R_2) be undefined otherwise. Similarly, we shall let C_1 (respectively C_2) denote the longest column in $\lambda\cap\mu$ which appears to the left of δ (respectively between δ and γ) if such a column exists, and let C_1 (respectively C_2) be undefined otherwise. This is depicted in Figure 8.

If γ has exactly one connected component and δ has exactly two connected components, then we can assume without loss of generality that either

- γ lies below δ' and δ'' on the diagram, as depicted in the middle diagram in Figure 8;

- γ lies between δ' and δ'' on the diagram, as depicted in the rightmost diagram in Figure 8.

We define the rows R_1, R_2, R_3 and C_1, C_2, C_3 by the obvious extension of the definition above, which is depicted in the two rightmost diagrams in Figure 8, below.

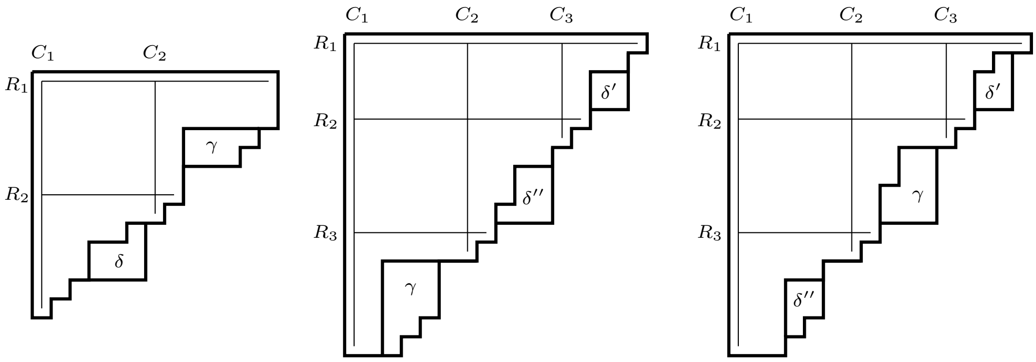

FIGURE 8. Extra rows and columns

Lemma 8.4. *If δ is a proper skew partition with one connected component, then $g(\lambda, \mu) > 1$.*

Proof. We assume, without loss of generality, that γ appears higher than δ in the diagram. By Theorem 2.3 and Remark 5.8, we know that δ is of the form $\delta = \sigma/\rho$ where σ is a partition, and ρ is a rectangle, or σ, ρ satisfy $|\text{rem}(\rho)| \geq 2$ and $|\text{rem}(\sigma)| = 2$.

We consider the exceptional case in which $\rho = (1)$ and $|\text{rem}(\sigma)| = 2$ and $\gamma = (k)$. By assumption, neither λ nor μ is a fat hook and so there exists at least one extra row or column R_1, R_2, C_1 or C_2 as in Figure 8. We remove all rows and columns common to both λ and μ with the exception of one of R_1 or R_2 or C_1 or C_2 to obtain $(\tilde{\mu}, \tilde{\lambda})$. In all other cases, we remove all rows and columns common to both λ and μ to obtain a pair of partition (λ, μ). The resulting pair $(\tilde{\lambda}, \tilde{\mu})$ are such that (i) $\tilde{\lambda} \neq \tilde{\mu}$ (ii) both $\tilde{\lambda}$ and $\tilde{\mu}$ are non-rectangular (iii) neither $[\tilde{\lambda}], [\tilde{\mu}]$ is equal to the standard character or its dual. Therefore $g(\tilde{\lambda}, \tilde{\mu}) > 1$ as required. □

Lemma 8.5. *If γ is linear and δ has two connected components, then $g(\lambda, \mu) > 1$.*

Proof. We first consider the two exceptional cases, in which δ' and δ'' are both linear.

Suppose that δ'' is below γ and γ is below δ' as depicted in the rightmost diagram in Figure 8. We assume without loss of generality that $\gamma = (k_1 + k_2)$ and δ' and δ'' are partitions of k_1 and k_2, respectively. The only exceptional case for such a shape is given by $\delta'' = (1^{k_2})$ and $\delta' = (k_1)$. We want to remove all but a single row or column from λ and μ depending on having a suitable row or column in one of the six cases illustrated in Figure 8; however, as we assume that μ is not a 2-row partition, we can ignore the two cases C_2 and C_3. It therefore remains to consider the cases where one of the columns or rows C_1, R_1, R_2, and R_3 exist, and we have reduced all other rows and columns common to both λ and μ to obtain $\tilde{\lambda}$, $\tilde{\mu}$. In each of these four cases, the partition $\tilde{\mu}$ is either a proper fat hook or a fat rectangle and $\tilde{\lambda}$ is a partition with $w(\tilde{\lambda}) \geqslant 4$, $\ell(\tilde{\lambda}) \geqslant 3$ and $|\text{rem}(\tilde{\lambda})| \geqslant 2$; the assertion follows from the result for fat hooks.

Suppose that γ is below δ'', and δ'' is below δ' as depicted in the central diagram in Figure 8. The only exceptional case for such a shape is given by $\gamma = (1^{k_1+k_2})$, $\delta'' = (1^{k_2})$ and $\delta' = (k_1)$. In this case, we need to consider each of the six possible cases given by removing all rows and columns common to λ and μ with the exception of one of R_1, R_2, R_3, C_1, C_2, or C_3 to hence obtain partititons $\tilde{\lambda}$ and $\tilde{\mu}$. In the case of R_1, R_3, C_1 or C_2, we have that one of the partitions $\tilde{\lambda}, \tilde{\mu}$ is a fat hook and the other has 3 removable nodes. In the case of R_2 or C_3, we have that $|\text{rem}(\tilde{\lambda})|, |\text{rem}(\tilde{\mu})| = 2$ and either $\tilde{\lambda}$ or $\tilde{\mu}$ has width and length at least 3. Therefore the claim follows from the result for fat hooks.

Having taken care of the exceptional cases, we now turn our attention to the generic case. By our inductive assumption, we have that one of δ' and δ'' is a rectangle and the other is a proper partition, up to rotation. Note that this covers all the pairs δ' and δ'' in Theorems 1.3 and 2.2. We let $\tilde\lambda$ and $\tilde\mu$ denote the partitions obtained by removing all row and columns common to both λ and μ.

We first cover the simplest case in which δ' and δ'' are both rectangles (and one may be linear). In this case, we remove all rows and columns common to λ and μ to obtain a pair of partitions $\tilde\lambda \neq \tilde\mu$ which are both fat hooks and do not give a pair on our list; the result follows.

We now assume that one of δ' and δ'' is a rectangle and the other is a proper non-rectangular partition up to rotation. If one of δ' and δ'' is a rectangle and the other is obtained by rotating a proper non-rectangular partition, then $\tilde\lambda$ is necessarily a proper fat hook and $\tilde\mu$ is either a proper fat hook or $|\mathrm{rem}(\tilde\mu)| > 2$, and the result follows. In the non-rotated case, $\tilde\mu$ is necessarily a fat hook and $|\mathrm{rem}(\tilde\lambda)| = |\mathrm{rem}(\delta')| + |\mathrm{rem}(\delta'')| \geqslant 2 + 1 = 3$, and the result follows. □

Lemma 8.6. *If δ is a proper skew partition, then $g(\lambda, \mu) > 1$.*

Proof. By Theorem 1.3 and Lemmas 8.4 and 8.5, it only remains to check the case where $\gamma = (a^b)$, for $a, b > 1$, and $[\delta] = [\delta'] \boxtimes [\delta'']$ with one of δ', δ'' being (1) and the other linear. We remove all rows and columns common to both λ and μ to obtain a pair or partitions $(\tilde\lambda, \tilde\mu)$.

We can assume without loss of generality that γ appears below δ' and δ'' or between δ' and δ''. In the former case, $\tilde\mu$ is a proper fat hook and $\tilde\lambda \supseteq (2^2)$; the result follows. In the latter case $\tilde\mu$ is a fat rectangle and $\tilde\lambda$ is a partition satisfying $|\mathrm{rem}(\tilde\lambda)| = 2$ and $\ell(\tilde\lambda), w(\tilde\lambda) \geqslant 4$; therefore the result holds. □

Lemma 8.7. *If either γ or δ is a rectangular partition, then $g(\lambda, \mu) > 1$.*

Proof. Given the previous results, we suppose without loss of generality that γ is a non-linear rectangle and δ is a non-linear fat hook up to rotation. We assume without loss of generality that γ appears above δ, as in Figure 8.

There are numerous exceptional small cases, however we do not need to list them all. Instead, we shall show that if there is a row or column R_1, R_2, C_1, or C_2 as in Figure 8 then the product contains multiplicities. If there is no such row or column in the diagram for λ and μ then if δ (respectively δ^{rot}) is a proper partition, then it follows that μ (respectively λ) is a rectangular partition (recall that λ, μ are not 2-line partitions, so the pair $(3^4), (6^2)$ does not occur), and so we are done.

Now suppose that the diagram has a row or column R_1, R_2, C_1, or C_2 and we let $(\tilde\lambda, \tilde\mu)$ denote the pair obtained by removing all common rows and columns except this single row or column (in each of the four cases); we now show that the product contains multiplicities.

If there is a row, R_1, in the diagram for λ and μ, then $\tilde\mu$ is either a proper fat hook or a fat rectangle and $\tilde\lambda$ is such that $w(\tilde\lambda), \ell(\tilde\lambda) \geqslant 3$. If there is a column, C_1, then $\tilde\mu$ is either a proper fat hook or $|\mathrm{rem}(\tilde\mu)| = 3$, and $\tilde\lambda$ is either a proper fat hook or a fat rectangle. If there is a row, R_2, and δ is a proper partition (respectively δ^{rot} is a proper partition and δ is not), then $\tilde\mu$ is a proper fat hook (respectively $|\mathrm{rem}(\tilde\mu)| = 3$) and $\tilde\lambda \supseteq (2^2)$ (respectively $\tilde\lambda$ is a rectangular partition). If there is a column, C_2, and δ is a proper partition (respectively δ^{rot} is a proper partition and δ is not) then $\tilde\mu$ is a non-linear rectangle (respectively is a proper fat hook) and in either case $\tilde\lambda$ is either a proper fat hook or $|\mathrm{rem}(\tilde\lambda)| \geqslant 3$. In each of these cases, the result follows by the result for rectangles and fat hooks. □

Lemma 8.8. *If γ and δ are both equal to $(k+1, k)$ up to conjugation and rotation, then $g(\lambda, \mu) > 1$.*

Proof. Without loss of generality we can reduce to three cases:

(*i*) γ^{rot} and δ^{rot} are both proper partitions;
(*ii*) γ and δ^{rot} are both proper partitions;
(*iii*) γ and δ are both proper partitions.

In case (i), we remove all rows and columns to obtain $\tilde{\lambda}$ and $\tilde{\mu}$ a pair of proper fat hooks and the result follows. In case (ii), we remove all rows and columns to obtain $\tilde{\lambda}$ a rectangular partition and $\tilde{\mu}$ such that $|\mathrm{rem}(\tilde{\mu})| = 3$, the result follows.

In case (iii), we first deal with the exceptional case, in which $\ell(\gamma) = w(\delta) = 2$. We remove all but one row or column R_1, R_2, C_1, C_2 to obtain a pair $(\tilde{\lambda}, \tilde{\mu})$. In the case of C_1 (respectively R_1) the partition $\tilde{\lambda}$ (respectively $\tilde{\mu}$) is a proper fat hook and $\tilde{\mu}$ (respectively $\tilde{\lambda}$) has 3 removable nodes, the result follows. In the case of C_2 (respectively R_2) the partition $\tilde{\lambda}$ (respectively $\tilde{\mu}$) is a 2-line partition and $\tilde{\mu}$ (respectively $\tilde{\lambda}$) has 3 removable nodes, the result follows. Now suppose $\ell(\gamma) > 2$ and $w(\delta) = 2$, in which case $\tilde{\lambda} = (2^{2k+1}, 1)$ and $\tilde{\mu} = (4^k, 3)$; the result follows from the case for 2-line partitions. If $\ell(\gamma), w(\delta) > 2$, then $(\tilde{\lambda}, \tilde{\mu})$ are a pair of proper fat hooks and the result follows. □

Lemma 8.9. *If up to rotation and conjugation, one of γ and δ is equal to $(k-1, 1)$ and the other is a fat hook, then $g(\lambda, \mu) > 1$.*

Proof. By the previous results, we can assume $k > 3$ and that neither γ nor δ is a rectangle. There are three cases to consider

(i) γ and δ^{rot} are both proper partitions;
(ii) γ^{rot} and δ^{rot} are both proper partitions;
(iii) γ and δ are both proper partitions.

In all three cases, let $(\tilde{\lambda}, \tilde{\mu})$ denote the pair of partitions obtained by removing all rows and columns common to both λ and μ. In case (i), we have that $\tilde{\lambda}$ is a rectangular partition and $\tilde{\mu}$ is such that $|\mathrm{rem}(\tilde{\lambda})| \geqslant 3$. In case (ii) we have that $(\tilde{\lambda}, \tilde{\mu})$ is a pair of proper fat hooks. In case (iii), we have that $|\mathrm{rem}(\tilde{\lambda})| = |\mathrm{rem}(\tilde{\mu})| = 2$ and the pair is not on the list of Theorem 1.1. The result follows. □

In summary, we have now proved

Corollary 8.10. *If λ and μ is a pair of partitions which does not belong to the list in Theorem 1.1, then $g(\lambda, \mu) > 1$.*

Hence the proof of Theorem 1.1 and thus also the proofs of Theorem 1.2 and Theorem 1.3 are now complete.

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