Robust Stabilisation of T-S Fuzzy Stochastic Descriptor Systems via Integral Sliding Modes

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***Abstract*—This paper addresses the robust stabilisation problem for T-S fuzzy stochastic descriptor systems using an integral sliding mode control paradigm. A classical integral sliding mode control scheme and a non-parallel distributed compensation (Non-PDC) integral sliding mode control scheme are presented. It is shown that two restrictive assumptions previously adopted developing sliding mode controllers for T-S fuzzy stochastic systems are not required with the proposed framework. A unified framework for sliding mode control of T-S fuzzy systems is formulated. The proposed Non-PDC integral sliding mode control scheme encompasses existing schemes when the previously imposed assumptions hold. Stability of the sliding motion is analysed and the sliding mode controller is parameterised in terms of the solutions of a set of linear matrix inequalities (LMIs) which facilitates design. The methodology is applied to an inverted pendulum model to validate the effectiveness of the results presented.**

***Index Terms*—T-S fuzzy stochastic descriptor systems, integral sliding mode control, robust stabilisation, non-parallel distributed compensation (Non-PDC), inverted pendulum.**

I. Introduction

The DESCRIPTOR system representation is an established approach to fully characterize physical systems and research on linear descriptor systems is mature [1]-[2]. Practically, many complex physical models, such as constrained mechanical systems, bio-economic singular systems, robotic systems, show nonlinear features. Although the nonlinear descriptor system can be linearized at a certain operating point so that linear theory can be applied, the resulting analysis and synthesis results are only local and may not be satisfactory. This motivates considering the original nonlinear descriptor system directly for the purpose of design. Recently, detailed qualitative analysis and control methods for several classes of singular biological system have been developed [3]. However, for general nonlinear descriptor systems, the methodology is laborious and it is difficult to derive global stability conditions. In 1985, Takagi and Sugeno presented the well-known T-S fuzzy model [4], which can represent exactly a nonlinear model in a compact set of the state space. One advantage of representing a nonlinear system by a T-S fuzzy model is that existing results on linear systems can be utilized. Early results on stability and stabilisation are frequently based on a common quadratic Lyapunov function which inevitably introduces conservatism. With the objective of decreasing this conservatism, several different classes of non-quadratic Lyapunov functions have been explored where piecewise Lyapunov functions [5], fuzzy Lyapunov functions [6] and line-integral Lyapunov functions [7] are the most typical. Parallel distributed compensation (PDC) is the classical control approach adopted for T-S fuzzy systems whereby the controller shares the same fuzzy inference rules with the controlled plant. However, when a non-quadratic Lyapunov function together is used in conjunction with the PDC control scheme, the solution to a set of bilinear matrix inequalities is often required. In addition, conservatism will always exist. For this reason, non-parallel distributed compensation (Non-PDC) is proposed in [8] and combined with a non-quadratic Lyapunov function to show the superiority of the approach when compared to PDC. Fuzzy controller designs for T-S fuzzy systems have been developed for both PDC and Non-PDC where [5], [9] provide a complete review of T-S fuzzy systems. As nonlinear descriptor systems are often encountered in the real world, stabilisation of T-S fuzzy descriptor systems has been considered [4]. Subsequently investigations on T-S fuzzy descriptor systems have attracted increasing attention from the control community [10]-[12]. Stochastic phenomena are known to arise in many branches of science and engineering [13]. This motivates introducing stochastic characteristics into the model representation. In recent years, many results have been reported on the study of T-S fuzzy stochastic descriptor systems, including passivity and passification [14], filtering [15], observer-based control [16] and guaranteed cost control [17]. Notice that in practice within control systems there always exist unknown disturbances and parameter uncertainties which increase the complexity of the system. It follows that the design of a suitably robust control to tolerate or attenuate disturbances is pertinent.

Sliding mode control is widely established as an effective robust control strategy in both theoretical research and practical applications [18]-[20]. One of its superior features is the insensitivity to parameter variations and disturbances arising in the same channel as the control input. The essence of sliding mode control is to design a suitably high-speed switching control law such that the resultant closed-loop system is attracted to a user-defined sliding surface in finite time and remains there for all subsequent time. Increasing attention

This work was supported by the National Natural Science Foundation of China under Grant 61273008 and 61673099.

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has been paid to the sliding mode control problem for T-S fuzzy descriptor systems [21]-[23], stochastic descriptor systems [24]-[25] and T-S fuzzy stochastic normal systems ($E = I$) [26]-[27]. However, sliding mode control for T-S fuzzy stochastic descriptor systems has not yet been studied and this provides motivation for this paper. In addition, as demonstrated in [27], there exist two restrictive assumptions for the development of sliding mode controllers for T-S fuzzy stochastic systems: the input matrices of each linear subsystem of the T-S fuzzy system are forced to be equal and the product of a parameter matrix in the sliding variable and the diffusion matrices of each linear subsystem must be zero. Without these two assumptions, an effective sliding mode control method for T-S fuzzy stochastic normal system with parameter uncertainties has been developed by introducing the state and input vectors into the sliding variable [27]. However, it is difficult to apply this method to counteract unknown disturbances which occur in the input channel for T-S fuzzy stochastic normal systems and the direct extension of the results to T-S fuzzy stochastic descriptor systems is problematic. As a consequence, removing these two assumptions completely and designing a suitable sliding mode control scheme for a T-S fuzzy stochastic descriptor system with unknown input disturbances is the second motivation for this paper.

In this paper, the robust stabilisation problem for T-S fuzzy stochastic descriptor systems is studied using an integral sliding mode control approach. Firstly, two novel integral sliding surfaces are constructed and the stability of the corresponding sliding motion is analysed. The design parameter matrices defining the sliding variable are obtained by solving LMIs. A classical integral sliding mode controller and a Non-PDC integral sliding mode controller are presented to guarantee that motion on the prescribed sliding surface is maintained. To show the validity of the proposed integral sliding mode method, simulation results of an inverted pendulum system are provided. The contributions of this paper are threefold: 1) the equality of the input matrices of each subsystem and the restrictive assumption on the parameter matrix in the sliding variable and the diffusion matrix of each subsystem are no longer a requirement of the approach; 2) a series of new sliding mode control schemes for T-S fuzzy stochastic systems are presented; 3) descriptor redundancy and property of fuzzy membership functions are exploited to decrease the conservatism.

The rest of this paper is organised as follows. Section II presents the problem description and some essential lemmas. Section III focuses on construction of the sliding surface, stability of the sliding motion, synthesis of a sliding mode controller and comparisons with the existing results. Section IV provides examples to illustrate the effectiveness of the proposed methods and Section V concludes the paper.

Notation: The notation used throughout this paper is quite standard. $\mathbb{R}^n$ represents the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ represents the set of all $m \times n$ real matrices. The superscripts T and -1 denote matrix transposition and matrix inverse respectively. The symbol $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathbb{F}_t\}$ satisfying the usual conditions (i.e. it is right continuous and contains all $\mathbb{P}$-null sets) and $\mathbb{E}\{\cdot\}$ is the expectation operator. $\mathbb{R}^+$ represents the set of positive real numbers. $\|\cdot\|$ denotes the Euclidean norm of a vector or the induced norm of a matrix. $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R})$ denotes the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}^+$ such that they are continuously twice differentiable in x and once in t. $\mathcal{L}^1(\mathbb{R}^+; \mathbb{R}^n)$ and $\mathcal{L}^2(\mathbb{R}^+; \mathbb{R}^{n\times m})$ respectively denote the family of all $\mathbb{R}^n$-valued measurable $\{\mathbb{F}_t\}$-adapted process $f = \{f(t)\}_{t\geq 0}$ and $n \times m$-matrix-valued measurable $\{\mathbb{F}_t\}$-adapted process $g = \{g(t)\}_{t\geq 0}$ such that $\int_0^T \|f(t)\|dt < \infty$ and $\int_0^T \|g(t)\|^2 dt < \infty$ a.s. for every $T > 0$. The notation $P > 0$ ($P \geq 0$) implies that P is a real symmetric and positive definite (semi-positive definite) matrix. For a symmetric matrix A, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum eigenvalue and the maximum eigenvalue of matrix A, respectively. $\mathbf{He}(A)$ stands for $A + A^T$. The star $\star$ in a matrix block implies that it can be induced by symmetric position. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation and Preliminaries

Consider the following T-S fuzzy stochastic descriptor system fixed for the probability space $(\Omega, \mathbb{F}, \mathbb{P})$:
Plant Rule i: IF $z_1(t)$ is $\mathbf{F}_{i1}$, $z_2(t)$ is $\mathbf{F}_{i2}$, $\cdots$, $z_p(t)$ is $\mathbf{F}_{ip}$, **THEN**

$$Edx(t) = [A_i x(t) + B_i (u(t) + w(t))] dt + J_i x(t) d\omega(t) \quad (1)$$

where $i \in \{1, 2, \cdots, r\}$, $z_1(t)$, $z_2(t)$, $\cdots$, $z_p(t)$ are the premise variables, $\mathbf{F}_{i1}$, $\mathbf{F}_{i2}$, $\cdots$, $\mathbf{F}_{ip}$ are the fuzzy sets, and r is the number of IF-THEN rules. $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $w(t) \in \mathbb{R}^m$ is the unknown disturbance which satisfies $\|w(t)\| \leq \bar{w}$. $\omega(t)$ is a one-dimensional Brownian motion defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. E, A_i, B_i, J_i, $i = 1, 2, \cdots, r$ are known real matrices with proper dimensions and matrix E has the property $\mathrm{rank}(E) = r_e \leq n$. Without loss of generality, it is assumed that $\mathrm{rank}[E \;\; J_i] = \mathrm{rank}E$, $i = 1, 2, \cdots, r$.

Based on the centre-average defuzzifier, product inference and the singleton fuzzifier, the overall T-S fuzzy stochastic descriptor system can be inferred as

$$\begin{aligned} Edx(t) = \sum_{i=1}^{r} & h_i(z(t)) \{[A_i x(t) + B_i (u(t) + w(t))] dt \\ & + J_i x(t) d\omega(t)\} \end{aligned} \quad (2)$$

where $z(t) = [z_1(t), z_2(t), \cdots, z_p(t)]$ and $h_i(z(t)) = \frac{\prod_{j=1}^{p} F_{ij}(z_j(t))}{\sum_{i=1}^{r}\prod_{j=1}^{p} F_{ij}(z_j(t))}$ is the normalized membership function with $F_{ij}(z_j(t))$ denoting the membership degrees of $z_j(t)$ in fuzzy set $\mathbf{F}_{ij}$. For all $t \geq 0$, the normalized membership function satisfies $h_i(z(t)) \geq 0$, $i = 1, 2, \cdots, r$, $\sum_{i=1}^{r} h_i(z(t)) = 1$. To ease the notation, in the sequel, $A(h)$ and $B(hh)$ are respectively used to denote the single sum $\sum_{i=1}^{r} h_i(z(t)) A_i$ and double sums $\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t)) h_j(z(t)) B_{ij}$.

Some basic definitions and essential lemmas are first recalled to facilitate development of the main results. To this end, the unforced T-S fuzzy stochastic descriptor system (2) is shown as follows

$$Edx(t) = A(h)x(t)dt + J(h)x(t)d\omega(t) \tag{3}$$

Definition 1: The T-S fuzzy stochastic descriptor system (3) is said to be asymptotically mean square stable if for any initial condition $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} \mathbb{E}\{\|x(t)\|^2\} = 0$.

Lemma 1 [13]: Let $x(t)$ be an n-dimensional Itô process on $t \geq 0$ with the stochastic differential

$$dx(t) = f(t)dt + g(t)d\omega(t)$$

where $f(t) \in \mathcal{L}^1(\mathbb{R}^+;\mathbb{R}^n)$ and $g(t) \in \mathcal{L}^2(\mathbb{R}^+;\mathbb{R}^{n\times m})$. Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+;\mathbb{R})$. Then $V(x(t),t)$ is a real-valued Itô process with its stochastic differential given by

$$\begin{aligned} dV(x(t),t) = &[V_t(x(t),t) + \frac{1}{2}\mathrm{trace}\left(g^T(t)V_{xx}(x(t),t)\,g(t)\right) \\ &+V_x(x(t),t)\,f(t)]dt + V_x(x(t),t)\,g(t)d\omega(t) \end{aligned}$$

Lemma 2 [28]: Suppose a piecewise continuous matrix $A(t) \in \mathbb{R}^{n\times n}$, and a matrix $X \in \mathbb{R}^{n\times n}$ satisfy the following inequality

$$A(t)^T X + X^T A(t) \leq -\alpha I$$

for all t and some positive number α. Then the followings hold:

1) $A(t)$ is invertible,
2) $\|A^{-1}(t)\| \leq a$ for some $a > 0$.

Lemma 3 (Finsler's Lemma) [29]: Let $x \in \mathbb{R}^n$, $\Omega = \Omega^T \in \mathbb{R}^{n\times n}$, $W \in \mathbb{R}^{m\times n}$. The followings are equivalent:

1) $x^T\Omega x < 0$, $\forall\, Wx = 0, x \neq 0$;
2) $\exists X \in \mathbb{R}^{n\times m}$: $\Omega + XW + W^T X^T < 0$.

It should be noted that in the sliding mode control of T-S fuzzy descriptor systems [21]-[22] and stochastic descriptor systems [24], the following assumptions are imposed respectively:

A1: The matrices $B_i,\ i = 1, 2, \cdots, r$ satisfy $B_1 = B_2 = \cdots = B_r = B$;

A2: There exists a matrix S such that $\det(SB_i) \neq 0$ and $SJ_i = 0,\ i = 1, 2, \cdots, r$.

These assumptions are restrictive and limit the applicability of the methods. As will be shown in Section IV, the model describing the balancing of the inverted pendulum on a cart does not satisfy these two assumptions and in this case, existing results [21]-[24] are not applicable. The design of an appropriate sliding mode scheme for T-S fuzzy stochastic descriptor systems without the two assumptions is a main focus of this paper.

III. Main results

First of all, a classical integral sliding mode control scheme is presented to remove the restrictive assumptions A1 and A2 for the T-S fuzzy stochastic descriptor system (1). A Non-PDC integral sliding mode control scheme will then be derived to decrease the conservatism stemming from the selection of the coefficient matrix which defines the sliding surface in the classical integral sliding mode control approach. Finally, comparison with the existing sliding mode control methods is undertaken to show the merits of the proposed method in this paper.

A. Classical Integral Sliding Mode Control Scheme

This subsection is divided into three parts: the first part considers construction of an appropriate sliding surface, the second part focuses on the stability analysis of the motion, and the final part presents the sliding mode controller design method. First consider the construction of the sliding surface.

1) Construction of Sliding Surface: The sliding surface is defined by $s(t) = 0$, where the sliding variable is constructed as follows

$$\begin{aligned} s(t) = &SEx(t) - SEx(0) \\ &- \int_0^t S(A(h) + B(h)K_1)x(\tau)d\tau \end{aligned} \tag{4}$$

where $K_1 \in \mathbb{R}^{m\times n}$ is the coefficient matrix to be determined in the sequel, and $S \in \mathbb{R}^{m\times n}$ is the parameter matrix ensuring the nonsingularity of $SB(h)$. To this end, the method in [30] can be adopted. By defining $B = \frac{1}{r}\sum_{i=1}^r B_i$, it follows that

$$B(h) = B + HF(\bar{h}(z(t)))G \tag{5}$$

where $\bar{h}(z(t)) = [h_1(z(t)), h_2(z(t)), \cdots, h_r(z(t))]$, $H = \frac{1}{2}[B - B_1, B - B_2, \cdots, B - B_r]$, $F(\bar{h}(z(t))) = \mathrm{diag}[(1 - 2h_1(z(t)))I, (1-2h_2(z(t)))I, \cdots, (1-2h_r(z(t)))I]$, and $G = [\underbrace{I, I, \cdots, I}_{r}]^T$. Thus, the following result can be derived by the approach in [30].

Lemma 4 [30]: If the following LMIs

$$\begin{aligned} &\begin{bmatrix} -I & \star \\ f_1 H & -I \end{bmatrix} < 0,\ \begin{bmatrix} Q & \star \\ I & f_2 I \end{bmatrix} > 0,\ Q < f_3 I, \\ &\begin{bmatrix} 2f_1\sqrt{\lambda_{min}(B^T B)} & \star & \star \\ rf_2 & rf_1 & \star \\ rf_3 & 0 & rf_1 \end{bmatrix} > 0 \end{aligned} \tag{6}$$

are solvable for (Q, f_1, f_2, f_3) with $Q > 0$, then there exists parameter matrix $S = \left(B^T Q^{-1} B\right)^{-1} B^T Q^{-1}$ such that $SB(h)$ is nonsingular.

Remark 1: More generally, matrix B can also be chosen as the convex combination of B_i, $i = 1, 2, \cdots, r$, that is, $B = \sum_{i=1}^r \xi_i B_i$ with $\xi_i \geq 0$ and $\sum_{i=1}^r \xi_i = 1$. From the property of convex combinations, it follows that if just one of the matrices B_i is nonsingular, then there must exist a set of scalars ξ_i, $i = 1, 2, \cdots, r$ such that the nonsingularity of B can be guaranteed. In this case, define

$$\mathcal{H} = \frac{1}{2}\left[\ B - r\xi_1 B_1, B - r\xi_2 B_2, \cdots, B - r\xi_r B_r, 2\zeta I\ \right],$$

$$\mathcal{F}(\bar{h}(z(t))) = \mathrm{diag}[(1 - 2h_1(z(t)))I, (1 - 2h_2(z(t)))I, \cdots,$$

$$(1-2h_r(z(t)))I, \frac{1}{\zeta}\sum_{i=1}^r h_i(z(t))(1-r\xi_i)B_i], \mathcal{G} = \left[\ G^T \quad I\ \right]^T$$

where $\zeta = \|\sum_{i=1}^r h_i(z(t))\,(1 - r\xi_i)\,B_i\|$. It can be shown that $B(h) = B + \mathcal{H}\mathcal{F}(\bar{h}(z(t)))\mathcal{G}$. Therefore, the result in Lemma 4 is also applicable with H, G, r replaced by $\mathcal{H}$, $\mathcal{G}$, $r + 1$, respectively.

Remark 2: Note that when $B_1 = B_2 = \cdots = B_r = B$, by choosing $Q = I$ and without solving the LMIs (6), the parameter matrix S can be given as $S = \left(B^T B\right)^{-1} B^T$, since it has been proved in [31] that this set is optimal in the sence that the Euclidean norm of the mismatched disturbances is minimized.

2) Stability of the Sliding Motion: Based on (2) and (4), it can be shown that

$$ds(t) = SB(h)(u(t) + w(t) - K_1x(t))dt + SJ(h)x(t)d\omega(t) \tag{7}$$

In the sliding phase, $E\{s(t)\} = 0$ holds. When the state trajectories of the system (2) reach and are confined to the sliding surface with sliding variable (4), from (7), it is necessary to satisfy

$$(SB(h))(u(t) + w(t) - K_1x(t)) = 0$$

Since $SB(h)$ is nonsingular, the equivalent control can be obtained as

$$u_{eq}(t) = K_1x(t) - w(t) \tag{8}$$

By substituting (8) into the system (2), the sliding mode dynamics are given as follows

$$Edx(t) = A_c(h)x(t)dt + J(h)x(t)d\omega(t) \tag{9}$$

where $A_c(h) = A(h) + B(h)K_1$.

Theorem 1: If the following matrix inequalities

$$\Delta_i = \begin{bmatrix} \Delta_{i1} & \star & \star \\ \Delta_{i2} & -\varepsilon\mathbf{He}(\mathcal{P}_i) & \star \\ \Delta_{i3} & 0 & -P_1 \end{bmatrix} < 0 \tag{10}$$

are solvable for $(P_1, P_{2i}, P_{3i}, X, \Phi_i, Z_1, \varepsilon)$, $i = 1, 2, \cdots, r$ where $P_1 > 0$, $\varepsilon > 0$, $P_{3i} = P_{3i}^T$, E^+ is the pseudoinverse of E in the Moore-Penrose sense, $\Delta_{i1} = \mathbf{He}(A_iX + B_iZ_1)$, $\Delta_{i2} = X - \mathcal{P}_i + \varepsilon(A_iX + B_iZ_1)$, $\Delta_{i3} = \begin{bmatrix} I_{r_e} & 0 \end{bmatrix} N^TE^+J_iX$, $\mathcal{P}_i = N\begin{bmatrix} P_1 & P_{i2}^T \\ P_{i2} & P_{i3} \end{bmatrix} N^TE^T + V\Phi_iU$, orthogonal matrices M and N satisfying $MEN = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{r_e}\} > 0$, $\lambda_1, \lambda_2, \cdots, \lambda_{r_e}$ are the singular values of matrix E, U and V are respectively the last $n - r_e$ rows and the last $n - r_e$ columns of M and N, then the sliding motion (9) is regular, impulse free and asymptotically mean square stable. Furthermore, the coefficient matrix K_1 in (4) can be expressed as $K_1 = Z_1X^{-1}$.

Proof: Suppose that matrix inequalities in (10) are solvable, pre- and post-multiplying Δ_i by $\begin{bmatrix} -\varepsilon I & I & 0 \end{bmatrix}$ and its transpose yield X is invertible. It can be shown that

$$\begin{aligned} \mathcal{P}(h) &= N\begin{bmatrix} P_1 & P_2(h)^T \\ P_2(h) & P_3(h) \end{bmatrix} N^TE^T + V\Phi(h)U \\ &= N\begin{bmatrix} P_1 & 0 \\ P_2(h) & 0 \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} M \\ &\quad + N\begin{bmatrix} 0 & 0 \\ 0 & \Phi(h) \end{bmatrix} M \\ &= N\begin{bmatrix} P_1 & 0 \\ P_2(h) & \Phi(h) \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix} M \end{aligned} \tag{11}$$

Furthermore

$$(\mathcal{P}(h))^{-1} = M^T\begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} P_1^{-1} & 0 \\ \bigstar & (\Phi(h))^{-1} \end{bmatrix} N^T \tag{12}$$

where $\bigstar$ represent terms that are unimportant within the current analysis.

From (12), it follows that

$$\begin{aligned} &E^T(\mathcal{P}(h))^{-1} \\ &= NN^TE^TM^T\begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} P_1^{-1} & 0 \\ \bigstar & \bigstar \end{bmatrix} N^T \\ &= N\begin{bmatrix} I_{r_e} \\ 0 \end{bmatrix} P_1^{-1}\begin{bmatrix} I_{r_e} & 0 \end{bmatrix} N^T = (\mathcal{P}(h))^{-T}E \geq 0 \end{aligned} \tag{13}$$

Summing (10) for all $i = 1, 2, \cdots, r$ and using the Schur complement Lemma, straightforward algebraic manipulation yields

$$\begin{bmatrix} \Delta_4(z(t)) & \star \\ \Delta_2(h) & -\varepsilon\mathbf{He}(P(h)) \end{bmatrix} < 0 \tag{14}$$

where $\Delta_4(z(t)) = \mathbf{He}(A(h)X + B(h)Z_1) + X^TJ(h)^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h)X$.

Pre- and post-multiplying (14) by $\mathrm{diag}\{X^{-T}, (\mathcal{P}(h))^{-1}\}$ and its transpose, the following can be obtained

$$\begin{aligned} &\begin{bmatrix} J(h)^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h) & \star \\ (\mathcal{P}(h))^{-1} & 0 \end{bmatrix} \\ &+ \mathbf{He}(\begin{bmatrix} X^{-T} \\ \varepsilon(\mathcal{P}(h))^{-1} \end{bmatrix}\begin{bmatrix} A_c(h) & -I \end{bmatrix}) < 0 \end{aligned} \tag{15}$$

By Finsler's Lemma, (15) can be guaranteed by the following inequality

$$y^T\begin{bmatrix} J(h)^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h) & \star \\ (\mathcal{P}(h))^{-1} & 0 \end{bmatrix} y < 0 \tag{16}$$

for any $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T \neq 0$ satisfying

$$\begin{bmatrix} A_c(h) & -I \end{bmatrix} y = 0 \tag{17}$$

Substituting (17) into (16), the following can be obtained

$$\begin{aligned} \tilde{\Delta}(z(t)) = &(J(h))^T(E^+)^TE^T(\mathcal{P}(h))^{-1}(E^+)J(h) \\ &+ \mathbf{He}\left((A_c(h))^T(\mathcal{P}(h))^{-1}\right) < 0 \end{aligned} \tag{18}$$

The regularity and absence of impulse in the system (9) can now be proved. Define

$$\begin{aligned} &\tilde{M} = \begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix} M, \ \tilde{M}J_iN = \begin{bmatrix} J_{1i} & J_{2i} \\ 0 & 0 \end{bmatrix} \\ &\tilde{M}A_{ci}N = \begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{bmatrix} \end{aligned} \tag{19}$$

Substituting (19) into (18), it follows that

$$\begin{bmatrix} \bigstar & \bigstar \\ \bigstar & \nabla(z(t)) \end{bmatrix} < 0$$

where $\nabla(z(t)) = (J_2(h))^TP_1^{-1}J_2(h) + \mathbf{He}((A_4(h))^T(\Phi(h))^{-1})$.

Note that $P_1 > 0$ and $\nabla(z(t)) < 0$. Then by Lemma 2, it follows that $A_4(h)$ is nonsingular and $\|(A_4(h))^{-1}\| \leq \rho_1$ with $\rho_1 > 0$. As a result, from [14], the sliding motion (9) is regular and impulse free.

Using the coordinate transformation $x(t) = N\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$, the sliding motion (9) is equivalent to

$$\begin{aligned} d\tilde{x}_1(t) = &[(A_1(h) - A_2(h)(A_4(h))^{-1}A_3(h))\tilde{x}_1(t)dt \\ &+ (J_1(h) - J_2(h)(A_4(h))^{-1}A_3(h))\tilde{x}_1(t)d\omega(t)] \\ \tilde{x}_2(t) = &-(A_4(h))^{-1}A_3(h)\tilde{x}_1(t) \end{aligned}$$

Next, the sliding motion (9) will be shown to be asymptotically mean square stable. Select the Lyapunov function candidate as follows

$$V\left(\tilde{x}_1(t)\right)=\tilde{x}_1^T(t)P_1^{-1}\tilde{x}_1(t)=x^T(t)E^T(\mathcal{P}(h))^{-1}x(t) \quad (20)$$

Let $\mathbb{L}$ be the diffusion operator associated with (20). Then by Lemma 1, it can be shown that

$$\begin{aligned} dV\left(\tilde{x}_1(t)\right)&=x^T(t)\tilde{\Delta}(z(t))x(t)dt\\ &\quad+2x^T(\mathcal{P}(h))^{-T}J(h)x(t)d\omega(t)\end{aligned} \quad (21)$$

Thus, $\mathbb{L}V\left(\tilde{x}_1(t)\right)=x^T(t)\tilde{\Delta}(z(t))x(t)$. From (18), there exists a positive constant ϱ such that

$$\mathbb{L}V\left(\tilde{x}_1(t)\right)<-\varrho\|x(t)\|^2 \quad (22)$$

According to (20)

$$\lambda_{min}(P_1^{-1})\|\tilde{x}_1(t)\|^2\le V(\tilde{x}_1(t))\le\lambda_{max}(P_1^{-1})\|\tilde{x}_1(t)\|^2 \quad (23)$$

Due to $\|(A_4(h))^{-1}\|\le\rho_1$, two positive constants ρ_2 and ρ_3 can be defined satisfying

$$\rho_2\|\tilde{x}_1(t)\|\le\|\tilde{x}_2(t)\|\le\rho_3\|\tilde{x}_1(t)\|$$

which further implies

$$\rho_4\|\tilde{x}_1(t)\|^2\le\|x(t)\|^2\le\rho_5\|\tilde{x}_1(t)\|^2 \quad (24)$$

where $\rho_4=\rho_2^2+1$ and $\rho_5=\rho_3^2+1$.

Using Lemma 1 and (21), it can be calculated that

$$\begin{aligned} d\left[e^{\epsilon t}V\left(\tilde{x}_1(t)\right)\right]&=\epsilon e^{\epsilon t}V\left(\tilde{x}_1(t)\right)dt+e^{\epsilon t}\mathbb{L}V\left(\tilde{x}_1(t)\right)dt\\ &\quad+e^{\epsilon t}V_x\left(\tilde{x}_1(t)\right)J_ix(t)d\omega(t)\end{aligned} \quad (25)$$

Integrating and taking expectations on both sides of (25), it follows that

$$\begin{aligned} e^{\epsilon t}\mathbb{E}\left\{V\left(\tilde{x}_1(t)\right)\right\}&=\mathbb{E}\left\{V\left(\tilde{x}_1(0)\right)\right\}\\ &\quad+\mathbb{E}\int_0^t e^{\epsilon\tau}\mathbb{L}V\left(\tilde{x}_1(\tau)\right)d\tau\\ &\quad+\mathbb{E}\int_0^t e^{\epsilon\tau}\epsilon V\left(\tilde{x}_1(\tau)\right)d\tau\end{aligned} \quad (26)$$

Substituting (22), (23) and (24) into (26), it can be established that

$$\begin{aligned}\mathbb{E}\left\{V\left(\tilde{x}_1(t)\right)\right\}\quad\le\quad& e^{-\epsilon t}\mathbb{E}\left\{V\left(\tilde{x}_1(0)\right)\right\}\\ &+\mathbb{E}\int_0^t\tilde{\epsilon}e^{-\epsilon(t-\tau)}\|\tilde{x}_1(\tau)\|^2d\tau\end{aligned}$$

where $\tilde{\epsilon}=\epsilon\lambda_{max}\left(P_1^{-1}\right)-\varrho\rho_4$.

Assign $0<\epsilon\le\frac{\varrho\rho_4}{\lambda_{max}\left(P_1^{-1}\right)}$, and note (23), then

$$\mathbb{E}\{\|\tilde{x}_1(t)\|^2\}\le\lambda_{max}(P_1)\mathbb{E}\left\{V\left(\tilde{x}_1(0)\right)\right\}e^{-\epsilon t} \quad (27)$$

As t tends to ∞, (27) yields $\lim_{t\to\infty}\mathbb{E}\{\|\tilde{x}_1(t)\|^2\}=0$. By (24), it follows that $\lim_{t\to\infty}\mathbb{E}\{\|x(t)\|^2\}=0$. As a consequence, based on Definition 1, the sliding motion (9) is asymptotically mean square stable. ■

Remark 3: The existence of an asymptotically mean square stable sliding motion (9) is proved in Theorem 1 and the coefficient matrix K_1 in the sliding varible (4) is obtained in terms of a set of matrix inequalities. Due to the redundancy in the derivative coefficient matrix E, some slack matrices P_{i2}, P_{i3}, Φ_i, $i=1,2,\cdots,r$ are introduced and the matrix $E^T(\mathcal{P}(h))^{-1}$ is only dependent on the orthogonal matrix N and positive definite matrix P_1. Moreover, based on the property of fuzzy membership functions, P_1 is set to be independent of the fuzzy membership functions to avoid the derivative of the fuzzy membership functions appearing. Therefore, the conservatism of the common quadratic Lyapunov function is reduced by exploiting the properties of fuzzy membership functions and descriptor redundancy.

3) Design of the Sliding Mode Controller: Theorem 2: Assume that matrices S and K_1 satisfy Lemma 4 and Theorem 1. The sliding mode controller

$$u(t)=K_1x(t)-(SB(h))^{-1}Qs(t)-\varsigma\frac{(SB(h))^T s(t)}{\|(SB(h))^T s(t)\|} \quad (28)$$

can confine the state trajectories of the resultant closed-loop system in a sufficiently small band around the sliding surface with sliding variable (4) if Q is a positive definite matrix and $\varsigma>\bar{w}$ where $\bar{w}$ is defined by the upper bound on the norm of the disturbance $w(t)$.

Proof: Select the Lyapunov function candidate as $\tilde{V}\left(s(t)\right)=\frac{1}{2}s^T(t)s(t)$. By the Itô formula, it follows that

$$\begin{aligned} d\tilde{V}\left(s(t)\right)&=s^T(t)SB(h)\left(u(t)+w(t)-K_1x(t)\right)dt\\ &\quad+x^T(t)\Upsilon(hh)x(t)dt\\ &\quad+s^T(t)SJ(h)x(t)d\omega(t)\\ &=\mathbb{L}\tilde{V}\left(s(t)\right)dt+s^T(t)SJ(h)x(t)d\omega(t)\end{aligned}$$

where $\Upsilon(hh)=\frac{1}{2}(J(h))^TS^TSJ(h)$.

By (28), it can be computed that

$$\begin{aligned}\mathbb{L}\tilde{V}\left(s(t)\right)\le&-\lambda_{min}\left(Q\right)\|s(t)\|^2+\lambda_{max}\left(\Upsilon(hh)\right)\|x(t)\|^2\\ &+\left(\bar{w}-\varsigma\right)\left\|(SB(h))^Ts(t)\right\|\end{aligned} \quad (29)$$

To achieve the sliding mode, the following condition should be satisfied

$$\mathbb{L}\tilde{V}\left(s(t)\right)\le-\zeta\left\|(SB(h))^Ts(t)\right\| \quad (30)$$

where $\zeta>0$. Without loss of generality, ς can be selected to satisfy $\varsigma=\zeta+\bar{w}$.

Combining (29) with (30), (30) holds if the following is satisfied

$$-\lambda_{min}\left(Q\right)\|s(t)\|^2+\lambda_{max}\left(\Upsilon(hh)\right)\|x(t)\|^2\le 0$$

which means that for $\|s(t)\|\ge\sqrt{\frac{\lambda_{max}(\Upsilon(hh))\|x(t)\|^2}{\lambda_{min}(Q)}}$, (30) is true. Similar to [27], [32], define the following small band around the sliding surface

$$\mathcal{B}\left(s(t)\right)=\left\{s(t)\,\middle|\,\|s(t)\|\le\sqrt{\frac{\lambda_{max}\left(\Upsilon(hh)\right)\|x(t)\|^2}{\lambda_{min}\left(Q\right)}}\right\}$$

It can be concluded that the sliding variable remains in the band $\mathcal{B}\left(s(t)\right)$ as in [27], [32]-[34]. It follows directly from theorem 3.1 in [32] that the state trajectories of the resultant closed-loop system are generally not kept on the sliding surface, but will remain in a sufficiently small bounded region surrounding the sliding surface. ■

Remark 4: It should be noted that a term proportional to the sliding variable is introduced into the sliding mode controller (28). This removes the rigorous assumption A2 by defining a small band around the sliding surface as in [27], [32] and it is proved that the sliding variable is restricted to a small neighbourhood of the sliding surface. Note that when the assumption $SJ(h) = 0$ holds, by assigning $Q = 0$, the band $\mathcal{B}(s(t))$ is the sliding surface itself. In this case, the sliding mode controller (28) can maintain the state trajectories of the closed-loop system on the sliding surface.

B. Non-PDC Integral Sliding Mode Control Scheme

It should be noted that despite the tractability of the classical integral sliding mode control scheme presented above, some conservatism may be produced in solving matrix inequalities for the coefficient matrix K_1 since a common matrix K_1 is required to stabilise all the local subsystem (E, A_i, B_i, J_i), $i = 1, 2, \cdots, r$. As a consequence, a Non-PDC integral sliding mode control scheme will be proposed to further reduce this conservatism.

The sliding surface is defined by $s(t) = 0$, where the sliding variable is constructed as follows

$$
\begin{aligned}
s(t) &= SEx(t) - SEx(0) \\
&\quad - \int_0^t S(A(h) + B(h)K_2(h)(Y(h))^{-1})x(\tau)d\tau
\end{aligned} \tag{31}
$$

Here $S \in \mathbb{R}^{m\times n}$ is the same as that in (4) and $K_{2i} \in \mathbb{R}^{m\times n}$, $Y_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, \cdots, r$ are unknown coefficient matrices to be designed later.

Remark 5: The sliding variable in (31) introduces the nonlinear term $K_2(h)(Y(h))^{-1}x(\tau)$ to deal with the case when the coefficient matrix K_1 in (4) cannot be obtained by Theorem 1. In the case that $Y_1 = Y_2 = \cdots = Y_r$, the Non-PDC integral sliding mode control scheme reduces to the PDC integral sliding mode control scheme. Furthermore, when assumption A1 and $Y_1 = Y_2 = \cdots = Y_r$ hold, the sliding variable in (31) can recover the sliding variable presented in [21], [26] or in [22] by incorporating a delay term. In fact, when the matrix S is selected to ensure the invertibility of $SB(h)$, the nonlinear term $K_2(h)(Y(h))^{-1}x(\tau)$ in (31) can be replaced by other stabilising state feedback control laws [35]-[36] applicable for T-S fuzzy stochastic descriptor systems. This observation is similar to that seen for nonlinear normal systems in [37]. As a result, a new framework for the sliding mode control of T-S fuzzy stochastic descriptor system is proposed, even when assumptions A1 and A2 are not satisfied.

As in the previous subsection, the equivalent control law can be obtained as

$$
u_{eq}(t) = K_2(h)(Y(h))^{-1}x(t) - w(t) \tag{32}
$$

By substituting (32) into the system (2), the sliding mode dynamics are given by

$$
\begin{aligned}
Edx(t) &= (A(h) + B(h)K_2(h)(Y(h))^{-1})x(t)dt \\
&\quad + J(h)x(t)d\omega(t)
\end{aligned} \tag{33}
$$

To use the non-quadratic Lyapunov function, the following assumption in [38]-[39] is enforced.

Assumption 1: $\frac{\partial h_i(z(t))}{\partial t} \geq \phi_i$ ($\phi_i \leq 0$) for all $i = 1, 2, \cdots, r$, where ϕ_i, $i = 1, 2, \cdots, r$ are scalars.

Now, the following theorem will provide a method to solve the existence problem of sliding modes and the unknown coefficient matrices can also be obtained.

Theorem 3: If the following matrix inequalities

$$
P_{1i} + X \geq 0,\ i = 1, 2, \cdots, r \tag{34}
$$

$$
\begin{aligned}
&\Theta_{ii} < 0,\ i = 1, 2, \cdots, r \\
&\frac{1}{r-1}\Theta_{ii} + \frac{1}{2}(\Theta_{ij} + \Theta_{ji}) < 0,\ i, j = 1, 2, \cdots, r,\ i \neq j
\end{aligned} \tag{35}
$$

are solvable for $(P_{1i}, Y_i, \Phi_i, K_{2i}, X, P_{2i}, P_{3i}, \eta)$, $i = 1, 2, \cdots, r$ where $P_{1i} > 0$, $\eta > 0$, $P_{3i} = P_{3i}^T$ and

$$
\Theta_{ij} = \begin{bmatrix} \Theta_{ij1} & \star & \star \\ \Theta_{ij2} & -\eta\mathbf{He}(Y_i) & \star \\ \Theta_{ij3} & 0 & -P_{i1} \end{bmatrix}
$$

with $\Theta_{1ij} = \mathbf{He}(A_iY_j + B_iK_{2j}) - \sum_{k=1}^{r}\phi_k(EN\begin{bmatrix} P_{1k}+X & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T)$, $\Theta_{2ij} = \mathcal{P}_i - Y_i + \eta(A_iY_j + B_iK_{2j})^T$, $\Theta_{3ij} = \begin{bmatrix} I_{r_e} & 0 \end{bmatrix}N^TE^+J_i\mathcal{P}_j$, $\mathcal{P}_i = N\begin{bmatrix} P_{1i} & P_{2i}^T \\ P_{2i} & P_{3i}\end{bmatrix}N^TE^T + V\Phi_iU$, orthogonal matrices M and N satisfying $MEN = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{r_e}\} > 0$, $\lambda_1, \lambda_2, \cdots, \lambda_{r_e}$ are the singular values of matrix E, U and V are respectively the last $n - r_e$ rows and the last $n - r_e$ columns of M and N, then the sliding motion (33) is regular, impulse free and asymptotically mean square stable.

Proof: If the matrix inequalities (35) hold, then $\Theta(hh) < 0$. Based on (34) and $\phi_i \leq 0$, pre- and post-multiplying $\Theta(hh)$ by $\begin{bmatrix} -\eta I & I & 0 \end{bmatrix}$ and its transpose yield $\mathcal{P}(h)$ is invertible. It can be verified that

$$
\begin{aligned}
\mathcal{P}(h) &= N\begin{bmatrix} P_1(h) & 0 \\ P_2(h) & 0 \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}M \\
&\quad + N\begin{bmatrix} 0 & 0 \\ 0 & \Phi(h) \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}M \\
&= N\begin{bmatrix} P_1(h) & 0 \\ P_2(h) & \Phi(h) \end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & I \end{bmatrix}M
\end{aligned}
$$

From the invertibility of $\mathcal{P}(h)$, it follows that

$$
(\mathcal{P}(h))^{-1} = M^T\begin{bmatrix} \Lambda^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (P_1(h))^{-1} & 0 \\ \bigstar & (\Phi(h))^{-1} \end{bmatrix}N^T
$$

where $\bigstar$ represent terms that are unimportant within the current analysis.

Furthermore, it can be computed that

$$
\begin{aligned}
&E^T(\mathcal{P}(h))^{-1} = (\mathcal{P}(h))^{-T}E \\
&= N\begin{bmatrix} I_{r_e} \\ 0 \end{bmatrix}(P_1(h))^{-1}\begin{bmatrix} I_{r_e} & 0 \end{bmatrix}N^T \geq 0
\end{aligned} \tag{36}
$$

Since $\frac{\partial h_k(z(t))}{\partial t} \geq \phi_k$ and $\phi_k \leq 0$, it can be obtained that

$$\begin{aligned}
&-\sum_{k=1}^{r}\phi_k(EN\begin{bmatrix} P_{1k}+X & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T) \\
=&\ -\sum_{k=1}^{r}\phi_k(M^T\begin{bmatrix} \Lambda & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} P_{1k}+X & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} \Lambda & 0 \\ 0 & I\end{bmatrix}M) \\
\leq&\ -\sum_{k=1}^{r}\frac{\partial h_k(z(t))}{\partial t}EN\begin{bmatrix} P_{1k} & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T \\
&-\sum_{k=1}^{r}\frac{\partial h_k(z(t))}{\partial t}EN\begin{bmatrix} X & 0 \\ 0 & 0\end{bmatrix}N^TE^T
\end{aligned}$$

Note that $\sum_{k=1}^{r} h_k(z(t)) = 1$, then $\sum_{k=1}^{r}\frac{\partial h_k(z(t))}{\partial t} = 0$, it can be further shown that

$$\begin{aligned}
&-\sum_{k=1}^{r}\phi_k(EN\begin{bmatrix} P_{1k}+X & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T) \\
\leq&\ -\sum_{k=1}^{r}\frac{\partial h_k(z(t))}{\partial t}EN\begin{bmatrix} P_{1k} & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T \\
=&\ -\sum_{k=1}^{r}\frac{\partial h_k(z(t))}{\partial t}E\mathcal{P}_k = -E\frac{\partial}{\partial t}(\mathcal{P}(h))
\end{aligned} \tag{37}$$

Due to $\Theta(hh) < 0$, by using the Schur complement lemma, it follows from (36) and (37) that

$$\begin{aligned}
&\mathbf{He}(\begin{bmatrix} (Y(h))^T \\ \eta(Y(h))^T\end{bmatrix}\begin{bmatrix} A(h)+B(h)K_2(h)(Y(h))^{-1} & -I\end{bmatrix}) \\
&+\begin{bmatrix} \Theta_4(z(t)) & \star \\ \mathcal{P}(h) & 0\end{bmatrix} < 0
\end{aligned} \tag{38}$$

where $\Theta_4(z(t)) = -E\frac{\partial}{\partial t}(\mathcal{P}(h)) + (\mathcal{P}(h))^T(J(h))^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h)\mathcal{P}(h)$.

By Finsler's Lemma, (38) holds if the following is satisfied

$$z^T\begin{bmatrix} \Theta_4(z(t)) & \star \\ \mathcal{P}(h) & 0\end{bmatrix}z < 0 \tag{39}$$

for any $z = \begin{bmatrix} z_1^T & z_2^T\end{bmatrix}^T \neq 0$ satisfies $z_2 = (A(h) + B(h)K_2(h)(Y(h))^{-1})z_1$.

Furthermore, (39) implies that

$$\begin{aligned}
&\mathbf{He}((A(h)+B(h)K_2(h)(Y(h))^{-1})\mathcal{P}(h)) \\
&+(\mathcal{P}(h))^T(J(h))^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h)\mathcal{P}(h) \\
&-E\frac{\partial}{\partial t}(\mathcal{P}(h)) < 0
\end{aligned} \tag{40}$$

Similar to the proof of Theorem 1, the regularity and absence of impulse of the sliding motion (33) can be proved.

Due to $\mathcal{P}(h)(\mathcal{P}(h))^{-1} = I$, it can be obtained that

$$\frac{\partial}{\partial t}((\mathcal{P}(h))^{-1}) = -(\mathcal{P}(h))^{-1}\frac{\partial}{\partial t}(\mathcal{P}(h))(\mathcal{P}(h))^{-1} \tag{41}$$

Pre- and post-multiplying (40) by $(\mathcal{P}(h))^{-T}$ and its transpose, it follows from (41) that

$$\begin{aligned}
\Xi(z(t)) =&\ \mathbf{He}((A(h)+B(h)K_2(h)(Y(h))^{-1})^T(\mathcal{P}(h))^{-1}) \\
&+(J(h))^T(E^+)^TE^T(\mathcal{P}(h))^{-1}E^+J(h) \\
&-E^T\frac{\partial}{\partial t}((\mathcal{P}(h))^{-1}) < 0
\end{aligned}$$

Define $x(t) = N\begin{bmatrix}\tilde{x}_1(t) \\ \tilde{x}_2(t)\end{bmatrix}$ and choose the following Lyapunov function candidate

$$V(\tilde{x}_1(t)) = \tilde{x}_1^T(t)(P_1(h))^{-1}\tilde{x}_1(t) = x^T(t)E^T(\mathcal{P}(h))^{-1}x(t)$$

Then by Lemma 1, it can be calculated that

$$\mathbb{L}V(\tilde{x}_1(t)) = x^T(t)\Xi(z(t))x(t)$$

The subsequent proof can be directly obtained from that of Theorem 1 and thus is omitted. ■

Remark 6: If the conditions in Theorem 3 are solvable, an ideal sliding mode exists and a set of unknown coefficient matrices $K_{2i},\ Y_i,\ i = 1, 2, \cdots, r$ are obtained. Theorem 3 also provides an approach to solve the state feedback stabilising problems for a T-S fuzzy stochastic descriptor system based on the Non-PDC scheme. Since the non-quadratic Lyapunov function and Non-PDC scheme are used, some slack matrices are introduced, the conditions in Theorem 3 are expected to be less conservative than that in Theorem 1.

The sliding mode controller can be designed using the following result.

Theorem 4: Assume that matrices S and $K_{2i},\ Y_i,\ i = 1, 2, \cdots, r$ satisfy Lemma 4 and Theorem 3. The sliding mode controller

$$\begin{aligned}
u(t) =&\ K_2(h)(Y(h))^{-1}x(t) - (SB(h))^{-1}Qs(t) \\
&-\varsigma\frac{(SB(h))^Ts(t)}{\|(SB(h))^Ts(t)\|}
\end{aligned} \tag{42}$$

can confine the state trajectories of the resultant closed-loop system to a sufficiently small band around the sliding surface with sliding variable (31) if Q is a positive definite matrix and $\varsigma > \bar{w}$ where $\bar{w}$ is defined in Theorem 2.

When the coefficient matrices $Y_1 = Y_2 = \cdots = Y_r$, the sliding variable in (31) degenerates to

$$\begin{aligned}
s(t) =&\ SEx(t) - SEx(0) \\
&-\int_0^t S(A(h)+B(h)K_3(h))x(\tau)d\tau
\end{aligned} \tag{43}$$

Here $S \in \mathbb{R}^{m\times n}$ is the same as that in (4) and $K_{3i} \in \mathbb{R}^{m\times n}$, $i = 1, 2, \cdots, r$ are unknown coefficient matrices to be designed later.

In this case, the sliding mode dynamics are given by

$$Edx(t) = A_{c3}(hh)x(t)dt + J(h)x(t)d\omega(t) \tag{44}$$

where $A_{c3}(hh) = A(h) + B(h)K_3(h)$.

Corollary 1: If (34) and the following matrix inequalities

$$\begin{aligned}
&\digamma_{ii} < 0,\ i = 1, 2, \cdots, r \\
&\frac{1}{r-1}\digamma_{ii} + \frac{1}{2}(\digamma_{ij} + \digamma_{ji}) < 0,\ i, j = 1, 2, \cdots, r,\ i \neq j
\end{aligned}$$

are solvable for $(P_{1i}, Y, \Phi_i, Z_{3i}, X, P_{2i}, P_{3i}, \eta)$, $i = 1, 2, \cdots, r$ where $P_{1i} > 0$, $\eta > 0$, $P_{3i} = P_{3i}^T$ and

$$\digamma_{ij} = \begin{bmatrix} \digamma_{ij1} & \star & \star \\ \digamma_{ij2} & -\eta\mathbf{He}(Y) & \star \\ \digamma_{ij3} & 0 & -P_{i1}\end{bmatrix}$$

with $\Theta_{1ij} = \mathbf{He}(A_iY + B_iZ_{3j}) - \sum_{k=1}^{r}\phi_k(EN\begin{bmatrix} P_{1k}+X & P_{2k}^T \\ P_{2k} & P_{3k}\end{bmatrix}N^TE^T)$, $\Theta_{2ij} =$

$\mathcal{P}_i - Y + \eta(A_i Y + B_i Z_{3j})^T$, $\Theta_{3ij} = \left[\begin{array}{cc} I_{r_e} & 0 \end{array}\right] N^T E^+ J_i \mathcal{P}_j$, $\mathcal{P}_i = N \left[\begin{array}{cc} P_{1i} & P_{2i}^T \\ P_{2i} & P_{3i} \end{array}\right] N^T E^T + V\Phi_i U$, orthogonal matrices M and N satisfying $MEN = \left[\begin{array}{cc} \Lambda & 0 \\ 0 & 0 \end{array}\right]$, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_{r_e}\} > 0$, $\lambda_1, \lambda_2, \cdots, \lambda_{r_e}$ are the singular values of matrix E, U and V are respectively the last $n - r_e$ rows and the last $n - r_e$ columns of M and N, then the sliding motion (44) is regular, impulse free and asymptotically mean square stable. Furthermore, the coefficient matrix K_{3i} in (43) can be expressed as $K_{3i} = Z_{3i} Y^{-1}$.

Remark 7: As pointed out in [11], a logarithmically spaced search $\varepsilon, \eta \in \{10^{-6}, 10^{-5}, \cdots, 10^{6}\}$ is used to avoid optimization technique to search for ε and η. As a result, the conditions in Theorems 1 and 3, Corollary 1 are linear matrix inequalities.

The sliding mode controller can also be synthesized by a similar structure with that in Theorem 4. In this case, the PDC integral sliding mode control scheme can be obtained.

C. Comparison with Existing Sliding Mode Control Methods

Other authors have developed sliding mode control methods for T-S fuzzy normal systems ($E = I$) when each local subsystem does not share the same input matrix [30], [27]. Although such methods are effective for T-S fuzzy normal systems, some restrictions have been observed when the methods are applied to T-S fuzzy descriptor systems. The following discussion clarifies the differences between this existing literature and the method proposed in this paper.

1) Comparison with the Method in [30]:

- C1 the methods in [30] and in this paper are applicable to T-S fuzzy normal systems. The method in [30] requires a rigorous precondition that $(A_i, \frac{1}{r}\sum_{i=1}^{r} B_i)$ is stabilisable. The results presented in this paper have no such requirement;
- C2 the method in [30] is based on the assumption [18] that the system $(A_i, \frac{1}{r}\sum_{i=1}^{r} B_i)$ can be expressed in the regular form $\left(\left[\begin{array}{cc} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{array}\right], \left[\begin{array}{c} 0 \\ B \end{array}\right]\right)$ with $\det(B) \neq 0$ by an appropriate coordinate transformation. The sliding variables then appear as a distinct subsystem which is dependent of the control input. This facilitates the sliding mode design and the transformation to regular form is straightforward for any T-S fuzzy normal system where $\sum_{i=1}^{r} B_i$ is full column rank. However, due to the existence of the derivative coefficient matrix E, it can be difficult to express the T-S fuzzy descriptor system $(E, A_i, \frac{1}{r}\sum_{i=1}^{r} B_i)$ in regular form.

2) Comparison with the Method in [27]:

- C3 Although the method in [27] provides a very effective solution of the sliding mode control problem for T-S fuzzy systems with parameter uncertainties, when the system is subject to unknown matched nonlinearities or disturbances, the method is not applicable [27]. The method in this paper can be used;
- C4 When the method in [27] is applied to T-S fuzzy descriptor systems, it is required that each descriptor subsystem (E, A_i) is impulse free in order to determine the unknown coefficient matrices in the sliding variable. This restriction is not needed in this paper.

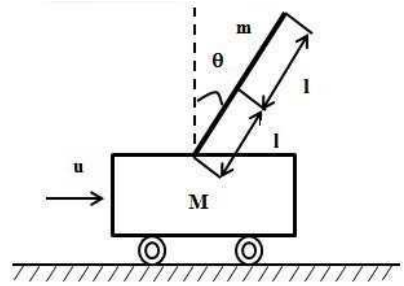

Fig. 1. Inverted pendulum on a cart

IV. Examples

In this section, three examples are considered to show the applicability and effectiveness of the results proposed in this paper. Example 1 is used to validate statements C2 and C3 in Subsection III-C and to show that the proposed method can be used to stabilize a T-S fuzzy stochastic descriptor system which does not satisfy A1 and A2. Example 2 compares the solvability of classical, PDC and Non-PDC integral sliding mode control schemes and also justifies the statement C4 in Subsection III-C. Example 3 is given to verify the statement C1 in Subsection III-C. In the simulation, the unit vector $\frac{s(t)}{\|s(t)\|}$ is replaced by $\frac{s(t)}{\|s(t)\|+0.005}$ as in [18].

Example 1: Consider the problem of balancing the inverted pendulum on a cart as shown in Fig. 1, where the pivot of the pendulum is mounted on the cart and the cart can move in a horizontal direction. By referring to [40]-[41], the state equation of the dynamic model is represented by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{kmlx_4\cos x_1 + (M+m)mgx_5}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\
&\quad - \frac{ml\cos x_1}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1}(u + mx_2^2x_5) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{-k(J+ml^2)x_4 - m^2lgx_5\cos x_1}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\
&\quad + \frac{J+ml^2}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1}(u + mx_2^2x_5) \\
0 &= l\sin x_1 - x_5
\end{aligned}$$

where x_1 is the angular rotation of the pendulum(measured clockwise); x_3 is the displacement of the pivot; x_5 is the the horizontal position of the pendulum centre relative to the pivot; m is the mass of the pendulum; M is the mass of the cart; l is the distance from the centre of gravity to the pivot; J is the moment of inertia of the pendulum with respect to the centre of gravity; k is a viscous damping coefficient; g is the acceleration due to gravity; $u(t)$ is the horizontal force exerted on the cart.

It is well known that the viscous damping coefficient is closely related to the shape of the cart and the air viscosity,

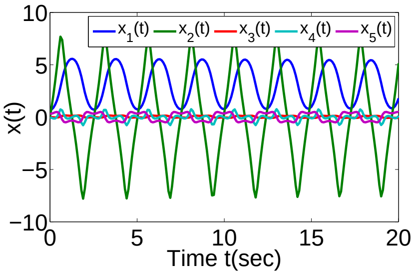

Fig. 2. Time responses of the unforce system (45)

and the air viscosity varies with changes to external environmental factors such as air density, wind, dryness and humidity, temperature and so forth. These environmental factors often feature random variation, which produces a stochastic fluctuation of the damping coefficient and this motivates considering stochastic noise in the environment within the model. Here it is assumed that the damping coefficient is subjected to white noise which is known as the derivative of Brownian motion. The damping coefficient is replaced by

$$k \longrightarrow k + \sigma\dot{\omega}$$

where ω is a one dimensional Brownian motion defined on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$. In addition, unknown disturbances may arise in the control input channel. As a result, the dynamics of the inverted pendulum on a cart are described by

$$\begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= \frac{(kmlx_4\cos x_1 + (M+m)mgx_5)\,dt}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\ &\quad - \frac{ml\cos x_1\left(u + mx_2^2x_5 + w\right)dt - \sigma mlx_4\cos x_1 d\omega}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\ dx_3 &= x_4 dt \\ dx_4 &= \frac{\left(-k(J+ml^2)x_4 - m^2lgx_5\cos x_1\right)dt}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\ &\quad + \frac{(J+ml^2)\left(u + mx_2^2x_5 + w\right)dt - \sigma(J+ml^2)x_4 d\omega}{(M+m)(J+ml^2) - m^2l^2\cos^2 x_1} \\ 0 &= \left[l\sin x_1 - x_5\right]dt \end{aligned} \tag{45}$$

where $w \in \mathbb{R}$ denotes an unknown disturbance or parameter variation. Taking $M = 8kg$, $m = 2kg$, $g = 9.8m/s^2$, $l = 0.5m$, $k = 0.5$, $\sigma = 0.1$.

The time responses of the open-loop system (45) are shown in Fig. 2, which shows that the unforced system (45) is unstable and oscillatory. Although the integral sliding mode control method in [37] may be generalized to a nonlinear stochastic descriptor system, it is required that there exists a nominal controller to stabilize the nominal nonlinear system. It should be noted that it may not be straightforward to find a nominal controller to stabilize the nonlinear stochastic descriptor system (45). This fact is true especially for complex nonlinear systems. In the sequel, it will be shown that it is convenient to apply the proposed fuzzy integral sliding mode control methods in this paper to stabilize the nonlinear stochastic descriptor system (45).

Define $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) & x_5(t) \end{bmatrix}^T$ and a compact set $\Omega = \{x(t) : \ |x_i(t)| \le \xi_i, \ i = 1, 2, \cdots, 5\}$ where $\xi_1 = \frac{5\pi}{18}$ and $\xi_2, \ \xi_3, \ \xi_4, \ \xi_5$ are appropriate positive constants. By sector nonlinear approach [4], the inverted pendulum system (45) can be represented in the compact set Ω by the following T-S fuzzy model:

$$\begin{aligned} Edx(t) = &\sum_{i=1}^{8} h_i\left(x_1(t)\right)\{[A_ix(t) + B_i(u(t) \\ &+ 2x_2^2(t)x_5(t) + w(t))]dt + J_ix(t)d\omega\} \end{aligned} \tag{46}$$

where the premise variables are $z_1(t) = \cos(x_1(t))$, $z_2(t) = \frac{1}{2-0.3\cos^2(x_1(t))}$ and $z_3(t) = \sin(x_1(t))$. The membership functions are $h_i(x_1(t)) = t_j(x_1(t))v_k(x_1(t))\mu_l(x_1(t))$, $i = l+2(k-1)+4(j-1)$, $j,k,l = 1,2$ with $t_1(x_1(t)) = \frac{z_1(t)-a_2}{a_1-a_2}$, $v_1(x_1(t)) = \frac{z_2(t)-b_2}{b_1-b_2}$, $\mu_1(x_1(t)) = \frac{z_3(t)-c_2\arcsin(z_3(t))}{(c_1-c_2)\arcsin(z_3(t))}$, $t_2(x_1(t)) = 1 - t_1(x_1(t))$, $v_2(x_1(t)) = 1 - v_1(x_1(t))$, $\mu_2(x_1(t)) = 1 - \mu_1(x_1(t))$, $a_1 = 1$, $a_2 = \cos(\xi_1)$, $b_1 = \frac{1}{1.7}$, $b_2 = \frac{1}{2-0.3\cos^2(\xi_1)}$, $c_1 = 1$, $c_2 = \frac{\sin(\xi_1)}{\xi_1}$. The matched disturbance is $w(t) = 0.5\sin(t)$. The coefficient matrices in system (46) are

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_i = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.15a_jb_k & 58.8b_k \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.1b_k & -5.88a_jb_k \\ 0.5c_l & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$J_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.03a_jb_k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.02b_k & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ -0.3a_jb_k \\ 0 \\ 0.2b_k \\ 0 \end{bmatrix}$$

where $i = l + 2(k-1) + 4(j-1)$, $j,k,l = 1,2$.

It should be noted that the methods in [21]-[24] cannot be applied, since $B_1 \ne B_3$ and there does not exist a matrix S such that $\det(SB_i) \ne 0$ and $SJ_i = 0$, $i = 1, 2, \cdots, 8$. It is noted that the regular form in [30] can not be obtained for the T-S fuzzy stochastic descriptor system (46) Therefore the sliding mode control method in [30] can not be applied, which validates the statement C2. Next, a classical integral sliding mode control scheme and a PDC integral sliding mode control scheme will be designed.

Classical integral sliding mode control scheme: By applying Lemma 4 and Theorem 1 with $\varepsilon = 0.1$, the coefficient matrices defining the sliding variable are obtained as

$$S = \begin{bmatrix} 0 & -4.3638 & 0 & 3.5418 & 0 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 245.0219 & 99.0627 & 10.7313 & 37.4414 & 302.9810 \end{bmatrix}$$

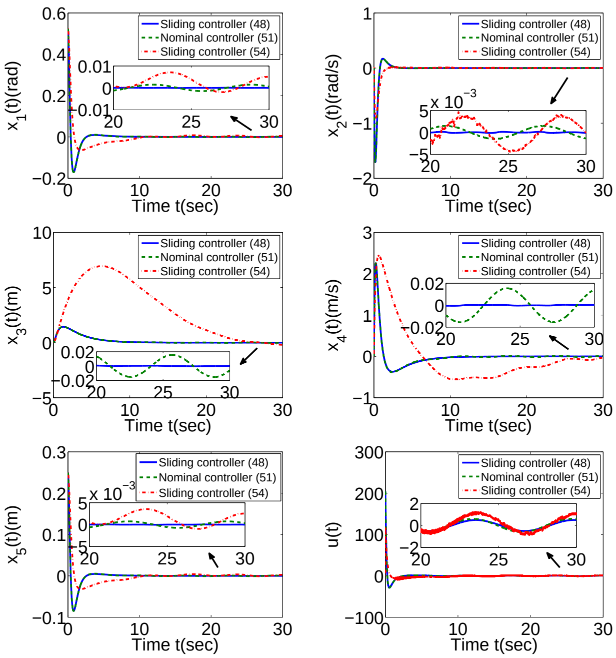

Fig. 3. Time responses of the system (45) using the classical integral sliding mode controller (48), classical nominal controller (51) and integral sliding mode controller (54)

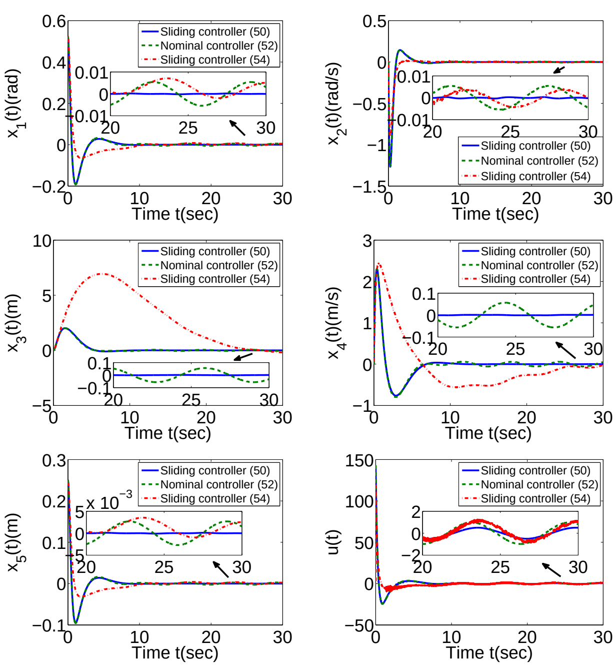

Fig. 4. Time responses of the system (45) using the PDC integral sliding mode controller (50), PDC nominal controller (52) and integral sliding mode controller (54)

The sliding variable is given by

$$s(t) = -4.3638x_2(t) + 3.5418x_4(t) - \int_0^t \sum_{i=1}^{8} h_i(x_1(\tau))S(A_i + B_iK_1)x(\tau)d\tau \quad (47)$$

By Theorem 2, the classical integral sliding mode controller can be obtained as follows

$$u(t) = -2x_2^2(t)x_5(t) + K_1x(t) - 2(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^{-1}s(t) - 0.1\frac{(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^T s(t)}{\|(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^T s(t)\|} \quad (48)$$

PDC integral sliding mode control scheme: Take $\eta = 1$, $\phi_i = -1000$, $i = 1, 2, \cdots, 8$. By Corollary 1, the coefficient matrices for the PDC integral sliding mode controller are obtained as

$$K_{31} = \begin{bmatrix} 58.4573 & 50.0312 & 5.1554 & 11.6981 & 326.4954 \end{bmatrix},$$
$$K_{32} = \begin{bmatrix} 57.9005 & 49.5838 & 5.0237 & 11.4594 & 323.5178 \end{bmatrix},$$
$$K_{33} = \begin{bmatrix} 65.5225 & 55.8304 & 5.7554 & 13.0006 & 343.8131 \end{bmatrix},$$
$$K_{34} = \begin{bmatrix} 64.9000 & 55.3832 & 5.6204 & 12.7576 & 340.7738 \end{bmatrix},$$
$$K_{35} = \begin{bmatrix} 96.1546 & 84.5041 & 9.4160 & 22.2747 & 548.4739 \end{bmatrix},$$
$$K_{36} = \begin{bmatrix} 94.8990 & 83.4823 & 9.1935 & 21.7344 & 541.3294 \end{bmatrix},$$
$$K_{37} = \begin{bmatrix} 103.2430 & 90.4962 & 9.9912 & 23.4546 & 554.8338 \end{bmatrix},$$
$$K_{38} = \begin{bmatrix} 102.7955 & 89.6102 & 9.8341 & 23.0596 & 546.9059 \end{bmatrix}.$$

The sliding variable is calculated as

$$s(t) = -4.3638x_2(t) + 3.5418x_4(t) - \int_0^t \sum_{i=1}^{8} h_i(x_1(\tau))S(A_i + B_i\sum_{i=1}^{8} h_i(x_1(\tau))K_{3i})x(\tau)d\tau \quad (49)$$

By (42), the PDC integral sliding mode controller can be obtained as

$$u(t) = -2x_2^2(t)x_5(t) + \sum_{i=1}^{8} h_i(x_1(t))K_{3i}x(t) - 2(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^{-1}s(t) - 0.1\frac{(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^T s(t)}{\|(\sum_{i=1}^{8} h_i(x_1(t))SB_i)^T s(t)\|} \quad (50)$$

Utilizing the classical integral sliding mode control scheme (47)-(48) and the PDC integral sliding mode control scheme (49)-(50), under the initial condition $x(0) = [\frac{\pi}{6}\ 0\ 0\ 0\ 0.25]^T$, the time responses of the resultant closed-loop system, and sliding mode controller are shown in Fig. 3 and Fig. 4. It shows that the resultant closed-loop system is asymptotically mean square stable. It is noted that the simulation results by the classical integral sliding mode control scheme and the PDC integral sliding mode control scheme are similar, whereas, 8 matrix inequalities in Theorem 1 and 72 matrix inequalities in Corollary 1 are needed to be checked to guarantee the existence of sliding mode. Therefore, if the matrix inequalities in Theorem 1 are solvable, the classical integral sliding mode control scheme is more desirable from the numerical aspect.

The above simulations validate the fact that the results in this paper can be applied to T-S fuzzy stochastic descriptor system that does not satisfy assumptions A1 and A2. Although the result in this paper is proposed for the system (46), it can be applied to the original system (45) in that the T-S fuzzy stochastic descriptor system (46) is an exact representation of the system (45) in the compact set Ω. The simulation results show that the proposed sliding mode control schemes are also applicable to the original system (45). The method proposed in this paper uses the integral sliding mode control concept [42] and thus it is possible to ensure the system initially starts close to the sliding surface and remains within a bounded region of the surface for all subsequent time.

In order to show the effect of the disturbance $w(t)$ on the system performance, the classical nominal controller

$$u(t) = -2x_2^2(t)x_5(t) + K_1x(t) \tag{51}$$

and the PDC nominal controller

$$u(t) = -2x_2^2(t)x_5(t) + \sum_{i=1}^{8} h_i(x_1(t))K_{3i}x(t) \tag{52}$$

are also used to control the T-S fuzzy stochastic descriptor system (46). It is noted that when the disturbance $w(t)$ is absent, the nominal controllers (51) and (52) can stabilize the T-S fuzzy stochastic descriptor system (46). The simulation results are shown in Fig. 3 and Fig. 4. It is seen that in the presence of the disturbance $w(t)$, the nominal controllers (51) and (52) can no longer stabilize the system (46). This means that the input disturbance degrades the system performance. Example 1 also shows that the proposed integral sliding mode controllers exhibit much better performance than the nominal controllers since the discontinuous term is added to reject the bounded input disturbance.

It should be noted that although the sliding mode control method in [27] can be generalized to control the T-S fuzzy stochastic descriptor system, when the matched disturbance $w(t)$ cannot be expressed as parameter uncertainty, the sliding mode control method [27] can not stabilize the T-S fuzzy stochastic descriptor system (46), which coincides with statement C3 in Subsection III-C. Infact, when the method in [27] is derived for a T-S fuzzy stochastic descriptor system, the sliding variable becomes

$$\begin{aligned} s(t) &= S_xEx(t) - S_xEx(0) + S_uu(t) - S_uu(0) \\ &\quad - S_x\int_0^t (A(h)x(\tau) + B(h)u(\tau))\,d\tau \\ &\quad - S_u\int_0^t (F(h)x(\tau) + G(h)u(\tau))\,d\tau \end{aligned} \tag{53}$$

and the sliding mode controller is

$$du(t) = \left(F(h)x(t) + G(h)u(t) - \eta(t)S_u^{-1}\frac{s(t)}{\|s(t)\|}\right)dt \tag{54}$$

$\eta(t) = \beta\frac{\|J(h)x(t)\|^2}{\|s(t)\|} + \alpha + \|S_xB(h)\|\,\bar{w}$, $\alpha > 0$, $\beta = \frac{1}{2}\lambda_{max}\left(S_x^TS_x\right)$.

With the parameters in Example 1, it can be calculated that
$F_1 = [466246.2973\ 118847.4115\ 813.1039\ 8110.1673\ {-1895.2390}]$, $G_1 = -2014.3961$,
$F_2 = [466338.3969\ 118871.0948\ 813.4386\ 8112.7965\ {-1893.7248}]$, $G_2 = -2014.3766$,
$F_3 = [411256.8052\ 104806.1379\ 696.6064\ 7032.0596\ {-1895.2377}]$, $G_3 = -1826.233$,
$F_4 = [411353.0760\ 104830.8839\ 696.9483\ 7034.7605\ {-1893.7411}]$, $G_4 = -1826.2316$,
$F_5 = [462626.9289\ 117925.0662\ 807.8164\ 8061.6496\ {-1895.2435}]$, $G_5 = -2001.3824$,
$F_6 = [462719.8338\ 117948.9530\ 808.1522\ 8064.2913\ {-1893.7346}]$, $G_6 = -2001.3668$,
$F_7 = [407976.5748\ 103970.2153\ 691.8141\ 6988.0922\ {-1895.2402}]$, $G_7 = -1814.4383$, $\alpha = 0.1$,
$F_8 = [408069.7028\ 103994.1591\ 692.1507\ 6990.7395\ {-1893.7303}]$, $G_8 = -1814.4232$, $S_u = 0.0928$,
$S_x = [-33.8185\ \ -17.5846\ \ -0.1193\ \ -1.2028\ \ -168.6425]$

By using the sliding mode control scheme (53) and (54), the initial condition $x(0) = [\frac{\pi}{6}\ 0\ 0\ 0\ 0.25]^T$, $u(0) = 0$, the time responses of the resultant closed-loop system and the sliding mode controller are show in Fig. 3 and Fig. 4. This shows that the sliding mode controller (54) cannot stabilize the T-S fuzzy stochastic descriptor system (46), which validates statement C3 in Subsection III-C. This means that the sliding mode control method proposed in this manuscript has certain advantages over existing methods [QGao].

Example 2: Consider the nonlinear stochastic descriptor system described by the T-S fuzzy model (2) with the following data

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & a \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, J_1 = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} b & 0 \\ -0.2 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, J_2 = \begin{bmatrix} -0.5 & 0.5 \\ 0 & 0 \end{bmatrix}$$

where a and b are tuning parameters. The membership functions are $h_1(x_1(t)) = \frac{1+\sin(x_1(t))}{2}$, $h_1(x_1(t)) = \frac{1-\sin(x_1(t))}{2}$ and the matched disturbance is $w = 0.05e^{-t}$. For different pairs (a, b) with $a \in [-1, 3]$ and $b \in [-1, 3]$, the solvability of the coefficient matrix K_1 using Theorem 1, coefficient matrices K_{2i}, Y_i, $i = 1, 2$ using Theorem 3 and coefficient matrices K_{3i}, $i = 1, 2$ using Corollary 1 is verified. The result shown in Fig. 5 which reveals that the Non-PDC integral sliding mode control scheme is much less conservative than the classical integral sliding mode control scheme and PDC integral sliding mode control scheme.

When $a = 0$, $b = 3$, the descriptor system (E, A_1) is not impulse free. Although the sliding mode control method in [27] can be generalized to a T-S fuzzy descriptor system, as pointed out in statements C3 and C4 in Subsection III-C, the method is invalid for this example since it is required that each descriptor subsystem is impulse free and the unknown disturbance can be expressed by parameter uncertainty. However, from Lemma 4, the parameter matrix S is computed as

$$S = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$$

Furthermore, the matrix inequalities in (10) in Theorem 1 are found to be infeasible. Nevertheless, using Theorem 3 with $\eta = 0.1$ and $\phi_1 = \phi_2 = -100$, the following coefficient matrices are obtained

$$K_{21} = \begin{bmatrix} -6.1012 & -4.7596 \end{bmatrix},$$

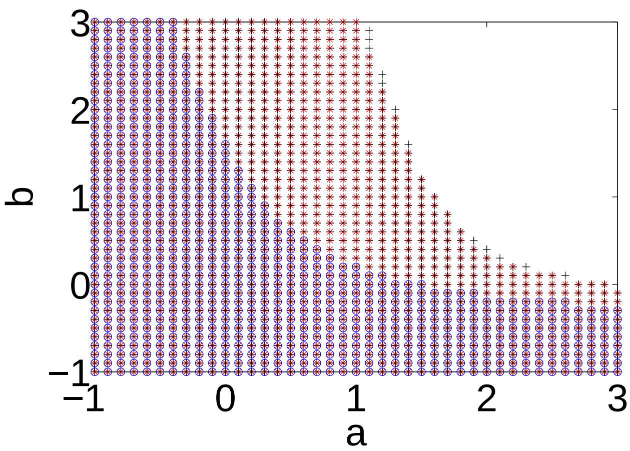

Fig. 5. Solvability of coefficient matricies with Theorem 1(O), Corollary 1(∗) and Theorem 3(+)

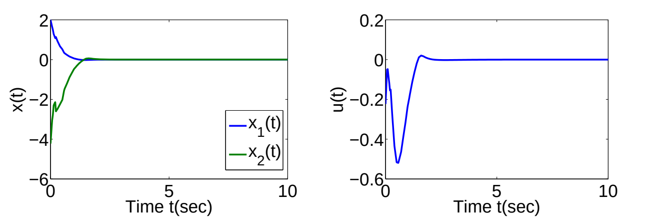

Fig. 6. Time responses of the system (2) under the Non-PDC integral sliding mode controller (55)

$$K_{22} = \begin{bmatrix} -15.9196 & 11.3776 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 7.5602 & -0.2557 \\ -1.7573 & 10.2554 \end{bmatrix}, Y_2 = \begin{bmatrix} 7.5625 & -0.9032 \\ -3.3748 & 19.2265 \end{bmatrix}$$

The Non-PDC integral sliding mode controller can be computed as

$$\begin{aligned} u(t) = & \sum_{i=1}^{2} h_i(x_1(t))K_{2i}(\sum_{i=1}^{2} h_i(x_1(t))Y_i)^{-1}x(t) \\ & - 2(\sum_{i=1}^{2} h_i(x_1(t))SB_i)^{-1}s(t) \\ & - 0.1\frac{(\sum_{i=1}^{2} h_i(x_1(t))SB_i)^T s(t)}{\|(\sum_{i=1}^{2} h_i(x_1(t))SB_i)^T s(t)\|} \end{aligned} \tag{55}$$

Using the Non-PDC integral sliding mode control scheme (55) with the initial condition $x(0) = \begin{bmatrix} 2 & -3.8 \end{bmatrix}^T$, the time responses of the resulting closed-loop system and sliding mode controller are shown in Fig. 6. They are asymptotically mean square stable. Example 2 shows that among the proposed integral sliding mode control schemes, the classical integral sliding mode control scheme is the most conservative and the Non-PDC integral sliding mode control scheme is the least conservative. It also shows that when one of the subsystems of the T-S fuzzy descriptor system is impulse free and the unknown disturbace is matched, as stated in statements C3 and C4 in Subsection III-C, the sliding mode control method in [27] cannot be generalized to the T-S fuzzy descriptor system, but the method proposed in this paper can be used.

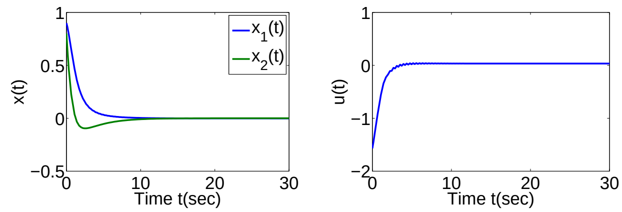

Fig. 7. Time responses of the system (56) under the classical integral sliding mode controller (57)

Example 3: Consider the T-S fuzzy model in the following form

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(x(t))(A_i x(t) + B_i(u(t) + w(t))) \tag{56}$$

where $w(t) = 0.05\sin(x_1(t))$, the membership functions are $h_1(x_1(t)) = \frac{1}{2}(1 + \sin(x_1(t)))$, $h_2(x_1(t)) = \frac{1}{2}(1 - \sin(x_1(t)))$, and the coefficient matrices are given as follows

$$A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0.2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & 0 \\ -0.2 & -1 \end{bmatrix}, B_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

It can be verified that the system (56) is unstable and the method in [30] is not applicable since the pair $\left(A_1, \frac{B_1+B_2}{2}\right)$ cannot be stabilised. This also verifies the statement C1 in Subsection III-C. However, using the integral sliding mode control schemes in this paper, $S = \begin{bmatrix} 1 & 1 \end{bmatrix}$ is selected to guarantee the nonsingularity of $S\sum_{i=1}^{2} h_i(x(t))B_i$. The coefficient matrices in the sliding variable can be solved by Theorems 1 and 3, Corollary 1. Since the classical integral sliding mode control scheme is much easier to be implemented than the Non-PDC integral sliding mode control scheme and PDC integral sliding mode control scheme, only the classical integral sliding mode control scheme is considered here. Using Theorem 1 with $\varepsilon = 1$, it follows that

$$K_1 = \begin{bmatrix} -1.1678 & -0.6487 \end{bmatrix}$$

The classical integral sliding mode controller is obtained as

$$u(t) = -1.1678x_1(t) - 0.6487x_2(t) - 0.1\frac{s(t)}{\|s(t)\|} \tag{57}$$

Under the initial condition $x(0) = \begin{bmatrix} 0.9 & 0.8 \end{bmatrix}^T$, the time responses of the resultant closed-loop system using the classical integral sliding mode controller are shown in Fig. 7. The simulation results show that the resultant closed-loop system is asymptotically mean square stable. Example 3 implies that the proposed integral sliding mode control method does not require the assumption that $\left(A_1, \frac{B_1+B_2}{2}\right)$ and $\left(A_2, \frac{B_1+B_2}{2}\right)$ are stabilisable. This coincides with statement C1 in Subsection III-C.

V. CONCLUSION

This paper has utilized integral sliding mode techniques to prescribe robust stability of T-S fuzzy stochastic descriptor systems. Two restrictive assumptions previously employed in the sliding mode control of stochastic and T-S fuzzy systems have been removed by the proposed classical integral sliding mode control scheme and the Non-PDC integral sliding mode control scheme. In fact, the proposed sliding mode control scheme can be generalized to the more general case as explained in Remark 5. Finally, a few examples including an inverted pendulum model were simulated to support the theoretical results obtained in this paper.

ACKNOWLEDGMENT

The authors would like to express sincere thanks to the Associate Editor and reviewers for their kind help and valuable comments which improved the quality of this paper.

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