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$\mathbb{Z}_N$ graded discrete Lax pairs and Yang-Baxter maps

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Abstract

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems (lattice equations). In this paper we introduce the corresponding Yang-Baxter maps. Many well known examples belong to this scheme for $N = 2$, so, for $N \geq 3$, our systems may be regarded as generalisations of these.

In particular, for each $N$ we introduce a class of multi-component Yang-Baxter maps, which include $H_{III}^0$ (of [6]), when $N = 2$, and that associated with the discrete modified Boussinesq equation, for $N = 3$. For $N \geq 5$ we introduce a new families of Yang-Baxter maps, which have no lower dimensional analogue. We also present new multi-component versions of the Yang-Baxter maps $F_{IV}$ and $F_V$ (given in the classification of [2]).

Keywords: Discrete integrable system, Lax pair, symmetry, Yang-Baxter map.

1 Introduction

The term “Yang-Baxter map” was introduced by Veselov [10] as an abbreviation for Drinfeld’s notion of “set-theoretical solutions to the quantum Yang-Baxter equation”. The basic ingredient is a map $R : X \times X \to X \times X$, where $X$ is some algebraic variety. For the case $X = \mathbb{C}P^1$, these were partially classified in [2, 6]. In [8] a symmetry approach was introduced to relate Yang-Baxter equations with 3D consistent equations on quad-graphs, which had been classified in [1]. Starting with any symmetry of an integrable equation on a quad-graph, the authors introduce invariant functions, which are then used to define a map. The Yang-Baxter relation was shown to be a consequence of 3D consistency. Multi-component Yang-Baxter maps are not yet classified, but several are known (see, for example, [9, 8, 7, 5, 3]).

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems [4]. Many well known examples belong to that scheme for $N = 2$, so, for $N \geq 3$, our systems may be regarded as generalisations of these. As mentioned above, the quad systems for $N = 2$ can be related to Yang-Baxter maps. In this paper we construct generalisations of these, associated with our generalised lattice equations.

In Section 2 we present the basic background theory of Yang-Baxter maps and their relationship to lattice equations on a quadrilateral lattice. In Section 3, we introduce the $\mathbb{Z}_N$-graded Lax pairs of [4] and derive the reduction to Yang-Baxter maps. We show that all such maps are equivalent to ones with “level structure” $(0, \delta; 0, \delta)$. For each $N$ and $\delta$, with $1 \leq \delta \leq \frac{N}{2}$, we present a Yang-Baxter map $R^{(\delta)}(a, b)$ with $2N - 2$ components (see Section 4). For $\delta = 1$, this

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includes the map $H^R_{11}$ of [6], when $N = 2$, and the Yang-Baxter map associated with the discrete modified Boussinesq equation, for $N = 3$. The general map for $\delta = 1$ is known [5], but for $\delta \geq 2$ this is a new class of Yang-Baxter maps. In Section 5 we present a new multi-component generalisation of the Yang-Baxter maps $F_{IV}$ and $F_V$ (given in the classification of [2])

2 Basic Definitions

Let $X$ be an algebraic variety. A parametric Yang-Baxter map $R(a, b)$, depending upon parameters $(a, b)$, is a map

$$R(a, b) : X \times X \to X \times X,$$

satisfying:

$$R_{23}(a_2, a_3) \circ R_{13}(a_1, a_3) \circ R_{12}(a_1, a_2) = R_{12}(a_1, a_2) \circ R_{13}(a_1, a_3) \circ R_{23}(a_2, a_3), \quad (2.1)$$

where $R_{ij}(a_i, a_j)$ is the map that acts as $R(a, b)$ on the $i$ and $j$ factor of $X \times X \times X$, and identically on the other.

**Definition 2.1 (Reversibility)** Let $P$ be the involution given by $P(x, y; a, b) = (y, x; b, a)$. If $P \circ R(a, b)$ is also an involution, then the map $R(a, b)$ is said to be reversible.

**Remark 2.2** An alternative way of writing this is that the map $P \circ R(a, b) \circ P$ is the inverse of $R(a, b)$.

Lax pairs were defined for Yang-Baxter maps in [10, 9]. A matrix $L(x, a)$, with $x \in X$, depending upon the YB parameter $a$ and the spectral parameter $\lambda$ is used to define the equation:

$$L(x', a)L(y', b) = L(y, b)L(x, a). \quad (2.2)$$

It was shown in [10] that if $L$ satisfies this, then the map $(x, y) \mapsto (x', y')$ satisfies the parametric Yang-Baxter equation (2.1) and is reversible.

**Definition 2.3 (The Companion Map)** The companion map $(x, y') \mapsto (x', y)$ is obtained by solving equation (2.2) for the variables $(x', y)$.

2.1 Travelling Wave Reductions of a Lattice Equation

Suppose we have a square lattice with vertices labelled $(m, n)$. At each vertex we have functions

$$u_{m,n} = \left( u^{(0)}_{m,n}, \ldots, u^{(N-1)}_{m,n} \right), \quad v_{m,n} = \left( v^{(0)}_{m,n}, \ldots, v^{(N-1)}_{m,n} \right),$$

and vector function $\Psi_{m,n}$, satisfying

$$\Psi_{m+1,n} = L(u_{m,n}, a) \Psi_{m,n}, \quad \Psi_{m,n+1} = L(v_{m,n}, b) \Psi_{m,n}, \quad (2.3)$$

with compatibility conditions

$$L(u_{m,n+1}, a)L(v_{m,n}, b) = L(v_{m+1,n}, b)L(u_{m,n}, a). \quad (2.4)$$

If we now consider the reduction

$$u_{m,n} = x_p, \quad v_{m,n} = y_{p+1}, \quad \text{where } p = n - m, \quad (2.5)$$

then (2.4) reduces to (2.2), with $x = x_p, x' = x_{p+1}, y = y_p, y' = y_{p+1}$, with the map $(x, y) \mapsto (x', y')$ being Yang-Baxter.

**Remark 2.4** Notice that this does not rely on any underlying Lie point symmetry of the lattice equation. It is just a “travelling wave” solution of the lattice equation.
3 \( \mathbb{Z}_N \)-Graded Lax Pairs

We now consider the specific discrete Lax pairs, which we introduced in [4]. Consider a pair of matrix equations of the form

\[
\Psi_{m+1,n} = L_{m,n} \Psi_{m,n} \equiv \left( U_{m,n} + \lambda \Omega^k \right) \Psi_{m,n}, \quad (3.1a)
\]

\[
\Psi_{m,n+1} = M_{m,n} \Psi_{m,n} \equiv \left( V_{m,n} + \lambda \Omega^\ell \right) \Psi_{m,n}, \quad (3.1b)
\]

where

\[
U_{m,n} = \text{diag} \left( u_{m,n}^{(0)}, \ldots, u_{m,n}^{(N-1)} \right) \Omega^k, \quad V_{m,n} = \text{diag} \left( v_{m,n}^{(0)}, \ldots, v_{m,n}^{(N-1)} \right) \Omega^\ell \quad (3.1c)
\]

and

\[
(\Omega)_{i,j} = \delta_{j-i,1} + \delta_{j-i,N-1}.
\]

The matrix \( \Omega \) defines a grading and the four matrices of (3.1) are said to be of respective levels \( k_i, \ell_i \), with \( \ell_i \neq k_i \) (for each \( i \)). The Lax pair is characterised by the quadruple \((k_1, \ell_1; k_2, \ell_2)\), which we refer to as the level structure of system, and for consistency, we require

\[
k_1 + \ell_2 \equiv k_2 + \ell_1 \pmod{N}. \quad (3.2)
\]

Since matrices \( U, V \) and \( \Omega \) are independent of \( \lambda \), the compatibility condition of (3.1),

\[
L_{m,n+1} M_{m,n} = M_{m+1,n} L_{m,n}, \quad (3.3)
\]

splits into the system

\[
U_{m,n+1} V_{m,n} = V_{m+1,n} U_{m,n}, \quad (3.4a)
\]

\[
U_{m,n+1} \Omega^\ell_2 - \Omega^\ell_2 U_{m,n} = V_{m+1,n} \Omega^k_1 - \Omega^k_1 V_{m,n}, \quad (3.4b)
\]

which can be written explicitly as

\[
u_{m,n+1}^{(i)} \Omega^{(i+k_1)} = \nu_{m+1,n}^{(i)} \nu_{m,n}^{(i+k_2)}, \quad (3.5a)
\]

\[
u_{m,n+1}^{(i)} - \nu_{m,n}^{(i+\ell_2)} = \nu_{m+1,n}^{(i)} - \nu_{m,n}^{(i+\ell_1)}, \quad (3.5b)
\]

or, in a solved form, as

\[
u_{m,n+1}^{(i)} = \frac{\nu_{m,n}^{(i+\ell_2)} - \nu_{m,n}^{(i+\ell_1)}}{\nu_{m,n}^{(i+k_2)} - \nu_{m,n}^{(i+k_1)}} \nu_{m,n}^{(i+k_1)}, \quad \nu_{m+1,n}^{(i)} = \frac{\nu_{m,n}^{(i+\ell_2)} - \nu_{m,n}^{(i+\ell_1)}}{\nu_{m,n}^{(i+k_2)} - \nu_{m,n}^{(i+k_1)}} \nu_{m,n}^{(i+k_1)}, \quad (3.6)
\]

assuming that \( u_{m,n}^{(i)} \neq v_{m,n}^{(j)} \) for all \( i, j \). In all the above formulae, \( i, j \) are taken \pmod{N}.

It is easily seen that the quantities

\[
a = \prod_{i=0}^{N-1} u_{m,n}^{(i)}, \quad b = \prod_{i=0}^{N-1} v_{m,n}^{(i)} \quad \text{satisfy} \quad \Delta_m(a) = \Delta_n(b) = 0, \quad (3.7)
\]

where

\[
\Delta_m = S_m - 1, \quad \Delta_n = S_n - 1, \quad \text{with} \quad S_m f_{m,n} = f_{m+1,n}, \quad S_n f_{m,n} = f_{m,n+1}.
\]
### 3.1 Reduction to Yang-Baxter Maps

We can now employ the reduction (2.5), using (3.7) to replace the components \( x_p^{(N-1)} \), \( y_p^{(N-1)} \). This introduces parameters \( a, b \) into the Lax matrices. If we define

\[
X_p = \text{diag} \left( x_p^{(0)}, \ldots, x_p^{(N-1)} \right), \quad Y_p = \text{diag} \left( y_p^{(0)}, \ldots, y_p^{(N-1)} \right),
\]

then the compatibility condition (3.3) takes the form

\[
(X_{p+1} + \lambda \Omega^k)(Y_{p+1} + \lambda \Omega^k) = (Y_p + \lambda \Omega^k)(X_p + \lambda \Omega^k),
\]

and equations (3.5) take the form

\[
x_p^{(i)} y_p^{(i+k_1)} = y_p^{(i)} x_p^{(i+k_2)}, \quad x_p^{(i)} + y_p^{(i+\ell_1)} = y_p^{(i)} + x_p^{(i+\ell_2)}.
\]

We can write (3.9) as

\[
(X_{p+1} + \lambda \Omega^\delta)(Y_{p+1} + \lambda \Omega^\delta) = (Y_p + \lambda \Omega^\delta)(X_p + \lambda \Omega^\delta),
\]

where \( 0 < \delta \leq N - 1 \), with \( \delta \equiv \ell_i - k_i \pmod{N} \). This allows us to reduce the general case with level structure \((k_1, \ell_1; k_2, \ell_2)\) to that with level structure \((0, \delta; 0, \delta)\). First, note that formula (3.11) can be written

\[
(\bar{X}_{p+1} + \lambda \Omega^\delta)(\bar{Y}_{p+1} + \lambda \Omega^\delta) = (\bar{Y}_p + \lambda \Omega^\delta)(\bar{X}_p + \lambda \Omega^\delta),
\]

where

\[
\bar{X}_p = \text{diag} \left( \bar{x}_p^{(0)}, \ldots, \bar{x}_p^{(N-1)} \right), \quad \bar{Y}_p = \text{diag} \left( \bar{y}_p^{(0)}, \ldots, \bar{y}_p^{(N-1)} \right).
\]

Comparing (3.12) and (3.11), we see that

\[
\bar{x}_p^{(i)} = x_p^{(i)}, \quad \bar{y}_p^{(i)} = y_p^{(i)}, \quad \bar{x}_p^{(i+k_1)}, \quad \bar{y}_p^{(i+k_2)}, \quad \bar{y}_p^{(i)} = y_p^{(i)},
\]

all taken \( \pmod{N} \). We see from (3.12) that the components \( \bar{x}_p^{(i)}, \bar{y}_p^{(i)} \) satisfy

\[
\bar{x}_p^{(i)} \bar{y}_p^{(i)} = \bar{y}_p^{(i)} \bar{x}_p^{(i)}, \quad \bar{x}_p^{(i)} + \bar{y}_p^{(i+\delta)} = \bar{y}_p^{(i)} + \bar{x}_p^{(i+\delta)}.
\]

which are just (3.10) with \((k_i, \ell_i) = (0, \delta)\). We summarise these results in:

**Proposition 3.1** In the Yang-Baxter reduction, all systems with level structure \((k_1, \ell_1; k_2, \ell_2)\), for which \( \ell_i - k_i \equiv \delta \pmod{N} \), are equivalent (up to point transformation) to the system with level structure \((0, \delta; 0, \delta)\).

### 4 The Yang-Baxter Map Corresponding to the Case \((0, \delta; 0, \delta)\)

In this section we consider the Lax equations with level structure \((0, \delta; 0, \delta)\), with \( 0 < \delta \leq N - 1 \). The resulting equations are quadrirational, with both the Yang-Baxter and companion maps being birational. We find that the Yang-Baxter maps corresponding to \( \delta \) and \( N - \delta \) are inverses to each other and that the companion map is periodic, with period \( N \).
4.1 The Equations and Maps

With Lax matrices
\[ L(x, a) = X_p + \lambda \Omega^\delta, \quad L(y, b) = Y_p + \lambda \Omega^\delta, \]
where \( X_p \) and \( Y_p \) are defined by (3.8), with
\[
x_p^{(N-1)} = \frac{a}{\prod_{i=0}^{N-2} x_p^{(i)}}, \quad y_p^{(N-1)} = \frac{b}{\prod_{i=0}^{N-2} y_p^{(i)}},
\]
the Lax equation (2.2) implies
\[
x_{p+1}^{(i)} y_{p+1}^{(i)} = y_p^{(i)} x_p^{(i)}, \quad x_p^{(i)} + y_{p+1}^{(i)} = y_p^{(i)} + x_p^{(i+\delta)}, \quad 0 \leq i \leq N - 1.
\]
Only the formulae with \( 0 \leq i \leq N - 2 \) are independent, but the full set is useful when discussing first integrals.

**Remark 4.1 (Level structure \( (\delta, 0; \delta, 0) \) vs \( (0, \delta; 0, \delta) \))** Under the point transformation
\[ x_p^{(i)} = \tilde{x}_p^{(i+\delta)}, \quad y_p^{(i)} = \tilde{y}_p^{(i+\delta)}, \quad y_p^{(i)} = y_{p+1}^{(i)}, \quad 0 \leq i \leq N - 1,
\]
equations (4.3) take the form
\[ \tilde{x}_{p+1}^{(i)} \tilde{y}_{p+1}^{(i)} = \tilde{y}_p^{(i)} \tilde{x}_p^{(i)}, \quad \tilde{x}_p^{(i)} + \tilde{y}_{p+1}^{(i)} = \tilde{y}_p^{(i)} + \tilde{x}_p^{(i+\delta)}, \quad 0 \leq i \leq N - 1,
\]
which are just the equations for level structure \( (\delta, 0; \delta, 0) \), so these structures are equivalent.

4.1.1 The Yang-Baxter map \( R^{(\delta)}(a, b) \)

Here we solve (4.3) for \( (x_{p+1}^{(i)}, y_{p+1}^{(i)}) \) as functions of \( (x_p^{(i)}, y_p^{(i)}) \) (with \( 0 \leq i \leq N - 2 \) and \( x_p^{(N-1)}, y_p^{(N-1)} \) replaced by (4.2)). We write this map as \( R^{(\delta)}(a, b) \), but when no ambiguity can arise, we suppress the parametric dependence by writing the map as \( R^{(\delta)} \).

Notice that by shifting \( i \mapsto i + N - \delta \equiv i - \delta \) (mod\( N \)), the second part of equation (4.3) takes the form
\[ x_{p+1}^{(i-\delta)} + y_{p+1}^{(i)} = y_{p+1}^{(i-\delta)} + x_p^{(i)}, \]
which leads to:

**Proposition 4.2 (Inverse Map)** The Yang-Baxter map \( R^{(-\delta)}(a, b) \) is just the inverse of the map \( R^{(\delta)}(a, b) \).

This means that we only need to consider \( \delta \leq \frac{N}{2} \) and that, when \( N = 2M \), the map \( R^{(M)}(a, b) \) is an involution.

**Proposition 4.3 (First Integrals)** The Yang-Baxter map \( R^{(\delta)}(a, b) \) has the following \( N \) first integrals:
\[ x_p^{(i)} y_p^{(i)} = c_i, \quad 0 \leq i \leq N - 2, \quad \sum_{i=0}^{N-1} (x_p^{(i+\delta)} + y_p^{(i)}) = c_{N-1}, \]
where, in the latter, \( x_p^{(N-1)} \) and \( y_p^{(N-1)} \) are replaced by (4.2).

The last of these integrals is obtained by summing the additive equations of (4.3).
4.1.2 The Companion Map $\varphi^{(\delta)}$

Here we solve (4.3) for $(x_p^{(i+1)}, y_p^{(i)})$ as functions of $(x_p^{(i)}, y_p^{(i)})$ (with $0 \leq i \leq N - 2$ and $x_p^{(N-1)}, y_p^{(N-1)}$ replaced by (4.2)). Since $p$ is no longer the evolution parameter, we relabel our variables as:

$$(x_p^{(i)}, y_p^{(i)}) = (x_q^{(i)}, y_q^{(i)}), \quad (x_p^{(i+1)}, y_p^{(i)}) = (x_q^{(i+1)}, y_q^{(i+1)}).$$

**Remark 4.4 (A second travelling wave reduction)** This labelling follows directly from the travelling wave reduction

$$u_{m,n} = x_q, \quad v_{m,n} = y_q, \quad \text{where } q = n + m$$

We can re-arrange the quadratic formulae in (4.3) (with this new labelling) to obtain $N - 1$ first integrals:

$$\frac{x_q^{(i)}}{y_q^{(i)}} = c_i, \quad 0 \leq i \leq N - 2. \quad (4.5)$$

We can also re-arrange the linear formulae of (4.3) to obtain

$$x_q^{(i+1)} - y_q^{(i+1)} = x_q^{(i+\delta)} - y_q^{(i+\delta)}, \quad 0 \leq i \leq N - 1. \quad (4.6)$$

If we define

$$f(x, y) = x - y,$$

then

$$f(x_q^{(i)}, y_q^{(i)}) = f(x_q^{(i+\delta)}, y_q^{(i+\delta)}), \quad 0 \leq i \leq N - 1. \quad (4.7)$$

We may use

$$\left(\frac{x_q^{(0)}}{y_q^{(0)}}, \ldots, \frac{x_q^{(N-2)}}{y_q^{(N-2)}}, f\left(x_q^{(0)}, y_q^{(0)}\right), \ldots, f\left(x_q^{(N-2)}, y_q^{(N-2)}\right)\right)$$

as coordinates and, in these coordinates, the map $\varphi^{(\delta)}$ just shifts the coordinates $f(x_q^{(i)}, y_q^{(i)})$ by $\delta$, whilst leaving the coordinates $\frac{x_q^{(i)}}{y_q^{(i)}}$ fixed. This leads to the following:

**Proposition 4.5 (Periodicity)** The map $\varphi^{(\delta)}$ is periodic with period $N$. When $(N, \delta) = 1$ this is the minimum period. Furthermore, we have that $\varphi^{(\delta)} = \varphi^{(1)} \circ \cdots \circ \varphi^{(1)}$ (the $\delta$-fold composition of $\varphi^{(1)}$).

This statement is, of course, independent of coordinates.

**Remark 4.6 ((2N − 2) first integrals)** Any cyclically symmetric function of $f(x_q^{(i)}, y_q^{(i)})$ is a first integral of the companion map, so it possesses $(2N - 2)$ first integrals. The common level set is then finite, corresponding to the periodicity of the map.

4.2 Examples of the map $R^{(\delta)}$

We can build hierarchies of Yang-Baxter maps for each $\delta$. It follows from Proposition 4.2 that we only need to consider $\delta \leq \frac{N}{2}$. However, as the value of $N$ increases, so does the number of different maps $R^{(\delta)}$. We have:
Case $\delta = 1$: At $N = 2$, we only have the case $\delta = 1$, and $R^{(1)}$ is just the map $H_{III}^B$ in the classification of scalar Yang-Baxter maps [6]. The map $R^{(1)}$ exists for all $N \geq 2$, which can therefore be considered as a multi-component generalisation of the scalar Yang-Baxter map $H_{III}^B$.

Case $\delta = 2$: For $N \geq 4$ we have the map $R^{(2)}$. When $N$ is even, this map degenerates to lower dimensional maps (see the case $N = 4$ below), but when $N$ is odd, we have a new sequence of Yang-Baxter maps which fully couple $2N - 2$ variables. The 8-component case can be seen in the case $N = 5$ below.

Case $\delta = 3$: For $N \geq 6$ we have the map $R^{(3)}$, but again, this map degenerates to lower dimensional maps when $N$ is a multiple of 3. The first fully coupled system is at $N = 7$.

Whilst the generalisation of $\delta = 1$ is already known [5], the maps $R^{(\delta)}$, for $\delta \geq 2$, are new classes of Yang-Baxter maps.

4.2.1 When $N = 2$

Here we only have the case $\delta = 1$, which leads to (with $x(0) = x$, $y(0) = y$, $x(1) = a/x$, $y(1) = b/y$)

$$x_{p+1} = y_p \left( \frac{a + xy}{b + xy} \right), \quad y_{p+1} = x_p \left( \frac{b + xy}{a + xy} \right),$$

(4.8)

which (up to a relabelling of parameters) is just the map $H_{III}^B$ in the classification of scalar Yang-Baxter maps [6].

The existence of the two invariant functions (4.4) implies (the well known fact) that this map is an involution.

4.2.2 When $N = 3$

Here we have $\delta = 1$ and $\delta = 2$, but since $N - 1 = 2 \equiv -1 \pmod{3}$, the map $R^{(2)}$ is just the inverse of $R^{(1)}$. In this case $R^{(1)}$ takes the form:

$$x_p^{(i)} = y_p^{(i)} \frac{A^{(i)}}{A^{(i+1)}}, \quad y_p^{(i)} = x_p^{(i)} \frac{A^{(i+1)}}{A^{(i)}} , \quad 0 \leq i \leq 1,$$

(4.9)

with upper indices taken (mod2) and where

$$A^{(0)} = a(x_p^{(1)} + y_p^{(0)}) + x_p^{(0)} y_p^{(0)}, \quad A^{(1)} = y_p^{(0)} + (b - a)x_p^{(1)}, \quad A^{(2)} = A^{(1)} + (b - a)y_p^{(0)}.$$

Remark 4.7 (Discrete Modified Boussinesq Equation) This is equivalent to the Yang-Baxter map derived in [8], associated with the discrete modified Boussinesq equation (see equation (67a-b) of [8]). They are related by a simple point transformation:

$$x(0) \mapsto \frac{c_0}{x^1}, \quad x(1) \mapsto c_0 x^2, \quad y(0) \mapsto \frac{c_0^2}{c_2 y^1}, \quad y(1) \mapsto \frac{c_0^2 y^2}{c_3}, \quad \text{where} \quad c_3 = \frac{\alpha_1}{\alpha_2}.$$

4.2.3 When $N = 4$

For $\delta = 1$: We obtain the 6-component version of (4.9).
For $\delta = 2$: Since $(N, \delta) = 2 \neq 1$, the map is reducible, with a 4-component subsystem:

\[
\begin{align*}
x_{p+1}^{(0)} &= x_{p}^{(0)} \frac{x_{p}^{(0)} x_{p}^{(2)} + y_{p}^{(0)}}{x_{p}^{(0)} + y_{p}^{(2)}}, & x_{p+1}^{(2)} &= x_{p}^{(2)} \frac{x_{p}^{(2)} x_{p}^{(0)} + y_{p}^{(2)}}{x_{p}^{(2)} + y_{p}^{(0)}}, \\
y_{p+1}^{(0)} &= y_{p}^{(0)} \frac{x_{p}^{(0)} x_{p}^{(2)} + y_{p}^{(0)}}{x_{p}^{(0)} + y_{p}^{(2)}}, & y_{p+1}^{(2)} &= y_{p}^{(2)} \frac{x_{p}^{(2)} x_{p}^{(0)} + y_{p}^{(2)}}{x_{p}^{(0)} + y_{p}^{(2)}}. \\
\end{align*}
\]

(4.10)

in which the parameters $(a, b)$ are absent.

The remaining pair of equations are a non-autonomous version of (4.8), with coefficients depending upon $(x_{p}^{(0)}, x_{p}^{(2)}, y_{p}^{(0)}, y_{p}^{(2)})$:

\[
\begin{align*}
x_{p+1}^{(1)} &= x_{p}^{(0)} x_{p}^{(2)} \frac{a + x_{p}^{(0)} x_{p}^{(2)} y_{p}^{(1)}}{x_{p}^{(0)} x_{p}^{(2)} b + y_{p}^{(0)} x_{p}^{(2)} y_{p}^{(1)}}, & y_{p+1}^{(1)} &= y_{p}^{(0)} x_{p}^{(2)} \frac{b + y_{p}^{(0)} y_{p}^{(2)} y_{p}^{(1)}}{a + y_{p}^{(0)} y_{p}^{(2)} y_{p}^{(1)}}, \\
\end{align*}
\]

(4.11)

Notice that this last pair could also be written

\[
\begin{align*}
x_{p+1}^{(1)} &= x_{p}^{(1)} \frac{x_{p}^{(3)} + y_{p}^{(1)}}{x_{p}^{(1)} + y_{p}^{(3)}}, & y_{p+1}^{(1)} &= y_{p}^{(1)} \frac{x_{p}^{(3)} + y_{p}^{(1)}}{x_{p}^{(3)} + y_{p}^{(1)}}, \\
\end{align*}
\]

which, with the constraint (4.2), explains the formulae in (4.11).

The 4-component system (4.10) has 4 independent first integrals

\[
I_1 = x_{p}^{(0)} y_{p}^{(0)}, \quad I_2 = x_{p}^{(2)} y_{p}^{(2)}, \quad I_3 = x_{p}^{(0)} x_{p}^{(2)}, \quad I_4 = x_{p}^{(0)} + x_{p}^{(2)} + y_{p}^{(0)} + y_{p}^{(2)},
\]

so is periodic (and has period 2).

The remaining two equations (4.11) cannot be taken alone, but only as part of the 6-component system. This system has two more first integrals,

\[
I_5 = x_{p}^{(1)} y_{p}^{(1)}, \quad I_6 = x_{p}^{(1)} + \frac{a}{x_{p}^{(0)} x_{p}^{(1)} + y_{p}^{(1)}} + \frac{b}{y_{p}^{(0)} y_{p}^{(1)} y_{p}^{(2)}},
\]

so is also periodic (of period 2). As commented after Proposition 4.2, this involutive property follows from $\delta = N - \delta$ for this case.

**Remark 4.8 (Non-Coprime Case)** This decoupling, when $(N, \delta) \neq 1$, is a general feature.

4.2.4 When $N = 5$

Here $\delta = 1$ and $\delta = 2$ give genuinely different maps.

For $\delta = 1$: The map $R^{(1)}$ takes the same form as (4.9):

\[
\begin{align*}
x_{p+1}^{(i)} &= y_{p}^{(i)} \frac{A^{(i)}}{A^{(i+1)}}, & y_{p+1}^{(i)} &= x_{p}^{(i)} \frac{A^{(i+1)}}{A^{(i)}}, \quad 0 \leq i \leq 3, \\
\end{align*}
\]

(4.12)

with upper indices taken (mod4) and where

\[
\begin{align*}
A^{(0)} &= a(x_{p}^{(1)} x_{p}^{(2)} x_{p}^{(3)} + x_{p}^{(2)} x_{p}^{(3)} y_{p}^{(0)} + x_{p}^{(3)} y_{p}^{(0)} y_{p}^{(1)} + y_{p}^{(0)} y_{p}^{(1)} y_{p}^{(2)}) + \prod_{i=0}^{3} x_{p}^{(i)} y_{p}^{(i)}, \\
A^{(1)} &= A^{(0)} + (b - a)x_{p}^{(1)} x_{p}^{(2)} x_{p}^{(3)}, \quad A^{(2)} = A^{(1)} + (b - a)x_{p}^{(2)} x_{p}^{(3)} y_{p}^{(0)}, \\
A^{(3)} &= A^{(2)} + (b - a)x_{p}^{(3)} y_{p}^{(0)} y_{p}^{(1)}, \quad A^{(4)} = A^{(3)} + (b - a)y_{p}^{(0)} y_{p}^{(1)} y_{p}^{(2)}.
\end{align*}
\]

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For \( \delta = 2 \): The map \( R^{(2)} \) takes the form:

\[
x_{p+1}^{(0)} = y_p^{(0)} \frac{A^{(2)}}{A^{(3)}}, \quad x_{p+1}^{(1)} = y_p^{(1)} \frac{A^{(0)}}{A^{(1)}}, \quad x_{p+1}^{(2)} = y_p^{(2)} \frac{A^{(3)}}{A^{(4)}}, \quad x_{p+1}^{(3)} = y_p^{(3)} \frac{A^{(1)}}{A^{(2)}},
\]

and \( y_{p+1}^{(i)} = \frac{x_p^{(i)} y_p^{(i)}}{x_{p+1}^{(i)}} \), with upper indices taken \((\text{mod}4)\) and where

\[
A^{(0)} = a(x_p^{(3)}x_p^{(0)}x_p^{(2)}x_p^{(0)}x_p^{(2)} + x_p^{(2)}y_p^{(1)}y_p^{(3)} + y_p^{(1)}y_p^{(3)}y_p^{(0)}) + \prod_{i=0}^{3} x_i^{(i)} y_i^{(i)},
\]

\[
A^{(1)} = A^{(0)} + (b-a)x_p^{(3)}y_p^{(1)}y_p^{(3)}, \quad A^{(2)} = A^{(1)} + (b-a)x_p^{(0)}x_p^{(2)}y_p^{(1)},
\]

\[
A^{(3)} = A^{(2)} + (b-a)x_p^{(2)}y_p^{(1)}y_p^{(3)}, \quad A^{(4)} = A^{(3)} + (b-a)y_p^{(1)}y_p^{(3)}y_p^{(0)}.
\]

4.2.5 The Structure of the Formulae

The order of appearance of \( A^{(i)} \) in (4.13) and the combination of variables appearing in the definition of \( A^{(0)} \) is controlled by the following ordering of the variables \( x_p^{(i)}, y_p^{(i)} \):

\[
\{x^{(\delta-1)}, x^{(2\delta-1)}, \ldots, x^{((N-1)\delta-1)}, y^{(\delta-1)}, y^{(2\delta-1)}, \ldots, y^{((N-1)\delta-1)}\}.
\]

When \((N, \delta) = 1\), the numbers \(\{(m\delta - 1)\}_{m=1}^{N-1}\) form a permutation of the numbers \(0, \ldots, N - 2\), so all the variables are included in this list. The formulae (4.13) are just

\[
x_{p+1}^{(m\delta-1)} = y_p^{(m\delta-1)} \frac{A^{(m-1)}}{A^{(m)}}, \quad 1 \leq m \leq N - 1.
\]

The coefficient of the parameter \( a \) in function \( A^{(0)} \) is constructed as follows: the first term is \( \prod_{x^{(\delta-1)}}^{x^{(2\delta-1)}} \). We then repeatedly act by the permutation

\[
x^{(\delta-1)} \rightarrow x^{(2\delta-1)} \rightarrow \ldots \rightarrow x^{((N-1)\delta-1)} \rightarrow y^{(\delta-1)} \rightarrow y^{(2\delta-1)} \rightarrow \ldots \rightarrow y^{((N-1)\delta-1)} \rightarrow x^{(\delta-1)},
\]

for \((N-2)\) times, which ends with \( \prod_{x^{(\delta-1)}}^{y^{((N-1)\delta-1)}} \). The coefficient of \( a \) is then just the sum of these \((N-1)\) terms. The remaining term in \( A^{(0)} \) is just \( \prod_{i=0}^{N-2} x^{(i)} y^{(i)} \).

The functions \( A^{(i)} \) are formed by successively changing the coefficient \( a \) to \( b \) at each of the terms in the above sum.

Example 4.9 (The case \( N = 5, \delta = 2 \)) Here we have

\[
x^{(1)} \rightarrow x^{(3)} \rightarrow x^{(0)} \rightarrow x^{(2)} \rightarrow y^{(1)} \rightarrow y^{(3)} \rightarrow y^{(0)} \rightarrow y^{(2)},
\]

and

\[
x_p^{(3)}x_p^{(0)}x_p^{(2)} \rightarrow x_p^{(0)}x_p^{(2)}y_p^{(1)} \rightarrow x_p^{(2)}y_p^{(1)}y_p^{(3)} \rightarrow y_p^{(1)}y_p^{(3)}y_p^{(0)},
\]

giving the expression for \( A^{(0)} \), given in the case of (4.13).

Example 4.10 (The case \( N = 7, \delta = 3 \)) Here we have

\[
x^{(2)} \rightarrow x^{(5)} \rightarrow x^{(1)} \rightarrow x^{(4)} \rightarrow x^{(0)} \rightarrow x^{(3)} \rightarrow y^{(2)} \rightarrow y^{(5)} \rightarrow y^{(1)} \rightarrow y^{(4)} \rightarrow y^{(0)} \rightarrow y^{(3)},
\]

and

\[
x_p^{(5)}x_p^{(1)}x_p^{(4)}x_p^{(0)}x_p^{(3)} \rightarrow x_p^{(1)}x_p^{(4)}x_p^{(0)}x_p^{(3)}y_p^{(2)} \rightarrow \ldots \rightarrow y_p^{(2)}y_p^{(1)}y_p^{(4)}y_p^{(0)},
\]

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Equations (3.5a) then take the form

\[ A^{(0)} = a(x_p^{(5)} x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} + x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} y_p^{(2)} + \cdots + y_p^{(2)} y_p^{(1)} y_p^{(4)} y_p^{(0)}) + \prod_{i=0}^{5} x_p^{(i)} y_p^{(i)}. \]

The remaining \( A^{(i)} \) are then constructed by the above prescription and the map \( R^{(3)} \), for \( N = 7 \) is given by (4.14), for \( \delta = 3 \).

### 4.3 The Quotient Potential Case and Symmetries

In [4] we introduced two potential forms of our equations (3.5). Here we briefly mention the “quotient potential”, leaving the “additive potential” to Section 5.

Equations (3.5a) hold identically if we set

\[ u^{(i)}_{m,n} = \alpha \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i)}_{m,n}}, \quad v^{(i)}_{m,n} = \beta \frac{\phi^{(i)}_{m,n+1}}{\phi^{(i)}_{m,n}}, \tag{4.15} \]

where \( a = \alpha^N, b = \beta^N \). Equations (3.5b) then take the form

\[ \alpha \left( \frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i)}_{m,n+1}} - \frac{\phi^{(i+\ell_2)}_{m+1,n}}{\phi^{(i+\ell_2)}_{m,n}} \right) = \beta \left( \frac{\phi^{(i)}_{m,n+1}}{\phi^{(i)}_{m,n}} - \frac{\phi^{(i+\ell_1)}_{m+1,n}}{\phi^{(i+\ell_1)}_{m,n}} \right), \tag{4.16} \]

where indices are taken (mod \( N \)).

These equations have a weighted scaling symmetry, whose invariants are given exactly by the formulae (4.15), leading us back to equations (3.5) and therefore to our previous Yang-Baxter maps.

### 5 The Additive Potential

Equations (3.5b) hold identically if we set

\[ u^{(i)}_{m,n} = \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}, \quad v^{(i)}_{m,n} = \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}. \tag{5.1} \]

Equations (3.5a) then take the form

\[ \frac{\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_1)}_{m+1,n}}{\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_2)}_{m+1,n}} = \frac{\chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m+1,n}}{\chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}}, \tag{5.2} \]

and the first integrals (3.7) take the form

\[ \prod_{i=0}^{N-1} \left( \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n} \right) = a, \quad \prod_{i=0}^{N-1} \left( \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n} \right) = b. \tag{5.3} \]

**Remark 5.1 (Reduction)** It is not always possible to use these first integrals to explicitly reduce (5.2) to a system with \( N - 1 \) components (eliminating \( \chi^{(N-1)}_{m,n} \)), and even when this is possible the spectral problem (3.1) cannot be written in terms of the reduced variables.
In [4] we showed that it is possible to explicitly reduce the system with \((k_i, \ell_i) = (0, 1)\), which takes the form

\[
\frac{\chi^{(i)}_{m+1, n+1} - \chi^{(i+1)}_{m, n+1}}{\chi^{(i)}_{m+1, n+1} - \chi^{(i+1)}_{m, n+1}} = \frac{\chi^{(i)}_{m+1, n} - \chi^{(i+1)}_{m, n}}{\chi^{(i)}_{m, n+1} - \chi^{(i+1)}_{m+1, n}}, \quad i = 0, \ldots, N - 3,
\]  
\tag{5.4a}

\[
\chi^{(N-2)}_{m+1, n+1} = \chi^{(0)}_{m, n} + \frac{1}{\chi^{(N-2)}_{m+1, n} - \chi^{(N-2)}_{m, n+1}} \left( \frac{a}{X} - \frac{b}{Y} \right),
\]  
\tag{5.4b}

where \(X = \prod_{j=0}^{N-3} (\chi^{(j)}_{m+1, n} - \chi^{(j+1)}_{m, n})\) and \(Y = \prod_{j=0}^{N-3} (\chi^{(j)}_{m, n+1} - \chi^{(j+1)}_{m+1, n})\).

**Remark 5.2** This is a direct generalisation of equation \(H1\) in the ABS classification [1].

It is easy to see that the system (5.4) has the following pair of symmetry generators:

\[
X_t = \sum_{i=0}^{N-2} \omega^{m+n+i} \partial_{\chi^{(i)}_{m, n}},
\]  
\tag{5.5a}

\[
X_s = \sum_{i=0}^{N-2} \omega^{m+n+i} \chi^{(i)}_{m, n} \partial_{\chi^{(i)}_{m, n}}, \quad \omega \neq 1,
\]  
\tag{5.5b}

where \(\omega^N = 1\). It is therefore possible to write equations (5.4) in terms of the invariants of these symmetries. We can then reduce this form of the lattice equations to Yang-Baxter maps.

### 5.1 The Invariants of \(X_t\)

It is straightforward to write a suitable “basis” for the invariants of \(X_t\). The formulae are more symmetric if we write “too many” invariants, which then satisfy some additional identities. We therefore define \(4(N - 1)\) invariants, satisfying \((N - 1)\) identities. Furthermore, we make the reduction (2.5), so that we derive a map. Following [8], we denote these invariants by

\[
x^{(i)} = x^{(i)}_p, \quad y^{(i)} = y^{(i)}_p, \quad u^{(i)} = u^{(i)}_{p+1}, \quad v^{(i)} = v^{(i)}_{p+1}, \quad \text{where} \quad p = n - m,
\]  
\tag{5.6}

corresponding to specific edges of the lattice square, as shown in Figure 1 and noting that the shifts \(m \mapsto m - 1\) and \(n \mapsto n + 1\) both correspond to \(p \mapsto p + 1\).

![Figure 1: Invariants defined on edges](image-url)
The $4(N-1)$ invariants:

\[ x^{(i)} = \chi^{(i)}_{m+1,n} - \chi^{(i+1)}_{m,n}, \quad i = 0, \ldots, N - 3, \quad x^{(N-2)} = \chi^{(N-2)}_{m+1,n} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n}, \]

\[ y^{(i)} = \chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m,n+1}, \quad i = 0, \ldots, N - 3, \quad y^{(N-2)} = \chi^{(N-2)}_{m+1,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m+1,n}, \]

\[ u^{(i)} = \chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m,n}, \quad i = 0, \ldots, N - 3, \quad u^{(N-2)} = \chi^{(N-2)}_{m+1,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n+1}, \]

\[ v^{(i)} = \chi^{(i)}_{m,n+1} - \chi^{(i+1)}_{m,n+1}, \quad i = 0, \ldots, N - 3, \quad v^{(N-2)} = \chi^{(N-2)}_{m,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n}. \]

satisfy $(N - 1)$ identities:

\[ x^{(i+1)} + y^{(i)} = u^{(i)} + v^{(i+1)}, \quad i = 0, \ldots, N - 3, \quad (5.7a) \]

\[ y^{(N-2)} + \sum_{j=0}^{N-2} v^{(j)} = u^{(N-2)} + \sum_{j=0}^{N-2} x^{(j)}, \quad (5.7b) \]

and equations (5.4) take the form

\[ u^{(i)} v^{(i)} = x^{(i)} y^{(i)}, \quad i = 0, \ldots, N - 3, \quad (5.7c) \]

\[ u^{(N-2)} = \sum_{j=0}^{N-2} v^{(j)} + \frac{1}{x^{(N-2)} - v^{(N-2)}} \left( \frac{a}{\prod_{j=0}^{N-3} x^{(j)} - \prod_{j=0}^{N-3} v^{(j)}} \right). \quad (5.7d) \]

The Yang-Baxter map corresponds to the solution of equations (5.7) for $(u^{(i)}, v^{(i)})$. We do not have an explicit form of the solution in general, but for any given value of $N$, this can be found.

**Remark 5.3 (The Case $N = 2$)** We already remarked that for $N = 2$ the lattice equation is just $H1$ in the ABS classification [1]. Using the symmetry $X_4$, with $\omega = -1$ leads to the Yang-Baxter map

\[ u = y + \frac{a - b}{x - y}, \quad v = x + \frac{a - b}{x - y}, \]

which is just $F_Y$ of the ABS classification of quadrirational maps [2] (the Adler map). Clearly, we may consider this whole family of maps as multi-component generalisations of $F_Y$.

**Example 5.4 (The Case $N = 3$)** In this case, we find

\[ u^{(0)} = y^{(0)} + \frac{(a - b)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \]

\[ u^{(1)} = y^{(1)} + \frac{(b - a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1))}} + \frac{(b - a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \]

\[ v^{(0)} = x^{(0)} + \frac{(b - a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \]

\[ v^{(1)} = x^{(1)} + \frac{(b - a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}. \]
5.2 The Invariants of $X_s$

Again we denote invariants as in (5.6) and Figure 1. The $4(N - 1)$ invariants:

$$x^{(i)} = \frac{\chi_{m+1,n}}{\chi_{m,n}^{(i+1)}}, \quad i = 0, \ldots, N - 3$$

$$y^{(i)} = \frac{\chi_{m+1,n+1}^{(i)}}{\chi_{m,n+1}^{(i+1)}}, \quad i = 0, \ldots, N - 3$$

$$u^{(i)} = \frac{\chi_{m+1,n+1}}{\chi_{m,n}^{(i+1)}}, \quad i = 0, \ldots, N - 3$$

$$v^{(i)} = \frac{\chi_{m,n+1}}{\chi_{m,n}^{(i+1)}}, \quad i = 0, \ldots, N - 3$$

satisfy $(N - 1)$ identities:

$$u^{(i)}v^{(i+1)} = x^{(i+1)}y^{(i)}, \quad i = 0, \ldots, N - 3$$

$$u^{(N-2)} \prod_{j=0}^{N-2} x^{(j)} = y^{(N-2)} \prod_{j=0}^{N-2} v^{(j)}$$

and equations (5.4) take the form

$$u^{(i)}v^{(i+1)} = \frac{(v^{(i)} - 1)v^{(i+1)} - (x^{(i)} - 1)x^{(i+1)}}{v^{(i)} - x^{(i)}}, \quad i = 0, \ldots, N - 3$$

$$u^{(N-2)} = \left(1 + \frac{1}{x^{(N-2)} - v^{(N-2)}} \left(\frac{a}{X} - \frac{b}{Y}\right)\right) \prod_{j=0}^{N-2} v^{(j)}$$

where $X = \prod_{j=0}^{N-3} (x^{(j)} - 1)$, $Y = \prod_{j=0}^{N-3} (v^{(j)} - 1)$.

**Remark 5.5 (The Case $N = 2$)** Again, since the lattice equation is just $H_1$ in the ABS classification [1], the symmetry $X_s$, with $\omega = -1$, leads to the Yang-Baxter map

$$u = y \left(1 + \frac{a - b}{x - y}\right), \quad v = x \left(1 + \frac{a - b}{x - y}\right),$$

which is just $F_{IV}$ of the ABS classification of quadrirational maps [2].Clearly, we may consider this whole family of maps as multi-component generalisations of $F_{IV}$.

**Example 5.6 (The Case $N = 3$)** In this case, we first define

$$P_a = ax^{(0)} - (x^{(0)} - 1)(y^{(0)} - 1)(x^{(0)}x^{(1)} - y^{(1)}), \quad P_b = bx^{(0)} - (x^{(0)} - 1)(y^{(0)} - 1)(x^{(0)}x^{(1)} - y^{(1)})$$
We then have the map

\[
u^{(0)} = y^{(0)} \left( 1 - \frac{(a-b)x^{(0)}(y^{(0)} - 1)}{(y^{(0)} - 1)P_a - y^{(0)}P_b} \right),
\]

\[
u^{(1)} = y^{(1)} \left( 1 - (a-b) \left( \frac{(x^{(0)} - 1)y^{(0)}}{P_a} + \frac{(y^{(0)} - 1)}{P_b} \right) \right),
\]

\[
v^{(0)} = x^{(0)} \left( 1 - \frac{(a-b)(x^{(0)} - 1)}{P_a} \right),
\]

\[
v^{(1)} = x^{(1)} \left( 1 - \frac{(a-b)(y^{(0)} - 1)x^{(0)}}{P_b} \right).
\]

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References


