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# $\mathbb{Z}_N$ graded discrete Lax pairs and Yang-Baxter maps

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## Abstract

We recently introduced a class of  $\mathbb{Z}_N$  graded discrete Lax pairs and studied the associated discrete integrable systems (lattice equations). In this paper we introduce the corresponding Yang-Baxter maps. Many well known examples belong to this scheme for  $N = 2$ , so, for  $N \geq 3$ , our systems may be regarded as generalisations of these.

In particular, for each  $N$  we introduce a class of multi-component Yang-Baxter maps, which include  $H_{III}^B$  (of [6]), when  $N = 2$ , and that associated with the discrete modified Boussinesq equation, for  $N = 3$ . For  $N \geq 5$  we introduce a new families of Yang-Baxter maps, which have no lower dimensional analogue. We also present new multi-component versions of the Yang-Baxter maps  $F_{IV}$  and  $F_V$  (given in the classification of [2]).

*Keywords:* Discrete integrable system, Lax pair, symmetry, Yang-Baxter map.

## 1 Introduction

The term “Yang-Baxter map” was introduced by Veselov [10] as an abbreviation for Drinfeld’s notion of “set-theoretical solutions to the quantum Yang-Baxter equation”. The basic ingredient is a map  $R : X \times X \rightarrow X \times X$ , where  $X$  is some algebraic variety. For the case  $X = \mathbb{CP}^1$ , these were partially classified in [2, 6]. In [8] a symmetry approach was introduced to relate Yang-Baxter equations with 3D consistent equations on quad-graphs, which had been classified in [1]. Starting with any symmetry of an integrable equation on a quad-graph, the authors introduce invariant functions, which are then used to define a map. The Yang-Baxter relation was shown to be a *consequence* of 3D consistency. Multi-component Yang-Baxter maps are not yet classified, but several are known (see, for example, [9, 8, 7, 5, 3]).

We recently introduced a class of  $\mathbb{Z}_N$  graded discrete Lax pairs and studied the associated discrete integrable systems [4]. Many well known examples belong to that scheme for  $N = 2$ , so, for  $N \geq 3$ , our systems may be regarded as generalisations of these. As mentioned above, the quad systems for  $N = 2$  can be related to Yang-Baxter maps. In this paper we construct generalisations of these, associated with our generalised lattice equations.

In Section 2 we present the basic background theory of Yang-Baxter maps and their relationship to lattice equations on a quadrilateral lattice. In Section 3, we introduce the  $\mathbb{Z}_N$ -graded Lax pairs of [4] and derive the reduction to Yang-Baxter maps. We show that all such maps are equivalent to ones with “level structure”  $(0, \delta; 0, \delta)$ . For each  $N$  and  $\delta$ , with  $1 \leq \delta \leq \frac{N}{2}$ , we present a Yang-Baxter map  $R^{(\delta)}(a, b)$  with  $2N - 2$  components (see Section 4). For  $\delta = 1$ , this

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includes the map  $H_{III}^B$  of [6], when  $N = 2$ , and the Yang-Baxter map associated with the discrete modified Boussinesq equation, for  $N = 3$ . The general map for  $\delta = 1$  is known [5], but for  $\delta \geq 2$  this is a new class of Yang-Baxter maps. In Section 5 we present a new multi-component generalisation of the Yang-Baxter maps  $F_{IV}$  and  $F_V$  (given in the classification of [2])

## 2 Basic Definitions

Let  $X$  be an algebraic variety. A *parametric Yang-Baxter map*  $R(a, b)$ , depending upon parameters  $(a, b)$ , is a map

$$R(a, b) : X \times X \rightarrow X \times X,$$

satisfying:

$$R_{23}(a_2, a_3) \circ R_{13}(a_1, a_3) \circ R_{12}(a_1, a_2) = R_{12}(a_1, a_2) \circ R_{13}(a_1, a_3) \circ R_{23}(a_2, a_3), \quad (2.1)$$

where  $R_{ij}(a_i, a_j)$  is the map that acts as  $R(a, b)$  on the  $i$  and  $j$  factor of  $X \times X \times X$ , and identically on the other.

**Definition 2.1 (Reversibility)** *Let  $P$  be the involution given by  $P(\mathbf{x}, \mathbf{y}; a, b) = (\mathbf{y}, \mathbf{x}; b, a)$ . If  $P \circ R(a, b)$  is also an involution, then the map  $R(a, b)$  is said to be reversible.*

**Remark 2.2** *An alternative way of writing this is that the map  $P \circ R(a, b) \circ P$  is the inverse of  $R(a, b)$ .*

Lax pairs were defined for Yang-Baxter maps in [10, 9]. A matrix  $L(\mathbf{x}, a)$ , with  $\mathbf{x} \in X$ , depending upon the YB parameter  $a$  and the spectral parameter  $\lambda$  is used to define the equation:

$$L(\mathbf{x}', a)L(\mathbf{y}', b) = L(\mathbf{y}, b)L(\mathbf{x}, a). \quad (2.2)$$

It was shown in [10] that if  $L$  satisfies this, then the map  $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$  satisfies the parametric Yang-Baxter equation (2.1) and is *reversible*.

**Definition 2.3 (The Companion Map)** *The companion map  $(\mathbf{x}, \mathbf{y}') \mapsto (\mathbf{x}', \mathbf{y})$  is obtained by solving equation (2.2) for the variables  $(\mathbf{x}', \mathbf{y})$ .*

### 2.1 Travelling Wave Reductions of a Lattice Equation

Suppose we have a square lattice with vertices labelled  $(m, n)$ . At each vertex we have functions

$$\mathbf{u}_{m,n} = \left( u_{m,n}^{(0)}, \dots, u_{m,n}^{(N-1)} \right), \quad \mathbf{v}_{m,n} = \left( v_{m,n}^{(0)}, \dots, v_{m,n}^{(N-1)} \right),$$

and vector function  $\Psi_{m,n}$ , satisfying

$$\Psi_{m+1,n} = L(\mathbf{u}_{m,n}, a) \Psi_{m,n}, \quad \Psi_{m,n+1} = L(\mathbf{v}_{m,n}, b) \Psi_{m,n}, \quad (2.3)$$

with compatibility conditions

$$L(\mathbf{u}_{m,n+1}, a)L(\mathbf{v}_{m,n}, b) = L(\mathbf{v}_{m+1,n}, b)L(\mathbf{u}_{m,n}, a). \quad (2.4)$$

If we now consider the reduction

$$\mathbf{u}_{m,n} = \mathbf{x}_p, \quad \mathbf{v}_{m,n} = \mathbf{y}_{p+1}, \quad \text{where } p = n - m, \quad (2.5)$$

then (2.4) reduces to (2.2), with  $\mathbf{x} = \mathbf{x}_p$ ,  $\mathbf{x}' = \mathbf{x}_{p+1}$ ,  $\mathbf{y} = \mathbf{y}_p$ ,  $\mathbf{y}' = \mathbf{y}_{p+1}$ , with the map  $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$  being Yang-Baxter.

**Remark 2.4** *Notice that this does not rely on any underlying Lie point symmetry of the lattice equation. It is just a ‘‘travelling wave’’ solution of the lattice equation.*

### 3 $\mathbb{Z}_N$ -Graded Lax Pairs

We now consider the specific discrete Lax pairs, which we introduced in [4]. Consider a pair of matrix equations of the form

$$\Psi_{m+1,n} = L_{m,n} \Psi_{m,n} \equiv \left( U_{m,n} + \lambda \Omega^{\ell_1} \right) \Psi_{m,n}, \quad (3.1a)$$

$$\Psi_{m,n+1} = M_{m,n} \Psi_{m,n} \equiv \left( V_{m,n} + \lambda \Omega^{\ell_2} \right) \Psi_{m,n}, \quad (3.1b)$$

where

$$U_{m,n} = \text{diag} \left( u_{m,n}^{(0)}, \dots, u_{m,n}^{(N-1)} \right) \Omega^{k_1}, \quad V_{m,n} = \text{diag} \left( v_{m,n}^{(0)}, \dots, v_{m,n}^{(N-1)} \right) \Omega^{k_2}, \quad (3.1c)$$

and

$$(\Omega)_{i,j} = \delta_{j-i,1} + \delta_{i-j,N-1}.$$

The matrix  $\Omega$  defines a grading and the four matrices of (3.1) are said to be of respective levels  $k_i, \ell_i$ , with  $\ell_i \neq k_i$  (for each  $i$ ). The Lax pair is characterised by the quadruple  $(k_1, \ell_1; k_2, \ell_2)$ , which we refer to as *the level structure* of system, and for consistency, we require

$$k_1 + \ell_2 \equiv k_2 + \ell_1 \pmod{N}. \quad (3.2)$$

Since matrices  $U, V$  and  $\Omega$  are independent of  $\lambda$ , the compatibility condition of (3.1),

$$L_{m,n+1} M_{m,n} = M_{m+1,n} L_{m,n}, \quad (3.3)$$

splits into the system

$$U_{m,n+1} V_{m,n} = V_{m+1,n} U_{m,n}, \quad (3.4a)$$

$$U_{m,n+1} \Omega^{\ell_2} - \Omega^{\ell_2} U_{m,n} = V_{m+1,n} \Omega^{\ell_1} - \Omega^{\ell_1} V_{m,n}, \quad (3.4b)$$

which can be written explicitly as

$$u_{m,n+1}^{(i)} v_{m,n}^{(i+k_1)} = v_{m+1,n}^{(i)} u_{m,n}^{(i+k_2)}, \quad (3.5a)$$

$$u_{m,n+1}^{(i)} - u_{m,n}^{(i+\ell_2)} = v_{m+1,n}^{(i)} - v_{m,n}^{(i+\ell_1)}, \quad (3.5b)$$

or, in a solved form, as

$$u_{m,n+1}^{(i)} = \frac{u_{m,n}^{(i+\ell_2)} - v_{m,n}^{(i+\ell_1)}}{u_{m,n}^{(i+k_2)} - v_{m,n}^{(i+k_1)}} u_{m,n}^{(i+k_2)}, \quad v_{m+1,n}^{(i)} = \frac{u_{m,n}^{(i+\ell_2)} - v_{m,n}^{(i+\ell_1)}}{u_{m,n}^{(i+k_2)} - v_{m,n}^{(i+k_1)}} v_{m,n}^{(i+k_1)}, \quad (3.6)$$

assuming that  $u_{m,n}^{(i)} \neq v_{m,n}^{(j)}$  for all  $i, j$ . In all the above formulae,  $i, j$  are taken  $\pmod{N}$ .

It is easily seen that the quantities

$$a = \prod_{i=0}^{N-1} u_{m,n}^{(i)}, \quad b = \prod_{i=0}^{N-1} v_{m,n}^{(i)} \quad \text{satisfy} \quad \Delta_n(a) = \Delta_m(b) = 0, \quad (3.7)$$

where

$$\Delta_m = \mathcal{S}_m - 1, \quad \Delta_n = \mathcal{S}_n - 1, \quad \text{with} \quad \mathcal{S}_m f_{m,n} = f_{m+1,n}, \quad \mathcal{S}_n f_{m,n} = f_{m,n+1}.$$

### 3.1 Reduction to Yang-Baxter Maps

We can now employ the reduction (2.5), using (3.7) to replace the components  $x_p^{(N-1)}, y_p^{(N-1)}$ . This introduces parameters  $a, b$  into the Lax matrices. If we define

$$X_p = \text{diag} \left( x_p^{(0)}, \dots, x_p^{(N-1)} \right), \quad Y_p = \text{diag} \left( y_p^{(0)}, \dots, y_p^{(N-1)} \right), \quad (3.8)$$

then the compatibility condition (3.3) takes the form

$$(X_{p+1}\Omega^{k_1} + \lambda\Omega^{\ell_1})(Y_{p+1}\Omega^{k_2} + \lambda\Omega^{\ell_2}) = (Y_p\Omega^{k_2} + \lambda\Omega^{\ell_2})(X_p\Omega^{k_1} + \lambda\Omega^{\ell_1}), \quad (3.9)$$

and equations (3.5) take the form

$$x_{p+1}^{(i)} y_{p+1}^{(i+k_1)} = y_p^{(i)} x_p^{(i+k_2)}, \quad x_{p+1}^{(i)} + y_{p+1}^{(i+\ell_1)} = y_p^{(i)} + x_p^{(i+\ell_2)}. \quad (3.10)$$

We can write (3.9) as

$$(X_{p+1} + \lambda\Omega^\delta)(\Omega^{k_1} Y_{p+1} \Omega^{-k_1} + \lambda\Omega^\delta) = (Y_p + \lambda\Omega^\delta)(\Omega^{k_2} X_p \Omega^{-k_2} + \lambda\Omega^\delta), \quad (3.11)$$

where  $0 < \delta \leq N-1$ , with  $\delta \equiv \ell_i - k_i \pmod{N}$ . This allows us to reduce the general case with level structure  $(k_1, \ell_1; k_2, \ell_2)$  to that with level structure  $(0, \delta; 0, \delta)$ . First, note that formula (3.11) can be written

$$(\bar{X}_{p+1} + \lambda\Omega^\delta)(\bar{Y}_{p+1} + \lambda\Omega^\delta) = (\bar{Y}_p + \lambda\Omega^\delta)(\bar{X}_p + \lambda\Omega^\delta), \quad (3.12)$$

where

$$\bar{X}_p = \text{diag} \left( \bar{x}_p^{(0)}, \dots, \bar{x}_p^{(N-1)} \right), \quad \bar{Y}_p = \text{diag} \left( \bar{y}_p^{(0)}, \dots, \bar{y}_p^{(N-1)} \right).$$

Comparing (3.12) and (3.11), we see that

$$\bar{x}_{p+1}^{(i)} = x_{p+1}^{(i)}, \quad \bar{y}_{p+1}^{(i)} = y_{p+1}^{(i+k_1)}, \quad \bar{x}_p^{(i)} = x_p^{(i+k_2)}, \quad \bar{y}_p^{(i)} = y_p^{(i)},$$

all taken  $\pmod{N}$ . We see from (3.12) that the components  $(\bar{x}_p^{(i)}, \bar{y}_p^{(i)})$  satisfy

$$\bar{x}_{p+1}^{(i)} \bar{y}_{p+1}^{(i)} = \bar{y}_p^{(i)} \bar{x}_p^{(i)}, \quad \bar{x}_{p+1}^{(i)} + \bar{y}_{p+1}^{(i+\delta)} = \bar{y}_p^{(i)} + \bar{x}_p^{(i+\delta)},$$

which are just (3.10) with  $(k_i, \ell_i) = (0, \delta)$ . We summarise these results in:

**Proposition 3.1** *In the Yang-Baxter reduction, all systems with level structure  $(k_1, \ell_1; k_2, \ell_2)$ , for which  $\ell_i - k_i \equiv \delta \pmod{N}$ , are equivalent (up to point transformation) to the system with level structure  $(0, \delta; 0, \delta)$ .*

## 4 The Yang-Baxter Map Corresponding to the Case $(0, \delta; 0, \delta)$

In this section we consider the Lax equations with level structure  $(0, \delta; 0, \delta)$ , with  $0 < \delta \leq N-1$ . The resulting equations are *quadrirational*, with both the Yang-Baxter and companion maps being *birational*. We find that the Yang-Baxter maps corresponding to  $\delta$  and  $N-\delta$  are inverses to each other and that the companion map is periodic, with period  $N$ .

## 4.1 The Equations and Maps

With Lax matrices

$$L(\mathbf{x}, a) = X_p + \lambda \Omega^\delta, \quad L(\mathbf{y}, b) = Y_p + \lambda \Omega^\delta, \quad (4.1)$$

where  $X_p$  and  $Y_p$  are defined by (3.8), with

$$x_p^{(N-1)} = \frac{a}{\prod_{i=0}^{N-2} x_p^{(i)}}, \quad y_p^{(N-1)} = \frac{b}{\prod_{i=0}^{N-2} y_p^{(i)}}, \quad (4.2)$$

the Lax equation (2.2) implies

$$x_{p+1}^{(i)} y_{p+1}^{(i)} = y_p^{(i)} x_p^{(i)}, \quad x_{p+1}^{(i)} + y_{p+1}^{(i+\delta)} = y_p^{(i)} + x_p^{(i+\delta)}, \quad 0 \leq i \leq N-1. \quad (4.3)$$

Only the formulae with  $0 \leq i \leq N-2$  are independent, but the full set is useful when discussing first integrals.

**Remark 4.1 (Level structure  $(\delta, 0; \delta, 0)$  vs  $(0, \delta; 0, \delta)$ )** *Under the point transformation*

$$x_{p+1}^{(i)} = \tilde{x}_p^{(i+\delta)}, \quad x_p^{(i)} = \tilde{x}_{p+1}^{(i)}, \quad y_{p+1}^{(i)} = \tilde{y}_p^{(i)}, \quad y_p^{(i)} = \tilde{y}_{p+1}^{(i+\delta)},$$

equations (4.3) take the form

$$\tilde{x}_{p+1}^{(i)} \tilde{y}_{p+1}^{(i+\delta)} = \tilde{y}_p^{(i)} \tilde{x}_p^{(i+\delta)}, \quad \tilde{x}_{p+1}^{(i)} + \tilde{y}_{p+1}^{(i)} = \tilde{y}_p^{(i)} + \tilde{x}_p^{(i)}, \quad 0 \leq i \leq N-1,$$

which are just the equations for level structure  $(\delta, 0; \delta, 0)$ , so these structures are equivalent.

### 4.1.1 The Yang-Baxter map $R^{(\delta)}(a, b)$

Here we solve (4.3) for  $(x_{p+1}^{(i)}, y_{p+1}^{(i)})$  as functions of  $(x_p^{(i)}, y_p^{(i)})$  (with  $0 \leq i \leq N-2$  and  $x_p^{(N-1)}, y_p^{(N-1)}$  replaced by (4.2)). We write this map as  $R^{(\delta)}(a, b)$ , but when no ambiguity can arise, we suppress the parametric dependence by writing the map as  $R^{(\delta)}$ .

Notice that by shifting  $i \mapsto i + N - \delta \equiv i - \delta \pmod{N}$ , the second part of equation (4.3) takes the form

$$x_{p+1}^{(i-\delta)} + y_{p+1}^{(i)} = y_p^{(i-\delta)} + x_p^{(i)},$$

which leads to:

**Proposition 4.2 (Inverse Map)** *The Yang-Baxter map  $R^{(-\delta)}(a, b)$  is just the inverse of the map  $R^{(\delta)}(a, b)$ .*

This means that we only need to consider  $\delta \leq \frac{N}{2}$  and that, when  $N = 2M$ , the map  $R^{(M)}(a, b)$  is an involution.

**Proposition 4.3 (First Integrals)** *The Yang-Baxter map  $R^{(\delta)}(a, b)$  has the following  $N$  first integrals:*

$$x_p^{(i)} y_p^{(i)} = c_i, \quad 0 \leq i \leq N-2, \quad \sum_{i=0}^{N-1} (x_p^{(i+\delta)} + y_p^{(i)}) = c_{N-1}, \quad (4.4)$$

where, in the latter,  $x_p^{(N-1)}$  and  $y_p^{(N-1)}$  are replaced by (4.2).

The last of these integrals is obtained by summing the additive equations of (4.3).

### 4.1.2 The Companion Map $\varphi^{(\delta)}$

Here we solve (4.3) for  $(x_{p+1}^{(i)}, y_p^{(i)})$  as functions of  $(x_p^{(i)}, y_{p+1}^{(i)})$  (with  $0 \leq i \leq N-2$  and  $x_p^{(N-1)}, y_p^{(N-1)}$  replaced by (4.2)). Since  $p$  is no longer the evolution parameter, we relabel our variables as:

$$(x_p^{(i)}, y_{p+1}^{(i)}) = (x_q^{(i)}, y_q^{(i)}), \quad (x_{p+1}^{(i)}, y_p^{(i)}) = (x_{q+1}^{(i)}, y_{q+1}^{(i)}).$$

**Remark 4.4 (A second travelling wave reduction)** *This labelling follows directly from the travelling wave reduction*

$$\mathbf{u}_{m,n} = \mathbf{x}_q, \quad \mathbf{v}_{m,n} = \mathbf{y}_q, \quad \text{where } q = n + m$$

We can re-arrange the quadratic formulae in (4.3) (with this new labelling) to obtain  $N-1$  first integrals:

$$\frac{x_q^{(i)}}{y_q^{(i)}} = c_i, \quad 0 \leq i \leq N-2. \quad (4.5)$$

We can also re-arrange the linear formulae of (4.3) to obtain

$$x_{q+1}^{(i)} - y_{q+1}^{(i)} = x_q^{(i+\delta)} - y_q^{(i+\delta)}, \quad 0 \leq i \leq N-1.$$

If we define

$$f(x, y) = x - y, \quad (4.6)$$

then

$$f(x_{q+1}^{(i)}, y_{q+1}^{(i)}) = f(x_q^{(i+\delta)}, y_q^{(i+\delta)}), \quad 0 \leq i \leq N-1. \quad (4.7)$$

We may use

$$\left( \frac{x_q^{(0)}}{y_q^{(0)}}, \dots, \frac{x_q^{(N-2)}}{y_q^{(N-2)}}, f(x_q^{(0)}, y_q^{(0)}), \dots, f(x_q^{(N-2)}, y_q^{(N-2)}) \right)$$

as coordinates and, in these coordinates, the map  $\varphi^{(\delta)}$  just shifts the coordinates  $f(x_q^{(i)}, y_q^{(i)})$  by  $\delta$ , whilst leaving the coordinates  $\frac{x_q^{(i)}}{y_q^{(i)}}$  fixed. This leads to the following:

**Proposition 4.5 (Periodicity)** *The map  $\varphi^{(\delta)}$  is periodic with period  $N$ . When  $(N, \delta) = 1$  this is the minimum period. Furthermore, we have that  $\varphi^{(\delta)} = \varphi^{(1)} \circ \dots \circ \varphi^{(1)}$  (the  $\delta$ -fold composition of  $\varphi^{(1)}$ ).*

This statement is, of course, independent of coordinates.

**Remark 4.6 (( $2N-2$ ) first integrals)** *Any cyclically symmetric function of  $f(x_q^{(i)}, y_q^{(i)})$  is a first integral of the companion map, so it possesses  $(2N-2)$  first integrals. The common level set is then finite, corresponding to the periodicity of the map.*

## 4.2 Examples of the map $R^{(\delta)}$

We can build hierarchies of Yang-Baxter maps for each  $\delta$ . It follows from Proposition 4.2 that we only need to consider  $\delta \leq \frac{N}{2}$ . However, as the value of  $N$  increases, so does the number of different maps  $R^{(\delta)}$ . We have:

**Case  $\delta = 1$ :** At  $N = 2$ , we only have the case  $\delta = 1$ , and  $R^{(1)}$  is just the map  $H_{III}^B$  in the classification of scalar Yang-Baxter maps [6]. The map  $R^{(1)}$  exists for all  $N \geq 2$ , which can therefore be considered as a multi-component generalisation of the scalar Yang-Baxter map  $H_{III}^B$ .

**Case  $\delta = 2$ :** For  $N \geq 4$  we have the map  $R^{(2)}$ . When  $N$  is even, this map degenerates to lower dimensional maps (see the case  $N = 4$  below), but when  $N$  is odd, we have a new sequence of Yang-Baxter maps which fully couple  $2N - 2$  variables. The 8–component case can be seen in the case  $N = 5$  below.

**Case  $\delta = 3$ :** For  $N \geq 6$  we have the map  $R^{(3)}$ , but again, this map degenerates to lower dimensional maps when  $N$  is a multiple of 3. The first fully coupled system is at  $N = 7$ .

Whilst the generalisation of  $\delta = 1$  is already known [5], the maps  $R^{(\delta)}$ , for  $\delta \geq 2$ , are new classes of Yang-Baxter maps.

#### 4.2.1 When $N = 2$

Here we only have the case  $\delta = 1$ , which leads to (with  $x^{(0)} = x$ ,  $y^{(0)} = y$ ,  $x^{(1)} = a/x$ ,  $y^{(1)} = b/y$ )

$$x_{p+1} = y_p \left( \frac{a + xy}{b + xy} \right), \quad y_{p+1} = x_p \left( \frac{b + xy}{a + xy} \right), \quad (4.8)$$

which (up to a relabelling of parameters) is just the map  $H_{III}^B$  in the classification of scalar Yang-Baxter maps [6].

The existence of the two invariant functions (4.4) implies (the well known fact) that this map is an involution.

#### 4.2.2 When $N = 3$

Here we have  $\delta = 1$  and  $\delta = 2$ , but since  $N - 1 = 2 \equiv -1 \pmod{3}$ , the map  $R^{(2)}$  is just the inverse of  $R^{(1)}$ . In this case  $R^{(1)}$  takes the form:

$$x_{p+1}^{(i)} = y_p^{(i)} \frac{A^{(i)}}{A^{(i+1)}}, \quad y_{p+1}^{(i)} = x_p^{(i)} \frac{A^{(i+1)}}{A^{(i)}}, \quad 0 \leq i \leq 1, \quad (4.9)$$

with upper indices taken  $\pmod{2}$  and where

$$A^{(0)} = a(x_p^{(1)} + y_p^{(0)}) + x_p^{(0)}x_p^{(1)}y_p^{(0)}y_p^{(1)}, \quad A^{(1)} = A^{(0)} + (b - a)x_p^{(1)}, \quad A^{(2)} = A^{(1)} + (b - a)y_p^{(0)}.$$

**Remark 4.7 (Discrete Modified Boussinesq Equation)** *This is equivalent to the Yang-Baxter map derived in [8], associated with the discrete modified Boussinesq equation (see equation (67a-b) of [8]). They are related by a simple point transformation:*

$$x^{(0)} \mapsto \frac{c_0}{x^1}, \quad x^{(1)} \mapsto c_0 x^2, \quad y^{(0)} \mapsto \frac{c_0 \alpha_1}{\alpha_2 y^1}, \quad y^{(1)} \mapsto \frac{\alpha_1^2 y^2}{c_0^3}, \quad \text{where} \quad c_0^4 = \frac{\alpha_1^3}{\alpha_2}.$$

#### 4.2.3 When $N = 4$

**For  $\delta = 1$ :** We obtain the 6–component version of (4.9).



**For  $\delta = 2$ :** Since  $(N, \delta) = 2 \neq 1$ , the map is reducible, with a 4–component subsystem:

$$\begin{aligned} x_{p+1}^{(0)} &= \frac{x_p^{(0)}(x_p^{(2)} + y_p^{(0)})}{x_p^{(0)} + y_p^{(2)}}, & x_{p+1}^{(2)} &= \frac{x_p^{(2)}(x_p^{(0)} + y_p^{(2)})}{x_p^{(2)} + y_p^{(0)}}, \\ y_{p+1}^{(0)} &= \frac{y_p^{(0)}(x_p^{(0)} + y_p^{(2)})}{x_p^{(2)} + y_p^{(0)}}, & y_{p+1}^{(2)} &= \frac{y_p^{(2)}(x_p^{(2)} + y_p^{(0)})}{x_p^{(0)} + y_p^{(2)}}, \end{aligned} \quad (4.10)$$

in which the parameters  $(a, b)$  are absent.

The remaining pair of equations are a non-autonomous version of (4.8), with coefficients depending upon  $(x_p^{(0)}, x_p^{(2)}, y_p^{(0)}, y_p^{(2)})$ :

$$x_{p+1}^{(1)} = \frac{y_p^{(0)} y_p^{(2)} y_p^{(1)} \left( a + x_p^{(0)} x_p^{(2)} x_p^{(1)} y_p^{(1)} \right)}{x_p^{(0)} x_p^{(2)} \left( b + y_p^{(0)} y_p^{(2)} x_p^{(1)} y_p^{(1)} \right)}, \quad y_{p+1}^{(1)} = \frac{x_p^{(0)} x_p^{(2)} x_p^{(1)} \left( b + y_p^{(0)} y_p^{(2)} x_p^{(1)} y_p^{(1)} \right)}{x_p^{(0)} x_p^{(2)} \left( a + x_p^{(0)} x_p^{(2)} x_p^{(1)} y_p^{(1)} \right)}. \quad (4.11)$$

Notice that this last pair could also be written

$$x_{p+1}^{(1)} = \frac{x_p^{(1)}(x_p^{(3)} + y_p^{(1)})}{x_p^{(1)} + y_p^{(3)}}, \quad y_{p+1}^{(1)} = \frac{y_p^{(1)}(x_p^{(1)} + y_p^{(3)})}{x_p^{(3)} + y_p^{(1)}},$$

which, with the constraint (4.2), explains the formulae in (4.11).

The 4–component system (4.10) has 4 independent first integrals

$$I_1 = x_p^{(0)} y_p^{(0)}, \quad I_2 = x_p^{(2)} y_p^{(2)}, \quad I_3 = x_p^{(0)} x_p^{(2)}, \quad I_4 = x_p^{(0)} + x_p^{(2)} + y_p^{(0)} + y_p^{(2)},$$

so is periodic (and has period 2).

The remaining two equations (4.11) cannot be taken alone, but only as part of the 6–component system. This system has two more first integrals,

$$I_5 = x_p^{(1)} y_p^{(1)}, \quad I_6 = x_p^{(1)} + \frac{a}{x_p^{(0)} x_p^{(1)} x_p^{(2)}} + y_p^{(1)} + \frac{b}{y_p^{(0)} y_p^{(1)} y_p^{(2)}},$$

so is also periodic (of period 2). As commented after Proposition 4.2, this involutive property follows from  $\delta = N - \delta$  for this case.

**Remark 4.8 (Non-Coprime Case)** *This decoupling, when  $(N, \delta) \neq 1$ , is a general feature.*

#### 4.2.4 When $N = 5$

Here  $\delta = 1$  and  $\delta = 2$  give genuinely different maps.

**For  $\delta = 1$ :** The map  $R^{(1)}$  takes the same form as (4.9):

$$x_{p+1}^{(i)} = y_p^{(i)} \frac{A^{(i)}}{A^{(i+1)}}, \quad y_{p+1}^{(i)} = x_p^{(i)} \frac{A^{(i+1)}}{A^{(i)}}, \quad 0 \leq i \leq 3, \quad (4.12)$$

with upper indices taken (mod 4) and where

$$\begin{aligned} A^{(0)} &= a(x_p^{(1)} x_p^{(2)} x_p^{(3)} + x_p^{(2)} x_p^{(3)} y_p^{(0)} + x_p^{(3)} y_p^{(0)} y_p^{(1)} + y_p^{(0)} y_p^{(1)} y_p^{(2)}) + \prod_{i=0}^3 x_p^{(i)} y_p^{(i)}, \\ A^{(1)} &= A^{(0)} + (b - a)x_p^{(1)} x_p^{(2)} x_p^{(3)}, \quad A^{(2)} = A^{(1)} + (b - a)x_p^{(2)} x_p^{(3)} y_p^{(0)}, \\ A^{(3)} &= A^{(2)} + (b - a)x_p^{(3)} y_p^{(0)} y_p^{(1)}, \quad A^{(4)} = A^{(3)} + (b - a)y_p^{(0)} y_p^{(1)} y_p^{(2)}. \end{aligned}$$

**For  $\delta = 2$ :** The map  $R^{(2)}$  takes the form:

$$x_{p+1}^{(0)} = y_p^{(0)} \frac{A^{(2)}}{A^{(3)}}, \quad x_{p+1}^{(1)} = y_p^{(1)} \frac{A^{(0)}}{A^{(1)}}, \quad x_{p+1}^{(2)} = y_p^{(2)} \frac{A^{(3)}}{A^{(4)}}, \quad x_{p+1}^{(3)} = y_p^{(3)} \frac{A^{(1)}}{A^{(2)}}, \quad (4.13)$$

and  $y_{p+1}^{(i)} = \frac{x_p^{(i)} y_p^{(i)}}{x_{p+1}^{(i)}}$ , with upper indices taken (mod 4) and where

$$\begin{aligned} A^{(0)} &= a(x_p^{(3)} x_p^{(0)} x_p^{(2)} + x_p^{(0)} x_p^{(2)} y_p^{(1)} + x_p^{(2)} y_p^{(1)} y_p^{(3)} + y_p^{(1)} y_p^{(3)} y_p^{(0)}) + \prod_{i=0}^3 x_p^{(i)} y_p^{(i)}, \\ A^{(1)} &= A^{(0)} + (b-a)x_p^{(3)} x_p^{(0)} x_p^{(2)}, \quad A^{(2)} = A^{(1)} + (b-a)x_p^{(0)} x_p^{(2)} y_p^{(1)}, \\ A^{(3)} &= A^{(2)} + (b-a)x_p^{(2)} y_p^{(1)} y_p^{(3)}, \quad A^{(4)} = A^{(3)} + (b-a)y_p^{(1)} y_p^{(3)} y_p^{(0)}. \end{aligned}$$

#### 4.2.5 The Structure of the Formulae

The order of appearance of  $A^{(i)}$  in (4.13) and the combination of variables appearing in the definition of  $A^{(0)}$  is controlled by the following ordering of the variables  $x_p^{(i)}, y_p^{(i)}$ :

$$\{x^{(\delta-1)}, x^{(2\delta-1)}, \dots, x^{((N-1)\delta-1)}, y^{(\delta-1)}, y^{(2\delta-1)}, \dots, y^{((N-1)\delta-1)}\}.$$

When  $(N, \delta) = 1$ , the numbers  $\{(m\delta - 1)\}_{m=1}^{N-1}$  form a permutation of the numbers  $0, \dots, N-2$ , so all the variables are included in this list. The formulae (4.13) are just

$$x_{p+1}^{(m\delta-1)} = y_p^{(m\delta-1)} \frac{A^{(m-1)}}{A^{(m)}}, \quad 1 \leq m \leq N-1. \quad (4.14)$$

The coefficient of the parameter  $a$  in function  $A^{(0)}$  is constructed as follows: the first term is  $\frac{\prod_{i=0}^{N-2} x^{(i)}}{x^{(\delta-1)}}$ . We then repeatedly act by the permutation

$$x^{(\delta-1)} \rightarrow x^{(2\delta-1)} \rightarrow \dots \rightarrow x^{((N-1)\delta-1)} \rightarrow y^{(\delta-1)} \rightarrow y^{(2\delta-1)} \rightarrow \dots \rightarrow y^{((N-1)\delta-1)} \rightarrow x^{(\delta-1)},$$

for  $(N-2)$  times, which ends with  $\frac{\prod_{i=0}^{N-2} y^{(i)}}{y^{(N-1-\delta)}}$ . The coefficient of  $a$  is then just the sum of these  $(N-1)$  terms. The remaining term in  $A^{(0)}$  is just  $\prod_{i=0}^{N-2} x^{(i)} y^{(i)}$ .

The functions  $A^{(i)}$  are formed by successively changing the coefficient  $a$  to  $b$  at each of the terms in the above sum.

**Example 4.9 (The case  $N = 5, \delta = 2$ )** Here we have

$$x^{(1)} \rightarrow x^{(3)} \rightarrow x^{(0)} \rightarrow x^{(2)} \rightarrow y^{(1)} \rightarrow y^{(3)} \rightarrow y^{(0)} \rightarrow y^{(2)},$$

and

$$x_p^{(3)} x_p^{(0)} x_p^{(2)} \rightarrow x_p^{(0)} x_p^{(2)} y_p^{(1)} \rightarrow x_p^{(2)} y_p^{(1)} y_p^{(3)} \rightarrow y_p^{(1)} y_p^{(3)} y_p^{(0)},$$

giving the expression for  $A^{(0)}$ , given in the case of (4.13).

**Example 4.10 (The case  $N = 7, \delta = 3$ )** Here we have

$$x^{(2)} \rightarrow x^{(5)} \rightarrow x^{(1)} \rightarrow x^{(4)} \rightarrow x^{(0)} \rightarrow x^{(3)} \rightarrow y^{(2)} \rightarrow y^{(5)} \rightarrow y^{(1)} \rightarrow y^{(4)} \rightarrow y^{(0)} \rightarrow y^{(3)},$$

and

$$x_p^{(5)} x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} \rightarrow x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} y_p^{(2)} \rightarrow \dots \rightarrow y_p^{(2)} y_p^{(5)} y_p^{(1)} y_p^{(4)} y_p^{(0)},$$

giving

$$A^{(0)} = a(x_p^{(5)} x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} + x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} y_p^{(2)} + \dots + y_p^{(2)} y_p^{(5)} y_p^{(1)} y_p^{(4)} y_p^{(0)}) + \prod_{i=0}^5 x_p^{(i)} y_p^{(i)}.$$

The remaining  $A^{(i)}$  are then constructed by the above prescription and the map  $R^{(3)}$ , for  $N = 7$  is given by (4.14), for  $\delta = 3$ .

### 4.3 The Quotient Potential Case and Symmetries

In [4] we introduced two potential forms of our equations (3.5). Here we briefly mention the “quotient potential”, leaving the “additive potential” to Section 5.

Equations (3.5a) hold identically if we set

$$u_{m,n}^{(i)} = \alpha \frac{\phi_{m+1,n}^{(i)}}{\phi_{m,n}^{(i+k_1)}}, \quad v_{m,n}^{(i)} = \beta \frac{\phi_{m,n+1}^{(i)}}{\phi_{m,n}^{(i+k_2)}}, \quad (4.15)$$

where  $a = \alpha^N$ ,  $b = \beta^N$ . Equations (3.5b) then take the form

$$\alpha \left( \frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m,n+1}^{(i+k_1)}} - \frac{\phi_{m+1,n}^{(i+\ell_2)}}{\phi_{m,n}^{(i+\ell_2+k_1)}} \right) = \beta \left( \frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m+1,n}^{(i+k_2)}} - \frac{\phi_{m,n+1}^{(i+\ell_1)}}{\phi_{m,n}^{(i+\ell_1+k_2)}} \right), \quad (4.16)$$

where indices are taken (mod  $N$ ).

These equations have a weighted scaling symmetry, whose invariants are given exactly by the formulae (4.15), leading us back to equations (3.5) and therefore to our previous Yang-Baxter maps.

## 5 The Additive Potential

Equations (3.5b) hold identically if we set

$$u_{m,n}^{(i)} = \chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+\ell_1)}, \quad v_{m,n}^{(i)} = \chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+\ell_2)}. \quad (5.1)$$

Equations (3.5a) then take the form

$$\frac{\left( \chi_{m+1,n+1}^{(i)} - \chi_{m,n+1}^{(i+\ell_1)} \right)}{\left( \chi_{m+1,n+1}^{(i)} - \chi_{m+1,n}^{(i+\ell_2)} \right)} = \frac{\left( \chi_{m+1,n}^{(i+k_2)} - \chi_{m,n}^{(i+k_2+\ell_1)} \right)}{\left( \chi_{m,n+1}^{(i+k_1)} - \chi_{m,n}^{(i+k_1+\ell_2)} \right)}, \quad (5.2)$$

and the first integrals (3.7) take the form

$$\prod_{i=0}^{N-1} \left( \chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+\ell_1)} \right) = a, \quad \prod_{i=0}^{N-1} \left( \chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+\ell_2)} \right) = b. \quad (5.3)$$

**Remark 5.1 (Reduction)** *It is not always possible to use these first integrals to explicitly reduce (5.2) to a system with  $N - 1$  components (eliminating  $\chi_{m,n}^{(N-1)}$ ), and even when this is possible the spectral problem (3.1) cannot be written in terms of the reduced variables.*

In [4] we showed that it is possible to explicitly reduce the system with  $(k_i, \ell_i) = (0, 1)$ , which takes the form

$$\frac{\left(\chi_{m+1,n+1}^{(i)} - \chi_{m,n+1}^{(i+1)}\right)}{\left(\chi_{m+1,n+1}^{(i)} - \chi_{m+1,n}^{(i+1)}\right)} = \frac{\left(\chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+1)}\right)}{\left(\chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+1)}\right)}, \quad i = 0, \dots, N-3, \quad (5.4a)$$

$$\chi_{m+1,n+1}^{(N-2)} = \chi_{m,n}^{(0)} + \frac{1}{\chi_{m+1,n}^{(N-2)} - \chi_{m,n+1}^{(N-2)}} \left( \frac{a}{X} - \frac{b}{Y} \right), \quad (5.4b)$$

where  $X = \prod_{j=0}^{N-3} (\chi_{m+1,n}^{(j)} - \chi_{m,n}^{(j+1)})$  and  $Y = \prod_{j=0}^{N-3} (\chi_{m,n+1}^{(j)} - \chi_{m,n}^{(j+1)})$ .

**Remark 5.2** *This is a direct generalisation of equation H1 in the ABS classification [1].*

It is easy to see that the system (5.4) has the following pair of symmetry generators:

$$\mathbf{X}_t = \sum_{i=0}^{N-2} \omega^{m+n+i} \partial_{\chi_{m,n}^{(i)}}, \quad (5.5a)$$

$$\mathbf{X}_s = \sum_{i=0}^{N-2} \omega^{m+n+i} \chi_{m,n}^{(i)} \partial_{\chi_{m,n}^{(i)}}, \quad \omega \neq 1, \quad (5.5b)$$

where  $\omega^N = 1$ . It is therefore possible to write equations (5.4) in terms of the invariants of these symmetries. We can then reduce this form of the lattice equations to Yang-Baxter maps.

### 5.1 The Invariants of $\mathbf{X}_t$

It is straightforward to write a suitable “basis” for the invariants of  $\mathbf{X}_t$ . The formulae are more symmetric if we write “too many” invariants, which then satisfy some additional identities. We therefore define  $4(N-1)$  invariants, satisfying  $(N-1)$  identities. Furthermore, we make the reduction (2.5), so that we derive a map. Following [8], we denote these invariants by

$$x^{(i)} \equiv x_p^{(i)}, \quad y^{(i)} \equiv y_p^{(i)}, \quad u^{(i)} \equiv x_{p+1}^{(i)}, \quad v^{(i)} \equiv y_{p+1}^{(i)}, \quad \text{where } p = n - m, \quad (5.6)$$

corresponding to specific edges of the lattice square, as shown in Figure 1 and noting that the shifts  $m \mapsto m-1$  and  $n \mapsto n+1$  both correspond to  $p \mapsto p+1$ .

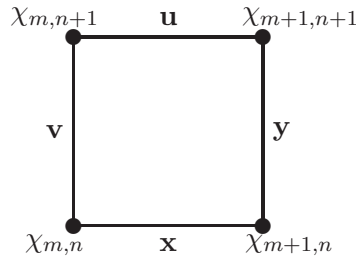


Figure 1: Invariants defined on edges

The  $4(N - 1)$  invariants:

$$\begin{aligned}
x^{(i)} &= \chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+1)}, & i = 0, \dots, N-3, & \quad x^{(N-2)} = \chi_{m+1,n}^{(N-2)} + \sum_{j=0}^{N-2} \chi_{m,n}^{(j)}, \\
y^{(i)} &= \chi_{m+1,n+1}^{(i)} - \chi_{m+1,n}^{(i+1)}, & i = 0, \dots, N-3, & \quad y^{(N-2)} = \chi_{m+1,n+1}^{(N-2)} + \sum_{j=0}^{N-2} \chi_{m+1,n}^{(j)}, \\
u^{(i)} &= \chi_{m+1,n+1}^{(i)} - \chi_{m,n+1}^{(i+1)}, & i = 0, \dots, N-3, & \quad u^{(N-2)} = \chi_{m+1,n+1}^{(N-2)} + \sum_{j=0}^{N-2} \chi_{m,n+1}^{(j)}, \\
v^{(i)} &= \chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+1)}, & i = 0, \dots, N-3, & \quad v^{(N-2)} = \chi_{m,n+1}^{(N-2)} + \sum_{j=0}^{N-2} \chi_{m,n}^{(j)},
\end{aligned}$$

satisfy  $(N - 1)$  identities:

$$x^{(i+1)} + y^{(i)} = u^{(i)} + v^{(i+1)}, \quad i = 0, \dots, N-3, \quad (5.7a)$$

$$y^{(N-2)} + \sum_{j=0}^{N-2} v^{(j)} = u^{(N-2)} + \sum_{j=0}^{N-2} x^{(j)}, \quad (5.7b)$$

and equations (5.4) take the form

$$u^{(i)}v^{(i)} = x^{(i)}y^{(i)}, \quad i = 0, \dots, N-3, \quad (5.7c)$$

$$u^{(N-2)} = \sum_{j=0}^{N-2} v^{(j)} + \frac{1}{x^{(N-2)} - v^{(N-2)}} \left( \frac{a}{\prod_{j=0}^{N-3} x^{(j)}} - \frac{b}{\prod_{j=0}^{N-3} v^{(j)}} \right). \quad (5.7d)$$

The Yang-Baxter map corresponds to the solution of equations (5.7) for  $(u^{(i)}, v^{(i)})$ . We do not have an explicit form of the solution in general, but for any given value of  $N$ , this can be found.

**Remark 5.3 (The Case  $N = 2$ )** We already remarked that for  $N = 2$  the lattice equation is just H1 in the ABS classification [1]. Using the symmetry  $\mathbf{X}_t$ , with  $\omega = -1$  leads to the Yang-Baxter map

$$u = y + \frac{a-b}{x-y}, \quad v = x + \frac{a-b}{x-y},$$

which is just  $F_V$  of the ABS classification of quadrirational maps [2] (the Adler map). Clearly, we may consider this whole family of maps as multi-component generalisations of  $F_V$ .

**Example 5.4 (The Case  $N = 3$ )** In this case, we find

$$\begin{aligned}
u^{(0)} &= y^{(0)} + \frac{(a-b)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
u^{(1)} &= y^{(1)} + \frac{(b-a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})} + \frac{(b-a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
v^{(0)} &= x^{(0)} + \frac{(b-a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
v^{(1)} &= x^{(1)} + \frac{(b-a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}.
\end{aligned}$$

## 5.2 The Invariants of $\mathbf{X}_s$

Again we denote invariants as in (5.6) and Figure 1. The  $4(N-1)$  invariants:

$$\begin{aligned} x^{(i)} &= \frac{\chi_{m+1,n}^{(i)}}{\chi_{m,n}^{(i+1)}}, \quad i = 0, \dots, N-3, & x^{(N-2)} &= \chi_{m+1,n}^{(N-2)} \prod_{j=0}^{N-2} \chi_{m,n}^{(j)}, \\ y^{(i)} &= \frac{\chi_{m+1,n+1}^{(i)}}{\chi_{m+1,n}^{(i+1)}}, \quad i = 0, \dots, N-3, & y^{(N-2)} &= \chi_{m+1,n+1}^{(N-2)} \prod_{j=0}^{N-2} \chi_{m+1,n}^{(j)}, \\ u^{(i)} &= \frac{\chi_{m+1,n+1}^{(i)}}{\chi_{m,n+1}^{(i+1)}}, \quad i = 0, \dots, N-3, & u^{(N-2)} &= \chi_{m+1,n+1}^{(N-2)} \prod_{j=0}^{N-2} \chi_{m,n+1}^{(j)}, \\ v^{(i)} &= \frac{\chi_{m,n+1}^{(i)}}{\chi_{m,n}^{(i+1)}}, \quad i = 0, \dots, N-3, & v^{(N-2)} &= \chi_{m,n+1}^{(N-2)} \prod_{j=0}^{N-2} \chi_{m,n}^{(j)}, \end{aligned}$$

satisfy  $(N-1)$  identities:

$$u^{(i)}v^{(i+1)} = x^{(i+1)}y^{(i)}, \quad i = 0, \dots, N-3, \quad (5.8a)$$

$$u^{(N-2)} \prod_{j=0}^{N-2} x^{(j)} = y^{(N-2)} \prod_{j=0}^{N-2} v^{(j)}, \quad (5.8b)$$

and equations (5.4) take the form

$$u^{(i)}v^{(i+1)} = \frac{(v^{(i)} - 1)v^{(i+1)} - (x^{(i)} - 1)x^{(i+1)}}{v^{(i)} - x^{(i)}}, \quad i = 0, \dots, N-3, \quad (5.8c)$$

$$u^{(N-2)} = \left( 1 + \frac{1}{x^{(N-2)} - v^{(N-2)}} \left( \frac{a}{X} - \frac{b}{Y} \right) \right) \prod_{j=0}^{N-2} v^{(j)}, \quad (5.8d)$$

where  $X = \prod_{j=0}^{N-3} (x^{(j)} - 1)$ ,  $Y = \prod_{j=0}^{N-3} (v^{(j)} - 1)$ .

**Remark 5.5 (The Case  $N = 2$ )** Again, since the lattice equation is just H1 in the ABS classification [1], the symmetry  $\mathbf{X}_s$ , with  $\omega = -1$ , leads to the Yang-Baxter map

$$u = y \left( 1 + \frac{a-b}{x-y} \right), \quad v = x \left( 1 + \frac{a-b}{x-y} \right),$$

which is just  $F_{IV}$  of the ABS classification of quadrirational maps [2]. Clearly, we may consider this whole family of maps as multi-component generalisations of  $F_{IV}$ .

**Example 5.6 (The Case  $N = 3$ )** In this case, we first define

$$P_a = ax^{(0)} - (x^{(0)} - 1)(y^{(0)} - 1)(x^{(0)}x^{(1)} - y^{(1)}), \quad P_b = bx^{(0)} - (x^{(0)} - 1)(y^{(0)} - 1)(x^{(0)}x^{(1)} - y^{(1)}).$$

We then have the map

$$\begin{aligned}
u^{(0)} &= y^{(0)} \left( 1 - \frac{(a-b)x^{(0)}(y^{(0)}-1)}{(y^{(0)}-1)P_a - y^{(0)}P_b} \right), \\
u^{(1)} &= y^{(1)} \left( 1 - (a-b) \left( \frac{(x^{(0)}-1)y^{(0)}}{P_a} + \frac{(y^{(0)}-1)}{P_b} \right) \right), \\
v^{(0)} &= x^{(0)} \left( 1 - \frac{(a-b)(x^{(0)}-1)}{P_a} \right), \\
v^{(1)} &= x^{(1)} \left( 1 - \frac{(a-b)(y^{(0)}-1)x^{(0)}}{P_b} \right).
\end{aligned}$$

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