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An Improved Least Squares Monte Carlo Valuation Method Based on Heteroscedasticity

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Abstract

Longstaff-Schwartz's least squares Monte Carlo method is one of the most applied numerical methods for pricing American-style derivatives. We examine the algorithms regression step, demonstrating that the OLS regression is not the best linear unbiased estimator because of heteroscedasticity. We prove the existence of heteroscedasticity for single-asset and multi-asset payoffs numerically and theoretically, and propose weighted-least squares MC valuation method to correct for it. An extensive numerical study shows that the proposed method produces significantly smaller pricing bias than the Longstaff-Schwartz method under several well-known price dynamics. An empirical pricing exercise using market data confirms the advantages of the improved method.

Keywords: Finance, American options, Heteroscedasticity, Weighted least squares, Least squares Monte Carlo pricing method

JEL: G13, C51, C63

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1. Introduction

The problem of pricing American-style derivatives has been extensively examined over the past 40 years[1]. A long series of papers have focused on the approximation of the conditional expected payoff to the option holder from continuation. While they all use regression methods in a dynamic programming context, they have distinctive features. Carriere (1996) estimates the continuation value along each simulated path by employing spline regressions and regressions with a local polynomial smoother, while Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001) employ the ordinary least squares (OLS) regression.

The regression-based methods for pricing American options are centered on the least squares Monte Carlo (LSMC) method described in Longstaff and Schwartz (2001). Stentoft (2014) justified the widespread use of LSMC by noting that it has the smallest absolute bias and less error accumulation when compared to other regression-based algorithms. Longstaff and Schwartz (2001) proved the convergence for problems with one state variable and only one exercise date (except maturity). Clement et al. (2002) showed that, for a given set of basis functions, the error resulting from Monte Carlo simulation goes to zero when the number of paths goes to infinity. Within

[1]Recently, several important applications emerged that have successfully employed the LSMC algorithm in fields other than the American option pricing problem. Jarrow et al. (2010) priced callable bonds via the LSMC method, showing that the same technique can be applied to mortgage-backed securities. Carmona and Ludkovski (2010) utilised the LSMC for optimal switching models with inventory to evaluate energy storage facilities. Broadie and Detemple (1996, 2004), Glasserman (2003) and Detemple (2005) provide reviews of the literature related to this issue.

a multi-dimensional and multi-period setting, Stentoft (2004b) proved convergence as the number of basis functions M and the number of paths n_s go to infinity with $M^3/n_s \to 0$. Studying the LSMC near the beginning of the contract when the time-step size approaches zero, Mostovyi (2013) found that the regression problem is ill-posed, making the LSMC unstable.

On the computational side, Moreno and Navas (2003), Stentoft (2004a) and Areal et al. (2008) assessed the pricing performance of LSMC under different numbers of simulated paths, payoff structures and polynomial families in the regressions. They argued that the performance of LSMC is virtually the same for vanilla options when different polynomial families are employed but that their selection has a major impact in the case of exotic options. Wang and Caflisch (2010) modified LSMC to calculate directly the delta and the gamma parameters.

The literature has shown that the pricing bias in the LSMC is a combination of the downward bias caused by the approximation of this curve by a finite low-dimensional polynomial, and the upward bias caused by using the same paths to estimate the optimal stopping time (see, among others, Létourneau and Stentoft (2014)). Létourneau and Stentoft (2014) employed the linear inequality constrained least squares (ICLS) method to impose monotonicity and convexity properties on the continuation value curve, as the theoretical results suggest. They showed that the ICLS algorithm is less prone to curve-overfitting compared to LSMC and thus the upward pricing bias is substantially reduced.

Another improvement for the Monte Carlo regression used in American option pricing has been described in Belomestny (2011) and Belomestny et al. (2015) where they apply local polynomial kernel regression to the problem of pricing Bermudan options. The idea is to generate an additional independent set of Monte Carlo sample paths to the sample already used in the prior regression step and then average the payoffs stopped according to simple rule that, although is suboptimal, is capable of producing a low-biased estimate for the option price that has improved convergence properties, as discussed in Zanger (2016). Other authors that used methods for generating a new set of independent random paths corresponding to the underlying process at each exercise time increment, independent of all the other sets of paths generated at all other time-steps, were Glasserman and Yu (2004), Egloff et al. (2007) and Zanger (2013). A survey of regression-based Monte Carlo methods for pricing American options can be found in Kohler (2010) and an excellent discussion of the convergence of various algorithms proposed for pricing American options is contained in Zanger (2013).

The calibration and parameter estimation processes can be sometimes impossible to separate and model misspecification may be difficult to disentangle given the information available in the options market, as pointed out by Jarrow and Kwok (2015). Even when the data generating process is fully known, Monte Carlo pricing methods coupled with least-squares algorithms may be subject to inefficient parameter estimation. This seems to be the case in the literature employing LSMC, as we highlight here, and this phenomenon goes beyond the geometric Brownian motion standard assumption widely utilized when pricing American options.

In this paper, we propose an improved pricing method which we refer to as the weighted least squares Monte Carlo (wLSMC) for American put option pricing. The wLSMC, similar in structure to the LSMC, employs the weighted least squares regression (WLS) method instead of the OLS method. We proceed by proving that the homoscedasticity of the errors, one of the assumptions underpinning the OLS method, does not hold for the regressions in the LSMC. Consequently, the errors of the regressions of LSMC are heteroscedastic, a condition which makes the OLS estimators not the best linear unbiased estimators (BLUE). We show that in LSMC, the OLS estimators tend to exhibit large pricing bias because they are more prone to overfitting the continuation value curve. Our analysis extends to the multi-asset American option pricing case, where similar results are valid. Here we also emphasize the importance of an improved estimation and we provide numerical evidence that correcting for heteroscedasticity in our proposed wLSMC method also improves the option price estimators.

The outline of this paper is as follows. In Section 2 we review the American option pricing problem and the LSMC method in Longstaff and Schwartz (2001). Section 3 provides substantial evidence on the existence of heteroscedasticity in each regression step of the LSMC method. After introducing the wLSMC method in Section 4, we compare the pricing performances of the LSMC, ICLS and wLSMC methods under several price dynamics and show how the wLSMC reduces significantly the upward pricing bias of LSMC and ICLS. Section 5 expands the results in the previous sections to multi-asset payoffs. Section 6 highlights the application of our method for stochastic volatility models. A detailed numerical and empirical analysis based of the performance of the new method is provided in the Online Appendix. Section 7 concludes our paper.

2. American Options and the LSMC Method

Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ associated to a financial market consisting of three assets: a bank account $dM_t = rM_t dt$, where the risk-free interest rate r is assumed constant over time, a risky asset with the dynamics $\{S_t\}_{t\geq 0}$ given under the risk-neutral measure $\mathbb{Q}$ as

$S_t = S_0 e^{s_t}$, where $S_0 > 0$ and $\{s_t\}_{t \geq 0}$ is a Markovian process with $s_0 = 0$, and an American put option written on the risky asset (usually referred to as the underlying asset) with strike price K and maturity date T. The pricing problem for the American put option can be formulated as the problem to find the optimal expected discounted payoff given by $\sup_{\tau \in \Gamma} E_0 \left[h(S_\tau) | S_0 \right]$, where $h(S_t) = e^{-rt} \max\{0, K - S_t\}$ is the payoff in time-0 dollars to the option holder from exercise at time t and Γ is the class of admissible stopping times in $(0, T]$. The numerical applications we carry out in this paper are for the four different dynamics outlined in the Online Appendix: geometric Brownian motion, exponential Ornstein-Uhlenbeck process, log-normal jump-diffusion process and double exponential jump diffusion process. In addition, we also investigate two stochastic volatility models.

Numerical methods usually restrict the pricing of American options to contracts that can be exercised only at a fixed set of exercise opportunities $t_1 < t_2 < \ldots < t_m = T$ and $t_0 = 0$, the time of evaluation, is not usually part of this set. Without loss of generality, we can assume that $\Delta_{t_i} = t_{i+1} - t_i = T/m = \Delta_t$, for any $i = 0, \ldots, m-1$. Henceforth, to simplify the notation under the discrete-time settings, we denote the underlying asset price at the ith exercise opportunity (the one at time t_i), simply as S_i so the logarithmic return over the period (t_i, t_{i+1}) will be $s_{i+1} - s_i$; the payoff function in time-0 dollars for exercise at time t_i when current state is $S_i = \mathcal{X}$ as

$$h_i(\mathcal{X}) = r_{0,i} \max\{0, K - \mathcal{X}\} \tag{1}$$

where $r_{0,i} = e^{-ri\Delta_t}$ and $V_i(\mathcal{X})$ is the value in time-0 dollars of the option at time t_i given $S_i = \mathcal{X}$, which is calculated with the dynamic programming:[2]

$$\begin{cases} V_m(\mathcal{X}) = h_m(\mathcal{X}) & (2) \\ V_i(\mathcal{X}) = \max\{h_i(\mathcal{X}), C_i(\mathcal{X})\},\ i = 0, \cdots, m-1 & (3) \end{cases}$$

where

$$C_i(\mathcal{X}) = E_i\left[V_{i+1}(S_{i+1})|S_i = \mathcal{X}\right] \tag{4}$$

is the continuation value of the American put option measured in time-0 dollars conditioned on the current state $\mathcal{X}$ and $E_i[\cdot]$ is the expectation operator under the risk-neutral measure $\mathbb{Q}$. One is ultimately interested in $V_0(S_0)$. Furthermore, let us define S_{f_i} as the underling asset price such that $h_i(S_{f_i}) = C_i(S_{f_i})$ which is commonly referred to as the optimal exercise price (OEP). By employing the optimal exercise price[3], an equivalent formulation of problem (2)-(3) for American put options is

$$\begin{cases} V_m(\mathcal{X}) = h_m(\mathcal{X}) & (5) \\ V_i(\mathcal{X}) = \begin{cases} h_i(\mathcal{X}) & \text{if } \mathcal{X} \leq S_{f_i} \\ C_i(\mathcal{X}) & \text{if } \mathcal{X} > S_{f_i} \end{cases}, \quad i = 0, \ldots, m-1. & (6) \end{cases}$$

The LSMC method in Longstaff and Schwartz (2001) solves the dynamic programming problem in (2)-(3) by combining Monte Carlo simulation and OLS regression method. Given a set of n_s simulated paths of the Markovian

[2]Note that time t_0 is excluded from the set of exercise opportunities by choosing $h_0(S_0) = 0$.

[3]Chockalingam and Muthuraman (2015) introduced the approximate moving boundaries method which iteratively finds an approximation of the OEP while Chockalingam and Feng (2015) extended Ibanez and Paraskevopoulos (2011) and investigated the cost of using suboptimal OEP. For long-term American options Fabozzi et al. (2016) designed a construction method for the OEP based on an approximation of the optimal exercise price near the beginning of the contract combined with existing quasi-analytical pricing approaches for the remaining part.

process $\{S_t\}_{t\geq 0}$, a set of $M+1$ basis functions[4] $\psi_l(\cdot):\Re\mapsto\Re$ and a set of $M+1$ parameters $\beta_{i,l}\in\Re$, $l=0,\cdots,M$, for any time t_i with $i=1,\ldots,m-1$, Longstaff and Schwartz employ the following approximation for the continuation value (4):

$$\hat{C}_i(\mathcal{X})=\sum_{l=0}^{M}\beta_{i,l}\psi_l(\mathcal{X}), \tag{7}$$

The OLS method is applied to calculate the parameters $\beta_{i,l}$ from the pairs $\left(S_{i^{(j)}},V_{i+1}\left(S_{i+1^{(j)}}\right)\right), j=1\ldots,n_s$ where $S_{i^{(j)}}$ indicates the value of process $\{S_t\}_{t\geq 0}$ at time t_i for the j-th simulated path. The LSMC has the steps:

1. Simulate n_s independent paths $\{S_{1^{(j)}},\cdots,S_{m^{(j)}}\}, j=1,\cdots,n_s$,
2. Set the option terminal-value equal to $V_m(S_{m^{(j)}})=h_m(S_{m^{(j)}})$, $j=1,\ldots,n_s$,
3. Using backward dynamic programming for $i=m-1,\cdots,1$,
   (a) Select the set $\tilde{J}_i=\{j|h_i(S_{i^{(j)}})>0\}$ of in-the-money paths at time-step i
   (b) Run an OLS regression on the pairs $\left(S_{i^{(j)}},V_{i+1}\left(S_{i+1^{(j)}}\right)\right)$ for $j\in\tilde{J}_i$, with basis functions $\psi_l(\cdot)$, to determine $\beta_{i,l}$,
   (c) For each $j\in\tilde{J}_i$ set

$$V_i(S_{i^{(j)}})=\begin{cases}h_i(S_{i^{(j)}}), & h_i(S_{i^{(j)}})\geq\hat{C}_i(S_{i^{(j)}});\\ V_{i+1}\left(S_{i+1^{(j)}}\right) & h_i(S_{i^{(j)}})<\hat{C}_i(S_{i^{(j)}})\end{cases} \tag{8}$$

   with $\hat{C}_i(\cdot)$ as in (7) and $\beta_{i,l}$ found in step 3b. For $j\in\{1,\cdots,n_s\}\setminus\tilde{J}_i$

[4] It is usually required that $\psi_0(\cdot)=1$.

(out-of-the money paths), set $V_i(S_{i^{(j)}}) = V_{i+1}\big(S_{i+1^{(j)}}\big)$,

4. Set $V_0(S_0) = \frac{1}{n_s}\sum_{j=1}^{n_s} V_1(S_{1^{(j)}})$.

3. Heteroscedastic Errors in LSMC

In this section, we prove that the assumption of homoscedastic errors for the OLS regressions in the LSMC method (step 3b above) does not hold when the method is applied to the estimation of the continuation value of American put options.[5] This provides the foundation for the wLSMC method proposed in Section 4, which corrects LSMC for heteroscedasticity. Let us consider the regression at any given time-step $i = 1, \ldots, m-1$ and define u_i as the error of the time-t_i regression given the current price S_i:

$$u_i = V_{i+1}(S_{i+1}) - C_i(S_i) \tag{9}$$

which is a random variable dependent on $S_{i+1} = S_i e^{s_{i+1}-s_i}$. These errors are homoscedastic if

$$\mathrm{Var}[u_i|S_i = \mathcal{X}] = c, c \in \Re^+, \quad \text{for all } \mathcal{X} \in (0, K]. \tag{10}$$

We note that in (10) it is required that u_i, the variance of the error of the OLS regression at time-step i, is equal to a constant c whatever the value $\mathcal{X}$ taken by the underlying asset S_i. However, we shall show that for the regressions in step 3b there exist some values $\mathcal{X}_1$ and $\mathcal{X}_2$ of the underlying asset such that[6] $\mathrm{Var}[u_i|S_i = \mathcal{X}_1] \neq \mathrm{Var}[u_i|S_i = \mathcal{X}_2]$. This means that

[5] Similar results are found for American call options but are not reported here due to space constraints. Section 5 extends the results in Section 3 to American basket options.

[6] Note that the subscript i denotes a time period along the simulated paths whereas the subscript j is for different observation values associated with different paths. Moreover, the construction presented in this paper is feasible for any time period i.

the conditional variance of errors changes with the underlying spot price, a condition usually defined as heteroscedasticity of the errors.

To start with, we construct a set of American put options written on assets whose risk-neutral price dynamics are given by one of the following: geometric Brownian motion, exponential Ornstein-Uhlenbeck process, log-normal jump diffusion process and double exponential jump diffusion process. Overall, the set is generated from 160 scenarios spanned by the values for the parameters associated with the underlying processes. The values are similar to those shown in Longstaff and Schwartz (2001). Under geometric Brownian motion, the study is carried out on the 20 scenarios (rescaled for the strike price) described in (Longstaff and Schwartz, 2001, Table 1). The parameters are $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $T = \{1, 2\}$ year(s), $r = 6\%$ and $K = 1$. For the other three processes, we use the following additional parameters: the exponential Ornstein-Uhlenbeck process has $\eta \in \{0.15, 0.3\}$, $\mu = \{0, \log(0.9)\}$ and $T = 1$ year; the log-normal jump diffusion process has $\lambda \in \{0.5, 1\}$, $\alpha_J \in \{-0.25, 0.25\}$, $\sigma_J \in \{0.2, 0.4\}$ and $T = 1$ year, and the double exponential jump diffusion process has $q = 0.5$, $\lambda = 0.5$, $(\eta_1, \eta_2) \in \{(2, 3), (4, 6)\}$ and $T = 1$ year. In Table 1 we provide evidence on three main statistical tests, rejecting overall the null hypothesis of homoscedasticity resulting from the application of LSMC and highlighting the improvement achieved by the wLSMC correction.

3.1. Heuristic Evidence of Heteroscedasticity

This section provides a heuristic proof that the conditional standard deviation of the errors as a function of the values of the underlying asset prices is not constant and, consequently, provides evidence of the existence of heteroscedasticity in the LSMC algorithm. This argument is valid for any regression step of the LSMC procedure and our evidence and conclu-

Table 1: Results of statistical tests for heteroscedasticity: American put options

	Statistical test under LSMC					
	Park's		**White's**		**BPG**	
Critical Value	1%	5%	1%	5%	1%	5%
GBM	2.30%	2.03%	4.05%	2.84%	9.19%	7.36%
Exp. Ornstein-Uhlenbeck	4.18%	3.27%	5.77%	3.93%	13.01%	11.02%
Log-normal jumps	2.19%	1.43%	10.38%	6.61%	9.54%	7.17%
Double Exp. jumps	0.56%	0.36%	12.45%	6.79%	6.94%	4.08%
	Statistical test under wLSMC					
	Park's		**White's**		**BPG**	
Critical Value	1%	5%	1%	5%	1%	5%
GBM	21.69%	16.46%	36.51%	30.34%	49.50%	40.44%
Exp. Ornstein-Uhlenbeck	13.19%	10.16%	42.45%	35.81%	44.16%	35.83%
Log-normal jumps	30.07%	23.48%	79.94%	71.88%	83.16%	75.22%
Double Exp. jumps	20.27%	14.85%	76.48%	63.29%	86.17%	76.82%

Note: The entries in the table are the percentages of time for which it is not possible to reject the null hypothesis of homoscedasticity for the regressions in the LSMC and the wLSMC algorithm, respectively, for the 160 put option scenarios considered. A low percentage indicates serious evidence of heteroscedasticity among the option scenarios considered.

sions hold the same for all four models. The reason for considering such an alternative method is that the statistical tests considered above are based on the residuals rather than the (theoretical) errors in (9) and one may erroneously conclude that heteroscedasticity depends on the selection of the basis functions $\psi_l(\cdot)$ for the regressions at step 3b.

We start by using the formula for the standard deviation,

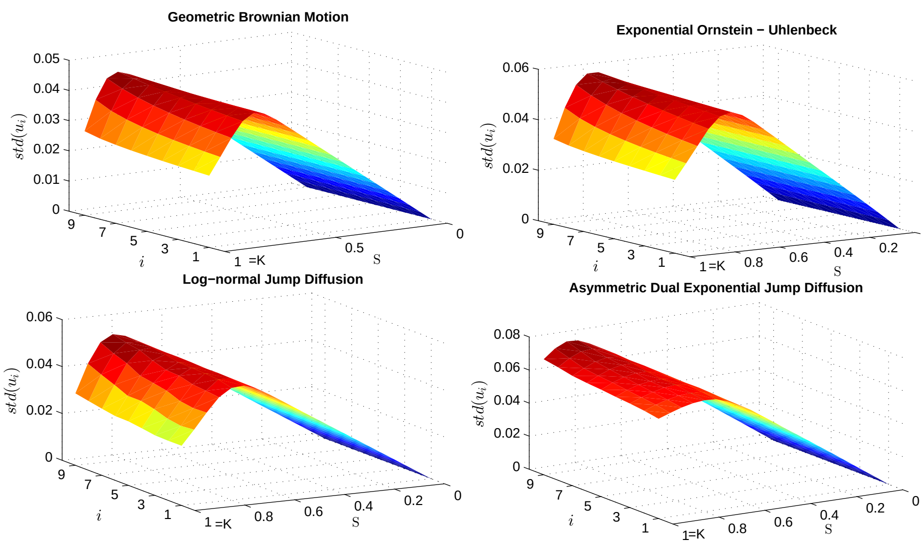
$$\mathrm{std}[u_i|S_i = \mathcal{X}] = \mathrm{std}[V_{i+1}(S_{i+1})|S_i = \mathcal{X}], \tag{11}$$

and estimate the conditional standard deviation on the right-hand side via the MC simulation technique. We consider the time-steps $i = 1, \ldots, T/\Delta_t$ with $\Delta_t = 0.1$ years and discretize[7] the underlying spot price range as

[7]As in the LSMC algorithm, we consider in-the-money paths only.

$S = 0, \Delta_S, \ldots, K$ with step size $\Delta_S = 0.05$. For each point on the grid (i, S), we simulate $N_h = 100$ stock prices at the next time-step $(i+1)$ conditional on the spot price at the current time-step $\mathcal{X}$. We then price the American options for each of the N_h underlying spot prices at time t_{i+1} using the appropriate binomial tree method, see the Online Appendix. Then, we calculate the standard deviation of the N_h prices. The plots in Figure 1

Figure 1: Heteroscedasticity in the regressions of the LSMC algorithm via simulation: American put options



Note: The plots report the conditional standard deviations $\text{std}(S) = \sqrt{\text{Var}[u_i|S_i = \mathcal{X}]}$, $i = 1, \ldots, 9$ calculated using formula (12) for American put options under four price dynamics. All price dynamics have $\sigma = 20\%$, $r = 6\%$, $K = 1$ and $T = 1$. Moreover, the exponential Ornstein-Uhlenbeck process has $\eta = 0.15$ and $\mu = 0$; the log-normal jump-diffusion has $\alpha_J = -0.25$, $\sigma_J = 0.2$ and $\lambda = 0.5$; the double exponential jump-diffusion has $\lambda = 0.5$, $\eta_1 = 2$, $\eta_2 = 3$ and $q = 0.5$. The plots are created for a grid with $\Delta_S = 0.05$ and $\Delta_T = 0.1$ years. For each point on the grid (i, S), $N_h = 100$ (50+50 antithetic) simulations of S_{i+1} (conditional on $S_i = \mathcal{X}$) are calculated together with the price of the option with time-to-maturity $T - t_{i+1}$ and underlying spot price S_{i+1}. The option prices are calculated using the binomial tree method. The plots represent the standard deviation of the N_h prices.

illustrate the standard deviations of the errors for American options priced under different models. Each cross section along the i axis is the standard deviation of the regression errors conditional on the underlying asset price being S at time-step i. Figure 1 indicates that the conditional standard deviation changes with the level of price S and hence, for the eight selected options, the errors are heteroscedastic. These patterns can be observed for all the other scenarios and there is graphical evidence of heteroscedasticity for the 160 option scenarios considered.

In the next section we provide a formal proof of heteroscedasticity for an underlying asset whose price follows the general dynamics $S_t = S_0 e^{s_t}$ where $\{s_t\}_{t\geq 0}$ is a Markovian process. In what follows, we make use of the equality

$$\mathrm{Var}[u_i|S_i = \mathcal{X}] = \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathcal{X}], \tag{12}$$

which follows from (9).

3.2. Correcting for Heteroscedasticity

Let us consider the American put option written on the risky asset whose price dynamics $S_t = S_0 e^{s_t}$ has the first two moments finite and is defined such that $S_0 > 0$ and $\{s_t\}_{t\geq 0}$ is a Markovian process with $s_0 = 0$. In the following proposition, we prove the main result of the paper.

Proposition 3.1. *The errors of the regressions in the LSMC algorithm for the American put option are heteroscedastic.*

A proof of this proposition is detailed in the Online Appendix. In Section 5 we also offer a general proof for the multi-asset case.

The proposition and the numerical evidence highlighted above indicate that the $\beta_{i,l}$ estimated via OLS regression in LSMC is not BLUE. Section 4 introduces a new pricing method which retains the BLUE property even in the presence of heteroscedastic errors and shows the positive effect of this new pricing method on the pricing bias.

4. The Weighted LSMC Valuation Method

It is well known that under all classical linear regression assumptions with the exception of homoscedasticity of the errors, the WLS estimators are BLUE, see Greene (2012). This theorem and the results in Section 3 lay the foundation for us to introduce the new valuation method for American option pricing, the weighted least squared Monte Carlo method.

4.1. Description of the valuation method

The new method is equivalent to the LSMC whose step 3b is substituted by

$3b^w$ Run a WLS regression on the pairs $\big(S_{i(j)}, V_{i+1}\big(S_{i+1(j)}\big)\big)$ for $j \in \tilde{J}_i$, with basis functions $\psi_l\left(\cdot\right)$, to determine $\beta^w_{i,l}$.

Fitting a WLS regression corresponds to computing an OLS regression for the transformed variables

$$\frac{\psi_l\left(S_{i(j)}\right)}{\mathrm{std}(S_{i(j)})} \rightarrow \psi^w_l\left(S_{i(j)}\right), \quad \frac{V_{i+1}\big(S_{i+1(j)}\big)}{\mathrm{std}(S_{i(j)})} \rightarrow V^w_{i+1}\big(S_{i+1(j)}\big) \tag{13}$$

where $\mathrm{std}(S_{i(j)}) = \sqrt{\mathrm{Var}[u_i|S_i = S_{i(j)}]}$ is the conditional standard deviation of the errors and $\psi_l\left(\cdot\right)$ are the basis functions in (7). Thus, the continuation value is now calculated as: $\hat{C}_i(\mathcal{X}) = \sum_{l=0}^{M} \beta^w_{i,l}\psi_l\left(\mathcal{X}\right)$. For the estimation of $\mathrm{std}(\cdot)$, we resort to the two-step algorithm outlined in Greene (2012). This algorithm first computes the conditional standard deviation using the OLS residuals; then, in the second step, it employs this standard deviation as a weighting function in the WLS regression. We assume the structure:

$$\hat{u}^2_{i,j} = \varphi_0 + \varphi_1 S_{i(j)} + \varphi_2 S^2_{i(j)} \tag{14}$$

where $\hat{u}_{i,j}$ is the residual resulting from the OLS regression in correspondence to state $S_{i(j)}$ and $\hat{u}_{i,j}^2$ is used as a proxy for the variances of the residuals.

5. The Case of Multi-asset Payoffs

In this section, we extend the newly improved LSMC valuation method based on heteroscedasticity to the case of multi-asset payoffs. For this purpose, let us expand the financial market considered in Section 2 by considering Υ risky assets whose price dynamics under the risk-neutral measure $\mathbb{Q}$ are of the type $S_t^{(k)} = S_0^{(k)} e^{s_t^{(k)}}, k = 1, \ldots, \Upsilon$ where $S_0^{(k)} > 0$ and $\left\{ s_t^{(k)} \right\}_{t \geq 0}$ is a Markovian process with $s_0^{(k)} = 0$. We indicate by $\mathbf{S}_t = [S_t^{(1)}, S_t^{(2)}, \ldots, S_t^{(\Upsilon)}]$ the column vector of the underlying asset prices at time t.

Under the discrete settings introduced in Section 2, the quantities defined above will be indicated as $S_i^{(k)}$, $s_i^{(k)}$ and $\mathbf{S}_i$ where i stands for the early exercise date at t_i. Additionally the correlation between any couple of assets (k_1, k_2) is given as $\mathrm{cov}(e^{s_{i+1}^{(k_1)}}, e^{s_{i+1}^{(k_2)}} | S_i^{(k_1)}, S_i^{(k_2)}) = \sigma_{k_1,k_2} \in \Re$ and we indicate by Ξ the variance-covariance matrix. The multi-asset payoff function in time 0 dollars for exercise at time t_i when current state is $\mathbf{S}_i = \boldsymbol{\mathcal{X}} = [\mathcal{X}^{(1)}, \mathcal{X}^{(2)}, \ldots, \mathcal{X}^{(\Upsilon)}]$ is defined as

$$h_i(\boldsymbol{\mathcal{X}}) = r_{0,i+1} \max \{0, K - B(\boldsymbol{\mathcal{X}})\} \tag{15}$$

This represents an American basket put option, where

$$B(\boldsymbol{\mathcal{X}}) = \sum_{k=1}^{\Upsilon} a_k \mathcal{X}^{(k)} \tag{16}$$

is the basket value and the asset weights, a_k, are deterministic and can be both positive and negative.

Equivalently, we also consider the multi-asset continuation value function $C_i(\cdot) : \Re^{\Upsilon} \mapsto \Re$ as

$$C_i(\boldsymbol{\mathcal{X}}) = E_i \left[V_{i+1}(\mathbf{S}_{i+1}) | \mathbf{S}_i = \boldsymbol{\mathcal{X}} \right], \tag{17}$$

where $V_i(\cdot) : \Re^{\Upsilon} \mapsto \Re$ is the value in time-0 dollars of the multi-asset option at time t_i. Under these multi-asset settings, the LSMC algorithm follows the same steps as defined in Section 2 and the set of basis functions $\psi_l(\cdot) : \Re^{\Upsilon} \mapsto \Re$ also includes the cross product terms. Finally, we define the error of the time-t_i regression given the current prices $\mathbf{S}_i$ as

$$u_i = V_{i+1}(\mathbf{S}_{i+1}) - C_i(\mathbf{S}_i),$$

which, similar to the unidimensional case in (9), is a random variable dependent on the Υ random variables $S_{i+1}^{(k)} = S_i^{(k)} e^{s_{i+1}^{(k)} - s_i^{(k)}}, i = 1, \ldots, \Upsilon$.

As in Section 3, we start identifying heteroscedasticity over a set of simulated option scenarios. In particular, we considered the five American basket option scenarios in (Kovalov et al., 2007, Table 1). The underlying baskets are composed of assets whose prices follow a geometric Brownian motion. The number of assets in the five scenarios are $\Upsilon = \{2, 3, 4, 5, 6\}$, the risk-free rate $r = 3\%$, the strike price $K = 100$ and maturity is at $T = 0.25$ year. Additionally, the underlying spot prices are $S_0^{(k)} = 100$, the asset weights $a_k = \frac{1}{\Upsilon}$, the variances of the log-returns are 20% and the correlation among each couple of assets is 0.5.

The results of the statistical tests are reported in Table 2 and show that heteroscedasticity is also an issue for the vast majority (more than 87%) of the regressions of the LSMC applied to American basket options. Moreover, considering the graphical test proposed in Section 3.1, the graphs in

Table 2: Statistical tests for heteroscedasticity: American basket options

		Park's		**White's**		**BPG**	
	Critical value	1%	5%	1%	5%	1%	5%
No. of Assets	2	14.29%	12.24%	10.20%	6.12%	6.12%	2.04%
	3	12.24%	12.24%	8.16%	8.16%	18.37%	10.20%
	4	22.45%	12.24%	16.37%	8.16%	14.29%	8.16%
	5	16.33%	14.29%	2.04%	2.04%	18.37%	16.33%
	6	12.24%	10.20%	8.16%	6.12%	20.41%	18.37%

Note: The entries in the table are the percentages of time for which it is not possible to reject the null hypothesis of homoscedasticity for the regressions in the LSMC algorithm for the 5 basket option scenarios in (Kovalov et al., 2007, Table 1). A low percentage indicates serious evidence of heteroscedasticity among the option scenarios considered. The 'No. of assets' represents the number of assets in the basket of the option scenario.

Figure 2 show, for a regression of the LSMC applied to each basket option with $\Upsilon = 2$, that the conditional standard deviation of errors is not constant for any of the three different multi-asset American options, suggesting heteroscedasticity. Finally, in the following proposition, we provide a theoretical proof for generic asset prices of the type $S_t^{(k)} = S_0^{(k)} e^{s_t^{(k)}}$ with the first two moments finite and defined such that $S_0^{(k)} > 0$ and $\{s_t^{(k)}\}_{t\geq 0}$ is a Markovian process with $s_0^{(k)} = 0$. These dynamics include but are not limited to correlated geometric Brownian motion, exponential Ornstein-Uhlenbeck process, various jump processes and also a mixture of them.

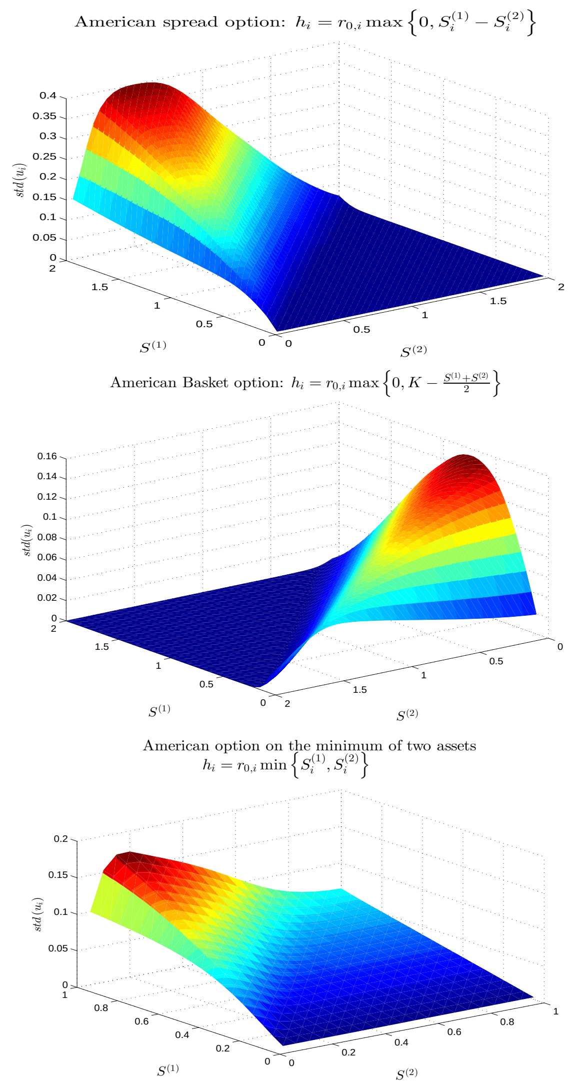
Proposition 5.1. *The errors of the regressions in the LSMC method for the American basket put option are heteroscedastic.*

The proof consists in showing that for any $\boldsymbol{\mathcal{X}}_2$, there is $\boldsymbol{\mathcal{X}}_1$ such that

$$\mathrm{Var}[u_i|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_1] < \mathrm{Var}[u_i|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2] \tag{18}$$

is satisfied. We consider the regression at any given time-step $i = 1, \ldots, m-1$ and we calculate a lower bound of the conditional variance in correspondence

Figure 2: Heteroscedasticity in the regressions of the LSMC algorithm via simulations: multi-asset American options



Note: The three plots show the conditional standard deviation of the errors $\text{std}(u_i) = \sqrt{\text{Var}[u_i|\mathbf{S}_i = \boldsymbol{\mathcal{X}}]}$ for i such that $t_i = 0.15$ year, as a function of the two underlying asset values, $S_i^{(1)} = S^{(1)}$ and $S_i^{(2)} = S^{(2)}$. The basket considered each time has $\Upsilon = 2$ assets whose price dynamics have $\sigma_k = 20\%$, $r = 3\%$, $K = 1$ (rescaled) and $T = 0.25$ year, $\Delta_S = 0.05$. For each point on the grid $(\mathcal{X}^{(1)}, \mathcal{X}^{(2)})$, $N_h = 100$ (50+50 antithetic) simulations of $\mathbf{S}_{i+1}$ (conditional on $\mathbf{S}_i = \boldsymbol{\mathcal{X}}$) are calculated together with the price of the option with time-to-maturity $T - t_{i+1}$ and underlying spot price $\mathbf{S}_{i+1}$. The option prices are calculated using the binomial tree method. The plotted points represent the standard deviation of the N_h prices.

of $\boldsymbol{\mathcal{X}}_2$ and an upper bound in correspondence of $\boldsymbol{\mathcal{X}}_1$ and then show that the upper bound can be made smaller than the lower bound. We calculate the lower bound for the variance of the error u_i for the right-hand-side of (18):

$$
\begin{aligned}
\mathrm{Var}[u_i|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2] &\geq \mathrm{Var}[V_{i+1}(\mathbf{S}_{i+1})|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2, X_{i+1} = 1]\mathbb{Q}(X_{i+1} = 1|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2) \\
&= \mathrm{Var}[h_{i+1}(\mathbf{S}_{i+1})|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2, X_{i+1} = 1]\mathbb{Q}(X_{i+1} = 1|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2) \\
&= r_{0,i+1}^2 \mathrm{Var}\left[\sum_{k=1}^{\Upsilon} a_k S_{i+1}^{(k)} |\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2, X_{i+1} = 1\right]\mathbb{Q}(X_{i+1} = 1|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2) \\
&= r_{0,i+1}^2 \mathbb{Q}(X_{i+1} = 1|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2)\times \\
&\times \sum_{k_1=1}^{\Upsilon}\sum_{k_2=1}^{\Upsilon} \tilde{\mathcal{X}}_2^{(k_1)}\tilde{\mathcal{X}}_2^{(k_2)} \mathrm{cov}\left[e^{\Delta s_{i+1}^{(k_1)}}, e^{\Delta s_{i+1}^{(k_2)}}|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2, X_{i+1} = 1\right] \\
&= r_{0,i+1}^2 \mathbb{Q}(X_{i+1} = 1|\mathbf{S}_i = \boldsymbol{\mathcal{X}}_2)\tilde{\boldsymbol{\mathcal{X}}}_2^T\, \Xi\, \tilde{\boldsymbol{\mathcal{X}}}_2
\end{aligned}
$$

where $\tilde{\boldsymbol{\mathcal{X}}}_2$ is the vector whose components are $\tilde{\mathcal{X}}_2^{(k)} = a_k \mathcal{X}_2^{(k)}$, Δ denotes the first difference operator, and X_{i+1} indicates whether the option is exercised at time-step $i+1$. The last term in (5) is strictly positive for any $\boldsymbol{\mathcal{X}}_2$ since Ξ is positive-definite[8] and the probability of an early exercise for an American-style derivative is strictly positive for at least one state $\boldsymbol{\mathcal{X}}_2$.

Conversely, we derive an upper bound for the conditional variance on the left-hand-side of (18). Let $g(\mathbf{S}_{i+1}) = \frac{V_{i+1}(\mathbf{S}_{i+1}) - V_{i+1}(\mathbf{0})}{B(\mathbf{S}_{i+1})}$ where $B(\cdot)$ is as defined in (16) and $\mathbf{0}$ is the null vector in $\Re^{\Upsilon}$. It can be proved that, for any $\mathbf{S}_{i+1}$, $g(\mathbf{S}_{i+1}) \in (-r_{0,i+1}, 0)$ since it correspond to the delta of the

[8] This follows from the fact that the quadratic form $\tilde{\boldsymbol{\mathcal{X}}}_2^T\, \Xi\, \tilde{\boldsymbol{\mathcal{X}}}_2$ corresponds to the variance of the basket.

discounted option price[9] $V_{i+1}(\cdot)$. Consequently,

$$E\left[\left(V_{i+1}(\mathbf{S}_{i+1})-V_{i+1}(\mathbf{0})\right)^2|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1\right]=E\left[B(\mathbf{S}_{i+1})^2g^2(\mathbf{S}_{i+1})|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1\right] \tag{19}$$

$$=E\left[\left(\sum_{k=1}^{\Upsilon}a_kS_{i+1}^{(k)}\right)^2g^2(\mathbf{S}_{i+1})|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1\right]\leq r_{0,i+1}^2E\left[\left(\sum_{k=1}^{\Upsilon}a_kS_{i+1}^{(k)}\right)^2|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1\right]$$

$$=r_{0,i+1}^2\sum_{k_1=1}^{\Upsilon}\sum_{k_2=1}^{\Upsilon}a_{k_1}a_{k_2}\mathcal{X}_1^{(k_1)}\mathcal{X}_1^{(k_2)}E\left[e^{\Delta s_{i+1}^{(k_1)}}e^{\Delta s_{i+1}^{(k_2)}}\right]=r_{0,i+1}^2\tilde{\boldsymbol{\mathcal{X}}}_1^T\Theta\tilde{\boldsymbol{\mathcal{X}}}_1,$$

where $\tilde{\mathcal{X}}_1^{(k)}=a_k\mathcal{X}_1^{(k)}$ and $\Theta\in\Re^{\Upsilon\times\Upsilon}$ is a symmetrical matrix with entries $E\left[e^{\Delta s_{i+1}^{(k_1)}}e^{\Delta s_{i+1}^{(k_2)}}\right]>0$. Then,

$$\mathrm{Var}[u_i|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1]=\mathrm{Var}[V_{i+1}(\boldsymbol{S}_{i+1})|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1] \tag{20}$$

$$=\mathrm{Var}[V_{i+1}(\boldsymbol{S}_{i+1})-V_{i+1}(\mathbf{0})|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1]\leq r_{0,i+1}^2\tilde{\boldsymbol{\mathcal{X}}}_1^T\Theta\tilde{\boldsymbol{\mathcal{X}}}_1.$$

since $E[V_{i+1}(\boldsymbol{S}_{i+1})-V_{i+1}(\mathbf{0})|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1]^2\geq 0$. The last term on the right hand side in (19) will be non-negative since $E\left[\left(V_{i+1}(\mathbf{S}_{i+1})-V_{i+1}(\mathbf{0})\right)^2|\mathbf{S}_i=\boldsymbol{\mathcal{X}}_1\right]$ is non-negative. Since the matrix Θ is symmetrical, it is known that

$$\tilde{\boldsymbol{\mathcal{X}}}_1^T\Theta\tilde{\boldsymbol{\mathcal{X}}}_1\leq\lambda_{max}\sum_{k=1}^{\Upsilon}(\tilde{\mathcal{X}}_1^{(k)})^2, \tag{21}$$

where λ_{max} is the maximum eigenvalue Θ.

Notice that the last term of (5) (lower bound) is strictly greater than the term on the right-hand side of (21) (upper bound), if there is a state

[9] In case of negative strike price, $g(\mathbf{S}_{i+1})\in(0,r_{0,i+1})$ and this proof is still valid.

vector $\boldsymbol{\mathcal{X}}_1$ such that

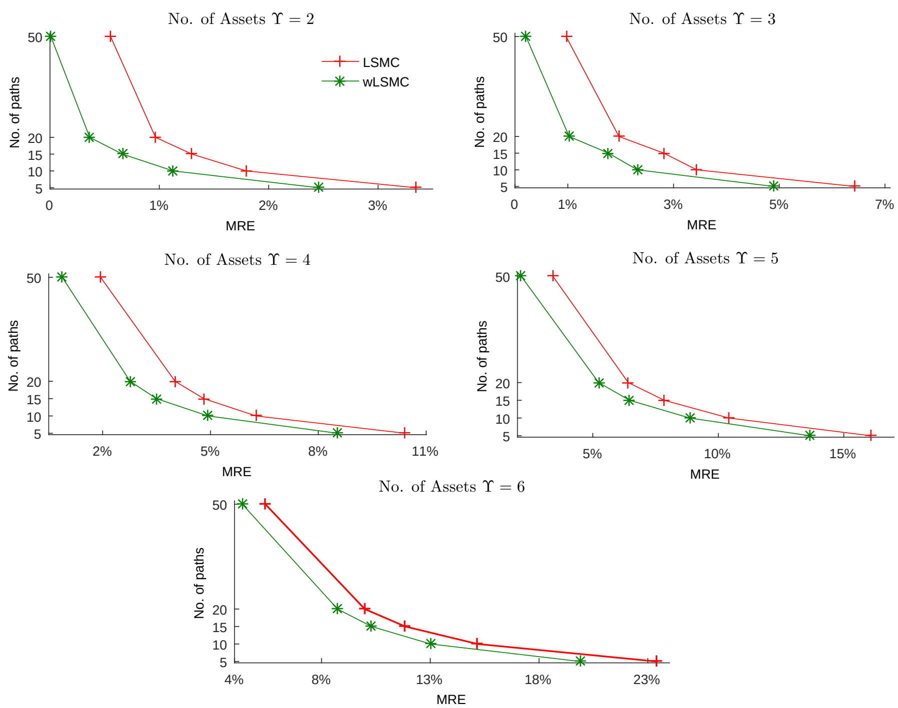
$$r_{0,i+1}^2 \lambda_{max} \sum_{k=1}^{\Upsilon} (\tilde{\mathcal{X}}_1^{(k)})^2 < r_{0,i+1}^2 \mathbb{Q}(X_{i+1} = 1 | \mathbf{S}_i = \boldsymbol{\mathcal{X}}_2) \tilde{\boldsymbol{\mathcal{X}}}_2^T \, \Xi \, \tilde{\boldsymbol{\mathcal{X}}}_2. \qquad (22)$$

The left-hand side term of (22) is a positive constant $r_{0,i+1}^2 \lambda_{max}$ times the square of the Euclidian norm $\| \cdot \|$ of the vector $\tilde{\mathcal{X}}_1$. There will always be at least one $\boldsymbol{\mathcal{X}}_1$ such that the left-hand side in (22) is equal to half the positive value on the right-hand side of (22).

The wLSMC method can be applied to correct for heteroscedasticity. We highlight how the option pricing with the LSMC can be improved. Table 3 shows the pricing performances of the LSMC and the wLSMC[10] for an increasing number of the polynomial order over the five basket option scenarios described above. As in the univariate case, the upper bias in the LSMC (MRE) and the price dispersion (RMSE) increase with the polynomial order. The correction for heteroscedasticity by using our wLSMC method has a significant impact in reducing both MRE and RMSE. Additionally, Figure 3 shows the evolution of the upper bias in pricing a for various number of paths and all basis functions polynomials of order 5. As for the univariate case, the wLSMC has a substantial impact in reducing the pricing bias. The figure also reveals that the improvements in the error performance measure for the wLSMC are substantial. More evidence is presented in the Online Appendix.

[10]For the variance of the residual, similarly to (14), we assume the following structure: $\hat{u}_{i,j}^2 = \varphi_0 + \sum_{k=1}^{\Upsilon} \varphi_k S_{i(j)}^{(k)} + \sum_{k_1=1}^{\Upsilon} \sum_{k_2=k_1}^{\Upsilon} \varphi_{k_1 k_2} S_{i(j)}^{(k_1)} S_{i(j)}^{(k_2)}$

Figure 3: Pricing comparison for American basket option scenarios



Note: The methods compared are the least squares Monte Carlo (LSMC) and the weighted least squares Monte Carlo (wLSMC). The mean relative errors (MRE) are based on the mean over 100 independent simulations. The five option scenarios considered ($\Upsilon = 2, 3, 4, 5, 6$) are the American basket options in (Kovalov et al., 2007, Table 1). Fifty exercise dates per year are used and the label of each data point indicates the number of paths n_s (in thousands). The results are based on number of regressors equal to M=5.

Table 3: Pricing performance comparison for American basket options

			LSMC				wLSMC			
			Polynomial Order				Polynomial Order			
		Benchmark	2	3	4	5	2	3	4	5
No. Assets	2	3.1396	3.178	3.120	3.218	3.237	3.162	3.191	3.207	3.221
	3	2.944	3.001	3.040	3.071	3.124	2.981	3.018	3.041	3.079
	4	2.840	2.917	2.970	3.042	3.141	2.882	2.936	2.997	3.075
	5	2.772	2.865	2.940	3.055	3.2407	2.828	2.895	2.996	3.178
	6	2.718	2.829	2.938	3.120	3.514	2.776	2.883	3.056	3.477
		RMSE	0.082	0.144	0.248	0.444	**0.044**	**0.107**	**0.198**	**0.408**
		MRE	0.075	0.134	0.218	0.368	**0.043**	**0.102**	**0.176**	**0.323**

Note: The entries represent the prices for the five option scenarios in (Kovalov et al., 2007, Table 1) calculated by the LSMC algorithm of Longstaff and Schwartz (2001) and our wLSMC algorithm as average of 100 simulations each with n_S =1,000 simulated paths. The benchmark prices are calculated by using the FEM method in Kovalov et al. (2007). Per column, it is indicated the maximum polynomial order of the basis functions considered. The bottom two rows summarise the root mean squared relative error (RMSE) and the mean relative error (MRE).

6. Stochastic Volatility Models and Other Methods

Our adjustment based on wLSMC works well also with stochastic volatility models. Here we consider the stochastic volatility model Heston-type with Merton-type jumps Bates (1996) that will be called '*stochastic volatility with jumps*' and it is described by

$$\begin{aligned} S_t &= S_0 e^{X_t}, \quad dX_t = \left(r - \delta - \lambda\kappa - \frac{1}{2}Y_t\right) dt + \sqrt{Y_t} dW_t^1 + dZ_t, X_0 = 0 \\ dY_t &= \epsilon(\eta - Y_t)dt + \theta\sqrt{Y_t} dW_t^2 \end{aligned} \tag{23}$$

where W_t^1 and W_t^2 are standard Wiener processes with constant volatility, Z is a compound Poisson process with intensity λ and the jumps J are Gaussian distributed with mean γ and standard deviation δ_J.

In addition, we analyse the mean reverting stochastic volatility (MRSV) model described in Rambharat and Brockwell (2010) as

$$\begin{aligned} dS_t &= rS_t dt + \sigma(Y_t) S_t dW_t^1 \\ \sigma(Y_t) &= exp(Y_t) \\ dY_t &= \alpha(\beta - \frac{\lambda\gamma}{\alpha} - Y_t)dt + \gamma dW_t^2 \end{aligned} \quad (24)$$

where, as before, the two Wiener processes W_t^1 and W_t^2 have correlation ρ.

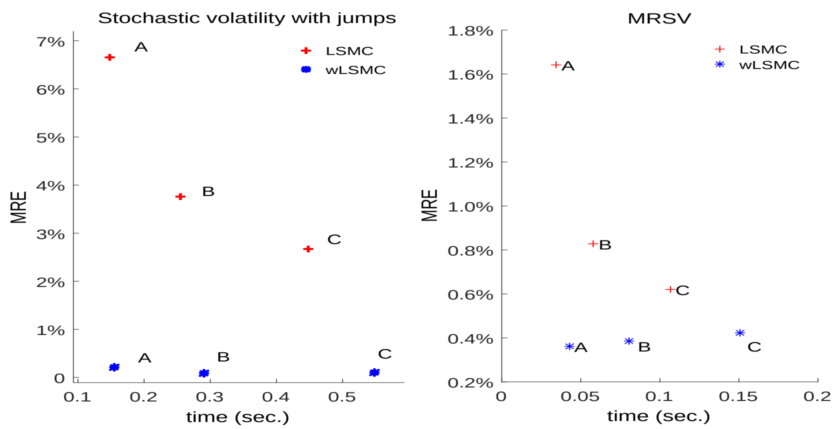
For the stochastic volatility model with jumps we run the comparison of LSMC and wLSMC over the 10 scenarios (normalized for strike): $K = 1$, $T = 0.5$ year, $r = 0.03$, $\delta = 0.05$, S_0 in the set $\{0.8, 0.9, 1, 1.1, 1.2\}$, $\epsilon = 2$, $\eta = 0.04$, $\theta = 0.4$, $\lambda = 5$, $\delta_J = 0.1$, $\gamma = -\frac{\delta_J^2}{2}$ and $\rho = \{0.5, -0.5\}$. $\kappa = \exp\gamma + \sigma_J^2/2 - 1$. The benchmark prices are taken from Chiarella et al. (2009) calculated by using finite difference approximations and Monte Carlo simulations on an extremely fine mesh. For the mean reverting stochastic volatility model we run the comparison over the nine scenarios described in Rambharat and Brockwell (2010) and summarised in Table 4.

Table 4: American option pricing scenarios for the mean reverting stochastic volatility model.

	ρ	α	β	γ	λ	T	r	S_0	σ_0	Benchmark
1	-0.055	3.3	log(0.55)	0.5	-0.1	0.040	0.055	0.86957	0.5	0.1328
2	-0.035	0.25	log(0.2)	2.1	-1	0.079	0.0255	0.88235	0.35	0.1279
3	-0.09	0.95	log(0.25)	3.95	-0.025	0.056	0.0325	0.9375	0.3	0.0806
4	-0.01	0.02	log(0.25)	2.95	-0.021	0.198	0.03	0.92593	0.5	0.1864
5	-0.03	0.015	log(0.35)	3	-0.02	0.198	0.0225	0.9	0.35	0.1695
6	-0.017	0.019	log(0.7)	2.5	-0.015	0.218	0.0325	0.89474	0.75	0.2574
7	-0.075	0.015	log(0.75)	6.25	0	0.067	0.0325	0.9375	0.35	0.1317
8	-0.025	0.035	log(0.15)	5.07	-0.015	0.060	0.055	1.11111	0.2	0.0115
9	-0.05	0.025	log(0.25)	4.5	-0.015	0.099	0.025	0.89474	0.35	0.1065

We report the results per unit of strike. Since we are interested in the convergence of the method, the benchmark prices are calculated by the standard LSMC method with 150000 paths. We consider the original scenarios plus the scenarios for maturity 10 times longer.

Figure 4: Stochastic volatility with jumps



Note: The two plots show the performance of wLSMC versus LSMC for three different sets of paths, where "A" is 2000 paths in the simulation, "B" is 5000 paths and "C" is 10000 paths. Performance is measured in time to complete the simulation on the horizontal axis and the MRE percentage error from benchmark prices on the vertical axis. The wLSMC values are in blue while the LSMC values are in red.

The results described in Figure 4 indicate that using the wLSMC method improves the MRE and RMSE measures at a small cost in terms of computational speed. In the online appendix we report in Table 8 the heteroscedasticity tests for the residuals of both LSMC and wLSMC under stochastic volatility. For both the mean-reverting stochastic volatility and the stochastic volatility with jumps, there is an improvement when applying wLSMC for models with stochastic volatility. Thus the argument for using the wLSMC method even in the stochastic volatility case becomes stronger[11].

7. Conclusion

In this paper we studied in detail the regression step of the least squares Monte Carlo algorithm for pricing American-style options. We evidenced

[11]We thank an anonymous referee for suggesting this line of investigation.

both numerically and theoretically that there exists heteroscedasticity in the regressions performed for pricing American put options, for several well known models of the underlying asset prices.

As a solution to this problem we proposed the weighted least squares Monte Carlo method that retains all the original steps of the (ordinary) least squares Monte Carlo method described in Longstaff and Schwartz (2001) but substitutes the ordinary least squares regression by its weighted version in order to account for heteroscedasticity. In our numerical study, we find that for each of the six considered underlying price dynamics the wLSMC produces a smaller pricing error than the LSMC. Also comparing the wLSMC with other competing methods, we show that in many cases the wLSMC produces similar or better results.

The improvements that can be gained by using our proposed method is demonstrated for multi-asset American options as well. In addition, we provide ample evidence in the Online Appendix that wLSMC produces the lowest error measures or is highly comparable with the ICLS method, under various polynomial regression orders, for various numbers of assets in a basket, maturities and across moneyness.

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