Conservation laws and integral relations for the Boussinesq equation

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Abstract

We are concerned with conservation laws and integral relations associated with rational solutions of the Boussinesq equation, a soliton equation solvable by inverse scattering which was first introduced by Boussinesq in 1871. The rational solutions are logarithmic derivatives of a polynomial, are algebraically decaying and have a similar appearance to rogue-wave solutions of the focusing nonlinear Schrödinger equation. For these rational solutions the constants of motion associated with the conserved quantities are zero and they have some interesting integral relations which depend on the total degree of the associated polynomial.

This paper is dedicated to the memory of Professor David J. Benney

1 Introduction

In this paper we discuss conservation laws and integral relations associated with algebraically decaying rational solutions $u = u(x,t)$ of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0, \tag{1.1}$$

where subscripts denote partial derivatives. Equation (1.1) was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water [19, 20]; see also [60, 62]. Benney and Luke [15] showed that certain classical equations derived by mathematicians in the late 1800s, such as the Boussinesq equation (1.1) and the Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.2}$$

actually were generic approximations of weakly nonlinear-weakly dispersive wave phenomena. The Boussinesq equation (1.1) is also a soliton equation solvable by inverse scattering [4, 18, 25, 65] which arises in several other physical applications including one-dimensional nonlinear lattice-waves [59, 64], vibrations in a nonlinear string [65], and ion sound waves in a plasma [37, 53]. We remark that equation (1.1) is sometimes referred to as the "bad" Boussinesq equation, i.e. when the ratio of the u_{tt} and u_{xxxx} terms is negative. If the sign of the u_{xxxx} term is reversed in (1.1), then the equation is sometimes called the "good" Boussinesq equation. The coefficients of the u_{xx} and $(u^2)_{xx}$ terms can be changed by scaling and translation of the dependent variable u. For example, letting $u \to u+1$ in (1.1) gives

$$u_{tt} - u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0, \tag{1.3}$$

which is the non-dimensionalised form of the equation originally written down by Boussinesq [19, 20].

Recently Clarkson and Dowie [22] studied rational solutions $u_n(x,t)$ of the Boussinesq equation (1.1). These rational solutions, which are algebraically decaying and can depend on two arbitrary parameters, have the form

$$u_n(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln F_n(x,t;\alpha,\beta), \tag{1.4}$$

where $F_n(x,t;\alpha,\beta)$ is a polynomial of degree $n(n+1)$ in both x and t, with total degree $n(n+1)$, with α and β parameters. The polynomial $F_n(x,t;\alpha,\beta)$ satisfies a fourth-order, bilinear equation – see (2.3) below. These rational solutions have a similar appearance to rogue-wave solutions of the focusing nonlinear Schrödinger (NLS) equation, cf. [9, 11, 12, 40, 41, 42]

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0, \tag{1.5}$$

which also is a soliton equation solvable by inverse scattering [66]. Benney and Newell [16] showed that the NLS equation arises universally in diverse applications in nonlinear dispersive waves.

In §2, we review the results in [22] concerning algebraically decaying rational solutions $u_n(x,t;\alpha,\beta)$ of the Boussinesq equation (1.1). In §3, we discuss conservation laws for the Boussinesq equation (1.1), in particular showing that the constants of motion for the rational solutions $u_n(x,t;\alpha,\beta)$ given by (1.4) are all zero. In §4, we discuss integral relations for these rational solutions of the Boussinesq equation (1.1). Specifically we prove the following result:

Theorem 1.1. *Suppose that $u_n(x,t;\alpha,\beta)$ is an algebraically decaying rational solution of the Boussinesq equation* (1.1) *of the form* (1.4)*, then*

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^2(x,t;\alpha,\beta)\,\mathrm{d}x\,\mathrm{d}t = \tfrac{1}{2}n(n+1), \tag{1.6}$$

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^3(x,t;\alpha,\beta)\,\mathrm{d}x\,\mathrm{d}t = n(n+1). \tag{1.7}$$

Theorem 1.1 shows a relationship between the integrals and the total degree of the polynomial $F_n(x,t;\alpha,\beta)$ associated with the rational solution $u_n(x,t;\alpha,\beta)$. An analogous result to (1.6) has been conjectured for rogue-wave solutions of the NLS equation (1.5) [10]. In §5 we discuss rational solutions of the Kadomtsev-Petviashvili I (KPI) equation

$$(v_\tau + 6vv_\xi + v_{\xi\xi\xi})_\xi = 3v_{\eta\eta}, \tag{1.8}$$

specifically how the approach of Ablowitz *et al.* [2, 8, 61] might elucidate the results obtained here. In §6 we discuss our results.

2 Rational solutions of the Boussinesq equation

Clarkson and Kruskal [23] showed that Boussinesq equation (1.1) has symmetry reductions to the first, second and fourth Painlevé equations ($\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{IV}}$). Since $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$ themselves have rational solutions, symmetry reductions were used in [21] to derive rational solutions of the Boussinesq equation (1.1). Further more general rational solutions of (1.1) are also given in [21]. Unfortunately none of these rational solutions are bounded for all real x and t, and so it is unlikely that they will have any physical significance.

However it is known that there are additional rational solutions of the Boussinesq equation (1.1) which do not arise from the above construction. For example, Ablowitz and Satsuma [5] derived the rational solution

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln(1+x^2+t^2) = \frac{4(1-x^2+t^2)}{(1+x^2+t^2)^2}, \tag{2.1}$$

by taking a long-wave limit of the two-soliton solution, see also [57, 58]. This solution is bounded for real x and t, and tends to zero algebraically as $|x| \to \infty$ and $|t| \to \infty$.

If in the Boussinesq equation (1.1), we make the transformation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln F(x,t), \tag{2.2}$$

then we obtain the bilinear equation [35, 36]

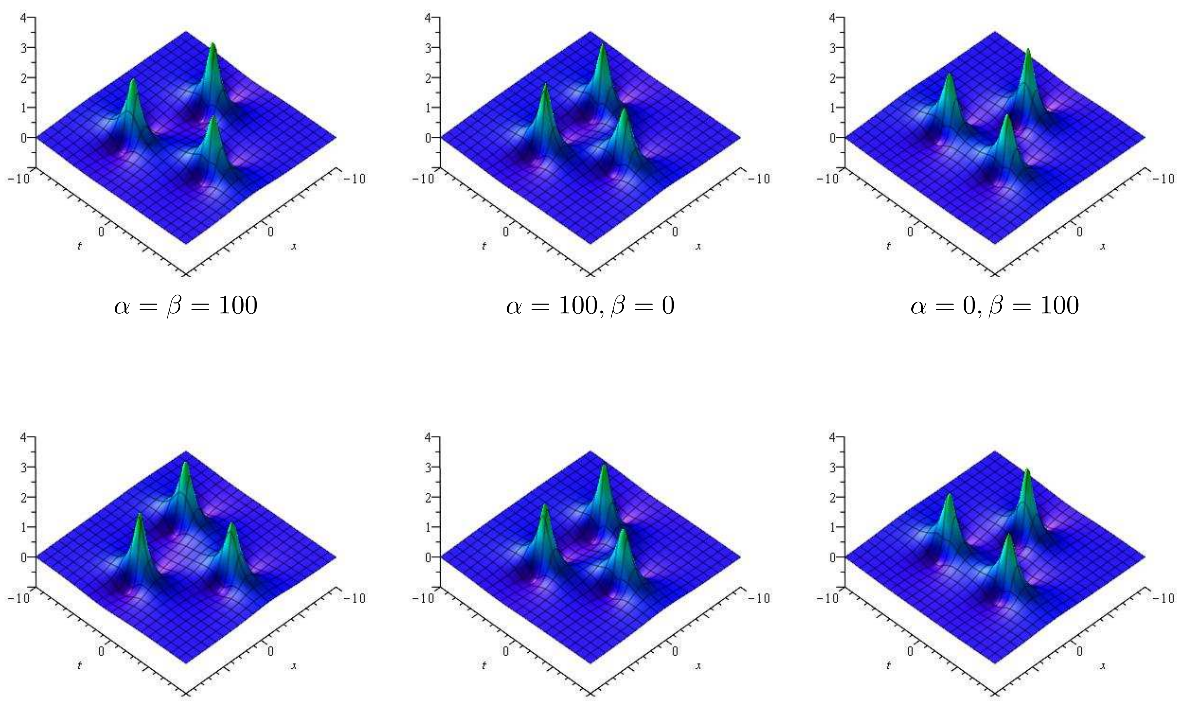
$$FF_{tt} - F_t^2 + FF_{xx} - F_x^2 - \tfrac{1}{3}\left(FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2\right) = 0, \tag{2.3}$$

which can be written in the form

$$(\mathrm{D}_t^2 + \mathrm{D}_x^2 - \tfrac{1}{3}\mathrm{D}_x^4)F \bullet F = 0, \tag{2.4}$$

where D_x and D_t are Hirota operators

$$\mathrm{D}_x^m\,\mathrm{D}_t^n F(x,t) \bullet F(x,t) = \left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n F(x,t)F(x',t')\right]_{x'=x,t'=t}. \tag{2.5}$$



$\alpha = \beta = 100$ $\alpha = 100, \beta = 0$ $\alpha = 0, \beta = 100$

$\alpha = 100, \beta = -100$ $\alpha = 100, \beta = 10$ $\alpha = 10, \beta = 100$

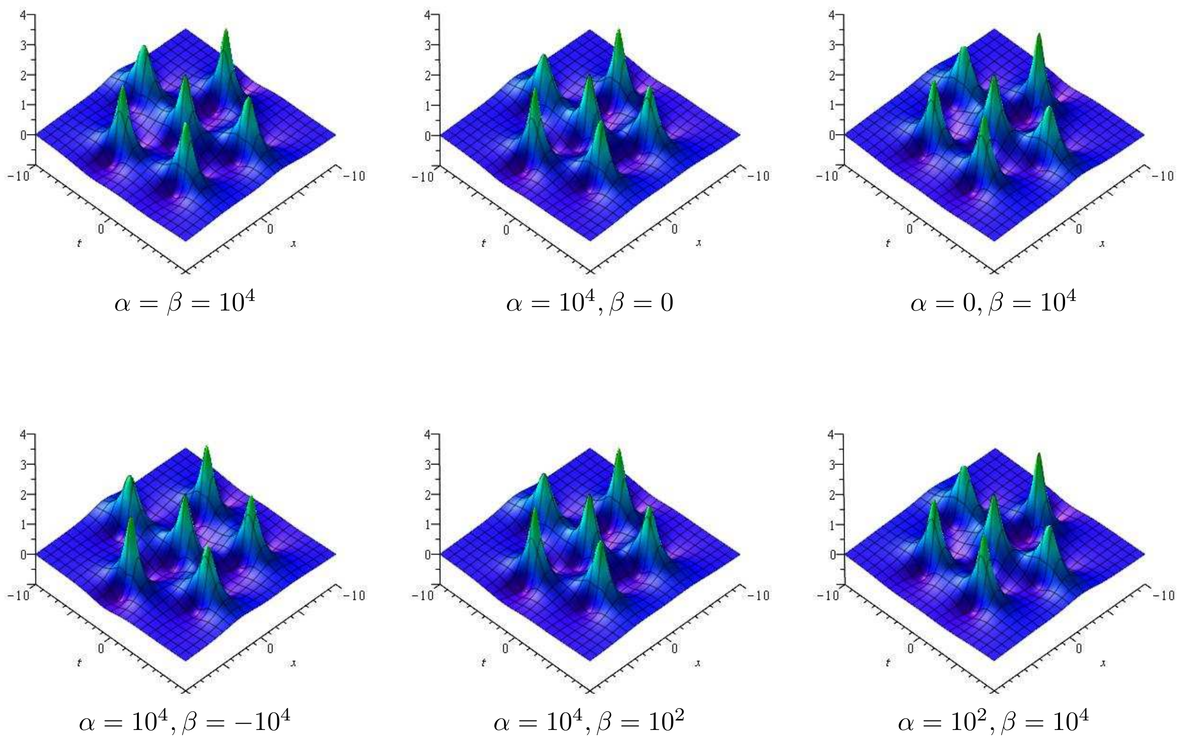
Figure 2.1: Plots of the solution $u_2(x,t;\alpha,\beta)$ for various choices of α and β.

Since the Boussinesq equation (1.1) admits the rational solution (2.1), Clarkson and Dowie [22] sought solutions in the form

$$u_n(x,t) = 2\frac{\partial^2}{\partial x^2}\ln f_n(x,t), \qquad n \geq 1, \tag{2.6}$$

where $F_n(x,t)$ is a polynomial of degree $n(n+1)$ in x and t, with total degree $n(n+1)$. In particular

$$f_n(x,t) = \sum_{m=0}^{n(n+1)/2} \sum_{j=0}^{m} a_{j,m} x^{2j} t^{2(m-j)}, \tag{2.7}$$



$\alpha = \beta = 10^4$ $\quad$ $\alpha = 10^4, \beta = 0$ $\quad$ $\alpha = 0, \beta = 10^4$

$\alpha = 10^4, \beta = -10^4$ $\quad$ $\alpha = 10^4, \beta = 10^2$ $\quad$ $\alpha = 10^2, \beta = 10^4$

Figure 2.2: Plots of the solution $u_3(x, t; \alpha, \beta)$ for various choices of α and β.

with $a_{j,m}$ constants, which are determined using (2.3) and equating powers of x and t. Using this ansatz the following polynomials were obtained:

$$f_1(x,t) = x^2 + t^2 + 1, \tag{2.8a}$$

$$f_2(x,t) = x^6 + \left(3t^2 + \tfrac{25}{3}\right)x^4 + \left(3t^4 + 30t^2 - \tfrac{125}{9}\right)x^2 + t^6 + \tfrac{17}{3}t^4 + \tfrac{475}{9}t^2 + \tfrac{625}{9}, \tag{2.8b}$$

$$\begin{aligned} f_3(x,t) = {} & x^{12} + \left(6t^2 + \tfrac{98}{3}\right)x^{10} + \left(15t^4 + 230t^2 + \tfrac{245}{3}\right)x^8 + \left(20t^6 + \tfrac{1540}{3}t^4 + \tfrac{18620}{9}t^2 + \tfrac{75460}{81}\right)x^6 \\ & + \left(15t^8 + \tfrac{1460}{3}t^6 + \tfrac{37450}{9}t^4 + \tfrac{24500}{3}t^2 - \tfrac{5187875}{243}\right)x^4 \\ & + \left(6t^{10} + 190t^8 + \tfrac{35420}{9}t^6 - \tfrac{4900}{9}t^4 + \tfrac{188650}{27}t^2 + \tfrac{159786550}{729}\right)x^2 \\ & + t^{12} + \tfrac{58}{3}t^{10} + \tfrac{1445}{3}t^8 + \tfrac{798980}{81}t^6 + \tfrac{16391725}{243}t^4 + \tfrac{300896750}{729}t^2 + \tfrac{878826025}{6561}. \end{aligned} \tag{2.8c}$$

(The lengthy polynomials $f_4(x,t)$ and $f_5(x,t)$ are also given in [22].)

Clarkson and Dowie [22] further showed that the Boussinesq equation (1.1) possesses generalised rational solutions of the form

$$u_n(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln F_n(x,t;\alpha,\beta), \tag{2.9}$$

for $n \geq 1$, with α and β arbitrary constants, where

$$F_{n+1}(x,t;\alpha,\beta) = f_{n+1}(x,t) + 2\alpha t P_n(x,t) + 2\beta x Q_n(x,t) + \left(\alpha^2 + \beta^2\right)f_{n-1}(x,t), \tag{2.10}$$

with $f_n(x,t)$ given by (2.8), and $P_n(x,t)$ and $Q_n(x,t)$ polynomials of degree $n(n+1)$ in x and t. Specifically

$$P_n(x,t) = \sum_{m=0}^{n(n+1)/2}\sum_{j=0}^{m} b_{j,m}x^{2j}t^{2(m-j)}, \qquad Q_n(x,t) = \sum_{m=0}^{n(n+1)/2}\sum_{j=0}^{m} c_{j,m}x^{2j}t^{2(m-j)}, \tag{2.11}$$

with the constants $b_{j,m}$ and $c_{j,m}$ determined by equating powers of x and t. By substituting (2.10) into the bilinear equation (2.3) with $f_j(x,t)$, for $j = 1, 2, \ldots, 5$, given by (2.8), then it is shown in [22] that

$$P_1(x,t) = 3x^2 - t^2 + \tfrac{5}{3}, \tag{2.12a}$$
$$Q_1(x,t) = x^2 - 3t^2 - \tfrac{1}{3}, \tag{2.12b}$$
$$P_2(x,t) = 5x^6 - \left(5t^2 - 35\right)x^4 - \left(9t^4 + \tfrac{190}{3}t^2 + \tfrac{665}{9}\right)x^2 + t^6 - \tfrac{7}{3}t^4 - \tfrac{245}{9}t^2 + \tfrac{18865}{81}, \tag{2.12c}$$
$$Q_2(x,t) = x^6 - \left(9t^2 - \tfrac{13}{3}\right)x^4 - \left(5t^4 + \tfrac{230}{3}t^2 + \tfrac{245}{9}\right)x^2 + 5t^6 + 15t^4 + \tfrac{535}{9}t^2 + \tfrac{12005}{81}, \tag{2.12d}$$

with α and β arbitrary constants; the polynomials $P_3(x,t)$, $Q_3(x,t)$, $P_4(x,t)$ and $Q_4(x,t)$ are given in [22]. The polynomials have the form

$$F_n(x,t;\alpha,\beta) = \left(x^2 + t^2\right)^{n(n+1)/2} + G_n(x,t;\alpha,\beta), \tag{2.13}$$

where $G_n(x,t;\alpha,\beta)$ is a polynomial of degree $(n+2)(n-1)$ in both x and t. In Figures 2.1 and 2.2, plots of $u_2(x,t;\alpha,\beta)$ and $u_3(x,t;\alpha,\beta)$ are given for various choices of α and β, respectively.

3 Conservation laws

A conservation law is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{3.1}$$

where $T(x,t)$ is the *conserved density* and $X(x,t)$ the *associated flux*. The integral

$$\int_{-\infty}^{\infty} T(x,t)\,\mathrm{d}x = c, \tag{3.2}$$

with c a constant, is called a *constant of motion*, with t interpreted as a timelike variable. It follows that

$$\int_{-\infty}^{\infty} X(x,t)\,\mathrm{d}t = k, \tag{3.3}$$

with k also a constant.

In order to study conservation laws for the Boussinesq equation (1.1) we cast it as the system

$$u_t + v_x = 0, \tag{3.4a}$$
$$v_t + (u^2)_x - u_x + \tfrac{1}{3}u_{xxx} = 0. \tag{3.4b}$$

If $u(x,t)$ has the form $u(x,t) = 2[\ln F(x,t)]_{xx}$, where $F(x,t)$ satisfies the bilinear equation (2.3) then equation (3.4a), shows that

$$v(x,t) = -2\frac{\partial^2}{\partial x \partial t}\ln F(x,t). \tag{3.5}$$

The first few conserved densities $T_j(x,t)$ and associated fluxes $X_j(x,t)$ for the system (3.4) are

$$\begin{aligned}
&T_1(x,t) = u, && X_1(x,t) = v,\\
&T_2(x,t) = v, && X_2(x,t) = u^2 - u + \tfrac{1}{3}u_{xx},\\
&T_3(x,t) = uv, && X_3(x,t) = \tfrac{2}{3}u^3 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 - \tfrac{1}{6}u_x^2 + \tfrac{1}{3}uu_{xx},\\
&T_4(x,t) = \tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2, && X_4(x,t) = 2u^2v - 2uv + \tfrac{2}{3}vu_{xx} - \tfrac{2}{3}u_xv_x;
\end{aligned}$$

see Hereman *et al.* [34] for details. Hence the first few constants of the motion for the system (3.4) are

$$\int_{-\infty}^{\infty} u(x,t)\,\mathrm{d}x = c_1, \tag{3.6a}$$
$$\int_{-\infty}^{\infty} v(x,t)\,\mathrm{d}x = c_2, \tag{3.6b}$$
$$\int_{-\infty}^{\infty} u(x,t)v(x,t)\,\mathrm{d}x = c_3, \tag{3.6c}$$
$$\int_{-\infty}^{\infty} \left(\tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2\right)\mathrm{d}x = c_4, \tag{3.6d}$$

where c_1, c_2, c_3 and c_4 are constants. The integral (3.6c) corresponds to the conservation of momentum and (3.6d) to the conservation of energy. Further from the associated fluxes we have

$$\int_{-\infty}^{\infty} v(x,t)\,\mathrm{d}t = k_1, \tag{3.7a}$$

$$\int_{-\infty}^{\infty} \left(u^2 - u + \tfrac{1}{3}u_{xx}\right)\mathrm{d}t = k_2, \tag{3.7b}$$

$$\int_{-\infty}^{\infty} \left(\tfrac{2}{3}u^3 + \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 - \tfrac{1}{6}u_x^2 + \tfrac{1}{3}uu_{xx}\right)\mathrm{d}t = k_3, \tag{3.7c}$$

$$\int_{-\infty}^{\infty} \left(2u^2v - 2uv + \tfrac{2}{3}vu_{xx} - \tfrac{2}{3}u_xv_x\right)\mathrm{d}t = k_4, \tag{3.7d}$$

where k_1, k_2, k_3 and k_4 are constants.

It is easily shown that for the algebraically decaying rational solutions of the Boussinesq equation (1.1) described in §2, then $c_j = 0$ and $k_j = 0$, for $j = 1, \ldots, 4$.

4 Integral relations of rational solutions

In this section we examine Theorem 1.1. The results hold for the generalised rational solutions $u_n(x,t;\alpha,\beta)$ given by (2.9), though in this section we will suppress the explicit dependence of the parameters α and β.

4.1 Integral of $u_n^2(x,t)$.

First we shall consider the integral of $u_n^2(x,t)$, i.e. result (1.6). Setting $u = U_{xx}$ in the Boussinesq equation (1.1) and then integrating twice w.r.t. x, assuming that u and its derivatives vanish sufficiently rapidly as $|x| \to \infty$, yields

$$u^2 = U_{tt} + U_{xx} - \tfrac{1}{3}U_{xxxx}. \tag{4.1}$$

We integrate (4.1) w.r.t. x and t, for $-R < x < R$ and $-R < t < R$, with R large, but finite, with a view to later letting $R \to \infty$. This gives

$$\int_{-R}^{R}\int_{-R}^{R} u^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-R}^{R}\int_{-R}^{R} \left\{U_{tt}(x,t) + U_{xx}(x,t) - \tfrac{1}{3}U_{xxxx}(x,t)\right\}\mathrm{d}x\,\mathrm{d}t.$$

The rationale for considering $(x,t) \in [-R,R] \times [-R,R]$, for R large but finite, rather than considering $(x,t) \in \mathbb{R}^2$ from the outset is that for the rational solutions given in §2, $U_{xx}(x,t) \notin L^1(\mathbb{R}^2)$ and $U_{tt}(x,t) \notin L^1(\mathbb{R}^2)$, though $U_{xx}(x,t) + U_{tt}(x,t) \in L^1(\mathbb{R}^2)$.

For the rational solutions described in §2, $U(x,t) = 2\ln F_n(x,t)$, where

$$F_n(x,t) = (x^2 + t^2)^{n(n+1)/2} + G_n(x,t), \tag{4.2}$$

with $G_n(x,t)$ a polynomial of degree $(n+2)(n-1)$ in both x and t. Therefore

$$\begin{aligned}
\frac{1}{8\pi}\int_{-R}^{R}\int_{-R}^{R} \{U_{xx} + U_{tt}\}\,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{4\pi}\int_{-R}^{R}\int_{-R}^{R} \left\{\left(\frac{F_{n,x}(x,t)}{F_n(x,t)}\right)_x + \left(\frac{F_{n,t}(x,t)}{F_n(x,t)}\right)_t\right\}\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{4\pi}\int_{-R}^{R} \left\{\frac{F_{n,x}(R,t)}{F_n(R,t)} - \frac{F_{n,t}(-R,t)}{F_n(-R,t)}\right\}\mathrm{d}t \\
&\quad + \frac{1}{4\pi}\int_{-R}^{R} \left\{\frac{F_{n,x}(x,R)}{F_n(x,R)} - \frac{F_{n,t}(x,-R)}{F_n(x,-R)}\right\}\mathrm{d}x,
\end{aligned}$$

where the order of integration has been reversed for the second integral. Next consider

$$\begin{aligned}
\frac{F_{n,x}(R,t)}{F_n(R,t)} &= \frac{n(n+1)R(R^2+t^2)^{n(n+1)/2-1} + G_{n,x}(R,t)}{(R^2+t^2)^{n(n+1)/2} + G_n(R,t)} \\
&= \frac{n(n+1)R}{R^2+t^2}\left\{1 + \frac{G_{n,x}(R,t)}{(R^2+t^2)^{n(n+1)/2}}\right\}^{-1} + \frac{G_{n,x}(R,t)}{(R^2+t^2)^{n(n+1)/2} + G_n(R,t)},
\end{aligned}$$

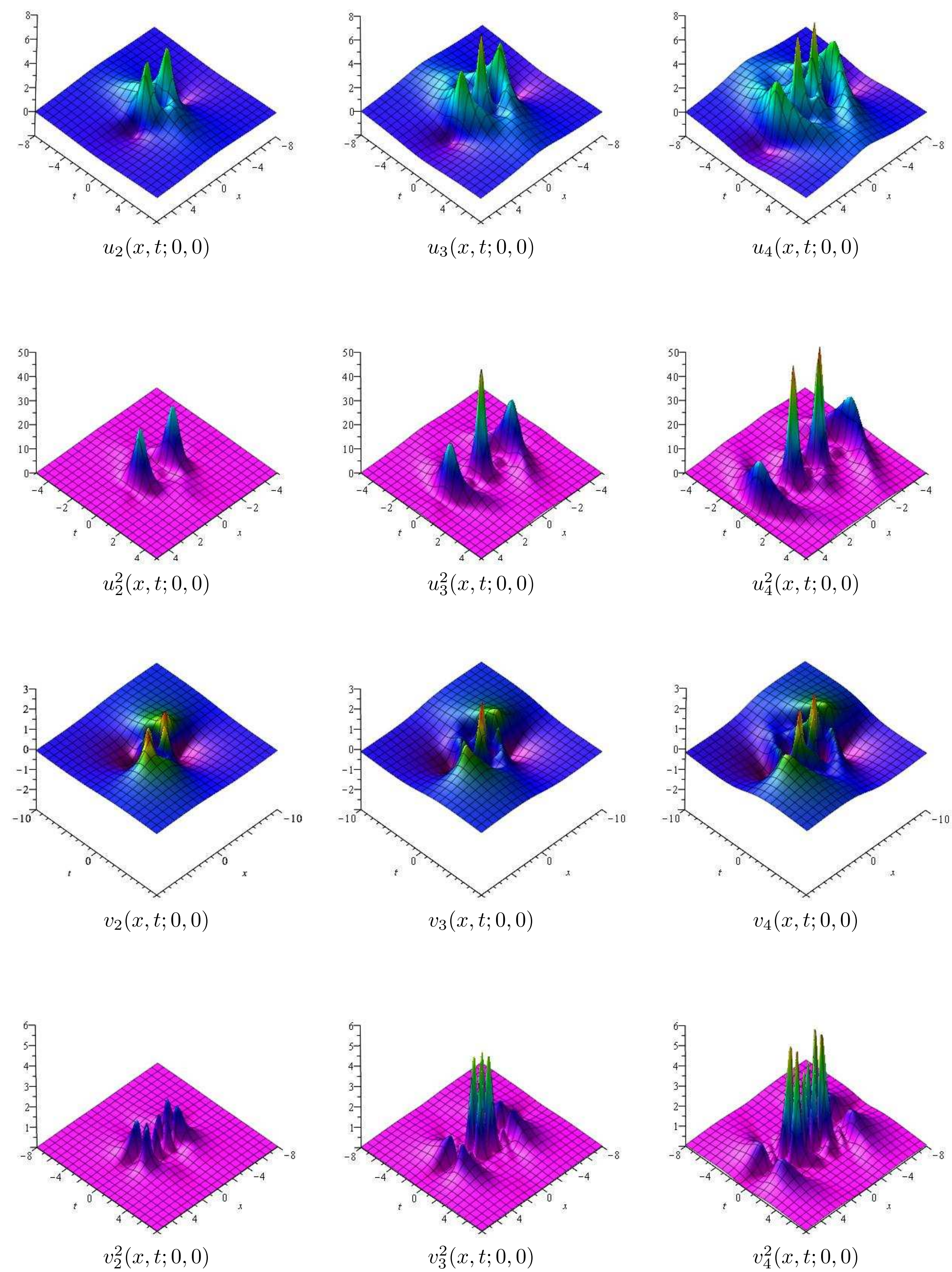


Figure 4.1: Plots of the solutions $u_j(x, t; 0, 0)$, $u_j^2(x, t; 0, 0)$, $v_j(x, t; 0, 0)$ and $v_j^2(x, t; 0, 0)$ for $j = 2, 3, 4$.

then letting $t = \tau R$ gives

$$\begin{aligned}
\frac{1}{4\pi}\int_{-R}^{R}\left\{\frac{F_{n,x}(R,t)}{F_n(R,t)}-\frac{F_{n,x}(-R,t)}{F_n(-R,t)}\right\}\mathrm{d}t &= \frac{R}{4\pi}\int_{-1}^{1}\left\{\frac{F_{n,x}(R,R\tau)}{F_n(R,R\tau)}-\frac{F_{n,x}(-R,R\tau)}{F_n(-R,R\tau)}\right\}\mathrm{d}\tau\\
&= \frac{n(n+1)}{2\pi}\int_{-1}^{1}\frac{1}{1+\tau^2}\left\{1+\mathcal{O}\left(R^{-2}\right)\right\}\mathrm{d}\tau\\
&= \tfrac14 n(n+1)\left\{1+\mathcal{O}\left(R^{-2}\right)\right\},
\end{aligned}$$

since

$$\int_{-1}^{1}\frac{\mathrm{d}\,\tau}{1+\tau^2}=\big[\arctan(\tau)\big]_{-1}^{1}=\tfrac12\pi.$$

Similarly

$$\frac{F_{n,t}(x,R)}{F_n(x,R)}=\frac{n(n+1)R}{x^2+R^2}\left\{1+\frac{G_{n,t}(x,R)}{(x^2+R^2)^{n(n+1)/2}}\right\}^{-1}+\frac{G_{n,t}(x,R)}{(x^2+R^2)^{n(n+1)/2}+G_n(x,R)}$$

and letting $x = \xi R$ gives

$$\begin{aligned}
\frac{1}{4\pi}\int_{-R}^{R}\left\{\frac{F_{n,t}(x,R)}{F_n(x,R)}-\frac{F_{n,t}(x,-R)}{F_n(x,-R)}\right\}\mathrm{d}x &= \frac{n(n+1)}{2\pi}\int_{-1}^{1}\frac{1}{1+\xi^2}\left\{1+\mathcal{O}\left(R^{-2}\right)\right\}\mathrm{d}\xi\\
&= \tfrac14 n(n+1)\left\{1+\mathcal{O}\left(R^{-2}\right)\right\}
\end{aligned}$$

Hence we have shown that

$$\frac{1}{8\pi}\int_{-R}^{R}\int_{-R}^{R}\left\{u^2(x,t)+\tfrac13 U_{xxxx}(x,t)\right\}\mathrm{d}x\,\mathrm{d}t=\tfrac12 n(n+1)\left\{1+\mathcal{O}\left(R^{-2}\right)\right\},$$

We note that

$$\lim_{R\to\infty}\int_{-R}^{R}\int_{-R}^{R}U_{xxxx}(x,t)\,\mathrm{d}x\,\mathrm{d}t=0,$$

since $\lim_{|R|\to\infty}U_{xxx}(R,t)=0$, and so in the limit as $R\to\infty$ we obtain the following result.

Theorem 4.1. *If $u_n(x,t)$ is an algebraically decaying rational solution of the Boussinesq equation* (1.1) *given by* (2.9)*, then*

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t=\tfrac12 n(n+1). \qquad (4.3)$$

For $n = 2$, the result (4.3) equals the number of peaks, i.e. 3, as shown in Figure 2.1, and for $n = 3$, it equals the number of peaks, i.e. 6, as shown in Figure 2.2.

In Figure 4.1, plots are given of the solutions $u_j(x,t)$, $u_j^2(x,t)$, $v_j(x,t)$ and $v_j^2(x,t)$ for $j = 2,3,4$, with $\alpha=\beta=0$.

4.2 Integral of $u_n^3(x,t)$.

Now we shall consider the integral of $u_n^3(x,t)$, i.e. result (1.7). From the third conservation law

$$(uv)_t+\left(\tfrac23 u^3+\tfrac12 v^2-\tfrac12 u^2-\tfrac16 u_x^2+\tfrac13 uu_{xx}\right)_x=0, \qquad (4.4)$$

we have

$$\int_{-\infty}^{\infty}\left(\tfrac23 u^3+\tfrac12 v^2-\tfrac12 u^2-\tfrac16 u_x^2+\tfrac13 uu_{xx}\right)\mathrm{d}t=0;$$

recall (3.7c) with $k_3 = 0$. Integrating this result w.r.t. x and interchanging the order of integration gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u^3\,\mathrm{d}x\,\mathrm{d}t=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tfrac34 u^2-\tfrac34 v^2+\tfrac14 u_x^2-\tfrac12 uu_{xx}\right)\mathrm{d}x\,\mathrm{d}t.$$

One integration by parts yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = \tfrac{3}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^2 - v^2 + u_x^2\right)\mathrm{d}x\,\mathrm{d}t. \tag{4.5}$$

From the fourth conservation law

$$\left(\tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2\right)_t + \left(2u^2v - 2uv + \tfrac{2}{3}vu_{xx} - \tfrac{2}{3}u_xv_x\right)_x = 0, \tag{4.6}$$

we have

$$\int_{-\infty}^{\infty} \left(\tfrac{2}{3}u^3 + v^2 - u^2 - \tfrac{1}{3}u_x^2\right)\mathrm{d}x = 0;$$

recall (3.6d) with $c_4 = 0$. Integrating this w.r.t. t gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tfrac{3}{2}u^2 - \tfrac{3}{2}v^2 + \tfrac{1}{2}u_x^2\right)\mathrm{d}x\,\mathrm{d}t. \tag{4.7}$$

Therefore equations (4.5) and (4.7) give

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_x^2(x,t)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.8a}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3(x,t)\,\mathrm{d}x\,\mathrm{d}t = 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{u^2(x,t) - v^2(x,t)\right\}\mathrm{d}x\,\mathrm{d}t. \tag{4.8b}$$

From the Boussinesq equation (1.1), we have

$$u^2 = u + U_{tt} - \tfrac{1}{3}u_{xx},$$

where $u = U_{xx}$, so

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^2 + u\,U_{tt} - \tfrac{1}{3}uu_{xx}\right)\mathrm{d}x\,\mathrm{d}t.$$

Using integration by parts gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^2 + u\,U_{tt} + \tfrac{1}{3}u_x^2\right)\mathrm{d}x\,\mathrm{d}t, \tag{4.9}$$

and so from (4.8) we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = \tfrac{3}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(u^2 + u\,U_{tt}\right)\mathrm{d}x\,\mathrm{d}t, \tag{4.10a}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(v^2 + u\,U_{tt}\right)\mathrm{d}x\,\mathrm{d}t. \tag{4.10b}$$

Consequently, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(2v^2 + u\,U_{tt}\right)\mathrm{d}x\,\mathrm{d}t. \tag{4.11}$$

Lemma 4.2. *Suppose that* $u(x,t)$ *and* $v(x,t)$ *are solutions of the system* (3.4)*, and* $u(x,t) = U_{xx}(x,t)$ *with*

$$\lim_{|x|\to\infty} U_x(x,t) = 0, \qquad \lim_{|t|\to\infty} U_x(x,t) = 0,$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\,U_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.12}$$

Proof. Since $u = U_{xx}$ and $v = -U_{xt}$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} U_{xt}^2\,\mathrm{d}t\right)\mathrm{d}x = -\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} U_x U_{xtt}\,\mathrm{d}t\right)\mathrm{d}x,$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\,U_{tt}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} U_{xx}\,U_{tt}\,\mathrm{d}x\right)\mathrm{d}t = -\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} U_x U_{xtt}\,\mathrm{d}x\right)\mathrm{d}t,$$

so the result follows, since the order of integration can be switched. □

Consequently we have the following result.

Theorem 4.3. *If $u_n(x,t)$ is an algebraically decaying rational solution of the Boussinesq equation* (1.1) *given by* (2.9)*, then*

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^3(x,t)\,\mathrm{d}x\,\mathrm{d}t = n(n+1). \tag{4.13}$$

Proof. From equation (4.11) and Lemma 4.2 we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(2v^2 + u\,U_{tt}\right)\,\mathrm{d}x\,\mathrm{d}t = 3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.14}$$

Then from equations (4.8b) and (4.14) we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^3\,\mathrm{d}x\,\mathrm{d}t = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u^2\,\mathrm{d}x\,\mathrm{d}t, \tag{4.15}$$

and so using Theorem 4.1 we obtain the result. □

Corollary 4.4. *If $v_n(x,t)$ is an algebraically decaying rational solution of the system* (3.4)*, then*

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_n^2(x,t)\,\mathrm{d}x\,\mathrm{d}t = \tfrac{1}{6}n(n+1). \tag{4.16}$$

Proof. The result follows immediately from equation (4.14) and Theorem 4.1. □

4.3 Integrals of $u_1^m(x,t)$.

Finally we consider the integral of $u_1^m(x,t)$, for $m \geq 2$.

Theorem 4.5. *Consider the rational solution of the Boussinesq equation given by*

$$u_1(x,t) = 2\frac{\partial^2}{\partial x^2}\ln(x^2+t^2+1) = \frac{4(1-x^2+t^2)}{(1+x^2+t^2)^2}. \tag{4.17}$$

Then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1^m(x,t)\,\mathrm{d}x\,\mathrm{d}t = \frac{m!}{(2m-1)!}\sum_{\ell=0}^{\lfloor m/2\rfloor}\frac{(2\ell)!(2m-2\ell-2)!}{2^{2\ell-2m-3}(\ell!)^2(m-2\ell)!}, \tag{4.18}$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x, for m an integer with $m \geq 2$.

Proof. We need to evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1^m(x,t)\,\mathrm{d}x\,\mathrm{d}t = 2^{2m}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(1-x^2+t^2)^m}{(1+x^2+t^2)^{2m}}\,\mathrm{d}x\,\mathrm{d}t,$$

for all integers $m \geq 2$. If we make the transformation $x = \sqrt{\rho}\,\cos\varphi$, $t = \sqrt{\rho}\,\sin\varphi$ then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1^m(x,t)\,\mathrm{d}x\,\mathrm{d}t = 2^{2m-1}\int_0^{\infty}\int_0^{2\pi}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\frac{\rho^k\cos^k(2\varphi)}{(1+\rho)^{2m}}\,\mathrm{d}\varphi\,\mathrm{d}\rho.$$

Elementary results give that

$$\int_0^{2\pi} \cos^{2\ell}(2\varphi)\,\mathrm{d}\varphi = \frac{2\sqrt{\pi}\,\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)} \equiv \frac{\pi}{2^{2\ell-1}}\frac{(2\ell)!}{(\ell!)^2}, \qquad \int_0^{2\pi} \cos^{2\ell+1}(2\varphi)\,\mathrm{d}\varphi = 0,$$

for integer ℓ, which, when combined with the knowledge that

$$\int_0^{\infty} \frac{\rho^k}{(1+\rho)^{2m}}\,\mathrm{d}\rho = \frac{\Gamma(k+1)\,\Gamma(2m-k-1)}{\Gamma(2m)} \equiv \frac{k!(2m-k-2)!}{(2m-1)!}, \qquad \text{if} \quad k<2m-1,$$

leads to the desired result (4.18). □

Remark 4.6. We note that for the rational solution $u_1(x,t)$

$$\frac{1}{8\pi}\int_{-\infty}^{\infty} u_1(x,t)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-x^2+t^2}{(1+x^2+t^2)^2}\,\mathrm{d}x = 0,$$

and therefore

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u_1(x,t)\,\mathrm{d}x\right)\mathrm{d}t = 0.$$

On the other hand

$$\frac{1}{8\pi}\int_{-\infty}^{\infty} u_1(x,t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-x^2+t^2}{(1+x^2+t^2)^2}\,\mathrm{d}t = \frac{1}{2(x^2+1)^{3/2}},$$

and then

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u_1(x,t)\,\mathrm{d}t\right)\mathrm{d}x = 1.$$

So for $u_1(x,t)$ the order of integration is important since $u_1(x,t) \notin L^1(\mathbb{R}^2)$. In fact more generally $u_n(x,t) \notin L^1(\mathbb{R}^2)$.

5 Rational solutions of the Kadomtsev-Petviashvili I equation

The Kadomtsev-Petviashvili (KP) equation

$$(v_\tau + 6vv_\xi + v_{\xi\xi\xi})_\xi + 3\sigma^2 v_{\eta\eta} = 0, \qquad \sigma^2 = \pm 1, \tag{5.1}$$

which is known as KPI if $\sigma^2 = -1$, i.e. (1.8), and KPII if $\sigma^2 = 1$, was derived by Kadomtsev and Petviashvili [39] to model ion-acoustic waves of small amplitude propagating in plasmas and is a two-dimensional generalisation of the KdV equation (1.2). The KP equation arises in many physical applications including weakly two-dimensional long waves in shallow water [6, 54], where the sign of σ^2 depends upon the relevant magnitudes of gravity and surface tension. The KP equation (5.1) is also a completely integrable soliton equation solvable by inverse scattering and again the sign of σ^2 is critical since if $\sigma^2 = -1$, then the inverse scattering problem is formulated in terms of a Riemann-Hilbert problem [29, 44], whereas for $\sigma^2 = 1$, it is formulated in terms of a $\overline{\partial}$ ("DBAR") problem [1].

The first rational solution of KPI (1.8), is the so-called "lump solution"

$$v(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial\xi^2}\ln[(\xi-3\tau)^2+\eta^2+1] = -4\frac{(\xi-3\tau)^2-\eta^2-1}{[(\xi-3\tau)^2+\eta^2+1]^2}, \tag{5.2}$$

which was found by Manakov *et al.* [45]. Subsequent studies of rational solutions, also known as lump solutions, of KPI (1.8) include Ablowitz *et al.* [2], Ablowitz and Villarroel [8, 61], Dubard and Matveev [27, 28], Gaillard [31, 32], Johnson and Thompson [38], Ma [43], Pelinovsky [49, 50], Pelinovsky and Stepanyants [51], Satsuma and Ablowitz [52], and Singh and Stepanyants [55].

Dubard and Matveev [27, 28] derive rational solutions of KPI (1.8) from generalised rational solutions of the focusing NLS equation (1.5) which are discussed in Appendix A below; see also [26, 31, 32].

The Boussinesq equation (1.1) is a symmetry reduction of KPI (1.8) and so the rational solutions of the Boussinesq equation can be used to generate rational solutions of KPI. If in KPI (1.8) we make the travelling wave reduction

$$v(\xi,\eta,\tau) = u(x,t), \qquad x = \xi - 3\tau, \quad t = \eta,$$

then $u(x,t)$ satisfies the Boussinesq equation (1.1). Consequently given a solution of the Boussinesq equation (1.1), then we can derive a solution of KPI (1.8). In particular, if

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln F(x,t),$$

for some known $F(x,t)$, is a solution of the Boussinesq equation (1.1), then

$$v(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial \xi^2}\ln F(\xi-3\tau,\eta),$$

is a solution of KPI (1.8). For example the choice $F(x,t) = x^2+t^2+1$ gives the lump solution (5.2) of KPI. In [22] Clarkson and Dowie show that the family of rational solutions for KPI (1.8) derived through this approach are fundamentally different from those derived by Dubard and Matveev [27, 28] from the rational solutions of the focusing NLS equation (1.5).

Ablowitz *et al.* [2, 8, 61] derived a hierarchy of algebraically decaying rational solutions, or lump solutions, of KPI (1.8) which have the form

$$v_m(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial \xi^2}\ln G_m(\xi,\eta,\tau), \tag{5.3}$$

where $G_m(\xi,\eta,\tau)$ is a polynomial of degree $2m$ in ξ, η and τ. These rational solutions are derived in terms of the eigenfunctions of the non-stationary Schrödinger equation

$$\mathrm{i}\varphi_\eta + \varphi_{\xi\xi} + v\varphi = 0, \tag{5.4}$$

with potential $v = v(\xi,\eta,\tau)$, which is used in the solution of KPI (1.8) by inverse scattering; equation (1.8) is obtained from the compatibility of (5.4) and

$$\varphi_\tau + 4\varphi_{\xi\xi\xi} + 6v\varphi_\xi + w\varphi = 0, \qquad w_\xi = v. \tag{5.5}$$

This is a different hierarchy of rational solutions of KPI (1.8) compared to those discussed above, not least because it involves polynomials of *all* even degrees, not just of degree $n(n+1)$, with $n\in\mathbb{N}$. These rational solutions of KPI (1.8) are deeply connected with an integer called the “charge” or “index”, and this number is related to the degree of the polynomial that generates the rational solution [2, 8, 61]. It would be interesting to investigate whether there is an analogous result for the rational solutions of the Boussinesq equation (1.1) discussed here which might explain the results. However we shall not pursue this further here.

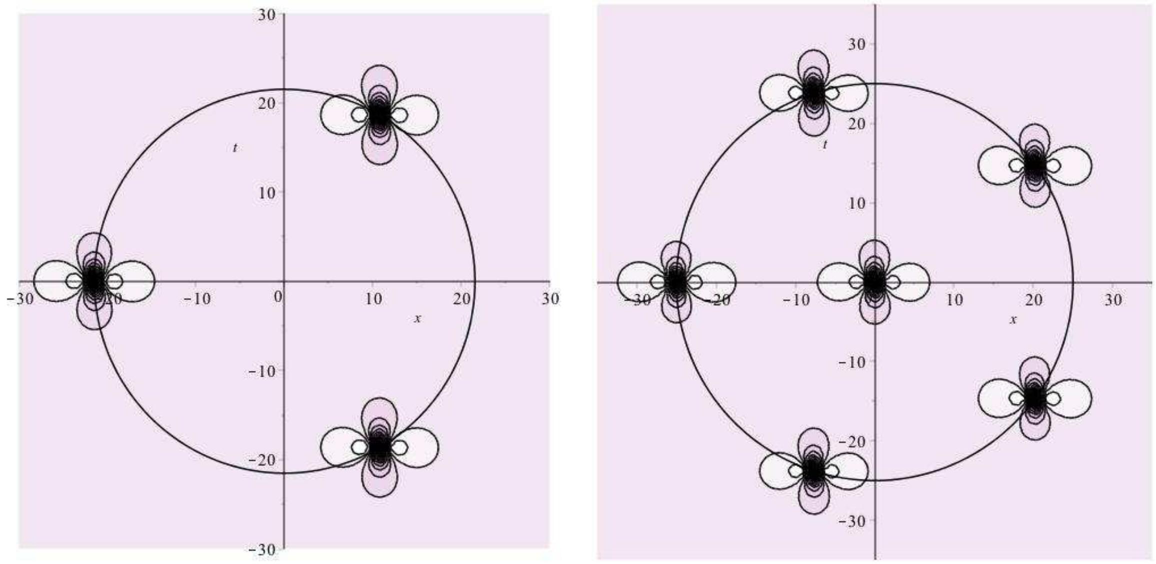
6 Discussion

Amongst his extensive contributions to fluid mechanics, David Benney conducted many studies of nonlinear wave equations. Long waves were the topic of interest in [13, 14, 17, 63] while lump solutions of the modified Zakharov-Kuznetsov equation are studied in [56]. A recurring theme in Benney's work was that of conservation laws and in this spirit here we have been concerned with conservation laws and integral relations associated with algebraically decaying rational solution of the Boussinesq equation (1.1) given by (2.9). In addition to the results discussed above, we have performed numerical investigations of higher integral relations, i.e.

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_n^m(x,t;\alpha,\beta)\,\mathrm{d}x\,\mathrm{d}t,$$

for $m\geq 4$, where $u_n(x,t;\alpha,\beta)$ is an algebraically decaying rational solution of the Boussinesq equation given by (2.9), with $n\geq 2$. However for $m\geq 4$ there appears to be no pattern analogous to the results for $m=2$ and $m=3$ given in Theorem 1.1.

It is interesting to compare the results described here with analogous conservation laws and integral relations for rogue wave solutions of the NLS equation (1.5). It is straightforward to show that the first few constants of motion and associated fluxes for rogue wave solutions of NLS equation (1.5) are zero, which was the case for the Boussinesq equation (1.1); the first few rogue wave solutions and



$u_2(x, t; 0, 10^4)$ $u_3(x, t; 0, 10^7)$

Figure 6.1: Contour plots of the solutions $u_2(x, t; 0, 10^4)$ and $u_3(x, t; 0, 10^7)$ of the Boussinesq equation.

conservation laws for the NLS equation (1.5) are given in Appendix A below. Ankiewicz and Akhmediev [10] conjectured that the number of components in any NLS rogue wave is a triangle number which can be calculated as the integral of the squared deviation from the background level across the space-time plane.

Conjecture 6.1. *Suppose that $\psi_n(x, t)$ is a rogue wave solution of the focusing NLS equation* (1.5) *then*

$$Q_n = \frac{1}{32\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[|\psi_n(x, t)|^2 - 1 \right]^2 \mathrm{d}x \, \mathrm{d}t = \tfrac{1}{2} n(n + 1). \tag{6.1}$$

This conjecture is the analogue of (1.6); the factor of 4 is due to the fact that Ankiewicz and Akhmediev [10] consider an NLS equation which is obtained from (1.5) by letting $t \to \frac{1}{2}x$ and $x \to \frac{1}{2}t$.

We note that if the parameters α and β sufficiently large, then the rational solution $u_2(x, t; \alpha, \beta)$ has three lumps which are essentially copies of the lowest-order solution, of approximately the same height and equally spaced on a circle. An analogous situation arises for the second generalised rational solution of the NLS equation [41]. Further, for α and β sufficiently large, the rational solution $u_3(x, t; \alpha, \beta)$ has six lumps, again essentially copies of the lowest-order solution, which are of approximately the same height and with five of these equally spaced on a circle. Again an analogous situation arises for the third generalised rational solution of the NLS equation [40]. For both cases we conjecture that the radius of the circle is equal to $(\alpha^2 + \beta^2)^{1/h}$, for some h which depends on n, as appears to be the case for the NLS equation [40, 41], though we shall not investigate this further here. Contour plots of the solutions $u_2(x, t; 0, \beta)$ and $u_3(x, t; 0, \beta)$, for $\beta = 10^4$ and $\beta = 10^7$ respectively, of the Boussinesq equation (1.1) illustrating this behaviour are given in Figure 6.1. Contour plots of $v_2(x, t; 0, \beta)$ and $v_3(x, t; 0, \beta)$, for $\beta = 10^4$ and $\beta = 10^7$ respectively, where

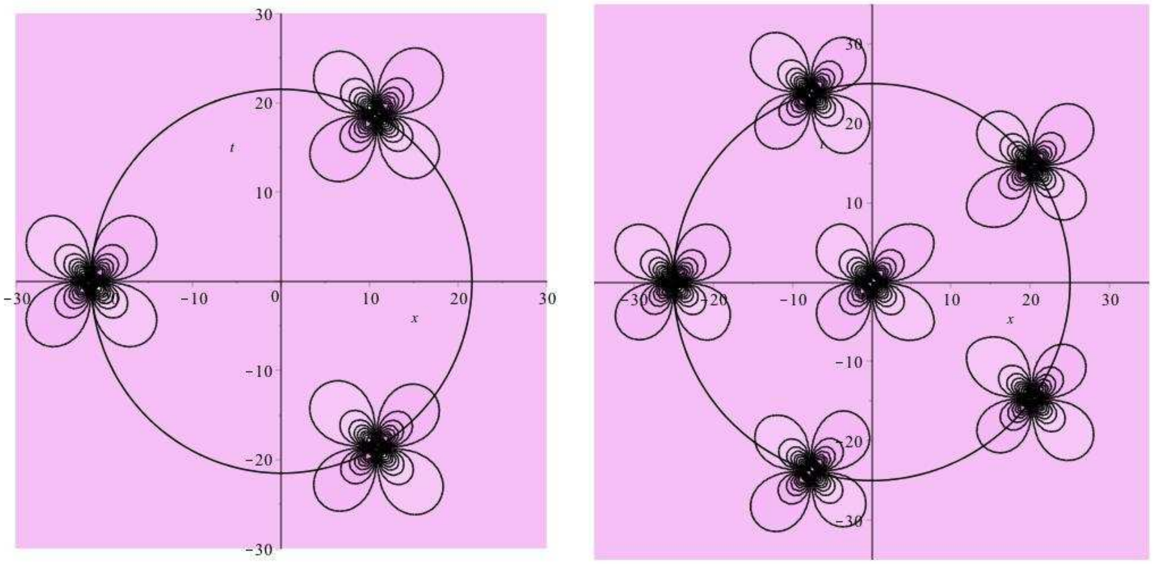
$$v_n(x, t; \alpha, \beta) = -2 \frac{\partial^2}{\partial x \partial t} \ln F_n(x, t; \alpha, \beta),$$

with $F_n(x, t; \alpha, \beta)$ given by (2.10) are given in Figure 6.2.

Rogue wave solutions have also been derived for $(2+1)$-dimensional equations such as the Benney-Roskes equation [17], also known as the Davey-Stewartson equation [24] (see also [3, 7])

$$\mathrm{i}q_t = q_{xx} + \sigma^2 q_{yy} + (\varepsilon |q|^2 - 2\phi) q, \tag{6.2a}$$

$$\phi_{xx} - \sigma^2 \phi_{yy} = \varepsilon (|q|^2)_{xx}, \tag{6.2b}$$



$v_2(x,t;0,10^4)$ $v_3(x,t;0,10^7)$

Figure 6.2: Contour plots of the solutions $v_2(x,t;0,10^4)$ and $v_3(x,t;0,10^7)$ of the Boussinesq system.

where $\sigma^2 = \pm 1$ and $\varepsilon = \pm 1$, independently; see [47, 48] for details of rogue wave solutions of (6.2). It would be interesting to see if there are analogous results to those given in this paper for $(2+1)$-dimensional equations such as equation (6.2).

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A Rational solutions and conservation laws for the focusing NLS equation

Rational solutions of the focusing NLS equation (1.5) have the general form

$$\psi_n(x,t) = \left\{1 - 4\frac{G_n(x,t) + \mathrm{i}tH_n(x,t)}{D_n(x,t)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{A.1}$$

where $G_n(x,t)$ and $H_n(x,t)$ are polynomials of degree $(n+2)(n-1)$ in both x and t, with total degree $(n+2)(n-1)$, and $D_n(x,t)$ is a polynomial of degree $n(n+1)$ in both x and t, with total degree $n(n+1)$ and has no real zeros. The first two rational solutions of the focusing NLS equation (1.5) have the form [9, 11]

$$\psi_1(x,t) = \left\{1 - \frac{4(1+\mathrm{i}t)}{x^2+t^2+1}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{A.2}$$

$$\psi_2(x,t) = \left\{1 - 12\,\frac{G_2(x,t) + \mathrm{i}tH_2(x,t)}{D_2(x,t)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{A.3}$$

where

$$G_2(x,t) = x^4 + 6(t^2+1)x^2 + 5t^4 + 18t^2 - 3,$$
$$H_2(x,t) = x^4 + 2(t^2-3)x^2 + (t^2+5)(t^2-3),$$
$$D_2(x,t) = x^6 + 3(t^2+1)x^4 + 3(t^2-3)^2x^2 + t^6 + 27t^4 + 99t^2 + 9.$$

Further

$$|\psi_n(x,t)|^2 = 1 + 4\frac{\partial^2}{\partial x^2}\ln D_n(x,t). \tag{A.4}$$

Dubard *et al.* [26] show that the rational solutions of the focusing NLS equation (1.5) can be generalised to include some arbitrary parameters. The first of these generalised solutions has the form

$$\widehat{\psi}_2(x,t;\alpha,\beta) = \left\{1 - 12\frac{\widehat{G}_2(x,t;\alpha,\beta) + \mathrm{i}\widehat{H}_2(x,t;\alpha,\beta)}{\widehat{D}_2(x,t;\alpha,\beta)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{A.5}$$

where

$$\widehat{G}_2(x,t;\alpha,\beta) = G_2(x,t) - 2\alpha t + 2\beta x,$$
$$\widehat{H}_2(x,t;\alpha,\beta) = tH_2(x,t) + \alpha(x^2 - t^2 + 1) + 2\beta xt,$$
$$\widehat{D}_2(x,t;\alpha,\beta) = D_2(x,t) + 2\alpha t(3x^2 - t^2 - 9) - 2\beta x(x^2 - 3t^2 - 3) + \alpha^2 + \beta^2,$$

with α and β arbitrary constants, see also [12, 27, 28, 30, 33, 40, 41, 42, 46].

The first few conservation laws for the focusing NLS equation (1.5) are

$$(|\psi|^2 - 1)_t + \mathrm{i}(\psi\psi_x^* - \psi^*\psi_x)_x = 0, \tag{A.6a}$$
$$(\psi\psi_x^* - \psi^*\psi_x)_t + \mathrm{i}\left((\psi^*\psi_{xx} + \psi\psi_{xx}^*) + \tfrac{1}{2}|\psi|^4 - 2\psi_x\psi_x^*\right)_x = 0, \tag{A.6b}$$
$$\left(4\psi_x\psi_x^* - |\psi|^4 + 1\right)_t + 2\mathrm{i}\left(|\psi|^2(\psi^*\psi_x - \psi\psi_x^*) + 2(\psi_x\psi_{xx}^* - \psi_x^*\psi_{xx})\right)_x = 0. \tag{A.6c}$$

We remark that the conserved quantities in (A.6) appear in [63], see equations (4.21)–(4.23) in that paper.

References

[1] M.J. Ablowitz, D. Bar Yaacov, and A.S. Fokas, On the inverse scattering transform for the Kadomtsev-Petviashvili equation, *Stud. Appl. Math.*, **69** (1983) 135–143.

[2] M.J. Ablowitz, S. Chakravarty, A.D. Trubatch, and J. Villarroel, A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev-Petviashvili I equations, *Phys. Lett. A*, **267** (2000) 132–146.

[3] M.J. Ablowitz and P.A. Clarkson, "*Solitons, Nonlinear Evolution Equations and Inverse Scattering*," *L.M.S. Lect. Notes Math.*, vol. **149**, C.U.P., Cambridge (1991).

[4] M.J. Ablowitz and R. Haberman, Resonantly coupled nonlinear evolution equations, *J. Math. Phys.*, **16** (1975) 2301–2305.

[5] M.J. Ablowitz and J. Satsuma, Solitons and rational solutions of nonlinear evolution equations, *J. Math. Phys.*, **19** (1978) 2180–2186.

[6] M.J. Ablowitz and H. Segur, On the evolution of packets of water waves, *J. Fluid Mech.*, **92** (1979) 691–715.

[7] M.J. Ablowitz and H. Segur, "*Solitons and the Inverse Scattering Transform*," SIAM, Philadelphia (1981).

[8] M.J. Ablowitz and J. Villarroel, Solutions to the time dependent Schrödinger and the Kadomtsev-Petviashvili equations, *Phys. Rev. Lett.*, **78** (1997) 570–573.

[9] N. Akhmediev, A. Ankiewicz, and J.M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation, *Phys. Rev. E*, **80** (2009) 026601.

[10] A. Ankiewicz and N. Akhmediev, Multi-rogue waves and triangle numbers, Australian and New Zealand Conference on Optics and Photonics (2015, Adelaide, South Australia) Engineers Australia/Australian Optical Society.

[11] A. Ankiewicz, P.A. Clarkson, and N. Akhmediev, Rogue waves, rational solutions, the patterns of their zeros and integral relations, *J. Phys. A*, **43** (2010) 122002.

[12] A. Ankiewicz, D.J. Kedziora, and N. Akhmediev, Rogue wave triplets, *Phys. Lett. A*, **375** (2011) 2782–2785.

[13] D.J. Benney, Some properties of long nonlinear waves, *Stud. Appl. Math.*, **52** (1973) 45–50.

[14] D.J. Benney, Conservation laws associated with long surface waves, *J. Comput. Appl. Math.*, **190** (2006) 136–141.

[15] D.J. Benney and J.C. Luke, On the interactions of permanent waves of finite amplitude, *J. Math. & Phys. (Stud. Appl. Math.)*, **43** (1964) 309–313.

[16] D.J. Benney and A.C. Newell, The propagation of nonlinear wave envelopes, *J. Math. & Phys. (Stud. Appl. Math.)*, **46** (1967) 133–139.

[17] D.J. Benney and G.J. Roskes, Wave instabilities, *Stud. Appl. Math.*, **48** (1969) 377–385.

[18] L.V. Bogdanov and V.E. Zakharov, The Boussinesq equation revisited, *Physica D*, **165** (2002) 137–162.

[19] J. Boussinesq, Théorie de l'intumescence liquide, appelée onde solitaire ou de translation se propagente dans un canal rectangulaire, *Comptes Rendus*, **72** (1871) 755–759.

[20] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblemant parielles de la surface au fond, *J. Pure Appl.*, **17** (1872) 55–108.

[21] P.A. Clarkson, Rational solutions of the Boussinesq equation, *Anal. Appl.*, **6** (2008) 349–369.

[22] P.A. Clarkson and E. Dowie, Rational solutions of the Boussinesq equation and applications to rogue waves, arXiv:1609.00503.

[23] P.A. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.*, **30** (1989) 2201–2213.

[24] A. Davey and K. Stewartson, On three dimensional packets of surface waves, *Proc. Roy. Soc. London Ser. A*, **338** (1974) 101–110.

[25] P. Deift, C. Tomei, and E. Trubowitz, Inverse scattering and the Boussinesq equation, *Commun. Pure Appl. Math.*, **35** (1982) 567–628.

[26] P. Dubard, P. Gaillard, C. Klein, and V.B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spec. Top.*, **185** (2010) 247–258.

[27] P. Dubard and V.B. Matveev, Multi-rogue waves solutions to the focusing NLS equation and the KP-I equation, *Nat. Hazards Earth. Syst. Sci.*, **11** (2011) 667–672.

[28] P. Dubard and V.B. Matveev, Multi-rogue waves solutions: from the NLS to the KP-I equation, *Nonlinearity*, **26** (2013) R93–R125.

[29] A.S. Fokas and M.J. Ablowitz, On the inverse scattering of the time-dependent Schrödinger equation and the associated Kadomtsev-Petviashvili equation, *Stud. Appl. Math.*, **69** (1983) 211–228.

[30] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A*, **44** (2011) 435204.

[31] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Ann. Phys.*, **367** (2016) 1–5.

[32] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *J. Math. Phys.*, **57** (2016) 063505.

[33] B. Guo, L. Ling and Q. P. Liu, Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions, *Phys. Rev. E*, **85** (2012) 026607.

[34] W. Hereman, P.J. Adams, H.L. Eklund, M.S. Hickman, and B.M. Herbst, Direct methods and symbolic software for conservation laws of nonlinear equations, in "*Advances in Nonlinear Waves and Symbolic Computation*" [Z. Yan, Editor], Nova Science Publishers, New York (2009) pp. 19–79.

[35] R. Hirota, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, *J. Math. Phys.*, **14** (1973) 810–814.

[36] R. Hirota and J. Satsuma, Nonlinear evolution equations generated from the Bäcklund transformation for the Boussinesq equation, *Progr. Theoret. Phys.*, **57** (1997) 797–807.

[37] E. Infeld and G. Rowlands, "*Nonlinear Waves, Solitons and Chaos*," C.U.P., Cambridge (1990).

[38] R.S. Johnson and S. Thompson, A solution of the inverse scattering problem for the Kadomtsev-Petviashvili equation by the method of separation of variables, *Phys. Lett. A*, **66** (1978) 279–281.

[39] B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.*, **15** (1970) 539–541.

[40] D.J. Kedziora, A. Ankiewicz, and N. Akhmediev, Circular rogue wave clusters, *Phys. Rev. E*, **84** (2011) 056611.

[41] D.J. Kedziora, A. Ankiewicz, and N. Akhmediev, Triangular rogue wave cascades, *Phys. Rev. E*, **86** (2012) 056602.

[42] D.J. Kedziora, A. Ankiewicz, and N. Akhmediev, Classifying the hierarchy of nonlinear Schrödinger equation rogue-wave solutions, *Phys. Rev. E*, **88** (2013) 132207.

[43] W.-X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **379** (2015) 1975–1978.

[44] S.V. Manakov, The inverse scattering transform for the time-dependent Schrödinger equation and Kadomtsev-Petviashvili equation, *Physica*, **3D** (1981) 420–427.

[45] S.V. Manakov, V.E. Zakharov, L.A. Bordag, A.R. Its and V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Lett.*, **63A** (1977) 205–206.

[46] Y. Ohta and J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, *Proc. R. Soc. London, Ser. A*, **468** (2012) 1716–1740.

[47] Y. Ohta and J. Yang, Rogue waves in the Davey-Stewartson I equation, *Phys. Rev. E*, **86** (2012) 036604.

[48] Y. Ohta and J. Yang, Dynamics of rogue waves in the Davey-Stewartson II equation, *J. Phys. A*, **46** (2013) 105202.

[49] D.E. Pelinovsky, Rational solutions of the KP hierarchy and the dynamics of their poles. I. New form of a general rational solution, *J. Math. Phys.*, **35** (1994) 5820–5830.

[50] D.E. Pelinovsky, Rational solutions of the KP hierarchy and the dynamics of their poles. II. Construction of the degenerate polynomial solutions, *J. Math. Phys.*, **39** (1998) 5377–5395.

[51] D.E. Pelinovsky and Yu.A. Stepanyants, New multi-soliton solutions of the Kadomtsev-Petviashvili equation, *JETP Lett.*, **57** (1993) 24–28.

[52] J. Satsuma and M.J. Ablowitz, Two-dimensional lumps in non-linear dispersive systems, *J. Math. Phys.*, **20** (1979) 1496–1503.

[53] A.C. Scott, The application of Bäcklund transforms to physical problems, in "*Bäcklund Transformations*" [R.M. Miura, Editor], *Lect. Notes Math.*, vol. **515**, Springer-Verlag, Berlin (1975) pp. 80–105.

[54] H. Segur and A. Finkel, An analytical model of periodic waves in shallow water, *Stud. Appl. Math.*, **73** (1985) 183–220.

[55] N. Singh and Y. Stepanyants, Obliquely propagating skew KP lumps, *Wave Motion*, **64** (2016) 92–102.

[56] R. Sipcic and D.J. Benney, Lump interactions and collapse in the modified Zakharov-Kuznetsov equation, *Stud. Appl. Math.*, **105** (2000) 385–403.

[57] M. Tajiri and Y. Murakami, Rational growing mode: exact solutions to the Boussinesq equation, *J. Phys. Soc. Japan*, **60** (1991) 2791–2792.

[58] M. Tajiri and Y. Watanabe, Periodic wave solutions as imbricate series of rational growing modes: solutions to the Boussinesq equation, *J. Phys. Soc. Japan*, **66** (1997) 1943–1949.

[59] M. Toda, Studies of a nonlinear lattice, *Phys. Rep.*, **8** (1975) 1–125.

[60] F. Ursell, The long-wave paradox in the theory of gravity waves, *Proc. Camb. Phil. Soc.*, **49** (1953) 685–694.

[61] J. Villarroel and M.J. Ablowitz, On the discrete spectrum of the nonstationary Schrödinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation, *Comm. Math. Phys.*, **207** (1999) 1–42.

[62] G.B. Whitham, "*Linear and Nonlinear Waves*," Wiley, New York (1974).

[63] J. Yang and D.J. Benney, Some properties of nonlinear wave systems, *Stud. Appl. Math.*, **96** (1996) 111–139.

[64] N.J. Zabusky, A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, in "*Nonlinear Partial Differential Equations*" [W. F. Ames, Editor], Academic, New York (1967) pp. 233–258.

[65] V.E. Zakharov, On stocastization of one-dimensional chains of nonlinear oscillations, *Sov. Phys. JETP*, **38** (1974) 108–110.

[66] V.E. Zakharov and A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional of waves in nonlinear media, *Sov. Phys. JETP*, **34** (1972) 62–69.